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Almost global existence of classical discontinuous solutions to genuinely nonlinear hyperbolic systems of conservation laws with small BV initial data

Zhi-Qiang Shao [1]

Department of Mathematics, Fuzhou University, Fuzhou 350002, China

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A B S T R A C T

In the present paper, the author investigates the initial problem for genuinely nonlinear quasilinear hyperbolic systems of conservation laws under small BV perturbations of the Riemann initial data, where the perturbations are in BV but they are assumed to be C^1-smooth, with bounded and possibly large C^1-norms. Combining the techniques employed by Li and Kong with the modified Glimm's functional, the author obtains the almost global existence and lifespan of classical discontinuous solutions to the above problem. This result is also applied to the system of one-dimensional isentropic flow.

1. Introduction and main result

Consider the following quasilinear hyperbolic system of conservation laws:

$$\partial_t u + \partial_x f(u) = 0, \quad x \in \mathbf{R},\ t > 0, \tag{1.1}$$

E-mail address: zqshao_fzu@yahoo.com.cn.

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where $u = (u_1, \dots, u_n)^T$ is the unknown vector-valued function of (t, x), $f : \mathbf{R}^n \to \mathbf{R}^n$ is a given C^3 vector function of u.

It is assumed that system (1.1) is strictly hyperbolic, i.e., for any given u on the domain under consideration, the Jacobian $A(u) = \nabla f(u)$ has n real distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{1.2}$$

Let $l_i(u) = (l_{i1}(u), \dots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \dots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$ $(i = 1, \dots, n)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad \big(\text{resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)\big). \tag{1.3}$$

We have

$$\det\big|l_{ij}(u)\big| \neq 0 \quad \big(\text{equivalently, } \det\big|r_{ij}(u)\big| \neq 0\big). \tag{1.4}$$

Without loss of generality, we may assume that on the domain under consideration

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \dots, n) \tag{1.5}$$

and

$$r_i^T(u)r_i(u) \equiv 1 \quad (i = 1, \dots, n), \tag{1.6}$$

where δ_{ij} stands for the Kronecker symbol.

Clearly, all $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)$ $(i, j = 1, \dots, n)$ have the same regularity as $A(u)$, i.e., C^2 regularity.

We also assume that on the domain under consideration, each characteristic field is genuinely nonlinear in the sense of Lax (cf. [1]):

$$\nabla\lambda_i(u)r_i(u) \neq 0. \tag{1.7}$$

We are interested in the generalized Riemann problem for system (1.1), which is a Cauchy problem with a piecewise C^1 initial data of the form:

$$t = 0: \quad u = \begin{cases} u_0^-(x), & x \leqslant 0, \\ u_0^+(x), & x \geqslant 0, \end{cases} \tag{1.8}$$

where $u_0^-(x)$ and $u_0^+(x)$ are C^1 vector functions defined for $x \leqslant 0$ and $x \geqslant 0$ respectively with

$$u_0^-(0) \neq u_0^+(0). \tag{1.9}$$

Problem (1.1) and (1.8) may be regarded as a perturbation of the corresponding Riemann problem (1.1) and

$$t = 0: \quad u = \begin{cases} \hat{u}_-, & x \leqslant 0, \\ \hat{u}_+, & x \geqslant 0, \end{cases} \tag{1.10}$$

in which

$$\hat{u}_\pm = u_0^\pm(0). \tag{1.11}$$

Let

$$\theta = |\hat{u}_- - \hat{u}_+|. \tag{1.12}$$

When $\theta > 0$ is suitably small, by Lax [1], the Riemann problem (1.1) and (1.10) admits a unique self-similar solution composed of $n+1$ constant states $\hat{u}^{(0)} = \hat{u}_-, \hat{u}^{(1)}, \ldots, \hat{u}^{(n-1)}, \hat{u}^{(n)} = \hat{u}_+$ separated by shocks or centered rarefaction waves. As in [2], this kind of solution is simply called the Lax's Riemann solution of the system (1.1).

For the self-similar solution of the Riemann problem of general quasilinear hyperbolic systems of conservation laws, the local nonlinear structure stability has been proved by Li and Yu [3] for one-dimensional case, and by Majda [4] for multidimensional case. If system (1.1) is strictly hyperbolic and genuinely nonlinear, Li and Zhao [5] proved the global structure stability of the self-similar solution containing only n shocks under perturbation (1.8) satisfying (1.10). In their work they do not require the amplitude of the self-similar solution is small, although the existence of the self-similar solution with non-small amplitude still remains open. If system (1.1) is strictly hyperbolic and linearly degenerate, Li and Kong [6] proved the global structure stability of the self-similar solution with small amplitude under perturbation (1.8) satisfying (1.10). In this case the self-similar solution contains only n contact discontinuities. Precisely speaking, under certain reasonable hypotheses Li and Zhao [5] obtained the following well-known result.

Theorem 1.1. *Suppose that system* (1.1) *is strictly hyperbolic and genuinely nonlinear. Suppose furthermore that $u_0^-(x)$ and $u_0^+(x)$ are all C^1 vector functions on $x \leqslant 0$ and on $x \geqslant 0$ respectively, $f(u)$ is a C^2 vector function and*

$$\theta \triangleq |\hat{u}_+ - \hat{u}_-| = \left|u_0^+(0) - u_0^-(0)\right| > 0$$

is suitably small. Suppose finally that the self-similar solution $u = U(\frac{x}{t})$ of the Riemann problem (1.1) *and* (1.10) *is composed of $n+1$ constant states $\hat{u}^{(0)} = \hat{u}_-, \hat{u}^{(1)}, \ldots, \hat{u}^{(n-1)}, \hat{u}^{(n)} = \hat{u}_+$ and n non-degenerate typical shocks $x = \widehat{F}^i t$ $(i = 1, \ldots, n)$:*

$$u = U\left(\frac{x}{t}\right) = \begin{cases} \hat{u}^{(0)}, & x \leqslant \widehat{F}^1 t, \\ \hat{u}^{(i)}, & \widehat{F}^i t \leqslant x \leqslant \widehat{F}^{i+1} t \ (i = 1, \ldots, n-1), \\ \hat{u}^{(n)}, & x \geqslant \widehat{F}^n t. \end{cases}$$

Then there exists a positive constant ε so small that if

$$\left|u_0^-(x) - u_0^-(0)\right|, \left|u_0^{-\prime}(x)\right| \leqslant \frac{\varepsilon}{1+|x|}, \quad \forall x \leqslant 0,$$

$$\left|u_0^+(x) - u_0^+(0)\right|, \left|u_0^{+\prime}(x)\right| \leqslant \frac{\varepsilon}{1+|x|}, \quad \forall x \geqslant 0,$$

then problem (1.1) *and* (1.8) *admits a unique global classical discontinuous solution $u = u(t,x)$ only containing n shocks $x = x_i(t)$ $(x_i(0) = 0)$ $(i = 1, \ldots, n)$, such that $u(t,x)$ belongs to C^1 on each domain D^i $(i = 0, 1, \ldots, n)$ and $x_i(t)$ $(i = 1, \ldots, n)$ to C^2 on $t \geqslant 0$ with*

$$\left|u(t,x) - u(0,0)\right| \leqslant \frac{K\varepsilon}{1+t}, \quad \forall (t,x) \in D^i \ (i = 0, 1, \ldots, n),$$

$$\left|\frac{\partial u}{\partial x}(t,x)\right|, \left|\frac{\partial u}{\partial t}(t,x)\right| \leqslant \frac{K\varepsilon}{1+t}, \quad \forall (t,x) \in D^i \ (i = 0, 1, \ldots, n),$$

$$\left|x_i'(t) - x_i'(0)\right| \leqslant \frac{K\varepsilon}{1+t}, \quad \forall t \geqslant 0 \ (i = 1, \ldots, n),$$

where

$$D^0 = \{(t,x) \mid t \geqslant 0,\ x \leqslant x_1(t)\},$$
$$D^i = \{(t,x) \mid t \geqslant 0,\ x_i(t) \leqslant x \leqslant x_{i+1}(t)\} \quad (i = 1, \dots, n-1),$$
$$D^n = \{(t,x) \mid t \geqslant 0,\ x \geqslant x_n(t)\}$$

and K is a positive constant independent of t. Moreover, $u(0,0) = \hat{u}^{(i)}$ on the domain D^i $(i = 0, 1, \dots, n)$ and $x_i'(0) = \widehat{F}^i$ $(i = 1, \dots, n)$. Therefore, as a global perturbation, $u(t,x)$ possesses a similar structure to that of the self-similar solution to Riemann problem (1.1) and (1.10) on $t \geqslant 0$.

Remark 1.1. Recently, under certain reasonable hypotheses Kong [2,7] proved that the Lax's Riemann solution of general $n \times n$ quasilinear hyperbolic system of conservation laws is globally structurally stable if and only if it contains only non-degenerate shocks and contact discontinuities, but no rarefaction waves and other weak discontinuities. Shao [8,9] also studied the global structure stability and instability of this kind of Lax's Riemann solution with small amplitude in a half space.

However, it is well known that the BV space is a suitable framework for one-dimensional Cauchy problem for the hyperbolic systems of conservation laws (see Bressan [10], Glimm [11]), the result in Bressan [12] suggests that one may achieve global smoothness even if the C^1-norm of the initial data is large. So the following question arises naturally: can we obtain the global existence and uniqueness of piecewise C^1 solution containing only shocks to a class of the generalized Riemann problem, which can be regarded as a small BV perturbation of the corresponding Riemann problem, for system (1.1) with the following piecewise C^1 initial data:

$$t = 0: \quad u = \begin{cases} \hat{u}_- + u_-(x), & x \leqslant 0, \\ \hat{u}_+ + u_+(x), & x \geqslant 0, \end{cases} \tag{1.13}$$

where $u_\pm(x) \in C^1$ with bounded and possibly large C^1-norm, but of small bounded variation, such that

$$\|u_-(x)\|_{C^1}, \|u_+(x)\|_{C^1} \leqslant M, \tag{1.14}$$

for some $M > 0$ bounded but possibly large, and also such that

$$\int_0^{+\infty} |u_+'(x)|\, dx, \int_{-\infty}^0 |u_-'(x)|\, dx \leqslant \varepsilon, \tag{1.15}$$

for some $\varepsilon > 0$ sufficiently small? Here, it is important to mention that the global existence of weak solutions to a strictly hyperbolic system of conservation laws in one space dimension when the initial data is a small BV perturbation of a solvable Riemann problem has been proved by Schochet [13], unfortunately his method is not useful to show that the solutions are still shock waves. An analogous result on stability of a strong shock wave under perturbations of small bounded variation is stated by Corli and Sable-Tougeron [14]. In this paper we exploit to some extent the ideas of Bressan [12], we will develop the method of using continuous Glimm's functional to provide a new, concise proof of an estimate on the lifespan of the piecewise C^1 solution containing only shocks to the generalized Riemann problem under consideration mentioned above. The basic idea we will use here is to combine the techniques employed by Li and Kong [5], especially both the decomposition of waves and the global behavior of waves on the shock curves, with the method of using continuous Glimm's functional. However, we must modify Glimm's functional in order to take care of the presence of shock waves. This makes our new analysis more complicated than those for the C^1 solutions of the Cauchy problem for linearly degenerate quasilinear hyperbolic systems in Bressan [12], Zhou [15], Dai and Kong [16].

As in [17], the aim of this paper is to study the global structure stability of Lax's Riemann solution containing only shocks. In this case, we shall first get a lower bound of the lifespan of the piecewise C^1 solution containing only shocks to the generalized Riemann problem.

To do so, we consider the generalized Riemann problem for the system (1.1) with the following piecewise C^1 initial data:

$$t=0: \quad u=\begin{cases} \hat{u}_- + \varepsilon u_-(x), & x \leqslant 0, \\ \hat{u}_+ + \varepsilon u_+(x), & x \geqslant 0, \end{cases} \tag{1.16}$$

where ε $(0<\varepsilon \ll |\hat{u}_+ - \hat{u}_-|)$ is a small parameter, $u_-(x)$ and $u_+(x)$ are C^1 vector functions defined on $x \leqslant 0$ and $x \geqslant 0$ respectively, which satisfy

$$u_-(0)=u_+(0)=0, \tag{1.17}$$

$$\left\|u_-(x)\right\|_{C^1}, \left\|u_+(x)\right\|_{C^1} \leqslant K_1 \tag{1.18}$$

and

$$\int_0^{+\infty} \left|u'_+(x)\right| dx, \int_{-\infty}^{0} \left|u'_-(x)\right| dx \leqslant K_2, \tag{1.19}$$

where K_1 and K_2 are positive constants independent of ε.

To state our result precisely, we introduce the concept of the lifespan of the piecewise C^1 solution to the generalized Riemann problem (1.1) and (1.16) as follows.

Definition 1.1. The existence of piecewise C^1 local solutions to the generalized Riemann problem (1.1) and (1.16) is guaranteed by the monograph of Li and Yu [3]. The lifespan is defined to be the supremum of the time T such that a Li–Yu solution exists for $0<t \leqslant T$. This definition will coincide with the usual definition of the lifespan of C^1 solution (without shocks).

Our main results can be summarized as follows.

Theorem 1.2. *Suppose that system* (1.1) *is strictly hyperbolic and each characteristic field is genuinely nonlinear. Suppose furthermore that $u_-(x)$ and $u_+(x)$ are all C^1 vector functions on $x \leqslant 0$ and on $x \geqslant 0$ respectively satisfying* (1.17)–(1.19) *and*

$$\theta = |\hat{u}_+ - \hat{u}_-| = \left|u_0^+(0) - u_0^-(0)\right| > 0 \tag{1.20}$$

is suitably small. Suppose finally that the self-similar solution $u=U(\frac{x}{t})$ of the Riemann problem (1.1) *and* (1.10) *is composed of $n+1$ constant states $\hat{u}^{(0)}=\hat{u}_-, \hat{u}^{(1)}, \dots, \hat{u}^{(n-1)}, \hat{u}^{(n)}=\hat{u}_+$ and n non-degenerate typical shocks $x=s_i t$ $(i=1,\dots,n)$:*

$$u=U\left(\frac{x}{t}\right)=\begin{cases} \hat{u}^{(0)}, & x \leqslant s_1 t, \\ \hat{u}^{(i)}, & s_i t \leqslant x \leqslant s_{i+1} t \ (i=1,\dots,n-1), \\ \hat{u}^{(n)}, & x \geqslant s_n t. \end{cases} \tag{1.21}$$

Then for small $\theta>0$, there exists a constant $\varepsilon_0>0$ so small that for any fixed $\varepsilon \in (0,\varepsilon_0]$, the lifespan $\widetilde{T}(\varepsilon)$ of the piecewise C^1 solution to the generalized Riemann problem (1.1) *and* (1.16) *satisfies*

$$\widetilde{T}(\varepsilon) \geqslant K_3 \varepsilon^{-1}, \tag{1.22}$$

where K_3 is a positive constant independent of ε. Moreover, when $u = u(t, x)$ blows up in a finite time, $u = u(t, x)$ itself is bounded on the domain $[0, \widetilde{T}(\varepsilon)) \times \mathbf{R}$, while the first-order derivatives of $u = u(t, x)$ tend to be unbounded as $t \nearrow \widetilde{T}(\varepsilon)$.

Remark 1.2. Our result implies that classical discontinuous solutions to the generalized Riemann problem under consideration exist almost globally in time. We refer to Kong [18] for the definition of an almost global solution.

Remark 1.3. Suppose that (1.1) is a non-strictly hyperbolic system with characteristics with constant multiplicity, say, on the domain under consideration,

$$\lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u) \quad (1 \leqslant p \leqslant n). \tag{1.23}$$

Then the conclusion of Theorem 1.2 still holds (cf. [16]).

Remark 1.4. The relation to the Li–Kong paper is as follows: the Li–Kong initial data will be shown to satisfy (1.16)–(1.19). Because, the Li–Kong solution is a global solution and naturally satisfies the conditions for almost global solutions.

Some of the results related to these topics are listed below. Chen et al. [19–22] investigated the asymptotic stability of Riemann waves for hyperbolic conservation laws. Hsiao and Tang [23] investigated the construction and qualitative behavior of the solution of the perturbed Riemann problem for the system of one-dimensional isentropic flow with damping. Xin et al. [24,25] proved the nonlinear stability of contact discontinuities in systems of conservation laws. Smoller et al. [26] investigated the instability of rarefaction shocks in systems of conservation laws. For the overcompressive shock waves, Liu [27] proved the nonlinear stability and instability. Bressan and LeFloch [28] investigated the structural stability and regularity of entropy solutions to hyperbolic systems of conservation laws. Lions et al. [29] proved the existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates. Recently, L^1 stability for hyperbolic systems of conservation laws was proved by Bressan, Liu and Yang [30], within the class of solutions with small total variation (see also [10,31–33]). Their results were extended by Lewicka in [34] to the case where the initial data is a BV perturbation of a possibly large Riemann data. Liu and Xin [35] proved the nonlinear stability of discrete shocks for systems of conservation laws. Dafermos [36] studied the entropy and the stability of classical solutions of hyperbolic systems of conservation laws. For a relaxation system in several space dimensions, Luo and Xin [37] proved the nonlinear stability of shock fronts. Liu and Xin [38] investigated the nonlinear stability of rarefaction waves for compressible Navier–Stokes equations. Hsiao and Pan [39] investigated the nonlinear stability of rarefaction waves for a rate-type viscoelastic system. Moreover, the nonlinear stability of an undercompressive shock for complex Burgers equation was studied by Liu and Zumbrun [40]. For the viscous conservation laws, the theory of nonlinear stability of shock waves was established (see [41,42] and the references therein).

The rest of this paper is organized as follows. For the sake of completeness, in Section 2, we briefly recall John's formula on the decomposition of waves with some supplements and give a generalized Hörmander Lemma. In Section 3, we first review the definition of shock waves and then analyze some properties of waves on the shock curves, which will play an important role in our proof. The main result, Theorem 1.2 is proved in Section 4. Some applications with physical interest will be given in Section 5.

2. John's formula, generalized Hörmander Lemma

For the sake of completeness, in this section we briefly recall John's formula on the decomposition of waves with some supplements, which will play an important role in our proof.

Let

$$v_i = l_i(u)u \quad (i = 1, \dots, n) \tag{2.1}$$

and

$$w_i = l_i(u)u_x \quad (i = 1, \dots, n), \tag{2.2}$$

where $l_i(u) = (l_{i1}(u), \dots, l_{in}(u))$ denotes the i-th left eigenvector.

By (1.5), it is easy to see that

$$u = \sum_{k=1}^{n} v_k r_k(u) \tag{2.3}$$

and

$$u_x = \sum_{k=1}^{n} w_k r_k(u). \tag{2.4}$$

Let

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x} \tag{2.5}$$

be the directional derivative along the i-th characteristic. We have (cf. [43,2,6])

$$\frac{dv_i}{d_i t} = \sum_{j,k=1}^{n} \beta_{ijk}(u) v_j w_k \quad (i = 1, \dots, n), \tag{2.6}$$

where

$$\beta_{ijk}(u) = \big(\lambda_k(u) - \lambda_i(u)\big) l_i(u) \nabla r_j(u) r_k(u). \tag{2.7}$$

Hence, we have

$$\beta_{iji}(u) \equiv 0, \quad \forall i, j. \tag{2.8}$$

On the other hand, we have (cf. [43,2,6])

$$\frac{dw_i}{d_i t} = \sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k \quad (i = 1, \dots, n), \tag{2.9}$$

where

$$\gamma_{ijk}(u) = \frac{1}{2}\big\{\big(\lambda_j(u) - \lambda_k(u)\big) l_i(u) \nabla r_k(u) r_j(u) - \nabla \lambda_i(u) r_j(u) \delta_{ik} + (j|k)\big\}, \tag{2.10}$$

in which $(j|k)$ denotes all the terms obtained by changing j and k in the previous terms. We have

$$\gamma_{ijj}(u) \equiv 0, \quad \forall j \neq i \ (i, j = 1, \dots, n) \tag{2.11}$$

and

$$\gamma_{iii}(u) \equiv -\nabla\lambda_i(u) r_i(u) \quad (i = 1, \dots, n). \tag{2.12}$$

Noting (2.4), by (2.9) we have (cf. [16])

$$\frac{\partial w_i}{\partial t} + \frac{\partial(\lambda_i(u) w_i)}{\partial x} = \sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k \stackrel{\text{def}}{=} G_i(t,x), \tag{2.13}$$

equivalently,

$$d\big[w_i\big(dx - \lambda_i(u)\,dt\big)\big] = \sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k\, dt \wedge dx = G_i(t,x)\, dt \wedge dx, \tag{2.14}$$

where

$$\Gamma_{ijk}(u) = \frac{1}{2}\big(\lambda_j(u) - \lambda_k(u)\big) l_i(u)\big[\nabla r_k(u) r_j(u) - \nabla r_j(u) r_k(u)\big]. \tag{2.15}$$

Hence, we have

$$\Gamma_{ijj}(u) \equiv 0, \quad \forall i, j. \tag{2.16}$$

Lemma 2.1 *(Generalized Hörmander Lemma). Suppose that $u = u(t,x)$ is a piecewise C^1 solution to system (1.1), τ_1 and τ_2 are two C^1 arcs which are never tangent to the i-th characteristic direction, and $\mathcal{D}$ is the domain bounded by τ_1, τ_2 and two i-th characteristic curves L_i^- and L_i^+. Suppose furthermore that the domain $\mathcal{D}$ contains m C^1 curves of discontinuity of u, denoted by $\widehat{C}_j$: $x = x_j(t)$ $(j = 1, \dots, m)$, which are never tangent to the i-th characteristic direction. Then we have*

$$\begin{aligned}\int_{\tau_1} \big|w_i\big(dx - \lambda_i(u)\,dt\big)\big| \leqslant{}& \int_{\tau_2} \big|w_i\big(dx - \lambda_i(u)\,dt\big)\big| + \sum_{j=1}^{m} \int_{\widehat{C}_j} \big|[w_i]\,dx - \big[w_i\lambda_i(u)\big]\,dt\big| \\ &+ \iint_{\mathcal{D}} \left| \sum_{j,k=1}^{n} \Gamma_{ijk}(u) w_j w_k \right| dt\,dx,\end{aligned} \tag{2.17}$$

where $\Gamma_{ijk}(u)$ is given by (2.15) and $[w_i] = w_i^+ - w_i^-$ denotes the jump of w_i over the curve of discontinuity $\widehat{C}_j$ $(j = 1, \dots, m)$, etc.

The proof can be found in Li and Kong [6].

3. Shock waves

In this section, we first review the definition of shocks and then analyze some properties of waves on the shock curves, which will play an important role in our proof.

Definition 3.1. A piecewise C^1 vector function $u = u(t,x)$ is called a classical discontinuous solution containing a k-th shock $x = x_k(t)$ $(x_k(0) = 0)$ for system (1.1), if $u = u(t,x)$ satisfies system (1.1) away from $x = x_k(t)$ in the classical sense and satisfies on $x = x_k(t)$ the following Rankine–Hugoniot condition:

$$f(u^+) - f(u^-) = s(u^+ - u^-) \tag{3.1}$$

and the Lax entropy condition:

$$\lambda_k(u^+) < s < \lambda_k(u^-), \qquad \lambda_{k+1}(u^+) > s > \lambda_{k-1}(u^-), \tag{3.2}$$

where $u^\pm = u^\pm(t, x_k(t)) \triangleq u(t, x_k(t) \pm 0)$ and $s = \frac{dx_k(t)}{dt}$ (when $k = 1$ (resp. $k = n$), the term $\lambda_{k-1}(u^-)$ (resp. $\lambda_{k+1}(u^+)$) disappears in (3.2)).

Definition 3.1 can be found in [1] or [3].

The following lemmas give some properties of waves on the shock curves.

Lemma 3.1. *On the k-th shock $x = x_k(t)$, it holds that*

$$v_i^+ = v_i^- + O(|v^\pm|^2) \quad (i = 1, \dots, k-1, k+1, \dots, n), \tag{3.3}$$

provided that $|u^\pm|$ is suitably small, where v_i is defined by (2.1) and $v_i^\pm \triangleq v_i(t, x_k(t) \pm 0)$, etc.

Introduce

$$A(u^-, u^+) = \int_0^1 \nabla f(u^- + \sigma(u^+ - u^-))\, d\sigma. \tag{3.4}$$

It follows from (1.2) that if $|u^+ - u^-|$ is suitably small, then the matrix $A(u^-, u^+)$ has n distinct real eigenvalues:

$$\lambda_1(u^-, u^+) < \lambda_2(u^-, u^+) < \cdots < \lambda_n(u^-, u^+). \tag{3.5}$$

Lemma 3.2. *On the k-th shock $x = x_k(t)$, it holds that*

$$\begin{aligned} w_i^- = w_i^+ &+ O\Big(|u^+ - u^-| \cdot \sum_{j\neq k} |w_j^\pm|\Big) + O\big(|u^+ - u^-| \cdot |(\lambda_k(u^-, u^+) - \lambda_k(u^+))w_k^+|\big) \\ &+ O\big(|u^+ - u^-| \cdot |(\lambda_k(u^-, u^+) - \lambda_k(u^-))w_k^-|\big) \quad (i = 1, \dots, k-1, k+1, \dots, n), \end{aligned} \tag{3.6}$$

provided that $|u^+ - u^-|$ is suitably small.

The proofs of Lemmas 3.1–3.2 can be found in Kong [2].

Corollary 3.1. *On the k-th shock $x = x_k(t)$, it holds that*

$$\begin{aligned} (w_i\lambda_i(u))^+ = (w_i\lambda_i(u))^- &+ O\Big(|u^+ - u^-| \cdot \sum_{j\neq k} |w_j^\pm|\Big) \\ &+ O\big(|u^+ - u^-| \cdot |(\lambda_k(u^-, u^+) - \lambda_k(u^+))w_k^+|\big) \\ &+ O\big(|u^+ - u^-| \cdot |(\lambda_k(u^-, u^+) - \lambda_k(u^-))w_k^-|\big) \quad (i = 1, \dots, k-1, k+1, \dots, n), \end{aligned} \tag{3.7}$$

provided that $|u^+ - u^-|$ is suitably small.

Proof. Noting

$$\big(w_i\lambda_i(u)\big)^+ - \big(w_i\lambda_i(u)\big)^- = \big[w_i^+ - w_i^-\big]\big(\lambda_i(u)\big)^+ + w_i^-\big[\big(\lambda_i(u)\big)^+ - \big(\lambda_i(u)\big)^-\big], \tag{3.8}$$

from (3.6), we immediately get (3.7). □

4. Proof of Theorem 1.2

For the sake of simplicity and without loss of generality, we may suppose that

$$0 < \lambda_1(0) < \lambda_2(0) < \cdots < \lambda_n(0) \tag{4.1}$$

and

$$|\hat{u}_\pm| \leqslant \theta. \tag{4.2}$$

By the existence and uniqueness of local classical discontinuous solutions of quasilinear hyperbolic systems of conservation laws (see [3]), when $\theta > 0$ is suitably small, the generalized Riemann problem (1.1) and (1.16) admits a unique piecewise C^1 solution $u = u(t,x)$ containing only n shocks $x = x_i(t)$ $(i = 1,\dots,n)$ on the strip $[0,h]\times \mathbf{R}$, where $h > 0$ is a small number; moreover, this solution has a local structure similar to the one of the self-similar solution to the corresponding Riemann problem. In order to prove Theorem 1.2, we first establish some uniform a priori estimates on u and u_x on the domain of existence of the piecewise C^1 solution $u = u(t,x)$.

By (4.1), there exist sufficiently small positive constants δ and δ_0 such that

$$\lambda_{i+1}(u) - \lambda_i(v) \geqslant \delta_0, \quad \forall |u|, |v| \leqslant \delta \ (i = 1,\dots,n-1). \tag{4.3}$$

For the time being it is supposed that on the domain of existence of the piecewise C^1 solution $u = u(t,x)$ to the generalized Riemann problem (1.1) and (1.16), we have

$$\big|u(t,x)\big| \leqslant \delta. \tag{4.4}$$

At the end of the proof of Lemma 4.5, we will explain that this hypothesis is reasonable.

For any fixed $T > 0$, let

$$U_\infty(T) = \sup_{0\leqslant t\leqslant T}\ \sup_{x\in\mathbf{R}}\big|u(t,x)\big|, \tag{4.5}$$

$$V_\infty(T) = \sup_{0\leqslant t\leqslant T}\ \sup_{x\in\mathbf{R}}\big|v(t,x)\big|, \tag{4.6}$$

$$W_\infty(T) = \sup_{0\leqslant t\leqslant T}\ \sup_{x\in\mathbf{R}}\big|w(t,x)\big|, \tag{4.7}$$

$$\widetilde{W}_1(T) = \max_{i=1,\dots,n}\ \max_{j\neq i}\ \sup_{\widetilde{C}_j}\int_{\widetilde{C}_j}\big|w_i(t,x)\big|\,dt, \tag{4.8}$$

$$W_1(T) = \max_{i=1,\dots,n}\int_0^T\big|\big(x_i'(t) - \lambda_i\big(u\big(t,x_i(t)\pm 0\big)\big)\big)w_i\big(t,x_i(t)\pm 0\big)\big|\,dt, \tag{4.9}$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbf{R}^n$, $v = (v_1,\dots,v_n)^T$ and $w = (w_1,\dots,w_n)^T$ in which v_i and w_i are defined by (2.1) and (2.2) respectively, while $\widetilde{C}_j$ stands for any given j-th characteristic on the domain $[0,T]\times\mathbf{R}$. In (4.4)–(4.7), on any shock $x=x_k(t)$ the values of $u(t,x)$, $v(t,x)$ and $w(t,x)$ are taken to be $u^{\pm}(t,x)=u(t,x_k(t)\pm 0)$, $v^{\pm}(t,x)=v(t,x_k(t)\pm 0)$ and $w^{\pm}(t,x)=w(t,x_k(t)\pm 0)$. Clearly, $V_\infty(T)$ is equivalent to $U_\infty(T)$.

First we recall some basic L^1 estimates. They are essentially due to Schatzman [44,45] and Zhou [15].

Lemma 4.1. *Let $\phi=\phi(t,x)\in C^1$ satisfy*

$$\phi_t+\big(\lambda(t,x)\phi\big)_x=F(t,x),\quad 0\leqslant t\leqslant T,\ x\in R,\quad \phi(0,x)=g(x),$$

where $\lambda\in C^1$. Then

$$\int_{-\infty}^{+\infty}\big|\phi(t,x)\big|\,dx\leqslant\int_{-\infty}^{+\infty}\big|g(x)\big|\,dx+\int_0^T\int_{-\infty}^{+\infty}\big|F(t,x)\big|\,dx\,dt,\quad \forall t\leqslant T,\tag{4.10}$$

provided that the right-hand side of the inequality is bounded.

Lemma 4.2. *Let $\phi=\phi(t,x)$ and $\psi=\psi(t,x)$ be C^1 functions satisfying*

$$\phi_t+\big(\lambda(t,x)\phi\big)_x=F_1(t,x),\quad 0\leqslant t\leqslant T,\ x\in R,\quad \phi(0,x)=g_1(x),$$

and

$$\psi_t+\big(\mu(t,x)\psi\big)_x=F_2(t,x),\quad 0\leqslant t\leqslant T,\ x\in R,\quad \psi(0,x)=g_2(x),$$

respectively, where $\lambda,\mu\in C^1$ such that there exists a positive constant δ_0 independent of T verifying

$$\mu(t,x)-\lambda(t,x)\geqslant\delta_0,\quad 0\leqslant t\leqslant T,\ x\in R.$$

Then

$$\begin{aligned}&\int_0^T\int_{-\infty}^{+\infty}\big|\phi(t,x)\big|\big|\psi(t,x)\big|\,dx\,dt\\&\quad\leqslant C\left(\int_{-\infty}^{+\infty}\big|g_1(x)\big|\,dx+\int_0^T\int_{-\infty}^{+\infty}\big|F_1(t,x)\big|\,dx\,dt\right)\left(\int_{-\infty}^{+\infty}\big|g_2(x)\big|\,dx+\int_0^T\int_{-\infty}^{+\infty}\big|F_2(t,x)\big|\,dx\,dt\right),\end{aligned}\tag{4.11}$$

provided that the two factors on the right-hand side of the inequality are bounded.

In the present situation, similar to the above basic L^1 estimates (4.10)–(4.11), we have

Lemma 4.3. *Under the assumptions of Theorem 1.2, on any given domain of existence $[0,T]\times\mathbf{R}$ of the piecewise C^1 solution $u=u(t,x)$ to the generalized Riemann problem (1.1) and (1.16), there exists a positive constant k_1 independent of θ, ε and T such that*

$$\begin{aligned}&\int_{-\infty}^{+\infty}\big|w_i(t,x)\big|\,dx\leqslant k_1\left\{\varepsilon+W_1(T)+V_\infty(T)\big(\widetilde{W}_1(T)+W_1(T)\big)+\int_0^T\int_{-\infty}^{+\infty}\big|G_i(t,x)\big|\,dx\,dt\right\},\\&\quad\forall t\leqslant T,\end{aligned}\tag{4.12}$$

provided that the right-hand side of the inequality is bounded.

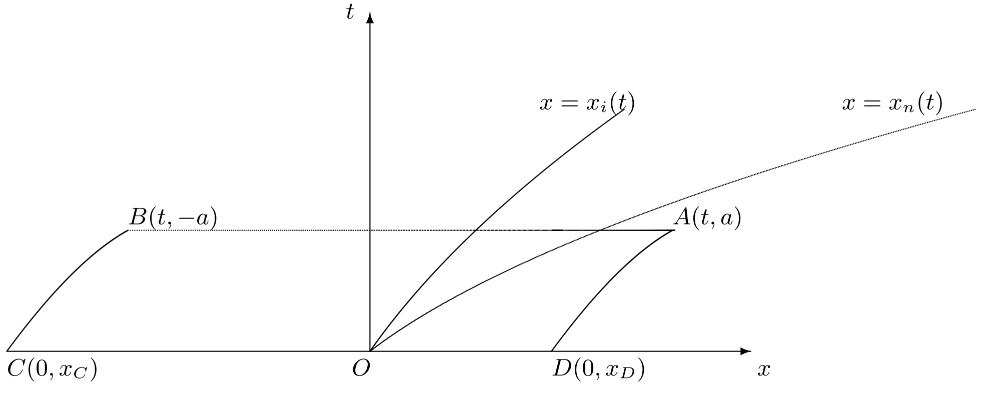

Fig. 1. The domain $ABCD$ in (t,x)-plane.

Proof. To estimate $\int_{-\infty}^{+\infty}|w_i(t,x)|\,dx$, we need only to estimate

$$\int_{-a}^{a}\bigl|w_i(t,x)\bigr|\,dx \tag{4.13}$$

for any given $a>0$ and then let $a\to+\infty$.

For $i=1,\dots,n$, for any given t with $0\leqslant t\leqslant T$, passing through point $A(t,a)$ $(a>x_n(t))$ (resp. $B(t,-a)$), we draw the i-th backward characteristic which intersects the x-axis at a point $D(0,x_D)$ (resp. $C(0,x_C)$), see Fig. 1.

Then, applying (2.17) on the domain ABCD, we have

$$\begin{aligned}\int_{B}^{A}\bigl|w_i(t,x)\bigr|\,dx\leqslant{}&\int_{x_C}^{x_D}\bigl|w_i(0,x)\bigr|\,dx+\sum_{k=1}^{n}\int_{\widehat{C}_k}\bigl|\bigl([w_i]x_k'(t)-\bigl[w_i\lambda_i(u)\bigr]\bigr)\,dt\bigr|\\&+\iint_{ABCD}|G_i|\,dx\,dt,\end{aligned}\tag{4.14}$$

where $\widehat{C}_k$: $x=x_k(t)$ stands for the k-th shock passing through the origin, which is contained in the region $ABCD$. Thus, we get

$$\begin{aligned}\int_{-a}^{a}\bigl|w_i(t,x)\bigr|\,dx\leqslant{}&\int_{-\infty}^{+\infty}\bigl|w_i(0,x)\bigr|\,dx+\int_{0}^{T}\bigl|\bigl(x_i'(t)-\lambda_i\bigl(u\bigl(t,x_i(t)\pm0\bigr)\bigr)\bigr)w_i\bigl(t,x_i(t)\pm0\bigr)\bigr|\,dt\\&+\sum_{k=1,\,k\neq i}^{n}\int_{\widehat{C}_k}\bigl|\bigl([w_i]x_k'(t)-\bigl[w_i\lambda_i(u)\bigr]\bigr)\,dt\bigr|+\int_{0}^{T}\int_{-\infty}^{+\infty}|G_i|\,dx\,dt.\end{aligned}\tag{4.15}$$

Using (3.6), (3.7) and (4.4), and noting (4.9), it is easy to see that

$$\int_{-a}^{a} \left|w_i(t,x)\right| dx \leqslant \int_{-\infty}^{+\infty} \left|w_i(0,x)\right| dx + W_1(T) + c_1 V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} |G_i|\, dx\, dt, \tag{4.16}$$

where here and henceforth, c_i $(i = 1, 2, \ldots)$ will denote positive constants independent of θ, ε and T.

Noting (1.19), we have

$$\int_{-a}^{a} \left|w_i(t,x)\right| dx \leqslant c_2 \left\{ \varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} |G_i|\, dx\, dt \right\}. \tag{4.17}$$

Letting $a \to +\infty$, we immediately get the assertion in (4.12). The proof of Lemma 4.3 is finished. $\square$

Lemma 4.4. *Under the assumptions of Theorem* 1.2*, on any given domain of existence* $[0,T] \times \mathbf{R}$ *of the piecewise* C^1 *solution* $u = u(t,x)$ *to the generalized Riemann problem* (1.1) *and* (1.16)*, there exists a positive constant* k_2 *independent of* θ*,* ε *and* T *such that*

$$\begin{aligned}
&\int_0^T \int_{-\infty}^{+\infty} \left|w_i(t,x)\right|\left|w_j(t,x)\right| dx\, dt \\
&\quad \leqslant k_2 \left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} \left|G_i(t,x)\right| dx\, dt \right) \\
&\qquad \times \left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} \left|G_j(t,x)\right| dx\, dt \right), \\
&\quad \forall i \neq j \ (i, j = 1, \ldots, n),
\end{aligned} \tag{4.18}$$

provided that the right-hand side of the inequality is bounded.

Proof. To estimate

$$\int_0^T \int_{-\infty}^{+\infty} \left|w_i(t,x)\right|\left|w_j(t,x)\right| dx\, dt, \tag{4.19}$$

it is enough to estimate

$$\int_0^T \int_{-L}^{L} \left|w_i(t,x)\right|\left|w_j(t,x)\right| dx\, dt \tag{4.20}$$

for any given $L > 0$ and then let $L \to +\infty$.

For $i, j \in \{1, \ldots, n\}$ and $i \neq j$, without loss of generality, we suppose that $i < j$. Let $x = x_i(t, L)$ $(0 \leqslant t \leqslant T)$ be the i-th forward characteristic passing through point $(0, L)$ $(L > x_n(T))$. Then, we draw the i-th backward characteristic $x = s_i(t)$ $(0 \leqslant t \leqslant T)$ passing through point (T, a) $(a > x_i(T, L))$. In the meantime, passing through the point $(T, -L)$, we draw the j-th characteristic $x = s_j(t)$ $(0 \leqslant t \leqslant T)$ which intersects the x-axis at a point.

We introduce the "continuous Glimm's functional" (cf. [12,46,15])

$$Q(t) = \iint\limits_{s_j(t)<x<y<s_i(t)} \left|w_j(t,x)\right|\left|w_i(t,y)\right| dx\,dy. \tag{4.21}$$

Because of the piecewise C^1 solution $u = u(t,x)$ containing only n shocks $x = x_k(t)$ $(x_k(0) = 0)$ $(k = 1, \dots, n)$, we divide the bounded domain $\widetilde{\Omega} \triangleq \{(x,y) \mid s_j(t) < x < y < s_i(t)\}$ by the straight lines $y = x_k(t)$ $(k = 1, \dots, n)$ into some parts. Then, the straightforward calculations on all parts of the domain $\widetilde{\Omega}$ reveal that

$$\begin{aligned}
\frac{dQ(t)}{dt} &= s_i'(t)\left|w_i\big(t, s_i(t)\big)\right| \int_{s_j(t)}^{s_i(t)} \left|w_j(t,x)\right| dx - s_j'(t)\left|w_j\big(t, s_j(t)\big)\right| \int_{s_j(t)}^{s_i(t)} \left|w_i(t,x)\right| dx \\
&\quad + \sum_{k=1}^{n} x_k'(t)\left\{\left|w_i\big(t, x_k(t)-0\big)\right| - \left|w_i\big(t, x_k(t)+0\big)\right|\right\} \int_{s_j(t)}^{x_k(t)} \left|w_j(t,x)\right| dx \\
&\quad + \iint\limits_{s_j(t)<x<y<s_i(t)} \frac{\partial}{\partial t}\big(\left|w_j(t,x)\right|\big)\left|w_i(t,y)\right| dx\,dy \\
&\quad + \iint\limits_{s_j(t)<x<y<s_i(t)} \left|w_j(t,x)\right| \frac{\partial}{\partial t}\big(\left|w_i(t,y)\right|\big)\, dx\,dy \\
&= s_i'(t)\left|w_i\big(t, s_i(t)\big)\right| \int_{s_j(t)}^{s_i(t)} \left|w_j(t,x)\right| dx - s_j'(t)\left|w_j\big(t, s_j(t)\big)\right| \int_{s_j(t)}^{s_i(t)} \left|w_i(t,x)\right| dx \\
&\quad + \sum_{k=1}^{n} x_k'(t)\left\{\left|w_i\big(t, x_k(t)-0\big)\right| - \left|w_i\big(t, x_k(t)+0\big)\right|\right\} \int_{s_j(t)}^{x_k(t)} \left|w_j(t,x)\right| dx \\
&\quad - \iint\limits_{s_j(t)<x<y<s_i(t)} \frac{\partial}{\partial x}\big(\lambda_j(u)\left|w_j(t,x)\right|\big)\left|w_i(t,y)\right| dx\,dy \\
&\quad - \iint\limits_{s_j(t)<x<y<s_i(t)} \left|w_j(t,x)\right| \frac{\partial}{\partial y}\big(\lambda_i(u)\left|w_i(t,y)\right|\big)\, dx\,dy \\
&\quad + \iint\limits_{s_j(t)<x<y<s_i(t)} \operatorname{sgn}(w_j) G_j(t,x)\left|w_i(t,y)\right| dx\,dy \\
&\quad + \iint\limits_{s_j(t)<x<y<s_i(t)} \left|w_j(t,x)\right| \operatorname{sgn}(w_i) G_i(t,y)\, dx\,dy
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{s_j(t)}^{s_i(t)} \big(\lambda_j\big(u(t,x)\big) - \lambda_i\big(u(t,x)\big)\big)\big|w_i(t,x)\big|\big|w_j(t,x)\big|\,dx \\
&\quad + \big(s_i'(t) - \lambda_i\big(u\big(t,s_i(t)\big)\big)\big)\big|w_i\big(t,s_i(t)\big)\big| \int_{s_j(t)}^{s_i(t)} \big|w_j(t,x)\big|\,dx \\
&\quad + \big(\lambda_j\big(u\big(t,s_j(t)\big)\big) - s_j'(t)\big)\big|w_j\big(t,s_j(t)\big)\big| \int_{s_j(t)}^{s_i(t)} \big|w_i(t,x)\big|\,dx \\
&\quad + \big(x_i'(t) - \lambda_i\big(u\big(t,x_i(t)-0\big)\big)\big)\big|w_i\big(t,x_i(t)-0\big)\big| \int_{s_j(t)}^{x_i(t)} \big|w_j(t,x)\big|\,dx \\
&\quad + \big(\lambda_i\big(u\big(t,x_i(t)+0\big)\big) - x_i'(t)\big)\big|w_i\big(t,x_i(t)+0\big)\big| \int_{s_j(t)}^{x_i(t)} \big|w_j(t,x)\big|\,dx \\
&\quad + \sum_{k=1,\,k\neq i}^{n} x_k'(t)\big\{\big|w_i\big(t,x_k(t)-0\big)\big| - \big|w_i\big(t,x_k(t)+0\big)\big|\big\} \int_{s_j(t)}^{x_k(t)} \big|w_j(t,x)\big|\,dx \\
&\quad + \sum_{k=1,\,k\neq i}^{n} \big\{\lambda_i\big(u\big(t,x_k(t)+0\big)\big)\big|w_i\big(t,x_k(t)+0\big)\big| - \lambda_i\big(u\big(t,x_k(t)-0\big)\big)\big|w_i\big(t,x_k(t)-0\big)\big|\big\} \\
&\quad \times \int_{s_j(t)}^{x_k(t)} \big|w_j(t,x)\big|\,dx + \iint_{s_j(t)<x<y<s_i(t)} \operatorname{sgn}(w_j)G_j(t,x)\big|w_i(t,y)\big|\,dx\,dy \\
&\quad + \iint_{s_j(t)<x<y<s_i(t)} \big|w_j(t,x)\big|\operatorname{sgn}(w_i)G_i(t,y)\,dx\,dy.
\end{aligned}
\tag{4.22}
$$

Noting (4.1) and (4.3), we get from (4.22) that

$$
\begin{aligned}
\frac{dQ(t)}{dt} &\leqslant -\delta_0 \int_{s_j(t)}^{s_i(t)} \big|w_i(t,x)\big|\big|w_j(t,x)\big|\,dx \\
&\quad + \big|\big(x_i'(t) - \lambda_i\big(u\big(t,x_i(t)\pm 0\big)\big)\big)w_i\big(t,x_i(t)\pm 0\big)\big| \int_{s_j(t)}^{x_i(t)} \big|w_j(t,x)\big|\,dx \\
&\quad + \sum_{k\neq i} x_k'(t)\big\{\big|w_i\big(t,x_k(t)-0\big) - w_i\big(t,x_k(t)+0\big)\big|\big\} \int_{s_j(t)}^{x_k(t)} \big|w_j(t,x)\big|\,dx
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k\neq i}\big\{\big|\lambda_i\big(u\big(t,x_k(t)+0\big)\big)w_i\big(t,x_k(t)+0\big)-\lambda_i\big(u\big(t,x_k(t)-0\big)\big)w_i\big(t,x_k(t)-0\big)\big|\big\}\\
&\times\int\limits_{s_j(t)}^{x_k(t)}\big|w_j(t,x)\big|\,dx+\int\limits_{s_j(t)}^{s_i(t)}\big|G_j(t,x)\big|\,dx\int\limits_{s_j(t)}^{s_i(t)}\big|w_i(t,x)\big|\,dx+\int\limits_{s_j(t)}^{s_i(t)}\big|G_i(t,x)\big|\,dx\int\limits_{s_j(t)}^{s_i(t)}\big|w_j(t,x)\big|\,dx\\
\leqslant{}&-\delta_0\int\limits_{s_j(t)}^{s_i(t)}\big|w_i(t,x)\big|\big|w_j(t,x)\big|\,dx\\
&+\big|\big(x_i'(t)-\lambda_i\big(u\big(t,x_i(t)\pm 0\big)\big)\big)w_i\big(t,x_i(t)\pm 0\big)\big|\int\limits_{-\infty}^{+\infty}\big|w_j(t,x)\big|\,dx\\
&+\sum_{k\neq i}x_k'(t)\big\{\big|w_i\big(t,x_k(t)-0\big)-w_i\big(t,x_k(t)+0\big)\big|\big\}\int\limits_{-\infty}^{+\infty}\big|w_j(t,x)\big|\,dx\\
&+\sum_{k\neq i}\big\{\big|\lambda_i\big(u\big(t,x_k(t)+0\big)\big)w_i\big(t,x_k(t)+0\big)-\lambda_i\big(u\big(t,x_k(t)-0\big)\big)w_i\big(t,x_k(t)-0\big)\big|\big\}\\
&\times\int\limits_{-\infty}^{+\infty}\big|w_j(t,x)\big|\,dx+\int\limits_{-\infty}^{+\infty}\big|G_j(t,x)\big|\,dx\int\limits_{-\infty}^{+\infty}\big|w_i(t,x)\big|\,dx\\
&+\int\limits_{-\infty}^{+\infty}\big|G_i(t,x)\big|\,dx\int\limits_{-\infty}^{+\infty}\big|w_j(t,x)\big|\,dx.
\end{aligned}\tag{4.23}
$$

It then follows from Lemma 4.3 that

$$
\begin{aligned}
&\frac{dQ(t)}{dt}+\delta_0\int\limits_{s_j(t)}^{s_i(t)}\big|w_i(t,x)\big|\big|w_j(t,x)\big|\,dx\\
&\leqslant k_1\int\limits_{-\infty}^{+\infty}\big|G_j(t,x)\big|\,dx\left(\varepsilon+W_1(T)+V_\infty(T)\big(\widetilde{W}_1(T)+W_1(T)\big)+\int\limits_0^T\int\limits_{-\infty}^{+\infty}\big|G_i(t,x)\big|\,dx\,dt\right)\\
&\quad+k_1\Bigg(\big|\big(x_i'(t)-\lambda_i\big(u\big(t,x_i(t)\pm 0\big)\big)\big)w_i\big(t,x_i(t)\pm 0\big)\big|\\
&\quad+\sum_{k\neq i}x_k'(t)\big\{\big|w_i\big(t,x_k(t)-0\big)-w_i\big(t,x_k(t)+0\big)\big|\big\}\\
&\quad+\sum_{k\neq i}\big\{\big|\lambda_i\big(u\big(t,x_k(t)+0\big)\big)w_i\big(t,x_k(t)+0\big)-\lambda_i\big(u\big(t,x_k(t)-0\big)\big)w_i\big(t,x_k(t)-0\big)\big|\big\}\\
&\quad+\int\limits_{-\infty}^{+\infty}\big|G_i(t,x)\big|\,dx\Bigg)\left(\varepsilon+W_1(T)+V_\infty(T)\big(\widetilde{W}_1(T)+W_1(T)\big)+\int\limits_0^T\int\limits_{-\infty}^{+\infty}\big|G_j(t,x)\big|\,dx\,dt\right).
\end{aligned}\tag{4.24}
$$

Therefore

$$
\begin{aligned}
&\delta_0 \int_0^T \int_{s_j(t)}^{s_i(t)} \left|w_i(t,x)\right|\left|w_j(t,x)\right| dx\,dt \\
&\leqslant Q(0) + k_1 \int_0^T \int_{-\infty}^{+\infty} \left|G_j(t,x)\right| dx\,dt \Bigg(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) \\
&\quad + \int_0^T \int_{-\infty}^{+\infty} \left|G_i(t,x)\right| dx\,dt\Bigg) + k_1 \Bigg(\int_0^T \left|\big(x_i'(t) - \lambda_i\big(u\big(t, x_i(t) \pm 0\big)\big)\big) w_i\big(t, x_i(t) \pm 0\big)\right| dt \\
&\quad + \sum_{k \neq i} \int_{\widehat{C}_k} \left|[w_i]\right| \lambda_k\big(u^{\pm}\big)\, dt + \sum_{k \neq i} \int_{\widehat{C}_k} \left|\big[w_i \lambda_i(u)\big]\right| dt + \int_0^T \int_{-\infty}^{+\infty} \left|G_i(t,x)\right| dx\,dt\Bigg) \\
&\quad \times \Bigg(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} \left|G_j(t,x)\right| dx\,dt\Bigg).
\end{aligned}
\tag{4.25}
$$

Using (3.6)–(3.7) and noting (4.4), we obtain

$$
\begin{aligned}
&\delta_0 \int_0^T \int_{s_j(t)}^{s_i(t)} \left|w_i(t,x)\right|\left|w_j(t,x)\right| dx\,dt \\
&\leqslant Q(0) + c_3 \Bigg(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} \left|G_i(t,x)\right| dx\,dt\Bigg) \\
&\quad \times \Bigg(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} \left|G_j(t,x)\right| dx\,dt\Bigg).
\end{aligned}
\tag{4.26}
$$

Noting

$$
Q(0) \leqslant \int_{-\infty}^{+\infty} \left|w_i(0,x)\right| dx \int_{-\infty}^{+\infty} \left|w_j(0,x)\right| dx
\tag{4.27}
$$

and (1.19), we get

$$
\delta_0 \int_0^T \int_{s_j(t)}^{s_i(t)} \left|w_i(t,x)\right|\left|w_j(t,x)\right| dx\,dt
$$

$$\leqslant c_4\left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} \big|G_i(t,x)\big|\,dx\,dt\right)$$
$$\times\left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} \big|G_j(t,x)\big|\,dx\,dt\right). \tag{4.28}$$

It then follows

$$\int_0^T \int_{s_j(t)}^{s_i(t)} \big|w_i(t,x)\big|\big|w_j(t,x)\big|\,dx\,dt$$
$$\leqslant \frac{c_4}{\delta_0}\left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} \big|G_i(t,x)\big|\,dx\,dt\right)$$
$$\times\left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} \big|G_j(t,x)\big|\,dx\,dt\right). \tag{4.29}$$

Therefore

$$\int_0^T \int_{-L}^{L} \big|w_i(t,x)\big|\big|w_j(t,x)\big|\,dx\,dt$$
$$\leqslant \frac{c_4}{\delta_0}\left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} \big|G_i(t,x)\big|\,dx\,dt\right)$$
$$\times\left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} \big|G_j(t,x)\big|\,dx\,dt\right) \tag{4.30}$$

and the desired conclusion follows by taking $L \to +\infty$. The proof of Lemma 4.4 is finished. □

Lemma 4.5. *Under the assumptions of Theorem* 1.2*, for small* $\theta > 0$ *there exists a constant* $\varepsilon > 0$ *so small that on any given domain of existence* $[0, T] \times \mathbf{R}$ *of the piecewise* C^1 *solution* $u = u(t,x)$ *to the generalized Riemann problem* (1.1) *and* (1.16)*, there exist positive constants* k_i $(i = 3, \dots, 7)$ *independent of* θ*,* ε *and* T*, such that the following uniform a priori estimates hold*:

$$W_1(T) \leqslant k_3\varepsilon, \tag{4.31}$$

$$\widetilde{W}_1(T) \leqslant k_4\varepsilon, \tag{4.32}$$

$$U_\infty(T), V_\infty(T) \leqslant k_5\theta \tag{4.33}$$

and

$$W_\infty(T) \leqslant k_6\varepsilon, \tag{4.34}$$

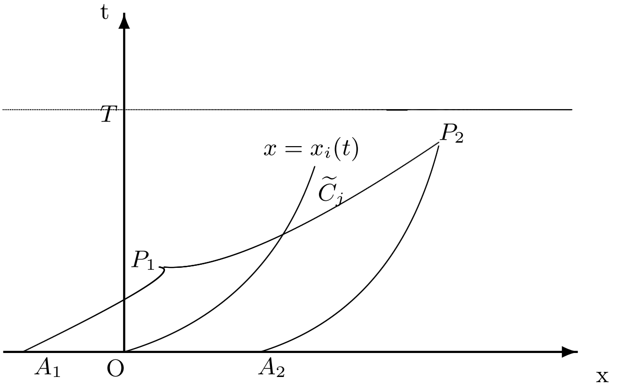

Fig. 2. The domain $P_1A_1A_2P_2$ in (t,x)-plane.

where T satisfies

$$T\varepsilon \leqslant k_7. \tag{4.35}$$

Proof. We introduce

$$Q_W(T) = \sum_{j=1}^{n} \sum_{i\neq j} \int_0^T \int_{-\infty}^{+\infty} \left|w_i(t,x)\right|\left|w_j(t,x)\right| dx\,dt. \tag{4.36}$$

By (2.13), it follows from Lemma 4.4 that

$$Q_W(T) \leqslant c_5\left(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + \int_0^T \int_{-\infty}^{+\infty} \left|G(t,x)\right| dx\,dt\right)^2, \tag{4.37}$$

where $G = (G_1, G_2, \dots, G_n)$.

Noting (2.16), we have

$$\int_0^T \int_{-\infty}^{+\infty} \left|G(t,x)\right| dx\,dt \leqslant c_6 Q_W(T). \tag{4.38}$$

Substituting (4.38) into (4.37), we obtain

$$Q_W(T) \leqslant c_7\big(\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + Q_W(T)\big)^2. \tag{4.39}$$

We next estimate $\widetilde{W}_1(T)$.

Let

$$\widetilde{C}_j: \quad x = x_j(t) \quad (0 \leqslant t_1 \leqslant t \leqslant t_2 \leqslant T) \tag{4.40}$$

be any given j-th characteristic on the domain $[0,T] \times \mathbf{R}$. Then, passing through the point $P_1(t_1, x_j(t_1))$ (resp. $P_2(t_2, x_j(t_2))$) we draw the i-th characteristic which intersects the x-axis at a point $A_1(0, y_1)$ (resp. $A_2(0, y_2)$), see Fig. 2.

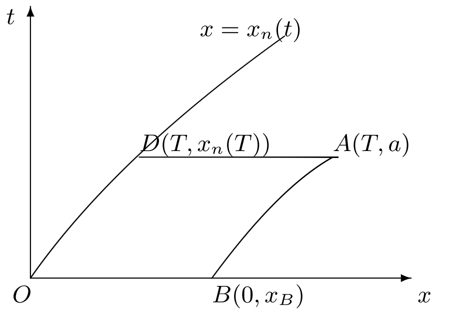

Fig. 3. The domain $ABOD$ in (t, x)-plane.

Without loss of generality, we assume that the i-th shock $x = x_i(t)$ passing through $O(0, 0)$ is partly contained in the domain $P_1A_1A_2P_2$. Then, applying (2.17) on the domain $P_1A_1A_2P_2$ and noting (2.16), it is easy to see that

$$\begin{aligned}
&\int_{t_1}^{t_2} \big|w_i\big(t, x_j(t)\big)\big|\big|\lambda_j\big(u\big(t, x_j(t)\big)\big) - \lambda_i\big(u\big(t, x_j(t)\big)\big)\big|\,dt \\
&\quad\leqslant \int_{y_1}^{y_2} \big|w_i(0, x)\big|\,dx + \sum_{k\in S_1} \int_{\widehat{C}_k} \big|\big([w_i]x_k'(t) - \big[w_i\lambda_i(u)\big]\big)\,dt\big| + \iint_{P_1A_1A_2P_2} \sum_{j\neq k} \big|\Gamma_{ijk}(u)w_jw_k\big|\,dt\,dx \\
&\quad\leqslant \int_{y_1}^{y_2} \big|w_i(0, x)\big|\,dx + \int_0^T \big|\big(x_i'(t) - \lambda_i\big(u\big(t, x_i(t) \pm 0\big)\big)\big)w_i\big(t, x_i(t) \pm 0\big)\big|\,dt \\
&\qquad + \sum_{k\neq i,\, k\in S_1} \int_{\widehat{C}_k} \big|\big([w_i]x_k'(t) - \big[w_i\lambda_i(u)\big]\big)\,dt\big| + \iint_{P_1A_1A_2P_2} \sum_{j\neq k} \big|\Gamma_{ijk}(u)w_jw_k\big|\,dt\,dx,
\end{aligned} \tag{4.41}$$

where S_1 stands for the set of all indices k such that the k-th shock curve $\widehat{C}_k$: $x = x_k(t)$ is partly contained in the domain $P_1A_1A_2P_2$. Using (1.19), (3.6), (3.7), (4.3) and (4.4), we have

$$\int_{t_1}^{t_2} \big|w_i\big(t, x_j(t)\big)\big|\,dt \leqslant c_8\big\{\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + Q_W(T)\big\}. \tag{4.42}$$

Thus, we get

$$\widetilde{W}_1(T) \leqslant c_8\big\{\varepsilon + W_1(T) + V_\infty(T)\big(\widetilde{W}_1(T) + W_1(T)\big) + Q_W(T)\big\}. \tag{4.43}$$

We next estimate $W_1(T)$.

(i) For $i = n$, passing through any fixed point $A(T, a)$ $(a > x_n(T))$, we draw the n-th backward characteristic which intersects the x-axis at a point $B(0, x_B)$, see Fig. 3.

We rewrite (2.14) as

$$d\big(\big|w_i(t, x)\big|\big(dx - \lambda_i(u)\,dt\big)\big) = \operatorname{sgn}(w_i)G_i\,dx\,dt. \tag{4.44}$$

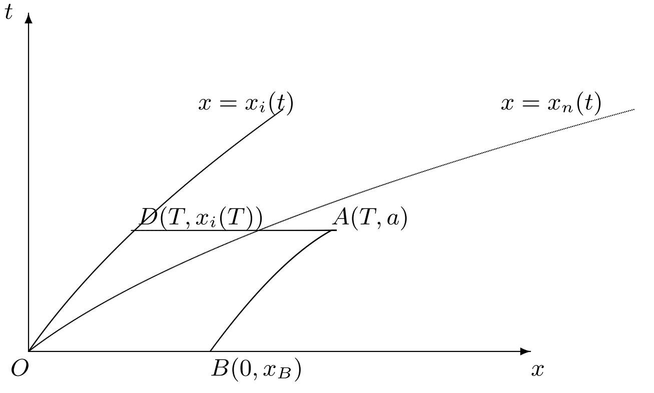

Fig. 4. The i-th backward characteristic passing through $A(T, a)$.

Let D denote the point $(T, x_n(T))$. Then, integrating (4.44) (in which we take $i = n$) on the domain $ABOD$ gives

$$\int_0^T \big(x_n'(t) - \lambda_n\big(u\big(t, x_n(t)+0\big)\big)\big)\big|w_n\big(t, x_n(t)+0\big)\big|\,dt + \int_D^A \big|w_n(T, x)\big|\,dx$$
$$\leqslant \int_0^{x_B} \big|w_n(0, x)\big|\,dx + \iint\limits_{ABOD} |G_n|\,dx\,dt. \tag{4.45}$$

Using (1.19) and (2.16), it is easy to see that

$$\int_0^T \big(x_n'(t) - \lambda_n\big(u\big(t, x_n(t)+0\big)\big)\big)\big|w_n\big(t, x_n(t)+0\big)\big|\,dt$$
$$\leqslant \int_0^{+\infty} \big|w_n(0, x)\big|\,dx + \int_0^T \int_{-\infty}^{+\infty} |G_n|\,dx\,dt \leqslant c_9\big\{\varepsilon + Q_W(T)\big\}. \tag{4.46}$$

Noting (3.2), we have

$$\int_0^T \big|\big(x_n'(t) - \lambda_n\big(u\big(t, x_n(t)+0\big)\big)\big)w_n\big(t, x_n(t)+0\big)\big|\,dt \leqslant c_9\big\{\varepsilon + Q_W(T)\big\}. \tag{4.47}$$

(ii) For $i = 1, \dots, n-1$, passing through point $A(T, a)$ $(a > x_n(T))$, we draw the i-th backward characteristic which intersects the x-axis at a point $B(0, x_B)$, see Fig. 4.

Let D denote the point $(T, x_i(T))$. Then, we divide the bounded domain $ABOD$ by the shock curves $x = x_k(t)$ $(x_k(0) = 0)$ $(k = i+1, \dots, n)$ into some parts. Thus, integrating (4.44) on all parts of the domain $ABOD$ gives

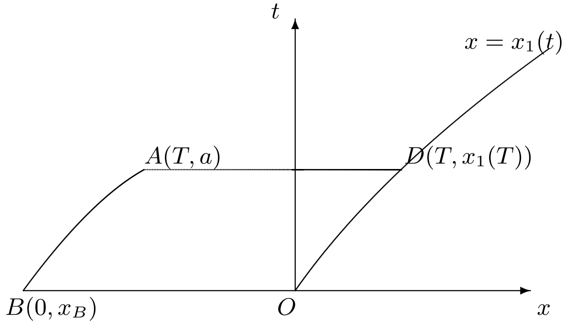

Fig. 5. The 1-st backward characteristic passing through $A(T,a)$.

$$\int_0^T \big(x_i'(t)-\lambda_i\big(u\big(t,x_i(t)+0\big)\big)\big)\big|w_i\big(t,x_i(t)+0\big)\big|\,dt+\int_D^A \big|w_i(T,x)\big|\,dx$$
$$\leqslant \int_0^{x_B}\big|w_i(0,x)\big|\,dx+\sum_{k=i+1}^{n}\int_{\widehat{C}_k}\big|[w_i]x_k'(t)-\big[w_i\lambda_i(u)\big]\big|\,dt+\iint_{ABOD}|G_i|\,dx\,dt, \tag{4.48}$$

where $\widehat{C}_k$: $x=x_k(t)$ stands for the k-th shock passing through the origin, which is contained in the domain $ABOD$. Then, using (1.19), (2.16), (3.6) and (3.7), we obtain

$$\begin{aligned}
&\int_0^T \big(x_i'(t)-\lambda_i\big(u\big(t,x_i(t)+0\big)\big)\big)\big|w_i\big(t,x_i(t)+0\big)\big|\,dt\\
&\quad\leqslant \int_0^{+\infty}\big|w_i(0,x)\big|\,dx+c_{10}V_\infty(T)\big(\widetilde{W}_1(T)+W_1(T)\big)+\int_0^T\int_{-\infty}^{+\infty}|G_i|\,dx\,dt\\
&\quad\leqslant c_{11}\big\{\varepsilon+V_\infty(T)\big(\widetilde{W}_1(T)+W_1(T)\big)+Q_W(T)\big\}.
\end{aligned} \tag{4.49}$$

Noting (3.2), it is easy to see that

$$\begin{aligned}
&\int_0^T \big|\big(x_i'(t)-\lambda_i\big(u\big(t,x_i(t)+0\big)\big)\big)w_i\big(t,x_i(t)+0\big)\big|\,dt\\
&\quad\leqslant c_{11}\big\{\varepsilon+V_\infty(T)\big(\widetilde{W}_1(T)+W_1(T)\big)+Q_W(T)\big\}.
\end{aligned} \tag{4.50}$$

(iii) For $i=1$, passing through any fixed point $A(T,a)$ $(a<x_1(T))$, we draw the 1-st backward characteristic which intersects the x-axis at a point $B(0,x_B)$, see Fig. 5.

Let D denote the point $(T,x_1(T))$. Then, integrating (4.44) (in which we take $i=1$) on the domain $ABOD$ gives

$$\int_A^D \big|w_1(T,x)\big|\,dx+\int_T^0 \big(x_1'(t)-\lambda_1\big(u\big(t,x_1(t)-0\big)\big)\big)\big|w_1\big(t,x_1(t)-0\big)\big|\,dt$$
$$\leqslant \int_{x_B}^0 \big|w_1(0,x)\big|\,dx+\iint_{ABOD}|G_1|\,dx\,dt. \tag{4.51}$$

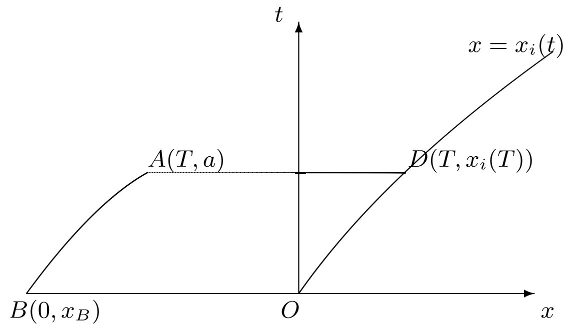

Fig. 6. The domain $ABOD$ in (t,x)-plane.

Using (1.19) and (2.16), it is easy to see that

$$\int_0^T \big(\lambda_1\big(u\big(t,x_1(t)-0\big)\big)-x_1'(t)\big)\big|w_1\big(t,x_1(t)-0\big)\big|\,dt$$
$$\leqslant \int_{-\infty}^{0} \big|w_1(0,x)\big|\,dx+\int_0^T\int_{-\infty}^{+\infty}|G_1|\,dx\,dt \leqslant c_{12}\big\{\varepsilon+Q_W(T)\big\}. \tag{4.52}$$

Noting (3.2), we have

$$\int_0^T \big|\big(x_1'(t)-\lambda_1\big(u\big(t,x_1(t)-0\big)\big)\big)w_1\big(t,x_1(t)-0\big)\big|\,dt \leqslant c_{12}\big\{\varepsilon+Q_W(T)\big\}. \tag{4.53}$$

(iv) For $i=2,\dots,n$, passing through point $A(T,a)$ $(a<x_i(T))$, we draw the i-th backward characteristic which intersects the x-axis at a point $B(0,x_B)$, see Fig. 6.

Let D denote the point $(T,x_i(T))$. Then, we divide the bounded domain $ABOD$ by the shock curves $x=x_k(t)$ $(x_k(0)=0)$ $(k=1,\dots,i-1)$ into some parts. Thus, integrating (4.44) on all parts of the domain $ABOD$ gives

$$\int_0^T \big(\lambda_i\big(u\big(t,x_i(t)-0\big)\big)-x_i'(t)\big)\big|w_i\big(t,x_i(t)-0\big)\big|\,dt+\int_A^D\big|w_i(T,x)\big|\,dx$$
$$\leqslant \int_{x_B}^{0}\big|w_i(0,x)\big|\,dx+\sum_{k=1}^{i-1}\int_{\widehat{C}_k}\big|[w_i]x_k'(t)-\big[w_i\lambda_i(u)\big]\big|\,dt+\iint_{ABOD}|G_i|\,dx\,dt, \tag{4.54}$$

where $\widehat{C}_k$: $x=x_k(t)$ stands for the k-th shock passing through the origin, which is contained in the domain $ABOD$. Then, using (1.19), (2.16), (3.6) and (3.7), we obtain

$$\int_0^T \big(\lambda_i\big(u\big(t,x_i(t)-0\big)\big)-x_i'(t)\big)\big|w_i\big(t,x_i(t)-0\big)\big|\,dt$$

$$\leqslant \int_{-\infty}^{0} \big|w_i(0,x)\big|\,dx + c_{13}V_\infty(T)\big(\widetilde{W}_1(T)+W_1(T)\big) + \int_0^T\int_{-\infty}^{+\infty} |G_i|\,dx\,dt$$

$$\leqslant c_{14}\big\{\varepsilon + V_\infty(T)\big(\widetilde{W}_1(T)+W_1(T)\big)+Q_W(T)\big\}. \tag{4.55}$$

Noting (3.2), it is easy to see that

$$\int_0^T \big|\big(x_i'(t)-\lambda_i\big(u\big(t,x_i(t)-0\big)\big)\big)w_i\big(t,x_i(t)-0\big)\big|\,dt$$
$$\leqslant c_{14}\big\{\varepsilon + V_\infty(T)\big(\widetilde{W}_1(T)+W_1(T)\big)+Q_W(T)\big\}. \tag{4.56}$$

Combining (4.47), (4.50), (4.53) and (4.56) all together, we have

$$W_1(T) \leqslant c_{15}\big\{\varepsilon + V_\infty(T)\big(\widetilde{W}_1(T)+W_1(T)\big)+Q_W(T)\big\}. \tag{4.57}$$

We next estimate $U_\infty(T)$ and $V_\infty(T)$.

Passing through any fixed point $(t,x)\in[0,T]\times\mathbf{R}$, we draw the i-th backward characteristic C_i which intersects the x-axis at a point $(0,y)$. Integrating (2.6) along this characteristic C_i and noting (2.8) yields

$$v_i(t,x) = v_i(0,y) + \sum_{k\in S_2}[v_i]_k + \int_{C_i} \sum_{j,k=1,\,k\neq i}^{n} \beta_{ijk}(u)v_jw_k\,dt, \tag{4.58}$$

where S_2 denotes the set of all indices k such that this characteristic C_i intersects the k-th shock $x=x_k(t)$ at a point $(t_k,x_k(t_k))$, and $[v_i]_k = v_i(t_k,x_k(t_k)+0) - v_i(t_k,x_k(t_k)-0)$. Noting (1.19) and using (1.17), we have

$$\big|u_+(x)\big| \leqslant \int_0^{+\infty}\big|u_+'(x)\big|\,dx \leqslant K_2, \quad \forall x\in\mathbf{R}^+, \tag{4.59}$$

and

$$\big|u_-(x)\big| \leqslant \int_{-\infty}^{0}\big|u_-'(x)\big|\,dx \leqslant K_2, \quad \forall x\in\mathbf{R}^-. \tag{4.60}$$

Therefore, noting the fact that $i\notin S_2$, and using (1.16), (2.1), (3.3), (4.2) and (4.4), we get from (4.58)–(4.60) that

$$V_\infty(T) \leqslant c_{16}\big\{\theta+\varepsilon+V_\infty(T)\big(V_\infty(T)+\widetilde{W}_1(T)\big)\big\}. \tag{4.61}$$

We now prove (4.31)–(4.33) and

$$Q_W(T) \leqslant k_8\varepsilon^2, \tag{4.62}$$

where k_8 is a positive constant independent of θ, ε and T.

Recalling (4.2), (4.59) and (4.60), evidently we have

$$U_\infty(0), V_\infty(0) \leqslant c_{17}\theta \tag{4.63}$$

and

$$Q_W(0) = W_1(0) = \widetilde{W}_1(0) = 0, \tag{4.64}$$

provided that $\varepsilon \ll \theta$. Thus, by continuity there exist positive constants k_3, k_4, k_5 and k_8 independent of θ, ε and T such that (4.31)–(4.33) and (4.62) hold at least for $0 \leqslant T \leqslant \tau_0$, where τ_0 is a small positive number. Hence, in order to prove (4.31)–(4.33) and (4.62) it suffices to show that we can choose k_3, k_4, k_5 and k_8 in such a way that for any fixed T_0 $(0 < T_0 \leqslant T)$ such that

$$W_1(T_0) \leqslant 2k_3\varepsilon, \tag{4.65}$$

$$\widetilde{W}_1(T_0) \leqslant 2k_4\varepsilon, \tag{4.66}$$

$$V_\infty(T_0) \leqslant 2k_5\theta, \tag{4.67}$$

$$Q_W(T_0) \leqslant 2k_8\varepsilon^2, \tag{4.68}$$

we have

$$W_1(T_0) \leqslant k_3\varepsilon, \tag{4.69}$$

$$\widetilde{W}_1(T_0) \leqslant k_4\varepsilon, \tag{4.70}$$

$$V_\infty(T_0) \leqslant k_5\theta, \tag{4.71}$$

$$Q_W(T_0) \leqslant k_8\varepsilon^2. \tag{4.72}$$

To this end, substituting (4.65)–(4.68) into the right-hand side of (4.39), (4.43), (4.57) and (4.61) (in which we take $T = T_0$), it is easy to see that, when $\theta > 0$ is suitably small, we have

$$Q_W(T_0) \leqslant 4(1+k_3)^2 c_7 \varepsilon^2, \tag{4.73}$$

$$\widetilde{W}_1(T_0) \leqslant 2(1+k_3)c_8\varepsilon, \tag{4.74}$$

$$W_1(T_0) \leqslant 2c_{15}\varepsilon, \tag{4.75}$$

$$V_\infty(T_0) \leqslant 3c_{16}\theta, \tag{4.76}$$

provided that $\varepsilon \ll \theta$.

Hence, if $k_3 \geqslant 2c_{15}$, $k_4 \geqslant 2(1+k_3)c_8$, $k_5 \geqslant 3c_{16}$ and $k_8 \geqslant 4(1+k_3)^2 c_7$, then we get (4.69)–(4.72), provided that θ is suitably small. This proves (4.31)–(4.33) and (4.62).

We finally estimate $W_\infty(T)$.

For any fixed point $(t, x) \in [0, T] \times \mathbf{R}$, we draw the i-th backward characteristic C_i passing through the point (t, x), which intersects the x-axis at a point $(0, y)$. Integrating (2.9) along this characteristic C_i and noting (2.11) yields

$$w_i(t,x) = w_i(0,y) + \sum_{k \in S_3} [w_i]_k + \int_{C_i} \left[\sum_{j,k=1,\, j \neq k}^{n} \gamma_{ijk}(u) w_j w_k + \gamma_{iii}(u) w_i^2 \right] dt, \tag{4.77}$$

where S_3 denotes the set of all indices k such that this characteristic C_i intersects the k-th shock $x = x_k(t)$ at a point $(t_k, x_k(t_k))$, and $[w_i]_k = w_i(t_k, x_k(t_k)+0) - w_i(t_k, x_k(t_k)-0)$. Using (3.6) and (4.4) and noting the fact that $i \notin S_3$, we have

$$W_\infty(T) \leqslant c_{18}\left\{\varepsilon + V_\infty(T)W_\infty(T) + W_\infty(T)\widetilde{W}_1(T) + T\left(W_\infty(T)\right)^2\right\}. \tag{4.78}$$

Noting (1.18), by continuity there exists a positive constant k_6 independent of θ, ε and T such that (4.34) holds at least for $T > 0$ suitably small. Thus, in order to prove (4.34) it suffices to show that we can choose k_6 and k_7 in such a way that for any fixed T_0 $(0 < T_0 \leqslant T)$ with $T_0\varepsilon \leqslant k_7$ such that

$$W_\infty(T_0) \leqslant 2k_6\varepsilon, \tag{4.79}$$

we have

$$W_\infty(T_0) \leqslant k_6\varepsilon. \tag{4.80}$$

For this purpose, substituting (4.79) into the right-hand side of (4.78) (in which we take $T = T_0$) and noting (4.32)–(4.33), it is easy to see that, when $\theta > 0$ is suitably small, we have

$$W_\infty(T_0) \leqslant 2c_{18}\left(1 + 2k_6^2 k_7\right)\varepsilon. \tag{4.81}$$

Hence, if $k_6 \geqslant 6c_{18}$ and $k_6^2 k_7 = 1$, then we have (4.80), provided that θ is suitably small. Therefore (4.34) is proved.

Finally, we observe that when $\theta > 0$ is suitably small, by (4.33) we have

$$U_\infty(T) \leqslant k_5\theta \leqslant \frac{1}{2}\delta. \tag{4.82}$$

This implies the validity of hypothesis (4.4). The proof of Lemma 4.5 is finished. □

Proof of Theorem 1.2. By (4.33)–(4.34), we know that for small $\theta > 0$ there exists $\varepsilon > 0$ suitably small such that the generalized Riemann problem (1.1) and (1.16) admits a unique piecewise C^1 solution $u = u(t, x)$ containing only shocks on the strip $[0, T] \times \mathbf{R}$, where T satisfies (4.35). Therefore, the lifespan $\widetilde{T}(\varepsilon)$ of the piecewise C^1 solution satisfies

$$\widetilde{T}(\varepsilon) \geqslant K_3\varepsilon^{-1}, \tag{4.83}$$

where $K_3 (= k_7)$ is a positive constant independent of ε. Moreover, by Lemma 4.5, when the piecewise C^1 solution $u = u(t, x)$ blows up in a finite time, $u = u(t, x)$ itself must be bounded on the domain $[0, \widetilde{T}(\varepsilon)) \times \mathbf{R}$. Hence, the first-order derivative u_x of $u = u(t, x)$ should tend to be unbounded as $t \nearrow \widetilde{T}(\varepsilon)$. The proof of Theorem 1.2 is finished. □

5. Applications

5.1. System of one-dimensional isentropic flows

Consider the following Cauchy problem for the system of one-dimensional isentropic flows in Lagrangian coordinates (cf. [47]):

$$\begin{cases} \dfrac{\partial \tau}{\partial t} - \dfrac{\partial u}{\partial x} = 0, \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial p(\tau)}{\partial x} = 0, \end{cases} \tag{5.1}$$

$$t = 0: \quad (\tau, u) = \begin{cases} (\tilde{\tau}_0 + \varepsilon\tau_-(x), \tilde{u}_0 + \varepsilon u_-(x)), & x \leqslant 0, \\ (\tilde{\tau}_0 + \varepsilon\tau_+(x), \tilde{u}_0 + \varepsilon u_+(x)), & x \geqslant 0, \end{cases} \tag{5.2}$$

where $\tau > 0$ is the specific volume, u is the velocity and $p = p(\tau)$ is the pressure; the pressure $p = p(\tau)$ is a suitably smooth function of τ such that

$$p'(\tau) < 0 \quad \text{and} \quad p''(\tau) > 0, \quad \forall \tau > 0, \tag{5.3}$$

moreover, $\varepsilon > 0$ is a small parameter, $\tilde{\tau}_0 > 0$ and $\tilde{u}_0$ are constants, $(\tau_\pm(x), u_\pm(x)) \in C^1$ satisfy

$$\tau_\pm(0) = u_\pm(0) = 0, \tag{5.4}$$

$$\left\| \tau_\pm(x) \right\|_{C^1}, \left\| u_\pm(x) \right\|_{C^1} \leqslant K_4 \tag{5.5}$$

and

$$\int_0^{+\infty} \left| \tau'_+(x) \right| dx, \int_0^{+\infty} \left| u'_+(x) \right| dx, \int_{-\infty}^{0} \left| \tau'_-(x) \right| dx, \int_{-\infty}^{0} \left| u'_-(x) \right| dx \leqslant K_5, \tag{5.6}$$

where K_4 and K_5 are positive constants independent of ε.

Let

$$U = \begin{pmatrix} \tau \\ u \end{pmatrix}. \tag{5.7}$$

By (5.3), it is easy to see that system (5.1) is strictly hyperbolic and genuinely nonlinear with the following two distinct real eigenvalues:

$$\lambda_1(U) = -\sqrt{-p'(\tau)} < \lambda_2(U) = \sqrt{-p'(\tau)}. \tag{5.8}$$

By Theorem 1.2 we get

Theorem 5.1. *Suppose that* (5.3) *holds. Suppose furthermore that the corresponding Riemann problem has a self-similar solution consisting of only non-degenerate shocks but no centered rarefaction waves. Suppose finally that $\tau_-(x)$, $u_-(x)$, $\tau_+(x)$ and $u_+(x)$ are all C^1 vector functions on $x \leqslant 0$ and on $x \geqslant 0$, respectively, satisfying* (5.4)–(5.6) *and*

$$\theta \triangleq \left| \big(\tau_+(0), u_+(0)\big) - \big(\tau_-(0), u_-(0)\big) \right| > 0 \quad \textit{is suitably small.} \tag{5.9}$$

Then for small $\theta > 0$, there exists a constant $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, the lifespan $\widetilde{T}(\varepsilon)$ of the piecewise C^1 solution to the generalized Riemann problem (5.1)–(5.2) *satisfies*

$$\widetilde{T}(\varepsilon) \geqslant K_6 \varepsilon^{-1}, \tag{5.10}$$

where K_6 is a positive constant independent of ε.

5.2. The relativistic Euler equations

Consider the following Cauchy problem for the Euler system of conservation laws of energy and momentum in special relativity (cf. [22,48]):

$$\begin{cases} \partial_t\left((p+\rho c^2)\dfrac{v^2}{c^2(c^2-v^2)}+\rho\right)+\partial_x\left((p+\rho c^2)\dfrac{v}{c^2-v^2}\right)=0, \\ \partial_t\left((p+\rho c^2)\dfrac{v}{c^2-v^2}\right)+\partial_x\left((p+\rho c^2)\dfrac{v^2}{c^2-v^2}+p\right)=0, \end{cases} \tag{5.11}$$

$$t=0: \quad (\rho, v)=\begin{cases} (\rho_0+\varepsilon\rho_-(x), v_0+\varepsilon v_-(x)), & x \leqslant 0, \\ (\rho_0+\varepsilon\rho_+(x), v_0+\varepsilon v_+(x)), & x \geqslant 0, \end{cases} \tag{5.12}$$

where ρ, p and v represent the proper energy density, the pressure and the particle speed, and the constant c is the speed of light; the equation of state is

$$p=p(\rho),$$

in which $p(\rho)$ is a suitably smooth function of ρ and satisfies

$$p(\rho)>0, \qquad p'(\rho)>0 \quad \text{and} \quad p''(\rho)>0, \quad \forall\rho>0. \tag{5.13}$$

Moreover, $\varepsilon>0$ is a small parameter, $\rho_0>0$ and v_0 are constants such that

$$\sqrt{p'(\rho_0)}<c \quad \text{and} \quad v_0^2<c^2, \tag{5.14}$$

$\rho_\pm(x)$ and $v_\pm(x)\in C^1$ satisfy

$$\rho_\pm(0)=v_\pm(0)=0, \tag{5.15}$$

$$\left\|\rho_\pm(x)\right\|_{C^1}, \left\|v_\pm(x)\right\|_{C^1} \leqslant K_7 \tag{5.16}$$

and

$$\int_0^{+\infty}\left|\rho'_+(x)\right|dx, \int_0^{+\infty}\left|v'_+(x)\right|dx, \int_{-\infty}^0\left|\rho'_-(x)\right|dx, \int_{-\infty}^0\left|v'_-(x)\right|dx\,dx \leqslant K_8, \tag{5.17}$$

where K_7 and K_8 are positive constants independent of ε.

Let

$$u=\begin{pmatrix}\rho \\ v\end{pmatrix}. \tag{5.18}$$

Then, we rewrite system (5.11) as

$$u_t+A(u)u_x=0, \tag{5.19}$$

where

$$A(u)=\begin{pmatrix} \frac{c^2v(c^2-p'(\rho))}{c^4-p'(\rho)v^2} & \frac{c^2(p+\rho c^2)}{c^4-p'(\rho)v^2} \\ \frac{c^2p'(\rho)(c^2-v^2)^2}{(p+\rho c^2)(c^4-p'(\rho)v^2)} & \frac{c^2v(c^2-p'(\rho))}{c^4-p'(\rho)v^2} \end{pmatrix}. \tag{5.20}$$

By (5.13) and (5.14), it is easy to see that in a neighborhood of $u_0 = (\rho_0, v_0)^T$, system (5.11) is strictly hyperbolic and has the following two distinct real eigenvalues:

$$\lambda_1(u) = \frac{c^2(v - \sqrt{p'(\rho)})}{c^2 - v\sqrt{p'(\rho)}} < \lambda_2(u) = \frac{c^2(v + \sqrt{p'(\rho)})}{c^2 + v\sqrt{p'(\rho)}}. \tag{5.21}$$

The corresponding right eigenvectors are

$$r_j(u)//\left(\frac{(-1)^j}{c^2 - v^2}, \frac{\sqrt{p'(\rho)}}{p + \rho c^2}\right)^T, \quad j = 1, 2. \tag{5.22}$$

It is easy to see that both characteristic fields are genuinely nonlinear, i.e.,

$$\nabla\lambda_j(u) r_j(u) \neq 0, \quad j = 1, 2. \tag{5.23}$$

By Theorem 1.2 we get

Theorem 5.2. *Suppose that* (5.13) *and* (5.14) *hold. Suppose furthermore that the corresponding Riemann problem has a self-similar solution consisting of only non-degenerate shocks but no centered rarefaction waves. Suppose finally that $\rho_-(x)$, $v_-(x)$, $\rho_+(x)$ and $v_+(x)$ are all C^1 vector functions on $x \leqslant 0$ and on $x \geqslant 0$, respectively, satisfying* (5.15)–(5.17) *and*

$$\theta \triangleq \left|\big(\rho_+(0), v_+(0)\big) - \big(\rho_-(0), v_-(0)\big)\right| > 0 \quad \text{is suitably small.} \tag{5.24}$$

Then for small $\theta > 0$, there exists a constant $\varepsilon_0 > 0$ so small that for any fixed $\varepsilon \in (0, \varepsilon_0]$, the lifespan $\widetilde{T}(\varepsilon)$ of the piecewise C^1 solution to the generalized Riemann problem (5.11)–(5.12) *satisfies*

$$\widetilde{T}(\varepsilon) \geqslant K_9 \varepsilon^{-1}, \tag{5.25}$$

where K_9 is a positive constant independent of ε.

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