J. Differential Equations 248 (2010) 1711–1755

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Reachability problems for a class of integro-differential equations

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ARTICLE INFO

Article history:
Received 5 June 2009
Revised 21 September 2009
Available online 28 October 2009

MSC:
42A38
45K05
93C20

Keywords:
Convolution kernel
Reachability
Fourier series
Ingham estimates
Haraux estimates
Hilbert Uniqueness Method

ABSTRACT

We study control problems for some integro-differential equations using the Hilbert Uniqueness Method. To do that we follow a harmonic analysis approach. Our results can be applied to concrete examples in viscoelasticity theory.

1. Introduction

In this paper we investigate control problems for a class of integro-differential equations

$$u_{tt}(t,x) - u_{xx}(t,x) + \beta \int_0^t e^{-\eta(t-s)} u_{xx}(s,x)\,ds = 0, \quad t \in (0,T),\ x \in (0,\pi) \tag{1.1}$$

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2009.09.016

$(0 < \beta < \eta)$ with null initial data

$$u(0,x) = u_t(0,x) = 0, \quad x \in (0,\pi), \tag{1.2}$$

and boundary conditions

$$u(t,x) = \begin{cases} 0, & \text{if } x = 0, \\ g(t), & \text{if } x = \pi. \end{cases} \tag{1.3}$$

If we regard g as a control function, our reachability problem consists in proving the existence of $g \in L^2(0,T)$ such that a weak solution of Eq. (1.1), subject to boundary conditions (1.3), moves from the null state to a given one in finite control time. To be more precise, we adopt the same definition of reachability problems for systems with memory given by several authors in the literature, see for example [19,8,9,13,15,16,20,21]. Indeed, we mean the following: given $T > 0$, $u_0 \in L^2(0,\pi)$ and $u_1 \in H^{-1}(0,\pi)$, find $g \in L^2(0,T)$ such that the weak solution u of problem (1.1)–(1.3) verifies the final conditions

$$u(T,x) = u_0(x), \qquad u_t(T,x) = u_1(x), \quad x \in (0,\pi). \tag{1.4}$$

Our goal is to achieve such result without any smallness assumption on the convolution kernel, as suggested by J.-L. Lions in [19, p. 258]. Moreover, due to the finite speed of propagation, we expect that the controllability time T will be sufficiently large. Indeed, we will find that $T > 2\pi/\gamma$, where γ is the gap of a branch of eigenvalues related to the integro-differential operator, see Theorem 6.1.

As it is well-known, a common way for studying exact controllability problems is the so-called Hilbert Uniqueness Method, introduced by Lagnese–Lions, see [12,17–19]. We will apply this method to Eq. (1.1). The HUM method is based on a "uniqueness theorem" for the adjoint problem. To prove such uniqueness theorem we employ some typical techniques of harmonic analysis, see [25]. This approach relies on Fourier series development for the solution v of the adjoint problem, that exhibits an expansion in the variable t like this

$$v(t) = \sum_{n=-\infty}^{\infty} \left(C_n e^{i\omega_n t} + R_n e^{r_n t}\right), \tag{1.5}$$

where $\omega_n, C_n \in \mathbb{C}$ and $r_n, R_n \in \mathbb{R}$. In this framework Ingham type estimates [7] play an important role. We need to establish for functions of the type (1.5) inverse and direct inequalities, obtaining them in the same sharp time of the nonintegral case.

Theorem 1.1. *Let $\{\omega_n\}_{n\in\mathbb{Z}}$ and $\{r_n\}_{n\in\mathbb{Z}}$ be sequences of pairwise distinct numbers such that $r_n \neq i\omega_m$ for any $n, m \in \mathbb{Z}$. Assume that $\{\Im\omega_n\}$, $\{r_n\}$ are bounded sequences and*

$$\Re\omega_n - \Re\omega_{n-1} \geqslant \gamma > 0 \quad \forall |n| \geqslant n',$$

$$\lim_{|n|\to\infty} \Im\omega_n = \alpha, \qquad r_n \leqslant -\Im\omega_n \quad \forall |n| \geqslant n',$$

$$|R_n| \leqslant \frac{\mu}{|n|^\nu}|C_n| \quad \forall |n| \geqslant n', \qquad |R_n| \leqslant \mu|C_n| \quad \forall |n| \leqslant n',$$

for some $n' \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\mu > 0$ and $\nu > 1/2$. Then, for any $T > 2\pi/\gamma$ we have

$$c_1(T) \sum_{n=-\infty}^{\infty} |C_n|^2 \leqslant \int_0^T \left| \sum_{n=-\infty}^{\infty} \left(C_n e^{i\omega_n t} + R_n e^{r_n t}\right)\right|^2 dt,$$

where $c_1(T)$ is a positive constant.

Theorem 1.2. *Assume that $\{\Im\omega_n\}$, $\{r_n\}$ are bounded sequences and*

$$\Re\omega_n - \Re\omega_{n-1} \geqslant \gamma > 0 \quad \forall |n| \geqslant n', \qquad \lim_{|n|\to\infty} \Im\omega_n = \alpha,$$

$$|R_n| \leqslant \frac{\mu}{|n|^{\nu}} |C_n| \quad \forall |n| \geqslant n', \qquad |R_n| \leqslant \mu |C_n| \quad \forall |n| \leqslant n',$$

for some $n' \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\mu > 0$ and $\nu > 1/2$. Then, for any $T > \pi/\gamma$ we have

$$\int_{-T}^{T} \left| \sum_{n=-\infty}^{\infty} \left(C_n e^{i\omega_n t} + R_n e^{r_n t} \right) \right|^2 dt \leqslant c_2(T) \sum_{n=-\infty}^{\infty} |C_n|^2,$$

where $c_2(T)$ is a positive constant.

To prove the previous results, we need Haraux type estimates [6] for functions defined as in (1.5).

Proposition 1.3. *Let $\{\omega_n\}_{n\in\mathbb{Z}}$ be such that $\lim_{|n|\to\infty} |\omega_n| = +\infty$. Assume that $\{\Im\omega_n\}$, $\{r_n\}$ are bounded sequences and there exists a finite set $\mathcal{F}$ of integers such that for any sequences $\{C_n\}$ and $\{R_n\}$ with $C_n = R_n = 0$ for $n \in \mathcal{F}$, the estimates*

$$c_1' \sum_{n\notin\mathcal{F}} |C_n|^2 \leqslant \int_0^T \left| \sum_{n\notin\mathcal{F}} \left(C_n e^{i\omega_n t} + R_n e^{r_n t} \right) \right|^2 dt \leqslant c_2' \sum_{n\notin\mathcal{F}} |C_n|^2$$

are satisfied for some constants $c_1', c_2' > 0$. Then, there exists $c_1 > 0$ such that for any sequences $\{C_n\}$ and $\{R_n\}$ the estimate

$$c_1 \sum_{n=-\infty}^{\infty} |C_n|^2 \leqslant \int_0^T \left| \sum_{n=-\infty}^{\infty} \left(C_n e^{i\omega_n t} + R_n e^{r_n t} \right) \right|^2 dt$$

holds.

Proposition 1.4. *Assume that there exists a finite set $\mathcal{F}$ of integers such that for any sequences $\{C_n\}$ and $\{R_n\}$ with $C_n = R_n = 0$ for $n \in \mathcal{F}$, the estimate*

$$\int_{-T}^{T} \left| \sum_{n\notin\mathcal{F}} \left(C_n e^{i\omega_n t} + R_n e^{r_n t} \right) \right|^2 dt \leqslant c_2' \sum_{n\notin\mathcal{F}} |C_n|^2$$

is satisfied for some $c_2' > 0$. Then, for any sequences $\{C_n\}$ and $\{R_n\}$ verifying

$$|R_n| \leqslant \mu |C_n| \quad \textit{for any } n \in \mathcal{F},$$

for some $\mu > 0$, the estimate

$$\int_{-T}^{T} \left| \sum_{n=-\infty}^{\infty} \left(C_n e^{i\omega_n t} + R_n e^{r_n t} \right) \right|^2 dt \leqslant c_2 \sum_{n=-\infty}^{\infty} |C_n|^2$$

holds for some $c_2 > 0$.

The proofs of these results are rather technical, see Sections 4, 5 and Appendix A. In particular, to prove the inverse inequality we need to introduce a family of operators, which annihilate a finite number of terms in the Fourier series. Our operators are slightly different from those proposed in [6] and [10]. Given $\delta > 0$, $\omega \in \mathbb{C}$ and $r \in \mathbb{R}$ arbitrarily, we define the linear operators $I_{\delta,\omega}$ and $I_{\delta,\omega,r}$ as follows: for every continuous function $u : \mathbb{R} \to \mathbb{C}$ the function $I_{\delta,\omega}u : \mathbb{R} \to \mathbb{C}$ is given by the formula

$$I_{\delta,\omega}u(t) := u(t) - \frac{1}{\delta}\int_0^\delta e^{-i\omega s}u(t+s)\,ds, \quad t \in \mathbb{R},$$

and

$$I_{\delta,\omega,r} := I_{\delta,\omega} \circ I_{\delta,-ir}.$$

In [9] Eq. (1.1) has been studied for more general memory kernels, but the control is put on the whole boundary, so that the abstract results obtained there cannot be directly applied to our problem.

Our non-harmonic analysis techniques, on the other hand, allow us to give precise estimates on controllability time. The inverse and direct inequalities we get (see Theorems 1.1 and 1.2 respectively) are interesting in themselves, because they could also be applied to other examples.

For Ingham's type estimates, our results can be compared with those proved in [20], where functions of the type

$$v(t) = \sum_{n=-\infty}^{\infty} \left(A_n^+ e^{-ia_n^+ t} + A_n^- e^{-ia_n^- t}\right), \quad t \geqslant 0 \tag{1.6}$$

($a_n^+, a_n^- \in \mathbb{R}$, $A_n^+, A_n^- \in \mathbb{C}$) are considered. Our analysis is different from that of [20], because our admissible integral kernels are exponential functions. We have to prove Ingham estimates for functions (1.5) under the assumptions of Theorems 1.1 and 1.2, so they are not of the type (1.6). The main difference is that the exponents in (1.5) have a nonvanishing real part. To overcome that problem, we need two technical results (see Theorems 4.2 and 5.3 below).

Exponential kernels arise in linear viscoelasticity theory, such as in the analysis of Maxwell fluids or Poynting–Thomson solids, see e.g. [22,24]. For other references in viscoelasticity theory see the seminal papers of Dafermos [1,2] and [23,14].

Concerning Haraux's type estimates, in [10] functions of the type

$$v(t) = \sum_{n=-\infty}^{\infty} C_n e^{i\omega_n t}, \quad t \geqslant 0$$

($\omega_n, C_n \in \mathbb{C}$) have been studied.

Our analysis of the estimates changes completely with respect to that of cited papers, because the functions under study are different. Indeed, as we shall see in Section 6, exponential kernels lead to a new form (1.5) of the functions, where the exponents $i\omega_n$ have also a nonvanishing real part and some other real terms $R_n e^{r_n t}$ appear in the sum. Moreover, in the proofs of Ingham estimates the choice of weight function is fundamental and we borrow from [3] the idea of a different weight function with respect to the classical case [7], see also [11]. Other papers related to our problem are [4,16,26,27], where the approach is different to that of Ingham type.

The plan of our paper is the following. In Section 2 we give some preliminary results. In Section 3 we describe the HUM method in an abstract setting. In Section 4 we prove Theorem 1.2 and Proposition 1.4 and in Section 5 we prove Theorem 1.1. In Section 6 we give a reachability result for an integro-differential equation. Finally, in Appendix A we prove some technical results and Proposition 1.3.

2. Preliminaries

Let X be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. For any $T \in (0, \infty]$ we denote by $L^1(0, T; X)$ the usual spaces of measurable functions $v : (0, T) \to X$ such that one has

$$\|v\|_{1,T} := \int_0^T \|v(t)\|\, dt < \infty.$$

We shall use the shorter notation $\|v\|_1$ for $\|v\|_{1,\infty}$. We denote by $L^1_{loc}(0, \infty; X)$ the space of functions belonging to $L^1(0, T; X)$ for any $T \in (0, \infty)$. In the case of $X = \mathbb{R}$, we will use the abbreviations $L^1(0, T)$ and $L^1_{loc}(0, \infty)$ to denote the spaces $L^1(0, T; \mathbb{R})$ and $L^1_{loc}(0, \infty; \mathbb{R})$, respectively.

Classical results for integral equations (see e.g. [5, Theorem 2.3.5]) ensure that, for any kernel $H \in L^1_{loc}(0, \infty)$ and any $g \in L^1_{loc}(0, \infty; X)$, the problem

$$\varphi(t) - H * \varphi(t) = g(t), \quad t \geqslant 0, \tag{2.1}$$

admits a unique solution $\varphi \in L^1_{loc}(0, \infty; X)$. In particular, there is a unique solution $\varrho \in L^1_{loc}(0, \infty)$ of

$$\varrho(t) - H * \varrho(t) = H(t), \quad t \geqslant 0. \tag{2.2}$$

Such a solution is called the *resolvent kernel* of H. Furthermore, the solution φ of (2.1) is given by the variation of constants formula

$$\varphi(t) = g(t) + \varrho * g(t), \quad t \geqslant 0, \tag{2.3}$$

where ϱ is the resolvent kernel of H.

Lemma 2.1. *Given $H \in L^1_{loc}(0, \infty)$ and $g \in L^1_{loc}(-\infty, T; X)$, a function $f \in L^1_{loc}(-\infty, T; X)$ is a solution of*

$$f(t) - \int_t^T H(s - t) f(s)\, ds = g(t), \quad t \leqslant T, \tag{2.4}$$

if and only if

$$f(t) = g(t) + \int_t^T \varrho(s - t) g(s)\, ds \quad t \leqslant T, \tag{2.5}$$

where ϱ is the resolvent kernel of H.

Proof. If f is a solution of (2.4), then, substituting t with $T - \tau$, $\tau \geqslant 0$, we get

$$f(T - \tau) - \int_{T-\tau}^T H(s - T + \tau) f(s)\, ds = g(T - \tau), \quad \tau \geqslant 0.$$

Set $p(\tau) = f(T - \tau)$ and $q(\tau) = g(T - \tau)$, we have

$$p(\tau) - \int_0^\tau H(\tau - s) p(s)\, ds = q(\tau), \quad \tau \geqslant 0.$$

Thanks to (2.3) one gets

$$p(\tau)=q(\tau)+\int_0^\tau \varrho(\tau-s)q(s)\,ds,$$

where ϱ is the resolvent kernel of H. Recalling that $p(\tau)=f(T-\tau)$ and $q(\tau)=g(T-\tau)$, we have

$$\begin{aligned} f(T-\tau)&=g(T-\tau)+\int_0^\tau \varrho(\tau-s)g(T-s)\,ds\\ &=g(T-\tau)+\int_{T-\tau}^{T} \varrho(\tau-T+s)g(s)\,ds,\quad \tau\geqslant 0. \end{aligned}$$

Finally, substituting $T-\tau$ with t, $t\leqslant T$, we obtain

$$f(t)=g(t)+\int_t^T \varrho(s-t)g(s)\,ds,$$

that is (2.5) holds true.

Repeating the reasoning backward, we have that if f verifies (2.5), then (2.4) is satisfied. □

Corollary 2.2. *The following are true.*

(i) *The resolvent kernel of $t\mapsto \beta e^{-\eta t}$ is $t\mapsto \beta e^{(\beta-\eta)t}$.*
(ii) *Given $g\in L^1_{loc}(-\infty,T;X)$, a function $f\in L^1_{loc}(-\infty,T;X)$ is a solution of*

$$f(t)-\beta\int_t^T e^{-\eta(s-t)}f(s)\,ds=g(t),\quad t\leqslant T,$$

if and only if

$$f(t)=g(t)+\beta\int_t^T e^{(\beta-\eta)(s-t)}g(s)\,ds,\quad t\leqslant T.$$

Proof. (i) The resolvent kernel of $t\mapsto \beta e^{-\eta t}$ is the solution of the integral equation

$$\varrho(t)-\beta\int_0^t e^{-\eta(t-s)}\varrho(s)\,ds=\beta e^{-\eta t},\quad t\geqslant 0,$$

whence, multiplying by $e^{\eta t}$, we obtain

$$e^{\eta t}\varrho(t)=\beta+\beta\int_0^t e^{\eta s}\varrho(s)\,ds, \quad t\geqslant 0.$$

Differentiating yields

$$\begin{cases} \dfrac{d}{dt}\big[e^{\eta t}\varrho(t)\big]=\beta e^{\eta t}\varrho(t), \quad t\geqslant 0,\\ \varrho(0)=\beta. \end{cases}$$

Solving the above Cauchy problem gives

$$e^{\eta t}\varrho(t)=\beta e^{\beta t},$$

whence, multiplying by $e^{-\eta t}$, one gets

$$\varrho(t)=\beta e^{(\beta-\eta)t}.$$

The point (ii) follows from Lemma 2.1. □

Lemma 2.3. *Given $\lambda,\beta,\eta\in\mathbb{R}$, a function $f\in C^2([0,\infty))$ is a solution of the integro-differential equation*

$$f''(t)+\lambda f(t)-\lambda\beta\int_0^t e^{-\eta(t-s)}f(s)\,ds=0, \quad t\geqslant 0, \tag{2.6}$$

if and only if f is a solution of the problem

$$\begin{cases} f'''(t)+\eta f''(t)+\lambda f'(t)+\lambda(\eta-\beta)f(t)=0, \quad t\geqslant 0,\\ f''(0)=-\lambda f(0). \end{cases} \tag{2.7}$$

Proof. Let f be a solution of (2.6). It follows that $f''(0)+\lambda f(0)=0$ and $f\in C^3([0,\infty))$. Differentiating (2.6), we get

$$f'''(t)+\lambda f'(t)+\eta\lambda\beta\int_0^t e^{-\eta(t-s)}f(s)\,ds-\lambda\beta f(t)=0.$$

Substituting in the above equation the identity

$$\lambda\beta\int_0^t e^{-\eta(t-s)}f(s)\,ds=f''(t)+\lambda f(t),$$

we obtain

$$f'''(t)+\lambda f'(t)+\eta f''(t)+\eta\lambda f(t)-\lambda\beta f(t)=0,$$

whence f is a solution of (2.7).

On the other hand, if f is a solution of (2.7), multiplying the differential equation in (2.7) by $e^{\eta t}$ and integrating from 0 to t, we obtain

$$\int_0^t e^{\eta s} f'''(s)\,ds + \eta \int_0^t e^{\eta s} f''(s)\,ds + \lambda \int_0^t e^{\eta s} f'(s)\,ds + \lambda(\eta - \beta) \int_0^t e^{\eta s} f(s)\,ds = 0.$$

Integrating by parts the first term and the third one, we have

$$e^{\eta t} f''(t) - f''(0) + \lambda e^{\eta t} f(t) - \lambda f(0) - \lambda\beta \int_0^t e^{\eta s} f(s)\,ds = 0.$$

Using $f''(0) = -\lambda f(0)$ and multiplying by $e^{-\eta t}$, we obtain (2.6). $\square$

It is easy to verify the following result.

Lemma 2.4. *The third degree polynomial*

$$F(t) := -32t^3 + 108t^2 - \frac{243}{2}t + \frac{729}{16} \tag{2.8}$$

is strictly decreasing in $[0, \infty)$. *Moreover, the unique real zero of* $F(t)$ *is* $\frac{9}{8}$.

3. Hilbert Uniqueness Method

In this section we formally describe the method in an abstract setting.

We introduce a linear operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ on X with domain $D(\mathcal{A})$ and $H \in L^1_{loc}(0, \infty)$. Let Y be another real Hilbert space with scalar product $\langle \cdot, \cdot \rangle_Y$ and norm $\| \cdot \|_Y$ and $\mathcal{B} \in L(X_0; Y)$, where X_0 is a space such that $D(\mathcal{A}) \subset X_0 \subset X$. We consider the integro-differential equation

$$u''(t) + \mathcal{A}u(t) - \int_0^t H(t-s)\mathcal{A}u(s)\,ds = 0, \quad t \in (0, T), \tag{3.1}$$

with null initial conditions

$$u(0) = u'(0) = 0, \tag{3.2}$$

and

$$\mathcal{B}u(t) = g(t), \quad t \in (0, T). \tag{3.3}$$

In the applications $\mathcal{B}$ can be, for example, a trace operator. For a reachability problem we mean the following: given $T > 0$, $u_0 \in X$ and $u_1 \in (Ker(\mathcal{B}))'$, find $g \in L^2(0, T; Y)$ such that the weak solution u of problem (3.1)–(3.3) verifies the final conditions

$$u(T) = u_0, \qquad u'(T) = u_1. \tag{3.4}$$

To explain how the HUM method can be used to solve a reachability problem, we proceed dividing the reasoning into several steps.

Step 1. $A : D(A) \subset X \to X$ denotes a self-adjoint positive linear operator on X with dense domain $D(A) \subset D(\mathcal{A})$ such that for any $x \in D(A)$ $\mathcal{A}x = Ax$ and $D(\sqrt{A}) = Ker(\mathcal{B})$. We define by induction

$$D\big(A^k\big) := \big\{x \in D\big(A^{k-1}\big):\ A^{k-1}x \in D(A)\big\}, \quad k \in \mathbb{N}.$$

Given $z_0 \in D(A^k)$ and $z_1 \in D(A^k)$, we consider the *adjoint* equation of (3.1), that is

$$z''(t) + Az(t) - \int_t^T H(s-t)Az(s)\,ds = 0, \quad t \in [0,T], \tag{3.5}$$

with final data

$$z(T) = z_0, \qquad z'(T) = z_1. \tag{3.6}$$

Problem (3.5)–(3.6) admits a unique solution $z \in C^{k-j}([0,T]; D(A^j))$, $j = 0, 1, \ldots, k$. Indeed, set $v(t) = z(T-t)$, problem (3.5)–(3.6) is equivalent to

$$\begin{cases} v''(t) + Av(t) - \displaystyle\int_0^t H(t-s)Av(s)\,ds = 0, & t \in [0,T], \\ v(0) = z_0, \quad v'(0) = -z_1, \end{cases} \tag{3.7}$$

and the above problem is well-posed, see e.g. [22]. We take k large enough to have the function z sufficiently regular.

Step 2. We introduce another operator $D_\nu : X_0 \to Y$ such that the following identity holds

$$\langle \mathcal{A}\varphi, \xi\rangle = \langle \varphi, A\xi\rangle - \langle \mathcal{B}\varphi, D_\nu \xi\rangle_Y, \quad \forall \varphi \in D(\mathcal{A}),\ \xi \in D(A), \tag{3.8}$$

and the problem

$$\begin{cases} \phi''(t) + \mathcal{A}\phi(t) - \displaystyle\int_0^t H(t-s)\mathcal{A}\phi(s)\,ds = 0, & t \in [0,T], \\ \mathcal{B}\phi(t) = D_\nu z(t) - \displaystyle\int_t^T H(s-t)D_\nu z(s)\,ds, & t \in [0,T], \\ \phi(0) = \phi'(0) = 0, \end{cases} \tag{3.9}$$

admits a unique solution ϕ. Then, we define the linear operator

$$\Psi(z_0, z_1) = \big(\phi'(T), -\phi(T)\big), \quad (z_0, z_1) \in D\big(A^k\big) \times D\big(A^k\big).$$

Step 3. Let $(\xi_0, \xi_1) \in D(A^k) \times D(A^k)$ and ξ the solution of

$$\begin{cases} \xi''(t) + A\xi(t) - \displaystyle\int_t^T H(s-t)A\xi(s)\,ds = 0, & t \in [0,T], \\ \xi(T) = \xi_0, \quad \xi'(T) = \xi_1. \end{cases} \tag{3.10}$$

We prove that

$$\begin{aligned}&\big\langle \Psi(z_0, z_1), (\xi_0, \xi_1)\big\rangle_{X\times X}\\&\quad = \int_0^T \bigg\langle \mathcal{B}\phi(t), D_\nu \xi(t) - \int_t^T H(s-t) D_\nu \xi(s)\,ds\bigg\rangle_Y dt.\end{aligned} \tag{3.11}$$

Indeed, multiplying the equation in (3.9) by $\xi(t)$ and integrating on $[0, T]$ we have

$$\int_0^T \big\langle \phi''(t), \xi(t)\big\rangle dt + \int_0^T \big\langle \mathcal{A}\phi(t), \xi(t)\big\rangle dt - \int_0^T \int_0^t H(t-s)\big\langle \mathcal{A}\phi(s),\ \xi(t)\big\rangle ds\,dt = 0.$$

Integrating by parts twice, in view also of (3.8) we have

$$\begin{aligned}&\big\langle \phi'(T), \xi(T)\big\rangle - \big\langle \phi(T), \xi'(T)\big\rangle + \int_0^T \bigg\langle \phi(t), \xi''(t) + A\xi(t) - \int_t^T H(s-t) A\xi(s)\,ds\bigg\rangle dt\\&\quad - \int_0^T \big\langle \mathcal{B}\phi(t), D_\nu \xi(t)\big\rangle_Y dt + \int_0^T \bigg\langle \mathcal{B}\phi(t), \int_t^T H(s-t) D_\nu \xi(s)\,ds\bigg\rangle_Y dt = 0.\end{aligned}$$

Since ξ is the solution of (3.10), we have that (3.11) holds.
Now, taking $(\xi_0, \xi_1) = (z_0, z_1)$ in (3.11), we have

$$\big\langle \Psi(z_0, z_1), (z_0, z_1)\big\rangle_{X\times X} = \int_0^T \bigg\| D_\nu z(t) - \int_t^T H(s-t) D_\nu z(s)\,ds\bigg\|_Y^2 dt. \tag{3.12}$$

So, we can introduce the semi-norm

$$\big\|(z_0, z_1)\big\|_F := \left(\int_0^T \bigg\| D_\nu z(t) - \int_t^T H(s-t) D_\nu z(s)\,ds\bigg\|_Y^2 dt\right)^{1/2} \tag{3.13}$$

for any $(z_0, z_1) \in D(A^k) \times D(A^k)$.

Step 4. In view of Lemma 2.1, $\|\cdot\|_F$ is a norm if and only if the following uniqueness theorem holds.

Theorem 3.1. *If z is the solution of problem* (3.5)–(3.6) *such that*

$$D_\nu z(t) = 0 \quad \forall t \in [0, T],$$

then

$$z(t) = 0 \quad \forall t \in [0, T].$$

If Theorem 3.1 holds true, then we can define the Hilbert space F as the completion of $D(A^k) \times D(A^k)$ for the norm (3.13). Moreover, the operator Ψ extends uniquely to a continuous operator, denoted again by Ψ, from F to the dual space F' in such a way that $\Psi : F \to F'$ is an isomorphism.

In conclusion, if we prove a result similar to Theorem 3.1 and $F = D(\sqrt{A}) \times X$, then we can solve the reachability problem (3.1)–(3.4).

4. Ingham type direct inequality

In this section, we consider functions of the type

$$f(t) := \sum_{n=-\infty}^{\infty} \left(C_n e^{i\omega_n t} + R_n e^{r_n t}\right), \quad t \geqslant 0,$$

with $\omega_n, C_n \in \mathbb{C}$ and $r_n, R_n \in \mathbb{R}$ such that the sequences $\{\Im\omega_n\}$, $\{r_n\}$ are bounded and

$$\sum_{n=-\infty}^{\infty} |C_n|^2 < +\infty, \qquad \sum_{n=-\infty}^{\infty} |R_n|^2 < +\infty.$$

Let $T > 0$.

Theorem 4.1. *Assume*

$$\Re\omega_n - \Re\omega_{n-1} \geqslant \gamma > 0 \quad \forall |n| \geqslant n', \tag{4.1}$$

$$\lim_{|n|\to\infty} \Im\omega_n = \alpha, \tag{4.2}$$

$$|R_n| \leqslant \frac{\mu}{|n|^\nu}|C_n| \quad \forall |n| \geqslant n', \qquad |R_n| \leqslant \mu|C_n| \quad \forall |n| \leqslant n', \tag{4.3}$$

for some $n' \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\mu > 0$ and $\nu > 1/2$. Then, for any $T > \pi/\gamma$ we have

$$\int_{-T}^{T} \left| \sum_{n=-\infty}^{\infty} \left(C_n e^{i\omega_n t} + R_n e^{r_n t}\right)\right|^2 dt \leqslant c_2(T) \sum_{n=-\infty}^{\infty} |C_n|^2, \tag{4.4}$$

where $c_2(T)$ is a positive constant.

To proceed with the proof, we state the following two results, but the proof of the first one can be found in Appendix A, as it is quite long and complex.

Theorem 4.2. *Under assumptions* (4.1)–(4.3)*, for any $0 < \varepsilon < 1$ and for any $T > \frac{\pi}{\gamma\sqrt{1-\varepsilon}}$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that if $C_n = 0$ for $|n| \leqslant n_0$, then we have*

$$\int_{-T}^{T} \left|f(t)\right|^2 dt \leqslant c_2(T) \sum_{|n|\geqslant n_0} |C_n|^2, \tag{4.5}$$

where $c_2(T) > 0$.

Proposition 4.3. *Assume that there exists a finite set $\mathcal{F}$ of integers such that for any sequences $\{C_n\}$ and $\{R_n\}$ verifying*

$$C_n = R_n = 0 \quad \text{for any } n \in \mathcal{F}, \tag{4.6}$$

the estimate

$$\int_{-T}^{T} \Big| \sum_{n \notin \mathcal{F}} \big(C_n e^{i\omega_n t} + R_n e^{r_n t} \big) \Big|^2 \, dt \leqslant c_2' \sum_{n \notin \mathcal{F}} |C_n|^2 \tag{4.7}$$

is satisfied for some $c_2' > 0$. Then, for any sequences $\{C_n\}$ and $\{R_n\}$ verifying

$$|R_n| \leqslant \mu |C_n| \quad \text{for any } n \in \mathcal{F}, \tag{4.8}$$

for some $\mu > 0$, the estimate

$$\int_{-T}^{T} \Big| \sum_{n=-\infty}^{\infty} \big(C_n e^{i\omega_n t} + R_n e^{r_n t} \big) \Big|^2 \, dt \leqslant c_2 \sum_{n=-\infty}^{\infty} |C_n|^2 \tag{4.9}$$

holds for some $c_2 > 0$.

Proof. Assume that $\{C_n\}$ and $\{R_n\}$ verify (4.8). If we use (4.7), then we have

$$\int_{-T}^{T} \Big| \sum_{n \notin \mathcal{F}} \big(C_n e^{i\omega_n t} + R_n e^{r_n t} \big) \Big|^2 \, dt \leqslant c_2' \sum_{n \notin \mathcal{F}} |C_n|^2. \tag{4.10}$$

Now, we prove that

$$\int_{-T}^{T} \Big| \sum_{n \in \mathcal{F}} \big(C_n e^{i\omega_n t} + R_n e^{r_n t} \big) \Big|^2 \, dt \leqslant c_2'' \sum_{n \in \mathcal{F}} |C_n|^2, \tag{4.11}$$

for some constant $c_2'' > 0$. Indeed, applying the Cauchy–Schwarz inequality we get

$$\begin{aligned} \Big| \sum_{n \in \mathcal{F}} \big(C_n e^{i\omega_n t} + R_n e^{r_n t} \big) \Big|^2 &\leqslant \Big(\sum_{n \in \mathcal{F}} \big(|C_n| e^{-\Im\omega_n t} + |R_n| e^{r_n t} \big) \Big)^2 \\ &\leqslant 2|\mathcal{F}| \sum_{n \in \mathcal{F}} \big(|C_n|^2 e^{-2\Im\omega_n t} + |R_n|^2 e^{2r_n t} \big), \end{aligned}$$

where $|\mathcal{F}|$ denotes the number of elements in the set $\mathcal{F}$. If we use the previous inequality and (4.8), then we get

$$\int_{-T}^{T} \Big| \sum_{n \in \mathcal{F}} \big(C_n e^{i\omega_n t} + R_n e^{r_n t} \big) \Big|^2 \, dt \leqslant 2|\mathcal{F}| \sum_{n \in \mathcal{F}} |C_n|^2 \int_{-T}^{T} \big(e^{-2\Im\omega_n t} + \mu^2 e^{2r_n t} \big) \, dt,$$

whence (4.11) follows with $c_2'' = 2|\mathcal{F}| \max_{n \in \mathcal{F}} \{ \int_{-T}^{T} (e^{-2\Im\omega_n t} + \mu^2 e^{2r_n t}) \, dt \}$.

Finally, from (4.10) and (4.11) we deduce that

$$\int_{-T}^{T}\left|\sum_{n=-\infty}^{\infty}\left(C_n e^{i\omega_n t}+R_n e^{r_n t}\right)\right|^2 dt$$

$$\leqslant 2\int_{-T}^{T}\left|\sum_{n\notin\mathcal{F}}\left(C_n e^{i\omega_n t}+R_n e^{r_n t}\right)\right|^2 dt+2\int_{-T}^{T}\left|\sum_{n\in\mathcal{F}}\left(C_n e^{i\omega_n t}+R_n e^{r_n t}\right)\right|^2 dt$$

$$\leqslant 2c_2'\sum_{n\notin\mathcal{F}}|C_n|^2+2c_2''\sum_{n\in\mathcal{F}}|C_n|^2,$$

so (4.9) holds with $c_2=2\max\{c_2',c_2''\}$. □

Proof of Theorem 4.1. Since $T>\pi/\gamma$, there exists $0<\varepsilon<1$ such that $T>\frac{\pi}{\gamma\sqrt{1-\varepsilon}}$. By applying Theorem 4.2, there exist $n_0\in\mathbb{N}$ and $c_2(T)>0$ such that if $C_n=0$ for $|n|\leqslant n_0$, then we have

$$\int_{-T}^{T}\left|f(t)\right|^2 dt\leqslant c_2(T)\sum_{|n|\geqslant n_0}|C_n|^2.$$

Finally, thanks also to (4.3) we can use Proposition 4.3 to conclude. □

5. Ingham type inverse inequality

Again in this section, we consider functions of the type

$$f(t):=\sum_{n=-\infty}^{\infty}\left(C_n e^{i\omega_n t}+R_n e^{r_n t}\right),\quad t\geqslant 0,$$

with $\omega_n, C_n\in\mathbb{C}$ and $r_n, R_n\in\mathbb{R}$ such that the sequences $\{\Im\omega_n\}$, $\{r_n\}$ are bounded and

$$\sum_{n=-\infty}^{\infty}|C_n|^2<+\infty,\qquad \sum_{n=-\infty}^{\infty}|R_n|^2<+\infty.$$

In addition, the sequences $\{\omega_n\}_{n\in\mathbb{Z}}$ and $\{r_n\}_{n\in\mathbb{Z}}$ are composed of pairwise distinct numbers such that $r_n\neq i\omega_m$ for any $n,m\in\mathbb{Z}$.

Let $T>0$.

Theorem 5.1. *Assume*

$$\Re\omega_n-\Re\omega_{n-1}\geqslant\gamma>0\quad\forall|n|\geqslant n',\tag{5.1}$$

$$\lim_{|n|\to\infty}\Im\omega_n=\alpha,\qquad r_n\leqslant-\Im\omega_n\quad\forall|n|\geqslant n',\tag{5.2}$$

$$|R_n|\leqslant\frac{\mu}{|n|^{\nu}}|C_n|\quad\forall|n|\geqslant n',\qquad |R_n|\leqslant\mu|C_n|\quad\forall|n|\leqslant n',\tag{5.3}$$

for some $n' \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\mu > 0$ and $\nu > 1/2$. Then, for any $T > 2\pi/\gamma$ we have

$$c_1(T) \sum_{n=-\infty}^{\infty} |C_n|^2 \leqslant \int_0^T \left| \sum_{n=-\infty}^{\infty} \left(C_n e^{i\omega_n t} + R_n e^{r_n t}\right)\right|^2 dt, \tag{5.4}$$

where $c_1(T)$ is a positive constant.

Remark 5.2. Since the sequence $\{\Im\omega_n\}$ is bounded the inverse inequality (5.4) can be written in the form

$$c_1(T) \sum_{n=-\infty}^{\infty} \left(1 + e^{-2(\Im\omega_n - \alpha)T}\right)|C_n|^2 \leqslant \int_0^T \left| \sum_{n=-\infty}^{\infty} \left(C_n e^{i\omega_n t} + R_n e^{r_n t}\right)\right|^2 dt,$$

which is similar to that proved in [27, Lemma 4.1] by different techniques.

We note that the direct inequality holds under weaker assumptions respect to the inverse one.

To prove Theorem 5.1, we need the following results, whose proofs are given in Appendix A, as they are quite long and complex.

Theorem 5.3. *Under assumptions* (5.1)–(5.3)*, for any $0 < \varepsilon < 1$ and $T > \frac{2\pi}{\gamma}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}$ there exist $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $c_1(T, \varepsilon) > 0$ such that if $C_n = 0$ for any $|n| \leqslant n_0$, then we have*

$$c_1(T, \varepsilon) \sum_{|n| \geqslant n_0} \left(1 + e^{-2(\Im\omega_n - \alpha)T}\right)|C_n|^2 \leqslant \int_0^T \left|f(t)\right|^2 dt. \tag{5.5}$$

In addition, the constant $c_1(T, \varepsilon)$ is given by

$$c_1(T, \varepsilon) = \min\left(1, e^{-2\alpha T}\right)\left(\frac{T\pi}{\pi^2 + T^2\gamma^2\varepsilon/8} - \frac{4\pi}{T\gamma^2}(1+\varepsilon)\right).$$

Proposition 5.4. *Let $\{\omega_n\}_{n\in\mathbb{Z}}$ be such that*

$$\lim_{|n|\to\infty} |\omega_n| = +\infty. \tag{5.6}$$

Assume that there exists a finite set $\mathcal{F}$ of integers such that for any sequences $\{C_n\}$ and $\{R_n\}$ verifying

$$C_n = R_n = 0 \quad \text{for any } n \in \mathcal{F}, \tag{5.7}$$

the estimates

$$c_1' \sum_{n\notin\mathcal{F}} |C_n|^2 \leqslant \int_0^T \left| \sum_{n\notin\mathcal{F}} \left(C_n e^{i\omega_n t} + R_n e^{r_n t}\right)\right|^2 dt, \tag{5.8}$$

$$\int_0^T \left| \sum_{n\notin\mathcal{F}} \left(C_n e^{i\omega_n t} + R_n e^{r_n t}\right)\right|^2 dt \leqslant c_2' \sum_{n\notin\mathcal{F}} |C_n|^2 \tag{5.9}$$

are satisfied for some constants $c_1', c_2' > 0$. Then, there exists $c_1 > 0$ such that for any sequences $\{C_n\}$ and $\{R_n\}$ the estimate

$$c_1 \sum_{n=-\infty}^{\infty} |C_n|^2 \leqslant \int_0^T \left| \sum_{n=-\infty}^{\infty} \left(C_n e^{i\omega_n t} + R_n e^{r_n t}\right) \right|^2 dt \tag{5.10}$$

holds.

Proof of Theorem 5.1. Since $T > 2\pi/\gamma$, there exists $0 < \varepsilon < 1$ such that $T > \frac{2\pi}{\gamma}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}$. By applying Theorems 5.3 and 4.1, there exist $n_0 \in \mathbb{N}$, $c_1(T, \varepsilon) > 0$ and $c_2(T) > 0$ such that if $C_n = 0$ for $|n| \leqslant n_0$, then we have

$$c_1(T,\varepsilon) \sum_{|n| \geqslant n_0} |C_n|^2 \leqslant \int_0^T \left|f(t)\right|^2 dt \leqslant c_2(T) \sum_{|n| \geqslant n_0} |C_n|^2.$$

Finally, we can use Proposition 5.4 to conclude. □

6. A reachability result

Before giving the result announced in the introduction concerning reachability problems for a class of systems with memory, we first need to develop a detailed spectral analysis.

Let $A : D(A) \subset X \to X$ be a self-adjoint positive linear operator on X with dense domain $D(A)$ and let $\{\lambda_j\}_{j\geqslant 1}$ be a strictly increasing sequence of eigenvalues for the operator A with $\lambda_j > 0$ and $\lambda_j \to \infty$ such that the sequence of the corresponding eigenvectors $\{w_j\}_{j\geqslant 1}$ constitutes a Hilbert basis for X.

For any $v_0 \in D(\sqrt{A})$ and $v_1 \in X$ there exists a unique weak solution v belonging to $C([0,\infty); D(\sqrt{A})) \cap C^1([0,\infty); X)$ of equation

$$v''(t) + Av(t) - \beta \int_0^t e^{-\eta(t-s)} Av(s)\, ds = 0, \quad t \geqslant 0, \tag{6.1}$$

verifying the initial conditions

$$v(0) = v_0, \qquad v'(0) = v_1. \tag{6.2}$$

We have

$$v_0 = \sum_{j=1}^{\infty} \alpha_j w_j, \qquad \alpha_j = \langle v_0, w_j\rangle, \qquad \sum_{j=1}^{\infty} \alpha_j^2 \lambda_j < \infty, \tag{6.3}$$

$$v_1 = \sum_{j=1}^{\infty} \gamma_j w_j, \qquad \gamma_j = \langle v_1, w_j\rangle, \qquad \sum_{j=1}^{\infty} \gamma_j^2 < \infty. \tag{6.4}$$

First, we observe that we can approximate the initial data v_0 and v_1 by sequences $\{v_{0n}\}$ in $D(A)$ and $\{v_{1n}\}$ in $D(\sqrt{A})$ respectively. So, the sequence of strong solutions $v_n(t)$ of (6.1), corresponding to the initial conditions v_{0n} and v_{1n}, approximates $v(t)$. Thanks to this remark, we can make our computations considering $v(t)$ as a strong solution, and then we go back to weak solutions by standard approximation arguments.

We want to write the solution $v(t)$ as a sum of series, that is

$$v(t)=\sum_{j=1}^{\infty} f_j(t)w_j, \qquad f_j(t)=\langle v(t), w_j\rangle.$$

Substituting the above expression of v in (6.1) and multiplying the equation by w_j, $j\in\mathbb{N}$, we have that $f_j(t)$ is the solution of

$$f_j''(t)+\lambda_j f_j(t)-\lambda_j\beta\int_0^t e^{-\eta(t-s)} f_j(s)\,ds=0, \tag{6.5}$$

with initial conditions given by

$$f_j(0)=\alpha_j \qquad f_j'(0)=\gamma_j. \tag{6.6}$$

Thanks to Lemma 2.3, problem (6.5)–(6.6) is equivalent to the Cauchy problem

$$\begin{cases} f_j'''(t)+\eta f_j''(t)+\lambda_j f_j'(t)+\lambda_j(\eta-\beta) f_j(t)=0, & t\geqslant 0,\\ f_j(0)=\alpha_j, \quad f_j'(0)=\gamma_j, \quad f_j''(0)=-\lambda_j\alpha_j. \end{cases} \tag{6.7}$$

Therefore, we proceed to solve (6.7). To this end, we must compute the solutions of the characteristic equation

$$\Lambda^3+\eta\Lambda^2+\lambda_j\Lambda+\lambda_j(\eta-\beta)=0, \tag{6.8}$$

following the well-known Scipione Del Ferro's method to obtain the Cardano formula.

First, we transform equation (6.8) into one without second degree term. For this reason, we will make a suitable change of variable. Indeed, set

$$\Lambda=\sigma-\frac{\eta}{3},$$

we have

$$\sigma^3+p_j\sigma+q_j=0, \tag{6.9}$$

where

$$p_j=\lambda_j-\frac{\eta^2}{3}, \qquad q_j=\frac{2}{27}\eta^3+2\left(\frac{\eta}{3}-\frac{\beta}{2}\right)\lambda_j.$$

To solve (6.9), we look for solutions in the form

$$\sigma=y+z.$$

We observe that the cube of $\sigma=y+z$ satisfies the following equation

$$\sigma^3-3yz\sigma-\left(y^3+z^3\right)=0. \tag{6.10}$$

Equalling the coefficients of similar terms in Eqs. (6.9) and (6.10), we have

$$yz = -p_j/3, \qquad y^3 + z^3 = -q_j.$$

Since $y^3z^3 = -p_j^3/27$ and $y^3 + z^3 = -q_j$, it follows that y^3 and z^3 are solutions of the second degree equation

$$r^2 + q_j r - \frac{p_j^3}{27} = 0. \tag{6.11}$$

Now, defining the discriminant Δ_j of Eq. (6.9) as the $\frac{1}{4}$-discriminant of the above equation, that is

$$\Delta_j := \frac{q_j^2}{4} + \frac{p_j^3}{27},$$

we note that

$$\frac{q_j^2}{4} = \frac{\eta^6}{(27)^2} + \left(\frac{\eta}{3} - \frac{\beta}{2}\right)^2 \lambda_j^2 + \frac{\eta^3}{27}\left(\frac{2}{3}\eta - \beta\right)\lambda_j,$$

$$\frac{p_j^3}{27} = \frac{\lambda_j^3}{27} - \frac{\eta^2}{27}\lambda_j^2 + \frac{\eta^4}{81}\lambda_j - \frac{\eta^6}{(27)^2},$$

so we have

$$\Delta_j = \frac{\lambda_j}{27}\left(\lambda_j^2 + \left(2\eta^2 - 9\eta\beta + \frac{27}{4}\beta^2\right)\lambda_j + \eta^3(\eta - \beta)\right). \tag{6.12}$$

Now, to have $\Delta_j > 0$ it is sufficient that $(2\eta^2 - 9\eta\beta + \frac{27}{4}\beta^2)^2 - 4\eta^3(\eta - \beta) < 0$, that is

$$F\left(\frac{\eta}{\beta}\right) = -32\left(\frac{\eta}{\beta}\right)^3 + 108\left(\frac{\eta}{\beta}\right)^2 - \frac{243\eta}{2\beta} + \frac{729}{16} < 0,$$

where F is the polynomial defined in (2.8). Thanks to Lemma 2.4 the above condition is satisfied for $\eta > \frac{9}{8}\beta$, and hence $\Delta_j > 0$ for $\eta > \frac{9}{8}\beta$.

If $\beta < \eta \leqslant \frac{9}{8}\beta$, then we can write $\eta = t\beta$, with $1 < t \leqslant \frac{9}{8}$. So, we have $\Delta_j > 0$ for $\lambda_j > \beta^2(9t - 2t^2 - \frac{27}{4} + F(t)^{1/2})/2$. Since $9t - 2t^2 - \frac{27}{4} > 0$ for $1 < t \leqslant \frac{9}{8}$, we get $\Delta_j > 0$ if

$$\beta < \left(\frac{2\lambda_1}{9t - 2t^2 - \frac{27}{4} + \left(-32t^3 + 108t^2 - \frac{243}{2}t + \frac{729}{16}\right)^{1/2}}\right)^{1/2}. \tag{6.13}$$

Therefore, the solutions of Eq. (6.11) are given by

$$r_{1/2} = -\frac{q_j}{2} \pm \sqrt{\frac{q_j^2}{4} + \frac{p_j^3}{27}}.$$

Now, to write the solutions $\sigma = y + z$ of (6.9), we keep in mind not only that y^3 and z^3 are solutions of (6.11), but also that y and z must satisfy the condition $yz = -p_j/3$. Accordingly, if we consider the following real numbers,

$$y_j = \left(-\frac{q_j}{2} + \sqrt{\frac{q_j^2}{4} + \frac{p_j^3}{27}}\right)^{1/3}, \qquad z_j = \left(-\frac{q_j}{2} - \sqrt{\frac{q_j^2}{4} + \frac{p_j^3}{27}}\right)^{1/3},$$

then the solutions of (6.9) are given by

$$\sigma_{1,j} = y_j + z_j, \tag{6.14}$$

$$\sigma_{2,j} = y_j e^{2\pi i/3} + z_j e^{-2\pi i/3} = -\frac{1}{2}(y_j + z_j) + i\frac{\sqrt{3}}{2}(y_j - z_j), \tag{6.15}$$

$$\sigma_{3,j} = y_j e^{-2\pi i/3} + z_j e^{2\pi i/3} = -\frac{1}{2}(y_j + z_j) - i\frac{\sqrt{3}}{2}(y_j - z_j). \tag{6.16}$$

We note that the numbers $\sigma_{1,j}, \sigma_{2,j}, \sigma_{3,j}$ are all distinct.

Now, in view of (6.12) we evaluate the quantity

$$\begin{aligned}
&-\frac{q_j}{2} + \sqrt{\frac{q_j^2}{4} + \frac{p_j^3}{27}} \\
&= -\frac{\eta^3}{27} - \left(\frac{\eta}{3} - \frac{\beta}{2}\right)\lambda_j + \sqrt{\frac{\lambda_j^3}{27} + \left(\frac{2}{27}\eta^2 + \frac{\beta^2}{4} - \frac{\eta\beta}{3}\right)\lambda_j^2 + \frac{\eta^3}{27}(\eta - \beta)\lambda_j} \\
&= -\frac{\eta^3}{27} - \left(\frac{\eta}{3} - \frac{\beta}{2}\right)\lambda_j + \left(\frac{\lambda_j}{3}\right)^{3/2}\sqrt{1 + \left(\frac{2}{27}\eta^2 + \frac{\beta^2}{4} - \frac{\eta\beta}{3}\right)\frac{27}{\lambda_j} + \eta^3(\eta - \beta)\frac{1}{\lambda_j^2}} \\
&= \left(\frac{\lambda_j}{3}\right)^{3/2}\left[-\frac{\eta^3}{\sqrt{27}\lambda_j^{3/2}} - \left(\frac{\eta}{3} - \frac{\beta}{2}\right)\frac{\sqrt{27}}{\sqrt{\lambda_j}} + 1 + \frac{27}{2}\left(\frac{2}{27}\eta^2 + \frac{\beta^2}{4} - \frac{\eta\beta}{3}\right)\frac{1}{\lambda_j} + O\left(\frac{1}{\lambda_j^2}\right)\right] \\
&= \left(\frac{\lambda_j}{3}\right)^{3/2}\left[1 - \left(\frac{\eta}{3} - \frac{\beta}{2}\right)\frac{\sqrt{27}}{\sqrt{\lambda_j}} + \left(\eta^2 + \frac{27}{8}\beta^2 - \frac{9}{2}\eta\beta\right)\frac{1}{\lambda_j} - \frac{\eta^3}{\sqrt{27}\lambda_j^{3/2}} + O\left(\frac{1}{\lambda_j^2}\right)\right],
\end{aligned}$$

for $j \to \infty$. Therefore, using also the well-known formula

$$(1 + x)^{1/3} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + o\big(x^2\big), \quad x \to 0, \tag{6.17}$$

we obtain

$$\begin{aligned}
y_j &= \left(-\frac{q_j}{2} + \sqrt{\frac{q_j^2}{4} + \frac{p_j^3}{27}}\right)^{1/3} \\
&= \sqrt{\frac{\lambda_j}{3}}\left[1 - \left(\frac{\eta}{3} - \frac{\beta}{2}\right)\frac{3^{3/2}}{\sqrt{\lambda_j}} + \left(\eta^2 + \frac{27}{8}\beta^2 - \frac{9}{2}\eta\beta\right)\frac{1}{\lambda_j} - \frac{\eta^3}{(3\lambda_j)^{3/2}} + O\left(\frac{1}{\lambda_j^2}\right)\right]^{1/3}
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\frac{\lambda_j}{3}}\bigg[1 - \left(\frac{\eta}{3} - \frac{\beta}{2}\right)\sqrt{\frac{3}{\lambda_j}} + \frac{1}{3}\left(\eta^2 + \frac{27}{8}\beta^2 - \frac{9}{2}\eta\beta\right)\frac{1}{\lambda_j} - \frac{\eta^3}{3^{5/2}\lambda_j^{3/2}} - 3\left(\frac{\eta}{3} - \frac{\beta}{2}\right)^2\frac{1}{\lambda_j} \\
&\quad + \frac{2}{\sqrt{3}}\left(\frac{\eta}{3} - \frac{\beta}{2}\right)\left(\eta^2 + \frac{27}{8}\beta^2 - \frac{9}{2}\eta\beta\right)\frac{1}{\lambda_j^{3/2}} + O\left(\frac{1}{\lambda_j^2}\right)\bigg] \\
&= \sqrt{\frac{\lambda_j}{3}} - \frac{\eta}{3} + \frac{\beta}{2} + \frac{\beta}{2\sqrt{3}}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{\sqrt{\lambda_j}} + \left(\frac{5}{27}\eta^3 - \frac{4}{3}\eta^2\beta + \frac{9}{4}\eta\beta^2 - \frac{9}{8}\beta^3\right)\frac{1}{\lambda_j} \\
&\quad + O\left(\frac{1}{\lambda_j^{3/2}}\right), \quad j \to \infty.
\end{aligned}
\tag{6.18}
$$

In a similar way we get

$$
\begin{aligned}
&-\frac{q_j}{2} - \sqrt{\frac{q_j^2}{4} + \frac{p_j^3}{27}} \\
&\quad = -\frac{\eta^3}{27} - \left(\frac{\eta}{3} - \frac{\beta}{2}\right)\lambda_j - \sqrt{\frac{\lambda_j^3}{27} + \left(\frac{2}{27}\eta^2 + \frac{\beta^2}{4} - \frac{\eta\beta}{3}\right)\lambda_j^2 + \frac{\eta^3}{27}(\eta - \beta)\lambda_j} \\
&\quad = -\frac{\eta^3}{27} - \left(\frac{\eta}{3} - \frac{\beta}{2}\right)\lambda_j - \left(\frac{\lambda_j}{3}\right)^{3/2}\sqrt{1 + \left(\frac{2}{27}\eta^2 + \frac{\beta^2}{4} - \frac{\eta\beta}{3}\right)\frac{27}{\lambda_j} + \eta^3(\eta - \beta)\frac{1}{\lambda_j^2}} \\
&\quad = -\left(\frac{\lambda_j}{3}\right)^{3/2}\left[\frac{\eta^3}{\sqrt{27}\lambda_j^{3/2}} + \left(\frac{\eta}{3} - \frac{\beta}{2}\right)\frac{\sqrt{27}}{\sqrt{\lambda_j}} + 1 + \frac{27}{2}\left(\frac{2}{27}\eta^2 + \frac{\beta^2}{4} - \frac{\eta\beta}{3}\right)\frac{1}{\lambda_j} + O\left(\frac{1}{\lambda_j^2}\right)\right] \\
&\quad = -\left(\frac{\lambda_j}{3}\right)^{3/2}\left[1 + \left(\frac{\eta}{3} - \frac{\beta}{2}\right)\frac{\sqrt{27}}{\sqrt{\lambda_j}} + \left(\eta^2 + \frac{27}{8}\beta^2 - \frac{9}{2}\eta\beta\right)\frac{1}{\lambda_j} + \frac{\eta^3}{\sqrt{27}\lambda_j^{3/2}} + O\left(\frac{1}{\lambda_j^2}\right)\right],
\end{aligned}
$$

for $j \to \infty$. Therefore, using again (6.17), we have

$$
\begin{aligned}
z_j &= \left(-\frac{q_j}{2} - \sqrt{\frac{q_j^2}{4} + \frac{p_j^3}{27}}\right)^{1/3} \\
&= -\sqrt{\frac{\lambda_j}{3}}\left[1 + \left(\frac{\eta}{3} - \frac{\beta}{2}\right)\frac{3^{3/2}}{\sqrt{\lambda_j}} + \left(\eta^2 + \frac{27}{8}\beta^2 - \frac{9}{2}\eta\beta\right)\frac{1}{\lambda_j} + \frac{\eta^3}{(3\lambda_j)^{3/2}} + O\left(\frac{1}{\lambda_j^2}\right)\right]^{1/3} \\
&= -\sqrt{\frac{\lambda_j}{3}}\bigg[1 + \left(\frac{\eta}{3} - \frac{\beta}{2}\right)\sqrt{\frac{3}{\lambda_j}} + \frac{1}{3}\left(\eta^2 + \frac{27}{8}\beta^2 - \frac{9}{2}\eta\beta\right)\frac{1}{\lambda_j} + \frac{\eta^3}{3^{5/2}\lambda_j^{3/2}} - 3\left(\frac{\eta}{3} - \frac{\beta}{2}\right)^2\frac{1}{\lambda_j} \\
&\quad - \frac{2}{\sqrt{3}}\left(\frac{\eta}{3} - \frac{\beta}{2}\right)\left(\eta^2 + \frac{27}{8}\beta^2 - \frac{9}{2}\eta\beta\right)\frac{1}{\lambda_j^{3/2}} + O\left(\frac{1}{\lambda_j^2}\right)\bigg] \\
&= -\sqrt{\frac{\lambda_j}{3}} - \frac{\eta}{3} + \frac{\beta}{2} - \frac{\beta}{2\sqrt{3}}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{\sqrt{\lambda_j}} + \left(\frac{5}{27}\eta^3 - \frac{4}{3}\eta^2\beta + \frac{9}{4}\eta\beta^2 - \frac{9}{8}\beta^3\right)\frac{1}{\lambda_j} \\
&\quad + O\left(\frac{1}{\lambda_j^{3/2}}\right), \quad j \to \infty.
\end{aligned}
\tag{6.19}
$$

By (6.18) and (6.19) it follows

$$y_j + z_j = -\frac{2}{3}\eta + \beta + 2\left(\frac{5}{27}\eta^3 - \frac{4}{3}\eta^2\beta + \frac{9}{4}\eta\beta^2 - \frac{9}{8}\beta^3\right)\frac{1}{\lambda_j} + O\left(\frac{1}{\lambda_j^{3/2}}\right), \quad j \to \infty,$$

$$y_j - z_j = \frac{2}{\sqrt{3}}\sqrt{\lambda_j} + \frac{\beta}{\sqrt{3}}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{\sqrt{\lambda_j}} + O\left(\frac{1}{\lambda_j^{3/2}}\right), \quad j \to \infty.$$

In virtue of (6.14)–(6.16), the above relationships yield

$$\sigma_{1,j} = -\frac{2}{3}\eta + \beta + 2\left(\frac{5}{27}\eta^3 - \frac{4}{3}\eta^2\beta + \frac{9}{4}\eta\beta^2 - \frac{9}{8}\beta^3\right)\frac{1}{\lambda_j} + O\left(\frac{1}{\lambda_j^{3/2}}\right), \quad j \to \infty,$$

$$\begin{aligned}\sigma_{2,j} = {} & \frac{\eta}{3} - \frac{\beta}{2} - \left(\frac{5}{27}\eta^3 - \frac{4}{3}\eta^2\beta + \frac{9}{4}\eta\beta^2 - \frac{9}{8}\beta^3\right)\frac{1}{\lambda_j} + O\left(\frac{1}{\lambda_j^{3/2}}\right)\\ & + i\left[\sqrt{\lambda_j} + \frac{\beta}{2}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{\sqrt{\lambda_j}} + O\left(\frac{1}{\lambda_j^{3/2}}\right)\right], \quad j \to \infty,\end{aligned}$$

$$\begin{aligned}\sigma_{3,j} = {} & \frac{\eta}{3} - \frac{\beta}{2} - \left(\frac{5}{27}\eta^3 - \frac{4}{3}\eta^2\beta + \frac{9}{4}\eta\beta^2 - \frac{9}{8}\beta^3\right)\frac{1}{\lambda_j} + O\left(\frac{1}{\lambda_j^{3/2}}\right)\\ & - i\left[\sqrt{\lambda_j} + \frac{\beta}{2}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{\sqrt{\lambda_j}} + O\left(\frac{1}{\lambda_j^{3/2}}\right)\right], \quad j \to \infty.\end{aligned}$$

Finally, by using also the condition $\Lambda = \sigma - \eta/3$, we are able to write the solutions of Eq. (6.8), that is

$$\begin{aligned}\Lambda_{1,j} &= \beta - \eta + 2\left(\frac{5}{27}\eta^3 - \frac{4}{3}\eta^2\beta + \frac{9}{4}\eta\beta^2 - \frac{9}{8}\beta^3\right)\frac{1}{\lambda_j} + O\left(\frac{1}{\lambda_j^{3/2}}\right)\\ &= \beta - \eta + O\left(\frac{1}{\lambda_j}\right), \quad j \to \infty, \end{aligned} \tag{6.20}$$

$$\begin{aligned}\Lambda_{2,j} &= -\frac{\beta}{2} - \left(\frac{5}{27}\eta^3 - \frac{4}{3}\eta^2\beta + \frac{9}{4}\eta\beta^2 - \frac{9}{8}\beta^3\right)\frac{1}{\lambda_j} + O\left(\frac{1}{\lambda_j^{3/2}}\right)\\ &\quad + i\left[\sqrt{\lambda_j} + \frac{\beta}{2}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{\sqrt{\lambda_j}} + O\left(\frac{1}{\lambda_j^{3/2}}\right)\right]\\ &= -\frac{\beta}{2} + O\left(\frac{1}{\lambda_j}\right) + i\left[\sqrt{\lambda_j} + \frac{\beta}{2}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{\sqrt{\lambda_j}} + O\left(\frac{1}{\lambda_j^{3/2}}\right)\right], \quad j \to \infty, \end{aligned} \tag{6.21}$$

$$\begin{aligned}\Lambda_{3,j} &= -\frac{\beta}{2} - \left(\frac{5}{27}\eta^3 - \frac{4}{3}\eta^2\beta + \frac{9}{4}\eta\beta^2 - \frac{9}{8}\beta^3\right)\frac{1}{\lambda_j} + O\left(\frac{1}{\lambda_j^{3/2}}\right)\\ &\quad - i\left[\sqrt{\lambda_j} + \frac{\beta}{2}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{\sqrt{\lambda_j}} + O\left(\frac{1}{\lambda_j^{3/2}}\right)\right]\\ &= -\frac{\beta}{2} + O\left(\frac{1}{\lambda_j}\right) - i\left[\sqrt{\lambda_j} + \frac{\beta}{2}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{\sqrt{\lambda_j}} + O\left(\frac{1}{\lambda_j^{3/2}}\right)\right], \quad j \to \infty. \end{aligned} \tag{6.22}$$

Therefore, we can write the solution of (6.7) in the following way

$$f_j(t) = C_{1,j}e^{t\Lambda_{1,j}} + C_{2,j}e^{t\Lambda_{2,j}} + C_{3,j}e^{t\Lambda_{3,j}}, \tag{6.23}$$

where $C_{k,j}$, $k = 1, 2, 3$, are complex numbers to determine. To find the coefficients $C_{k,j}$ we impose that f_j verifies the initial conditions

$$f_j(0) = \alpha_j, \qquad f_j'(0) = \gamma_j, \qquad f_j''(0) = -\alpha_j\lambda_j, \tag{6.24}$$

so we obtain the system

$$\begin{cases} C_{1,j} + C_{2,j} + C_{3,j} = \alpha_j, \\ \Lambda_{1,j}C_{1,j} + \Lambda_{2,j}C_{2,j} + \Lambda_{3,j}C_{3,j} = \gamma_j, \\ \Lambda_{1,j}^2C_{1,j} + \Lambda_{2,j}^2C_{2,j} + \Lambda_{3,j}^2C_{3,j} = -\alpha_j\lambda_j. \end{cases} \tag{6.25}$$

The matrix C of the coefficients of system (6.25) has determinant given by

$$\det(C) = (\Lambda_{2,j} - \Lambda_{1,j})(\Lambda_{3,j} - \Lambda_{1,j})(\Lambda_{3,j} - \Lambda_{2,j}),$$

so we obtain

$$\begin{aligned} C_{1,j} &= \frac{\alpha_j\Lambda_{2,j}\Lambda_{3,j}(\Lambda_{3,j} - \Lambda_{2,j}) - \gamma_j(\Lambda_{3,j}^2 - \Lambda_{2,j}^2) - \alpha_j\lambda_j(\Lambda_{3,j} - \Lambda_{2,j})}{(\Lambda_{2,j} - \Lambda_{1,j})(\Lambda_{3,j} - \Lambda_{1,j})(\Lambda_{3,j} - \Lambda_{2,j})} \\ &= \frac{\alpha_j\Lambda_{2,j}\Lambda_{3,j} - \gamma_j(\Lambda_{3,j} + \Lambda_{2,j}) - \alpha_j\lambda_j}{(\Lambda_{2,j} - \Lambda_{1,j})(\Lambda_{3,j} - \Lambda_{1,j})}, \\ C_{2,j} &= \frac{\gamma_j(\Lambda_{3,j}^2 - \Lambda_{1,j}^2) + \alpha_j\lambda_j(\Lambda_{3,j} - \Lambda_{1,j}) - \alpha_j\Lambda_{1,j}\Lambda_{3,j}(\Lambda_{3,j} - \Lambda_{1,j})}{(\Lambda_{2,j} - \Lambda_{1,j})(\Lambda_{3,j} - \Lambda_{1,j})(\Lambda_{3,j} - \Lambda_{2,j})} \\ &= \frac{\gamma_j(\Lambda_{3,j} + \Lambda_{1,j}) + \alpha_j\lambda_j - \alpha_j\Lambda_{1,j}\Lambda_{3,j}}{(\Lambda_{2,j} - \Lambda_{1,j})(\Lambda_{3,j} - \Lambda_{2,j})}, \\ C_{3,j} &= \frac{-\alpha_j\lambda_j(\Lambda_{2,j} - \Lambda_{1,j}) - \gamma_j(\Lambda_{2,j}^2 - \Lambda_{1,j}^2) + \alpha_j\Lambda_{1,j}\Lambda_{2,j}(\Lambda_{2,j} - \Lambda_{1,j})}{(\Lambda_{2,j} - \Lambda_{1,j})(\Lambda_{3,j} - \Lambda_{1,j})(\Lambda_{3,j} - \Lambda_{2,j})} \\ &= \frac{-\alpha_j\lambda_j - \gamma_j(\Lambda_{2,j} + \Lambda_{1,j}) + \alpha_j\Lambda_{1,j}\Lambda_{2,j}}{(\Lambda_{3,j} - \Lambda_{1,j})(\Lambda_{3,j} - \Lambda_{2,j})}. \end{aligned}$$

Plugging (6.20)–(6.22) into the above identities, we obtain the expressions of coefficients $C_{k,j}$. Indeed,

$$\begin{aligned} C_{1,j} = {} & \frac{\alpha_j\{[-\frac{\beta}{2} + O(\frac{1}{\lambda_j})]^2 + [\sqrt{\lambda_j} + \frac{\beta}{2}(\frac{3}{4}\beta - \eta)\frac{1}{\sqrt{\lambda_j}} + O(\frac{1}{\lambda_j^{3/2}})]^2\}}{[\eta - \frac{3}{2}\beta + O(\frac{1}{\lambda_j})]^2 + [\sqrt{\lambda_j} + \frac{\beta}{2}(\frac{3}{4}\beta - \eta)\frac{1}{\sqrt{\lambda_j}} + O(\frac{1}{\lambda_j^{3/2}})]^2} \\ & + \frac{\gamma_j\beta + \gamma_j O(\frac{1}{\lambda_j}) - \alpha_j\lambda_j}{[\eta - \frac{3}{2}\beta + O(\frac{1}{\lambda_j})]^2 + [\sqrt{\lambda_j} + \frac{\beta}{2}(\frac{3}{4}\beta - \eta)\frac{1}{\sqrt{\lambda_j}} + O(\frac{1}{\lambda_j^{3/2}})]^2} \end{aligned}$$

$$= \frac{\alpha_j\beta^2 - \alpha_j\eta\beta + \gamma_j\beta + O\left(\frac{1}{\lambda_j}\right)}{\left(\eta - \frac{3}{2}\beta\right)^2 + \lambda_j + \frac{3}{4}\beta^2 - \eta\beta + O\left(\frac{1}{\lambda_j}\right)} = \frac{\alpha_j\beta^2 - \alpha_j\eta\beta + \gamma_j\beta + O\left(\frac{1}{\lambda_j}\right)}{\lambda_j + \eta^2 + 3\beta^2 - 4\eta\beta + O\left(\frac{1}{\lambda_j}\right)}$$

$$= \frac{\alpha_j\beta^2 - \alpha_j\eta\beta + \gamma_j\beta + O\left(\frac{1}{\lambda_j}\right)}{1 + (\eta^2 + 3\beta^2 - 4\eta\beta)\frac{1}{\lambda_j} + O\left(\frac{1}{\lambda_j^2}\right)}\,\frac{1}{\lambda_j}. \tag{6.26}$$

We note that $C_{1,j} \in \mathbb{R}$. To write explicitly $C_{2,j}$ we observe that

$$\begin{aligned}(\Lambda_{2,j} - \Lambda_{1,j})(\Lambda_{3,j} - \Lambda_{2,j}) &= -2i\left[\sqrt{\lambda_j} + \frac{\beta}{2}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{\sqrt{\lambda_j}} + O\left(\frac{1}{\lambda_j^{3/2}}\right)\right]\left\{\eta - \frac{3}{2}\beta + O\left(\frac{1}{\lambda_j}\right)\right. \\ &\quad \left. + i\left[\sqrt{\lambda_j} + \frac{\beta}{2}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{\sqrt{\lambda_j}} + O\left(\frac{1}{\lambda_j^{3/2}}\right)\right]\right\} \\ &= 2\lambda_j + \frac{3}{2}\beta^2 - 2\beta\eta + O\left(\frac{1}{\lambda_j}\right) - i\left[(2\eta - 3\beta)\sqrt{\lambda_j} + O\left(\frac{1}{\sqrt{\lambda_j}}\right)\right],\end{aligned}$$

whence

$$\begin{aligned}C_{2,j} &= \frac{\gamma_j\left\{\frac{\beta}{2} - \eta + O\left(\frac{1}{\lambda_j}\right) - i\left[\sqrt{\lambda_j} + \frac{\beta}{2}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{\sqrt{\lambda_j}} + O\left(\frac{1}{\lambda_j^{3/2}}\right)\right]\right\} + \alpha_j\lambda_j}{2\lambda_j + \frac{3}{2}\beta^2 - 2\beta\eta + O\left(\frac{1}{\lambda_j}\right) - i\left[(2\eta - 3\beta)\sqrt{\lambda_j} + O\left(\frac{1}{\sqrt{\lambda_j}}\right)\right]} \\ &\quad - \frac{\alpha_j\left[\beta - \eta + O\left(\frac{1}{\lambda_j}\right)\right]\left\{-\frac{\beta}{2} + O\left(\frac{1}{\lambda_j}\right) - i\left[\sqrt{\lambda_j} + \frac{\beta}{2}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{\sqrt{\lambda_j}} + O\left(\frac{1}{\lambda_j^{3/2}}\right)\right]\right\}}{2\lambda_j + \frac{3}{2}\beta^2 - 2\beta\eta + O\left(\frac{1}{\lambda_j}\right) - i\left[(2\eta - 3\beta)\sqrt{\lambda_j} + O\left(\frac{1}{\sqrt{\lambda_j}}\right)\right]}.\end{aligned}$$

Therefore,

$$\begin{aligned}C_{2,j} &= \frac{\alpha_j\lambda_j + \gamma_j\left(\frac{\beta}{2} - \eta\right) + \alpha_j(\beta - \eta)\frac{\beta}{2} + \alpha_j O\left(\frac{1}{\lambda_j}\right) + \gamma_j O\left(\frac{1}{\lambda_j}\right)}{2\lambda_j + \frac{3}{2}\beta^2 - 2\beta\eta + O\left(\frac{1}{\lambda_j}\right) - i\left[(2\eta - 3\beta)\sqrt{\lambda_j} + O\left(\frac{1}{\sqrt{\lambda_j}}\right)\right]} \\ &\quad - \frac{i\left[(\gamma_j - \alpha_j\beta + \alpha_j\eta)\sqrt{\lambda_j} + \alpha_j O\left(\frac{1}{\sqrt{\lambda_j}}\right) + \gamma_j O\left(\frac{1}{\sqrt{\lambda_j}}\right)\right]}{2\lambda_j + \frac{3}{2}\beta^2 - 2\beta\eta + O\left(\frac{1}{\lambda_j}\right) - i\left[(2\eta - 3\beta)\sqrt{\lambda_j} + O\left(\frac{1}{\sqrt{\lambda_j}}\right)\right]}.\end{aligned} \tag{6.27}$$

Divide now numerator and denominator by λ_j and take the square modulus to get

$$\begin{aligned}|C_{2,j}|^2 &= \frac{\left[\alpha_j + \gamma_j\frac{\beta - 2\eta}{2\lambda_j} + \gamma_j O\left(\frac{1}{\lambda_j^2}\right) + \alpha_j O\left(\frac{1}{\lambda_j}\right)\right]^2}{\left[2 + O\left(\frac{1}{\lambda_j}\right)\right]^2 + \left[\frac{2\eta - 3\beta}{\sqrt{\lambda_j}} + O\left(\frac{1}{\lambda_j^{3/2}}\right)\right]^2} \\ &\quad + \frac{\left[\frac{\gamma_j - \alpha_j\beta + \alpha_j\eta}{\sqrt{\lambda_j}} + \alpha_j O\left(\frac{1}{\lambda_j^{3/2}}\right) + \gamma_j O\left(\frac{1}{\lambda_j^{3/2}}\right)\right]^2}{\left[2 + O\left(\frac{1}{\lambda_j}\right)\right]^2 + \left[\frac{2\eta - 3\beta}{\sqrt{\lambda_j}} + O\left(\frac{1}{\lambda_j^{3/2}}\right)\right]^2},\end{aligned}$$

whence

$$
\begin{aligned}
|C_{2,j}|^2 &= \frac{1}{4}\left(1+O\left(\frac{1}{\lambda_j}\right)\right)\left(\alpha_j^2-\alpha_j\gamma_j\frac{\beta}{\lambda_j}+\alpha_j\gamma_j O\left(\frac{1}{\lambda_j^2}\right)+\alpha_j^2 O\left(\frac{1}{\lambda_j}\right)+\frac{\gamma_j^2}{\lambda_j}+\gamma_j^2 O\left(\frac{1}{\lambda_j^2}\right)\right)\\
&= \frac{1}{4}\alpha_j^2-\alpha_j\gamma_j\frac{\beta}{4\lambda_j}+\alpha_j\gamma_j O\left(\frac{1}{\lambda_j^2}\right)+\alpha_j^2 O\left(\frac{1}{\lambda_j}\right)+\frac{\gamma_j^2}{4\lambda_j}+\gamma_j^2 O\left(\frac{1}{\lambda_j^2}\right).
\end{aligned}
$$

Taking into account

$$
\frac{\alpha_j\gamma_j}{\lambda_j}=\frac{\alpha_j}{\lambda_j^{1/3}}\frac{\gamma_j}{\lambda_j^{2/3}}\leqslant\frac{1}{2}\frac{\alpha_j^2}{\lambda_j^{2/3}}+\frac{1}{2}\frac{\gamma_j^2}{\lambda_j^{4/3}},
$$

it follows that there exist some constants $c_1, c_2 > 0$ such that, for sufficiently large j,

$$
c_1\left(\alpha_j^2+\frac{\gamma_j^2}{\lambda_j}\right)\leqslant|C_{2,j}|^2\leqslant c_2\left(\alpha_j^2+\frac{\gamma_j^2}{\lambda_j}\right). \tag{6.28}
$$

Similarly,

$$
\begin{aligned}
&(\Lambda_{3,j}-\Lambda_{1,j})(\Lambda_{3,j}-\Lambda_{2,j})\\
&\quad=2\lambda_j+\frac{3}{2}\beta^2-2\beta\eta+O\left(\frac{1}{\lambda_j}\right)+i\left[(2\eta-3\beta)\sqrt{\lambda_j}+O\left(\frac{1}{\sqrt{\lambda_j}}\right)\right],
\end{aligned}
$$

and hence

$$
\begin{aligned}
C_{3,j} &= \frac{-\alpha_j\lambda_j-\gamma_j\{\frac{\beta}{2}-\eta+O(\frac{1}{\lambda_j})+i[\sqrt{\lambda_j}+\frac{\beta}{2}(\frac{3}{4}\beta-\eta)\frac{1}{\sqrt{\lambda_j}}+O(\frac{1}{\lambda_j^{3/2}})]\}}{2\lambda_j+\frac{3}{2}\beta^2-2\beta\eta+O(\frac{1}{\lambda_j})+i[(2\eta-3\beta)\sqrt{\lambda_j}+O(\frac{1}{\sqrt{\lambda_j}})]}\\
&\quad+\frac{\alpha_j[\beta-\eta+O(\frac{1}{\lambda_j})]\{-\frac{\beta}{2}+O(\frac{1}{\lambda_j})+i[\sqrt{\lambda_j}+\frac{\beta}{2}(\frac{3}{4}\beta-\eta)\frac{1}{\sqrt{\lambda_j}}+O(\frac{1}{\lambda_j^{3/2}})]\}}{2\lambda_j+\frac{3}{2}\beta^2-2\beta\eta+O(\frac{1}{\lambda_j})+i[(2\eta-3\beta)\sqrt{\lambda_j}+O(\frac{1}{\sqrt{\lambda_j}})]}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
C_{3,j} &= -\frac{\alpha_j\lambda_j+\gamma_j(\frac{\beta}{2}-\eta)+\alpha_j(\beta-\eta)\frac{\beta}{2}+\alpha_j O(\frac{1}{\lambda_j})+\gamma_j O(\frac{1}{\lambda_j})}{2\lambda_j+\frac{3}{2}\beta^2-2\beta\eta+O(\frac{1}{\lambda_j})+i[(2\eta-3\beta)\sqrt{\lambda_j}+O(\frac{1}{\sqrt{\lambda_j}})]}\\
&\quad-\frac{i[(\gamma_j-\alpha_j\beta+\alpha_j\eta)\sqrt{\lambda_j}+\alpha_j O(\frac{1}{\sqrt{\lambda_j}})+\gamma_j O(\frac{1}{\sqrt{\lambda_j}})]}{2\lambda_j+\frac{3}{2}\beta^2-2\beta\eta+O(\frac{1}{\lambda_j})+i[(2\eta-3\beta)\sqrt{\lambda_j}+O(\frac{1}{\sqrt{\lambda_j}})]}.
\end{aligned} \tag{6.29}
$$

Repeating the same steps as in the proof of (6.28), we have, for sufficiently large j,

$$
c_1\left(\alpha_j^2+\frac{\gamma_j^2}{\lambda_j}\right)\leqslant|C_{3,j}|^2\leqslant c_2\left(\alpha_j^2+\frac{\gamma_j^2}{\lambda_j}\right). \tag{6.30}
$$

By (6.26), (6.28) and (6.30), one deduces that there exists a positive constant c such that for any $j \in \mathbb{N}$ we have

$$\frac{|C_{1,j}|}{|C_{2,j}|} \leqslant \frac{c}{\lambda_j}, \qquad \frac{|C_{1,j}|}{|C_{3,j}|} \leqslant \frac{c}{\lambda_j}. \tag{6.31}$$

In conclusion, thanks into account (6.23) we have proved that the solution $v(t)$ of the Cauchy problem (6.1)–(6.2) can be written as

$$v(t) = \sum_{j=1}^{\infty} \bigl(C_{1,j} e^{t\Lambda_{1,j}} + C_{2,j} e^{t\Lambda_{2,j}} + C_{3,j} e^{t\Lambda_{3,j}}\bigr) w_j, \quad t \geqslant 0,$$

where $\Lambda_{k,j}$ and $C_{k,j}$ are given by (6.20)–(6.22) and (6.26)–(6.29) respectively, and condition (6.31) holds.

We will show as the function v can be written in the form

$$v(t) = \sum_{n=-\infty}^{\infty} \bigl(C_n e^{i\omega_n t} + R_n e^{r_n t}\bigr) w_{|n|}, \quad t \geqslant 0, \tag{6.32}$$

where $C_n, \omega_n \in \mathbb{C}$ and $R_n, r_n \in \mathbb{R}$. Indeed, we define ω_n as the complex numbers having real and imaginary parts given by

$$\Re\omega_n := \operatorname{sign}(n)\sqrt{\lambda_{|n|}} + \operatorname{sign}(n)\frac{\beta}{2}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{\sqrt{\lambda_{|n|}}} + O\left(\frac{1}{\lambda_{|n|}^{3/2}}\right), \quad |n| \geqslant 1,$$

$$\Im\omega_n := \frac{\beta}{2} + \left(\frac{5}{27}\eta^3 - \frac{4}{3}\eta^2\beta + \frac{9}{4}\eta\beta^2 - \frac{9}{8}\beta^3\right)\frac{1}{\lambda_{|n|}} + O\left(\frac{1}{\lambda_{|n|}^{3/2}}\right), \quad |n| \geqslant 1.$$

Moreover, we set

$$r_n := \beta - \eta + 2\left(\frac{5}{27}\eta^3 - \frac{4}{3}\eta^2\beta + \frac{9}{4}\eta\beta^2 - \frac{9}{8}\beta^3\right)\frac{1}{\lambda_{|n|}} + O\left(\frac{1}{\lambda_{|n|}^{3/2}}\right), \quad |n| \geqslant 1,$$

$$C_n := \begin{cases} C_{2,n} & \text{if } n \geqslant 1, \\ C_{3,-n} & \text{if } n \leqslant -1, \end{cases}$$

$$R_n := C_{1,n}, \quad n \geqslant 1, \qquad w_0 = C_0 = R_n = 0, \quad n \leqslant 0.$$

Finally, applying the abstract results of Sections 4 and 5 we can show our reachability result.

Theorem 6.1. *Let $\eta > 3\beta/2$. For any $T > 2\pi$, $u_0 \in L^2(0,\pi)$ and $u_1 \in H^{-1}(0,\pi)$ there exists $g \in L^2(0,T)$ such that the weak solution u of problem*

$$\begin{cases} u_{tt}(t,x) - u_{xx}(t,x) + \beta \displaystyle\int_0^t e^{-\eta(t-s)} u_{xx}(s,x)\,ds = 0, & t \in (0,T),\ x \in (0,\pi), \\ u(0,x) = u_t(0,x) = 0, & x \in (0,\pi), \\ u(t,0) = 0, \quad u(t,\pi) = g(t), & t \in (0,T), \end{cases} \tag{6.33}$$

verifies the final conditions

$$u(T,x) = u_0(x), \qquad u_t(T,x) = u_1(x), \quad x \in (0,\pi). \tag{6.34}$$

Proof. To prove our claim, we apply the HUM method described in Section 3. Let $X = L^2(0,\pi)$ be endowed with the usual scalar product and norm

$$\|u\| := \left(\int_0^\pi \big|u(x)\big|^2\,dx\right)^{1/2}, \quad u \in L^2(0,\pi).$$

We consider the operator $A : D(A) \subset X \to X$ defined by

$$D(A) = H^2(0,\pi) \cap H_0^1(0,\pi),$$

$$Au = -u_{xx}, \quad u \in D(A).$$

It is well known that A is a self-adjoint positive operator on X with dense domain $D(A)$, $\{j^2\}_{j\geqslant 1}$ is the sequence of eigenvalues for A and the sequence of the corresponding eigenvectors is $\{\sin(jx)\}_{j\geqslant 1}$. The fractional power $\sqrt{A}$ of A is well defined and $D(\sqrt{A}) = H_0^1(0,\pi)$. Therefore, we can apply our spectral analysis to the adjoint problem of (6.33). Indeed, the solution z of the adjoint problem can be written in the form (6.32), that is

$$z(t,x) = \sum_{n=-\infty}^{\infty} \big(C_n e^{i\omega_n(T-t)} + R_n e^{r_n(T-t)}\big)\sin\big(|n|x\big) \quad (t,x) \in [0,T]\times[0,\pi],$$

whence

$$z_x(t,\pi) = \sum_{n=-\infty}^{\infty} (-1)^n |n| \big(C_n e^{i\omega_n(T-t)} + R_n e^{r_n(T-t)}\big) \quad (t,x) \in [0,T]\times[0,\pi].$$

Since $\eta > 3\beta/2$ we can apply Theorems 4.1 and 5.1 to function $z_x(t,\pi)$. Therefore, thanks to inequalities (4.4) and (5.4) Theorem 3.1 holds true. In addition, by estimates (6.28) and (6.30) we have that

$$c_1\big(\|v_0\|^2_{H_0^1} + \|v_1\|^2\big) \leqslant \int_0^T \big|z_x(t,\pi)\big|^2\,dt \leqslant c_2\big(\|v_0\|^2_{H_0^1} + \|v_1\|^2\big),$$

whence the space F introduced at the end of Section 3 is $H_0^1(0,\pi) \times L^2(0,\pi)$. So, our proof is complete. □

Acknowledgments

The authors wish to thank the anonymous referee for her/his valuable suggestions.

Appendix A

To prove Theorem 4.2 we need to introduce an auxiliary function. Let $T > 0$. We define

$$k^*(t) := \begin{cases} \cos\frac{\pi t}{2T} & \text{if } |t| \leqslant T, \\ 0 & \text{if } |t| > T. \end{cases} \tag{A.1}$$

For the reader's convenience, we list some easy to check properties of k^* in the following lemma.

Lemma A.1. *Set*

$$K^*(u) := \frac{4T\pi}{\pi^2 - 4T^2u^2}, \quad u \in \mathbb{C},$$

the following properties hold for any $u \in \mathbb{C}$

$$\int_{-\infty}^{\infty} k^*(t)e^{iut}\,dt = \cos(uT)K^*(u), \tag{A.2}$$

$$\overline{K^*(u)} = K^*(\bar{u}), \tag{A.3}$$

$$\left|K^*(u)\right| = \left|K^*(\bar{u})\right|, \tag{A.4}$$

$$\left|K^*(u)\right| \leqslant \frac{4T\pi}{|4T^2(\Re u)^2 - 4T^2(\Im u)^2 - \pi^2|}. \tag{A.5}$$

Proof of Theorem 4.2. First, without loss of generality, it may be assumed that the sequence $\{\Im\omega_n\}$ converges to 0, that is

$$\alpha = 0. \tag{A.6}$$

Indeed, suppose for a moment that we have proved inequality (4.5) under this extra condition. For the general case $\alpha \neq 0$, we consider the function

$$g(t) := e^{\alpha t} f(t) = \sum_{n=-\infty}^{\infty} \left(C_n e^{i\omega'_n t} + R_n e^{(r_n+\alpha)t}\right),$$

where $\omega'_n = \omega_n - i\alpha$ and $\lim_{|n|\to\infty} \Im\omega'_n = 0$. So, inequality (4.5) holds for g, that is

$$\int_{-T}^{T} \left|g(t)\right|^2 dt \leqslant c_2(T) \sum_{|n|\geqslant n_0} |C_n|^2.$$

Since $f(t) = e^{-\alpha t} g(t)$, we have

$$\left|f(t)\right| \leqslant \max\left\{e^{\alpha T}, e^{-\alpha T}\right\}\left|g(t)\right| \quad \forall t \in [-T, T],$$

whence it follows

$$\int_{-T}^{T} \left|f(t)\right|^2 dt \leqslant \max\left\{e^{2\alpha T}, e^{-2\alpha T}\right\} \int_{-T}^{T} \left|g(t)\right|^2 dt$$

$$\leqslant \max\left\{e^{2\alpha T}, e^{-2\alpha T}\right\} c_2(T) \sum_{|n|\geqslant n_0} |C_n|^2,$$

that is (4.5) also holds for f.

Let $k^*(t)$ be the function defined by (A.1). If we use (A.2), then we have

$$\begin{aligned}
\int_{-\infty}^{\infty} k^*(t)\left|f(t)\right|^2 dt &= \int_{-\infty}^{\infty} k^*(t) \sum_n \left(C_n e^{i\omega_n t} + R_n e^{r_n t}\right) \sum_m \left(\overline{C}_m e^{-i\overline{\omega}_m t} + R_m e^{r_m t}\right) dt \\
&= \sum_{n,m} C_n \overline{C}_m \cos\bigl((\omega_n - \overline{\omega}_m)T\bigr) K^*(\omega_n - \overline{\omega}_m) \\
&\quad + \sum_{n,m} C_n R_m \cosh\bigl((i\omega_n + r_m)T\bigr) K^*(\omega_n - i r_m) \\
&\quad + \sum_{n,m} R_n \overline{C}_m \cosh\bigl((r_n - i\overline{\omega}_m)T\bigr) K^*(i r_n + \overline{\omega}_m) \\
&\quad + \sum_{n,m} R_n R_m \cosh\bigl((r_n + r_m)T\bigr) K^*\bigl(i(r_n + r_m)\bigr).
\end{aligned} \tag{A.7}$$

We may write the first sum on the right-hand side as follows

$$\begin{aligned}
\sum_{n,m} C_n \overline{C}_m \cos\bigl((\omega_n - \overline{\omega}_m)T\bigr) K^*(\omega_n - \overline{\omega}_m) &= \sum_n |C_n|^2 \cosh(2\Im\omega_n T) K^*(\omega_n - \overline{\omega}_n) \\
&\quad + \sum_{n,m,\, n\neq m} C_n \overline{C}_m \cos\bigl((\omega_n - \overline{\omega}_m)T\bigr) K^*(\omega_n - \overline{\omega}_m).
\end{aligned}$$

Plugging the above identity into (A.7) and using (A.3), we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} k^*(t)\left|f(t)\right|^2 dt &= \sum_n |C_n|^2 \cosh(2\Im\omega_n T) K^*(\omega_n - \overline{\omega}_n) \\
&\quad + \sum_{n,m,\, n\neq m} C_n \overline{C}_m \cos\bigl((\omega_n - \overline{\omega}_m)T\bigr) K^*(\omega_n - \overline{\omega}_m) \\
&\quad + 2\sum_{n,m} R_m \Re\bigl[C_n \cosh\bigl((i\omega_n + r_m)T\bigr) K^*(\omega_n - i r_m)\bigr] \\
&\quad + \sum_{n,m} R_n R_m \cosh\bigl((r_n + r_m)T\bigr) K^*\bigl(i(r_n + r_m)\bigr).
\end{aligned}$$

Notice that the terms on the right-hand side of the previous identity are real. Therefore, applying the elementary estimates $\theta \leqslant |\theta|$, $\theta \in \mathbb{R}$, and $|\cosh z| \leqslant \cosh(\Re z)$, $z \in \mathbb{C}$, we obtain

$$\int_{-\infty}^{\infty} k^*(t)\big|f(t)\big|^2\,dt \leqslant \sum_n |C_n|^2\cosh(2\Im\omega_n T)K^*(\omega_n-\overline{\omega}_n) + \sum_{n,m,\,n\neq m}|C_n||C_m|\cosh\big((\Im\omega_n+\Im\omega_m)T\big)\big|K^*(\omega_n-\overline{\omega}_m)\big| + 2\sum_{n,m}|C_n||R_m|\cosh\big((\Im\omega_n-r_m)T\big)\big|K^*(\omega_n-ir_m)\big| + \sum_{n,m}|R_n||R_m|\cosh\big((r_n+r_m)T\big)K^*\big(i(r_n+r_m)\big). \tag{A.8}$$

Since the sequences $\{\Im\omega_n\}$ and $\{r_n\}$ are bounded, there exists a positive constant $c(T)$ such that, for any $n,m\in\mathbb{Z}$, we have

$$\cosh\big((\Im\omega_n+\Im\omega_m)T\big)+\cosh\big((\Im\omega_n-r_m)T\big)+\cosh\big((r_n+r_m)T\big)\leqslant c(T),$$

and hence from (A.8) it follows

$$\int_{-\infty}^{\infty} k^*(t)\big|f(t)\big|^2\,dt \leqslant c(T)\sum_n |C_n|^2K^*(\omega_n-\overline{\omega}_n) + c(T)\sum_{n,m,\,n\neq m}|C_n||C_m|\big|K^*(\omega_n-\overline{\omega}_m)\big| + 2c(T)\sum_{n,m}|C_n||R_m|\big|K^*(\omega_n-ir_m)\big| + c(T)\sum_{n,m}|R_n||R_m|K^*\big(i(r_n+r_m)\big).$$

In virtue of the definition of K^* we have

$$K^*(\omega_n-\overline{\omega}_n)=\frac{4T\pi}{\pi^2+16T^2(\Im\omega_n)^2}\leqslant\frac{4T}{\pi},$$

whence

$$\int_{-\infty}^{\infty} k^*(t)\big|f(t)\big|^2\,dt \leqslant \frac{4T}{\pi}c(T)\sum_n |C_n|^2 + c(T)\sum_{n,m,\,n\neq m}|C_n||C_m|\big|K^*(\omega_n-\overline{\omega}_m)\big| + 2c(T)\sum_{n,m}|C_n||R_m|\big|K^*(\omega_n-ir_m)\big| + c(T)\sum_{n,m}|R_n||R_m|K^*\big(i(r_n+r_m)\big). \tag{A.9}$$

To evaluate the second sum on the right-hand side of the above inequality, we note that, in virtue of (A.4), we have

$$\big|K^*(\omega_n-\overline{\omega}_m)\big|=\big|K^*(\overline{\omega}_n-\omega_m)\big|,$$

whence

$$\begin{aligned}
&\sum_{n,m,\,n\neq m} |C_n||C_m|\big|K^*(\omega_n-\overline{\omega}_m)\big| \\
&\quad\leqslant \frac{1}{2}\sum_{n,m,\,n\neq m}\big(|C_n|^2+|C_m|^2\big)\big|K^*(\omega_n-\overline{\omega}_m)\big| \\
&\quad= \frac{1}{2}\sum_n |C_n|^2 \sum_{m,m\neq n}\big|K^*(\omega_n-\overline{\omega}_m)\big| + \frac{1}{2}\sum_m |C_m|^2 \sum_{n,n\neq m}\big|K^*(\omega_n-\overline{\omega}_m)\big| \\
&\quad= \frac{1}{2}\sum_n |C_n|^2 \sum_{m,m\neq n}\big|K^*(\omega_n-\overline{\omega}_m)\big| + \frac{1}{2}\sum_m |C_m|^2 \sum_{n,n\neq m}\big|K^*(\omega_m-\overline{\omega}_n)\big| \\
&\quad= \sum_n |C_n|^2 \sum_{m,m\neq n}\big|K^*(\omega_n-\overline{\omega}_m)\big|. \qquad (A.10)
\end{aligned}$$

Now, using (A.5) we get

$$\sum_{m,m\neq n}\big|K^*(\omega_n-\overline{\omega}_m)\big| \leqslant 4T\pi \sum_{m,m\neq n}\frac{1}{|4T^2(\Re\omega_n-\Re\omega_m)^2-4T^2(\Im\omega_n+\Im\omega_m)^2-\pi^2|}. \qquad (A.11)$$

From assumption (4.1) it follows

$$|\Re\omega_n-\Re\omega_m| \geqslant \gamma|n-m| \quad \forall |n|,|m|\geqslant n'. \qquad (A.12)$$

Fix $0<\varepsilon<1$, thanks to (A.6), there exists $n_1\in\mathbb{N}$, $n_1\geqslant n'$, such that for any $n\in\mathbb{Z}$, $|n|\geqslant n_1$,

$$|\Im\omega_n| < \frac{\gamma\sqrt{\varepsilon}}{4}.$$

Therefore, for any $n,m\in\mathbb{Z}$, $|n|,|m|\geqslant n_1$, we have

$$4T^2(\Re\omega_n-\Re\omega_m)^2-4T^2(\Im\omega_n+\Im\omega_m)^2-\pi^2 \geqslant 4T^2\gamma^2(n-m)^2-T^2\gamma^2\varepsilon-\pi^2.$$

Now, for any $T>\frac{\pi}{\gamma\sqrt{1-\varepsilon}}$ we have $T^2\gamma^2\varepsilon+\pi^2<T^2\gamma^2$, so from the above inequality it follows, for $m\neq n$,

$$4T^2(\Re\omega_n-\Re\omega_m)^2-4T^2(\Im\omega_n+\Im\omega_m)^2-\pi^2 \geqslant 4T^2\gamma^2(n-m)^2-T^2\gamma^2>0.$$

Putting the previous formula into (A.11), we obtain

$$\begin{aligned}
\sum_{|m|\geqslant n_1,m\neq n}\big|K^*(\omega_n-\overline{\omega}_m)\big| &\leqslant 4T\pi\sum_{m,m\neq n}\frac{1}{4T^2\gamma^2(m-n)^2-T^2\gamma^2} = \frac{4\pi}{T\gamma^2}\sum_{m,m\neq n}\frac{1}{4(m-n)^2-1} \\
&\leqslant \frac{8\pi}{T\gamma^2}\sum_{j=1}^{\infty}\frac{1}{4j^2-1} = \frac{4\pi}{T\gamma^2}\sum_{j=1}^{\infty}\left(\frac{1}{2j-1}-\frac{1}{2j+1}\right) = \frac{4\pi}{T\gamma^2}.
\end{aligned}$$

Assuming $C_n=0$ for $|n|\leqslant n_1$ and putting the above formula into (A.10), we get

$$\sum_{|n|,|m|\geqslant n_1,\,n\neq m}|C_n||C_m|\big|K^*(\omega_n-\overline{\omega}_m)\big| \leqslant \frac{4\pi}{T\gamma^2}\sum_{|n|\geqslant n_1}|C_n|^2. \qquad (A.13)$$

Notice that, thanks to (4.3), we have $R_n = 0$ for $|n| \leqslant n_1$. Therefore, from (A.9) and (A.13) it follows

$$\int_{-\infty}^{\infty} k^*(t)\big|f(t)\big|^2\,dt \leqslant c(T)\left(\frac{4T}{\pi}+\frac{4\pi}{T\gamma^2}\right)\sum_{|n|\geqslant n_1}|C_n|^2 + 2c(T)\sum_{|n|,|m|\geqslant n_1}|C_n||R_m|\big|K^*(\omega_n - ir_m)\big|$$
$$+\,c(T)\sum_{|n|,|m|\geqslant n_1}|R_n||R_m|K^*\big(i(r_n+r_m)\big). \tag{A.14}$$

To estimate the second term on the right-hand side, we use (4.3) to obtain

$$\begin{aligned}2\sum_{|n|,|m|\geqslant n_1}|C_n||R_m|\big|K^*(\omega_n - ir_m)\big| &\leqslant 2\mu\sum_{|n|,|m|\geqslant n_1}|C_n|\frac{|C_m|}{|m|^\nu}\big|K^*(\omega_n - ir_m)\big| \\ &\leqslant \mu\sum_{|n|\geqslant n_1}|C_n|^2\sum_{|m|\geqslant n_1}\frac{|K^*(\omega_n - ir_m)|}{m^{2\nu}} \\ &\quad+\mu\sum_{|m|\geqslant n_1}|C_m|^2\sum_{|n|\geqslant n_1}\big|K^*(\omega_n - ir_m)\big|.\end{aligned} \tag{A.15}$$

Applying (A.5), one gets

$$\big|K^*(\omega_n - ir_m)\big| \leqslant \frac{4T\pi}{|4T^2(\Re\omega_n)^2 - 4T^2(\Im\omega_n - r_m)^2 - \pi^2|}. \tag{A.16}$$

Now, we observe that, by (A.12) it follows

$$|\Re\omega_n| \geqslant \gamma|n-n'| - |\Re\omega_{n'}| \quad \forall n\in\mathbb{Z}, \quad |n|\geqslant n',$$

whence

$$|\Re\omega_n| \geqslant \frac{\gamma}{2}|n| \quad \forall|n|\geqslant 2n' + 2\left[\frac{|\Re\omega_{n'}|}{\gamma}\right]+1.$$

Therefore, since the sequences $\{\Im\omega_n\}$, $\{r_n\}$ are bounded, there exists $n_0\in\mathbb{N}$,

$$n_0 \geqslant \max\left\{n_1,\, 2n' + 2\left[\frac{|\Re\omega_{n'}|}{\gamma}\right]+1\right\}$$

such that, for any $n,m\in\mathbb{Z}$, $|n|,|m|\geqslant n_0$, we have

$$4T^2(\Re\omega_n)^2 - 4T^2(\Im\omega_n - r_m)^2 - \pi^2 \geqslant \frac{1}{2}T^2\gamma^2 n^2;$$

so, plugging the above inequality into (A.16) we have

$$\big|K^*(\omega_n - ir_m)\big| \leqslant \frac{8\pi}{T\gamma^2 n^2}.$$

Assuming $C_n = 0$ for $|n|\leqslant n_0$, and hence also $R_n = 0$ for $|n|\leqslant n_0$, by (A.15) it follows

$$
\begin{aligned}
&2\sum_{|n|,|m|\geqslant n_0}|C_n||R_m|\left|K^*(\omega_n-ir_m)\right|\\
&\quad\leqslant\frac{8\pi\mu}{T\gamma^2}\sum_{|n|\geqslant n_0}|C_n|^2\sum_{m\neq0}\frac{1}{m^{2\nu}}+\frac{8\pi\mu}{T\gamma^2}\sum_{|m|\geqslant n_0}|C_m|^2\sum_{n\neq0}\frac{1}{n^2}\\
&\quad=\frac{16\pi\mu}{T\gamma^2}\left(\sum_{j=1}^{\infty}\frac{1}{j^{2\nu}}+\sum_{j=1}^{\infty}\frac{1}{j^2}\right)\sum_{|n|\geqslant n_0}|C_n|^2. \qquad (A.17)
\end{aligned}
$$

At last, we must consider the term

$$\sum_{|n|,|m|\geqslant n_0}|R_n||R_m|K^*\big(i(r_n+r_m)\big).$$

Recalling the definition of K^* we have

$$K^*\big(i(r_n+r_m)\big)=\frac{4T\pi}{\pi^2+4T^2(r_n+r_m)^2}\leqslant\frac{4T}{\pi},$$

so, in virtue of (4.3) we get

$$
\begin{aligned}
&\sum_{|n|,|m|\geqslant n_0}|R_n||R_m|K^*\big(i(r_n+r_m)\big)\\
&\quad\leqslant\frac{4T\mu^2}{\pi}\sum_{|n|,|m|\geqslant n_0}\frac{|C_n|}{|m|^{\nu}}\frac{|C_m|}{|n|^{\nu}}\\
&\quad\leqslant\frac{2T\mu^2}{\pi}\sum_{m\neq0}\frac{1}{m^{2\nu}}\sum_{|n|\geqslant n_0}|C_n|^2+\frac{2T\mu^2}{\pi}\sum_{n\neq0}\frac{1}{n^{2\nu}}\sum_{|m|\geqslant n_0}|C_m|^2\\
&\quad=\frac{4T\mu^2}{\pi}\sum_{n\neq0}\frac{1}{n^{2\nu}}\sum_{|n|\geqslant n_0}|C_n|^2=\frac{8T\mu^2}{\pi}\sum_{j=1}^{\infty}\frac{1}{j^{2\nu}}\sum_{|n|\geqslant n_0}|C_n|^2. \qquad (A.18)
\end{aligned}
$$

Putting (A.17) and (A.18) into (A.14), we obtain

$$\int_{-\infty}^{\infty}k^*(t)\left|f(t)\right|^2dt\leqslant c(T)\left(\frac{4T}{\pi}+\frac{4\pi}{T\gamma^2}+\frac{16\pi\mu}{T\gamma^2}\sum_{j=1}^{\infty}\frac{1}{j^2}+8\mu\left(\frac{2\pi}{T\gamma^2}+\frac{T\mu}{\pi}\right)\sum_{j=1}^{\infty}\frac{1}{j^{2\nu}}\right)\sum_{|n|\geqslant n_0}|C_n|^2.$$

Now, if we consider the auxiliary function k^* defined by (A.1) with T replaced by $2T$, then from the above inequality we get

$$
\begin{aligned}
&\int_{-2T}^{2T}\cos\frac{\pi t}{4T}\left|f(t)\right|^2dt\\
&\quad\leqslant c(2T)\left(\frac{8T}{\pi}+\frac{2\pi}{T\gamma^2}+\frac{8\pi\mu}{T\gamma^2}\sum_{j=1}^{\infty}\frac{1}{j^2}+8\mu\left(\frac{\pi}{T\gamma^2}+\frac{2T\mu}{\pi}\right)\sum_{j=1}^{\infty}\frac{1}{j^{2\nu}}\right)\sum_{|n|\geqslant n_0}|C_n|^2,
\end{aligned}
$$

whence

$$\int_{-T}^{T} \left|f(t)\right|^2 dt \leqslant \sqrt{2}c(2T)\left(\frac{8T}{\pi} + \frac{2\pi}{T\gamma^2} + \frac{8\pi\mu}{T\gamma^2}\sum_{j=1}^{\infty}\frac{1}{j^2} + 8\mu\left(\frac{\pi}{T\gamma^2} + \frac{2T\mu}{\pi}\right)\sum_{j=1}^{\infty}\frac{1}{j^{2\nu}}\right)\sum_{|n|\geqslant n_0}|C_n|^2.$$

So, the proof is complete. □

As for the direct inequality, to prove Theorem 5.3 we need to introduce an auxiliary function. We define

$$k(t) := \begin{cases} \sin\frac{\pi t}{T} & \text{if } t \in [0,T], \\ 0 & \text{otherwise.} \end{cases} \tag{A.19}$$

For the reader's convenience, we list some easy to check properties of k in the following lemma.

Lemma A.2. *Set*

$$K(u) := \frac{T\pi}{\pi^2 - T^2u^2}, \quad u \in \mathbb{C}, \tag{A.20}$$

the following properties hold for any $u \in \mathbb{C}$

$$\int_{-\infty}^{\infty} k(t)e^{iut}\,dt = \left(1 + e^{iuT}\right)K(u), \tag{A.21}$$

$$\overline{K(u)} = K(\bar{u}), \tag{A.22}$$

$$\left|K(u)\right| = \left|K(\bar{u})\right|, \tag{A.23}$$

$$\left|K(u)\right| \leqslant \frac{T\pi}{|T^2(\Re u)^2 - T^2(\Im u)^2 - \pi^2|}. \tag{A.24}$$

Proof of Theorem 5.3. As in the proof of Theorem 4.2, without loss of generality, it may be assumed that

$$\alpha = 0. \tag{A.25}$$

Indeed, suppose for a moment that we have proved inequality (5.5) under this extra condition. For the general case $\alpha \neq 0$, we consider the function

$$g(t) := e^{\alpha t} f(t) = \sum_{n=-\infty}^{\infty}\left(C_n e^{i\omega_n' t} + R_n e^{(r_n+\alpha)t}\right),$$

where $\omega_n' = \omega_n - i\alpha$ and $\lim_{|n|\to\infty}\Im\omega_n' = 0$. So, inequality (5.5) holds for g, that is

$$\int_0^T \left|g(t)\right|^2 dt \geqslant \left(\frac{T\pi}{\pi^2 + T^2\gamma^2\varepsilon/8} - \frac{4\pi}{T\gamma^2}(1+\varepsilon)\right)\sum_{|n|\geqslant n_0}\left(1 + e^{-2\Im\omega_n' T}\right)|C_n|^2.$$

Since $f(t) = e^{-\alpha t} g(t)$, we have

$$\left|f(t)\right| \geqslant \min\left\{1, e^{-\alpha T}\right\}\left|g(t)\right|, \quad \forall t \in [0, T],$$

whence it follows

$$\int_0^T \left|f(t)\right|^2 dt \geqslant \min\left\{1, e^{-2\alpha T}\right\} \int_0^T \left|g(t)\right|^2 dt \geqslant c_1(T, \varepsilon) \sum_{|n| \geqslant n_0} \left(1 + e^{-2(\Im\omega_n - \alpha)T}\right)|C_n|^2,$$

that is (5.5) also holds for f.

Let $k(t)$ be the function defined by (A.19). If we use (A.21), then we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} k(t)\left|f(t)\right|^2 dt \\
&\quad = \int_{-\infty}^{\infty} k(t) \sum_n \left(C_n e^{i\omega_n t} + R_n e^{r_n t}\right) \sum_m \left(\overline{C}_m e^{-i\overline{\omega}_m t} + R_m e^{r_m t}\right) dt \\
&\quad = \sum_{n,m} C_n \overline{C}_m \left(1 + e^{i(\omega_n - \overline{\omega}_m)T}\right) K(\omega_n - \overline{\omega}_m) + \sum_{n,m} C_n R_m \left(1 + e^{(i\omega_n + r_m)T}\right) K(\omega_n - i r_m) \\
&\qquad + \sum_{n,m} R_n \overline{C}_m \left(1 + e^{(r_n - i\overline{\omega}_m)T}\right) K(i r_n + \overline{\omega}_m) \\
&\qquad + \int_{-\infty}^{\infty} k(t) \left| \sum_n R_n e^{r_n t} \right|^2 dt. \qquad \text{(A.26)}
\end{aligned}$$

We may write the first sum on the right-hand side as follows

$$\begin{aligned}
&\sum_{n,m} C_n \overline{C}_m \left(1 + e^{i(\omega_n - \overline{\omega}_m)T}\right) K(\omega_n - \overline{\omega}_m) \\
&\quad = \sum_n |C_n|^2 \left(1 + e^{-2\Im\omega_n T}\right) K(\omega_n - \overline{\omega}_n) + \sum_{n,m,\, n \neq m} C_n \overline{C}_m \left(1 + e^{i(\omega_n - \overline{\omega}_m)T}\right) K(\omega_n - \overline{\omega}_m).
\end{aligned}$$

Plugging the above identity into (A.26) and using (A.22), we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} k(t)\left|f(t)\right|^2 dt = {} & \sum_n |C_n|^2 \left(1 + e^{-2\Im\omega_n T}\right) K(\omega_n - \overline{\omega}_n) \\
& + \sum_{n,m,\, n \neq m} C_n \overline{C}_m \left(1 + e^{i(\omega_n - \overline{\omega}_m)T}\right) K(\omega_n - \overline{\omega}_m) \\
& + 2 \sum_{n,m} R_m \Re\left[C_n \left(1 + e^{(i\omega_n + r_m)T}\right) K(\omega_n - i r_m)\right] \\
& + \int_{-\infty}^{\infty} k(t) \left| \sum_n R_n e^{r_n t} \right|^2 dt.
\end{aligned}$$

Notice that, by difference, the second term on the right-hand side of the previous identity is real. Therefore, using the elementary estimate $\theta \geqslant -|\theta|$, $\theta \in \mathbb{R}$, we obtain

$$\int_{-\infty}^{\infty} k(t)\big|f(t)\big|^2\,dt \geqslant \sum_n |C_n|^2\big(1+e^{-2\Im\omega_n T}\big)K(\omega_n-\overline{\omega}_n)$$
$$-\sum_{n,m,\,n\neq m}|C_n||C_m|\big(1+e^{-(\Im\omega_n+\Im\omega_m)T}\big)\big|K(\omega_n-\overline{\omega}_m)\big|$$
$$-2\sum_{n,m}|C_n||R_m|\big(1+e^{(r_m-\Im\omega_n)T}\big)\big|K(\omega_n-ir_m)\big|$$
$$+\int_{-\infty}^{\infty} k(t)\Big|\sum_n R_n e^{r_n t}\Big|^2\,dt. \tag{A.27}$$

Now, arguing as in the proof of (A.10) and using $|K(\omega_n-\overline{\omega}_m)| = |K(\overline{\omega}_n-\omega_m)|$, we have

$$\sum_{n,m,\,n\neq m}|C_n||C_m|\big|K(\omega_n-\overline{\omega}_m)\big| \leqslant \sum_n |C_n|^2\sum_{m,m\neq n}\big|K(\omega_n-\overline{\omega}_m)\big|. \tag{A.28}$$

Similarly, we get

$$\sum_{n,m,\,n\neq m}|C_n||C_m|e^{-(\Im\omega_n+\Im\omega_m)T}\big|K(\omega_n-\overline{\omega}_m)\big|$$
$$\leqslant \sum_n |C_n|^2 e^{-2\Im\omega_n T}\sum_{m,m\neq n}\big|K(\omega_n-\overline{\omega}_m)\big|. \tag{A.29}$$

Therefore, plugging (A.28) and (A.29) into (A.27) and being k a non-negative function, we have

$$\int_{-\infty}^{\infty} k(t)\big|f(t)\big|^2\,dt \geqslant \sum_n |C_n|^2\big(1+e^{-2\Im\omega_n T}\big)\Big(K(\omega_n-\overline{\omega}_n)-\sum_{m,m\neq n}\big|K(\omega_n-\overline{\omega}_m)\big|\Big)$$
$$-2\sum_{n,m}|C_n||R_m|\big(1+e^{(r_m-\Im\omega_n)T}\big)\big|K(\omega_n-ir_m)\big|. \tag{A.30}$$

Now, fixed $n\in\mathbb{Z}$, we have to estimate the sum

$$\sum_{m,m\neq n}\big|K(\omega_n-\overline{\omega}_m)\big|.$$

Using (A.24), we get

$$\sum_{m,m\neq n}\big|K(\omega_n-\overline{\omega}_m)\big| \leqslant T\pi\sum_{m,m\neq n}\frac{1}{|T^2(\Re\omega_n-\Re\omega_m)^2-T^2(\Im\omega_n+\Im\omega_m)^2-\pi^2|}. \tag{A.31}$$

From assumption (5.1) it follows

$$|\Re\omega_n-\Re\omega_m| \geqslant \gamma|n-m|, \quad \forall |n|,|m|\geqslant n'. \tag{A.32}$$

Moreover, if we fix $0<\varepsilon<1$, then, thanks to (A.25), there exists $n_1\in\mathbb{N}$, $n_1\geqslant n'$, such that for any $n\in\mathbb{Z}$, $|n|\geqslant n_1$,

$$|\Im\omega_n|<\frac{\gamma}{4}\sqrt{\frac{\varepsilon}{2}}. \tag{A.33}$$

Therefore, for any $n,m\in\mathbb{Z}$, $|n|,|m|\geqslant n_1$, we have

$$T^2(\Re\omega_n-\Re\omega_m)^2-T^2(\Im\omega_n+\Im\omega_m)^2-\pi^2\geqslant T^2\gamma^2(n-m)^2-T^2\gamma^2\frac{\varepsilon}{4}-\pi^2.$$

Now, for any $T>\frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$ we have $T^2\gamma^2\varepsilon+4\pi^2<T^2\gamma^2$, so from the above inequality it follows, for $m\neq n$,

$$T^2(\Re\omega_n-\Re\omega_m)^2-T^2(\Im\omega_n+\Im\omega_m)^2-\pi^2\geqslant T^2\gamma^2(n-m)^2-\frac{1}{4}T^2\gamma^2>0.$$

Putting the previous formula into (A.31), we obtain

$$\begin{aligned}\sum_{|m|\geqslant n_1,\, m\neq n}\big|K(\omega_n-\overline{\omega}_m)\big|&\leqslant 4T\pi\sum_{m,m\neq n}\frac{1}{4T^2\gamma^2(m-n)^2-T^2\gamma^2}=\frac{4\pi}{T\gamma^2}\sum_{m,m\neq n}\frac{1}{4(m-n)^2-1}\\&\leqslant\frac{8\pi}{T\gamma^2}\sum_{j=1}^{\infty}\frac{1}{4j^2-1}=\frac{4\pi}{T\gamma^2}\sum_{j=1}^{\infty}\left(\frac{1}{2j-1}-\frac{1}{2j+1}\right)=\frac{4\pi}{T\gamma^2}.\end{aligned}$$

If we assume $C_n=0$ for $|n|\leqslant n_1$, then due to (5.3) we also have $R_n=0$ for $|n|\leqslant n_1$. Therefore, putting the above estimate into (A.30), for $T>\frac{2\pi}{\gamma\sqrt{1-\varepsilon}}$ we get

$$\begin{aligned}\int_{-\infty}^{\infty}k(t)\big|f(t)\big|^2\,dt&\geqslant\sum_{|n|\geqslant n_1}|C_n|^2\big(1+e^{-2\Im\omega_nT}\big)\left(K(\omega_n-\overline{\omega}_n)-\frac{4\pi}{T\gamma^2}\right)\\&\quad-2\sum_{|n|,|m|\geqslant n_1}|C_n||R_m|\big(1+e^{(r_m-\Im\omega_n)T}\big)\big|K(\omega_n-ir_m)\big|.\end{aligned}\tag{A.34}$$

It remains to estimate the second sum on the right-hand side. Thanks to (5.3) we have

$$\begin{aligned}2\sum_{|n|,|m|\geqslant n_1}|C_n||R_m|\big|K(\omega_n-ir_m)\big|&\leqslant 2\mu\sum_{|n|,|m|\geqslant n_1}|C_n|\frac{|C_m|}{|m|^\nu}\big|K(\omega_n-ir_m)\big|\\&\leqslant\mu\sum_{|n|\geqslant n_1}|C_n|^2\sum_{|m|\geqslant n_1}\frac{|K(\omega_n-ir_m)|}{|m|^{2\nu}}\\&\quad+\mu\sum_{|m|\geqslant n_1}|C_m|^2\sum_{|n|\geqslant n_1}\big|K(\omega_n-ir_m)\big|.\end{aligned}\tag{A.35}$$

Again by (A.24) we have

$$\big|K(\omega_n-ir_m)\big|\leqslant\frac{T\pi}{|T^2(\Re\omega_n)^2-T^2(\Im\omega_n-r_m)^2-\pi^2|}. \tag{A.36}$$

Now, we observe that, by (A.32) it follows

$$|\Re\omega_n| \geqslant \gamma|n-n'| - |\Re\omega_{n'}|, \quad \forall n\in\mathbb{Z},\ |n|\geqslant n',$$

whence

$$|\Re\omega_n| \geqslant \frac{\gamma}{\sqrt{2}}|n|, \quad \forall |n| \geqslant \left[\frac{\gamma n' + |\Re\omega_{n'}|}{\gamma(1-1/\sqrt{2})}\right] + 1 =: n_2.$$

Therefore, for any $n\in\mathbb{Z}$, $|n|\geqslant n_2$, we get

$$\begin{aligned} T^2(\Re\omega_n)^2 - T^2(\Im\omega_n - r_m)^2 - \pi^2 &\geqslant T^2\left(\frac{1}{2}\gamma^2 n^2 - (\Im\omega_n - r_m)^2\right) - \pi^2 \\ &\geqslant T^2\gamma^2 n^2\left(\frac{1}{2} - \frac{(\Im\omega_n - r_m)^2}{\gamma^2 n^2}\right) - \pi^2. \end{aligned} \tag{A.37}$$

Since the sequences $\{\Im\omega_n\}$ and $\{r_n\}$ are bounded, there exists $n_3\in\mathbb{N}$, such that

$$\frac{1}{2} - \frac{(\Im\omega_n - r_m)^2}{\gamma^2 n^2} \geqslant \frac{1}{4}, \quad \forall |n|, |m| \geqslant n_3. \tag{A.38}$$

Choosing $n_0\in\mathbb{N}$ such that

$$n_0 \geqslant \max\{n_1, n_2, n_3, 2\}, \tag{A.39}$$

and putting (A.38) into (A.37), for any $|n|, |m|\geqslant n_0$ we have

$$T^2(\Re\omega_n)^2 - T^2(\Im\omega_n - r_m)^2 - \pi^2 \geqslant \frac{1}{4}\big(T^2\gamma^2 n^2 - 4\pi^2\big).$$

Moreover, since $T > 2\pi/\gamma$ we have $4\pi^2 < T^2\gamma^2 n_0^{1/2}$, so

$$T^2(\Re\omega_n)^2 - T^2(\Im\omega_n - r_m)^2 - \pi^2 \geqslant \frac{1}{4}T^2\gamma^2\big(n^2 - n_0^{1/2}\big) \geqslant \frac{1}{4}T^2\gamma^2 n_0^{1/2}\big(|n|^{3/2} - 1\big).$$

Therefore from (A.36), thanks to the above inequality, we get

$$\big|K(\omega_n - ir_m)\big| \leqslant \frac{4\pi}{T\gamma^2 n_0^{1/2}(|n|^{3/2}-1)}, \quad \forall |n|, |m| \geqslant n_0, \tag{A.40}$$

and hence, assuming $C_n = 0$ for $|n|\leqslant n_0$, (A.35) can be written as

$$\begin{aligned} &2\sum_{|n|,|m|\geqslant n_0} |C_n||R_m|\big|K(\omega_n - ir_m)\big| \\ &\quad\leqslant \frac{4\pi\mu}{T\gamma^2 n_0^{1/2}}\sum_{|n|\geqslant n_0}|C_n|^2\sum_{m\neq 0}\frac{1}{|m|^{2\nu}} + \frac{4\pi\mu}{T\gamma^2 n_0^{1/2}}\sum_{|m|\geqslant n_0}|C_m|^2\sum_{|n|\geqslant 2}\frac{1}{|n|^{3/2}-1} \\ &\quad= \frac{8\pi\mu}{T\gamma^2 n_0^{1/2}}\left(\sum_{j=1}^{\infty}\frac{1}{j^{2\nu}} + \sum_{j=2}^{\infty}\frac{1}{j^{3/2}-1}\right)\sum_{|n|\geqslant n_0}|C_n|^2. \end{aligned} \tag{A.41}$$

Moreover, by (5.2) and (5.3) we have

$$\begin{aligned}
&2\sum_{|n|,|m|\geqslant n_0}|C_n||R_m|e^{(r_m-\Im\omega_n)T}\big|K(\omega_n-ir_m)\big| \\
&\quad\leqslant 2\mu\sum_{|n|,|m|\geqslant n_0}|C_n|e^{-\Im\omega_n T}\frac{|C_m|e^{-\Im\omega_m T}}{|m|^{\nu}}\big|K(\omega_n-ir_m)\big| \\
&\quad\leqslant \mu\sum_{|n|\geqslant n_0}|C_n|^2e^{-2\Im\omega_n T}\sum_{|m|\geqslant n_0}\frac{|K(\omega_n-ir_m)|}{|m|^{2\nu}} \\
&\qquad+\mu\sum_{|m|\geqslant n_0}|C_m|^2e^{-2\Im\omega_m T}\sum_{|n|\geqslant n_0}\big|K(\omega_n-ir_m)\big|.
\end{aligned}$$

If we use again (A.40), then, reasoning as in (A.41), we obtain

$$\begin{aligned}
&2\sum_{|n|,|m|\geqslant n_0}|C_n||R_m|e^{(r_m-\Im\omega_n)T}\big|K(\omega_n-ir_m)\big| \\
&\quad\leqslant \frac{8\pi\mu}{T\gamma^2n_0^{1/2}}\left(\sum_{j=1}^{\infty}\frac{1}{j^{2\nu}}+\sum_{j=2}^{\infty}\frac{1}{j^{3/2}-1}\right)\sum_{|n|\geqslant n_0}|C_n|^2e^{-2\Im\omega_n T}.
\end{aligned}\tag{A.42}$$

Set

$$S:=2\mu\left(\sum_{j=1}^{\infty}\frac{1}{j^{2\nu}}+\sum_{j=2}^{\infty}\frac{1}{j^{3/2}-1}\right),$$

(A.41) and (A.42) yield

$$2\sum_{|n|,|m|\geqslant n_0}|C_n||R_m|\big(1+e^{(-\Im\omega_n+r_m)T}\big)\big|K(\omega_n-ir_m)\big|\leqslant\frac{4\pi S}{T\gamma^2n_0^{1/2}}\sum_{|n|\geqslant n_0}|C_n|^2\big(1+e^{-2\Im\omega_n T}\big).$$

Plugging the above formula into (A.34), we get

$$\int_{-\infty}^{\infty}k(t)\big|f(t)\big|^2\,dt\geqslant\sum_{|n|\geqslant n_0}|C_n|^2\big(1+e^{-2\Im\omega_n T}\big)\left(K(\omega_n-\overline{\omega}_n)-\frac{4\pi}{T\gamma^2}\left(1+\frac{S}{n_0^{1/2}}\right)\right).$$

Now, in virtue of (A.20) we note that

$$K(\omega_n-\overline{\omega}_n)=\frac{T\pi}{\pi^2+4T^2(\Im\omega_n)^2},$$

so

$$\int_{-\infty}^{\infty}k(t)\big|f(t)\big|^2\,dt\geqslant\sum_{|n|\geqslant n_0}|C_n|^2\big(1+e^{-2\Im\omega_n T}\big)\left(\frac{T\pi}{\pi^2+4T^2(\Im\omega_n)^2}-\frac{4\pi}{T\gamma^2}\left(1+\frac{S}{n_0^{1/2}}\right)\right).\tag{A.43}$$

If we use (A.33) and take

$$n_0 \geqslant S^2/\varepsilon^2,$$

then we get, for any $|n| \geqslant n_0$,

$$\frac{T\pi}{\pi^2 + 4T^2(\Im\omega_n)^2} - \frac{4\pi}{T\gamma^2}\left(1 + \frac{S}{n_0^{1/2}}\right) \geqslant \frac{T\pi}{\pi^2 + T^2\gamma^2\varepsilon/8} - \frac{4\pi}{T\gamma^2}(1+\varepsilon). \tag{A.44}$$

Now, we prove that for $T > \frac{2\pi}{\gamma}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}$

$$\frac{T\pi}{\pi^2 + T^2\gamma^2\varepsilon/8} - \frac{4\pi}{T\gamma^2}(1+\varepsilon) > 0.$$

Indeed,

$$\begin{aligned}
&\frac{T\pi}{\pi^2 + T^2\gamma^2\varepsilon/8} - \frac{4\pi}{T\gamma^2}(1+\varepsilon) \\
&\quad = \pi\,\frac{T^2\gamma^2 - 4(1+\varepsilon)(\pi^2 + T^2\gamma^2\varepsilon/8)}{(\pi^2 + T^2\gamma^2\varepsilon/8)T\gamma^2} = \pi\,\frac{T^2\gamma^2(1-(1+\varepsilon)\varepsilon/2) - 4\pi^2(1+\varepsilon)}{(\pi^2 + T^2\gamma^2\varepsilon/8)T\gamma^2}.
\end{aligned}$$

Since $\varepsilon < 1$, we have $(1+\varepsilon)\varepsilon/2 < \varepsilon$, whence for $T > \frac{2\pi}{\gamma}\sqrt{\frac{1+\varepsilon}{1-\varepsilon}}$

$$\frac{T\pi}{\pi^2 + T^2\gamma^2\varepsilon/8} - \frac{4\pi}{T\gamma^2}(1+\varepsilon) > \pi\,\frac{T^2\gamma^2(1-\varepsilon) - 4\pi^2(1+\varepsilon)}{(\pi^2 + T^2\gamma^2\varepsilon/8)T\gamma^2} > 0.$$

Finally, by (A.43), (A.44) and the definition of $k(t)$ we obtain

$$\int_0^T \left|f(t)\right|^2 dt \geqslant \left(\frac{T\pi}{\pi^2 + T^2\gamma^2\varepsilon/8} - \frac{4\pi}{T\gamma^2}(1+\varepsilon)\right) \sum_{|n|\geqslant n_0} |C_n|^2\left(1 + e^{-2\Im\omega_n T}\right),$$

so the proof is complete. □

To prove Proposition 5.4, we first introduce some auxiliary tools. Indeed, we introduce a family of operators, which will be needed to annihilate a finite number of terms in the Fourier series. Our operators are slightly different from those introduced in [6] and [10]. For that reason and for the reader's convenience, we then proceed to recall and prove some of their properties.

Given $\delta > 0$ and $\omega \in \mathbb{C}$ arbitrarily, we define the linear operator $I_{\delta,\omega}$ as follows: for every continuous function $u : \mathbb{R} \to \mathbb{C}$ the function $I_{\delta,\omega}u : \mathbb{R} \to \mathbb{C}$ is given by the formula

$$I_{\delta,\omega}u(t) := u(t) - \frac{1}{\delta}\int_0^\delta e^{-i\omega s}u(t+s)\,ds, \quad t \in \mathbb{R}. \tag{A.45}$$

The following result states some properties connected with operators $I_{\delta,\omega}$.

Lemma A.3.

(a) *If* $u(t)=e^{i\omega t}$, *then* $I_{\delta,\omega}u=0$.
(b) *If* $u(t)=e^{i\omega' t}$ *with* $\omega'\neq\omega$, *then*

$$I_{\delta,\omega}u(t)=\left(1-\frac{e^{i(\omega'-\omega)\delta}-1}{i(\omega'-\omega)\delta}\right)u(t).$$

(c) *The linear operators* $I_{\delta,\omega}$ *commute, that is*

$$I_{\delta,\omega}I_{\delta',\omega'}u=I_{\delta',\omega'}I_{\delta,\omega}u$$

for all $\delta,\omega,\delta',\omega'$ *and* u.

Proof. (a) By definition, we have

$$I_{\delta,\omega}u(t)=u(t)-\frac{1}{\delta}\int_0^\delta e^{-i\omega s}e^{i\omega(t+s)}\,ds=u(t)-e^{i\omega t}=0.$$

(b) Again by definition, we obtain

$$\begin{aligned}I_{\delta,\omega}u(t)&=u(t)-\frac{1}{\delta}\int_0^\delta e^{-i\omega s}e^{i\omega'(t+s)}\,ds=u(t)-\frac{1}{\delta}\left[\frac{e^{i(\omega'-\omega)s}}{i(\omega'-\omega)}\right]_0^\delta e^{i\omega' t}\\&=\left(1-\frac{e^{i(\omega'-\omega)\delta}-1}{i(\omega'-\omega)\delta}\right)u(t).\end{aligned}$$

(c) It follows at once by definition of operators $I_{\delta,\omega}$. □

Lemma A.4. *For any* $T>0$, $\delta\in(0,T)$, $\omega\in\mathbb{C}$ *and every continuous function* $u:\mathbb{R}\to\mathbb{C}$ *we have*

$$\int_0^T\left|I_{\delta,\omega}u(t)\right|^2dt\leqslant 2\left(1+e^{2|\Im\omega|\delta}\right)\int_0^{T+\delta}\left|u(t)\right|^2dt. \tag{A.46}$$

Proof. For every $t\in[0,T]$, by (A.45) one has

$$\begin{aligned}\left|I_{\delta,\omega}u(t)\right|^2&\leqslant 2\left|u(t)\right|^2+2\left|\frac{1}{\delta}\int_0^\delta e^{-i\omega s}u(t+s)\,ds\right|^2\\&\leqslant 2\left|u(t)\right|^2+\frac{2}{\delta^2}\int_0^\delta\left|e^{-i\omega s}\right|^2ds\int_0^\delta\left|u(t+s)\right|^2ds\\&\leqslant 2\left|u(t)\right|^2+\frac{2}{\delta^2}\int_0^\delta e^{2\Im\omega s}\,ds\int_0^\delta\left|u(t+s)\right|^2ds\\&\leqslant 2\left|u(t)\right|^2+\frac{2}{\delta}e^{2|\Im\omega|\delta}\int_t^{t+\delta}\left|u(x)\right|^2dx.\end{aligned}$$

Integrating the above inequality from 0 to T, we obtain

$$\int_0^T \left|I_{\delta,\omega}u(t)\right|^2 dt \leqslant 2\int_0^T \left|u(t)\right|^2 dt + \frac{2}{\delta}e^{2|\Im\omega|\delta}\int_0^T \int_t^{t+\delta} \left|u(x)\right|^2 dx\, dt. \tag{A.47}$$

Since $\delta \in (0,T)$ we have that

$$\begin{aligned}\int_0^T \int_t^{t+\delta} \left|u(x)\right|^2 dx\, dt &= \int_0^\delta \left|u(x)\right|^2 \int_0^x dt\, dx + \int_\delta^T \left|u(x)\right|^2 \int_{x-\delta}^x dt\, dx + \int_T^{T+\delta} \left|u(x)\right|^2 \int_{x-\delta}^T dt\, dx \\ &= \int_0^{T+\delta} \left|u(x)\right|^2 \int_{\max\{0,x-\delta\}}^{\min\{x,T\}} dt\, dx \\ &\leqslant \int_0^{T+\delta} \left|u(x)\right|^2 \int_{x-\delta}^{x} dt\, dx = \delta \int_0^{T+\delta} \left|u(x)\right|^2 dx.\end{aligned}$$

Plugging this inequality into (A.47), we get

$$\int_0^T \left|I_{\delta,\omega}u(t)\right|^2 dt \leqslant 2\int_0^T \left|u(t)\right|^2 dt + 2e^{2|\Im\omega|\delta}\int_0^{T+\delta} \left|u(x)\right|^2 dx \leqslant 2\left(1+e^{2|\Im\omega|\delta}\right)\int_0^{T+\delta} \left|u(t)\right|^2 dt,$$

that is (A.46). □

We now proceed to define another operator, namely:

$$I_{\delta,\omega,r} := I_{\delta,\omega} \circ I_{\delta,-ir}, \quad \delta > 0,\ \omega \in \mathbb{C},\ r \in \mathbb{R}. \tag{A.48}$$

Some properties of that operator are collected in the following results.

Lemma A.5.

(a) *If $u(t) = e^{i\omega t}$ or $u(t) = e^{rt}$, then $I_{\delta,\omega,r}u = 0$.*
(b) *If $u(t) = e^{i\omega' t}$ with $\omega' \neq \omega$ and $\omega' \neq -ir$, then*

$$I_{\delta,\omega,r}u(t) = \left(1 - \frac{e^{i(\omega'-\omega)\delta}-1}{i(\omega'-\omega)\delta}\right)\left(1 - \frac{e^{(i\omega'-r)\delta}-1}{(i\omega'-r)\delta}\right)u(t).$$

(c) *If $u(t) = e^{r't}$ with $r' \neq r$ and $r' \neq i\omega$, then*

$$I_{\delta,\omega,r}u(t) = \left(1 - \frac{e^{(r'-r)\delta}-1}{(r'-r)\delta}\right)\left(1 - \frac{e^{(r'-i\omega)\delta}-1}{(r'-i\omega)\delta}\right)u(t).$$

(d) *The linear operators $I_{\delta,\omega,r}$ commute, that is*

$$I_{\delta,\omega,r} I_{\delta',\omega',r'} u = I_{\delta',\omega',r'} I_{\delta,\omega,r} u$$

for all $\delta, \omega, r, \delta', \omega', r'$ and u.

Proof. (a) Thanks to (c) and (a) of Lemma A.3, we have

$$I_{\delta,\omega,r}\big(e^{i\omega t}\big) = I_{\delta,\omega}\big(I_{\delta,-ir}\big(e^{i\omega t}\big)\big) = I_{\delta,-ir}\big(I_{\delta,\omega}\big(e^{i\omega t}\big)\big) = I_{\delta,-ir}(0) = 0,$$
$$I_{\delta,\omega,r}\big(e^{rt}\big) = I_{\delta,\omega}\big(I_{\delta,-ir}\big(e^{rt}\big)\big) = I_{\delta,\omega}(0) = 0.$$

(b) By Lemma A.3(a) we get

$$\begin{aligned} I_{\delta,\omega,r} u(t) &= I_{\delta,-ir}\big(I_{\delta,\omega}\big(e^{i\omega' t}\big)\big) = \left(1 - \frac{e^{i(\omega'-\omega)\delta} - 1}{i(\omega'-\omega)\delta}\right) I_{\delta,-ir}\big(e^{i\omega' t}\big) \\ &= \left(1 - \frac{e^{i(\omega'-\omega)\delta} - 1}{i(\omega'-\omega)\delta}\right)\left(1 - \frac{e^{i(\omega'+ir)\delta} - 1}{i(\omega'+ir)\delta}\right) e^{i\omega' t} \\ &= \left(1 - \frac{e^{i(\omega'-\omega)\delta} - 1}{i(\omega'-\omega)\delta}\right)\left(1 - \frac{e^{(i\omega'-r)\delta} - 1}{(i\omega'-r)\delta}\right) e^{i\omega' t}. \end{aligned}$$

(c) It follows by (b) with $\omega' = -ir'$.

(d) It is a consequence of Lemma A.3(c). □

Corollary A.6. *For any $T > 0$, $\delta \in (0, T)$, $\omega \in \mathbb{C}$, $r \in \mathbb{R}$ and every continuous function $u : \mathbb{R} \to \mathbb{C}$ we have*

$$\int_0^T \big|I_{\delta,\omega,r} u(t)\big|^2\, dt \leqslant 4\big(1 + e^{2|\Im\omega|\delta}\big)\big(1 + e^{2|r|\delta}\big) \int_0^{T+2\delta} \big|u(t)\big|^2\, dt. \tag{A.49}$$

Proof. Applying (A.46) two times, first to function $I_{\delta,-ir}u(t)$ and next to $u(t)$, we obtain

$$\begin{aligned} \int_0^T \big|I_{\delta,\omega,r} u(t)\big|^2\, dt &= \int_0^T \big|I_{\delta,\omega} I_{\delta,-ir} u(t)\big|^2\, dt \leqslant 2\big(1 + e^{2|\Im\omega|\delta}\big) \int_0^{T+\delta} \big|I_{\delta,-ir} u(t)\big|^2\, dt \\ &\leqslant 4\big(1 + e^{2|\Im\omega|\delta}\big)\big(1 + e^{2|r|\delta}\big) \int_0^{T+2\delta} \big|u(t)\big|^2\, dt, \end{aligned}$$

that is (A.49). □

Proof of Proposition 5.4. To begin with, we will transform the function

$$f(t) = \sum_{n=-\infty}^{\infty} \big(C_n e^{i\omega_n t} + R_n e^{r_n t}\big)$$

in a series such that the terms corresponding to indices in $\mathcal{F}$ are null, so we can apply assumption (5.8).

To this end, we fix $\varepsilon > 0$ and choose $\delta \in (0, \frac{\varepsilon}{2|\mathcal{F}|} \wedge T)$, where $|\mathcal{F}|$ indicates the number of elements in the set $\mathcal{F}$. Let us denote by I the composition of all linear operators I_{δ,ω_j,r_j}, where $j \in \mathcal{F}$; by Lemma A.5(d) the definition of I does not depend on the order of the operators I_{δ,ω_j,r_j}. Therefore, we can use Lemma A.5 to get

$$If(t) = \sum_{n\notin\mathcal{F}} C_n \prod_{j\in\mathcal{F}} \left(1 - \frac{e^{i(\omega_n-\omega_j)\delta} - 1}{i(\omega_n - \omega_j)\delta}\right)\left(1 - \frac{e^{(i\omega_n-r_j)\delta} - 1}{(i\omega_n - r_j)\delta}\right) e^{i\omega_n t}$$
$$+ \sum_{n\notin\mathcal{F}} R_n \prod_{j\in\mathcal{F}} \left(1 - \frac{e^{(r_n-r_j)\delta} - 1}{(r_n - r_j)\delta}\right)\left(1 - \frac{e^{(r_n-i\omega_j)\delta} - 1}{(r_n - i\omega_j)\delta}\right) e^{r_n t}.$$

If we define for any $n \notin \mathcal{F}$

$$C'_n := C_n \prod_{j\in\mathcal{F}} \left(1 - \frac{e^{i(\omega_n-\omega_j)\delta} - 1}{i(\omega_n - \omega_j)\delta}\right)\left(1 - \frac{e^{(i\omega_n-r_j)\delta} - 1}{(i\omega_n - r_j)\delta}\right),$$
$$R'_n := R_n \prod_{j\in\mathcal{F}} \left(1 - \frac{e^{(r_n-r_j)\delta} - 1}{(r_n - r_j)\delta}\right)\left(1 - \frac{e^{(r_n-i\omega_j)\delta} - 1}{(r_n - i\omega_j)\delta}\right),$$

then we have

$$If(t) = \sum_{n\notin\mathcal{F}} \left(C'_n e^{i\omega_n t} + R'_n e^{r_n t}\right).$$

Therefore, applying estimate (5.8) to $If(t)$ we obtain

$$\int_0^T \left|If(t)\right|^2 dt \geqslant c'_1 \sum_{n\notin\mathcal{F}} \left|C'_n\right|^2. \tag{A.50}$$

Next, we choose $\delta \in (0, \frac{\varepsilon}{2|\mathcal{F}|} \wedge T)$ such that none of the products

$$\prod_{j\in\mathcal{F}} \left(1 - \frac{e^{i(\omega_n-\omega_j)\delta} - 1}{i(\omega_n - \omega_j)\delta}\right)\left(1 - \frac{e^{(i\omega_n-r_j)\delta} - 1}{(i\omega_n - r_j)\delta}\right), \quad n \notin \mathcal{F},$$

vanishes. This is possible because the analytic function $1 - \frac{e^z-1}{z}$ does not vanish identically and, since the numbers $\omega_n - \omega_j$ and $i\omega_n - r_j$ are all different from zero, we have to exclude only a countable set of values of δ.

Now, we note that there exists a constant $c' > 0$ such that

$$\left|\prod_{j\in\mathcal{F}} \left(1 - \frac{e^{i(\omega_n-\omega_j)\delta} - 1}{i(\omega_n - \omega_j)\delta}\right)\left(1 - \frac{e^{(i\omega_n-r_j)\delta} - 1}{(i\omega_n - r_j)\delta}\right)\right|^2 \geqslant c' \quad \forall n \notin \mathcal{F}. \tag{A.51}$$

Indeed, it is sufficient to observe that for any fixed $j \in \mathcal{F}$ we have

$$\left|\frac{e^{i(\omega_n-\omega_j)\delta}-1}{i(\omega_n-\omega_j)\delta}\right| \leqslant \frac{e^{-\Im(\omega_n-\omega_j)\delta}+1}{|\omega_n-\omega_j|\delta} \to 0 \quad \text{as } |n|\to\infty,$$
$$\left|\frac{e^{(i\omega_n-r_j)\delta}-1}{(i\omega_n-r_j)\delta}\right| \leqslant \frac{e^{-(\Im\omega_n+r_j)\delta}+1}{|\omega_n+ir_j|\delta} \to 0 \quad \text{as } |n|\to\infty,$$

in view of (5.6). As a result, the product

$$\prod_{j\in\mathcal{F}}\left(1-\frac{e^{i(\omega_n-\omega_j)\delta}-1}{i(\omega_n-\omega_j)\delta}\right)\left(1-\frac{e^{(i\omega_n-r_j)\delta}-1}{(i\omega_n-r_j)\delta}\right)$$

tends to 1 as $|n|\to\infty$, so that it is minorized, e.g., by $1/2$ for all sufficiently large $|n|$. Therefore, (A.50) and (A.51) yield

$$\int_0^T \left|If(t)\right|^2 dt \geqslant c_1'c' \sum_{n\notin\mathcal{F}} |C_n|^2. \tag{A.52}$$

On the other hand, applying (A.49) repeatedly with $\omega=\omega_j$ and $r=r_j$, $j\in\mathcal{F}$, we have

$$\int_0^T \left|If(t)\right|^2 dt \leqslant 4^{|\mathcal{F}|}\prod_{j\in\mathcal{F}}\left(1+e^{2|\Im\omega_j|\delta}\right)\left(1+e^{2|r_j|\delta}\right)\int_0^{T+2|\mathcal{F}|\delta}\left|f(t)\right|^2 dt,$$

from which, using (A.52) and $2|\mathcal{F}|\delta<\varepsilon$, it follows

$$\sum_{n\notin\mathcal{F}}|C_n|^2 \leqslant \frac{4^{|\mathcal{F}|}}{c_1'c'}\prod_{j\in\mathcal{F}}\left(1+e^{|\Im\omega_j|\varepsilon/|\mathcal{F}|}\right)\left(1+e^{|r_j|\varepsilon/|\mathcal{F}|}\right)\int_0^{T+\varepsilon}\left|f(t)\right|^2 dt,$$

whence

$$\sum_{n\notin\mathcal{F}}|C_n|^2 \leqslant \frac{4^{2|\mathcal{F}|}}{c_1'c'}\int_0^T\left|f(t)\right|^2 dt. \tag{A.53}$$

In addition, thanks to the triangle inequality, (5.9) and (A.53) we get

$$\begin{aligned}
\int_0^T \left|\sum_{n\in\mathcal{F}}\left(C_ne^{i\omega_nt}+R_ne^{r_nt}\right)\right|^2 dt &= \int_0^T\left|f(t)-\sum_{n\notin\mathcal{F}}\left(C_ne^{i\omega_nt}+R_ne^{r_nt}\right)\right|^2 dt\\
&\leqslant 2\int_0^T\left|f(t)\right|^2dt+2\int_0^T\left|\sum_{n\notin\mathcal{F}}\left(C_ne^{i\omega_nt}+R_ne^{r_nt}\right)\right|^2dt\\
&\leqslant 2\int_0^T\left|f(t)\right|^2dt+2c_2'\sum_{n\notin\mathcal{F}}|C_n|^2\\
&\leqslant 2\left(1+c_2'\frac{4^{2|\mathcal{F}|}}{c_1'c'}\right)\int_0^T\left|f(t)\right|^2dt.
\end{aligned} \tag{A.54}$$

Let us note that the expression

$$\int_0^T \left| \sum_{n\in\mathcal{F}} \left(C_n e^{i\omega_n t} + R_n e^{r_n t} \right) \right|^2 dt$$

is a positive semidefinite quadratic form of the variable $(\{C_n\}_{n\in\mathcal{F}}, \{R_n\}_{n\in\mathcal{F}}) \in \mathbb{C}^{|\mathcal{F}|} \times \mathbb{R}^{|\mathcal{F}|}$. Moreover, it is positive *definite*, because the functions $e^{i\omega_n t}$, $e^{r_n t}$, $n \in \mathcal{F}$, are linearly independent. Hence, there exists a constant $c'' > 0$ such that

$$\int_0^T \left| \sum_{n\in\mathcal{F}} \left(C_n e^{i\omega_n t} + R_n e^{r_n t} \right) \right|^2 dt \geqslant c'' \sum_{n\in\mathcal{F}} \left(|C_n|^2 + |R_n|^2 \right),$$

so, from (A.54) and the above inequality we deduce that

$$\sum_{n\in\mathcal{F}} |C_n|^2 \leqslant \frac{2}{c''} \left(1 + c_2' \frac{4^{2|\mathcal{F}|}}{c_1' c'} \right) \int_0^T \left| f(t) \right|^2 dt.$$

Finally, from the above estimate and (A.53) the desired inequality (5.10) follows. □

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