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Global regularity for ordinary differential operators with polynomial coefficients

Fabio Nicola [a,b,*], Luigi Rodino [a,b]

[a] Dipartimento di Scienze Matematiche, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy
[b] Dipartimento di Matematica, Università degli Studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy

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ABSTRACT

For a class of ordinary differential operators P with polynomial coefficients, we give a necessary and sufficient condition for P to be globally regular in $\mathbb{R}$, i.e. $u \in \mathcal{S}'(\mathbb{R})$ and $Pu \in \mathcal{S}(\mathbb{R})$ imply $u \in \mathcal{S}(\mathbb{R})$ (this can be regarded as a global version of the Schwartz' hypoellipticity notion). The condition involves the asymptotic behavior, at infinity, of the roots $\xi = \xi_j(x)$ of the equation $p(x, \xi) = 0$, where $p(x, \xi)$ is the (Weyl) symbol of P.

1. Introduction and statement of the result

Linear ordinary differential operators with polynomial coefficients play an important role in mathematics. On the one hand, they were source for the study of classes of special functions, with links all around to Applied Sciences. On the other hand, the general study of Fuchs points, irregular singular points and Stokes phenomena present deep connections with different branches of geometry.

We may write the generic operator in the form

$$P = \sum_{\alpha+\beta \leqslant m} a_{\alpha,\beta} x^\beta D^\alpha, \quad a_{\alpha,\beta} \in \mathbb{C}, \tag{1.1}$$

* Corresponding author at: Dipartimento di Scienze Matematiche, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy.

E-mail addresses: fabio.nicola@polito.it (F. Nicola), luigi.rodino@unito.it (L. Rodino).

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where we use the notation $D = -id/dx$. In this paper we deal with the Fourier analysis of (1.1). Namely, rather than analyzing the extension of the solution u of the corresponding equation in the complex domain $\mathbb{C}$, we shall address to their analysis in $\mathbb{R}^2_{x,\xi}$, with respect to the phase-space variables $x, \xi \in \mathbb{R}$. Basic function space in this order of ideas is the space $\mathcal{S}(\mathbb{R})$ of L. Schwartz [17], defined by imposing

$$\sup_{x\in\mathbb{R}} \left|x^\beta D^\alpha u(x)\right| < \infty, \quad \forall \alpha, \beta \in \mathbb{N}. \tag{1.2}$$

These functions are regular in $\mathbb{R}^2_{x,\xi}$, in the sense that both $u(x)$ and its Fourier transform $\widehat{u}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} u(x)\, dx$ present a rapid decay at infinity, beside local regularity. As universal set for our study we shall take the dual $\mathcal{S}'(\mathbb{R})$; note that this will exclude solutions which have exponential growth at infinity.

Aim of this paper is to characterize the operators P in (1.1) which are globally regular, according the following definition.

Definition 1.1. We say that P is globally regular if

$$u \in \mathcal{S}'(\mathbb{R}) \quad \text{and} \quad Pu \in \mathcal{S}(\mathbb{R}) \quad \Rightarrow \quad u \in \mathcal{S}(\mathbb{R}). \tag{1.3}$$

In particular, if (1.3) is satisfied, the solutions $u \in \mathcal{S}'(\mathbb{R})$ of $Pu = 0$ belong to $\mathcal{S}(\mathbb{R})$. Global regularity turns out to be basic information in many applications. So for example in connection with Quantum Mechanics, assuming P in (1.1) is self-adjoint, we may deduce that the eigenfunctions, intended as solutions $u \in L^2(\mathbb{R})$ of $Pu = 0$, are in $\mathcal{S}(\mathbb{R})$. In the Theory of Signals, where we may regard P in (1.1) as a filter reproduced by electronic devices, we have that the globally regular P are exactly the ineffective filters, i.e. filters which do not cancel any essential part of the signal.

The literature concerning global regularity, sometimes also called global hypoellipticity, in these last 30 years is extremely large, taking also into account the same problem for operators with smooth coefficients and pseudo-differential operators in $\mathbb{R}^n$, with $n \geqslant 1$. We address to the recent monograph of the authors [13] for a survey.

Let us begin with a simple example. The constant coefficient operator $p(D) = \sum_{\alpha\leqslant m} c_\alpha D^\alpha$, $c_\alpha \in \mathbb{C}$, is globally regular if and only if $p(\xi) \neq 0$ for all $\xi \in \mathbb{R}$. In fact, if $p(\xi_0) = 0$, then $u(x) = \exp[i\xi_0 x]$ provides a solution of $p(D)u = 0$, whereas by Fourier transform one gets easily that $p(D)u \in \mathcal{S}(\mathbb{R})$, $u \in \mathcal{S}'(\mathbb{R})$ imply $u \in \mathcal{S}(\mathbb{R})$ if $p(\xi) \neq 0$ for all $\xi \in \mathbb{R}$. The same result keeps valid for partial differential operators with constant coefficients.

Passing to operators with polynomial coefficients, a characterization of globally regular operators in $\mathbb{R}^n$ is certainly out of reach at this moment. However for the ordinary differential operator (1.1) a necessary and sufficient condition seems possible, and we shall give it in the following, under an additional algebraic condition.

Consider first the standard left-symbol of P in (1.1):

$$a(x, \xi) = \sum_{\alpha+\beta\leqslant m} a_{\alpha,\beta} x^\beta \xi^\alpha. \tag{1.4}$$

In our approach, it will be convenient to argue on the Weyl symbol, see for example [10, Chapter XVIII], given by

$$\begin{aligned} p(x, \xi) &= \sum_{\alpha+\beta\leqslant m} c_{\alpha,\beta} x^\beta \xi^\alpha \\ &= \sum_{\gamma\geqslant 0} \frac{1}{\gamma!} \left(-\frac{1}{2}\right)^\gamma \partial_\xi^\gamma D_x^\gamma a(x, \xi). \end{aligned} \tag{1.5}$$

We shall assume $c_{m,0} = 1$ in (1.5), i.e. $a_{m,0} = 1$ in (1.4). We have

$$p(x,\xi) = \prod_{j=1}^{m} \bigl(\xi - \xi_j(x)\bigr) \tag{1.6}$$

where $\xi_j(x)$, $j = 1, \dots, m$, are real-analytic functions defined for $x \in \mathbb{R}$, $|x|$ large enough. Let us denote by $\lambda_1, \dots, \lambda_m$ the complex roots of the polynomial

$$\sum_{\alpha=0}^{m} c_{\alpha,m-\alpha} \lambda^\alpha.$$

We have, possibly after relabeling,

$$\xi_j(x)/x \to \lambda_j \quad \text{as } x \in \mathbb{R},\ |x| \to +\infty. \tag{1.7}$$

In fact, the roots have a Puiseaux expansion at infinity [9, Lemma A.1.3, p. 363], namely

$$\xi_j(x) = \lambda_j x + \sum_{-\infty<k\leqslant p-1} c_{j,k}\bigl(x^{1/p}\bigr)^k \quad \text{for } |x| \text{ large}, \tag{1.8}$$

for some integer p, where the function $x^{1/p}$ is the positive pth root of x for $x > 0$ and, say, $x^{1/p} = |x|^{1/p} e^{i\pi/p}$ for $x < 0$ (by taking the lowest common multiple we can assume that the same integer p occurs for every j).

We suppose that the roots which approach the real axis at infinity are asymptotically separated, in the sense that

$$\begin{gathered} \text{If } \lambda_j = \lambda_k \in \mathbb{R}, \text{ with } j \neq k, \text{ then} \\ \bigl|\xi_j(x) - \xi_k(x)\bigr| \gtrsim \max\bigl\{\bigl|\xi_j(x) - \lambda_j x\bigr|, \bigl|\xi_k(x) - \lambda_k x\bigr|, |x|^{-1+\varepsilon}\bigr\} \end{gathered} \tag{1.9}$$

for some $\varepsilon > 0$.

Theorem 1.2. *Assume (1.9). Then P is globally regular, i.e. (1.3) holds, if and only if*

$$\bigl|x \operatorname{Im} \xi_j(x)\bigr| \to +\infty \quad \textit{as } x \in \mathbb{R},\ |x| \to +\infty. \tag{1.10}$$

For a better understanding of (1.9), (1.10), we may argue on the Puiseaux expansions (1.8). Let us emphasize the first non-vanishing term after $\lambda_j x$, namely write

$$\xi_j(x) = \lambda_j x + c_{j,r(j)}\bigl(x^{1/p}\bigr)^{r(j)} + \sum_{-\infty<k<r(j)} c_{j,k}\bigl(x^{1/p}\bigr)^k \tag{1.11}$$

with $c_{j,r(j)} \neq 0$. Arguing for simplicity for $x > 0$, condition (1.9) states that, if $\lambda_j = \lambda_k \in \mathbb{R}$, then one at least between $r(j)$ and $r(k)$ is strictly larger than -1 and moreover, in the case $r(j) = r(k) > -1$, we have $c_{j,r(j)} \neq c_{k,r(k)}$.

As for (1.10), arguing again for $x > 0$, it means that for all $j = 1, \dots, m$, we have $\operatorname{Im} \lambda_j \neq 0$ or else $\operatorname{Im} c_{j,k} \neq 0$ for some k with $k/p > -1$. To be definite: (1.10) is not satisfied when for some j the function

$$\lambda_j x + \sum_{-p<k<p-1} c_{j,k}\left(x^{1/p}\right)^k$$

is real-valued.

In Section 2, we recall some notation for the pseudodifferential calculus and we reduce the problem to the analysis of a global wave-front set. The proof of Theorem 1.2 is given in Section 4, by using the factorization of the operator presented in Section 3. In some sense, this factorization is the counterpart in phase-space of the classical methods of the asymptotic integration. In fact, we think that, in terms of classical asymptotic analysis, under condition (1.9), the proof of Theorem 1.2 would be as well possible, but much more difficult. Section 5 is devoted to remarks and examples. In particular, we recapture some existing results on the global regularity for operators of the form (1.1), giving corresponding references.

We finally observe that Theorem 1.2 keeps valid for relevant classes of operators of the form (1.1), independently of the asymptotic separation (1.9). For example, (1.9) is not satisfied for constant coefficients operators with a double root $\xi_j(x) = \xi_k(x) = \xi_0$, whereas in this class (1.10) is necessary and sufficient for the global regularity, as observed before. Regretfully, the assumption (1.9) will be essential in our proof of Theorem 1.2 for operators with polynomial coefficients.

2. Microlocal reduction

We recall that a pseudodifferential operator in $\mathbb{R}$, according to the standard quantization, is an integral operator of the form

$$p(x, D)u(x) = (2\pi)^{-1} \int_{\mathbb{R}} e^{ix\xi} p(x, \xi)\widehat{u}(\xi)\, d\xi,$$

for $u \in \mathcal{S}(\mathbb{R})$, where the so-called symbol $p(x, \xi)$ is a smooth function in $\mathbb{R}^2$ which satisfies suitable growth estimates at infinity, which will be detailed below. The corresponding operator $p(x, D)$ will define continuous maps $\mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$.

An important class of symbols is given by the space $\Gamma^m(\mathbb{R})$ of smooth functions $p(x, \xi)$ in $\mathbb{R}^2$ satisfying the estimates

$$\left|\partial_\xi^\alpha \partial_x^\beta p(x, \xi)\right| \leqslant C_{\alpha\beta}\left(1 + |x| + |\xi|\right)^{m-\alpha-\beta}, \quad \forall\alpha, \beta \in \mathbb{N},\ (x, \xi) \in \mathbb{R}^2,$$

for some $m \in \mathbb{R}$. This class arises, in particular, in the following definition of the global wave-front set of a temperate distribution, as introduced in [11].

A point $(x_0, \xi_0) \in \mathbb{R}^2 \setminus \{0\}$ is called *non-characteristic* for $p \in \Gamma^m(\mathbb{R})$, if there are $\varepsilon, C > 0$ such that

$$\left|p(x, \xi)\right| \geqslant C\left(1 + |x| + |\xi|\right)^m \quad \text{for } (x, \xi) \in V_{(x_0,\xi_0),\varepsilon} \tag{2.1}$$

where, for $z_0 \in \mathbb{R}^2$, $z_0 \neq 0$, $V_{z_0,\varepsilon}$ is the conic neighborhood

$$V_{z_0,\varepsilon} = \left\{z \in \mathbb{R}^2 \setminus \{0\}\colon \left|\frac{z}{|z|} - \frac{z_0}{|z_0|}\right| < \varepsilon,\ |z| > \varepsilon^{-1}\right\}.$$

Let $u \in \mathcal{S}'(\mathbb{R})$. We define its (global) wave-front set $WF(u) \subset \mathbb{R}^2 \setminus \{0\}$ by saying that $(x_0, \xi_0) \in \mathbb{R}^2$, $(x_0, \xi_0) \neq (0, 0)$, does not belong to $WF(u)$ if there exists $\psi \in \Gamma^0(\mathbb{R})$ which is non-characteristic at (x_0, ξ_0), such that $\psi(x, D)u \in \mathcal{S}(\mathbb{R})$. The set $WF(u)$ is a conic closed subset of $\mathbb{R}^2 \setminus \{0\}$.

Proposition 2.1. *(See [11].) If the point (x_0, ξ_0) is non-characteristic for p, and $(x_0, \xi_0) \notin WF(p(x, D)u)$, then $(x_0, \xi_0) \notin WF(u)$.*

Proposition 2.2. *(See [11].) For every $p \in \Gamma^m(\mathbb{R})$, and $u \in \mathcal{S}'(\mathbb{R})$, we have $WF(p(x,D)u) \subset WF(u)$.*

We will also use the following characterization of the Schwartz class.

Proposition 2.3. *(See [11].) For $u \in \mathcal{S}'(\mathbb{R})$ we have $WF(u) = \emptyset$ if and only if $u \in \mathcal{S}(\mathbb{R})$.*

In fact, results similar to those in Proposition 2.1 hold for more general classes of operators. The following case will be important in the following.

Consider the class $\tilde{\Gamma}^m_\delta(\mathbb{R})$ of symbols $p(x,\xi)$ satisfying the following estimates:

$$\left|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)\right| \leqslant C_{\alpha\beta}\langle\xi\rangle^{m-\alpha}\langle x\rangle^{m-\beta+\delta\alpha}, \quad \forall\alpha,\beta \in \mathbb{N},\ (x,\xi) \in \mathbb{R}^2, \tag{2.2}$$

for some $m \in \mathbb{R}$, $0 \leqslant \delta < 1$. For $t \in \mathbb{R}$, we write $\langle t\rangle = (1+t^2)^{1/2}$. Notice that $\Gamma^0(\mathbb{R}) \subset \tilde{\Gamma}^0(\mathbb{R})$. These classes are a special case of the Weyl–Hörmander classes $S(M,g)$, with weight $M(x,\xi) = \langle x\rangle^m\langle\xi\rangle^m$ and metric $g_{x,\xi} = \frac{dx^2}{\langle x\rangle^2} + \frac{d\xi^2}{\langle x\rangle^{-2\delta}\langle\xi\rangle^2}$. The usual symbolic calculus with full asymptotic expansions works for these classes (cf. [10, Chapter XVIII]), because the so-called Planck function $h(x,\xi) = \langle\xi\rangle^{-1}\langle x\rangle^{\delta-1}$ satisfies $h(x,\xi) \leqslant C(1+|x|+|\xi|)^{\delta-1}$ and $\delta - 1 < 0$.

A symbol $p \in \tilde{\Gamma}^m_\delta(\mathbb{R})$ is called hypoelliptic at $(x_0,\xi_0) \neq (0,0)$ if for some $\varepsilon > 0$ it satisfies

$$\left|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)\right| \leqslant C_{\alpha\beta}\left|p(x,\xi)\right|\langle\xi\rangle^{-\alpha}\langle x\rangle^{-\beta+\delta\alpha}, \quad \forall\alpha,\beta \in \mathbb{N},\ (x,\xi) \in V_{(x_0,\xi_0),\varepsilon}, \tag{2.3}$$

and

$$\left|p(x,\xi)\right| \geqslant C\langle x\rangle^{m'}\langle\xi\rangle^{m'}, \quad \forall (x,\xi) \in V_{(x_0,\xi_0),\varepsilon}, \tag{2.4}$$

for some $m' \in \mathbb{R}$, $C > 0$.

Proposition 2.4. *Let $p(x,\xi)$ satisfy (2.2), (2.3) and (2.4). If $(x_0,\xi_0) \notin WF(p(x,D)u)$ then $(x_0,\xi_0) \notin WF(u)$.*

The proof is standard, since the assumptions imply the existence of a microlocal parametrix $Q = q(x,D)$, $q \in \tilde{\Gamma}^{-m}_\delta(\mathbb{R})$ (see e.g. [8, Lemma 3.1] and [13, Lemma 1.1.4 and Proposition 1.1.6]).

Finally, we will also make use of the class $\Gamma^m_{1,\delta}(\mathbb{R})$ of smooth functions $p \in C^\infty(\mathbb{R}^2)$ such that

$$\left|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)\right| \leqslant C_{\alpha\beta}\left(1+|x|+|\xi|\right)^{m-\alpha+\delta\beta}, \quad \forall\alpha,\beta \in \mathbb{N},\ (x,\xi) \in \mathbb{R}^2, \tag{2.5}$$

for some $m \in \mathbb{R}$, $0 \leqslant \delta < 1$. We have the following result.

Proposition 2.5. *Let $p \in \Gamma^m_{1,\delta}(\mathbb{R})$. Then $WF(p(x,D)u) \subset WF(u)$.*

The proof is again standard and omitted for the sake of brevity.

Finally we define, for future references, the following class of functions in $\mathbb{R}$. We set $S^m(\mathbb{R})$, $m \in \mathbb{R}$, for the space of smooth functions satisfying the estimates

$$\left|f^{(\alpha)}(x)\right| \leqslant C_\alpha\langle x\rangle^{m-\alpha}, \quad \forall\alpha \in \mathbb{N},\ x \in \mathbb{R}. \tag{2.6}$$

As already observed in the Introduction, it is also useful to deal with the *Weyl quantization*, defined as

$$p^w(x,D) = (2\pi)^{-1}\int_{\mathbb{R}} e^{i(x-y)\xi} p\left(\frac{x+y}{2},\xi\right)u(y)\,dy\,d\xi.$$

For example, an operator with a symbol $a(x,\xi)$ as in (1.4) in the standard quantization can be always re-written as an operator with Weyl symbol $p(x,\xi)$ given by the formula (1.5).

The main feature of Weyl quantization is its symplectic invariance: if $\chi:\mathbb{R}^2\to\mathbb{R}^2$ is a linear symplectic map, there is a unitary operator U in $L^2(\mathbb{R})$ which is also an isomorphism of $\mathcal{S}(\mathbb{R})$ into itself and of $\mathcal{S}'(\mathbb{R})$ into itself (in fact a metaplectic operator), such that

$$(p\circ\chi)^w(x,D)=U^{-1}p^w(x,D)U. \tag{2.7}$$

Moreover, in the definition of the global wave-front set it would be equivalent to use the Weyl quantization, which implies at once that

$$(x_0,\xi_0)\notin WF(u) \quad\Longleftrightarrow\quad \chi(x_0,\xi_0)\notin WF(Uu), \tag{2.8}$$

where U is the operator associated to χ as in (2.7).

3. Factorization of the operator

In this section we provide a convenient microlocal factorization of the operator P in Theorem 1.2, which reduces the proof of that result to the case of first order operators.

Let $j\in\mathbb{N}$ and consider the elementary symmetric functions defined by

$$\begin{aligned}
&\sigma_0(\xi_1,\dots,\xi_j)=1,\\
&\sigma_1(\xi_1,\dots,\xi_j)=-\sum_{1\leqslant j\leqslant j}\xi_j,\\
&\sigma_2(\xi_1,\dots,\xi_j)=\sum_{1\leqslant j<k\leqslant j}\xi_j\xi_k,\\
&\quad\dots\\
&\sigma_j(\xi_1,\dots,\xi_j)=(-1)^j\xi_1\cdots\xi_j.
\end{aligned}$$

Proposition 3.1. *Let $r_1\geqslant 0$, $r_2\geqslant 0$ and $a_j(x)$, $j=0,\dots,r_1$, and $\xi_j(x)$, $j=r_1,\dots,r_1+r_2$, be smooth functions, defined in some open subset of $\mathbb{R}$. Then we have*

$$\begin{aligned}
&\sum_{k=0}^{r_1}a_k(x)D^{r_1-k}\prod_{j=r_1}^{r_1+r_2}\bigl(D-\xi_j(x)\bigr)\\
&\quad=\sum_{k=0}^{r_1+r_2}\Bigl(\sum_{\substack{l+h=k\\ l\leqslant r_1,h\leqslant r_2}}a_l(x)\sigma_h\bigl(\xi_{r_1+1}(x),\dots,\xi_{r_1+r_2}(x)\bigr)+R_k(x)\Bigr)D^{r_1+r_2-k}
\end{aligned}$$

where $R_0=R_1=0$ and, for $k\geqslant 2$,

$$\begin{aligned}
R_k\in\operatorname{span}_{\mathbb{C}}\bigl\{&a_l\xi_{j_1}^{(m_1)}\cdots\xi_{j_h}^{(m_h)},\ l+h+m_1+\dots+m_h=k,\ 1\leqslant h\leqslant k-1,\\
&m_1+\dots+m_h>0,\ r_1+1\leqslant j_1<\dots<j_h\leqslant r_1+r_2\bigr\}.
\end{aligned}\tag{3.1}$$

Proof. Let us apply induction on r_2. The conclusion obviously holds with $R_j=0$ for every j if $r_2=0$. Suppose it holds for r_2. Then we have

$$\sum_{k=0}^{r_1} a_j(x) D^{r_1-k}\big(D-\xi_{r_1+1}(x)\big)\cdots\big(D-\xi_{r_1+r_2+1}(x)\big)$$
$$= \sum_{k=0}^{r_1+r_2} \sum_{\substack{l+h=k\\ l\leqslant r_1, h\leqslant r_2}} a_l(x)\sigma_h\big(\xi_1(x),\dots,\xi_k(x)+R_k(x)\big)D^{r_1+r_2-k}\big(D-\xi_{r_1+r_2+1}(x)\big), \tag{3.2}$$

with R_k as in the statement. Let $b_j(x)=\sum_{\substack{l+h=j\\ l\leqslant r_1, h\leqslant r_2}} a_l(x)\sigma_h(\xi_{r_1+1}(x),\dots,\xi_{r_1+r_2}(x))$. By Leibniz' formula the expression in the right-hand side of (3.2) reads

$$\sum_{k=0}^{r_1+r_2+1}\Bigg(b_k(x)-\xi_{r_1+r_2+1}(x)b_{k-1}(x)+R_k(x)$$
$$-\sum_{j=1}^{k-1}\binom{r_1+r_2+1-k-j}{j}\big(b_{k-j-1}(x)+R_{k-j-1}(x)\big)D^j\xi_{r_1+r_2+1}(x)\Bigg)D^{r_1+r_2+1-k}. \tag{3.3}$$

Now, we have

$$b_k(x)-\xi_{r_1+r_2+1}(x)b_{k-1}(x)=\sum_{\substack{l+h=k\\ l\leqslant r_1, h\leqslant r_2+1}} a_l(x)\sigma_h\big(\xi_{r_1+1}(x),\dots,\xi_{r_1+r_2+1}(x)\big),$$

whereas the terms $R_k(x)$, $b_{k-j-1}(x)D^j\xi_{r_1+r_2+1}(x)$ and $R_{k-j-1}(x)D^j\xi_{r_1+r_2+1}(x)$ are admissible errors. $\square$

Proposition 3.2. *Let $r_1\geqslant 0$, $r_2\geqslant 0$ and $a_j(x)$, $j=0,\dots,r_1$, and $b_h(x)$, $h=0,\dots,r_2$ be smooth functions on $\mathbb{R}$. Consider the operator P with Weyl symbol*

$$p(x,\xi)=\sum_{j=0}^{r_1} a_j(x)\xi^{r_1-j}\cdot\sum_{h=0}^{r_2} b_h(x)\xi^{r_2-h}. \tag{3.4}$$

Then P in the standard quantization has symbol

$$q(x,\xi)=\sum_{k=0}^{r_1+r_2}\Bigg(R_k+\sum_{\substack{l+h=k\\ l\leqslant r_1, h\leqslant r_2}} a_l b_h\Bigg)\xi^{r_1+r_2-k},$$

where

$$R_k\in \operatorname{span}_{\mathbb{C}}\big\{(a_l b_h)^{(\nu)},\ 1\leqslant\nu\leqslant k,\ l+h=k-\nu\big\}.$$

Proof. From (3.4) we have $p(x,\xi)=\sum_{k=0}^{r_1+r_2}\sum_{l+h=k} a_l b_h \xi^{r_1+r_2-k}$. On the other hand, for the standard symbol $q(x,\xi)$ we have the formula

$$q(x,\xi)=\sum_{\nu=0}^{r_1+r_2-k}\frac{1}{\nu!}\left(-\frac{i}{2}\right)^{\nu}\partial_x^{\nu}\partial_\xi^{\nu}p(x,\xi),$$

so that the desired result follows at once. $\square$

Lemma 3.3. *Let $\xi_k \in \mathbb{C}$, $k = 1, \dots, r_1 + r_2$,*

$$Q(\xi) = \prod_{r_1+1 \leqslant k \leqslant r_1+r_2} (\xi - \xi_k)$$

and

$$Q_j(\xi) = \prod_{\substack{1 \leqslant k \leqslant r_1+r_2 \\ k \neq j}} (\xi - \xi_k),$$

for $j = r_1 + 1, \dots, r_1 + r_2$.

Consider the square matrix A of size $r_1 + r_2$, whose jth column for $1 \leqslant j \leqslant r_1$ is made of the coefficients of the polynomials $\xi^{r_1-j} Q(\xi)$, whereas the jth column for $r_1 + 1 \leqslant j \leqslant r_1 + r_2$ is made of the coefficients of the polynomial Q_j (hence in the jth row there are the coefficients of $x^{r_1+r_2-j}$ which appear in those polynomials). Assume that $\xi_k \neq \xi_j$ when $j \neq k$. Then the matrix A is invertible and its inverse $A^{-1} = B = (B_{jk})$ is given by the following formula:

$$B_{jk} = \sum_{h=1}^{r_1} \frac{\xi_h^{r_1+r_2-k}}{\prod_{\substack{1 \leqslant l \leqslant r_1+r_2 \\ l \neq h}} (\xi_h - \xi_l)} \sigma_{j-1}(\xi_1, \dots, \xi_{h-1}, \xi_{h+1}, \dots, \xi_{r_1}) \tag{3.5}$$

if $1 \leqslant j \leqslant r_1$, $1 \leqslant k \leqslant r_1 + r_2$, whereas

$$B_{jk} = \frac{\xi_j^{r_1+r_2-k}}{\prod_{\substack{1 \leqslant l \leqslant r_1+r_2 \\ l \neq j}} (\xi_j - \xi_l)} \tag{3.6}$$

if $r_1 + 1 \leqslant j \leqslant r_1 + r_2$, $1 \leqslant k \leqslant r_1 + r_2$.

Proof. To compute the kth column of the inverse matrix, we have to solve the system $AX = Y$, where $Y = [0, \dots, 0, 1, 0 \dots, 0]^t$ is the kth element of the canonical basis of $\mathbb{C}^{r_1+r_2-1}$. Now, the vector $X = [a_1, \dots, a_{r_1+r_2}]^t$ is a solution if and only if

$$R(\xi)Q(\xi) + \sum_{j=r_1+1}^{r_1+r_2} a_j Q_j(\xi) = \xi^{r_1+r_2-k}, \tag{3.7}$$

with $R(\xi) = \sum_{j=1}^{r_1} a_j \xi^{r_1-j}$.

To solve (3.7), we express the right-hand side in terms of the interpolating Lagrange polynomials associated to the points ξ_j, $j = 1, \dots, r_1 + r_2$, which are

$$L_h(\xi) := \prod_{\substack{1 \leqslant k \leqslant r_1+r_2 \\ k \neq h}} (\xi - \xi_k), \quad h = 1, \dots, r_1 + r_2$$

(hence $L_h(x) = Q_h(x)$ for $h = r_1 + 1, \dots, r_1 + r_2$). We have

$$\xi^{r_1+r_2-k} = \sum_{h=1}^{r_1+r_2-1} \frac{\xi_h^{r_1+r_2-k}}{L_h(\xi_h)} L_h(\xi), \tag{3.8}$$

hence we are reduced to solve (3.7) when the right-hand side is replaced by each Lagrange polynomial. On the other hand, one sees immediately that the equation

$$R(\xi)Q(\xi)+\sum_{j=r_1+1}^{r_1+r_2} a_j Q_j(\xi)=L_h(x),$$

when $1\leqslant h\leqslant r_1$ has the solution $R(\xi)=\prod_{\substack{1\leqslant l\leqslant r_1\\ l\neq h}}(\xi-\xi_l)$, and $a_j=0$ for every $j=r_1+1,\dots,r_1+r_2$, whereas when $r_1+1\leqslant h\leqslant r_1+r_2$ the solution is given by $R(x)=0$ and $a_h=1$, $a_j=0$ if $r_1+1\leqslant j\leqslant r_1+r_2$ with $j\neq h$.

By linearity and (3.8) we obtain that the solution of (3.7) is given by

$$\begin{cases} R(\xi)=\displaystyle\sum_{j=1}^{r_1}\frac{\xi_h^{r_1+r_2-k}}{L_h(\xi_h)}\prod_{\substack{1\leqslant l\leqslant r_1\\ l\neq h}}(\xi-\xi_l),\\ a_j=\dfrac{\xi_j^{r_1+r_2-k}}{L_j(\xi_j)},\qquad j=r_1+1,\dots,r_1+r_2. \end{cases}$$

We can now come back to the original system $AX=Y$, and we obtain the desired form for the inverse matrix. □

It follows from the proof of the following proposition that the result in Lemma 3.3 continues to hold if one just require that $\xi_j\neq\xi_k$ for the indices $j\neq k$ such that $j>r_1$ or $k>r_1$. In fact, as we will see, a simplification of the factors $\xi_h-\xi_l$, with $1\leqslant k\neq l\leqslant r_1$ occurs in (3.6).

Lemma 3.4. *Let $\xi_j(x)$, $j=1,\dots,r_1+r_2$ be smooth functions for x large, such that $\xi_j(x)\neq\xi_k(x)$ if $r_1+1\leqslant j\neq k\leqslant r_1+r_2$,*

$$\xi_j(x)=O(x),\quad \text{if } 1\leqslant j\leqslant r_1+r_2 \tag{3.9}$$

and

$$\bigl|\xi_j(x)-\xi_k(x)\bigr|\gtrsim x\quad \text{if } 1\leqslant j\leqslant r_1 \text{ and } r_1+1\leqslant k\leqslant r_1+r_2 \tag{3.10}$$

as $x\to+\infty$. If we set $\xi_j=\xi_j(x)$ in Lemma 3.3, for the inverse matrix $B_{jk}=B_{jk}(x)$ given there, the following asymptotic formulae hold as $x\to+\infty$:

$$B_{jk}(x)=O\bigl(x^{j-k}\bigr) \tag{3.11}$$

for $1\leqslant j\leqslant r_1$, $1\leqslant k\leqslant r_1+r_2$, whereas

$$B_{jk}(x)=O\left(\frac{|\xi_j(x)|^{r_1+r_2-k}}{x^{r_1}\prod_{\substack{r_1+1\leqslant l\leqslant r_1+r_2\\ l\neq j}}|\xi_j(x)-\xi_l(x)|}\right) \tag{3.12}$$

for $r_1+1\leqslant j\leqslant r_1+r_2$, $1\leqslant k\leqslant r_1+r_2$.

Proof. Consider first the entries with $1 \leqslant j \leqslant r_1$. With the notation of Lemma 3.3 we can re-write (3.5) as

$$B_{jk} = \frac{\sum_{h=1}^{r_1}(-1)^{h-1}\xi_h^{r_1+r_2-k}\sigma_{j-1}(\xi_1,\dots,\xi_{h-1},\xi_{h+1},\dots,\xi_{r_1})\prod_{\substack{1\leqslant l\leqslant r_1\\ l\neq h}} Q(\xi_l)\prod_{\substack{1\leqslant l<l'\leqslant r_1\\ l,l'\neq h}}(\xi_l-\xi_{l'})}{\prod_{1\leqslant l<l'\leqslant r_1}(\xi_l-\xi_{l'})\prod_{1\leqslant l\leqslant r_1} Q(\xi_l)}. \tag{3.13}$$

We claim that the numerator of this fraction, as a polynomial in $\xi_1,\dots,\xi_{r_1+r_2}$ is divisible by the product $\prod_{1\leqslant l<l'\leqslant r_1}(\xi_l-\xi_{l'})$ which arises in the denominator. To this end, it is sufficient to show that the numerator vanishes if $\xi_\mu=\xi_\nu$, for every $1\leqslant \mu<\nu\leqslant r_1$. It is clear that all the terms in the above sum with $h\neq\mu,\nu$ vanish if $\xi_\mu=\xi_\nu$, because of the presence of the factor $\xi_\mu-\xi_\nu$. Let us prove that also the sum of the two terms corresponding to $h=\mu$ and $h=\nu$ vanishes. Due to the symmetry of σ_{j-1}, it suffices to show that

$$(-1)^{\mu-1}\prod_{\substack{1\leqslant l<l'\leqslant r_1\\ l,l'\neq\mu}}(\xi_l-\xi_{l'})+(-1)^{\nu-1}\prod_{\substack{1\leqslant l<l'\leqslant r_1\\ l,l'\neq\nu}}(\xi_l-\xi_{l'})=0.$$

This amounts to prove that

$$(-1)^{\mu-1+r_1-\nu}\prod_{\substack{1\leqslant l\leqslant r_1\\ l\neq\mu,\nu}}(\xi_l-\xi_\nu)+(-1)^{\nu-1+r_1-\mu-1}\prod_{\substack{1\leqslant l\leqslant r_1\\ l\neq\mu,\nu}}(\xi_l-\xi_\mu)=0,$$

where we took into account that $\mu<\nu$. But this is true, because the two products that arise in the last formula coincide when $\xi_\mu=\xi_\nu$.

This proves the claim and shows, by a limiting argument, that the matrix A in Lemma 3.3 is still invertible if one just require $\xi_j\neq\xi_k$ for the indices $j\neq k$ such that $j>r_1$ or $k>r_1$.

Now, by dividing the numerator of (3.13) by $\prod_{1\leqslant l<l'\leqslant r_1}(\xi_l-\xi_{l'})$ we get a polynomial in $\xi_1,\dots,\xi_{r_1+r_2}$ of degree[1]

$$(r_1+r_2-k)+(j-1)+(r_1-1)r_2+\frac{(r_1-1)(r_1-2)}{2}-\frac{r_1(r_1-1)}{2}=r_1r_2-k+j.$$

Hence, when we compose such a polynomial with the functions $\xi_j=\xi_j(x)$, by (3.9) we get a function which is $O(x^{r_1r_2-k+j})$ as $x\to+\infty$. On the other hand, using (3.10) we have $|\prod_{1\leqslant l\leqslant r_1} Q(\xi_l)|\gtrsim x^{r_1r_2}$, so that we deduce (3.11).

Finally, let us prove (3.12). This follows at once from (3.6) taking into account that, if $r_1+1\leqslant j\leqslant r_1+r_2$,

$$\prod_{\substack{1\leqslant l\leqslant r_1+r_2\\ l\neq j}}\left|\xi_j(x)-\xi_l(x)\right|\gtrsim x^{r_1}\prod_{\substack{r_1+1\leqslant l\leqslant r_1+r_2\\ l\neq j}}\left|\xi_j(x)-\xi_l(x)\right|,$$

which is a consequence of (3.10). $\square$

[1] Observe that the polynomials $\prod_{\substack{1\leqslant l<l'\leqslant r_1\\ l,l'\neq h}}(\xi_l-\xi_{l'})$ and $\prod_{1\leqslant l<l'\leqslant r_1}(\xi_l-\xi_{l'})$ have degree $\frac{(r_1-1)(r_1-2)}{2}$ and $\frac{r_1(r_1-1)}{2}$ respectively.

Proposition 3.5. *Let $p(x,\xi)$ be as in (1.5). Let $\xi_1(x),\dots,\xi_{r_1+r_2}(x)$, $r_1+r_2=m$, be its roots, defined for x large, and assume (3.9), (3.10), as well as*

$$\bigl|\xi_j(x)-\xi_k(x)\bigr|\gtrsim \max\bigl\{\bigl|\xi_j(x)\bigr|,\bigl|\xi_k(x)\bigr|,x^{-1+\varepsilon}\bigr\},\quad \text{for } j,k=r_1+1,\dots,r_1+r_2, \tag{3.14}$$

for some $\varepsilon>0$, as $x\to+\infty$. Let P be the operator with Weyl symbol $p(x,\xi)$. We can write, for x large,

$$P=\sum_{k=0}^{r_1}\eta_k(x)D^{r_1-k}\prod_{j=r_1+1}^{r_1+r_2}\bigl(D-\eta_j(x)\bigr)+R, \tag{3.15}$$

where $\eta_k\in S^j(\mathbb{R})$, $k=1,\dots,r_1$, $b_0(x)=1$, $\eta_j\in S^1(\mathbb{R})$, $j=r_1+1,\dots r_1+r_2$ (see (2.6)),

$$\eta_j(x)=\sigma_j\bigl(\xi_1(x),\dots,\xi_{r_1}(x)\bigr)+O\bigl(x^{j-2}\bigr)\quad \text{for } j=1,\dots,r_1 \tag{3.16}$$

and

$$\eta_j(x)=\xi_j(x)+O\bigl(x^{-1}\bigr)\quad \text{for } j=r_1+1,\dots,r_1+r_2 \tag{3.17}$$

as $x\to+\infty$, whereas R is a differential operator of order r_1+r_2 whose coefficients are in $S^m(\mathbb{R})$ and rapidly decreasing as $x\to+\infty$, together with their derivatives.

Proof. The standard symbol of P is given in Proposition 3.2. On the other hand, the symbol of the first operator in the right-hand side of (3.15) is computed in Proposition 3.1. By equating the coefficients of the terms of the same order we deduce that, modulo rapidly decreasing functions as $x\to+\infty$, it must be

$$\sum_{k=0}^{r_1+r_2}\sum_{\substack{l+h=k\\ l\leqslant r_1,h\leqslant r_2}}a_l\sigma_h(\xi_{r_1+1},\dots,\xi_{r_1+r_2})+R'_k=\sum_{\substack{l+h=k\\ l\leqslant r_1,h\leqslant r_2}}\eta_l(x)\sigma_h(\eta_{r_1+1},\dots\eta_{r_1+r_2})+R''_k \tag{3.18}$$

for $k=1,\dots,r_1+r_2$, where the η_j's are unknown, $a_l=a_l(x)=\sigma_l(\xi_1(x),\dots,\xi_{r_1}(x))$,

$$R'_k\in \operatorname{span}_{\mathbb{C}}\bigl\{\bigl(a_l\sigma_h(\xi_{r_1+1},\dots,\xi_{r_1+r_2})\bigr)^{(\nu)},\ 1\leqslant\nu\leqslant k,\ l+h=k-\nu\bigr\},$$

and

$$\begin{aligned}R''_k\in \operatorname{span}_{\mathbb{C}}\bigl\{&a_l\eta_{j_1}^{(m_1)}\cdots\eta_{j_h}^{(m_h)},\ l+h+m_1+\dots+m_h=k,\ 1\leqslant h\leqslant k-1,\\ &m_1+\dots+m_h>0,\ r_1+1\leqslant j_1<\dots<j_h\leqslant r_1+r_2\bigr\}.\end{aligned}$$

We set

$$\begin{cases}\eta_j(x)=a_j(x)+\zeta_j(x) & \text{for } j=1,\dots,r_1,\\ \eta_j(x)=\xi_j(x)-\zeta_j(x) & \text{for } j=r_1+1,\dots,r_1+r_2,\end{cases} \tag{3.19}$$

in the right-hand side of (3.18). By isolating the terms which are of order 0 or 1 with respect to $\zeta=(\zeta_1,\dots,\zeta_{r_1+r_2})$ we can write

$$\sum_{\substack{l+h=k\\ l\leqslant r_1, h\leqslant r_2}} \eta_l\sigma_h(\eta_{r_1+1},\dots\eta_{r_1+r_2}) = \sum_{\substack{l+h=k\\ l\leqslant r_1, h\leqslant r_2}} a_l\sigma_h(\xi_{r_1+1},\dots,\xi_{r_1+r_2}) + (A\zeta)_k + X_k,$$

where A is exactly the matrix described in Lemma 3.3. By multi-linearity we can also write $R''_k = Y_k + Z_k$, where Z_k is the term constant with respect to the ζ_j's. The system (3.18) hence becomes

$$\zeta = -A^{-1}X - A^{-1}Y - A^{-1}Z - A^{-1}R'. \tag{3.20}$$

Let us prove that

$$\left(A^{-1}Z\right)_j = \begin{cases} O(x^{j-2}) & \text{if } j = 1,\dots,r_1,\\ O(x^{-1}) & \text{if } j = r_1+1,\dots,r_1+r_2, \end{cases} \tag{3.21}$$

as $x \to +\infty$. We have

$$\begin{aligned} Z_k \in \operatorname{span}_{\mathbb{C}}\bigl\{& a_l\xi_{j_1}^{(m_1)}\cdots\xi_{j_h}^{(m_h)},\ l+h+m_1+\cdots+m_h = k,\ 1\leqslant h\leqslant k-1,\\ & m_1+\cdots+m_h > 0,\ r_1+1\leqslant j_1<\cdots<j_h\leqslant r_1+r_2\bigr\}. \end{aligned} \tag{3.22}$$

Hence, if $j \leqslant r_1$ and $A^{-1} = (B_{jk})$, we have $(A^{-1}Z)_j = \sum_{k=1}^{r_1+r_2} B_{jk}Z_k$, and by (3.22), (3.9) and (3.11)

$$B_{jk}Z_k = O\left(x^{j-k}x^{l+h-m_1-\cdots-m_h}\right) = O\left(x^{j-2(k-l-h)}\right) = O\left(x^{j-2}\right),$$

because $k-l-h \geqslant 1$.

If $j \geqslant r_1+1$, since $\xi_{j_\nu}^{(m_\nu)} = O(\xi_{j_\nu}(x)x^{-m_\nu})$ and $m_1+\cdots+m_h = k-k-l$, by (3.22) it suffices to prove that

$$\frac{a_l(x)x^{-k+h+l}|\xi_j(x)|^{r_1+r_2-k}\xi_{j_1}(x)\cdots\xi_{j_h}(x)}{x^{r_1}\prod_{\substack{r_1+1\leqslant l\leqslant r_1+r_2\\ l\neq j}}|\xi_j(x)-\xi_l(x)|} = O\left(x^{-1}\right). \tag{3.23}$$

By (3.14) we have $\xi_{j_\nu}(x) = O(|\xi_{j_\nu}-\xi_j|)$ if $j_\nu \neq j$ and $\xi_j(x) = O(|\xi_j(x)-\xi_l(x)|)$ and $x|\xi_j(x)-\xi_l(x)| \to +\infty$ if $j\neq l$, $l \geqslant r_1+1$. Hence we obtain

$$x^{-r_1+k-h-1}\cdot\frac{|\xi_j(x)|^{r_1+r_2-k}\xi_{j_1}(x)\cdots\xi_{j_h}(x)}{\prod_{\substack{r_1+1\leqslant l\leqslant r_1+r_2\\ l\neq j}}|\xi_j(x)-\xi_l(x)|} = O(1) \tag{3.24}$$

(this is easily verified by considering separately the cases when the number of factors in the numerator is less or greater than that of the denominator). On the other hand, by (3.9) we have $|a_l(x)|x^{-k+k+l}x^{-k+h+1} = O(x^{-2(k-l-h)+1}) = O(x^{-1})$, which combined with (3.24) gives (3.23). This proves (3.21).

Similarly one proves that

$$\left(A^{-1}R'\right)_j = \begin{cases} O(x^{j-2}) & \text{if } j = 1,\dots,r_1,\\ O(x^{-1}) & \text{if } j = r_1+1,\dots,r_1+r_2 \end{cases} \tag{3.25}$$

as $x \to +\infty$.

Now, (3.21) and (3.25) suggest to look for functions ζ_j with asymptotic expansion

$$\zeta_j(x) \sim \sum_{\mu=0}^{+\infty} c_{j,\mu} x^{j-2-\frac{\mu}{p}}, \quad \text{for } j = 1, \dots, r_1, \tag{3.26}$$

$$\zeta_j(x) \sim \sum_{\mu=0}^{+\infty} c_{j,\mu} x^{-1-\frac{\mu}{p}}, \quad \text{for } j = r_1 + 1, \dots, r_1 + r_2, \tag{3.27}$$

where p is the integer for which (1.8) holds. In order for the system (3.20) to be solvable by an iterative argument (in the sense of formal power series), it suffices to prove that the coefficient in $(A^{-1}X)_j$, respectively $(A^{-1}Y)_j$, of $x^{j-2-\frac{k}{p}}$, respectively $x^{-1-\frac{k}{p}}$, depends only on the $c_{j,\mu}$ with $\mu < k$. To this end, consider first the term $(A^{-1}X)_j$. By the definition of X_k, it belongs to the complex span of

$$\delta_l(x)\gamma_{j_1}(x)\cdots\gamma_{j_h}(x),$$

with $l+h=k$, $l \leqslant r_1$, $h \leqslant r_2$, where $\delta_l = a_l$ or $\delta_l = \zeta_l$, and $\gamma_{j_\nu} = \xi_{j_\nu}$ or $\gamma_{j_\nu} = \zeta_{j_\nu}$, and the above product contains *at least two factors of type* ζ.

Let $1 \leqslant j \leqslant r_1$. We have $(A^{-1}X)_j = \sum_{k=1}^{r_1+r_2} B_{jk}X_k$. By (3.11), (3.9), (3.26) and (3.27) we get $B_{jk}X_k = O(x^{j-k}x^{l+h-4}) = O(x^{j-4})$. More generally, the same argument shows that for a fixed $\mu \geqslant 0$ the coefficients $c_{i,\mu}$, $i = 1, \dots, r_1 + r_2$, may appear in the asymptotic expansion of $(A^{-1}X)_j$ only in the terms of degree $\leqslant j - 2 - \frac{\mu}{p} - 2$.

Let now $r_1 + 1 \leqslant j \leqslant r_1 + r_2$. Let us prove $B_{jk}X_k = o(x^{-1})$. It suffices to prove that

$$\frac{\delta_l(x)|\xi_j(x)|^{r_1+r_2-k}\gamma_{j_1}(x)\cdots\gamma_{j_h}(x)}{x^{r_1}\prod_{\substack{r_1+1\leqslant l\leqslant r_1+r_2\\ l\neq j}} |\xi_j(x) - \xi_l(x)|} = o\big(x^{-1}\big), \tag{3.28}$$

if $l+h=k$, $l \leqslant r_1$, $h \leqslant r_2$. We can suppose that, say, $\gamma_{j_h} = \zeta_{j_h}$. Assume furthermore that there exists $1 \leqslant \nu \leqslant h-1$ such that $\gamma_{j_\nu} = \zeta_{j_\nu}$. Then we have $\delta_l(x)x^{-k+h}\gamma_{j_h}(x) = O(x^{-1})$ and

$$\frac{|\xi_j(x)|^{r_1+r_2-k}\gamma_{j_1}(x)\cdots\gamma_{j_{h-1}}(x)x^{-r_1+k-h}}{\prod_{\substack{r_1+1\leqslant l\leqslant r_1+r_2\\ l\neq j}} |\xi_j(x) - \xi_l(x)|} = o(1),$$

where we used the same arguments as in the proof of (3.24) and the fact that $\zeta_{j_\nu}(x)/|\xi_j(x) - \xi_{j_\nu}(x)| = O(x^{-1}/|\xi_j(x) - \xi_{j_\nu}(x)|) = o(1)$. This gives (3.28). If instead there is no such a ν, it must be $\delta_l(x) = \zeta_l(x)$. Hence $\delta_l(x)x^{-k+h+1} = O(x^{-1})$ and

$$\frac{|\xi_j(x)|^{r_1+r_2-k}\gamma_{j_1}(x)\cdots\gamma_{j_h}(x)x^{-r_1+k-h-1}}{\prod_{\substack{r_1+1\leqslant l\leqslant r_1+r_2\\ l\neq j}} |\xi_j(x) - \xi_l(x)|} = o(1),$$

which still gives (3.28). This proves that, if $r_1 + 1 \leqslant j \leqslant r_1 + r_2$, $(A^{-1}X)_j = o(x^{-1})$ as $x \to +\infty$. More generally, the same argument shows that for a fixed $\mu \geqslant 0$ the coefficients $c_{i,\mu}$, $i = 1, \dots, r_1 + r_2$, may appear in the asymptotic expansion of $(A^{-1}X)_j$ only in the terms of degree $< -1 - \frac{\mu}{p}$. Similarly one sees that the same is true for $(A^{-1}Y)_j$.

Hence, the system (3.20) has a formal power series solutions in the form (3.26), (3.27), as $x \to +\infty$. Now, by a classical Borel-type argument, see e.g. [10, Proposition 18.1.3], one can construct functions

$\zeta_j \in S^{j-2}(\mathbb{R})$ if $1 \leqslant j \leqslant r_1$, $\zeta_j \in S^{-1}(\mathbb{R})$ if $r_1+1 \leqslant j \leqslant r_1+r_2$, with the asymptotic expansions in (3.26) and (3.27) respectively.

The functions $\eta_j(x)$ defined in (3.19) for x large and extended smoothly to zero for $x<0$ then fulfill (3.15) for a convenient operator R having the desired properties. This concludes the proof. $\square$

4. Proof of the main result (Theorem 1.2)

4.1. Sufficient condition

Let u be in $\mathcal{S}'(\mathbb{R})$ with $Pu \in \mathcal{S}(\mathbb{R})$. Let us prove that, under the assumptions (1.9) and (1.10) we have $u \in \mathcal{S}(\mathbb{R})$, i.e. $WF(u)=\emptyset$ (Proposition 2.3).

We use the factorization (1.6) for the Weyl symbol $p(x,\xi)$, valid for large $|x|$, and define the complex constant λ_j, $j=1,\dots,m$ by (1.7). Let now $(x_0,\xi_0) \neq (0,0)$. If (x_0,ξ_0) does not belong to any of the rays $\{t(1,\lambda_j),\ t\in\mathbb{R}\setminus\{0\}\}$, with $\lambda_j\in\mathbb{R}$, we see from (1.6), (1.7) that p is non-characteristic at (x_0,ξ_0), i.e. it satisfies the estimate (2.1) for some $\varepsilon>0$. Hence $(x_0,\xi_0)\notin WF(u)$ by Proposition 2.1.

Let now (x_0,ξ_0) lie on a ray $\{t(1,\lambda_j),\ t\in\mathbb{R}\setminus\{0\}\}$, for some $\lambda_j\in\mathbb{R}$. We can suppose that $(x_0,\xi_0)=(1,\lambda_j)$ or $(x_0,\xi_0)=(-1,-\lambda_j)$. We can further reduce to the case when $(x_0,\xi_0)=(1,0)$. In fact, suppose that $(x_0,\xi_0)=(1,\lambda_j)$, and consider the linear symplectic transformation $\chi(x,\xi)=(x,\xi+\lambda_j x)$. Let U be the operator associated to χ via (2.7). By (2.8), in order to get $(x_0,\xi_0)\notin WF(u)$ it suffices to prove that $(1,0)\notin WF(U^{-1}u)$. Now, we have $(p\circ\chi)^w(x,D)U^{-1}u=U^{-1}Pu\in\mathcal{S}(\mathbb{R})$, so that our original problem is equivalent to a similar one with (x_0,ξ_0) replaced by $(1,0)$ and the symbol $p(x,\xi)$ replaced by $(p\circ\chi)(x,\xi)$. The same holds for $(x_0,\xi_0)=(-1,-\lambda)$ if we perform the preliminary symplectic transformation $\chi(x,\xi)=(-x,-\xi)$.

After these transformations we get a symbol, which we will continue to call $p(x,\xi)$, which has a factorization as in (1.6) for x large, where the new roots $\xi_j(x)$ satisfy (1.7) and (1.8) for other values of λ_j, as well as (1.9) and (1.10).

Suppose now $\lambda_j\neq 0$ for $j=1,\dots,r_1$, and $\lambda_j=0$ for $j=r_1+1,\dots,r_1+r_2$, with $r_1+r_2=m$. Accordingly, by (1.7), (1.9) we have

$$\bigl|\xi_j(x)\bigr|\asymp x \quad \text{for } 1\leqslant j\leqslant r_1, \qquad |\xi_j(x)|=o(x) \quad \text{for } r_1+1\leqslant j\leqslant r_1+r_2, \tag{4.1}$$

$$\bigl|\xi_j(x)-\xi_k(x)\bigr|\gtrsim \max\bigl\{\bigl|\xi_j(x)\bigr|,\bigl|\xi_k(x)\bigr|,x^{-1+\varepsilon}\bigr\} \quad \text{for } r_1+1\leqslant j\leqslant r_1+r_2, \tag{4.2}$$

as $x\to+\infty$. Let us verify that $(1,0)\notin WF(u)$ if $Pu\in\mathcal{S}(\mathbb{R})$.

We can apply Proposition 3.5 and use the factorization in (3.15). We claim that the symbol of the first factor, namely $\sum_{k=0}^{r_1}\eta_k(x)\xi^{r_1-k}$, is non-characteristic at $(1,0)$. In fact by (4.1) we have

$$\left|\sum_{k=0}^{r_1}\sigma_j\bigl(\xi_1(x),\dots,\xi_{r_1}(x)\bigr)\xi^{r_1-k}\right|=\prod_{j=1}^{r_1}\bigl|\xi-\xi_j(x)\bigr|\gtrsim x^{r_1}, \quad \text{for } (x,\xi)\in V_{(1,0),\varepsilon}$$

if ε is small enough. Now by (3.16) we deduce that

$$\left|\sum_{k=0}^{r_1}\eta_k(x)\xi^{r_1-k}\right|\gtrsim x^{r_1}, \quad \text{for } (x,\xi)\in V_{(1,0),\varepsilon}$$

which proves the claim. Hence, since $Pu\in\mathcal{S}(\mathbb{R})$ and $(1,0)\notin WF(Ru)$, by Proposition 2.1 and (3.15) we deduce that $(1,0)$ does not belong to the wave-front set of $\prod_{j=r_1+1}^{r_1+r_2}(D-\eta_j(x))u$. Hence, to finish the proof it is sufficient to prove that, for every $j=r_1+1,\dots,r_1+r_2$, if $(1,0)\notin WF((D-\eta_j(x))u)$ then $(1,0)\notin WF(u)$.

Fix such a j. We have from (1.8) that

$$\xi_j(x) = \sum_{-\infty<k\leqslant p-1} c_{j,k}x^{k/p} \quad \text{for } x \text{ large}. \tag{4.3}$$

By (3.17) we have an asymptotic expansion

$$\eta_j(x) = \sum_{-p<k\leqslant p-1} c_{j,k}x^{k/p} + O\left(x^{-1}\right) \quad \text{for } x \text{ large}. \tag{4.4}$$

By (1.10), there exists $k > -p$ such that $c_{j,k} \notin \mathbb{R}$. Let $\nu \leqslant p-1$ be the greatest index for which this holds. Let $Q(x)$ be a smooth real-valued function, $Q(x) = 0$ for $x < 0$, and $Q_j(x) = \int_0^x \sum_{\nu<k\leqslant p-1} c_{j,k}t^{k/p}\,dt$ for x large. We can write

$$D - \eta_j(x) = e^{iQ_j(x)}\left(D - \tilde{\eta}_j(x)\right) \circ e^{-iQ_j(x)}, \tag{4.5}$$

where $\tilde{\eta}_j \in S^1(\mathbb{R})$, and

$$\tilde{\eta}_j(x) = \eta_j(x) - \sum_{\nu<k\leqslant p-1} c_{j,k}x^{k/p} = c_{j,\nu}x^{\nu/p} + o\left(x^{\nu/p}\right) \tag{4.6}$$

for large x.

Now, we can regard $e^{\pm iQ_j(x)}$ as symbols (independent of ξ) in the class $\Gamma^0_{1,\delta}(\mathbb{R})$ with $\delta = 1 - 1/p$ (see (2.5)). Hence, if $(1,0) \notin WF((D - \eta_j(x))u)$, by Proposition 2.5 we get $(1,0) \notin WF((D - \tilde{\eta}_j(x))u)$. On the other hand, the symbol $\xi - \tilde{\eta}_j(x)$ belongs to $\tilde{\Gamma}^1_\delta(\mathbb{R})$ (see (2.2)), and it is hypoelliptic at $(1,0)$ (see (2.3), (2.4)). In fact, using (4.6) and the fact that $c_{j,\nu} \notin \mathbb{R}$ we have

$$\left|\partial_\xi^\alpha \partial_x^\beta \left(\xi - \tilde{\eta}_j(x)\right)\right| \leqslant C_{\alpha\beta}\left|\xi - \tilde{\eta}_j(x)\right| \langle\xi\rangle^{-\alpha} \langle x\rangle^{-\beta+\delta\alpha}, \quad \forall \alpha, \beta \in \mathbb{N},\ x > 1,\ \xi \in \mathbb{R},$$

with $\delta = \max\{0, -\nu/p\} \leqslant 1 - 1/p < 1$, and $|\xi - \tilde{\eta}_j(x)| \gtrsim |\xi| + x^{\nu/p}$ for x large and $\xi \in \mathbb{R}$. Hence, by Proposition 2.4 we obtain $(1,0) \notin WF(u)$, which concludes the proof.

4.2. *Necessary condition*

Let us assume (1.9) but suppose that (1.10) fails for some j. Let us prove that then there exists $u \in \mathcal{S}'(\mathbb{R})$, $u \notin \mathcal{S}(\mathbb{R})$ such that $Pu \in \mathcal{S}(\mathbb{R})$. By the applying the same arguments as in the proof of the sufficient condition we can assume that (1.10) fails, say, for $x \to +\infty$ and $j = r_1 + r_2$, with $\lambda_{r_1+r_2} = 0$. Hence (4.3) and therefore (4.4) hold with $j = r_1 + r_2$ and $c_{r_1+r_2,k} \in \mathbb{R}$ for every $-p < k \leqslant p-1$. Let us set

$$Q(x) = \int_0^x \eta_{r_1+r_2}(t)\,dt,$$

and let $u \in C^\infty(\mathbb{R})$ such that $u(x) = 0$ for $x < 0$, $u(x) = e^{iQ(x)}$ for $x > 1$. Then, by (4.4) with $j = r_1 + r_2$ we have $x^{-c} \lesssim |u(x)| \lesssim x^c$ when $x > 1$, for some $c > 1$. Hence $u \in \mathcal{S}'(\mathbb{R})$, $u \notin \mathcal{S}(\mathbb{R})$. Moreover, $Pu = 0$ for $x < 0$, whereas for x large we have $(D - \eta_{r_1+r_2}(x))u = 0$ and therefore, from (3.15), $Pu = Ru$. Now, we also have $|\partial^\alpha u(x)| \lesssim x^{c+|\alpha|}$ when $x > 1$, because $\eta_{r_1+r_2} \in S^1(\mathbb{R})$, so that Ru is rapidly decreasing as $x \to +\infty$ together with its derivatives. This implies that $Pu \in \mathcal{S}(\mathbb{R})$ and concludes the proof.

5. Examples and remarks

We give in the following some examples and remarks.

Remark 5.1. The assumption $c_{m,0} \neq 0$ for the Weyl symbol of P can be eliminated, provided we submit preliminarily P to the conjugation by a metaplectic operator U. Namely, let U be associated to the symplectic map $\chi(x,\xi) = (x - \lambda\xi, \xi)$, $\lambda \in \mathbb{R}$. In view of (2.7), the Weyl symbol of $\tilde{P} = U^{-1}PU$ is given by

$$(p \circ \chi)(x,\xi) = \sum_{\alpha+\beta \leqslant m} c_{\alpha,\beta}(x + \lambda\xi)^{\beta}\xi^{\alpha}$$

with new coefficient

$$\tilde{c}_{m,0} = \sum_{\alpha+\beta=m} c_{\alpha,\beta}\lambda^{\beta} \neq 0$$

for a generic choice of λ. On the other hand, the global regularity of P and $\tilde{P}$ are equivalent, because $U, U^{-1} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$, $U, U^{-1} : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$. We may correspondingly re-formulate condition (1.9) and Theorem 1.2.

Example 5.2 *(Globally elliptic operators).* (Cf. Grushin [6], Shubin [18], Helffer [7].) Assume all the roots λ_j, $j = 1, \dots, m$, in (1.5) satisfy $\operatorname{Im}\lambda_j \neq 0$. This is equivalent to the so-called global ellipticity of the symbol:

$$\bigl|p(x,\xi)\bigr| \geqslant \varepsilon\bigl(1 + |x| + |\xi|\bigr)^{m} \quad \text{for } |x| + |\xi| \geqslant R, \tag{5.1}$$

for some $\varepsilon > 0$, $R > 0$. In fact, if we write $p(x,\xi)$ as a sum of homogeneous terms

$$p(x,\xi) = \sum_{0 \leqslant j \leqslant m} p_j(x,\xi), \qquad p_j(x,\xi) = \sum_{\alpha+\beta=j} c_{\alpha,\beta}x^{\beta}\xi^{\alpha}, \tag{5.2}$$

the condition (5.1) is equivalent to

$$p_m(x,\xi) \neq 0 \quad \text{for } (x,\xi) \neq (0,0). \tag{5.3}$$

The corresponding operators are globally regular, since (1.10) is obviously satisfied. Basic example is the harmonic oscillator of Quantum Mechanics:

$$P = D^2 + x^2.$$

Example 5.3. Consider now the case when $\lambda_j \in \mathbb{R}$ for some j. Assume for the moment that all the real roots λ_j are distinct, that is $\partial_\xi p_m(x, \lambda_j x) \neq 0$ for $x \neq 0$. We may apply Theorem 1.2. By factorization we have

$$\xi_j(x) = \lambda_j x + c_j + O\bigl(x^{-1}\bigr)$$

with $c_j = p_{m-1}(x, \lambda_j x)/\partial_\xi p_m(x, \lambda_j x)$. Then P is globally regular if and only if $\operatorname{Im} c_j \neq 0$ for every real λ_j. Consider for example the elementary polynomial

$$p(x,\xi) = \xi - x + c.$$

Limiting attention to the corresponding homogeneous equation, a classical solution of $Du - xu + cu = 0$ is given by $u(x) = \exp[ix^2/2 - icx]$. If $\operatorname{Im} c \neq 0$, then $u \notin \mathcal{S}'(\mathbb{R})$, whereas if $c \in \mathbb{R}$, then $u \in \mathcal{S}'(\mathbb{R})$, $u \notin \mathcal{S}(\mathbb{R})$, contradicting the global regularity.

Example 5.4 *(Quasi-elliptic operators).* (Cf. Grushin [6], Boggiatto, Buzano, Rodino [2], Cappiello, Gramchev, Rodino [4].) We pass now to consider the case when two, or more, real roots coincide. For simplicity we shall assume $\lambda_j = 0$ for all $j = 1, \dots, m$, that is $c_{\alpha,\beta} = 0$ for all α, β with $\alpha + \beta = m$, apart from $c_{m,0} = 1$. We may then consider the largest rational number $q > 1$ such that $\alpha + q\beta \leqslant m$ for all (α, β) with $c_{\alpha,\beta} \neq 0$, and write

$$p(x,\xi) = \sum_{\alpha+q\beta\leqslant m} c_{\alpha,\beta} x^\beta \xi^\alpha, \tag{5.4}$$

with $c_{\alpha,\beta} \neq 0$ for some (α, β) with $\alpha + q\beta = m$, $\alpha \neq m$. For the moment, we understand $q < \infty$, that is a term with $\beta \neq 0$ actually exists. Note also that $q \geqslant \frac{m}{m-1}$. This symbol in (5.4) is called (globally) quasi-elliptic if

$$p_{m,q}(x,\xi) = \sum_{\alpha+q\beta=m} c_{\alpha,\beta} x^\beta \xi^\alpha \neq 0 \quad \text{for } (x,\xi) \neq (0,0). \tag{5.5}$$

This implies in particular $c_{0,m/q} \neq 0$ and m/q is a positive integer. The corresponding operators are globally regular, as proved for example in [6,2,4]. Computing the Puiseaux expansion (1.8), we have

$$\xi_j(x) = r_j^\pm |x|^{1/q} + o\big(|x|^{1/q}\big) \quad \text{for } x \to \pm\infty, \tag{5.6}$$

where $r_j^\pm$ are the roots in $\mathbb{C}$ of $p_{m,q}(\pm 1, r) = 0$. We deduce that (5.5) is satisfied if and only if $\operatorname{Im} r_j^\pm \neq 0$ for all the roots.

We then recapture the global regularity from Theorem 1.2, provided condition (1.9) is satisfied, i.e. the roots r_j^+, or equivalently r_j^-, are distinct, cf. (1.11). As example, consider the Airy-type operator

$$P = D^2 + cx,$$

which is globally regular if and only if $\operatorname{Im} c \neq 0$.

Example 5.5. Let the symbol $p(x,\xi)$ be of the form (5.4), with distinct roots $r_j^\pm$ in (5.6), but assume now $r_j^+ \in \mathbb{R}$, or $r_j^- \in \mathbb{R}$, for some j. The condition (1.9) is satisfied, and we are led to determine the subsequent terms in the Puiseaux expansion (1.11). We refer to Bliss [1] for general rules of computations, and we limit ourselves here to the example

$$p(x,\xi) = \xi^m - A\xi^r - x \tag{5.7}$$

with $0 \leqslant r < m$, $A \in \mathbb{R}$. The expansion in (5.6) reads in this case

$$\xi_j(x) = e_j x^{1/m} + o\big(|x|^{1/m}\big)$$

where e_j, $j = 1, \dots, m$, are the mth roots of 1. To fix ideas, assume m even; then we have for $x \to +\infty$ the two real roots ± 1. It is easy to compute

$$\xi_\pm(x) = \pm x^{1/m} + c_\pm x^s + o\big(x^s\big), \quad x \to +\infty,$$

with $s = (r + 1 - m)/m$ and $c_\pm = \pm A/m$. Note that $1/m > s > -1$. Hence if $\operatorname{Im} A \neq 0$, the condition (1.10) is satisfied and Theorem 1.2 gives global regularity. Let us test this result on the corresponding homogeneous equation

$$D^m u - AD^r u - xu = 0. \tag{5.8}$$

Every solution $u \in \mathcal{S}'(\mathbb{R})$, or $u \in \mathcal{S}(\mathbb{R})$ of the equation can be regarded as inverse Fourier transform of a solution $v \in \mathcal{S}'(\mathbb{R})$, respectively $v \in \mathcal{S}(\mathbb{R})$, of

$$Dv + \left(x^m - Ax^r\right)v = 0,$$

having the (classical) solution

$$v(x) = e^{iAx^{r+1}/(r+1) - ix^{m+1}/(m+1)}.$$

If $\operatorname{Im} A \neq 0$, then $v \notin \mathcal{S}'(\mathbb{R})$ or $v \in \mathcal{S}(\mathbb{R})$, depending on r and A, that agrees with the global regularity of the operator. If $A \in \mathbb{R}$, then $v \in \mathcal{S}'(\mathbb{R})$, $v \notin \mathcal{S}(\mathbb{R})$, contradicting global regularity.

Example 5.6 *(Multi-quasi-elliptic operators).* (Cf. Boggiatto, Buzano, Rodino [2].) By conjugation with Fourier transform, which we may consider as a metaplectic operator, we can treat operators with symbol of the form (5.4) where the role of x and ξ is exchanged. Global regularity is granted by (5.5) or (5.6) with $\operatorname{Im} r_j^\pm \neq 0$ where we exchange again x with ξ; relevant examples of the corresponding operators are

$$D + ix^m, \qquad D^2 + x^{2m},$$

for $m > 1$. Multi-quasi-elliptic symbols are products of the symbols of this form, the quasi-elliptic symbols in Example 5.4 and the globally elliptic symbols in Example 5.2, possibly perturbed by terms in the interior of the Newton polygon generated in this way, see [2] for details and equivalent definitions. Under the condition (1.9), we recapture from Theorem 1.2 the result of global regularity in [2]. Limiting again to an example, consider the symbol

$$\xi^3 + ix\xi^2 + x^2.$$

We have $\xi_1(x) = -ix + o(|x|)$, $\xi_2(x) = \sqrt{i}x^{1/2} + o(|x|^{1/2})$, $\xi_3(x) = -\sqrt{i}x^{1/2} + o(|x|^{1/2})$, writing $\pm\sqrt{i}$ for the two roots of i. Hence (1.10) is satisfied, and the corresponding operator is globally regular.

Example 5.7 *(SG-elliptic operators).* (Cf. Parenti [14], Cordes [5], Schrohe [15], Schulze [16, Section 1.4].) The case $q = \infty$ in Example 5.4 corresponds to the case of the operators with constant coefficients, that we treated in Section 1. More generally, we can consider symbols of the following form, with $m \geqslant 0, n \geqslant 0$:

$$p(x,\xi) = \sum_{\substack{\alpha \leqslant m,\\ \beta \leqslant n}} c_{\alpha,\beta} x^\beta \xi^\alpha \tag{5.9}$$

where we assume $c_{m,n} = 1$. We say that the symbol (5.9) is SG-elliptic if

$$\left|p(x,\xi)\right| \geqslant \varepsilon\left(1+|x|\right)^n\left(1+|\xi|\right)^m, \quad \text{for } |x|+|\xi| \geqslant R, \tag{5.10}$$

for some $\varepsilon > 0$, $R \geqslant 0$. In fact (5.10) is equivalent to the following couple of conditions:

$$\sum_{\alpha \leqslant m} c_{\alpha,n}\xi^{\alpha} \neq 0 \quad \text{for } \xi \neq 0, \tag{5.11}$$

$$\sum_{\beta \leqslant n} c_{m,\beta}x^{\beta} \neq 0 \quad \text{for } x \neq 0. \tag{5.12}$$

We know from [14,5,15,16] that SG-elliptic operators are globally regular. Willing to apply our Theorem 1.2, we first observe that factorizing $p(x,\xi)$ we obtain the roots

$$\xi_j(x) = r_j + o(1), \quad j = 1, \dots, m, \tag{5.13}$$

where r_j are exactly the roots of (5.11). Since (5.11) is equivalent to $\operatorname{Im} r_j \neq 0$ for every j, (5.11) implies (1.10) for (5.13). Similarly we can argue on the local ellipticity condition (5.12), by using Remark 5.1. So we recapture global regularity, in the case when (1.9) is satisfied (i.e. the roots of (5.11), (5.12) are distinct).

Example 5.8. When, for $p(x,\xi)$ as in (5.9), the equations (5.11), (5.12) admit real roots, we are led to study terms in the Puiseaux expansion with negative exponents. Consider for example the symbol

$$\left(1 + x^n\right)\xi^m + 1, \quad m > 0 \text{ and } n > 0 \text{ even integers}. \tag{5.14}$$

The local ellipticity condition (5.12) is satisfied, whereas (5.11) reduces to $\xi^m = 0$. We obtain

$$\xi_j(x) = e_j|x|^{-n/m} + o\left(|x|^{-n/m}\right), \quad j = 1, \dots, n$$

where e_j are the mth roots of -1. Since m is even, $\operatorname{Im} e_j \neq 0$, and condition (1.10) is satisfied if $-n/m > -1$, i.e. $n < m$. So we may conclude global regularity for the corresponding Weyl operator, in this case. When $n \geqslant m$, both (1.9) and (1.10) fail. In fact, the operator is Fuchsian at ∞ for $m = n$, and regular at ∞ if $n > m$. Hence the solutions of the homogeneous equation belong to $\mathcal{S}'(\mathbb{R})$ and do not belong to $\mathcal{S}(\mathbb{R})$. We address to Camperi [3] for a general class of symbols of the form (5.14).

Remark 5.9. According to the definition of Schwartz [17], a differential operator with polynomial coefficients P is *hypoelliptic* in $\mathbb{R}$ if for every open subset $\Omega \subset \mathbb{R}$ we have:

$$u \in \mathcal{D}'(\Omega) \quad \text{and} \quad Pu \in C^{\infty}(\Omega) \quad \Rightarrow \quad u \in C^{\infty}(\Omega). \tag{5.15}$$

We observe first that every operator of the form (1.1) with $a_{m,0} \neq 0$ is obviously hypoelliptic. In the general case, considered in Remark 5.1, we have to take into account the coefficient $Q(x)$ of the leading derivative, say of order $p < m$:

$$Q(x) = \sum_{\beta \leqslant m-p} c_{p,\beta}x^{\beta}.$$

Let us write $x_1, \dots, x_r$, $r \leqslant m - p$, for the real zeros of $Q(x)$. The hypoellipticity of P is granted in $\mathbb{R} \setminus \{x_1, \dots, x_r\}$. It is then easy to prove that *the global regularity in* (1.3) *implies hypoellipticity in* (5.15).

In fact, assume $u \in \mathcal{D}'(\Omega)$ and $Pu \in C^{\infty}(\Omega)$. Let $\phi \in C_0^{\infty}(\Omega)$, $\phi(x) = 1$ in a neighborhood of the points $x_1, \dots, x_r$ which belong to Ω. We know that $\operatorname{sing-supp} u \subset \{x_1, \dots, x_r\} \cap \Omega$, hence $(1-\phi)u \in C^{\infty}(\Omega)$. On the other hand $\phi u \in \mathcal{E}'(\Omega) \subset \mathcal{S}'(\mathbb{R})$ and $P(\phi u) = Pu - P(1-\phi)u \in C_0^{\infty}(\Omega) \subset \mathcal{S}(\mathbb{R})$, hence $\phi u \in C_0^{\infty}(\mathbb{R})$ by the global regularity, and therefore $u \in C^{\infty}(\Omega)$.

Note that this argument cannot be extended to operators with polynomial coefficients in $\mathbb{R}^n$, $n > 1$, because the manifold where the local ellipticity fails is in general non-compact.

Note also that Schwartz hypoellipticity does not imply global regularity. In dimension 1, consider for example the operators $P_1 = D_x$, $P_2 = D_x + x^h$, $h \geqslant 2$, which are obviously Schwartz hypoelliptic, but not globally regular. In particular, P_2 keeps the Schwartz hypoellipticity after any possible metaplectic conjugation. For a study of the Schwartz hypoellipticity at $x_1, \ldots, x_r$, we address to Kannai [12], where a necessary and sufficient condition was given under an asymptotic separation condition, similar to (1.9).

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