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Optimal time decay of the compressible micropolar fluids

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Abstract

This paper primarily studies the large-time behavior of solutions to the Cauchy problem on the compressible micropolar fluid system which is a generalization of the classical Navier–Stokes system. The asymptotic stability of the steady state with the strictly positive constant density, the vanishing velocity, and micro-rotational velocity is established under small perturbation in regular Sobolev space. Moreover, it turns out that both the density and the velocity tend time-asymptotically to the corresponding equilibrium state with rate $(1+t)^{-3/4}$ in L^2 and the micro-rotational velocity also tends to the equilibrium state with the faster rate $(1+t)^{-5/4}$ in L^2 norm. The proof is based on the spectrum analysis and time-weighted energy estimate.

Keywords: Compressible micropolar fluids; Global smooth solution; Optimal time decay

1. Introduction

The theory of micropolar fluids introduced by Eringen deals with a class of fluids consisting of dipole elements. Certain anisotropic fluids, such as liquid crystals, are of this type. Animal blood happens to fall into this category. Other polymeric fluids and fluids containing minute amount

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additives belong to micropolar fluids [13]. The motion of these viscous isentropic compressible micropolar fluids satisfies the following system [13]:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = (\mu + \zeta)\Delta u + (\mu + \lambda - \zeta)\nabla \operatorname{div} u + 2\zeta \nabla \times w, \\ (\rho w)_t + \operatorname{div}(\rho u \otimes w) + 4\zeta w = \mu' \Delta w + (\mu' + \lambda')\nabla \operatorname{div} w + 2\zeta \nabla \times u. \end{cases} \tag{1.1}$$

Here the unknown functions $\rho = \rho(t, x) \geq 0$, $u = u(t, x) \in \mathbb{R}^3$, $w = w(t, x) \in \mathbb{R}^3$ and $p(\rho)$ over $\{t > 0,\ x \in \mathbb{R}^3\}$ are density, velocity, micro-rotational velocity and pressure, respectively. The constants μ, λ are the shear and bulk viscosity coefficients of the flow, and they satisfy the physical restrictions $\mu > 0$ and $2\mu + 3\lambda - 4\zeta \geq 0$. The parameter $\zeta > 0$ means dynamics microrotation viscosity. μ' and λ' are the angular viscosity coefficients satisfying $\mu' > 0$ and $2\mu' + 3\lambda' \geq 0$.

Note that for $\mu' = \lambda' = \zeta = 0$, the material derivative of w is zero and roughly, the above micropolar fluid (1.1) reduces to the celebrated Navier–Stokes equation. In this sense, the micropolar theory can be viewed as a generalization of hydrodynamics. Note also for $\zeta = 0$ the velocity and the micro-rotational velocity are uncoupled and the global motion is unaffected by the micro-rotations [13].

The initial data are given by

$$(\rho, u, w)(0, x) = (\rho_0, u_0, w_0)(x) \quad \text{for} \quad x \in \mathbb{R}^3, \tag{1.2}$$

with the far field behavior:

$$(\rho, u, w)(t, x) \to (\bar{\rho}, 0, 0) \quad \text{as} \quad |x| \to \infty,\ t \geq 0, \tag{1.3}$$

without loss of generality, we set $\bar{\rho} = 1$ in the following.

Due to its importance in mathematics and physics, there is a lot of literature devoted to the mathematical theory of the micropolar fluid system. For an incompressible fluid $\rho = \text{const.}$, $\nabla \cdot u = 0$, we can refer to [1,4,14] as well as references cited therein. For the compressible equations of the micropolar fluids, Mujaković made a series of efforts for the model in one-dimensional space or with spherical symmetry in three dimensional space. Mujaković considered the local-in-time existence and uniqueness [20], the global existence [21] and regularity [22] of solutions to an initial–boundary value problem with homogenous boundary conditions of the compressible one-dimensional micropolar fluid system respectively. Similar results were proved in [25–27] for the non-homogenous boundary problems. Besides, Mujaković [23] analyzed large time behavior of the solutions based on *a priori* estimates independent of T. Stabilization of solutions to the Cauchy problem of the one-dimensional micropolar fluid system was established by Mujaković in [24]. Other authors, such as Chen [2] proved the global existence of strong solutions to the 1-D model with initial vacuum. For the three-dimensional model, Chen, Huang and Zhang [5] proved a blowup criterion of strong solutions to the Cauchy problem. Chen, Xu and Zhang [3] established the global weak solutions with discontinuous initial data and vacuum. Mujaković and her collaborator Dražić developed a compressible spherically symmetric flow of the isotropic, viscous and heat-conducting micropolar fluid, and for this model, they proved the local existence theorem for homogeneous boundary data, the global existence, the large time behavior of solution, uniqueness of solution respectively in [6–8,28].

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However, the global existence and uniqueness of solutions to the three-dimensional micropolar fluid system remains an open problem. In the especial case where $\mu' = \lambda' = \zeta = 0$, our model reduces to Navier–Stokes system. The global stability of the near-constant-equilibrium solutions to Cauchy problem or initial boundary value problem of the Navier–Stokes equations has been proved by Matsumura–Nishida [17–19] by using energy method. Meanwhile, Matsumura–Nishida [18] obtained the convergence rates of solutions towards the equilibrium $[1, 0]$ in $\mathbb{R}^3$ as

$$\|[\rho - 1, u]\|_{H^2} \leq C_1(1+t)^{-\frac{3}{4}}. \tag{1.4}$$

In this paper, we consider these problems for the three-dimensional micropolar fluid system.

Before stating the main results, we explain the notations and conventions throughout this paper. C denotes a positive (generally large) constant and c denotes a positive (generally small) constant, where both C and c may take different values in different places. For two quantities A and B, $A \sim B$ means $cA \leq B \leq CA$. For any integer $m \geq 0$, we use H^m, $\dot{H}^m$ to denote the usual Sobolev space $H^m(\mathbb{R}^3)$ and the corresponding m-order homogeneous Sobolev space, respectively. Set $L^2 = H^m$ when $m = 0$. For simplicity, the norm of H^m is denoted by $\|\cdot\|_m$ with $\|\cdot\| = \|\cdot\|_0$. We use $\langle \cdot, \cdot \rangle$ to denote the inner product over the Hilbert space $L^2(\mathbb{R}^3)$, i.e.

$$\langle f, g \rangle = \int_{\mathbb{R}^3} f(x)g(x)dx, \quad f = f(x), \quad g = g(x) \in L^2(\mathbb{R}^3).$$

For a nonnegative integer ℓ, we denote ∂_x^ℓ the total of all ℓ-order derivatives with respect to (x_1, x_2, x_3). For simplicity, we set

$$\|[A, B]\|_X = \|A\|_X + \|B\|_X.$$

Now, we state our main result about the global existence and decay properties of solution to the system (1.1)–(1.3) as follows.

Theorem 1.1. *Let $N \geq 4$. There are $\delta_0 > 0$, C_0 such that if*

$$\|[\rho_0 - 1, u_0, w_0]\|_N \leq \delta_0,$$

then, the Cauchy problem (1.1)–(1.3) of the micropolar fluid system admits a unique global solution $[\rho(t, x), u(t, x), w(t, x)]$ with

$$\begin{aligned}
&[\rho(t, x) - 1, u(t, x), w(t, x)] \in C([0, \infty); H^N(\mathbb{R}^3)),\\
&\nabla\rho \in L^2([0, \infty); H^{N-1}(\mathbb{R}^3)), \quad \nabla u \in L^2([0, \infty); H^N(\mathbb{R}^3)),\\
&\nabla w \in L^2([0, \infty); H^N(\mathbb{R}^3)),
\end{aligned}$$

and

$$\begin{aligned}
\|[\rho(t) - 1, u(t), w(t)]\|_N^2 + \int_0^t \left(\|\nabla\rho(s)\|_{N-1}^2 + \|\nabla u(s)\|_N^2 + \|\nabla w(s)\|_N^2\right) ds \\
\leq C_0\|[\rho_0 - 1, u_0, w_0]\|_N^2. \quad (1.5)
\end{aligned}$$

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Moreover, there are $\delta_1 > 0$, C_1 *such that if*

$$\|[\rho_0 - 1, u_0, w_0]\|_N + \|[\rho_0 - 1, u_0, w_0]\|_{L^1} \leq \delta_1,$$

then, the solution $[\rho(t,x), u(t,x), w(t,x)]$ *satisfies that for any* $t \geq 0$,

$$\|[\rho - 1, u]\| \leq C_1(1+t)^{-\frac{3}{4}}, \qquad \|w\| \leq C_1(1+t)^{-\frac{5}{4}}. \tag{1.6}$$

It is obvious that when N is large enough, the solution is classical belonging to $C^1([0,+\infty) \times \mathbb{R}^3)$ and particularly, the solution is smooth when initial perturbation is smooth. In general setting, damping term w usually leads to the time–space integrability of the function w. However, in our setting, (1.5) shows that micro-rotational velocity w itself is not time–space integrable because of the couple terms $2\zeta\nabla \times w$ in $(1.1)_2$ and $2\zeta\nabla \times u$ in $(1.1)_3$, see the proof of Theorem 4.1.

The time decay rates stated in Theorem 1.1 depend essentially on the spectral analysis of the linearized system around the constant steady state. In fact, the solution to the linearized homogenous system can be written as the sum of the fluid part and the electromagnetic part in the form of

$$\begin{bmatrix} n(t,x) \\ u(t,x) \\ w(t,x) \end{bmatrix} = \begin{bmatrix} n(t,x) \\ u_{\parallel}(t,x) \\ w_{\parallel}(t,x) \end{bmatrix} + \begin{bmatrix} 0 \\ u_{\perp}(t,x) \\ w_{\perp}(t,x) \end{bmatrix}.$$

The above decomposition is quite useful in dealing with complex linearized system containing curl, such as Euler–Maxwell and Navier–Stokes–Maxwell system. We can refer [10–12] for the detailed spectrum analysis with the use of a similar decomposition. Notice that the two terminologies, *fluid part* and *electromagnetic part* have been used in [10–12]. With the help of the above decomposition, we give explicit representations of solutions to the two eigenvalue problems, see more details in Section 3.

Notice that the micro-rotational velocity decays *one-half* faster than the time-decay rates for Navier–Stokes system in (1.4). The damping term $4\zeta w$ plays a crucial role to obtain the faster decay rate for w. The two decay rates in (1.6) are both optimal in the sense that they are the same as those in the linearized case. The general approach for obtaining optimal convergence rates of solutions in L^p with $p \geq 2$, developed by Kawashima [15,16], is to apply Fourier energy estimates to the linearized homogenous system. We will make Fourier energy estimates to the linearized homogenous micro-polar fluid system in Section 2. Its dissipative structure can be characterized by the following Lyapunov inequality

$$|\hat{U}(t,\xi)| \leq Ce^{\frac{c|\xi|^2 t}{1+|\xi|^2}} |\hat{U}_0(\xi)|,$$

which provides a clue to the more delicate spectral analysis.

The rest of the paper is organized as follows. In Section 2, we reformulate the Cauchy problem on the micropolar fluid system around the constant steady state, and study the decay structure of the linearized homogeneous system by the Fourier energy method. In Section 3, we present the spectral analysis of the linearized system by two parts. The fist part is for the fluid part, the second one for the electromagnetic part. In Section 4, we first prove the global existence of solutions by the energy method, and then show the time asymptotic rate of solutions around the constant states.

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2. Decay property of linearized system

In this section, we study the time-decay property of solutions to the linearized system based on the Fourier energy method. The main motivation to present this part is to understand the linear dissipative structure of such complex system in terms of the direct energy method and also provide a clue to the more delicate spectral analysis to be given later on.

2.1. Reformulation of the problem

We assume that the steady state of the micropolar fluids system (1.1) is trivial, taking the form of

$$\rho = 1,\ u = 0,\ w = 0.$$

Let $n = \rho - 1$. Then $U := [n, u, w]$ satisfies

$$n_t + \operatorname{div} u = S_1, \tag{2.1}$$

$$u_t + \gamma \nabla n - (\mu + \zeta)\Delta u - (\mu + \lambda - \zeta)\nabla \operatorname{div} u - 2\zeta \nabla \times w = S_2, \tag{2.2}$$

$$w_t + 4\zeta w - \mu' \Delta w - (\mu' + \lambda')\nabla \operatorname{div} w - 2\zeta \nabla \times u = S_3, \tag{2.3}$$

where the nonhomogeneous source terms S_i $(i = 1, 2, 3)$ are defined as

$$\begin{cases} S_1 = -n \operatorname{div} u - u \cdot \nabla n, \\ S_2 = -u \cdot \nabla u - f(n)[(\mu + \zeta)\Delta u + (\mu + \lambda - \zeta)\nabla \operatorname{div} u + 2\zeta \nabla \times w] - h(n)\nabla n, \\ S_3 = -u \cdot \nabla w - f(n)[\mu' \Delta w + (\mu' + \lambda')\nabla \operatorname{div} w - 4\zeta w + 2\zeta \nabla \times u], \end{cases} \tag{2.4}$$

and

$$\gamma = \frac{p'(1)}{1},\ f(n) = \frac{n}{n+1},\ h(n) = \frac{p'(n+1)}{n+1} - \frac{p'(1)}{1} \sim n.$$

The associated initial data is given by

$$(n, u, w)(x, 0) = (n_0, u_0, w_0)(x). \tag{2.5}$$

2.2. Linear decay structure

In this section, for brevity of presentation we still use $U = [n, u, w]$ to denote the solution to the linearized homogeneous system

$$\begin{cases} n_t + \operatorname{div} u = 0, \\ u_t + \gamma \nabla n - (\mu + \zeta)\Delta u - (\mu + \lambda - \zeta)\nabla \operatorname{div} u - 2\zeta \nabla \times w = 0, \\ w_t + 4\zeta w - \mu' \Delta w - (\mu' + \lambda')\nabla \operatorname{div} w - 2\zeta \nabla \times u = 0, \end{cases} \tag{2.6}$$

with the initial data

$$(n, u, w)(x, 0) = (n_0, u_0, w_0)(x). \tag{2.7}$$

The goal of this section is to apply the Fourier energy method to the Cauchy problem (2.6), (2.7) to show that there exists a time–frequency Lyapunov functional which is equivalent with $|\hat{U}(t,\xi)|^2$ and moreover its dissipation rate can also be characterized by the functional itself. Let us state the main result of this section as follows.

Theorem 2.1. *Let $U(t,x)$, $t>0$, $x\in\mathbb{R}^3$ be a well-defined solution to the system (2.6)–(2.7). There is a time–frequency Lyapunov functional $\mathcal{E}(\hat{U}(t,\xi))$ with*

$$\mathcal{E}(\hat{U}(t,\xi)) \sim |\hat{U}|^2 := |[\hat{n},\hat{u},\hat{w}]|^2 \tag{2.8}$$

satisfying that there is $c>0$ such that the Lyapunov inequality

$$\frac{d}{dt}\mathcal{E}(\hat{U}(t,\xi)) + \frac{c|\xi|^2}{1+|\xi|^2}\mathcal{E}(\hat{U}(t,\xi)) \le 0 \tag{2.9}$$

holds for any $t>0$ and $\xi\in\mathbb{R}^3$.

Proof. As in [10], we use the following notations. For an integrable function $f:\mathbb{R}^3\to\mathbb{R}$, its Fourier transform is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} \exp(-ix\cdot\xi) f(x)dx, \quad x\cdot\xi := \sum_{j=1}^{3} x_j\xi_j, \quad \xi\in\mathbb{R}^3,$$

where $i=\sqrt{-1}\in\mathbb{C}$ is the imaginary unit. For two complex numbers or vectors a and b, $(a|b)$ denotes the dot product of a with the complex conjugate of b. Taking the Fourier transform in x for (2.6), $\hat{U}=[\hat{n},\hat{u},\hat{w}]$ satisfies

$$\begin{cases} \hat{n}_t + i\xi\cdot\hat{u} = 0, \\ \hat{u}_t + \gamma i\xi\hat{n} + (\mu+\zeta)|\xi|^2\hat{u} + (\mu+\lambda-\zeta)\xi\xi\cdot\hat{u} - 2\zeta i\xi\times\hat{w} = 0, \\ \hat{w}_t + 4\zeta\hat{w} + \mu'|\xi|^2\hat{w} + (\mu'+\lambda')\xi\xi\cdot\hat{w} - 2\zeta i\xi\times\hat{u} = 0. \end{cases} \tag{2.10}$$

Multiplying $(2.10)_1$, $(2.10)_2$ and $(2.10)_3$ by $\hat{n}$, $\hat{u}$ and $\hat{w}$ respectively, taking real part and taking summation of three resultant equations, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left|(\sqrt{\gamma}\hat{n},\hat{u},\hat{w})\right|^2 + (\mu+\zeta)|\xi|^2|\hat{u}|^2 + (\mu+\lambda-\zeta)|\xi\cdot\hat{u}|^2 \\ &\quad + 4\zeta|\hat{w}|^2 + \mu'|\xi|^2|\hat{w}|^2 + (\mu'+\lambda')|\xi\cdot\hat{w}|^2 = 4\zeta\mathrm{Re}(i\xi\times\hat{u}|\hat{w}) \le \zeta|\xi|^2|\hat{u}|^2 + 4\zeta|\hat{w}|^2, \end{aligned} \tag{2.11}$$

which gives

$$\frac{1}{2}\frac{d}{dt}\left|(\sqrt{\gamma}\hat{n},\hat{u},\hat{w})\right|^2 + \mu|\xi|^2|\hat{u}|^2 + (\mu+\lambda-\zeta)|\xi\cdot\hat{u}|^2 + \mu'|\xi|^2|\hat{w}|^2 + (\mu'+\lambda')|\xi\cdot\hat{w}|^2 \le 0. \tag{2.12}$$

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By taking the complex dot product of the second equation of (2.10) with $i\xi\hat{n}$,

$$\frac{d}{dt}(\hat{u}|i\xi\hat{n}) + \gamma|\xi|^2|\hat{n}|^2 + ((\mu+\zeta)|\xi|^2\hat{u}|i\xi\hat{n}) + ((\mu+\lambda-\zeta)\xi\xi\cdot\hat{u}|i\xi\hat{n}) = |\xi\cdot\hat{u}|^2 + (2\zeta i\xi\times\hat{w}|i\xi\hat{n}), \tag{2.13}$$

where we have used the fact

$$\begin{aligned}(\hat{u}_t|i\xi\hat{n}) &= \frac{d}{dt}(\hat{u}|i\xi\hat{n}) - (\hat{u}|i\xi\hat{n}_t)\\ &= \frac{d}{dt}(\hat{u}|i\xi\hat{n}) + (i\xi\cdot\hat{u}|\hat{n}_t)\\ &= \frac{d}{dt}(\hat{u}|i\xi\hat{n}) - |\xi\cdot\hat{u}|^2.\end{aligned}$$

Taking the real part of (2.13), using Cauchy–Schwarz inequality, we have

$$\frac{d}{dt}\mathrm{Re}(\hat{u}|i\xi\hat{n}) + \gamma|\xi|^2|\hat{n}|^2 \le \frac{\gamma}{2}|\xi|^2|\hat{n}|^2 + C\left(|\xi|^2|\hat{u}|^2 + |\xi|^4|\hat{u}|^2 + |\xi|^2|\hat{w}|^2\right).$$

Dividing it by $1+|\xi|^2$ gives

$$\frac{d}{dt}\frac{\mathrm{Re}(\hat{u}|i\xi\hat{n})}{1+|\xi|^2} + \frac{\gamma}{2}\frac{|\xi|^2}{1+|\xi|^2}|\hat{n}|^2 \le C|\xi|^2|\hat{u}|^2 + \frac{|\xi|^2}{1+|\xi|^2}|\hat{w}|^2. \tag{2.14}$$

Finally, let's define

$$\mathcal{E}(\hat{U}(t,\xi)) = \left|(\sqrt{\gamma}\hat{n},\hat{u},\hat{w})\right|^2 + \kappa\frac{\mathrm{Re}(\hat{u}|i\xi\hat{n})}{1+|\xi|^2}$$

for constant $0<\kappa\ll 1$ to be determined later. Notice that as long as $0<\kappa\ll 1$ is small enough, then $\mathcal{E}(\hat{U}(t,\xi)) \sim |\hat{U}(t)|^2$ holds true and (2.8) is proved. The sum of (2.12), (2.14) $\times\,\kappa$ gives

$$\partial_t\mathcal{E}(\hat{U}(t,k)) + c|\xi|^2|[\hat{u},\hat{w}]|^2 + c\frac{|\xi|^2}{1+|\xi|^2}|\hat{n}|^2 \le 0, \tag{2.15}$$

by noticing

$$c|\xi|^2|[\hat{u},\hat{w}]|^2 + c\frac{|\xi|^2}{1+|\xi|^2}|\hat{n}|^2 \ge \frac{c|\xi|^2}{1+|\xi|^2}|\hat{U}|^2.$$

This completes the proof of Theorem 2.1. □

Theorem 2.1 directly leads to the pointwise time–frequency estimate on the modular $|\hat{U}(t,\xi)|$ in terms of initial data modular $|\hat{U}_0(\xi)|$, which is similar to [10, Corollary 4.1].

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Corollary 2.1. *Let $U(t,x)$, $t \geq 0$, $x \in \mathbb{R}^3$ be a well-defined solution to the system (2.6)–(2.7). Then, there are $c > 0$, $C > 0$ such that*

$$|\hat{U}(t,\xi)| \leq C\exp\left(-\frac{c|\xi|^2 t}{1+|\xi|^2}\right)|\hat{U}_0(\xi)| \tag{2.16}$$

holds for any $t \geq 0$ and $\xi \in \mathbb{R}^3$.

Based on the pointwise time–frequency estimate (2.16), it is also straightforward to obtain the L^p–L^q time-decay property to the Cauchy problem (2.6)–(2.7). Formally, the solution to the Cauchy problem (2.6)–(2.7) is denoted by

$$U(t) = [n,u,w] = e^{tL}U_0,$$

where e^{tL} for $t \geq 0$ is said to be the linearized solution operator corresponding to the linearized micropolar fluid system.

Corollary 2.2. *(See [10] for instance.) Let $1 \leq p, r \leq 2 \leq q \leq \infty$, $\ell \geq 0$ and let $m \geq 0$ be an integer. Define*

$$\left[\ell + 3\left(\frac{1}{r} - \frac{1}{q}\right)\right]_+ = \begin{cases} \ell, & \text{if } \ell \text{ is integer and } r = q = 2, \\ [\ell + 3(\frac{1}{r} - \frac{1}{q})]_- + 1, & \text{otherwise}, \end{cases} \tag{2.17}$$

where $[\cdot]_-$ denotes the integer part of the argument. Then e^{tL} satisfies the following time-decay property:

$$\|\nabla^m e^{Lt}U_0\|_{L^q} \leq C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m}{2}}\|U_0\|_{L^p} + Ce^{-ct}\|\nabla^{m+3[\frac{1}{r}-\frac{1}{q}]_+}U_0\|_{L^r}$$

for any $t \geq 0$, where $C = C(m,p,r,q)$.

3. Spectral representation

In fact, as in [10], the linearized micropolar fluid system (2.6) can be written as two decoupled subsystems which govern the time evolution of n, $\nabla \cdot u$, $\nabla \cdot w$ and $\nabla \times u$, $\nabla \times w$ respectively. We decompose the solution to (2.6)–(2.7) into two parts in the form of

$$\begin{bmatrix} n(t,x) \\ u(t,x) \\ w(t,x) \end{bmatrix} = \begin{bmatrix} n(t,x) \\ u_\parallel(t,x) \\ w_\parallel(t,x) \end{bmatrix} + \begin{bmatrix} 0 \\ u_\perp(t,x) \\ w_\perp(t,x) \end{bmatrix}, \tag{3.1}$$

where $u_\parallel$, $u_\perp$ are defined by

$$u_\parallel = \Delta^{-1}\nabla\nabla\cdot u, \quad u_\perp = -\Delta^{-1}\nabla \times (\nabla \times u),$$

and likewise for $w_\parallel$, $w_\perp$. For brevity, the first part on the right of (3.1) is called the fluid part and the second part is called the electromagnetic part, and we also write

$$U_\parallel = [n, u_\parallel, w_\parallel], \quad U_\perp = [u_\perp, w_\perp].$$

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We now derive the equations of $U_\parallel$ and $U_\perp$, respectively. Taking the divergence of the last two equations of (2.6), it follows that

$$\begin{cases} \partial_t n + \nabla \cdot u = 0, \\ (\nabla \cdot u)_t + \gamma \Delta n - (2\mu + \lambda)\Delta \nabla \cdot u = 0, \\ (\nabla \cdot w)_t + 4\zeta \nabla \cdot w - (2\mu' + \lambda')\Delta \nabla \cdot w = 0. \end{cases} \tag{3.2}$$

Applying $\Delta^{-1}\nabla$ to the last two equations of (3.2) and noticing $\nabla \cdot u = \nabla \cdot u_\parallel$, we see that the fluid part $U_\parallel$ satisfies

$$\begin{cases} \partial_t n + \mathrm{div} u_\parallel = 0, \\ \partial_t u_\parallel + \gamma \nabla n - (2\mu + \lambda)\Delta u_\parallel = 0, \\ \partial_t w_\parallel + 4\zeta w_\parallel - (2\mu' + \lambda')\Delta w_\parallel = 0. \end{cases} \tag{3.3}$$

Initial data is given by

$$[n, u_\parallel, w_\parallel]|_{t=0} = [n_0, u_{0\parallel}, w_{0\parallel}]. \tag{3.4}$$

Taking the curl of the last two equations of (2.6) and then replacing $-\nabla \times \nabla \times w$ by $\Delta w - \nabla\nabla \cdot w$, likewise for $-\nabla \times \nabla \times u$, it follows that

$$\begin{cases} (\nabla \times u)_t - (\mu + \zeta)\Delta \nabla \times u + 2\zeta(\Delta w - \nabla\nabla \cdot w) = 0, \\ (\nabla \times w)_t + 4\zeta \nabla \times w - \mu' \Delta \nabla \times w + 2\zeta(\Delta u - \nabla\nabla \cdot u) = 0. \end{cases} \tag{3.5}$$

Applying $-\Delta^{-1}\nabla\times$ to the above two equations and noticing $\nabla \times u = \nabla \times u_\perp$, we see that electromagnetic part $U_\perp$ satisfies

$$\begin{cases} \partial_t u_\perp - (\mu + \zeta)\Delta u_\perp - 2\zeta \nabla \times w_\perp = 0, \\ \partial_t w_\perp + 4\zeta w_\perp - \mu' \Delta w_\perp - 2\zeta \nabla \times u_\perp = 0, \end{cases} \tag{3.6}$$

with initial data

$$[u_\perp,\ w_\perp]|_{t=0} = [u_{0\perp},\ w_{0\perp}]. \tag{3.7}$$

3.1. Spectral representation for fluid part

Taking time derivative for the first equation of (3.3) and using the second equation of (3.3) to replace $\partial_t u_\parallel$, it follows that

$$\partial_{tt} n - \gamma \Delta n + (2\mu + \lambda)\Delta \nabla \cdot u_\parallel = 0.$$

Further noticing $\nabla \cdot u_\parallel = -\partial_t n$, one has

$$\partial_{tt} n - \gamma \Delta n - (2\mu + \lambda)\Delta \partial_t n = 0. \tag{3.8}$$

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Initial data is given by

$$n|_{t=0} = n_0, \qquad \partial_t n|_{t=0} = -\nabla \cdot u_{0\|}. \tag{3.9}$$

Taking Fourier transformation of the second order ODE (3.8)–(3.9), we have

$$\begin{cases} \partial_{tt}\hat{n} + (2\mu+\lambda)|\xi|^2\partial_t\hat{n} + \gamma|\xi|^2\hat{n} = 0, \\ \hat{n}|_{t=0} = \hat{n}_0, \\ \partial_t\hat{n}|_{t=0} = -i\xi\cdot\hat{u}_{0\|}. \end{cases} \tag{3.10}$$

By direct computation, the solution of (3.10) can be expressed as

$$\hat{n} = \frac{e^{\chi_+ t} - e^{\chi_- t}}{\chi_+ - \chi_-}(-i\xi\cdot\hat{u}_{0\|}) + \frac{\chi_+ e^{\chi_- t} - \chi_- e^{\chi_+ t}}{\chi_+ - \chi_-}\hat{n}_0, \tag{3.11}$$

where $\chi_\pm$ is defined as

$$\chi_\pm = -(\mu+\lambda/2)|\xi|^2 \pm \sqrt{(\mu+\lambda/2)^2|\xi|^4 - \gamma|\xi|^2}.$$

Similarly, taking time derivative for the second equation of (3.3), replacing n_t by $-\operatorname{div} u_\|$, it follows that

$$\partial_{tt}u_\| - \gamma\nabla\operatorname{div} u_\| - (2\mu+\lambda)\Delta\partial_t u_\| = 0.$$

Further noticing that $-\gamma\nabla\operatorname{div} u_\| = -\gamma\nabla\operatorname{div} u = -\gamma\Delta u_\|$, one has

$$\partial_{tt}u_\| - \gamma\Delta u_\| - (2\mu+\lambda)\Delta\partial_t u_\| = 0. \tag{3.12}$$

Initial data is given by

$$u_\||_{t=0} = u_{0\|}, \qquad \partial_t u_\||_{t=0} = -\gamma\nabla n_0 + (2\mu+\lambda)\Delta u_{0\|}. \tag{3.13}$$

Taking Fourier transformation of the second order PDE (3.12)–(3.13), we have

$$\begin{cases} \partial_{tt}\hat{u}_\| + (2\mu+\lambda)|\xi|^2\partial_t\hat{u}_\| + \gamma|\xi|^2\hat{u}_\| = 0, \\ \partial_t\hat{u}_\||_{t=0} = -\gamma i\xi\hat{n}_0 - (2\mu+\lambda)|\xi|^2\hat{u}_{0\|}, \\ \hat{u}_\||_{t=0} = \hat{u}_{0\|}. \end{cases} \tag{3.14}$$

By direct computation, the solution of (3.14) can be given as

$$\hat{u}_\| = \frac{e^{\chi_+ t} - e^{\chi_- t}}{\chi_+ - \chi_-}(-\gamma i\xi\hat{n}_0) + \frac{\chi_+ e^{\chi_+ t} - \chi_- e^{\chi_- t}}{\chi_+ - \chi_-}\hat{u}_{0\|}. \tag{3.15}$$

Taking Fourier transformation of the third equation of (3.3), we have

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$$\begin{cases} \partial_t \hat{w}_{\|} + 4\zeta \hat{w}_{\|} + (2\mu' + \lambda')|\xi|^2 \hat{w}_{\|} = 0, \\ \hat{w}_{\|}|_{t=0} = \hat{w}_{0\|}. \end{cases}$$

The solution of the above first order ODE is

$$\hat{w}_{\|} = e^{\mu_0 t} \hat{w}_{0\|} \tag{3.16}$$

with $\mu_0 = -\big(4\zeta + (2\mu' + \lambda')|\xi|^2\big)$.

In summary, we deduce from (3.11), (3.15) and (3.16) that

$$\begin{pmatrix} \hat{n} \\ \hat{u}_{\|} \\ \hat{w}_{\|} \end{pmatrix} = \hat{G}_1(t) \begin{pmatrix} \hat{n}_0 \\ \hat{u}_{\|0} \\ \hat{w}_{\|0}, \end{pmatrix}, \tag{3.17}$$

where the matrix $\hat{G}_1(t)$ is given by

$$\hat{G}_1(t) = \begin{pmatrix} \frac{\chi_+ e^{\chi_- t} - \chi_- e^{\chi_+ t}}{\chi_+ - \chi_-} & \frac{e^{\chi_+ t} - e^{\chi_- t}}{\chi_+ - \chi_-}(-i\xi^T) & 0 \\ \frac{e^{\chi_+ t} - e^{\chi_- t}}{\chi_+ - \chi_-}(-i\gamma\xi) & \frac{\chi_+ e^{\chi_+ t} - \chi_- e^{\chi_- t}}{\chi_+ - \chi_-} I_{3\times 3} & 0 \\ 0 & 0 & e^{\mu_0 t} I_{3\times 3} \end{pmatrix}. \tag{3.18}$$

Next, we use (3.18) to obtain some refined L^p–L^q time-decay properties for $U_{\|} = [n, u_{\|}, w_{\|}]$. For that, we first make the time–frequency pointwise estimates on $\hat{n}$, $\hat{u}_{\|}$, $\hat{w}_{\|}$ in the following

Lemma 3.1. *Let $U_{\|} = [n, u_{\|}, w_{\|}]$ be the solution to the linearized homogeneous system (3.3) with initial data $U_{\|0} = [n_0, u_{0\|}, w_{0\|}]$. Then, there exist constants $\varepsilon > 0$, $c > 0$, $C > 0$ such that for all $t > 0$, $|\xi| \le \varepsilon$,*

$$|[\hat{n}(t,\xi), \hat{u}_{\|}(t,\xi)]| \le C\exp(-c|\xi|^2 t)|[\hat{n}_0(\xi), \hat{u}_{0\|}(\xi)]|, \tag{3.19}$$

$$|\hat{w}_{\|}(t,\xi)| \le C\exp(-ct)|\hat{w}_{0\|}(\xi)|, \tag{3.20}$$

and for all $t > 0$, $|\xi| \ge \varepsilon$,

$$|[\hat{n}(t,\xi), \hat{u}_{\|}(t,\xi)]| \le C\exp(-ct)|[\hat{n}_0(\xi), \hat{u}_{0\|}(\xi)]|, \tag{3.21}$$

$$|\hat{w}_{\|}(t,\xi)| \le C\exp(-ct)|\hat{w}_{0\|}(\xi)|. \tag{3.22}$$

Proof. In order to get the upper bound of $\hat{n}(t,\xi)$ and $\hat{u}_{\|}(t,\xi)$, we have to estimate $\hat{G}_{11}$, $\hat{G}_{12}$, $\hat{G}_{21}$ and $\hat{G}_{22}$. Here we denote

$$\hat{G}_{11} = \frac{\chi_+ e^{\chi_- t} - \chi_- e^{\chi_+ t}}{\chi_+ - \chi_-}, \quad \hat{G}_{12} = \frac{e^{\chi_+ t} - e^{\chi_- t}}{\chi_+ - \chi_-}(-i\xi^T)$$

$$\hat{G}_{21} = \frac{e^{\chi_+ t} - e^{\chi_- t}}{\chi_+ - \chi_-}(-i\gamma\xi), \quad \hat{G}_{22} = \frac{\chi_+ e^{\chi_+ t} - \chi_- e^{\chi_- t}}{\chi_+ - \chi_-}.$$

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If $(\mu+\lambda/2)^2|\xi|^4-\gamma|\xi|^2 \geq 0$, then $\chi_\pm = -(\mu+\lambda/2)|\xi|^2 \pm \sqrt{(\mu+\lambda/2)^2|\xi|^4-\gamma|\xi|^2}$ are real. It is straightforward to obtain

$$\chi_- = -O(1)|\xi|^2, \ \chi_+ = -\frac{\gamma}{2\mu+\lambda} + O(1)\left(|\xi|^{-2}\right),$$
$$\chi_+ - \chi_- = 2\sqrt{(\mu+\lambda/2)^2|\xi|^4-\gamma|\xi|^2} = O(1)|\xi|^2,$$

as $|\xi| \to \infty$. And on the other hand, if $(\mu+\lambda/2)^2|\xi|^4-\gamma|\xi|^2 < 0$, then $\chi_\pm = -(\mu+\lambda/2)|\xi|^2 \pm i\sqrt{\gamma|\xi|^2-(\mu+\lambda/2)^2|\xi|^4}$ are complex conjugate. Moreover, one has

$$|\chi_\pm| = O(1)|\xi|,$$
$$\chi_+ - \chi_- = 2i\sqrt{\gamma|\xi|^2-(\mu+\lambda/2)^2|\xi|^4} = i\,O(1)|\xi|,$$

as $|\xi| \to 0$. Then, there exists $\varepsilon \leq \sqrt{\dfrac{4\gamma}{(2\mu+\lambda)^2}} \leq R$, with $0 < \varepsilon \ll 1 \ll R < \infty$ such that one can estimate $\hat{G}$ as follows:

$$|\hat{G}_{11}| + |\hat{G}_{12}| + |\hat{G}_{21}| + |G_{22}| \leq Ce^{-c|\xi|^2 t},$$

as $|\xi| \leq \varepsilon$, and

$$|\hat{G}_{11}| + |\hat{G}_{12}| + |\hat{G}_{21}| + |G_{22}| \leq Ce^{-ct},$$

as $|\xi| \geq R$.

In what follows we estimate only $\hat{G}_{12}$ over $\varepsilon \leq |\xi| \leq R$. When $|\xi| \leq \sqrt{\dfrac{4\gamma}{(2\mu+\lambda)^2}}$,

$$\frac{e^{\chi_+ t} - e^{\chi_- t}}{\chi_+ - \chi_-} = \frac{e^{-(\mu+\frac{\lambda}{2})|\xi|^2 t}\sin(\sqrt{\gamma|\xi|^2-(\mu+\lambda/2)^2|\xi|^4}t)}{\sqrt{\gamma|\xi|^2-(\mu+\lambda/2)^2|\xi|^4}},$$

and

$$\lim_{|\xi|\to\sqrt{\frac{4\gamma}{(2\mu+\lambda)^2}}} \frac{e^{-(\mu+\frac{\lambda}{2})|\xi|^2 t}\sin(\sqrt{\gamma|\xi|^2-(\mu+\lambda/2)^2|\xi|^4}t)}{\sqrt{\gamma|\xi|^2-(\mu+\lambda/2)^2|\xi|^4}} = te^{-\frac{2\gamma}{2\mu+\lambda}t} \leq Ce^{-\frac{\gamma}{2\mu+\lambda}t}.$$

When $|\xi| \geq \sqrt{\dfrac{4\gamma}{(2\mu+\lambda)^2}}$,

$$\lim_{|\xi|\to\sqrt{\frac{4\gamma}{(2\mu+\lambda)^2}}} \frac{e^{\chi_+ t} - e^{\chi_- t}}{\chi_+ - \chi_-} = te^{-\frac{2\gamma}{2\mu+\lambda}t} \leq Ce^{-\frac{\gamma}{2\mu+\lambda}t}.$$

Then there exists $\delta > 0$, if $\left||\xi| - \sqrt{\frac{4\gamma}{(2\mu+\lambda)^2}}\right| \leq \delta$, one has

$$\left|\frac{e^{\chi_+ t}-e^{\chi_- t}}{\chi_+-\chi_-}\right|\le Ce^{-ct},\qquad \left|\frac{e^{\chi_+ t}-e^{\chi_- t}}{\chi_+-\chi_-}(-i\xi^T)\right|\le Ce^{-ct}.$$

Next, let's consider $\hat{G}_{12}$ over $\{\varepsilon\le|\xi|\le R, \left||\xi|-\sqrt{\frac{4\gamma}{(2\mu+\lambda)^2}}\right|\ge\delta\}$. Notice that in this domain,

$$\left|\frac{e^{\chi_+ t}-e^{\chi_- t}}{\chi_+-\chi_-}(-i\xi^T)\right|\le Ce^{-ct},$$

where the fact that

$$\begin{cases}\chi_\pm<0 & \text{whenever } \chi_\pm \text{ real},\\ \mathrm{Re}\ \chi_\pm=-(\mu+\lambda/2)|\xi|^2 & \text{whenever } \chi_\pm \text{ non-real and conjugate},\end{cases}$$

has been used. Therefore, in the completely same way, we can get

$$|G_{11}|+|G_{21}|+|G_{22}|\le Ce^{-ct},$$

over $\varepsilon\le|\xi|\le R$. In summary,

$$|\hat{G}_{11}|+|\hat{G}_{12}|+|\hat{G}_{21}|+|G_{22}|\le Ce^{-c|\xi|^2 t}, \tag{3.23}$$

as $|\xi|\le\varepsilon$, and

$$|\hat{G}_{ij}|\le Ce^{-ct},\quad 1\le i,j\le 2, \tag{3.24}$$

as $|\xi|\ge\varepsilon$.

Now, in terms of (3.23), we can estimate $\hat{n}(t,\xi)$, $\hat{u}_\parallel(t,\xi)$ as

$$\begin{aligned}|\hat{n}(t,\xi)|&=|\hat{G}_{11}\hat{n}_0(\xi)+\hat{G}_{12}\hat{u}_{0\parallel}(\xi)|\\ &\le|\hat{G}_{11}||\hat{n}_0(\xi)|+|\hat{G}_{12}||\hat{u}_{0\parallel}(\xi)|\\ &\le Ce^{-c|\xi|^2 t}|[\hat{n}_0(\xi),\hat{u}_{0\parallel}(\xi)]|, \qquad (3.25)\\ |\hat{u}_\parallel(t,\xi)|&=|\hat{G}_{21}\hat{n}_0(\xi)+\hat{G}_{22}\hat{u}_{0\parallel}(\xi)|\\ &\le|\hat{G}_{21}||\hat{n}_0(\xi)|+|\hat{G}_{22}||\hat{u}_{0\parallel}(\xi)|\\ &\le Ce^{-c|\xi|^2 t}|[\hat{n}_0(\xi),\hat{u}_{0\parallel}(\xi)]|, \qquad (3.26)\end{aligned}$$

for $|\xi|\le\varepsilon$, which prove (3.19). Similarly, (3.21) directly follows from (3.24). Finally, (3.20) and (3.22) directly follow from the expression of $\hat{w}_\parallel$ in (3.16). This completes the proof of Lemma 3.1. □

As in [10,11], it is now a standard procedure to derive from Lemma 3.1 the L^p–L^q time decay property of the fluid part $U_\parallel$.

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Theorem 3.1. *Let $1 \le p, r \le 2 \le q \le \infty$ and let $m \ge 0$ be an integer. Suppose that $[n, u_\|, w_\|]$ is the solution to the Cauchy problem (3.3)–(3.4). Then $U_\| = [n, u_\|, w_\|]$ satisfies the following time-decay property:*

$$\|\nabla^m [n, u_\|]\|_{L^q} \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m}{2}} \|[n_0, u_{0\|}]\|_{L^p} + C\exp(-ct)\|\nabla^{m+[3(\frac{1}{r}-\frac{1}{q})]_+}[n_0, u_{0\|}]\|_{L^r},$$

$$\|\nabla^m w_\|\|_{L^q} \le C\exp(-ct)\|[n_0, u_{0\|}]\|_{L^p} + C\exp(-ct)\|\nabla^{m+[3(\frac{1}{r}-\frac{1}{q})]_+}[n_0, u_{0\|}]\|_{L^r},$$

for any $t \ge 0$, where $C = C(m, p, r, q)$ and $[3(\frac{1}{r} - \frac{1}{q})]_+$ is defined in (2.17).

3.2. Spectral representation for electromagnetic part

Recall that the electromagnetic part $U_\perp = [u_\perp, w_\perp]$ satisfies the following equation

$$\begin{cases} \partial_t u_\perp - (\mu+\zeta)\Delta u_\perp - 2\zeta \nabla \times w_\perp = 0, \\ \partial_t w_\perp + 4\zeta w_\perp - \mu' \Delta w_\perp - 2\zeta \nabla \times u_\perp = 0, \end{cases} \tag{3.27}$$

with initial data

$$[u_\perp,\ w_\perp]|_{t=0} = [u_{0\perp},\ w_{0\perp}]. \tag{3.28}$$

We first solve $\hat{u}_\perp$. Taking the time derivative of the first equation of (3.27), taking the curl for second equation, we obtain

$$\begin{cases} \partial_{tt} u_\perp - (\mu+\zeta)\Delta \partial_t u_\perp - 2\zeta \partial_t(\nabla \times w_\perp) = 0, \\ \partial_t(\nabla \times w_\perp) + 4\zeta(\nabla \times w_\perp) - \mu' \Delta(\nabla \times w_\perp) + 2\zeta \Delta u_\perp = 0, \end{cases} \tag{3.29}$$

where we have used the fact that $-\nabla \times \nabla \times u_\perp = \Delta u_\perp$. Using the second equation of (3.29) to replace $\partial_t \nabla \times w_\perp$ in the first equation of (3.29), and moreover, replacing $\nabla \times w_\perp$ by the first equation of (3.27), taking Fourier transformation, we get

$$\begin{cases} \partial_{tt}\hat{u}_\perp + \left[(\mu+\zeta)|\xi|^2 + (4\zeta + \mu'|\xi|^2)\right]\partial_t \hat{u}_\perp + \left[(4\zeta+\mu'|\xi|^2)(\mu+\zeta)|\xi|^2 - 4\zeta^2|\xi|^2\right]\hat{u}_\perp = 0, \\ \partial_t \hat{u}_\perp|_{t=0} = -(\mu+\zeta)|\xi|^2 \hat{u}_{0\perp} + 2\zeta i\xi \times \hat{w}_{0\perp}, \\ \hat{u}_\perp|_{t=0} = \hat{u}_{0\perp}. \end{cases} \tag{3.30}$$

Consider the character equation

$$\kappa^2 + \left[(\mu+\zeta)|\xi|^2 + (4\zeta+\mu'|\xi|^2)\right]\kappa + \left[4\zeta\mu|\xi|^2 + \mu'(\mu+\zeta)|\xi|^4\right] = 0,$$

and its roots $\kappa_\pm$ are denoted by

$$\begin{aligned} \kappa_\pm &= \frac{-\left[(\mu+\zeta)|\xi|^2 + 4\zeta + \mu'|\xi|^2\right] \pm \sqrt{\left[(\mu+\zeta)|\xi|^2 + 4\zeta + \mu'|\xi|^2\right]^2 - 4\left[4\zeta\mu|\xi|^2 + (\mu+\zeta)\mu'|\xi|^4\right]}}{2} \\ &= \frac{-\left[(\mu+\zeta+\mu')|\xi|^2 + 4\zeta\right] \pm \sqrt{(\mu+\zeta-\mu')^2|\xi|^4 + 16\zeta^2 + 8\zeta(\mu'+\zeta-\mu)|\xi|^2}}{2}. \end{aligned} \tag{3.31}$$

Notice that

$$(\mu+\zeta-\mu')^2|\xi|^4+16\zeta^2+8\zeta(\mu'+\zeta-\mu)|\xi|^2=\Big[(\mu+\zeta-\mu')|\xi|^2-4\zeta\Big]^2+16\zeta^2|\xi|^2>0.$$

Then $\kappa_\pm$ are both real roots. By direct computation, we give the solution to (3.30),

$$\hat{u}_\perp=\frac{e^{\kappa_+t}-e^{\kappa_-t}}{\kappa_+-\kappa_-}\Big[2\zeta i\xi\times\hat{w}_{\perp 0}-(\mu+\zeta)|\xi|^2\hat{u}_{\perp 0}\Big]+\frac{\kappa_+e^{\kappa_-t}-\kappa_-e^{\kappa_+t}}{\kappa_+-\kappa_-}\hat{u}_{\perp 0}. \tag{3.32}$$

We solve $\hat{w}_\perp$ in the following. Taking the time derivative of second equation of (3.27), taking the curl of the first equation, we get

$$\begin{cases}\partial_t(\nabla\times u_\perp)-(\mu+\zeta)\Delta\nabla\times u_\perp+2\zeta\Delta w_\perp=0,\\ \partial_{tt}w_\perp+4\zeta\partial_t w_\perp-\mu'\Delta\partial_t w_\perp-2\zeta\partial_t(\nabla\times u_\perp)=0.\end{cases} \tag{3.33}$$

Replacing $\partial_t(\nabla\times u_\perp)$ in the second equation by the first equation of (3.33), replacing $\nabla\times u_\perp$ by the second equation (3.27) and taking Fourier transformation of the resultant equation, we get

$$\begin{cases}\partial_{tt}\hat{w}_\perp+\big[(\mu+\zeta)|\xi|^2+(4\zeta+\mu'|\xi|^2)\big]\partial_t\hat{w}_\perp+\big[(4\zeta+\mu'|\xi|^2)(\mu+\zeta)|\xi|^2-4\zeta^2|\xi|^2\big]\hat{w}_\perp=0,\\ \partial_t\hat{w}_\perp|_{t=0}=-(\mu'|\xi|^2+4\zeta)\hat{w}_{0\perp}+2\zeta i\xi\times\hat{u}_{0\perp},\\ \hat{w}_\perp|_{t=0}=\hat{w}_{0\perp}.\end{cases} \tag{3.34}$$

In the same way as for solving $\hat{u}_\perp$, we give the solution of (3.34),

$$\hat{w}_\perp=\frac{e^{\kappa_+t}-e^{\kappa_-t}}{\kappa_+-\kappa_-}\Big[2\zeta i\xi\times\hat{u}_{0\perp}-(\mu'|\xi|^2+4\zeta)\hat{w}_{0\perp}\Big]+\frac{\kappa_+e^{\kappa_-t}-\kappa_-e^{\kappa_+t}}{\kappa_+-\kappa_-}\hat{w}_{0\perp}. \tag{3.35}$$

In summary, we deduce from (3.32) and (3.35) that

$$\begin{pmatrix}\hat{u}_\perp\\ \hat{w}_\perp\end{pmatrix}=\hat{G}_2(t)\begin{pmatrix}\hat{u}_{0\perp}\\ \hat{w}_{0\perp}\end{pmatrix}, \tag{3.36}$$

where the matrix $\hat{G}_2(t)$ is given by

$$\hat{G}_2(t)=\begin{pmatrix}\frac{\kappa_+e^{\kappa_-t}-\kappa_-e^{\kappa_+t}}{\kappa_+-\kappa_-}-(\mu+\zeta)|\xi|^2\frac{e^{\kappa_+t}-e^{\kappa_-t}}{\kappa_+-\kappa_-} & \frac{e^{\kappa_+t}-e^{\kappa_-t}}{\kappa_+-\kappa_-}2\zeta i\xi\times\\ \frac{e^{\kappa_+t}-e^{\kappa_-t}}{\kappa_+-\kappa_-}2\zeta i\xi\times & \frac{\kappa_+e^{\kappa_-t}-\kappa_-e^{\kappa_+t}}{\kappa_+-\kappa_-}-(\mu'|\xi|^2+4\zeta)\frac{e^{\kappa_+t}-e^{\kappa_-t}}{\kappa_+-\kappa_-}\end{pmatrix}.$$

Lemma 3.2. *Let $U_\perp=[u_\perp,w_\perp]$ be the solution to the linearized homogeneous system (3.27) with initial data $U_{\perp,0}=[u_{0\perp},w_{0\perp}]$. Then, there exist constants $\varepsilon>0$, $c>0$, $C>0$ such that for all $t>0$, $|\xi|\le\varepsilon$,*

$$|\hat{u}_\perp(t,\xi)|\le C\exp(-c|\xi|^2t)|[\hat{u}_{0\perp}(\xi),\hat{w}_{0\perp}]|, \tag{3.37}$$

$$|\hat{w}_\perp(t,\xi)|\le C|\xi|\exp(-c|\xi|^2t)|[\hat{u}_{0\perp}(\xi),\hat{w}_{0\perp}]|+C\exp(-ct)|[\hat{u}_{0\perp}(\xi),\hat{w}_{0\perp}]|, \tag{3.38}$$

and for all $t > 0$, $|\xi| \geq \varepsilon$,

$$|\hat{u}_\perp(t,\xi)| \leq C\exp(-ct)|[\hat{u}_{0\perp}(\xi), \hat{w}_{0\perp}]|, \tag{3.39}$$

$$|\hat{w}_\perp(t,\xi)| \leq C\exp(-ct)|[\hat{u}_{0\perp}(\xi), \hat{w}_{0\perp}]|. \tag{3.40}$$

Proof. Here we use the same notations $\hat{G}_{11}$, $\hat{G}_{12}$, $\hat{G}_{21}$ and $\hat{G}_{22}$ to denote the four elements of $\hat{G}_2$ for simplicity. By direct computation, we have

$$\begin{aligned}
\hat{G}_{11} &= \frac{\kappa_+ + (\mu+\zeta)|\xi|^2}{\kappa_+ - \kappa_-} e^{\kappa_- t} - \frac{\kappa_- + (\mu+\zeta)|\xi|^2}{\kappa_+ - \kappa_-} e^{\kappa_+ t},\\
\hat{G}_{12} &= \hat{G}_{21} = \frac{e^{\kappa_+ t} - e^{\kappa_- t}}{\kappa_+ - \kappa_-} 2\zeta i\xi \times,\\
\hat{G}_{22} &= \frac{\kappa_+ + \mu'|\xi|^2 + 4\zeta}{\kappa_+ - \kappa_-} e^{\kappa_- t} - \frac{\kappa_- + \mu'|\xi|^2 + 4\zeta}{\kappa_+ - \kappa_-} e^{\kappa_+ t}.
\end{aligned}$$

Recall the definition of $\kappa_\pm$ in (3.31), one can easily find that when $|\xi|$ is near 0,

$$\begin{aligned}
&\kappa_+ = -\mu|\xi|^2 + O(|\xi|^4), \;\; \kappa_- = -4\zeta + O(|\xi|^2),\\
&\kappa_+ - \kappa_- = \sqrt{(\mu+\zeta-\mu')^2|\xi|^4 + 16\zeta^2 + 8\zeta(\mu'+\zeta-\mu)|\xi|^2} = O(1),
\end{aligned}$$

which implies,

$$\begin{aligned}
&|G_{11}| \leq C|\xi|^2 e^{-ct} + Ce^{-c|\xi|^2 t},\\
&|G_{12}| + |G_{21}| \leq C|\xi| e^{-ct} + C|\xi| e^{-c|\xi|^2 t},\\
&|G_{22}| \leq Ce^{-ct} + C|\xi|^2 e^{-c|\xi|^2 t}.
\end{aligned}$$

When $|\xi|$ is near ∞,

$$\begin{aligned}
\kappa_+ &= \begin{cases} -\mu'|\xi|^2 + O(1), & \text{as } \mu+\zeta-\mu' > 0,\\ -(\mu+\zeta)|\xi|^2 + O(1), & \text{as } \mu+\zeta-\mu' \leq 0,\end{cases}\\
\kappa_- &= \begin{cases} -(\mu+\zeta)|\xi|^2 + O(1), & \text{as } \mu+\zeta-\mu' > 0,\\ -\mu'|\xi|^2 + O(1), & \text{as } \mu+\zeta-\mu' \leq 0,\end{cases}\\
\kappa_+ - \kappa_- &= \sqrt{(\mu+\zeta-\mu')^2|\xi|^4 + 16\zeta^2 + 8\zeta(\mu'+\zeta-\mu)|\xi|^2}\\
&= \begin{cases} O(|\xi|^2), & \text{as } \mu+\zeta-\mu' \neq 0,\\ O(|\xi|), & \text{as } \mu+\zeta-\mu' = 0,\end{cases}
\end{aligned}$$

which implies

$$|G_{11}| + |G_{12}| + |G_{21}| + |G_{22}| \leq Ce^{-ct}.$$

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When $|\xi|$ is far from 0 and ∞, $\kappa_\pm < 0$ implies that there exist a constant $c > 0$ such that $\kappa_\pm \le -c$ and $\kappa_+ - \kappa_- = O(1)$. Now, in terms of (3.36), one can estimate $|\hat{u}_\perp(t,\xi)|$ and $|\hat{w}_\perp(t,\xi)|$ directly from the above upper bounds of G_{ij}, $1 \le i, j \le 2$. This completes the proof of Lemma 3.2. $\square$

Analogous to Theorem 3.1, we can deduce the L^p–L^q time-decay property for the electromagnetic part $U_\perp$ directly from Lemma 3.2.

Theorem 3.2. *Let $1 \le p, r \le 2 \le q \le \infty$ and let $m \ge 0$ be an integer. Suppose that $[u_\perp, w_\perp]$ is the solution to the Cauchy problem (3.27)–(3.28). Then $U_\perp = [u_\perp, w_\perp]$ satisfies the following time-decay property:*

$$\|\nabla^m u_\perp\|_{L^q} \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m}{2}} \|[u_{0\perp}, w_{0\perp}]\|_{L^p} + C\exp(-ct)\|\nabla^{m+[3(\frac{1}{r}-\frac{1}{q})]_+}[u_{0\perp}, w_{0\perp}]\|_{L^r},$$

$$\|\nabla^m w_\perp\|_{L^q} \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m+1}{2}} \|[u_{0\perp}, w_{0\perp}]\|_{L^p} + C\exp(-ct)\|\nabla^{m+[3(\frac{1}{r}-\frac{1}{q})]_+}[u_{0\perp}, w_{0\perp}]\|_{L^r},$$

for any $t \ge 0$, where $C = C(m, p, r, q)$ and $[3(\frac{1}{r} - \frac{1}{q})]_+$ is defined in (2.17).

Based on the time-decay property for $[n, u_\parallel, w_\parallel]$ in Theorem 3.1 and the time-decay property for $[u_\perp, w_\perp]$ in Theorem 3.2, we have the following time-decay property for the full solution $[n, u, w]$.

Theorem 3.3. *Let $1 \le p, r \le 2 \le q \le \infty$ and let $m \ge 0$ be an integer. Suppose that $[n, u, w]$ is the solution to the Cauchy problem (2.6)–(2.7). Then $U = [n, u, w]$ satisfies the following time-decay property:*

$$\|\nabla^m n\|_{L^q} \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m}{2}} \|[n_0, u_0]\|_{L^p} + C\exp(-ct)\|\nabla^{m+[3(\frac{1}{r}-\frac{1}{q})]_+}[n_0, u_0]\|_{L^r},$$

$$\|\nabla^m u\|_{L^q} \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m}{2}} \|[n_0, u_0, w_0]\|_{L^p} + C\exp(-ct)\|\nabla^{m+[3(\frac{1}{r}-\frac{1}{q})]_+}[n_0, u_0, w_0]\|_{L^r},$$

$$\|\nabla^m w\|_{L^q} \le C(1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m+1}{2}} \|[u_0, w_0]\|_{L^p} + C\exp(-ct)\|\nabla^{m+[3(\frac{1}{r}-\frac{1}{q})]_+}[u_0, w_0]\|_{L^r},$$

for any $t \ge 0$, where $C = C(m, p, r, q)$ and $[3(\frac{1}{r} - \frac{1}{q})]_+$ is defined in (2.17).

4. Asymptotic behavior of the nonlinear system

4.1. Global existence

In this section, we will establish the global existence of solution to the compressible micropolar fluid system (1.1)–(1.3). For later use and clear reference, the following Sobolev inequality about the L^p estimate on any two product terms with the sum of the order of their derivatives equal to a given integer is listed as follows [9,29].

Lemma 4.1. *Let $n \ge 1$. Let $\alpha^1 = (\alpha_1^1, \cdots, \alpha_n^1)$ and $\alpha^2 = (\alpha_1^2, \cdots, \alpha_n^2)$ be two multi-indices with $|\alpha^1| = k_1$, $|\alpha^2| = k_2$ and set $k = k_1 + k_2$. Let $1 \le p, q, r \le \infty$ with $1/p = 1/q + 1/r$. Then, for $u_j : \mathbb{R}^n \to \mathbb{R}$ $(j = 1, 2)$, one has*

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$$\left\| \partial^{\alpha^1} u_1 \partial^{\alpha^2} u_2 \right\|_{L^p(\mathbb{R}^n)} \leq C \left(\|u_1\|_{L^q(\mathbb{R}^n)} \left\| \nabla^k u_2 \right\|_{L^r(\mathbb{R}^n)} + \|u_2\|_{L^q(\mathbb{R}^n)} \left\| \nabla^k u_1 \right\|_{L^r(\mathbb{R}^n)} \right) \tag{4.1}$$

for a constant C independent of u_1 and u_2.

To the end, we assume that $N \geq 4$. We define the full instant energy functional $\mathcal{E}_N(U(t))$ by

$$\mathcal{E}_N(U(t)) = \sum_{|l| \leq N} \int_{\mathbb{R}^3} \left(\frac{p'(n+1)}{n+1} |\partial_x^l n|^2 + (n+1)|\partial_x^l [u,w]|^2 \right) dx + \kappa \sum_{|l| \leq N-1} \langle \partial_x^l u, \partial_x^l \nabla n \rangle, \tag{4.2}$$

where κ is a constant to be properly chosen later. Notice that since the constant κ is small enough, one has

$$\mathcal{E}_N(U(t)) \sim \|[n,u,w]\|_N^2.$$

We further define the corresponding dissipation rate $\mathcal{D}_N(U(t))$ by

$$\mathcal{D}_N(U(t)) = \|\nabla n\|_{N-1} + \|\nabla [u,w]\|_N. \tag{4.3}$$

Then, the global existence of the reformulated Cauchy problem (2.1)–(2.5) with small smooth initial data can be stated as follows.

Theorem 4.1. *Let $N \geq 4$. If $\|U_0\|_N$ is small enough, then the Cauchy problem (2.1)–(2.5) admits a unique global solution $U = [n,u,w]$ with*

$$\begin{aligned} & U \in C([0,\infty); H^N(\mathbb{R}^3)), \\ & \nabla n \in L^2([0,\infty); H^{N-1}(\mathbb{R}^3)), \qquad \nabla u \in L^2([0,\infty); H^N(\mathbb{R}^3)), \\ & \nabla w \in L^2([0,\infty); H^N(\mathbb{R}^3)), \end{aligned}$$

and

$$\|U(t)\|_N^2 + \int_0^t \left(\|\nabla n(s)\|_{N-1}^2 + \|\nabla u(s)\|_N^2 + \|\nabla w(s)\|_N^2 \right) ds \leq C \|U_0\|_N^2.$$

To prove Theorem 4.1, it suffices to prove the following uniform-in-time *a priori* estimate, cf. [10].

Lemma 4.2 *(A priori estimates). Suppose that $U = [n,u,w] \in C([0,T); H^N(\mathbb{R}^3))$ is smooth for $T > 0$ with*

$$\sup_{0 \leq t < T} \|U(t)\|_N \leq 1,$$

and that U solves the system (2.1)–(2.5) over $0 \leq t < T$. Then, there is $\mathcal{E}_N(\cdot)$ in the form (4.2) such that

$$\frac{d}{dt}\mathcal{E}_N(U(t)) + c\mathcal{D}_N(U(t)) \le C[\mathcal{E}_N(U(t))^{\frac{1}{2}} + \mathcal{E}_N(U(t))]\mathcal{D}_N(U(t)) \tag{4.4}$$

for any $0 \le t < T$.

Proof. It is divided by three steps as follows.

Step 1. It holds that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{|l|\le N}\left(\left\langle \frac{p'(n+1)}{n+1}, |\partial_x^l n|^2\right\rangle + \left\langle n+1, |\partial_x^l u|^2\right\rangle + \left\langle n+1, |\partial_x^l w|^2\right\rangle\right)\\&+\mu\|\nabla u\|_N^2 + \mu'\|\nabla w\|_N^2 \le C\|[n,u,w]\|_N(\|\nabla n\|_{N-1}^2 + \|\nabla[u,w]\|_N^2).\end{aligned} \tag{4.5}$$

In fact, it is convenient to start from the following form of (2.1)–(2.3):

$$\begin{cases} n_t + (n+1)\operatorname{div} u = -u\cdot\nabla n,\\ u_t + \dfrac{p'(n+1)}{n+1}\nabla n - \dfrac{\mu+\zeta}{n+1}\Delta u - \dfrac{\mu+\lambda-\zeta}{n+1}\nabla\operatorname{div} u - \dfrac{2\zeta}{n+1}\nabla\times w = -u\cdot\nabla u,\\ w_t + \dfrac{4\zeta}{n+1}w - \dfrac{\mu'}{n+1}\Delta w - \dfrac{\mu'+\lambda'}{n+1}\nabla\operatorname{div} w - \dfrac{2\zeta}{n+1}\nabla\times u = -u\cdot\nabla w.\end{cases} \tag{4.6}$$

Zero-order energy estimates imply that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{\left\langle \frac{p'(n+1)}{n+1}, |n|^2\right\rangle + \left\langle n+1, |u|^2\right\rangle + \left\langle n+1, |w|^2\right\rangle\right\} + 4\zeta\|w\|^2\\&\quad+(\mu+\zeta)\|\nabla u\|^2 + (\mu+\lambda-\zeta)\|\operatorname{div} u\|^2 + \mu'\|\nabla w\|^2 + (\mu'+\lambda')\|\operatorname{div} w\|^2\\&=4\zeta\langle\nabla\times u, w\rangle + \left\langle p''(n+1)u\cdot\nabla n, n\right\rangle + \frac{1}{2}\left\langle\left(\frac{p'(n+1)}{n+1}\right)_t, |n|^2\right\rangle\\&\quad-\left\langle u\cdot\nabla n, \frac{p'(n+1)}{n+1}n,\right\rangle + \frac{1}{2}\left\langle (n+1)_t, |[u,w]|^2\right\rangle - \langle u\cdot\nabla u, (n+1)u\rangle - \langle u\cdot\nabla w, (n+1)w\rangle\\&\le\zeta\|\nabla u\|^2 + 4\zeta\|w\|^2 + C\|[n,u,w]\|_{H^1}\|\nabla[n,u,w]\|^2.\end{aligned}$$

Then, we obtain the zero-order energy estimate

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{\left\langle \frac{p'(n+1)}{n+1}, |n|^2\right\rangle + \left\langle n+1, |u|^2\right\rangle + \left\langle n+1, |w|^2\right\rangle\right\} + \mu\|\nabla u\|^2\\&\quad+(\mu+\lambda-\zeta)\|\operatorname{div} u\|^2 + \mu'\|\nabla w\|^2 + (\mu'+\lambda')\|\operatorname{div} w\|^2\\&\le C\|[n,u,w]\|_{H^1}\|\nabla[n,u,w]\|^2.\end{aligned} \tag{4.7}$$

Let $1 \le \ell \le N$. We apply ∂_x^ℓ to (4.6). The result is written as

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$$\begin{cases} \partial_x^\ell n_t + (n+1)\partial_x^\ell \operatorname{div} u = -u\cdot\partial_x^\ell \nabla n + f_1^\ell, \\ \partial_x^\ell u_t + \dfrac{p'(n+1)}{n+1}\partial_x^\ell \nabla n - \dfrac{\mu+\zeta}{n+1}\Delta\partial_x^\ell u - \dfrac{\mu+\lambda-\zeta}{n+1}\nabla\operatorname{div}\partial_x^\ell u \\ \qquad\qquad - \dfrac{2\zeta}{n+1}\nabla\times\partial_x^\ell w = -u\cdot\partial_x^\ell\nabla u + f_2^\ell, \\ \partial_x^\ell w_t + \dfrac{4\zeta}{n+1}\partial_x^\ell w - \dfrac{\mu'}{n+1}\Delta\partial_x^\ell w - \dfrac{\mu'+\lambda'}{n+1}\nabla\operatorname{div}\partial_x^\ell w - \dfrac{2\zeta}{n+1}\nabla\times\partial_x^\ell u = -u\cdot\nabla\partial_x^\ell w + f_3^\ell. \end{cases} \tag{4.8}$$

Here f_1^ℓ, f_2^ℓ and f_3^ℓ are defined as follows:

$$\begin{cases} f_1^\ell = -[\partial_x^\ell, u\cdot\nabla]n - [\partial_x^\ell, (n+1)\operatorname{div}]u, \\ f_2^\ell = -[\partial_x^\ell, u\cdot\nabla]u - \left[\partial_x^\ell, \dfrac{p'(n+1)}{n+1}\nabla\right]n + \left[\partial_x^\ell, \dfrac{\mu+\zeta}{n+1}\Delta\right]u, \\ \qquad + \left[\partial_x^\ell, \dfrac{\mu+\lambda-\zeta}{n+1}\nabla\operatorname{div}\right]u + \left[\partial_x^\ell, \dfrac{2\zeta}{n+1}\nabla\times\right]w, \\ f_3^\ell = -\left[\partial_x^\ell, \dfrac{4\zeta}{n+1}\right]w + \left[\partial_x^\ell, \dfrac{\mu'}{n+1}\Delta\right]w \\ \qquad + \left[\partial_x^\ell, \dfrac{\mu'+\lambda'}{n+1}\nabla\operatorname{div}\right]w + \left[\partial_x^\ell, \dfrac{2\zeta}{n+1}\nabla\times\right]u, \end{cases} \tag{4.9}$$

and [,] denotes the commutator defined by $[A, B] = AB - BA$. Multiplying the first equation of (4.8) by $\frac{p'(n+1)}{n+1}\partial_x^\ell n$, and taking integration in x give

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\langle \frac{p'(n+1)}{n+1}, |\partial_x^l n|^2\right\rangle - \left\langle p'(n+1)\partial_x^l u, \partial_x^l \nabla n\right\rangle - \left\langle p''(n+1)\partial_x^l u\cdot\nabla n, \partial_x^l n\right\rangle \\ &= \frac{1}{2}\left\langle \left(\frac{p'(n+1)}{n+1}\right)_t, |\partial_x^l n|^2\right\rangle + \frac{1}{2}\left\langle \nabla\cdot\left(\frac{p'(n+1)}{n+1}u\right), |\partial_x^\ell n|^2\right\rangle + \left\langle f_1^\ell, \frac{p'(n+1)}{n+1}\partial_x^\ell n\right\rangle. \end{aligned} \tag{4.10}$$

Multiplying the second equation of (4.8) by $(n+1)\partial_x^\ell u$, and taking integration in x give

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\langle n+1, |\partial_x^l u|^2\right\rangle + \left\langle p'(n+1)\partial_x^l\nabla n, \partial_x^l u\right\rangle + (\mu+\zeta)\|\nabla\partial_x^\ell u\|^2 + (\mu+\lambda-\zeta)\|\operatorname{div}\partial_x^\ell u\|^2 \\ &= 2\zeta\langle\nabla\times\partial_x^\ell w, \partial_x^\ell u\rangle + \frac{1}{2}\left\langle (n+1)_t, |\partial_x^l u|^2\right\rangle + \frac{1}{2}\left\langle \nabla\cdot((n+1)u), |\partial_x^\ell u|^2\right\rangle + \left\langle f_2^\ell, (n+1)\partial_x^\ell u\right\rangle. \end{aligned} \tag{4.11}$$

Multiplying the third equation of (4.8) by $(n+1)\partial_x^\ell w$, and taking integration in x give

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\langle n+1, |\partial_x^l w|^2\right\rangle + 4\zeta\|\partial_x^\ell w\|^2 + \mu'\|\nabla\partial_x^\ell w\|^2 + (\mu'+\lambda')\|\operatorname{div}\partial_x^\ell w\|^2 \\
&= 2\zeta\langle\nabla\times\partial_x^\ell u, \partial_x^\ell w\rangle + \frac{1}{2}\left\langle (n+1)_t, |\partial_x^l w|^2\right\rangle + \frac{1}{2}\left\langle \nabla\cdot((n+1)u), |\partial_x^\ell w|^2\right\rangle + \left\langle f_3^\ell, (n+1)\partial_x^\ell w\right\rangle.
\end{aligned} \tag{4.12}
$$

Taking the summation of (4.10)–(4.12), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\left\langle \frac{p'(n+1)}{n+1}, |\partial_x^l n|^2\right\rangle + \left\langle n+1, |\partial_x^l u|^2\right\rangle + \left\langle n+1, |\partial_x^l w|^2\right\rangle\right\} + 4\zeta\|\partial_x^\ell w\|^2 \\
&\quad + (\mu+\zeta)\|\nabla\partial_x^\ell u\|^2 + (\mu+\lambda-\zeta)\|\operatorname{div}\partial_x^\ell u\|^2 + \mu'\|\nabla\partial_x^\ell w\|^2 + (\mu'+\lambda')\|\operatorname{div}\partial_x^\ell w\|^2 \\
&= 4\zeta\langle\nabla\times\partial_x^\ell u, \partial_x^\ell w\rangle + \left\langle p''(n+1)\partial_x^l u\cdot\nabla n, \partial_x^l n\right\rangle + \frac{1}{2}\left\langle \left(\frac{p'(n+1)}{n+1}\right)_t, |\partial_x^l n|^2\right\rangle \\
&\quad + \frac{1}{2}\left\langle \nabla\cdot\left(\frac{p'(n+1)}{n+1}u\right), |\partial_x^\ell n|^2\right\rangle + \frac{1}{2}\left\langle (n+1)_t, |\partial_x^l [u,w]|^2\right\rangle + \frac{1}{2}\left\langle \nabla\cdot((n+1)u), |\partial_x^\ell [u,w]|^2\right\rangle \\
&\quad + \left\langle f_1^\ell, \frac{p'(n+1)}{n+1}\partial_x^\ell n\right\rangle + \left\langle f_2^\ell, (n+1)\partial_x^\ell u\right\rangle + \left\langle f_3^\ell, (n+1)\partial_x^\ell w\right\rangle \\
&\le \zeta\|\nabla\partial_x^\ell u\|^2 + 4\zeta\|\partial_x^\ell w\|^2 + \|\nabla[n,u]\|_{L^\infty}\|\partial_x^\ell[n,u,w]\|^2 + \|f_1^\ell\|\|\partial_x^\ell n\| + \|f_2^\ell\|\|\partial_x^\ell u\| \\
&\quad + \|f_3^\ell\|\|\partial_x^\ell w\|.
\end{aligned} \tag{4.13}
$$

Noticing the similarity of the quadratically nonlinear terms in f_1, f_2 and f_3, we only estimate four terms from Lemma 4.1 in the following,

$$
\begin{aligned}
\|[\partial_x^\ell, u\cdot\nabla]n\| &= \|\sum_{k<\ell} C_\ell^k \partial_x^{\ell-k-1}\nabla u\partial_x^k\nabla n\| \\
&\le C\|\nabla u\|_{L^\infty}\|\partial_x^{\ell-1}\nabla n\| + \|\nabla n\|_{L^\infty}\|\partial_x^{\ell-1}\nabla u\|, \\
\left\|\left[\partial_x^\ell, \frac{\mu+\zeta}{n+1}\Delta\right]u\right\| &\le C\|\sum_{k<\ell} C_\ell^k\partial_x^{\ell-k-1}\nabla n\partial_x^k\Delta u\| \\
&\le C\|\nabla n\|_{L^\infty}\|\partial_x^{\ell-1}\Delta u\| + \|\Delta u\|_{L^\infty}\|\partial_x^{\ell-1}\nabla n\|, \\
\left\|\left[\partial_x^\ell, \frac{4\zeta}{n+1}\right]w\right\| &\le C\|\sum_{k<\ell} C_\ell^k\partial_x^{\ell-k-1}\nabla n\partial_x^k w\| \\
&\le C\|\nabla n\|_{L^\infty}\|\partial_x^{\ell-1}w\| + \|w\|_{L^\infty}\|\partial_x^{\ell-1}\nabla n\|, \\
\left\|\left[\partial_x^\ell, \frac{2\zeta}{n+1}\nabla\times\right]w\right\| &\le C\|\sum_{k<\ell} C_\ell^k\partial_x^{\ell-k-1}\nabla n\partial_x^k\nabla\times w\| \\
&\le C\|\nabla n\|_{L^\infty}\|\partial_x^{\ell-1}\nabla\times w\| + \|\nabla\times w\|_{L^\infty}\|\partial_x^{\ell-1}\nabla n\|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\left\langle\frac{p'(n+1)}{n+1},|\partial_x^l n|^2\right\rangle+\left\langle n+1,|\partial_x^l u|^2\right\rangle+\left\langle n+1,|\partial_x^l w|^2\right\rangle\right\}+\mu\|\nabla\partial_x^\ell u\|^2\\
&\quad+(\mu+\lambda-\zeta)\|\operatorname{div}\partial_x^\ell u\|^2+\mu'\|\nabla\partial_x^\ell w\|^2+(\mu'+\lambda')\|\operatorname{div}\partial_x^\ell w\|^2\\
&\le\|\nabla[n,u,w]\|_{L^\infty}\|\partial_x^\ell[n,u,w]\|^2+\|\nabla n\|_{L^\infty}\|\partial_x^{\ell+1}[u,w]\|^2\\
&\quad+\|\partial_x^2[u,w]\|_{L^\infty}\|\partial_x^\ell[n,u,w]\|^2+\|\nabla n\|_{L^\infty}\|\partial_x^{\ell-1}w\|\|\partial_x^\ell w\|+\|w\|_{L^\infty}\|\partial_x^\ell n\|\|\partial_x^\ell w\|\\
&\le C\|[n,u,w]\|_N(\|\nabla n\|_{N-1}^2+\|\nabla[u,w]\|_N^2).
\end{aligned}\tag{4.14}
$$

Taking summation of (4.7) and (4.14) over $|\ell|\le N$, we obtain (4.5).

Step 2. It holds that

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|l|\le N-1}\langle\partial_x^l u,\partial_x^l\nabla n\rangle+c\|\nabla n\|_{N-1}^2\\
&\le C\|\nabla[u,w]\|_N^2+C\|[n,u,w]\|_N^2(\|\nabla n\|_{N-1}^2+\|\nabla[u,w]\|_N^2).
\end{aligned}\tag{4.15}
$$

In fact, recall the equations (2.1) and (2.2),

$$
\begin{cases}
n_t+\operatorname{div}u=S_1,\\
u_t+\gamma\nabla n-(\mu+\zeta)\Delta u-(\mu+\lambda-\zeta)\nabla\operatorname{div}u-2\zeta\nabla\times w=S_2,
\end{cases}\tag{4.16}
$$

with

$$
\begin{cases}
S_1=-\nabla\cdot(nu),\\
S_2=-u\cdot\nabla u-f(n)[(\mu+\zeta)\Delta u+(\mu+\lambda-\zeta)\nabla\operatorname{div}u+2\zeta\nabla\times w]-h(n)\nabla n.
\end{cases}
$$

Let $0\le l\le N-1$, applying ∂_x^l to the second momentum equation of (4.16), multiplying the resultant equation by $\partial_x^l\nabla n$, taking integrations in x, using integration by parts and replacing $\partial_t n$ from $(4.16)_1$, one has

$$
\begin{aligned}
&\frac{d}{dt}\langle\partial_x^l u,\partial_x^l\nabla n\rangle+\gamma\|\partial_x^l\nabla n\|^2\\
&=\|\nabla\cdot\partial_x^l u\|^2-\langle\nabla\cdot\partial_x^l u,\partial_x^l S_1\rangle+\langle(\mu+\zeta)\Delta\partial_x^l u,\partial_x^l\nabla n\rangle\\
&\quad+\langle(\mu+\lambda-\zeta)\nabla\operatorname{div}\partial_x^l u,\partial_x^l\nabla n\rangle+2\zeta\langle\partial_x^l\nabla\times w,\partial_x^l\nabla n\rangle+\langle\partial_x^l S_2,\partial_x^l\nabla n\rangle.
\end{aligned}
$$

Then, it follows from the Cauchy–Schwarz inequality that

$$
\begin{aligned}
&\frac{d}{dt}\langle\partial_x^l u,\partial_x^l\nabla n\rangle+\frac{\gamma}{2}\|\partial_x^l\nabla n\|^2\\
&\le C(\|\partial_x^{l+1}[u,w]\|^2+\|\partial_x^{l+2}u\|^2)+C(\|\partial_x^l S_1\|^2+\|\partial_x^l S_2\|^2).
\end{aligned}\tag{4.17}
$$

Noticing that S_1, S_2 are quadratically nonlinear, one has by using Lemma 4.1

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$$
\begin{aligned}
&\|\partial_x^l S_1\|^2 + \|\partial_x^l S_2\|^2 \\
&\quad \le C\|[n,u]\|_{L^\infty}^2 \left(\|\partial_x^{l+1}[n,u,w]\|^2 + \|\partial_x^{l+2}u\|^2\right) + \|\partial_x^l [n,u]\|^2 \left(\|\nabla[n,u,w]\|_{L^\infty}^2 + \|\Delta u\|_{L^\infty}^2\right) \\
&\quad \le C\|[n,u,w]\|_N^2 (\|\nabla n\|_{N-1}^2 + \|\nabla[u,w]\|_N^2).
\end{aligned}
$$

Substituting this into (4.17) and taking the summation over $|l| \le N-1$ imply (4.15).

Step 3. Let us define

$$
\mathcal{E}_N(U(t)) = \sum_{|l|\le N} \int_{\mathbb{R}^3} \left(\frac{p'(n+1)}{n+1}|\partial_x^l n|^2 + (n+1)|\partial_x^l [u,w]|^2\right) dx + \kappa \sum_{|l|\le N-1} \langle \partial_x^l u, \partial_x^l \nabla n\rangle,
$$

for constant $0<\kappa \ll 1$ to be determined. Notice that as long as $0<\kappa\ll 1$ is small enough, then $\mathcal{E}_N(U(t)) \sim \|U(t)\|_N^2$ holds true. Moreover, the sum of (4.5), (4.15) $\times\,\kappa$ implies that there are $c>0$, $C>0$ such that

$$
\frac{d}{dt}\mathcal{E}_N(U(t)) + c\mathcal{D}_N(U(s)) \le C\left(\|[n,u,w]\|_N^2 + \|[n,u,w]\|_N\right)(\|\nabla n\|_{N-1}^2 + \|\nabla[u,w]\|_N^2),
$$

which implies (4.4). The proof of Lemma 4.2 is completed. □

Since (4.6) is a quasi-linear symmetric hyperbolic–parabolic system, the local-in-time existence follows from much more general case showed in [15, Theorem 2.9, in Chapter II]. As long as the above estimate is proved, Theorem 4.1 follows in the standard way by combining the local-in-time existence and uniqueness as well as the continuity argument.

4.2. *Asymptotic rate to constant states*

Moreover, the solutions obtained in Theorem 4.1 indeed decay in time with some rates under some extra regularity and integrability conditions on initial data. For that, given $U_0 = [n_0,u_0,w_0]$, set $\epsilon_m(U_0)$ as

$$
\epsilon_m(U_0) = \|U_0\|_m + \|U_0\|_{L^1}, \tag{4.18}
$$

for the integer $m \ge 0$.

4.2.1. *Time rate for the full instant energy functional*

Under the smallness of $\|U_0\|_N$, (4.4) implies that

$$
\frac{d}{dt}\mathcal{E}_N(U(t)) + c\mathcal{D}_N(U(t)) \le 0, \tag{4.19}
$$

for any $t \ge 0$. We now apply the time-weighted energy estimate and iteration to the Lyapunov inequality (4.19). Let $\ell \ge 0$. Multiplying (4.19) by $(1+t)^\ell$ and taking integration over $[0,t]$ give

$$(1+t)^{\ell}\mathcal{E}_N(U(t)) + c\int_0^t (1+s)^{\ell}\mathcal{D}_N(U(s))ds$$
$$\leq \mathcal{E}_N(U_0) + \ell\int_0^t (1+s)^{\ell-1}\mathcal{E}_N(U(s))ds.$$

Noticing

$$\mathcal{E}_N(U(t)) \leq C(D_N(U(t)) + \|[n,u,w]\|^2),$$

it follows that

$$(1+t)^{\ell}\mathcal{E}_N(U(t)) + c\int_0^t (1+s)^{\ell}\mathcal{D}_N(U(s))ds$$
$$\leq \mathcal{E}_N(U_0) + C\ell\int_0^t (1+s)^{\ell-1}(\|[n,u,w]\|^2)ds$$
$$+ C\ell\int_0^t (1+s)^{\ell-1}\mathcal{D}_N(U(s))ds.$$

Similarly, it holds that

$$(1+t)^{\ell-1}\mathcal{E}_N(U(t)) + c\int_0^t (1+s)^{\ell-1}\mathcal{D}_N(U(s))ds$$
$$\leq \mathcal{E}_N(U_0) + C(\ell-1)\int_0^t (1+s)^{\ell-2}(\|[n,u,w]\|^2)ds$$
$$+ C(\ell-1)\int_0^t (1+s)^{\ell-2}\mathcal{D}_N(U(s))ds,$$

and

$$\mathcal{E}_N(U(t)) + c\int_0^t \mathcal{D}_N(U(s))ds \leq \mathcal{E}_N(U_0).$$

Then, for $1 < \ell < 2$, it follows by iterating the above estimates that

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$$(1+t)^{\ell}\mathcal{E}_N(U(t))+c\int_0^t(1+s)^{\ell}\mathcal{D}_N(U(s))ds$$

$$\le C\mathcal{E}_N(U_0)+C\int_0^t(1+s)^{\ell-1}(\|[n,u,w]\|^2)ds. \tag{4.20}$$

On the other hand, to estimate the integral term on the r.h.s. of (4.20), let's define

$$\mathcal{E}_{N,\infty}(U(t))=\sup_{0\le s\le t}(1+s)^{\frac{3}{2}}\mathcal{E}_N(U(s)). \tag{4.21}$$

Lemma 4.3. *For any $t\ge 0$, it holds that*

$$\begin{cases}\|n(t)\|^2\le C(1+t)^{-\frac{3}{2}}\left(\|[n_0,u_0]\|^2_{L^1\cap L^2}+C\mathcal{E}^2_{N,\infty}(U(t))\right),\\ \|u(t)\|^2\le C(1+t)^{-\frac{3}{2}}\left(\|[n_0,u_0,w_0]\|^2_{L^1\cap L^2}+C\mathcal{E}^2_{N,\infty}(U(t))\right),\\ \|w(t)\|^2\le C(1+t)^{-\frac{5}{2}}\left(\|[n_0,u_0,w_0]\|^2_{L^1\cap L^2}+C\mathcal{E}^2_{N,\infty}(U(t))\right).\end{cases} \tag{4.22}$$

Proof. Recall that the solution $U=[n,u,w]$ to the Cauchy problem (2.1)–(2.5) with initial data $U_0=[n_0,u_0,w_0]$ can be formally written as

$$U(t)=e^{tL}U_0+\int_0^t e^{(t-s)L}[S_1(s),S_2(s),S_3(s)]ds, \tag{4.23}$$

where e^{tL} is the linearized solution operator. By applying the linear estimates on $[n,u,w]$ in Theorem 3.3 with $m=0$, $q=r=2$, $p=1$ to the mild form (4.23) respectively, one has

$$\|n(t)\|\le C(1+t)^{-\frac{3}{4}}\|[n_0,u_0]\|_{L^1\cap L^2}+C\int_0^t(1+t-s)^{-\frac{3}{4}}\|[S_1(s),S_2(s)]\|_{L^1\cap L^2}ds, \tag{4.24}$$

$$\|u(t)\|\le C(1+t)^{-\frac{3}{4}}\|[n_0,u_0,w_0]\|_{L^1\cap L^2}+C\int_0^t(1+t-s)^{-\frac{3}{4}}\|[S_1(s),S_2(s),S_3(s)]\|_{L^1\cap L^2}ds, \tag{4.25}$$

and

$$\|w\|\le C(1+t)^{-\frac{5}{4}}\|[n_0,u_0,w_0]\|_{L^1\cap L^2}+C\int_0^t(1+t-s)^{-\frac{5}{4}}\|[S_1(s),S_2(s),S_3(s)]\|_{L^1\cap L^2}ds. \tag{4.26}$$

Recall the definition (2.4) of S_1, S_2 and S_3. It is straightforward to verify that for any $0 \le s \le t$,

$$\|[S_1(s), S_2(s), S_3(s)]\|_{L^1\cap L^2} \le C\mathcal{E}_N(U(s)) \le (1+s)^{-\frac{3}{2}}\mathcal{E}_{N,\infty}(U(t)).$$

Here we have used (4.21). Putting the above inequality into (4.24), (4.25) and (4.26) respectively gives

$$\begin{aligned}
\|n(t)\| &\le C(1+t)^{-\frac{3}{4}}\left(\|[n_0, u_0]\|_{L^1\cap L^2} + C\mathcal{E}_{N,\infty}(U(t))\right),\\
\|u(t)\| &\le C(1+t)^{-\frac{3}{4}}\left(\|[n_0, u_0, w_0]\|_{L^1\cap L^2} + C\mathcal{E}_{N,\infty}(U(t))\right),\\
\|w(t)\| &\le C(1+t)^{-\frac{5}{4}}\left(\|[n_0, u_0, w_0]\|_{L^1\cap L^2} + C\mathcal{E}_{N,\infty}(U(t))\right),
\end{aligned}$$

which imply (4.22). This completes the proof of Lemma 4.3. □

Now, the rest is to prove the uniform-in-time bound of $\mathcal{E}_{N,\infty}(U(t))$ which yields the time-decay rates of the Lyapunov functional $\mathcal{E}_N(U(t))$ and thus $\|U(t)\|_N^2$. In fact, by taking $\ell = \frac{3}{2} + \epsilon$ in (4.20) with $\epsilon > 0$ small enough, one has

$$\begin{aligned}
&(1+t)^{\frac{3}{2}+\epsilon}\mathcal{E}_N(U(t)) + c\int_0^t (1+s)^{\frac{3}{2}+\epsilon}\mathcal{D}_N(U(s))ds\\
&\le C\mathcal{E}_N(U_0) + C\int_0^t (1+s)^{\frac{1}{2}+\epsilon}\|[n(s), u(s), w(s)]\|^2 ds.
\end{aligned}$$

Here, using (4.22) and the fact that $\mathcal{E}_{N,\infty}(U(t))$ is non-decreasing in t, it further holds that

$$\begin{aligned}
&\int_0^t (1+s)^{\frac{1}{2}+\epsilon}\|[n(s), u(s), w(s)]\|^2 ds\\
&\le C(1+t)^{\epsilon}\left(\|[n_0, u_0, w_0]\|_{L^1\cap L^2}^2 + C\mathcal{E}_{N,\infty}^2(U(t))\right).
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
&(1+t)^{\frac{3}{2}+\epsilon}\mathcal{E}_N(U(t)) + c\int_0^t (1+s)^{\frac{3}{2}+\epsilon}\mathcal{D}_N(U(s))ds\\
&\le C\mathcal{E}_N(U_0) + C(1+t)^{\epsilon}\left(\|[n_0, u_0, w_0]\|_{L^1\cap L^2}^2 + C\mathcal{E}_{N,\infty}^2(U(t))\right),
\end{aligned}$$

which implies

$$(1+t)^{\frac{3}{2}}\mathcal{E}_N(U(t)) \le C\left(\mathcal{E}_N(U_0) + \mathcal{E}_{N,\infty}^2(U(t)) + \|[n_0, u_0, w_0]\|_{L^1\cap L^2}^2\right).$$

Thus, one has

$$\mathcal{E}_{N,\infty}(U(t)) \le C\left(\epsilon_N^2(U_0) + \mathcal{E}_{N,\infty}^2(U(t))\right).$$

Here, recall the definition of $\epsilon_N(U_0)$. Since $\epsilon_N(U_0) > 0$ is sufficiently small, $\mathcal{E}_{N,\infty}(U(t)) \le C\epsilon_N^2(U_0)$ holds true for any $t \ge 0$, which implies

$$\|U(t)\|_N \le C\mathcal{E}_N(U(t))^{1/2} \le C\epsilon_N(U_0)(1+t)^{-\frac{3}{4}},$$

for any $t \ge 0$. Combining Lemma 4.3 with $\mathcal{E}_{N,\infty}(U(t)) \le C\epsilon_N^2(U_0)$, one can immediately obtain (1.6) in our main Theorem 1.1.

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References

[1] Q. Chen, C. Miao, Global well-posedness for the micropolar fluid system in critical Besov spaces, J. Differential Equations 252 (2012) 2698–2724.
[2] M.T. Chen, Global strong solutions for the viscous, micropolar, compressible flow, J. Partial Differ. Equ. 24 (2) (2011) 158–164.
[3] M.T. Chen, X.Y. Xu, J.W. Zhang, Global weak solutions of 3D compressible micropolar fluids with discontinuous initial data and vacuum, Commun. Math. Sci. 13 (2015) 225–247.
[4] M.T. Chen, Global well-posedness of the 2D incompressible micropolar fluid flows with partial viscosity and angular viscosity, Acta Math. Sci. Ser. B Engl. Ed. 33 (2013) 929–935.
[5] M.T. Chen, B. Huang, J.W. Zhang, Blowup criterion for the three-dimensional equations of compressible viscous micropolar fluids with vacuum, Nonlinear Anal. 79 (2013) 1–11.
[6] I. Dražić, N. Mujaković, 3-D flow of a compressible viscous micropolar fluid with spherical symmetry: a local existence theorem, Bound. Value Probl. 2012 (2012) 69.
[7] I. Dražić, N. Mujaković, 3-D flow of a compressible viscous micropolar fluid with spherical symmetry: a global existence theorem, Bound. Value Probl. 2015 (2015) 98.
[8] I. Dražić, N. Mujaković, 3-D flow of a compressible viscous micropolar fluid with spherical symmetry: large time behavior of the solution, J. Math. Anal. Appl. 431 (2015) 545–568.
[9] R.J. Duan, L.Z. Ruan, C.J. Zhu, Optimal decay rates to conservation laws with diffusion-type terms of regularity-gain and regularity-loss, Math. Models Methods Appl. Sci. 22 (2012) 1250012.
[10] R.J. Duan, Global smooth flows for the compressible Euler–Maxwell system. The relaxation case, J. Hyperbolic Differ. Equ. 8 (2011) 375–413.
[11] R.J. Duan, Green's function and large time behavior of the Navier–Stokes–Maxwell system, Anal. Appl. (Singap.) 10 (2012) 133–197.
[12] R.J. Duan, Q.Q. Liu, C.J. Zhu, Darcy's law and diffusion for a two-fluid Euler–Maxwell system with dissipation, Math. Models Methods Appl. Sci. 25 (2015) 2089–2151.
[13] A.C. Eringen, Theory of micropolar fluids, J. Math. Mech. 16 (1966) 1–18.
[14] G.P. Galdi, S. Rionero, A note on the existence and uniqueness of solutions of the micropolar fluid equations, Internat. J. Engrg. Sci. 15 (1977) 105–108.
[15] S. Kawashima, Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics, Thesis, Kyoto University, 1983.

[16] S. Kawashima, Large-time behavior of solutions to hyperbolic-parabolic systems of conservation laws and applications, Proc. Roy. Soc. Edinburgh 106 (1/2) (1987) 169–194.
[17] A. Matsumura, T. Nishida, Initial boundary value problems for the equations of motion of compressible viscous and heat conductive fluids, Comm. Math. Phys. 89 (1983) 445–464.
[18] A. Matsumura, T. Nishida, The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids, Proc. Japan Acad. Ser. A Math. Sci. 55 (1979) 337–342.
[19] A. Matsumura, T. Nishida, The initial value problems for the equations of motion of viscous and heat-conductive gases, J. Math. Kyoto Univ. 20 (1980) 67–104.
[20] N. Mujaković, One-dimensional flow of a compressible viscous micropolar fluid: a local existence theorem, Glas. Mat. Ser. III 33(53) (2) (1998) 71–91.
[21] N. Mujaković, One-dimensional flow of a compressible viscous micropolar fluid: a global existence theorem, Glas. Mat. Ser. III 33(53) (2) (1998) 199–208.
[22] N. Mujaković, One-dimensional flow of a compressible viscous micropolar fluid: regularity of the solution, Rad. Mat. 10 (2) (2001) 181–193.
[23] N. Mujaković, Global in time estimates for one-dimensional compressible viscous micropolar fluid model, Glas. Mat. Ser. III 40(60) (1) (2005) 103–120.
[24] N. Mujaković, One-dimensional flow of a compressible viscous micropolar fluid: the Cauchy problem, Math. Commun. 10 (1) (2005) 1–14.
[25] N. Mujaković, Non-homogeneous boundary value problem for one-dimensional compressible viscous micropolar fluid model: a local existence theorem, Ann. Univ. Ferrara Sez. VII Sci. Mat. 53 (2) (2007) 361–379.
[26] N. Mujaković, Nonhomogeneous boundary value problem for one-dimensional compressible viscous micropolar fluid model: regularity of the solution, Bound. Value Probl. (2008) 189748.
[27] N. Mujaković, Non-homogeneous boundary value problem for one-dimensional compressible viscous micropolar fluid model: a global existence theorem, Math. Inequal. Appl. 12 (3) (2009) 651–662.
[28] N. Mujaković, 3-D flow of a compressible viscous micropolar fluid with spherical symmetry: uniqueness of a generalized solution, Bound. Value Probl. 2014 (2014) 226.
[29] M.E. Taylor, Partial Differential Equations, I. Basic Theory, Springer, New York, 1996.

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