ARTICLE IN PRESS YJDEQ:8282

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Global well-posedness for the Fokker–Planck–Boltzmann equation in Besov–Chemin–Lerner type spaces ☆

Zhengrong Liu [a], Hao Tang [b,*]

[a] Department of Mathematics, South China University of Technology, Guangzhou, Guangdong, PR China
[b] Department of Mathematics, City University of Hong Kong, Hong Kong, PR China

Received 4 February 2015; revised 8 January 2016

Abstract

In this paper, motivated by [16], we use the Littlewood–Paley theory to establish some estimates on the nonlinear collision term, which enable us to investigate the Cauchy problem of the Fokker–Planck–Boltzmann equation. When the initial data is a small perturbation of the Maxwellian equilibrium state, under the Grad's angular cutoff assumption, the unique global solution for the hard potential case is obtained in the Besov–Chemin–Lerner type spaces $C([0,\infty); \widetilde{L}^2_\xi(B^s_{2,r}))$ with $1 \le r \le 2$ and $s > 3/2$ or $s = 3/2$ and $r = 1$. Besides, we also obtain the uniform stability of the dependence on the initial data.

MSC: 35Q20; 35A01; 35B30

Keywords: Fokker–Planck–Boltzmann equation; Hard potential; Cutoff assumption; Cauchy problem; Littlewood–Paley theory

☆ This work is supported by the National Natural Science Foundation of China (No. 11571116).
* Corresponding author.
E-mail address: haotang4-c@my.cityu.edu.hk (H. Tang).

http://dx.doi.org/10.1016/j.jde.2016.02.031
0022-0396/© 2016 Elsevier Inc. All rights reserved.

Please cite this article in press as: Z. Liu, H. Tang, Global well-posedness for the Fokker–Planck–Boltzmann equation in Besov–Chemin–Lerner type spaces, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.02.031

1. Introduction and main results

The Fokker–Planck–Boltzmann equation describes the motion of particles in a thermal bath where the bilinear interaction is one of the main characters [6,7,31]. Mathematically, we consider the Cauchy problem for the Fokker–Planck–Boltzmann equation

$$\partial_t f + \xi \cdot \nabla_x f = Q(f, f) + \epsilon \nabla_\xi \cdot (\xi f) + \kappa \Delta_\xi f, \tag{1.1}$$

with initial data

$$f(0, \xi, x) = f_0(\xi, x), \tag{1.2}$$

where the unknown function $f = f(t, \xi, x)$ represents the density of gas molecules which are located at $x = (x_1, x_2, x_3) \in \mathbb{R}^3_x$ and have instantaneous velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3_\xi$ at time $t > 0$. Besides, ϵ, κ are nonnegative constants. The collision operator $Q(\cdot, \cdot)$ is a bilinear operator which acts only on the velocity variables ξ and is local in (t, x). Under the Grad's angular cutoff assumption [8,20,42], $Q(\cdot, \cdot)$ is defined as

$$Q(f, g) = \int\limits_{\mathbb{R}^3} \int\limits_{\mathbb{S}^2} |\xi - \xi_*|^\gamma B_0(\theta) \left[f(\xi'_*) g(\xi') - f(\xi_*) g(\xi) \right] d\xi_* d\omega. \tag{1.3}$$

Here ξ, ξ_* and ξ', ξ'_* are the velocities of a pair of particles before and after collision. These collisions are supposed to be elastic so that

$$\xi' = \xi - ((\xi - \xi_*) \cdot \omega)\, \omega, \quad \xi'_* = \xi_* + ((\xi - \xi_*) \cdot \omega)\, \omega, \quad \omega \in \mathbb{S}^2, \tag{1.4}$$

which come from the conservation of momentum and kinetic energy

$$\xi + \xi_* = \xi' + \xi'_*, \quad |\xi|^2 + |\xi_*|^2 = |\xi'|^2 + |\xi'_*|^2.$$

In (1.3), θ is given by $\cos\theta = \omega \cdot (\xi - \xi_*)/|(\xi - \xi_*)|$ and the collision kernel $|\xi - \xi_*|^\gamma B_0(\theta)$ is determined by the interaction law between particles.

All through this paper, we assume that

$$\epsilon = \kappa > 0,\ 0 \le \gamma \le 1,\ 0 \le B_0(\theta) \le C|\cos\theta|, \tag{1.5}$$

which include the hard potentials with angular cutoff as an example.

Let $\mathbf{M}(\xi)$ be the standard Maxwellian equilibrium state satisfying $Q(\mathbf{M}, \mathbf{M}) = 0$, i.e., $\mathbf{M}(\xi) = (2\pi)^{-\frac{3}{2}} \exp\left(-\frac{|\xi|^2}{2}\right)$. We aim at studying the solution to (1.1) (1.2) around $\mathbf{M}$. For this purpose, as in [20,40], we define the perturbation $u = u(t, \xi, x)$ by $f = \mathbf{M} + \sqrt{\mathbf{M}} u(t, \xi, x)$ and then we reformulate the Cauchy problem (1.1) (1.2) as

$$\begin{cases} \partial_t u + \xi \cdot \nabla_x u + \mathbf{L} u = \mathbf{\Gamma}(u, u) + \epsilon \mathbf{L}_{FP} u, \\ u(0, \xi, x) = u_0(\xi, x) = \mathbf{M}^{-\frac{1}{2}} (f_0 - \mathbf{M}). \end{cases} \tag{1.6}$$

Please cite this article in press as: Z. Liu, H. Tang, Global well-posedness for the Fokker–Planck–Boltzmann equation in Besov–Chemin–Lerner type spaces, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.02.031

Here the linearized collision operator $\mathbf{L}(\cdot)$, the nonlinear collision term $\mathbf{\Gamma}(\cdot,\cdot)$ and the linearized Fokker–Planck operator $\mathbf{L}_{FP}(\cdot)$ are defined by

$$\mathbf{L}u = -\mathbf{M}^{-\frac{1}{2}}\left[Q\left(\mathbf{M},\mathbf{M}^{\frac{1}{2}}u\right)+Q\left(\mathbf{M}^{\frac{1}{2}}u,\mathbf{M}\right)\right], \tag{1.7}$$

$$\mathbf{\Gamma}(u,v) = \mathbf{M}^{-\frac{1}{2}}Q\left(\mathbf{M}^{\frac{1}{2}}u,\mathbf{M}^{\frac{1}{2}}v\right), \tag{1.8}$$

$$\mathbf{L}_{FP}u = \Delta_\xi u + \frac{1}{4}(6-|\xi|^2)u, \tag{1.9}$$

respectively. It was pointed out, cf. [8,19,20,25,40], that the operator $\mathbf{L}$ is non-negative and the null space $\mathcal{N}$ of $\mathbf{L}$ is

$$\mathcal{N} = \mathrm{Span}\left\{\mathbf{M}^{\frac{1}{2}},\ \xi_i\mathbf{M}^{\frac{1}{2}}(1\le i\le 3),\left(|\xi|^2-3\right)\mathbf{M}^{\frac{1}{2}}\right\}.$$

Let $\mathbf{P}: L^2\left(\mathbb{R}^3_\xi\right) \to \mathcal{N}$ be the orthogonal projection, then one has

$$\mathbf{P}u = a(t,x)\mathbf{M}^{\frac{1}{2}} + b(t,x)\cdot\xi\mathbf{M}^{\frac{1}{2}} + c(t,x)\left(|\xi|^2-3\right)\mathbf{M}^{\frac{1}{2}}, \quad \forall\, u(t,\xi,x)\in L^2\left(\mathbb{R}^3_\xi\right), \tag{1.10}$$

with the coefficients functions $(a,b,c)=(a,b_i,c)$, $i=1,2,3$ as

$$\begin{cases} a = \int_{\mathbb{R}^3} \mathbf{M}^{\frac{1}{2}}u\,d\xi = \int_{\mathbb{R}^3}\mathbf{M}^{\frac{1}{2}}\mathbf{P}u\,d\xi, \\ b_i = \int_{\mathbb{R}^3}\xi_i\mathbf{M}^{\frac{1}{2}}u\,d\xi = \int_{\mathbb{R}^3}\xi_i\mathbf{M}^{\frac{1}{2}}\mathbf{P}u\,d\xi, \quad i=1,2,3, \\ c = \frac{1}{6}\int_{\mathbb{R}^3}\left(|\xi|^2-3\right)\mathbf{M}^{\frac{1}{2}}u\,d\xi = \frac{1}{6}\int_{\mathbb{R}^3}\left(|\xi|^2-3\right)\mathbf{M}^{\frac{1}{2}}\mathbf{P}u\,d\xi. \end{cases} \tag{1.11}$$

Consequently, we have the following macro–micro decomposition with respect to the given global Maxwellian $\mathbf{M}(\xi)$, cf. [22]:

$$u(t,\xi,x) = \mathbf{P}u(t,\xi,x) + \{\mathbf{I}-\mathbf{P}\}u(t,\xi,x),$$

where $\mathbf{I}$ is the identity operator. $\mathbf{P}u$ and $\{\mathbf{I}-\mathbf{P}\}u$ are called the macroscopic and the microscopic component of u, respectively. We refer to [22,29] for more details of the macro–micro decomposition. Besides, the operator $\mathbf{L}$ can be decomposed as [8,41],

$$\mathbf{L} = \nu(\xi) - K,$$

where $\nu(\xi)$ is the collision frequency given by

$$\nu(\xi) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |\xi-\xi_*|^\gamma B_0(\theta)\mathbf{M}(\xi_*)\,d\xi_*\,d\omega \sim (1+|\xi|)^\gamma. \tag{1.12}$$

Please cite this article in press as: Z. Liu, H. Tang, Global well-posedness for the Fokker–Planck–Boltzmann equation in Besov–Chemin–Lerner type spaces, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.02.031

The integral operator K can also be decomposed as

$$K = K_2 - K_1, \tag{1.13}$$

where

$$[K_2 f](\xi) = \int_{\mathbb{R}^3\times\mathbb{S}^2} |\xi-\xi_*|^\gamma B_0(\theta)\mathbf{M}^{\frac{1}{2}}(\xi_*)\left\{\mathbf{M}^{\frac{1}{2}}(\xi'_*)f(\xi') + \mathbf{M}^{\frac{1}{2}}(\xi')f(\xi'_*)\right\} d\xi_* d\omega, \tag{1.14}$$

and

$$[K_1 f](\xi) = \int_{\mathbb{R}^3\times\mathbb{S}^2} |\xi-\xi_*|^\gamma B_0(\theta)\mathbf{M}^{\frac{1}{2}}(\xi_*)\mathbf{M}^{\frac{1}{2}}(\xi) f(\xi_*) d\xi_* d\omega. \tag{1.15}$$

When $\epsilon = \kappa = 0$, (1.1) becomes the Boltzmann equation. The issue of well-posedness of the Boltzmann equation has been studied by many mathematicians and physicists. For brevity, we only recall the following results closely related to this manuscript.

The first global existence of the mild solution was given by Ukai [39,40] by using the spectrum method and the contraction mapping principle. The spectrum method was later improved by Ukai and Yang [41]. Recently, the nonlinear energy methods developed by Liu, Yu and Yang [30, 29] and independently by Guo [22,24], can be used to deal with this issue. By using the energy method, the well-posedness of classical solutions was established in Sobolev spaces which contain all the derivatives with respect to all variables t, ξ and x of the function. Duan [14] extended these results by showing that if the strong solution with the uniqueness property is considered, then the time differentiation can be disregarded.

For the Boltzmann equation without cutoff assumption, the global existence of small amplitude solutions for general hard and soft potentials was established by Gressman–Strain [21] and independently by Alexandre et al. [2]. In [21], the function space for the energy in the hard potential case can be taken as

$$C\left(0,\infty; L^2(\mathbb{R}^3_\xi; H^N(\mathbb{R}^3_x))\right), \quad N > 2.$$

Very recently, Alexandre et al. [3] proved the local existence in the function spaces which are significantly larger than those used in the previous works. For example, in cutoff case, the spaces can be

$$L^\infty\left(0,T_0; L^2(\mathbb{R}^3_\xi; H^s(\mathbb{R}^3_x))\right), \quad s > \frac{3}{2}.$$

We note that the regularity index $3/2$ in the above space is the critical value for the embedding $H^s(\mathbb{R}^3_x) \hookrightarrow L^\infty(\mathbb{R}^3_x)$. When $s = 3/2$, Duan et al. [16] proved a global well-posedness result in the spaces $\widetilde{L}^\infty_T \widetilde{L}^2_\xi(B^{3/2}_{2,1})$.

There have also been a lot of studies on the Fokker–Planck–Boltzmann equation and a series of achievements have been made. For instance, DiPerna and Lions in [12] proved the global existence of the renormalized solutions to the Cauchy problem (1.1)–(1.2). Hamdache [26] obtained the global existence near the vacuum state by direct construction. In [28], Li and Matsumura

Please cite this article in press as: Z. Liu, H. Tang, Global well-posedness for the Fokker–Planck–Boltzmann equation in Besov–Chemin–Lerner type spaces, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.02.031

showed that in the perturbation framework, a strong solution to equation (1.1) exists globally in time and tends asymptotically to another time-dependent self-similar Maxwellian in the large-time limit for the hard sphere case. When $-1 \le \gamma \le 1$, the long time behavior to the Cauchy problem (1.1)–(1.2) was studied in [44] by using the compensating functions.

In this paper, motivated by [16], we consider the Cauchy problem (1.1)–(1.2) in Besov–Chemin–Lerner type spaces. Now we explain some notations and give some preliminaries on the Besov–Chemin–Lerner type spaces.

1.1. Notations

We use $\lesssim$, $\gtrsim$ and $\approx$ to denote estimates that hold up to some universal constant which may change from line to line but whose meaning is clear from the context. $(\cdot,\cdot)_x$, $(\cdot,\cdot)_\xi$ and $(\cdot,\cdot)_{x,\xi}$ stand the inner product in L^2_x, L^2_ξ and $L^2_{x,\xi}$ respectively. $\mathscr{S}(\mathbb{R}^3_x)$ is the space of rapidly decreasing functions on $\mathbb{R}^3_x$ and $\mathscr{S}'(\mathbb{R}^3_x)$ is its dual space. $\mathscr{S}(\mathbb{R}^3_\xi)$ and $\mathscr{S}'(\mathbb{R}^3_\xi)$ can be defined in the same way. We define the mixed time-velocity-space $L^{p_1}_T L^{p_2}_\xi L^{p_3}_x$ for $0 < T \le \infty, 1 \le p_i \le \infty, i = 1, 2, 3$ with the norm

$$\|f\|_{L^{p_1}_T L^{p_2}_\xi L^{p_3}_x} = \left(\int_0^T \left(\int_{\mathbb{R}^3_\xi} \left(\int_{\mathbb{R}^3_x} |f(t,\xi,x)|^{p_3} dx \right)^{\frac{p_2}{p_3}} d\xi \right)^{\frac{p_1}{p_2}} dt \right)^{\frac{1}{p_1}} .$$

When $p_1 = \infty$, $p_2 = \infty$ or $p_3 = \infty$, we will obey the normal convention to define the above norm. Besides, we also define the space $L^{p_1}_T L^{p_2}_{\xi,\nu} L^{p_3}_x$ with the norm is given by

$$\|f\|_{L^{p_1}_T L^{p_2}_{\xi,\nu} L^{p_3}_x} = \left(\int_0^T \left(\int_{\mathbb{R}^3_\xi} \nu(\xi) \left(\int_{\mathbb{R}^3_x} |f(t,\xi,x)|^{p_3} dx \right)^{\frac{p_2}{p_3}} d\xi \right)^{\frac{p_1}{p_2}} dt \right)^{\frac{1}{p_1}} ,$$

where $\nu(\xi)$ is the velocity weight given by (1.12).

1.2. Littlewood–Paley decomposition and Besov–Chemin–Lerner type spaces

Now we recall some basic facts on the Littlewood–Paley decomposition. For the details, we refer to [5,32,38]. Let χ, φ be two smooth radial functions satisfying $0 \le \chi, \varphi \le 1$, $\operatorname{supp}\chi \subset \{y \in \mathbb{R}^3 : |y| \le \frac{4}{3}\}$, $\operatorname{supp}\varphi \subset \{y \in \mathbb{R}^3 : \frac{3}{4} \le |y| \le \frac{8}{3}\}$ and

$$\begin{aligned} \chi(y) + \sum_{j\in\mathbb{N}} \varphi(2^{-j}y) = 1, &\quad \forall y \in \mathbb{R}^3, \\ \sum_{j\in\mathbb{Z}} \varphi(2^{-j}y) = 1, &\quad \forall y \in \mathbb{R}^3 \backslash \{0\}, \\ \operatorname{supp}\varphi(2^{-j}\cdot) \bigcap \operatorname{supp}\varphi(2^{-j'}\cdot) = \varnothing &\quad \text{if} \quad |j-j'| \ge 2, \\ \operatorname{supp}\varphi(\cdot) \bigcap \operatorname{supp}\varphi(2^{-j}\cdot) = \varnothing &\quad \text{if} \quad j \ge 1. \end{aligned}$$

Please cite this article in press as: Z. Liu, H. Tang, Global well-posedness for the Fokker–Planck–Boltzmann equation in Besov–Chemin–Lerner type spaces, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.02.031

Let $\mathcal{F}_x f$ be the Fourier transform of f in x and $\mathcal{F}_x^{-1} f$ be the inverse transform. Then for any $u \in \mathscr{S}'(\mathbb{R}^3)$, we define the operators $\{\dot{\Delta}_j ; j \in \mathbb{Z}\}$ and $\{\dot{S}_j ; j \in \mathbb{Z}\}$ as

$$\dot{\Delta}_j u = \mathcal{F}_x^{-1}\Big[\varphi(2^{-j}\cdot)\,(\mathcal{F}_x u)\Big], \quad \dot{S}_j u = \mathcal{F}_x^{-1}\Big[\chi(2^{-j}\cdot)\,(\mathcal{F}_x u)\Big].$$

For the nonhomogeneous case, we define $\{\Delta_j ; j \geq -1\}$ and $\{S_j ; j \geq 0\}$ as

$$\Delta_{-1} u = \mathcal{F}_x^{-1}[\chi \mathcal{F}_x u], \quad \Delta_j u = \mathcal{F}_x^{-1}[\varphi(2^{-j}\cdot)\mathcal{F}_x u]\ (j \geq 0), \quad S_j u = \sum_{i=-1}^{j-1} \Delta_i u.$$

Then we have that

$$u = \sum_{j=-1}^{\infty} \Delta_j u \quad \text{converges in } \mathscr{S}'(\mathbb{R}^3) \text{ or in } H^s(\mathbb{R}^3).$$

In the next section, we will use the Bony decomposition. Let $u, v \in \mathscr{S}'(\mathbb{R}^3)$ and define

$$uv = \sum_{i,j} \Delta_i u \Delta_j v = \mathcal{T}_u v + \mathcal{T}_v u + \mathcal{R}(u, v), \tag{1.16}$$

where

$$\mathcal{T}_u v = \sum_j S_{j-1} u \Delta_j v, \quad \mathcal{R}(u, v) = \sum_i \sum_{|i-j| \leq 1} \Delta_i u \Delta_j v.$$

Direct computation implies that for any $1 \leq p \leq \infty$,

$$\Delta_i \Delta_j u \equiv 0, \quad \text{if} \quad |i-j| \geq 2, \ \ \forall\, u, v \in \mathscr{S}'(\mathbb{R}^3), \tag{1.17}$$

$$\Delta_j (S_{i-1} u \Delta_i v) \equiv 0, \quad \text{if} \quad |i-j| \geq 5, \quad \forall\, u, v \in \mathscr{S}'(\mathbb{R}^3), \tag{1.18}$$

$$\|\Delta_i u\|_{L^p} \leq C\|u\|_{L^p}, \ \ \|S_j u\|_{L^p} \leq C\|u\|_{L^p}, \quad \forall\, u \in L^p(\mathbb{R}^3). \tag{1.19}$$

We note that for $\mathcal{R}(u, v)$, there holds

$$\sum_i \sum_{|i-k| \leq 1} \Delta_j[\Delta_i u \Delta_k v] = \sum_{\max\{i,k\} \geq j-2} \sum_{|i-k| \leq 1} \Delta_j[\Delta_i u \Delta_k v], \ \ i, k, j \geq -1. \tag{1.20}$$

Let $\mathscr{P}(\mathbb{R}_x^3)$ denote the class of all polynomials on $\mathbb{R}_x^3$ and let $\mathscr{S}'/\mathscr{P}(\mathbb{R}_x^3)$ denote the tempered distributions on $\mathbb{R}_x^3$ modulo polynomials.

Definition 1.1 *(Besov spaces).* Let $s \in \mathbb{R}$, $1 \leq p, r \leq +\infty$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined by

$$\dot{B}^s_{p,r} = \Big\{ f \in \mathscr{S}'/\mathscr{P}(\mathbb{R}_x^3) : \|f\|_{\dot{B}^s_{p,r}} < \infty \Big\},$$

where $\|f\|_{\dot{B}^s_{p,r}} = \|2^{js}\dot{\Delta}_j f\|_{l^r(L^p_x)} = \Big\|2^{js}\|\dot{\Delta}_j f\|_{L^p_x}\Big\|_{l^r(j\in\mathbb{Z})}$. The nonhomogeneous Besov space $B^s_{p,r}$ is defined by

$$B^s_{p,r} = \left\{f \in \mathscr{S}'(\mathbb{R}^3_x) : \|f\|_{B^s_{p,r}} < \infty\right\},$$

where $\|f\|_{B^s_{p,r}} = \|2^{js}\Delta_j f\|_{l^r(L^p_x)} = \Big\|2^{js}\|\Delta_j f\|_{L^p_x}\Big\|_{l^2(j\geq -1)}$.

Lemma 1.1 *(Bernstein's lemma, [9]). Let $k \in \mathbb{N}$, $f \in L^p$ with $p \in [1,+\infty]$ and* supp $\mathcal{F}_x f \subset \left\{2^{j-2} \leq |\xi| \leq 2^j\right\}$. *Then there exists a constant C_k such that*

$$C_k^{-1}2^{jk}\|f\|_{L^p} \leq \|D^k f\|_{L^p} \leq C_k 2^{jk}\|f\|_{L^p}.$$

Lemma 1.1 gives rise to the equivalence of norms

$$\|f\|_{\dot{B}^{s+k}_{p,r}} \sim \|D^k f\|_{\dot{B}^s_{p,r}}. \tag{1.21}$$

Motivated by [10], we define the following spaces.

Definition 1.2 *(Besov–Chemin–Lerner type spaces).* Let $1 \leq \varrho,\ \varrho_i (i=1,2),\ p,\ r \leq \infty$ and $s \in \mathbb{R}$. Define the nonhomogeneous spaces $\widetilde{L}^\varrho_\xi(B^s_{p,r})$ as

$$\widetilde{L}^\varrho_\xi(B^s_{p,r}) = \left\{f \in \mathscr{S}'(\mathbb{R}^3_\xi \times \mathbb{R}^3_x) : \|f\|_{\widetilde{L}^\varrho_\xi(B^s_{p,r})} < \infty\right\},$$

where

$$\|f\|_{\widetilde{L}^\varrho_\xi(B^s_{p,r})} = \Big\|2^{js}\|\Delta_j f\|_{L^\varrho_\xi L^p_x}\Big\|_{l^r(j\geq -1)},$$

with the usual convention for $\varrho, p, r = \infty$. We also define

$$\|f\|_{\widetilde{L}^\varrho_{\xi,\nu}(B^s_{p,r})} = \Big\|2^{js}\|\Delta_j f\|_{L^\varrho_{\xi,\nu} L^p_x}\Big\|_{l^r(j\geq -1)}.$$

For $0 < T \leq \infty$, we define the nonhomogeneous spaces $\widetilde{L}^{\varrho_1}_T\widetilde{L}^{\varrho_2}_\xi(B^s_{p,r})$ as

$$\widetilde{L}^{\varrho_1}_T\widetilde{L}^{\varrho_2}_\xi(B^s_{p,r}) = \left\{f \in \mathscr{D}'[0,T] \times \mathscr{S}'(\mathbb{R}^3_\xi \times \mathbb{R}^3_x) : \|f\|_{\widetilde{L}^{\varrho_1}_T\widetilde{L}^{\varrho_2}_\xi(B^s_{p,r})} < \infty\right\},$$

where

$$\|f\|_{\widetilde{L}^{\varrho_1}_T\widetilde{L}^{\varrho_2}_\xi(B^s_{p,r})} = \Big\|2^{js}\|\Delta_j f\|_{L^{\varrho_1}_T L^{\varrho_2}_\xi L^p_x}\Big\|_{l^r(j\geq -1)},$$

with the usual convention for $\varrho_i\ (i=1,2),\ p, r = \infty$. Similarly, we define

$$\|f\|_{\widetilde{L}^{\varrho_1}_T\widetilde{L}^{\varrho_2}_{\xi,\nu}(B^s_{p,r})} = \Big\|2^{js}\|\Delta_j f\|_{L^{\varrho_1}_T L^{\varrho_2}_{\xi,\nu} L^p_x}\Big\|_{l^r(j\geq -1)}.$$

Please cite this article in press as: Z. Liu, H. Tang, Global well-posedness for the Fokker–Planck–Boltzmann equation in Besov–Chemin–Lerner type spaces, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.02.031

Similarly, one can also define the homogeneous spaces $\widetilde{L}^{\varrho}_{\xi}(\dot{B}^{s}_{p,r})$, $\widetilde{L}^{\varrho}_{\xi,\nu}(\dot{B}^{s}_{p,r})$, $\widetilde{L}^{\varrho_1}_{T}\widetilde{L}^{\varrho_2}_{\xi}(\dot{B}^{s}_{p,r})$, and $\widetilde{L}^{\varrho_1}_{T}\widetilde{L}^{\varrho_2}_{\xi,\nu}(\dot{B}^{s}_{p,r})$.

1.3. Main results and related remarks

With the definition of the Besov–Chemin–Lerner type spaces in hand, now we are in the position to state the main results of this paper.

Theorem 1.1. *Let $\epsilon > 0$, $1 \le r \le 2$ and $s > 3/2$ or $s = 3/2$ and $r = 1$ and $u_0 \in \widetilde{L}^2_{\xi}(B^s_{2,r})$. If $\|u_0\|_{\widetilde{L}^2_{\xi}(B^s_{2,r})}$ is sufficiently small, then (1.6) has a unique global solution*

$$u(t,\xi,x) \in C([0,\infty); \widetilde{L}^2_{\xi}(B^s_{2,r})),$$

which depends continuously on the initial data. Moreover, if we denote

$$\mathcal{E}_T(u) = \|u\|_{\widetilde{L}^{\infty}_T\widetilde{L}^2_{\xi}(B^s_{2,r})}, \tag{1.22}$$

$$\mathcal{D}_T(u) = \|\nabla_x(a,b,c)\|_{\widetilde{L}^2_T(B^{s-1}_{2,r})} + \|\{\mathbf{I}-\mathbf{P}\}u\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}, \tag{1.23}$$

then for any $T > 0$, we have

$$\sup_{0<t<T}\|u(t)\|_{\widetilde{L}^2_{\xi}(B^s_{2,r})} \le \mathcal{E}_T(u) + \mathcal{D}_T(u) \lesssim \|u_0\|_{\widetilde{L}^2_{\xi}(B^s_{2,r})}. \tag{1.24}$$

Moreover, if $f_0(\xi,x) = \mathbf{M} + \sqrt{\mathbf{M}}u_0(\xi,x) \ge 0$, then $f(t,\xi,x) = \mathbf{M} + \sqrt{\mathbf{M}}u(t,\xi,x) \ge 0$.

Theorem 1.2. *Let $\epsilon > 0$, $1 \le r \le 2$ and $s > 3/2$ or $s = 3/2$ and $r = 1$. If $u_0, v_0 \in \widetilde{L}^2_{\xi}(B^s_{2,r})$ and $\|u_0\|_{\widetilde{L}^2_{\xi}(B^s_{2,r})} + \|v_0\|_{\widetilde{L}^2_{\xi}(B^s_{2,r})}$ is sufficiently small, then for any $T > 0$ and $\epsilon > 0$, the associated solutions u, v to (1.6) satisfy*

$$\sup_{0<t<T}\|u(t)-v(t)\|_{\widetilde{L}^2_{\xi}(B^s_{2,r})} \le \mathcal{E}_T(u-v) + \mathcal{D}_T(u-v) \lesssim \|u_0-v_0\|_{\widetilde{L}^2_{\xi}(B^s_{2,r})}. \tag{1.25}$$

We give a few remarks on our main results as follows.

Remark 1.1. Let us recall some recent work on the Boltzmann equation related to Besov spaces and explain why Besov–Chemin–Lerner type spaces are relevant to the Cauchy problem (1.6). Then we can see what improvement they can achieve.

- We first notice that most of the previous results for the Boltzmann equation and the Fokker–Planck–Boltzmann equation are established in the (weighted) Sobolev spaces. Arsénio and Masmoudi [4] developed a new approach to velocity averaging lemmas in some Besov spaces. Sohinger and Strain [33] proved the optimal time decay rates in the whole space when the initial data belongs to some Besov space $B^{-s}_{2,\infty}$ with respect to x variable. In [18], Fournier obtained the Besov regularity for reasonable weak solution to the spatially homogeneous Boltzmann equation with some measurable initial data. Very recently, for the hard potential case and under the cutoff assumption, Duan et al. [16] proved a global well-posedness result for the Boltzmann equation in the spaces $\widetilde{L}^\infty_T \widetilde{L}^2_\xi(B^{3/2}_{2,1})$, which extends the result of Alexandre et al. [3], where in [3] the spaces in cutoff case can be fractional order Sobolev spaces

$$L^\infty\left(0, T_0; L^2(\mathbb{R}^3_\xi; H^s(\mathbb{R}^3_x))\right), \quad s > \frac{3}{2}. \tag{1.26}$$

The above results are part of our motivations to study the Fokker–Planck–Boltzmann in some Besov type spaces. Moreover, as pointed out in [41], it is a challenging problem to seek for large spaces in which the Cauchy problem of Boltzmann equation is well posed near Maxwellian and this also motives us to consider Besov type spaces.

- Besides, we notice that the regularity index $s = 3/2$ in (1.26) is the critical value for the embedding $H^s(\mathbb{R}^3_x) \hookrightarrow L^\infty(\mathbb{R}^3_x)$ to be true. Then it is natural to ask: *when $s = 3/2$, is (1.6) still well-posed, or in what kind of space with regularity index $s = 3/2$, (1.6) is well-posed?* This becomes another motivation to consider (1.6) in Besov type spaces. We see that the space $\widetilde{L}^2_\xi(B^s_{2,r})$ with $1 \le r \le 2$ is very suitable to fill in the gaps between Sobolev spaces and the usual Besov spaces and thus give an answer to the above question, that is, when $s = 3/2$ and $r = 1$, according to Theorem 1.1, (1.6) is still well posed in $\widetilde{L}^2_\xi(B^{3/2}_{2,1})$. So far, combining the results in [3,16], our global existence space $C([0,\infty); \widetilde{L}^2_\xi(B^s_{2,r}))$, $1 \le r \le 2$ is very suitable because it not only contains the Sobolev space $L^\infty\left(0, T_0; L^2_\xi(H^s_x)\right)$ in [3] but also covers the critical case $s = 3/2$ in [16]. To be more specific, when $r = 2$ in Theorem 1.1, $C([0,\infty); \widetilde{L}^2_\xi(B^s_{2,r}))$ becomes $C([0,\infty); L^2(\mathbb{R}^3_\xi; H^s(\mathbb{R}^3_x)))$ due to

$$\widetilde{L}^2_\xi(B^s_{2,2}) = L^2_\xi(B^s_{2,2}) = L^2(\mathbb{R}^3_\xi; H^s(\mathbb{R}^3_x)).$$

  Therefore in the hard potential case and under the cutoff assumption, our results extend the recent results on the Boltzmann equation in [3,16] and Fokker–Planck–Boltzmann [28,43]. In [28,43], the existence spaces are (weighted) Sobolev spaces with integer index.

Remark 1.2. Now we outline the proof of Theorem 1.1 and state the main difficulties we are confronted with. Following the approach in the recent work [16], we first construct the approximate solutions $\{u_k\}$ via the standard iteration method to obtain the local existence. Then we establish the global *a priori* estimate, which enables us to prove the global existence. The global *a priori* estimate mainly depends on the estimate for the macroscopic dissipations. We will overcome the following difficulties:

- The first difficulty appears in the *a priori* estimate of (1.6) in the framework of Besov–Chemin–Lerner type spaces. It seems difficult to obtain the suitable differential inequality for $\mathcal{E}_T(u)$ and $\mathcal{D}_T(u)$ in Besov–Chemin–Lerner type spaces. Actually, when we split (1.6) in dyadic block and take the energy estimates in $L^2_{\xi,x}$ for each $\Delta_j u$, we will have to take the

Please cite this article in press as: Z. Liu, H. Tang, Global well-posedness for the Fokker–Planck–Boltzmann equation in Besov–Chemin–Lerner type spaces, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.02.031

time integral, take the square root on both sides of the resulting estimate and then take the l^r norm for $j \geq -1$ with j-weight 2^{js}. Then we will have to estimate

$$\left\| 2^{js} \left(\int_0^T \left| \left(\Delta_j \mathbf{\Gamma}(u,u), \Delta_j u \right)_{\xi,x} \right| dt \right)^{\frac{1}{2}} \right\|_{l^r}.$$

In our Lemma 2.4, we consider the following general case

$$\left\| 2^{js} \left(\int_0^T \left| \left(\Delta_j \mathbf{\Gamma}(u,v), \Delta_j h \right)_{\xi,x} \right| dt \right)^{\frac{1}{2}} \right\|_{l^r}$$
$$\lesssim \left(\sqrt{\mathcal{E}_T(u)} \sqrt{\mathcal{H}_T(v)} + \sqrt{\mathcal{E}_T(v)} \sqrt{\mathcal{H}_T(u)} \right) \sqrt{\mathcal{H}_T(h)}, \tag{1.27}$$

where

$$\mathcal{H}_T(f) = \|f\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}. \tag{1.28}$$

This result extends the following important estimate in [22–24],

$$\left| \left(\partial_\xi^\beta \mathbf{\Gamma}(u,v), h \right)_{\xi,x} \right| \lesssim \sum_{\beta' \leq \beta} \int_{\mathbb{R}^3_x} \left[\left(\|\partial_\xi^{\beta'} u\|_{L^2_{\xi,\nu}} \|\partial_\xi^{\beta''} v\|_{L^2_\xi} + \|\partial_\xi^{\beta'} v\|_{L^2_{\xi,\nu}} \|\partial_\xi^{\beta''} u\|_{L^2_\xi} \right) \|h\|_{L^2_{\xi,\nu}} \right] dx,$$

where $\beta = (\beta_1, \beta_2, \beta_3)$ is multi-indices (i.e., $\partial_\xi^\beta = \partial_{\xi_1}^{\beta_1} \partial_{\xi_2}^{\beta_2} \partial_{\xi_3}^{\beta_3}$).

- To prove the global existence, we need to estimate the macroscopic dissipations in which $\mathcal{D}_T(u)$ will be involved. The second difficulty is that (1.27) can **not** be directly used to control the macroscopic dissipations because in the whole space we can **not** obtain $\mathcal{H}_T(u) \leq C\mathcal{D}_T(u)$ for some $0 < C < \infty$. Therefore we use the macro–micro decomposition to establish some more delicate estimates in Lemma 2.5 which enable us to obtain

$$\left\| 2^{js} \left(\int_0^T \left| \left(\Delta_j \mathbf{\Gamma}(u,u), \Delta_j \{\mathbf{I} - \mathbf{P}\} u \right)_{\xi,x} \right| dt \right)^{\frac{1}{2}} \right\|_{l^r(j \geq -1)} \lesssim \sqrt{\mathcal{E}_T(u)} \mathcal{D}_T(u). \tag{1.29}$$

- The third difficulty comes from the global *a priori* estimate. As mentioned before, to overcome the difficulty that (1.27) can **not** control the macroscopic dissipations, we follow the steps as in [13,17], and use the high-order velocity moment functions $\Theta = (\Theta_{mj}(\cdot))_{3\times 3}$ and $\Lambda = (\Lambda_j(\cdot))_{1 \leq j \leq 3}$ to estimate the macroscopic dissipations in Subsect. 4.1. We give some suitable inequalities in Lemma 2.6 to estimate

$$\left\| 2^{n(s-1)} \|\Lambda(\Delta_n \mathbb{h})\|_{L^2_T L^2_x} \right\|_{l^r} + \left\| 2^{n(s-1)} \|\Theta(\Delta_n \mathbb{h})\|_{L^2_T L^2_x} \right\|_{l^r},$$

where $\mathbb{h} = \mathbf{\Gamma}(u,u) + \epsilon \mathbf{L}_{FP}\{\mathbf{I}-\mathbf{P}\}u - \mathbf{L}\{\mathbf{I}-\mathbf{P}\}u$. Therefore in Lemma 4.1 we obtain the following estimate

$$\begin{aligned}\|\nabla_x(a,b,c)\|_{\widetilde{L}^2_T(B^{s-1}_{2,r})} \lesssim \|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})} + (1+\epsilon)\mathcal{E}_T(u) \\ + (2+\epsilon)\|\{\mathbf{I}-\mathbf{P}\}u\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})} + \mathcal{E}_T(u)\mathcal{D}_T(u).\end{aligned}$$

Using (1.29) and the above estimate, in Lemma 4.2 we obtain the global *a priori* estimate

$$\mathcal{E}_T(u) + \lambda\mathcal{D}_T(u) \le C\|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})} + C\left\{\sqrt{\mathcal{E}_T(u)} + \mathcal{E}_T(u)\right\}\mathcal{D}_T(u).$$

Remark 1.3. The issue of the stability of the dependence on initial data for nonlinear PDEs has been the subject of many papers. For the Burgers equation, Kato [27] considered the continuity of the solution map $u_0 \mapsto u$ in Sobolev spaces. For the results on the non-uniform stability and Hölder stability of the dependence on initial data in Besov spaces, we refer to [35–37] for the Camassa–Holm type equation and [34] for the incompressible Euler equation.

Remark 1.4. Notice that the singularities in the collision kernel is a necessary condition for a smoothing effect and this fact was initially observed by Desvillettes for the Kac equation [11]. In this paper, under the Grad's angular cutoff assumption [8,20,42], no smoothing effect is to be expected. When the issue of well-posedness is tackled without the cutoff assumption, some weight will be involved (see [2] and the references therein). Therefore in the non-cutoff case, the deduction in Besov–Chemin–Lerner type spaces becomes much more complicated, which is our work in the future. Moreover, we will also study the well-posedness in larger spaces $L^2_\xi(B^s_{2,r})$ or $\widetilde{L}^2_\xi(B^s_{2,r})$, $r > 2$, we notice that in this case $L^2_\xi(H^s) \hookrightarrow L^2_\xi(B^s_{2,r}) \hookrightarrow \widetilde{L}^2_\xi(B^s_{2,r})$.

The remainder of this paper is organized as follows. In Section 2, we give some preliminaries and basic estimates which will be used frequently later. In Section 3, we demonstrate the local well-posedness. In Section 4, we prove Theorem 1.1. In Section 5, we show Theorem 1.2.

2. Preliminaries

To begin with, we state the following Lemma which shows the relation between the Besov–Chemin–Lerner type spaces $\widetilde{L}^{\varrho_1}_T\widetilde{L}^{\varrho_2}_\xi(X)$ and the classical time-velocity Besov spaces $L^{\varrho_1}_T L^{\varrho_2}_\xi(X)$, where $X = B^s_{p,r}$ or $\dot{B}^s_{p,r}$.

Lemma 2.1. *If* $1 \le \varrho,\ \varrho_i (i=1,2),\ p,\ r \le \infty$ *and* $s \in \mathbb{R}$, *then for* $X = B^s_{p,r}$ *or* $\dot{B}^s_{p,r}$, *we have*

$$\begin{aligned}&\widetilde{L}^{\varrho_1}_T\widetilde{L}^{\varrho_2}_\xi(X) \hookrightarrow L^{\varrho_1}_T L^{\varrho_2}_\xi(X),\quad \widetilde{L}^{\varrho_1}_T\widetilde{L}^{\varrho_2}_{\xi,\nu}(X) \hookrightarrow L^{\varrho_1}_T L^{\varrho_2}_{\xi,\nu}(X),\quad \text{if } r \le \min\{\varrho_1,\varrho_2\},\\ &L^{\varrho_1}_T L^{\varrho_2}_\xi(X) \hookrightarrow \widetilde{L}^{\varrho_1}_T\widetilde{L}^{\varrho_2}_\xi(X),\quad L^{\varrho_1}_T L^{\varrho_2}_{\xi,\nu}(X) \hookrightarrow \widetilde{L}^{\varrho_1}_T\widetilde{L}^{\varrho_2}_{\xi,\nu}(X),\quad \text{if } r \ge \max\{\varrho_1,\varrho_2\}.\end{aligned}$$

Proof. When $r \le \min\{\varrho_1,\varrho_2\}$, we have $\frac{\varrho_i}{r} \ge 1$ $(i=1,2)$. For $X = \dot{B}^s_{p,r}$, according to Definitions 1.1 and 1.2, we use the Minkowski's inequality to exchange the order of integral and summation twice to find

Please cite this article in press as: Z. Liu, H. Tang, Global well-posedness for the Fokker–Planck–Boltzmann equation in Besov–Chemin–Lerner type spaces, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.02.031

$$\|u\|_{L_T^{\varrho_1} L_\xi^{\varrho_2}(\dot{B}^s_{p,r})} = \left(\int_0^T \left(\int_{\mathbb{R}^3_\xi} \left\| 2^{js} \|\dot{\Delta}_j u\|_{L^p_x} \right\|^{\varrho_2}_{l^r(j\in\mathbb{Z})} d\xi \right)^{\frac{\varrho_1}{\varrho_2}} dt \right)^{\frac{1}{\varrho_1}}$$

$$= \left(\int_0^T \left(\int_{\mathbb{R}^3_\xi} \left(\sum_{j\in\mathbb{Z}} 2^{rjs} \|\dot{\Delta}_j u\|^r_{L^p_x} \right)^{\frac{\varrho_2}{r}} d\xi \right)^{\frac{r}{\varrho_2}\cdot\frac{\varrho_1}{r}} dt \right)^{\frac{r}{\varrho_1}\cdot\frac{1}{r}}$$

$$\le \left(\int_0^T \left(\sum_{j\in\mathbb{Z}} \left(\int_{\mathbb{R}^3_\xi} 2^{\varrho_2 js} \|\dot{\Delta}_j u\|^{\varrho_2}_{L^p_x} d\xi \right)^{\frac{r}{\varrho_2}} \right)^{\frac{\varrho_1}{r}} dt \right)^{\frac{r}{\varrho_1}\cdot\frac{1}{r}}$$

$$\le \left(\sum_{j\in\mathbb{Z}} \left(\int_0^T 2^{\varrho_1 js} \|\dot{\Delta}_j u\|^{\varrho_1}_{L^{\varrho_2}_\xi L^p_x} dt \right)^{\frac{r}{\varrho_1}} \right)^{\frac{1}{r}} = \|u\|_{\widetilde{L}_T^{\varrho_1} \widetilde{L}_\xi^{\varrho_2}(\dot{B}^s_{p,r})},$$

which implies the embedding $\widetilde{L}_T^{\varrho_1} \widetilde{L}_\xi^{\varrho_2}(\dot{B}^s_{p,r}) \hookrightarrow L_T^{\varrho_1} L_\xi^{\varrho_2}(\dot{B}^s_{p,r})$. The proofs for the other cases are very similar and hence we omit the proof here. □

From the above Lemma, we see that if $1 \le r \le 2$ and $X = B^s_{p,r}$ or $\dot{B}^s_{p,r}$, the $\widetilde{L}^\infty_T \widetilde{L}^2_\xi(X)$-topology (resp. $\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(X)$-topology) is stronger than the $L^\infty_T L^2_\xi(X)$-topology (resp. $L^2_T L^2_{\xi,\nu}(X)$-topology) and

$$\|u\|_{L^\infty_T L^2_\xi(X)} \le \|u\|_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi(X)}, \quad \|u\|_{L^2_T L^2_{\xi,\nu}(X)} \le \|u\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(X)}.$$

If $s > 0$, $1 \le r \le 2$, then we can infer from Lemma 1.1 and the above inequalities that

$$\|u\|_{L^\infty_T L^2_\xi(\dot{B}^s_{2,r})} \le \|u\|_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi(B^s_{2,r})}, \quad \|u\|_{L^\infty_T L^2_{\xi,\nu}(\dot{B}^s_{2,r})} \le \|u\|_{\widetilde{L}^\infty_T \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}. \tag{2.1}$$

We will need the following Fatou property, and the proof is omitted since it is very similar to the one in [5].

Lemma 2.2. *Let $1 \le \varrho_i (i = 1, 2)$, $p, r \le \infty$ and $s \in \mathbb{R}$. For any given $T > 0$, if $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\widetilde{L}_T^{\varrho_1} \widetilde{L}_\xi^{\varrho_2}(B^s_{p,r})$ and u_n converges to u in $\mathscr{D}'[0, T] \times \mathscr{S}'(\mathbb{R}^3_\xi) \times \mathscr{S}'(\mathbb{R}^3_x)$, then $u \in \widetilde{L}_T^{\varrho_1} \widetilde{L}_\xi^{\varrho_2}(B^s_{p,r})$ and*

$$\|u\|_{\widetilde{L}_T^{\varrho_1} \widetilde{L}_\xi^{\varrho_2}(B^s_{p,r})} \le \liminf_{n\to\infty} \|u_n\|_{\widetilde{L}_T^{\varrho_1} \widetilde{L}_\xi^{\varrho_2}(B^s_{p,r})}.$$

Now we consider the behavior of the operators $\mathbf{L}$, $\mathbf{\Gamma}$ and $\mathbf{L}_{FP}$ in the Besov–Chemin–Lerner type spaces. From the coercivity property of the linearized collision operator $\mathbf{L}$, we see that there exists a $\lambda_0 > 0$ such that for $j \geq -1$,

$$\left(\Delta_j \mathbf{L}u, \Delta_j u\right)_{\xi,x} \geq \lambda_0 \|\{\mathbf{I}-\mathbf{P}\}\Delta_j u\|^2_{L^2_{\xi,\nu}L^2_x}. \tag{2.2}$$

For the operator $K = K_2 - K_1$ and $\mathbf{L}_{FP}$ defined in (1.13)–(1.15) and (1.9), since K and $\mathbf{L}_{FP}$ do not act on the variable x, we have

$$\Delta_j K = K\Delta_j, \quad \Delta_j \mathbf{L}_{FP} = \mathbf{L}_{FP}\Delta_j.$$

Furthermore, because K is a self-adjoint compact operator on L^2_ξ, it satisfies (see [8,16])

$$\left(\Delta_j Kf, \Delta_j g\right)_{\xi,x} \leq C\|\Delta_j f\|_{L^2_\xi L^2_x}\|\Delta_j g\|_{L^2_\xi L^2_x}. \tag{2.3}$$

For $\mathbf{L}_{FP}$, it is a linear self-adjoint operator with respect to the duality induced by the L^2_ξ-scalar product, and there is a $\lambda_{FP} > 0$ such that (see [1,15])

$$-(u, \mathbf{L}_{FP}u)_\xi \geq \lambda_{FP}\|\{\mathbf{I}-\mathbf{P}_0\}u\|^2_{L^2_\xi},$$

where

$$\begin{cases} \mathbf{P}u = \mathbf{P}_0 u \oplus \mathbf{P}_1 u, \\ \mathbf{P}_0 u = a(t,x)\mathbf{M}^{1/2}, \\ \mathbf{P}_1 u = \left\{b(t,x)\cdot\xi + c(t,x)(|\xi|^2-3)\right\}\mathbf{M}^{1/2}, \end{cases}$$

and $\mathbf{P}$ defined in (1.10). Therefore we can further deduce that

$$-(\Delta_j u, \Delta_j \mathbf{L}_{FP}u)_{\xi,x} \geq \lambda_{FP}\|\{\mathbf{I}-\mathbf{P}_0\}\Delta_j u\|^2_{L^2_\xi L^2_x}. \tag{2.4}$$

For the nonlinear collision operator $\mathbf{\Gamma}(u,v)$ in (1.8), it can be written as

$$\mathbf{\Gamma}(u,v) = \mathbf{\Gamma}_{gain}(u,v) - \mathbf{\Gamma}_{loss}(u,v),$$

where

$$\mathbf{\Gamma}_{gain}(u,v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |\xi-\xi_*|^\gamma B_0(\theta)\sqrt{\mathbf{M}}(\xi_*)u(\xi'_*)v(\xi')d\xi_* d\omega, \tag{2.5}$$

$$\mathbf{\Gamma}_{loss}(u,v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} |\xi-\xi_*|^\gamma B_0(\theta)\sqrt{\mathbf{M}}(\xi_*)u(\xi_*)v(\xi)d\xi_* d\omega. \tag{2.6}$$

Since $\mathbf{\Gamma}(u,v)$ does not act on the variable x, for $j \geq -1$, we have

Please cite this article in press as: Z. Liu, H. Tang, Global well-posedness for the Fokker–Planck–Boltzmann equation in Besov–Chemin–Lerner type spaces, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.02.031

$$\begin{aligned}\Delta_j\mathbf{\Gamma}(u,v)=&\Delta_j\mathbf{\Gamma}_{gain}(u,v)-\Delta_j\mathbf{\Gamma}_{loss}(u,v)\\ =&\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{S}^2}|\xi-\xi_*|^\gamma B_0(\theta)\sqrt{\mathbf{M}}(\xi_*)\Delta_j[u(\xi_*')v(\xi')]d\xi_*d\omega\\ &-\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{S}^2}|\xi-\xi_*|^\gamma B_0(\theta)\sqrt{\mathbf{M}}(\xi_*)\Delta_j[u(\xi_*)v(\xi)]d\xi_*d\omega.\end{aligned}$$

Next we give the key estimates in this paper.

Lemma 2.3. *Let $u(t,\xi,x)$, $v(t,\xi,x)$, $h(t,\xi,x)$ be three proper functions such that all norms in the following inequalities are well defined. Then for $T>0$, $s>0$ and $r\geq 1$, we have*

$$\begin{aligned}&\left\|2^{sj}\left(\int\limits_0^T\left|\left(\Delta_j\mathbf{\Gamma}(u,v),\Delta_jh\right)_{\xi,x}\right|dt\right)^{\frac{1}{2}}\right\|_{l^r(j\geq-1)}\\ \lesssim&\|h\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\|v\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\|u\|^{\frac12}_{L^\infty_TL^2_\xi L^\infty_x}+\|v\|^{\frac12}_{\widetilde{L}^\infty_T\widetilde{L}^2_{\xi}(B^s_{2,r})}\|u\|^{\frac12}_{L^2_TL^2_{\xi,\nu}L^\infty_x}\right]\\ &+\|h\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\|u\|^{\frac12}_{\widetilde{L}^\infty_T\widetilde{L}^2_{\xi}(B^s_{2,r})}\|v\|^{\frac12}_{L^2_TL^2_{\xi,\nu}L^\infty_x}+\|u\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\|v\|^{\frac12}_{L^\infty_TL^2_\xi L^\infty_x}\right].\end{aligned}$$

Proof. By applying the Cauchy–Schwarz inequality to both the terms $\mathbf{\Gamma}_{gain}$ and $\mathbf{\Gamma}_{loss}$ defined by (2.5) and (2.6) with respect to t, ξ, x, ξ_*, ω, making the change of variables $(\xi,\xi_*)\to(\xi',\xi_*')$ in $\mathbf{\Gamma}_{gain}$ and noticing that $0\leq B_0(\theta)\leq C|\cos\theta|\leq C$, $\int\limits_{\mathbb{S}^2}d\omega=4\pi$, $d\xi d\xi_*=d\xi'd\xi_*'$, $|\xi-\xi_*|=|\xi'-\xi_*'|$, we have

$$\begin{aligned}&\left(\int\limits_0^T\left|\left(\Delta_j\mathbf{\Gamma}_{gain}(u,v),\Delta_jh\right)_{\xi,x}\right|dt\right)^{\frac{r}{2}}\\ \lesssim&\left(\int\limits_0^T\int\limits_{\mathbb{R}^9}|\xi-\xi_*|^\gamma\left|\Delta_j[u_*v]\right|^2d\xi_*dxd\xi dt\right)^{\frac12\cdot\frac{r}{2}}\\ &\times\left(\int\limits_0^T\int\limits_{\mathbb{R}^9}|\xi-\xi_*|^\gamma\mathbf{M}_*\left|\Delta_jh\right|^2d\xi_*dxd\xi dt\right)^{\frac12\cdot\frac{r}{2}},\end{aligned}\tag{2.7}$$

and

$$\left(\int\limits_0^T\left|\left(\Delta_j\mathbf{\Gamma}_{loss}(u,v),\Delta_jh\right)_{\xi,x}\right|dt\right)^{\frac{r}{2}}$$

$$\lesssim \left(\int_0^T \int_{\mathbb{R}^9} |\xi - \xi_*|^\gamma \left|\Delta_j[u_* v]\right|^2 d\xi_* dx d\xi dt \right)^{\frac{1}{2}\cdot\frac{r}{2}}$$

$$\times \left(\int_0^T \int_{\mathbb{R}^9} |\xi - \xi_*|^\gamma \mathbf{M}_* \left|\Delta_j h\right|^2 d\xi_* dx d\xi dt \right)^{\frac{1}{2}\cdot\frac{r}{2}}. \tag{2.8}$$

For $\Phi = \mathbf{\Gamma}_{gain}(u, v)$ or $\mathbf{\Gamma}_{loss}(u, v)$, we apply the discrete Cauchy–Schwarz inequality to obtain

$$\sum_{j\geq -1} 2^{jsr} \left(\int_0^T \left| \left(\Delta_j \Phi, \Delta_j h\right)_{\xi,x} \right| dt \right)^{\frac{r}{2}}$$

$$\leq \left(\sum_{j\geq -1} 2^{jsr} \left(\int_0^T \int_{\mathbb{R}^9} |\xi - \xi_*|^\gamma \left|\Delta_j[u_* v]\right|^2 d\xi_* dx d\xi dt \right)^{\frac{r}{2}} \right)^{\frac{1}{2}}$$

$$\times \left(\sum_{j\geq -1} 2^{jsr} \left(\int_0^T \int_{\mathbb{R}^9} |\xi - \xi_*|^\gamma \mathbf{M}_* \left|\Delta_j h\right|^2 d\xi_* dx d\xi dt \right)^{\frac{r}{2}} \right)^{\frac{1}{2}}. \tag{2.9}$$

Therefore, we obtain

$$\left\| 2^{sj} \left(\int_0^T \left| \left(\Delta_j \mathbf{\Gamma}(u, v), \Delta_j h\right)_{\xi,x} \right| dt \right)^{\frac{1}{2}} \right\|_{l^r}$$

$$\leq \left\| 2^{sj} \left(\int_0^T \left| \left(\Delta_j \mathbf{\Gamma}_{gain}(u, v), \Delta_j h\right)_{\xi,x} \right| dt \right)^{\frac{1}{2}} \right\|_{l^r} + \left\| 2^{sj} \left(\int_0^T \left| \left(\Delta_j \mathbf{\Gamma}_{loss}(u, v), \Delta_j h\right)_{\xi,x} \right| dt \right)^{\frac{1}{2}} \right\|_{l^r}$$

$$\leq A^{\frac{1}{2}} \times B^{\frac{1}{2}}, \tag{2.10}$$

where

$$A = \left(\sum_{j\geq -1} 2^{jsr} \left(\int_0^T \int_{\mathbb{R}^9} |\xi - \xi_*|^\gamma \left|\Delta_j[u_* v]\right|^2 d\xi_* dx d\xi dt \right)^{\frac{r}{2}} \right)^{\frac{1}{r}},$$

$$B = \left(\sum_{j \geq -1} 2^{jsr} \left(\int_0^T \int_{\mathbb{R}^9} |\xi - \xi_*|^\gamma \mathbf{M}_* \left|\Delta_j h\right|^2 d\xi_* dx d\xi dt \right)^{\frac{r}{2}} \right)^{\frac{1}{r}}.$$

Since $\int_{\mathbb{R}^3} |\xi - \xi_*|^\gamma \mathbf{M}_* d\xi_* \sim \nu(\xi)$, we get

$$B \leq \|h\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}. \tag{2.11}$$

For the term A in (2.10), we use the Bony decomposition (1.16) to rewrite $\Delta_j[u_* v]$ as $\Delta_j[u_* v] = \Delta_j \left[\mathcal{T}_{u_*} v + \mathcal{T}_v u_* + \mathcal{R}(u_*, v)\right]$, where

$$\mathcal{T}_{u_*} v = \sum_q S_{q-1} u_* \Delta_q v, \quad \mathcal{T}_v u_* = \sum_q S_{q-1} v \Delta_q u_*, \quad \mathcal{R}(u_*, v) = \sum_q \sum_{|q'-q| \leq 1} \Delta_{q'} u_* \Delta_q v.$$

Using the Minkowski's inequality and (1.17)–(1.20) gives rise to

$$\begin{aligned}
A &= \left\| 2^{sj} \left(\int_0^T \int_{\mathbb{R}^6} |\xi - \xi_*|^\gamma \left\|\Delta_j[u_* v]\right\|^2_{L^2_x} d\xi_* d\xi dt \right)^{\frac{1}{2}} \right\|_{l^r} \\
&\lesssim \left\| 2^{sj} \sum_{|q-j| \leq 4} \left(\int_0^T \int_{\mathbb{R}^6} |\xi - \xi_*|^\gamma \left\|S_{q-1} u_* \Delta_q v\right\|^2_{L^2_x} d\xi_* d\xi dt \right)^{\frac{1}{2}} \right\|_{l^r} \\
&\quad + \left\| 2^{sj} \sum_{|q-j| \leq 4} \left(\int_0^T \int_{\mathbb{R}^6} |\xi - \xi_*|^\gamma \left\|S_{q-1} v \Delta_q u_*\right\|^2_{L^2_x} d\xi_* d\xi dt \right)^{\frac{1}{2}} \right\|_{l^r} \\
&\quad + \left\| 2^{sj} \sum_{q > j-3} \left(\int_0^T \int_{\mathbb{R}^6} |\xi - \xi_*|^\gamma \left\|\Delta_{q'} u_* \Delta_q v\right\|^2_{L^2_x} d\xi_* d\xi dt \right)^{\frac{1}{2}} \right\|_{l^r} \\
&\lesssim \sum_{i=1,2} \left(I_{1,i} + I_{2,i} + I_{3,i}\right).
\end{aligned} \tag{2.12}$$

In (2.12), for $j = 1, 2, 3$, $I_{j,i}$ $(i = 1, 2)$ come from the j-th l^r norm in the right hand side by exchanging the order of integral and then estimating the $L^2_T L^2_x$ norm. To be more specific, $I_{1,i}$ $(i = 1, 2)$ are as follows,

Please cite this article in press as: Z. Liu, H. Tang, Global well-posedness for the Fokker–Planck–Boltzmann equation in Besov–Chemin–Lerner type spaces, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.02.031

$$
\begin{aligned}
I_{1,1} &= \left[\sum_{j \ge -1} 2^{jsr} \sum_{|q-j|\le 4} \left(\int_{\mathbb{R}^6} |\xi|^\gamma \|u_*\|^2_{L^\infty_T L^\infty_x} \|\Delta_q v\|^2_{L^2_T L^2_x} d\xi_* d\xi \right)^{\frac{r}{2}} \right]^{\frac{1}{r}} \\
&\le \left[\sum_{j\ge -1} \sum_{|q-j|\le 4} 2^{s(j-q)r} \left(2^{qs} \|\Delta_q v\|_{L^2_T L^2_{\xi,\nu} L^2_x} \right)^r \right]^{\frac{1}{r}} \|u\|_{L^\infty_T L^2_\xi L^\infty_x} \\
&\le \left\| \sum_{|q-j|\le 4} 2^{(j-q)s} C_1(q) \right\|_{l^r(j\ge -1)} \|v\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})} \|u\|_{L^\infty_T L^2_\xi L^\infty_x}, \ C_1(q) = \frac{2^{qs}\|\Delta_q v\|_{L^2_T L^2_{\xi,\nu} L^2_x}}{\|v\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}},
\end{aligned}
$$

$$
\begin{aligned}
I_{1,2} &= \left[\sum_{j \ge -1} 2^{jsr} \sum_{|q-j|\le 4} \left(\int_{\mathbb{R}^6} |\xi_*|^\gamma \|u_*\|^2_{L^2_T L^\infty_x} \|\Delta_q v\|^2_{L^\infty_T L^2_x} d\xi_* d\xi \right)^{\frac{r}{2}} \right]^{\frac{1}{r}} \\
&\le \left[\sum_{j\ge -1} \sum_{|q-j|\le 4} 2^{s(j-q)r} \left(2^{qs} \|\Delta_q v\|_{L^\infty_T L^2_\xi L^2_x} \right)^r \right]^{\frac{1}{r}} \|u\|_{L^2_T L^2_{\xi,\nu} L^\infty_x} \\
&\le \left\| \sum_{|q-j|\le 4} 2^{(j-q)s} C_2(q) \right\|_{l^r(j\ge -1)} \|v\|_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi(B^s_{2,r})} \|u\|_{L^2_T L^2_{\xi,\nu} L^\infty_x}, \quad C_2(q) = \frac{2^{qs}\|\Delta_q v\|_{L^\infty_T L^2_\xi L^2_x}}{\|v\|^2_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi(B^s_{2,r})}}.
\end{aligned}
$$

And $I_{2,i}$ $(i = 1, 2)$ come from the second l^r norm in the right hand side of (2.12). The estimates for $I_{2,i}$ $(i = 1, 2)$ are similar to the ones for $I_{1,1}$, $I_{1,2}$ and we have

$$
\begin{aligned}
I_{2,1} &\le \left\| \sum_{|q-j|\le 4} 2^{(j-q)s} C_3(q) \right\|_{l^r(j\ge -1)} \|u\|_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi(B^s_{2,r})} \|v\|_{L^2_T L^2_{\xi,\nu} L^\infty_x}, \ C_3(q) = \frac{2^{qs}\|\Delta_q u\|_{L^\infty_T L^2_\xi L^2_x}}{\|u\|_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi(B^s_{2,r})}}, \\
I_{2,2} &\le \left\| \sum_{|q-j|\le 4} 2^{(j-q)s} C_4(q) \right\|_{l^r(j\ge -1)} \|u\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})} \|v\|_{L^\infty_T L^2_\xi L^\infty_x}, \ C_4(q) = \frac{2^{qs}\|\Delta_q u\|_{L^2_T L^2_{\xi,\nu} L^2_x}}{\|u\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}}.
\end{aligned}
$$

Finally, the terms $I_{3,i}$ $(i = 1, 2)$ in (2.12) are as follows,

$$
\begin{aligned}
I_{3,1} &= \left[\sum_{j \ge -1} 2^{jsr} \sum_{q > j-3} \left(\int_{\mathbb{R}^6} |\xi|^\gamma \|u_*\|^2_{L^\infty_T L^\infty_x} \|\Delta_q v\|^2_{L^2_T L^2_x} d\xi_* d\xi \right)^{\frac{r}{2}} \right]^{\frac{1}{r}} \\
&\le \left[\sum_{j\ge -1} \sum_{q > j-3} 2^{s(j-q)r} \left(2^{qs} \|\Delta_q v\|_{L^2_T L^2_{\xi,\nu} L^2_x} \right)^r \right]^{\frac{1}{r}} \|u\|_{L^\infty_T L^2_\xi L^\infty_x}
\end{aligned}
$$

$$\leq \left\| \sum_{q>j-3} 2^{(j-q)s} C_1(q) \right\|_{l^r(j\geq -1)} \|v\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})} \|u\|_{L^\infty_T L^2_\xi L^\infty_x},$$

$$I_{3,2} = \left[\sum_{j\geq -1} 2^{jsr} \sum_{q>j-3} \left(\int_{\mathbb{R}^6} |\xi_*|^\gamma \|\Delta_q u_*\|^2_{L^2_T L^2_x} \|v\|^2_{L^\infty_T L^\infty_x} d\xi_* d\xi \right)^{\frac{r}{2}} \right]^{\frac{1}{r}}$$

$$\leq \left[\sum_{j\geq -1} \sum_{q>j-3} 2^{s(j-q)r} \left(2^{qs} \|\Delta_q u\|_{L^2_T L^2_{\xi,\nu} L^2_x} \right)^r \right]^{\frac{1}{r}} \|v\|_{L^\infty_T L^2_\xi L^\infty_x}$$

$$\leq \left\| \sum_{q>j-3} 2^{(j-q)s} C_4(q) \right\|_{l^r(j\geq -1)} \|u\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})} \|v\|_{L^\infty_T L^2_\xi L^\infty_x}.$$

For $i = 1, 2, 3, 4$, we have $\|C_i(q)\|_{l^r(q)} = 1$. Then we obtain

$$\left\| \sum_{|q-j|\leq 4} 2^{(j-q)s} C_i(q) \right\|_{l^r(j\geq -1)} = \|\mathbf{1}_{|q|\leq 4} 2^{qs} * C_i(q)\|_{l^r} \leq \|\mathbf{1}_{|q|\leq 4} 2^{qs}\|_{l^1} \|C_i(q)\|_{l^r} < \infty.$$

For $i = 1, 4$, since $s > 0$, we have

$$\left\| \sum_{q>j-3} 2^{(j-q)s} C_i(q) \right\|_{l^r(j\geq -1)} \leq \|2^{-qs}\|_{l^1} \|C_i(q)\|_{l^r} < \infty.$$

Thus we have

$$I_{1,1} \lesssim \|v\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})} \|u\|_{L^\infty_T L^2_\xi L^\infty_x}, \quad I_{1,2} \lesssim \|v\|_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi(B^s_{2,r})} \|u\|_{L^2_T L^2_{\xi,\nu} L^\infty_x},$$
$$I_{2,1} \lesssim \|u\|_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi(B^s_{2,r})} \|v\|_{L^2_T L^2_{\xi,\nu} L^\infty_x}, \quad I_{2,2} \lesssim \|u\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})} \|v\|_{L^\infty_T L^2_\xi L^\infty_x},$$
$$I_{3,1} \lesssim \|v\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})} \|u\|_{L^\infty_T L^2_\xi L^\infty_x}, \quad I_{3,2} \lesssim \|u\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})} \|v\|_{L^\infty_T L^2_\xi L^\infty_x}.$$

Collecting the above estimates, (2.12) becomes

$$A \lesssim \|v\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})} \|u\|_{L^\infty_T L^2_\xi L^\infty_x} + \|v\|_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi(B^s_{2,r})} \|u\|_{L^2_T L^2_{\xi,\nu} L^\infty_x}$$
$$+ \|u\|_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi(B^s_{2,r})} \|v\|_{L^2_T L^2_{\xi,\nu} L^\infty_x} + \|u\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})} \|v\|_{L^\infty_T L^2_\xi L^\infty_x}. \tag{2.13}$$

Inserting (2.11) and (2.13) into (2.10) gives the desired result. □

Lemma 2.4. *Let $u(t,\xi,x), v(t,\xi,x), h(t,\xi,x)$ be three proper functions such that the following inequality is well defined. For any $0 \leq t_1 < t_2 < \infty$, $1 \leq r \leq 2$, $s \geq 3/2$ ($s = 3/2$ only if $r = 1$), let $\widetilde{T} = t_2 - t_1$, $X = B^s_{2,r}$ or $\dot{B}^s_{2,r}$ and $\Phi(t,\xi,x) = \mathbf{\Gamma}(u,v)$, $\mathbf{\Gamma}_{gain}(u,v)$ or $\mathbf{\Gamma}_{loss}(u,v)$, we have*

$$\left\| 2^{sj}\left(\int_{t_1}^{t_2}\left|\left(\Delta_j\Phi,\Delta_j h\right)_{\xi,x}\right|dt\right)^{\frac{1}{2}}\right\|_{l^r}$$

$$\lesssim \|h\|^{\frac{1}{2}}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\|v\|^{\frac{1}{2}}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\|u\|^{\frac{1}{2}}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(X)}+\|v\|^{\frac{1}{2}}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_{2,r})}\|u\|^{\frac{1}{2}}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(X)}\right]$$
$$+\|h\|^{\frac{1}{2}}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\|u\|^{\frac{1}{2}}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_{2,r})}\|v\|^{\frac{1}{2}}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(X)}+\|u\|^{\frac{1}{2}}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\|v\|^{\frac{1}{2}}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(X)}\right].$$

Proof. When $s\geq 3/2$ ($s=3/2$ only if $r=1$), we have $X\hookrightarrow L^\infty_x$. Hence Lemmas (2.1) and 2.3 lead to Lemma 2.4. □

Lemma 2.5. *Let $u(t,\xi,x)$, $v(t,\xi,x)$, $h(t,\xi,x)$ be three proper functions such that the following inequality is well defined. For $T>0$, $1\leq r\leq 2$, $s\geq 3/2$ ($s=3/2$ only if $r=1$) and $X=B^s_{2,r}$ or $\dot{B}^s_{2,r}$, we have*

$$\left\| 2^{sj}\left(\int_0^T\left|\left(\Delta_j\mathbf{\Gamma}(\mathbf{P}u,\mathbf{P}v),\Delta_j h\right)_{\xi,x}\right|dt\right)^{\frac{1}{2}}\right\|_{l^r}$$
$$\lesssim \|h\|^{\frac{1}{2}}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\sqrt{\mathcal{E}_T(v)}\sqrt{\mathcal{D}_T(u)}+\sqrt{\mathcal{D}_T(v)}\sqrt{\mathcal{E}_T(u)}\right], \tag{2.14}$$

$$\left\| 2^{sj}\left(\int_0^T\left|\left(\Delta_j\mathbf{\Gamma}(\mathbf{P}u,\{\mathbf{I}-\mathbf{P}\}v),\Delta_j h\right)_{\xi,x}\right|dt\right)^{\frac{1}{2}}\right\|_{l^r}\lesssim \|h\|^{\frac{1}{2}}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\sqrt{\mathcal{D}_T(v)}\sqrt{\mathcal{E}_T(u)}\right], \tag{2.15}$$

$$\left\| 2^{sj}\left(\int_0^T\left|\left(\Delta_j\mathbf{\Gamma}(\{\mathbf{I}-\mathbf{P}\}u,\mathbf{P}v),\Delta_j h\right)_{\xi,x}\right|dt\right)^{\frac{1}{2}}\right\|_{l^r}\lesssim \|h\|^{\frac{1}{2}}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\sqrt{\mathcal{E}_T(v)}\sqrt{\mathcal{D}_T(u)}\right], \tag{2.16}$$

$$\left\| 2^{sj}\left(\int_0^T\left|\left(\Delta_j\mathbf{\Gamma}(\{\mathbf{I}-\mathbf{P}\}u,\{\mathbf{I}-\mathbf{P}\}v),\Delta_j h\right)_{\xi,x}\right|dt\right)^{\frac{1}{2}}\right\|_{l^r}$$
$$\lesssim \|h\|^{\frac{1}{2}}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\sqrt{\mathcal{E}_T(v)}\sqrt{\mathcal{D}_T(u)}+\sqrt{\mathcal{D}_T(v)}\sqrt{\mathcal{E}_T(u)}\right], \tag{2.17}$$

where $\mathcal{E}_T(u)$ and $\mathcal{D}_T(u)$ are given in (1.22) and (1.23).

Proof. If we modify the proof of Lemmas 2.3 and 2.4 by exchanging the L^∞_T-norm and L^2_T-norm in the estimate of $I_{1,1}$, $I_{2,2}$ and $I_{3,2}$ with using the equivalent relation

$$\|\Delta_j \mathbf{P}u\|_{L^2_{\xi,\nu}L^2_x} \sim \|\Delta_j \mathbf{P}u\|_{L^2_\xi L^2_x}, \quad \|S_q\Delta_j \mathbf{P}u\|_{L^2_{\xi,\nu}L^2_x} \sim \|S_q\Delta_j \mathbf{P}u\|_{L^2_\xi L^2_x}, \tag{2.18}$$

then we obtain

$$\begin{aligned}
&\left\| 2^{sj}\left(\int_0^T \left|\left(\Delta_j \mathbf{\Gamma}(\mathbf{P}u,\mathbf{P}v),\Delta_j h\right)_{\xi,x}\right| dt\right)^{\frac12}\right\|_{l^r} \\
&\lesssim \|h\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\|\mathbf{P}v\|^{\frac12}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_{2,r})}\|\mathbf{P}u\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_\xi(X)}+\|\mathbf{P}v\|^{\frac12}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_{2,r})}\|\mathbf{P}u\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_\xi(X)}\right] \\
&\quad+\|h\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\|\mathbf{P}u\|^{\frac12}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_{2,r})}\|\mathbf{P}v\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_\xi(X)}+\|\mathbf{P}u\|^{\frac12}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_{2,r})}\|\mathbf{P}v\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_\xi(X)}\right] \\
&\lesssim \|h\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\|\mathbf{P}v\|^{\frac12}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_{2,r})}\|\mathbf{P}u\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_\xi(X)}+\|\mathbf{P}v\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_\xi(X)}\|\mathbf{P}u\|^{\frac12}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_{2,r})}\right].
\end{aligned}$$

Taking $X=\dot{B}^s_{2,r}$ and using (1.21) gives rise to

$$\|\mathbf{P}u\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_\xi(\dot{B}^s_{2,r})} \lesssim \|(a^u,b^u,c^u)\|^{\frac12}_{\widetilde{L}^2_T(\dot{B}^s_{2,r})} \sim \|\nabla_x(a^u,b^u,c^u)\|^{\frac12}_{\widetilde{L}^2_T(\dot{B}^{s-1}_{2,r})} \le \sqrt{\mathcal{D}_T(u)}.$$

Similarly, we have

$$\|\mathbf{P}v\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_\xi(\dot{B}^s_{2,r})} \lesssim \sqrt{\mathcal{D}_T(v)}.$$

From the above three estimates, we see that (2.14) holds true. Similarly, taking $X=B^s_{2,r}$, using (2.18) and exchanging the L^∞_T-norm and L^2_T-norm in the estimate of $I_{1,2}$, $I_{2,2}$ and $I_{3,2}$ in the procedure of the proof in Lemma 2.3 give rise to

$$\begin{aligned}
&\left\| 2^{sj}\left(\int_0^T \left|\left(\Delta_j \mathbf{\Gamma}(\mathbf{P}u,\{\mathbf{I}-\mathbf{P}\}v),\Delta_j h\right)_{\xi,x}\right| dt\right)^{\frac12}\right\|_{l^r} \\
&\lesssim \|h\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\|\{\mathbf{I}-\mathbf{P}\}v\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\|\mathbf{P}u\|^{\frac12}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(X)}+\|\{\mathbf{I}-\mathbf{P}\}v\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_\xi(B^s_{2,r})}\|\mathbf{P}u\|^{\frac12}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(X)}\right] \\
&\quad+\|h\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\|\mathbf{P}u\|^{\frac12}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_{2,r})}\|\{\mathbf{I}-\mathbf{P}\}v\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(X)}+\|\mathbf{P}u\|^{\frac12}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_{2,r})}\|\{\mathbf{I}-\mathbf{P}\}v\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_\xi(X)}\right] \\
&\lesssim \|h\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\|\{\mathbf{I}-\mathbf{P}\}v\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\|\mathbf{P}u\|^{\frac12}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_{2,r})}\right] \lesssim \|h\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\sqrt{\mathcal{D}_T(v)}\sqrt{\mathcal{E}_T(u)}\right],
\end{aligned}$$

which is (2.15). Then we take $X=B^s_{2,r}$, use (2.18) and exchange the L^∞_T-norm and L^2_T-norm in the estimate of $I_{1,1}$, $I_{2,1}$ and $I_{3,1}$ to obtain

$$\left\| 2^{sj}\left(\int_0^T \left|\left(\Delta_j\mathbf{\Gamma}(\{\mathbf{I}-\mathbf{P}\}u,\mathbf{P}v),\Delta_j h\right)_{\xi,x}\right| dt\right)^{\frac12}\right\|_{l^r}$$
$$\lesssim \|h\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\|\mathbf{P}v\|^{\frac12}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_{2,r})}\|\{\mathbf{I}-\mathbf{P}\}u\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_\xi(X)}+\|\mathbf{P}v\|^{\frac12}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_{2,r})}\|\{\mathbf{I}-\mathbf{P}\}u\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(X)}\right]$$
$$+\|h\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\|\{\mathbf{I}-\mathbf{P}\}u\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_\xi(B^s_{2,r})}\|\mathbf{P}v\|^{\frac12}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(X)}+\|\{\mathbf{I}-\mathbf{P}\}u\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\|\mathbf{P}v\|^{\frac12}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(X)}\right]$$
$$\lesssim \|h\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\|\mathbf{P}v\|^{\frac12}_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_{2,r})}\|\{\mathbf{I}-\mathbf{P}\}u\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\right]\lesssim \|h\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\sqrt{\mathcal{E}_T(v)}\sqrt{\mathcal{D}_T(u)}\right],$$

which is (2.16). Finally, due to Lemma 2.4 with $X=B^s_{2,r}$, it follows that

$$\left\| 2^{js}\left(\int_0^T \left|\left(\Delta_j\mathbf{\Gamma}(\{\mathbf{I}-\mathbf{P}\}u,\{\mathbf{I}-\mathbf{P}\}v),\Delta_j h\right)_{\xi,x}\right| dt\right)^{\frac12}\right\|_{l^r}$$
$$\lesssim \|h\|^{\frac12}_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\left[\sqrt{\mathcal{E}_T(v)}\sqrt{\mathcal{D}_T(u)}+\sqrt{\mathcal{D}_T(v)}\sqrt{\mathcal{E}_T(u)}\right],$$

which is (2.17). Hereto the proof is completed. □

As an immediate corollary of Lemma 2.5, we have

Corollary 2.1. *Let $0<T\le\infty$, $1\le r\le 2$, $s\ge 3/2$ ($s=3/2$ only if $r=1$). We have*

$$\left\| 2^{js}\left(\int_0^T \left|\left(\Delta_j\mathbf{\Gamma}(u,u),\Delta_j\{\mathbf{I}-\mathbf{P}\}u\right)_{\xi,x}\right| dt\right)^{\frac12}\right\|_{l^r(j\ge -1)}\lesssim\sqrt{\mathcal{E}_T(u)}\mathcal{D}_T(u), \qquad (2.19)$$

where $\mathcal{E}_T(u)$ and $\mathcal{D}_T(u)$ are given in (1.22) and (1.23).

Proof. We split $\mathbf{\Gamma}(u,u)$ as

$$\mathbf{\Gamma}(u,u)=\mathbf{\Gamma}(\mathbf{P}u,\mathbf{P}u)+\mathbf{\Gamma}(\mathbf{P}u,\{\mathbf{I}-\mathbf{P}\}u)+\mathbf{\Gamma}(\{\mathbf{I}-\mathbf{P}\}u,\mathbf{P}u)+\mathbf{\Gamma}(\{\mathbf{I}-\mathbf{P}\}u,\{\mathbf{I}-\mathbf{P}\}u). \qquad (2.20)$$

The desired result follows by using Lemma 2.5 with $v=u$. □

Lemma 2.6. *Let $\zeta=\zeta(\xi)\in\mathscr{S}(\mathbb{R}^3_\xi)$, $T>0$, $1\le r\le 2$ and $s\ge 3/2$ ($s=3/2$ only if $r=1$). We have the following estimates,*

$$\left\|2^{js_1}\left\|\left(\zeta,\Delta_j\mathbf{\Gamma}(u,u)\right)_\xi\right\|_{L^2_TL^2_x}\right\|_{l^r(j\ge -1)}\lesssim\mathcal{E}_T(u)\mathcal{D}_T(u),\quad s>3/2,\quad 0<s_1\le s-1, \qquad (2.21)$$

$$\left\|2^{js_2}\left\|\left(\zeta,\Delta_j\mathbf{L}\{\mathbf{I}-\mathbf{P}\}u\right)_\xi\right\|_{L^2_TL^2_x}\right\|_{l^r(j\ge -1)}\le\|\{\mathbf{I}-\mathbf{P}\}u\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^{s_2}_{2,r})}\quad s_2>0, \qquad (2.22)$$

Please cite this article in press as: Z. Liu, H. Tang, Global well-posedness for the Fokker–Planck–Boltzmann equation in Besov–Chemin–Lerner type spaces, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.02.031

$$\left\| 2^{js_2} \left\| \left(\zeta, \Delta_j \mathbf{L}_{FP}\{\mathbf{I}-\mathbf{P}\}u\right)_\xi \right\|_{L^2_T L^2_x} \right\|_{l^r(j\geq -1)} \leq \|\{\mathbf{I}-\mathbf{P}\}u\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^{s_2}_{2,r})} \quad s_2>0. \tag{2.23}$$

Proof. To prove (2.21), we first consider the general case $\left\| 2^{js_1} \left\| \left(\zeta, \Delta_j \mathbf{\Gamma}(u,v)\right)_\xi \right\|_{L^2_T L^2_x} \right\|_{l^r}$. In a very similar way that we obtain (2.10), we have

$$\left\| \left(\zeta, \Delta_j \mathbf{\Gamma}(u,v)\right)_\xi \right\|_{L^2_T L^2_x} \lesssim \left(\int_0^T \int_{\mathbb{R}^9} |\xi-\xi_*|^\gamma \left|\Delta_j[u_* v]\right|^2 d\xi_* dx d\xi dt \right)^{\frac{1}{2}}.$$

As what we have obtained in Lemmas 2.3 and 2.4, from the above inequality, we have

$$\begin{aligned} &\left\| 2^{js_1} \left\| \left(\Delta_j \mathbf{\Gamma}(u,v),\zeta\right)_\xi \right\|_{L^2_T L^2_x} \right\|_{l^r} \\ &\lesssim \left(\sum_{j\geq -1} 2^{js_1 r} \left(\int_0^T \int_{\mathbb{R}^9} |\xi-\xi_*|^\gamma \left|\Delta_j[u_* v]\right|^2 d\xi_* dx d\xi dt \right)^{\frac{r}{2}} \right)^{\frac{1}{r}} \\ &\lesssim \|v\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^{s_1}_{2,r})} \|u\|_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi(X)} + \|v\|_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi(B^{s_1}_{2,r})} \|u\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(X)} \\ &\quad + \|u\|_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi(B^{s_1}_{2,r})} \|v\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(X)} + \|u\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^{s_1}_{2,r})} \|v\|_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi(X)}, \end{aligned} \tag{2.24}$$

where X denotes $B^s_{2,r}$ or $\dot{B}^s_{2,r}$. Similar to Corollary 2.1, using (2.20) and Lemma 2.5 yields

$$\left\| 2^{js_1} \left\| \left(\zeta, \Delta_j \mathbf{\Gamma}(u,u)\right)_\xi \right\|_{L^2_T L^2_x} \right\|_{l^r} \lesssim \left\| 2^{js} \left\| \left(\zeta, \Delta_j \mathbf{\Gamma}(u,u)\right)_\xi \right\|_{L^2_T L^2_x} \right\|_{l^r} \lesssim \mathcal{E}_T(u)\mathcal{D}_T(u),$$

which is (2.21).

To prove (2.22), we notice that

$$\begin{aligned} \mathbf{L}\{\mathbf{I}-\mathbf{P}\}u &= \mathbf{M}^{-\frac{1}{2}} \left[Q\left(\mathbf{M}, \mathbf{M}^{\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\}u\right) + Q\left(\mathbf{M}^{\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\}u, \mathbf{M}\right) \right] \\ &= \mathbf{\Gamma}\left(\mathbf{M}^{\frac{1}{2}}, \{\mathbf{I}-\mathbf{P}\}u\right) + \mathbf{\Gamma}\left(\{\mathbf{I}-\mathbf{P}\}u, \mathbf{M}^{\frac{1}{2}}\right). \end{aligned}$$

Therefore the desired result comes from a similar estimate as for (2.17).

For (2.23), since the velocity-coordinate projector $\mathbf{P}$ is bounded uniformly in t and x, we can use (1.9) with integrating by part and then use $\zeta \in \mathscr{S}(\mathbb{R}^3_\xi)$ to absorb the velocity polynomials and velocity derivatives. In this way, we can obtain (2.23), and the details are omitted. □

3. Local well-posedness

In this section, we will prove the local existence of (1.6). We construct an iterating sequence $\{u_k\}_{k\geq 1}$ with $u_1(t,\xi,x)=0$ by solving the following equations iteratively:

$$\begin{cases} \partial_t u_{k+1} + \xi \cdot \nabla_x u_{k+1} + \nu(\xi) u_{k+1} - \epsilon \mathbf{L}_{FP} u_{k+1} \\ \qquad\qquad = K u_k + \mathbf{\Gamma}_{gain}(u_k, u_k) - \mathbf{\Gamma}_{loss}(u_k, u_{k+1}), \\ u_{k+1}(0, \xi, x) = u_0(\xi, x), \end{cases} \tag{3.1}$$

where $\nu(\xi)$ and K are given in (1.12)–(1.15). The proof includes the following steps.

3.1. Uniform bound of the approximate solutions

Lemma 3.1. *Let $1 \le r \le 2$ and $s \ge 3/2$ ($s = 3/2$ only if $r = 1$). For any $u_0(\xi, x) \in \widetilde{L}^2_\xi(B^s_{2,r})$ satisfying $\|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})} C^2 \ll 1$ for some $C \gg 1$, there is a $T_{u_0} = \|u_0\|^2_{\widetilde{L}^2_\xi(B^s_{2,r})} C^2 > 0$ such that for any $k \ge 1$, $\{u_k\} \subset \widetilde{L}^\infty_{T_{u_0}} \widetilde{L}^2_\xi(B^s_{2,r}) \bigcap \widetilde{L}^2_{T_{u_0}} \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})$ is uniformly bounded and satisfies*

$$\mathcal{E}_{T_{u_0}}(u_k) + \mathcal{H}_{T_{u_0}}(u_k) < 2\|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})}. \tag{3.2}$$

Proof. We first apply Δ_j $(j \ge -1)$ to (3.1) to obtain

$$\partial_t \Delta_j u_{k+1} + \xi \cdot \nabla_x \Delta_j u_{k+1} + \nu(\xi) \Delta_j u_{k+1} - \epsilon \mathbf{L}_{FP} \Delta_j u_{k+1} \\ = K \Delta_j u_k + \Delta_j \mathbf{\Gamma}_{gain}(u_k, u_k) - \Delta_j \mathbf{\Gamma}_{loss}(u_k, u_{k+1}).$$

Multiplying both sides of the above equation by $\Delta_j u_{k+1}$, integrating the resulting equation over $\mathbb{R}^3_\xi \times \mathbb{R}^3_x$ and using (2.4) give rise to

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j u_{k+1}\|^2_{L^2_\xi L^2_x} + \|\Delta_j u_{k+1}\|^2_{L^2_{\xi,\nu} L^2_x} \\ \le \left| \left(K \Delta_j u_k + \Delta_j \mathbf{\Gamma}_{gain}(u_k, u_k) - \Delta_j \mathbf{\Gamma}_{loss}(u_k, u_{k+1}), \Delta_j u_{k+1} \right)_{\xi,x} \right|.$$

For $T > 0$, integrating the above inequality on $[0, t]$ with $0 < t \le T$ and using (2.3) yield

$$\|\Delta_j u_{k+1}(t)\|^2_{L^2_\xi L^2_x} - \|\Delta_j u_{k+1}(0)\|^2_{L^2_\xi L^2_x} + 2\int_0^t \|\Delta_j u_{k+1}\|^2_{L^2_{\xi,\nu} L^2_x} dt' \\ \le 2\int_0^t \|\Delta_j u_k\|_{L^2_\xi L^2_x} \|\Delta_j u_{k+1}\|_{L^2_\xi L^2_x} dt' + 2\int_0^t \left| \left(\Delta_j \mathbf{\Gamma}_{gain}(u_k, u_k), \Delta_j u_{k+1} \right)_{\xi,x} \right| dt' \\ + 2\int_0^t \left| \left(\Delta_j \mathbf{\Gamma}_{loss}(u_k, u_{k+1}), \Delta_j u_{k+1} \right)_{\xi,x} \right| dt'. \tag{3.3}$$

Taking the square root on both sides of (3.3), multiplying both sides of the resulting inequality by 2^{js} and then taking l^r norm, we have

Please cite this article in press as: Z. Liu, H. Tang, Global well-posedness for the Fokker–Planck–Boltzmann equation in Besov–Chemin–Lerner type spaces, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.02.031

$$
\begin{aligned}
&\|u_{k+1}\|_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi(B^s_{2,r})} + \sqrt{2}\|u_{k+1}\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})} \\
\leq &\|u_{k+1}(0)\|_{\widetilde{L}^2_\xi(B^s_{2,r})} + \sqrt{T}C\left(\|u_k\|_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi(B^s_{2,r})} + \|u_{k+1}\|_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi(B^s_{2,r})}\right) \\
&+ C\left\| 2^{js}\left(\int_0^T \left|\left(\Delta_j \mathbf{\Gamma}_{gain}(u_k,u_k), \Delta_j u_{k+1}\right)_{\xi,x}\right| dt\right)^{\frac{1}{2}}\right\|_{l^r} \\
&+ C\left\| 2^{js}\left(\int_0^T \left|\left(\Delta_j \mathbf{\Gamma}_{loss}(u_k,u_{k+1}), \Delta_j u_{k+1}\right)_{\xi,x}\right| dt\right)^{\frac{1}{2}}\right\|_{l^r}.
\end{aligned}
\tag{3.4}
$$

Recalling (1.22) and (1.28), it follows from (3.4) and Lemma 2.4 that

$$
\begin{aligned}
&\mathcal{E}_T(u_{k+1}) + \mathcal{H}_T(u_{k+1}) \\
\leq &\|u(0)\|_{\widetilde{L}^2_\xi(B^s_{2,r})} + \sqrt{T}C\mathcal{E}_T(u_k) + \sqrt{T}C\mathcal{E}_T(u_{k+1}) + C\sqrt{\mathcal{E}_T(u_k)}\sqrt{\mathcal{H}_T(u_k)}\sqrt{\mathcal{H}_T(u_{k+1})} \\
&+ C\sqrt{\mathcal{E}_T(u_k)}\mathcal{H}_T(u_{k+1}) + C\sqrt{\mathcal{E}_T(u_{k+1})}\sqrt{\mathcal{H}_T(u_k)}\sqrt{\mathcal{H}_T(u_{k+1})} \\
\leq &\|u(0)\|_{\widetilde{L}^2_\xi(B^s_{2,r})} + \sqrt{T}C\mathcal{E}_T(u_k) + \sqrt{T}C\mathcal{E}_T(u_{k+1}) + \frac{C}{4\sqrt{T}}\mathcal{E}_T(u_k)\mathcal{H}_T(u_k) \\
&+ C\left(\sqrt{T} + \sqrt{\mathcal{E}_T(u_k)}\right)\mathcal{H}_T(u_{k+1}) + C\sqrt{\mathcal{H}_T(u_k)}\left\{\mathcal{H}_T(u_{k+1}) + \mathcal{E}_T(u_{k+1})\right\}.
\end{aligned}
\tag{3.5}
$$

If (3.2) is true when $k = l$ for $T = T_{u_0} = \|u_0\|^2_{\widetilde{L}^2_\xi(B^s_{2,r})}C^2$, we now prove that it also holds true for $k = l+1$. Due to the smallness of u_0, we have

$$
1 > \Omega \triangleq 1 - \sqrt{T_{u_0}}C - C\sqrt{\mathcal{E}_{T_{u_0}}(u_l)} - C\sqrt{\mathcal{H}_{T_{u_0}}(u_l)} > 1 - \|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})}C_1 - C_1\sqrt{\|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})}} > 0,
$$

and

$$
1 - \Omega = \sqrt{T_{u_0}}C + C\sqrt{\mathcal{E}_{T_{u_0}}(u_l)} + C\sqrt{\mathcal{H}_{T_{u_0}}(u_l)} \ll 1.
$$

For $0 < T_{u_0} = C^2\|u_0\|^2_{\widetilde{L}^2_\xi(B^s_{2,r})} \ll 1$, from (3.5) one can infer

$$
\begin{aligned}
&\Omega\left\{\mathcal{E}_{T_{u_0}}(u_{l+1}) + \mathcal{H}_{T_{u_0}}(u_{l+1})\right\} \\
\leq &\left(\Omega + C\sqrt{\mathcal{E}_{T_{u_0}}(u_l)}\right)\mathcal{E}_{T_{u_0}}(u_{l+1}) + \Omega\mathcal{H}_{T_{u_0}}(u_{l+1}) \\
\leq &\|u(0)\|_{\widetilde{L}^2_\xi(B^s_{2,r})} + \alpha\mathcal{E}_{T_{u_0}}(u_l) + \beta\mathcal{E}_{T_{u_0}}(u_l),
\end{aligned}
\tag{3.6}
$$

where $\alpha = C^2\|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})} \ll 1$ and $\beta = \frac{\mathcal{H}_{T_{u_0}}(u_l)}{4\|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})}} < 1/2$. Due to the smallness of u_0, we have

$$\mathcal{E}_{T_{u_0}}(u_{l+1})+\mathcal{H}_{T_{u_0}}(u_{l+1})\le\frac{1+\alpha+2\beta}{\Omega}\|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})}<2\|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})} \tag{3.7}$$

which is (3.2). □

3.2. Convergence of the approximate solutions

Now we prove that $\{u_k\}\subset L^\infty_{T_{u_0}}\left(\widetilde{L}^2_\xi(B^s_{2,r})\right)\bigcap L^2_{T_{u_0}}\left(\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})\right)$ is a Cauchy sequence. To show this, let $W_{k+m+1,k+1}=u_{k+m+1}-u_{k+1}$, which satisfies

$$\begin{cases}\partial_t W_{k+m+1,k+1}+\xi\cdot\nabla_x W_{k+m+1,k+1}+\nu(\xi)W_{k+m+1,k+1}-\epsilon\mathbf{L}_{FP}W_{k+m+1,k+1}\\ \qquad =KW_{k+m,k}+\mathbf{\Gamma}_{gain}(W_{k+m,k},u_{k+m})-\mathbf{\Gamma}_{loss}(W_{k+m,k},u_{k+1})\\ \qquad\quad +\mathbf{\Gamma}_{gain}(u_k,W_{k+m,k})-\mathbf{\Gamma}_{loss}(u_{k+m},W_{k+m+1,k+1}),\\ W_{k+1,m+1}(0,\xi,x)=0,\end{cases} \tag{3.8}$$

By taking the energy estimates in $L^2_{\xi,x}$ to (3.8) for each $\Delta_j W_{k+m+1,k+1}$ and using (2.4), Lemma 2.4 and (3.2), we find that

$$\begin{aligned}&\mathcal{E}_{T_{u_0}}(W_{k+m+1,k+1})+\mathcal{H}_{T_{u_0}}(W_{k+m+1,k+1})\\ &\le\sqrt{T_{u_0}}C\left(\mathcal{E}_{T_{u_0}}(W_{k+m,k})+\mathcal{E}_{T_{u_0}}(W_{k+m+1,k+1})\right)\\ &\quad+C\sqrt{\|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})}}\left(\sqrt{\mathcal{E}_{T_{u_0}}(W_{k+m,k})}+\sqrt{\mathcal{H}_{T_{u_0}}(W_{k+m,k})}\right)\sqrt{\mathcal{H}_{T_{u_0}}(W_{k+m+1,k+1})}\\ &\quad+C\sqrt{\|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})}}\left(\sqrt{\mathcal{E}_{T_{u_0}}(W_{k+m+1,k+1})}+\sqrt{\mathcal{H}_{T_{u_0}}(W_{k+m+1,k+1})}\right)\sqrt{\mathcal{H}_{T_{u_0}}(W_{k+m+1,k+1})}\\ &\le\delta\left[\mathcal{E}_{T_{u_0}}(W_{k+m,k})+\mathcal{H}_{T_{u_0}}(W_{k+m,k})\right]+\delta\left[\mathcal{E}_{T_{u_0}}(W_{k+m+1,k+1})+\mathcal{H}_{T_{u_0}}(W_{k+m+1,k+1})\right],\end{aligned} \tag{3.9}$$

where we have used the smallness of u_0 and T_{u_0} to deduce

$$\delta=\sqrt{T_{u_0}}C+2C\sqrt{\|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})}}\ll 1.$$

Equivalently, we have

$$\begin{aligned}&\mathcal{E}_{T_{u_0}}(W_{k+m+1,k+1})+\mathcal{H}_{T_{u_0}}(W_{k+m+1,k+1})\\ &\quad\le\frac{\delta}{1-\delta}\left[\mathcal{E}_{T_{u_0}}(W_{k+m,k})+\mathcal{H}_{T_{u_0}}(W_{k+m,k})\right].\end{aligned} \tag{3.10}$$

From the above estimate, for any $m\ge 1$, we have

$$\begin{aligned}\mathcal{E}_{T_{u_0}}(W_{k+m+1,k+1})+\mathcal{H}_{T_{u_0}}(W_{k+m+1,k+1})&\le\left(\frac{\delta}{1-\delta}\right)^{k-1}\left[\mathcal{E}_{T_{u_0}}(W_{2+m,2})+\mathcal{H}_{T_{u_0}}(W_{2+m,2})\right]\\ &\le C\left(\frac{\delta}{1-\delta}\right)^{k-1}\|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})}\xrightarrow{k\to\infty}0.\end{aligned}$$

For $1 \le r \le 2$ and any $m \ge 1$, applying the Minkowski's inequality yields

$$\|u_{k+m+1} - u_{k+1}\|_{L^\infty_{T_{u_0}}\left(\widetilde{L}^2_\xi(B^s_{2,r})\right)} + \|u_{k+m+1} - u_{k+1}\|_{L^2_{T_{u_0}}\left(\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})\right)} \xrightarrow{k\to\infty} 0. \tag{3.11}$$

In other words, $\{u_k\}$ is a Cauchy sequence in $L^\infty_{T_{u_0}}\left(\widetilde{L}^2_\xi(B^s_{2,r})\right) \bigcap L^2_{T_{u_0}}\left(\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})\right)$. Thus we can take $k \to \infty$ to obtain a local solution u to (1.6) in the sense of distribution.

3.3. Regularity and uniqueness of the solution

For the uniqueness of the solution, we can repeat the deduction as for (3.10) to obtain the estimate for two solutions u, v with initial data u_0, v_0, respectively. Actually, we have

$$\mathcal{E}_{T_{u_0}}(u - v) + \mathcal{H}_{T_{u_0}}(u - v) \lesssim \|u_0 - v_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})}, \tag{3.12}$$

which implies the uniqueness and the continuity of the solution map. From Lemma 2.2, we see that $u \in \widetilde{L}^\infty_{T_{u_0}} \widetilde{L}^2_\xi(B^s_{2,r}) \bigcap \widetilde{L}^2_{T_{u_0}} \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})$ and

$$\mathcal{E}_{T_{u_0}}(u) + \mathcal{H}_{T_{u_0}}(u) < 2\|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})}. \tag{3.13}$$

Now we consider the regularity of u with respect to time $t > 0$.

Lemma 3.2. *Let $1 \le r \le 2$ and $s \ge 3/2$ ($s = 3/2$ only if $r = 1$). Assume that $\|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})} \ll 1/C^2$ with $C \gg 1$. If $u(t, \xi, x)$ is the unique solution to (1.6) with $0 < t < T_{u_0} = C^2\|u_0\|^2_{\widetilde{L}^2_\xi(B^s_{2,r})}$, then we have the following properties:*

(i) *The function*

$$t \mapsto \left\| 2^{js} \left(\int_0^t \|\Delta_j u(t')\|^2_{L^2_{\xi,\nu} L^2_x} dt' \right)^{\frac{1}{2}} \right\|_{l^r}$$

is continuous for $0 < t < T_{u_0}$;

(ii) *The function $T \mapsto \mathcal{E}_T(u)$ is continuous for $0 < T < T_{u_0}$;*

(iii) *The function $T \mapsto \mathcal{D}_T(u)$ is continuous for $0 < T < T_{u_0}$;*

(iv) *The function $t \mapsto \|u(t)\|_{\widetilde{L}^2_\xi(B^s_{2,r})}$ is continuous for $0 < t < T_{u_0}$.*

Proof. To prove (i), for any $0 < t_1 < t_2 < T_{u_0}$, we need to show

$$\left\| 2^{js} \left(\int_{t_1}^{t_2} \|\Delta_j u(t')\|^2_{L^2_{\xi,\nu} L^2_x} dt' \right)^{\frac{1}{2}} \right\|_{l^r} \xrightarrow{t_2\to t_1} 0. \tag{3.14}$$

Because $\mathcal{H}_T(u) = \|u\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})} < \infty$ for any $0 < T < T_{u_0}$, we see that for any given $\eta > 0$, there exists a suitable $N > 0$ such that $M > N$ implies

$$\left[\sum_{j>M} 2^{jsr} \left(\int_{t_1}^{t_2} \|\Delta_j u(t')\|^2_{L^2_{\xi,\nu} L^2_x} dt'\right)^{\frac{r}{2}}\right]^{\frac{1}{r}} < \eta/2.$$

Hence for fixed $M > N$, we have

$$\left\| 2^{js} \left(\int_{t_1}^{t_2} \|\Delta_j u(t')\|^2_{L^2_{\xi,\nu} L^2_x} dt'\right)^{\frac{1}{2}} \right\|_{l^r} \le \left[\sum_{-1\le j\le M} 2^{jsr} \left(\int_{t_1}^{t_2} \|\Delta_j u(t')\|^2_{L^2_{\xi,\nu} L^2_x} dt'\right)^{\frac{r}{2}}\right]^{\frac{1}{r}} + \eta/2.$$

For the given η, there is a $\delta > 0$ such that when $0 < t_1 < t_2 < T_{u_0}$ and $|t_1 - t_2| < \delta$, there holds

$$\left[\sum_{-1\le j\le M} 2^{jsr} \left(\int_{t_2}^{t_1} \|\Delta_j u(t')\|^2_{L^2_{\xi,\nu} L^2_x} dt'\right)^{\frac{r}{2}}\right]^{\frac{1}{r}} \le \sum_{-1\le j\le M} 2^{js} \left(\int_{t_2}^{t_1} \|\Delta_j u(t')\|^2_{L^2_{\xi,\nu} L^2_x} dt'\right)^{\frac{1}{2}} < \eta/2.$$

Combining these inequalities, for any given $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < t_1 < t_2 < T_{u_0}$ and $|t_1 - t_2| < \delta$, we have

$$\left\| 2^{js} \left(\int_{t_1}^{t_2} \|\Delta_j u(t')\|^2_{L^2_{\xi,\nu} L^2_x} dt'\right)^{\frac{1}{2}} \right\|_{l^r} < \eta/2 + \eta/2 < \eta.$$

Thus we obtain (3.14).

To prove property (ii), we first show that the function $t \mapsto \|\Delta_j u\|_{L^2_\xi L^2_x}$ is continuous for $0 < t < T_{u_0}$. In fact, in a similar way that we obtain (3.3), from the equation in (1.6), we have

$$\begin{aligned}
&\|\Delta_j u(t_2)\|^2_{L^2_\xi L^2_x} - \|\Delta_j u(t_1)\|^2_{L^2_\xi L^2_x} + 2\int_{t_1}^{t_2} \|\Delta_j u\|^2_{L^2_{\xi,\nu} L^2_x} dt \\
&\quad \le 2C \int_{t_1}^{t_2} \|\Delta_j u\|^2_{L^2_\xi L^2_x} dt + 2\int_{t_1}^{t_2} \left|\left(\Delta_j \mathbf{\Gamma}(u,u), \Delta_j u\right)_{\xi,x}\right| dt,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|\Delta_j u(t_2)\|^2_{L^2_\xi L^2_x} - \|\Delta_j u(t_1)\|^2_{L^2_\xi L^2_x} \\
&\quad \le C \int_{t_1}^{t_2} \|\Delta_j u\|^2_{L^2_{\xi,\nu} L^2_x} dt + 2\int_{t_1}^{t_2} \left|\left(\Delta_j \mathbf{\Gamma}(u,u), \Delta_j u\right)_{\xi,x}\right| dt. \qquad (3.15)
\end{aligned}$$

Using (i) and Lemma 2.4 gives rise to the desired continuity. Now we prove (ii). We need to check that for any $0 < T_1 < T_2 < T_{u_0}$,

$$|\mathcal{E}_{T_2}(u) - \mathcal{E}_{T_1}(u)| \xrightarrow{T_2 \to T_1} 0. \tag{3.16}$$

By the definition of $\mathcal{E}_T(u)$ in (1.22), the Minkowski's inequality and the continuity of $t \mapsto \|\Delta_j u\|_{L^2_\xi L^2_x}$, we have

$$\begin{aligned}|\mathcal{E}_{T_2}(u) - \mathcal{E}_{T_1}(u)| &\le \left\| 2^{js}\left(\sup_{0<t<T_2} \|\Delta_j u(t)\|_{L^2_\xi L^2_x} - \sup_{0<t<T_1} \|\Delta_j u(t)\|_{L^2_\xi L^2_x}\right)\right\|_{l^r} \\ &\le \left\| 2^{js}\left(\sup_{T_1<t<T_2} \|\Delta_j u(t)\|_{L^2_\xi L^2_x} - \|\Delta_j u(T_1)\|_{L^2_\xi L^2_x}\right)\right\|_{l^r}.\end{aligned}$$

Obviously, (3.16) can be derived from the above inequality.

For (iii), since $\mathcal{D}_T(u)$ is nondecreasing for $0 < T < T_{u_0}$, we have

$$\begin{aligned}|\mathcal{D}_{T_2}(u) - \mathcal{D}_{T_1}(u)| &= \left\| 2^{js}\|\Delta_j u\|_{L^2_{T_2} L^2_{\xi,\nu} L^2_x}\right\|_{l^r} - \left\| 2^{js}\|\Delta_j u\|_{L^2_{T_1} L^2_{\xi,\nu} L^2_x}\right\|_{l^r} \\ &\le \left\| 2^{js}\left(\int_{T_1}^{T_2} \|\Delta_j u(t')\|^2_{L^2_{\xi,\nu} L^2_x} dt'\right)^{\frac{1}{2}}\right\|_{l^r} \xrightarrow[by (1)]{T_2 \to T_1} 0.\end{aligned}$$

Finally, we prove (iv). In view of (3.15) and Lemma 2.4, for $\widetilde{T} = t_2 - t_1$, we have

$$\begin{aligned}\|u(t_2)\|_{L^2_\xi(B^s_{2,r})} - \|u(t_1)\|_{L^2_\xi(B^s_{2,r})} &\lesssim \left\| 2^{js}\left(\int_{t_1}^{t_2} \|\Delta_j u(t)\|^2_{L^2_{\xi,\nu} L^2_x} dt\right)^{\frac{1}{2}}\right\|_{l^r} \left(1 + \sqrt{\mathcal{E}_{\widetilde{T}}(u)}\sqrt{\mathcal{D}_{\widetilde{T}}(u)}\right) \\ &\lesssim \left\| 2^{js}\left(\int_{t_1}^{t_2} \|\Delta_j u(t)\|^2_{L^2_{\xi,\nu} L^2_x} dt\right)^{\frac{1}{2}}\right\|_{l^r} \left(1 + \sqrt{\mathcal{E}_{T_{u_0}}(u)}\sqrt{\mathcal{D}_{T_{u_0}}(u)}\right).\end{aligned}$$

Therefore we obtain (iv) by combining (i) and (3.13). □

Using (3.11) and (iv) in Lemma 3.2, we see that the solution u to the problem (1.6) belongs to $C([0,T); \widetilde{L}^2_\xi(B^s_{2,r}))$. Finally, if $f_0(x,\xi) = \mathbf{M} + \sqrt{\mathbf{M}}u_0(x,\xi) \ge 0$, then the positivity of $f(t,\xi,x) = \mathbf{M} + \sqrt{\mathbf{M}}u(t,\xi,x)$ is standard (see [25] for example).

In conclusion, we get the following Theorem.

Theorem 3.1. *Let $1 \le r \le 2$ and $s \ge 3/2$ ($s = 3/2$ only if $r = 1$) and $u_0 \in \widetilde{L}^2_\xi(B^s_{2,r})$. If $\|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})} \ll 1/C^2$ for some $C \gg 1$, then there exists a $T_{u_0} = C^2\|u_0\|^2_{\widetilde{L}^2_\xi(B^s_{2,r})} > 0$ such that for any $\epsilon > 0$, (1.6) has a unique solution*

Please cite this article in press as: Z. Liu, H. Tang, Global well-posedness for the Fokker–Planck–Boltzmann equation in Besov–Chemin–Lerner type spaces, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.02.031

$$u \in C([0, T_{u_0}]; \widetilde{L}^2_\xi(B^s_{2,r}))$$

satisfying

$$\mathcal{E}_T(u) + \mathcal{H}_T(u) < 2\|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})}, \quad \textit{for} \ \ 0 \le T < T_{u_0}. \tag{3.17}$$

Moreover, if $f_0(x,\xi) = \mathbf{M} + \sqrt{\mathbf{M}}u_0(x,\xi) \ge 0$, *then* $f(t,\xi,x) = \mathbf{M} + \sqrt{\mathbf{M}}u(t,\xi,x) \ge 0$.

4. Global well-posedness

In this section, we give the proof of Theorem 1.1. To begin with, we estimate the macroscopic dissipation rate.

4.1. Estimate on macroscopic dissipation

Lemma 4.1. *Let* $1 \le r \le 2$ *and* $s \ge 3/2$ *(*$s = 3/2$ *only if* $r = 1$*). For any* $0 < T \le \infty$ *and* $\epsilon > 0$, *the macroscopic part* (a, b, c) *determined by* u *satisfies*

$$\begin{aligned} \|\nabla_x(a,b,c)\|_{\widetilde{L}^2_T(B^{s-1}_{2,r})} \lesssim & \|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})} + (1+\epsilon)\mathcal{E}_T(u) \\ & + (2+\epsilon)\|\{\mathbf{I}-\mathbf{P}\}u\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})} + \mathcal{E}_T(u)\mathcal{D}_T(u). \end{aligned} \tag{4.1}$$

Proof. Following the steps as in [43], we use the moment functions $\Theta = (\Theta_{mj}(\cdot))_{3\times 3}$ and $\Lambda = (\Lambda_j(\cdot))_{1\le j\le 3}$ defined as

$$\Theta_{mj}(u) = \Big((\xi_m\xi_j - 1)\mathbf{M}^{1/2}, u\Big)_\xi, \quad \Lambda_j(u) = \frac{1}{10}\Big((|\xi|^2 - 5)\xi_j\mathbf{M}^{1/2}, u\Big)_\xi \tag{4.2}$$

to obtain the fluid-type system for the coefficient functions (a, b, c) as

$$\begin{cases} \partial_t a + \nabla_x \cdot b = 0, \\ \partial_t b + \nabla_x(a+2c) + \nabla_x \cdot \Theta(\{\mathbf{I}-\mathbf{P}\}u) + \epsilon b = 0, \\ \partial_t c + \dfrac{1}{3}\nabla_x \cdot b + \dfrac{5}{3}\nabla_x \cdot \Lambda(\{\mathbf{I}-\mathbf{P}\}u) + 2\epsilon c = 0. \\ \partial_t\left[\Theta_{mj}(\{\mathbf{I}-\mathbf{P}\}u) + 2\delta_{mj}c\right] + \partial_{x_m}b_j + \partial_{x_j}b_m + 4\epsilon\delta_{mj}c = \Theta_{mj}(\mathbb{r} + \mathbb{h}), \\ \partial_t\Lambda_j(\{\mathbf{I}-\mathbf{P}\}u) + \partial_{x_j}c = \Lambda_j(\mathbb{r} + \mathbb{h}), \end{cases} \tag{4.3}$$

with

$$\mathbb{r} = -\xi \cdot \nabla_x\{\mathbf{I}-\mathbf{P}\}u, \quad \mathbb{h} = \mathbf{\Gamma}(u,u) + \epsilon\mathbf{L}_{FP}\{\mathbf{I}-\mathbf{P}\}u - \mathbf{L}\{\mathbf{I}-\mathbf{P}\}u. \tag{4.4}$$

Now we apply $\Delta_n (n \ge -1)$ to (4.3) to obtain

$$\begin{cases} \partial_t \Delta_n a + \nabla_x \cdot \Delta_n b = 0, \\ \partial_t \Delta_n b + \nabla_x \Delta_n (a + 2c) + \nabla_x \cdot \Theta(\{\mathbf{I} - \mathbf{P}\}\Delta_n u) + \epsilon \Delta_n b = 0, \\ \partial_t \Delta_n c + \frac{1}{3}\nabla_x \Delta_n \cdot b + \frac{5}{3}\nabla_x \cdot \Lambda(\{\mathbf{I} - \mathbf{P}\}\Delta_n u) + 2\epsilon \Delta_n c = 0, \\ \partial_t \left[\Theta_{mj}(\{\mathbf{I} - \mathbf{P}\}\Delta_n u) + 2\delta_{mj}\Delta_n c\right] + \partial_{x_m}\Delta_n b_j + \partial_{x_j}\Delta_n b_m + 4\epsilon\delta_{mj}\Delta_n c = \Theta_{mj}(\Delta_n \mathbb{r} + \Delta_n \mathbb{h}), \\ \partial_t \Lambda_j(\{\mathbf{I} - \mathbf{P}\}\Delta_n u) + \partial_{x_j}\Delta_n c = \Lambda_j(\Delta_n \mathbb{r} + \Delta_n \mathbb{h}). \end{cases}$$

For each $n \geq -1$, we define the temporal interactive functional $\mathcal{E}_n^{int}(u(t))$ as the linear combination of the following terms

$$\mathcal{E}_n^{int}(u(t)) = \mathcal{I}_n^c(u(t)) + \kappa_1 \mathcal{I}_n^b(u(t)) + \kappa_2 \mathcal{I}_n^{a,b}(u(t)), \quad \forall t > 0, \tag{4.5}$$

where

$$\begin{cases} \mathcal{I}_n^{a,b}(u(t)) = \sum_{m=1}^{3} \left(\partial_{x_m}\Delta_n a, \Delta_n b_m\right)_x, \\ \mathcal{I}_n^b(u(t)) = \sum_{m,j=1}^{3} \left(\partial_{x_m}\Delta_n b_j + \partial_{x_j}\Delta_n b_m, \Theta_{mj}(\{\mathbf{I} - \mathbf{P}\}\Delta_n u)\right)_x, \\ \mathcal{I}_n^c(u(t)) = \sum_{j=1}^{3} \left(\Delta_n \partial_{x_j} c, \Lambda_j(\{\mathbf{I} - \mathbf{P}\}\Delta_n u)\right)_x, \end{cases} \tag{4.6}$$

and κ_1, κ_2 will be chosen suitably later. Modifying the proof for Theorem 2.2 in [43], for any $\eta > 0$, we have:

$$\begin{aligned} \frac{d}{dt}\mathcal{I}_n^b(u(t)) + \frac{1}{2}\|\Delta_n \nabla_x b\|_{L_x^2}^2 &\leq C\eta \|\Delta_n \nabla_x (a, c)\|_{L_x^2}^2 + C\eta\epsilon^2 \|\Delta_n b\|_{L_x^2}^2 \\ &\quad + C_\eta \|\nabla_x \{\mathbf{I} - \mathbf{P}\}\Delta_n u\|_{L_{\xi,v}^2 L_x^2}^2 + C_\eta \sum_{m,j=1}^{3} \|\Theta_{m,j}(\Delta_n \mathbb{h})\|_{L_x^2}^2, \end{aligned} \tag{4.7}$$

$$\begin{aligned} \frac{d}{dt}\mathcal{I}_n^c(u(t)) + \frac{1}{2}\|\Delta_n \nabla_x c\|^2 &\leq 3\eta \|\Delta_n \nabla_x b\|_{L_x^2}^2 + 12\eta\epsilon^2 \|\Delta_n c\|_{L_x^2}^2 \\ &\quad + C_\eta \|\nabla_x \{\mathbf{I} - \mathbf{P}\}\Delta_n u\|_{L_{\xi,v}^2 L_x^2}^2 + C_\eta \sum_{j=1}^{3} \|\Lambda_j(\Delta_n \mathbb{h})\|_{L_x^2}^2, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} \frac{d}{dt}\mathcal{I}_n^{a,b}(u(t)) + \frac{1}{2}\|\Delta_n \nabla_x a\|_{L_x^2}^2 &\lesssim \|\Delta_n \nabla_x (b, c)\|_{L_x^2}^2 \\ &\quad + \epsilon^2 \|\Delta_n b\|_{L_x^2}^2 + C\|\nabla_x \{\mathbf{I} - \mathbf{P}\}\Delta_n u\|_{L_{\xi,v}^2 L_x^2}^2. \end{aligned} \tag{4.9}$$

Now we choose $0 < \eta \ll \kappa_2 \ll \kappa_1 \ll 1$ such that the first terms on the right-hand side of (4.7), (4.8) and (4.9) can be absorbed by the second terms on the left-hand side of (4.7), (4.8) and (4.9). Then we arrive at

$$\frac{d}{dt}\mathcal{E}_n^{int}(u(t)) + \lambda\|\Delta_n\nabla_x(a,b,c)\|_{L_x^2}^2 \lesssim \epsilon^2\|\Delta_n(b,c)\|_{L_x^2}^2 + \|\Delta_n\nabla_x\{\mathbf{I}-\mathbf{P}\}u\|_{L_{\xi,\nu}^2L_x^2}^2 + \sum_{j=1}^{3}\|\Lambda_j(\Delta_n\mathbb{h})\|_{L_x^2}^2 + \sum_{m,j=1}^{3}\|\Theta_{m,j}(\Delta_n\mathbb{h})\|_{L_x^2}^2.$$

For any $T > 0$, one can integrate the above inequality with respect to t over $[0, T]$ and take the square roots of both sides of the resulting inequality to obtain that

$$\begin{aligned}\left(\int_0^T \|\Delta_n\nabla_x(a,b,c)\|_{L_x^2}^2 dt\right)^{1/2} &\lesssim \epsilon\|\Delta_n(b,c)\|_{L_T^2L_x^2} + \sqrt{|\mathcal{E}_n^{int}(u(T))|} + \sqrt{|\mathcal{E}_n^{int}(u(0))|} \\ &\quad + \|\Delta_n\nabla_x\{\mathbf{I}-\mathbf{P}\}u\|_{L_T^2L_{\xi,\nu}^2L_x^2} + \sum_{j=1}^{3}\|\Lambda_j(\Delta_n\mathbb{h})\|_{L_T^2L_x^2} + \sum_{m,j=1}^{3}\|\Theta_{m,j}(\Delta_n\mathbb{h})\|_{L_T^2L_x^2}. \end{aligned} \tag{4.10}$$

Multiplying both sides of (4.10) by $2^{n(s-1)}$ and then taking l^r norm for $n \geq -1$ yield

$$\begin{aligned}\|\nabla_x(a,b,c)\|_{\widetilde{L}_T^2(B_{2,r}^{s-1})} &\lesssim \epsilon\mathcal{E}_T(u) + \left\|2^{n(s-1)}\sqrt{|\mathcal{E}_n^{int}(u(T))|}\right\|_{l^r} + \left\|2^{n(s-1)}\sqrt{|\mathcal{E}_n^{int}(u(0))|}\right\|_{l^r} \\ &\quad + \|\{\mathbf{I}-\mathbf{P}\}u\|_{\widetilde{L}_T^2\widetilde{L}_{\xi,\nu}^2(B_{2,r}^s)} + \left\|2^{n(s-1)}\|\Lambda(\Delta_n\mathbb{h})\|_{L_T^2L_x^2}\right\|_{l^r} + \left\|2^{n(s-1)}\|\Theta(\Delta_n\mathbb{h})\|_{L_T^2L_x^2}\right\|_{l^r}.\end{aligned}$$

From (4.5) and Cauchy–Schwarz inequality, we arrive at

$$|\mathcal{E}_n^{int}(u)| \lesssim \|\Delta_n\nabla_x(a,b,c)\|_{L_x^2}^2 + \|\Delta_n b\|_{L_x^2}^2 + \|\Theta(\{\mathbf{I}-\mathbf{P}\}\Delta_n u)\|_{L_x^2}^2 + \|\Lambda(\{\mathbf{I}-\mathbf{P}\}\Delta_n u)\|_{L_x^2}^2.$$

Using Cauchy–Schwarz inequality again in (4.2) and (4.6), for $0 \leq t \leq T$, we have

$$|\mathcal{E}_n^{int}(u)(t)| \lesssim \|\Delta_n\nabla_x(a,b,c)(t)\|_{L_x^2}^2 + \|\Delta_n b(t)\|_{L_x^2}^2 + \|\{\mathbf{I}-\mathbf{P}\}\Delta_n u(t)\|_{L_\xi^2L_x^2}^2.$$

Consequently, we have

$$\left\|2^{n(s-1)}\sqrt{|\mathcal{E}_n^{int}(u(T))|}\right\|_{l^r} \lesssim \mathcal{E}_T(u), \quad \left\|2^{n(s-1)}\sqrt{|\mathcal{E}_n^{int}(u(0))|}\right\|_{l^r} \lesssim \|u_0\|_{\widetilde{L}_\xi^2(B_{2,r}^s)}.$$

Using Lemma 2.6, we obtain

$$\begin{aligned}&\left\|2^{n(s-1)}\|\Lambda(\Delta_n\mathbb{h})\|_{L_T^2L_x^2}\right\|_{l^r} + \left\|2^{n(s-1)}\|\Theta(\Delta_n\mathbb{h})\|_{L_T^2L_x^2}\right\|_{l^r} \\ &\qquad \lesssim \mathcal{E}_T(u)\mathcal{D}_T(u) + (1+\epsilon)\,\|\{\mathbf{I}-\mathbf{P}\}u\|_{\widetilde{L}_T^2\widetilde{L}_{\xi,\nu}^2(B_{2,r}^s)}.\end{aligned}$$

Combining the above estimates, we finish the proof of Lemma 4.1. □

4.2. Global a priori estimate

With (4.1) in hand, we can establish the global *a priori* estimate for (1.6) as follows.

Lemma 4.2. *Let $1 \le r \le 2$ and $s \ge 3/2$ ($s = 3/2$ only if $r = 1$). For any $T > 0$ and $\epsilon > 0$, there are $C > 0$ and $\lambda > 0$, which do not depend on T and ϵ, such that*

$$\mathcal{E}_T(u) + \lambda \mathcal{D}_T(u) \le C\|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})} + C\left\{\sqrt{\mathcal{E}_T(u)} + \mathcal{E}_T(u)\right\}\mathcal{D}_T(u). \tag{4.11}$$

Proof. We apply the energy estimates in $L^2_{\xi,x}$ to (1.6) with using (2.2) and (2.4) for each $\Delta_j u$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_j u\|^2_{L^2_\xi L^2_x} + \lambda_0 \|\Delta_j \{\mathbf{I}-\mathbf{P}\}\Delta_j u\|^2_{L^2_{\xi,\nu} L^2_x} \le \left|\left(\Delta_j \mathbf{\Gamma}(u,u), \Delta_j\{\mathbf{I}-\mathbf{P}\}u\right)_{\xi,x}\right|.$$

Let $T > 0$. Integrating the above inequality on $[0, T]$ yields

$$\begin{aligned}
&\|\Delta_j u\|^2_{L^2_\xi L^2_x} + 2\lambda_0 \int_0^T \|\Delta_j\{\mathbf{I}-\mathbf{P}\}u\|^2_{L^2_{\xi,\nu}L^2_x}\,dt \\
&\qquad \le \|\Delta_j u(0)\|^2_{L^2_\xi L^2_x} + 2\int_0^T \left|\left(\Delta_j \mathbf{\Gamma}(u,u), \Delta_j\{\mathbf{I}-\mathbf{P}\}u\right)_{\xi,x}\right| dt.
\end{aligned}$$

Taking the square roots of both sides of the above inequality, multiplying both sides of the resulting inequality by 2^{js} and taking l^r norm with using (2.19), we have

$$\begin{aligned}
&\|u\|_{\widetilde{L}^\infty_T \widetilde{L}^2_\xi(B^s_{2,r})} + \sqrt{2}\sqrt{\lambda_0}\|\{\mathbf{I}-\mathbf{P}\}u\|_{\widetilde{L}^2_T \widetilde{L}^2_{\xi,\nu}(B^s_{2,r})} \\
&\quad \le C\|u(0)\|_{\widetilde{L}^2_\xi(B^s_{2,r})} + C\left\| 2^{js}\left(\int_0^T \left|\left(\Delta_j \mathbf{\Gamma}(u,u), \Delta_j\{\mathbf{I}-\mathbf{P}\}u\right)_{\xi,x}\right| dt\right)^{\frac{1}{2}}\right\|_{l^r} \\
&\quad \le C\|u(0)\|_{\widetilde{L}^2_\xi(B^s_{2,r})} + C\sqrt{\mathcal{E}_T(u)}\mathcal{D}_T(u), \quad \forall T > 0.
\end{aligned} \tag{4.12}$$

By setting $0 < \kappa_3 < \min\{\frac{\sqrt{2}\sqrt{\lambda_0}}{2(2+\epsilon)}, \frac{1}{2+2\epsilon}\}$ and using (4.1)$\times\kappa_3+$(4.12), we arrive at

$$\begin{aligned}
&(1-\kappa_3(1+\epsilon))\mathcal{E}_T(u) + \kappa_3\left(\|\nabla_x(a,b,c)\|_{\widetilde{L}^2_T(B^{s-1}_{2,r})} + \|\{\mathbf{I}-\mathbf{P}\}u\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\right) \\
&\qquad \le C\|u(0)\|_{\widetilde{L}^2_\xi(B^s_{2,r})} + \kappa_3\mathcal{E}_T(u)\mathcal{D}_T(u) + C\sqrt{\mathcal{E}_T(u)}\mathcal{D}_T(u) \\
&\qquad \le C\|u(0)\|_{\widetilde{L}^2_\xi(B^s_{2,r})} + C\left(\mathcal{E}_T(u) + \sqrt{\mathcal{E}_T(u)}\right)\mathcal{D}_T(u), \quad \forall T > 0,
\end{aligned}$$

which implies (4.11) due to the fact $\dfrac{\kappa_3}{1-\kappa_3(1+\epsilon)} < 2$. $\square$

4.3. Proof of global well-posedness

Now we are in the position to demonstrate Theorem 1.1. By Theorem 3.1, let u be the solution to (1.6). We define $T^{\star}_{u_0}$ to be its lifespan,

$$T^{\star}_{u_0} = \sup\left\{T > 0 : u \in C([0,T); \widetilde{L}^2_\xi(B^s_{2,r}))\right\}.$$

Proof for Theorem 1.1. Reset the constant C on the right of (4.11) to be $C_1 > 0$ (large enough). Fix C_1 and then choose u_0 small enough such that

$$2C_1\left\{\sqrt{\|u(0)\|_{L^2_\xi(B^s_{2,r})}} + \|u(0)\|_{L^2_\xi(B^s_{2,r})}\right\} < \frac{\lambda}{2}.$$

Define

$$\widetilde{T}_{u_0} = \sup\left\{T > 0 : \mathcal{E}_T(u) + \lambda\mathcal{D}_T(u) \le C_1\|u(0)\|_{\widetilde{L}^2_\xi(B^s_{2,r})}\right\}.$$

From Lemma 3.2 and (3.17) in Theorem 3.1, we have $\widetilde{T}_{u_0} > 0$. Using (4.11) and the standard continuity argument yields

$$\mathcal{E}_T(u) + \frac{\lambda}{2}\mathcal{D}_T(u) \le C_1\|u(0)\|_{\widetilde{L}^2_\xi(B^s_{2,r})}, \quad \forall T > 0.$$

Moreover, for any $T > 0$ and $0 \le t \le T$, we have

$$\sup_{0<t<T}\|u(t)\|_{\widetilde{L}^2_\xi(B^s_{2,r})} = \|u(t)\|_{L^\infty_T\widetilde{L}^2_\xi(B^s_{2,r})} \le \|u(t)\|_{\widetilde{L}^\infty_T\widetilde{L}^2_\xi(B^s_{2,r})} = \mathcal{E}_T(u),$$

and therefore (1.24) holds true for any $T > 0$. This means that $T^{\star}_{u_0} = \infty$. In other words, u exists globally. The proof is completed.

5. Uniform stability of the dependence on initial data

In this section, we prove Theorem 1.2. From Theorem 1.1, we let u, v be two solutions to (1.6) with initial data u_0, v_0, respectively. Then the Cauchy problem for difference $w = u - v$ is

$$\begin{cases} \partial_t w + \xi\cdot\nabla_x w + \mathbf{L}w = \mathbf{\Gamma}(w,u) + \mathbf{\Gamma}(v,w) + \epsilon\mathbf{L}_{FP}w, \\ w(0,x,\xi) = u_0 - v_0. \end{cases} \tag{5.1}$$

Let

$$\mathbf{P}w = a^w(t,x)\mathbf{M}^{\frac{1}{2}} + b^w(t,x)\cdot\xi\mathbf{M}^{\frac{1}{2}} + c^w(t,x)\left(|\xi|^2 - 3\right)\mathbf{M}^{\frac{1}{2}}.$$

Noticing that for any $\varphi \in \mathcal{N}$, there holds $(\mathbf{\Gamma}(w,u) + \mathbf{\Gamma}(v,w), \varphi)_\xi = 0$. Therefore for each $n \ge -1$, we can follow the same procedure as in Lemma 4.1 to obtain

Please cite this article in press as: Z. Liu, H. Tang, Global well-posedness for the Fokker–Planck–Boltzmann equation in Besov–Chemin–Lerner type spaces, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.02.031

$$\begin{cases} \partial_t \Delta_n a^w + \nabla_x \cdot \Delta_n b^w = 0, \\ \partial_t \Delta_n b^w + \nabla_x \Delta_n (a^w + 2c^w) + \nabla_x \cdot \Theta(\{\mathbf{I}-\mathbf{P}\}\Delta_n w) + \epsilon \Delta_n b^w = 0, \\ \partial_t \Delta_n c^w + \frac{1}{3}\nabla_x \Delta_n \cdot b^w + \frac{5}{3}\nabla_x \cdot \Lambda(\{\mathbf{I}-\mathbf{P}\}\Delta_n w) + 2\epsilon \Delta_n c^w = 0, \\ \partial_t \left[\Theta_{mj}(\{\mathbf{I}-\mathbf{P}\}\Delta_n w) + 2\delta_{mj}\Delta_n c^w\right] + \partial_{x_m}\Delta_n b_j^w + \partial_{x_j}\Delta_n b_m^w + 4\epsilon\delta_{mj}\Delta_n c^w \\ \quad = \Theta_{mj}(\Delta_n \mathbb{r}^w + \Delta_n \mathbb{h}^w), \\ \partial_t \Lambda_j(\{\mathbf{I}-\mathbf{P}\}\Delta_n w) + \partial_{x_j}\Delta_n c^w = \Lambda_j(\Delta_n \mathbb{r}^w + \Delta_n \mathbb{h}^w), \end{cases}$$

where

$$\mathbb{r}^w = -\xi \cdot \nabla_x \{\mathbf{I}-\mathbf{P}\}w, \quad \mathbb{h}^w = \mathbf{\Gamma}(w,u) + \mathbf{\Gamma}(v,w) + \epsilon \mathbf{L}_{FP}\{\mathbf{I}-\mathbf{P}\}w - \mathbf{L}\{\mathbf{I}-\mathbf{P}\}w,$$

and Θ, Λ are given in (4.2). Similarly, we define $\mathcal{E}_n^{int}(w(t))$ as (4.5) (4.6) with u replaced by w and a, b, c replaced by a^w, b^w, c^w. For any $T > 0$, we have

$$\begin{aligned} \|\nabla_x(a^w,b^w,c^w)\|_{\widetilde{L}_T^2(B_{2,r}^{s-1})} &\lesssim \|w_0\|_{\widetilde{L}_\xi^2(B_{2,r}^s)} + (1+\epsilon)\mathcal{E}_T(w) + \|\{\mathbf{I}-\mathbf{P}\}w\|_{\widetilde{L}_T^2\widetilde{L}_{\xi,\nu}^2(B_{2,r}^s)} \\ &\quad + \left\|2^{n(s-1)}\|\Lambda(\Delta_n \mathbb{h}^w)\|_{L_T^2 L_x^2}\right\|_{l^r} + \left\|2^{n(s-1)}\|\Theta(\Delta_n \mathbb{h}^w)\|_{L_T^2 L_x^2}\right\|_{l^r}. \end{aligned}$$

From Lemma 2.6, we have

$$\begin{aligned} &\left\|2^{n(s-1)}\|\Lambda(\Delta_n \mathbb{h}^w)\|_{L_T^2 L_x^2}\right\|_{l^r} + \left\|2^{n(s-1)}\|\Theta(\Delta_n \mathbb{h}^w)\|_{L_T^2 L_x^2}\right\|_{l^r} \\ &\lesssim \mathcal{E}_T(u)\mathcal{D}_T(w) + \mathcal{D}_T(u)\mathcal{E}_T(w) + \mathcal{E}_T(w)\mathcal{D}_T(v) + \mathcal{D}_T(w)\mathcal{E}_T(v) + (2+\epsilon)\|\{\mathbf{I}-\mathbf{P}\}w\|_{\widetilde{L}_T^2\widetilde{L}_{\xi,\nu}^2(B_{2,r}^s)} \\ &\lesssim [\mathcal{E}_T(u) + \mathcal{E}_T(v)]\mathcal{D}_T(w) + [\mathcal{D}_T(u) + \mathcal{D}_T(v)]\mathcal{E}_T(w) + (2+\epsilon)\|\{\mathbf{I}-\mathbf{P}\}w\|_{\widetilde{L}_T^2\widetilde{L}_{\xi,\nu}^2(B_{2,r}^s)}. \end{aligned}$$

Choose ρ such that

$$C\|u_0\|_{\widetilde{L}_\xi^2(B_{2,r}^s)} + C\|v_0\|_{\widetilde{L}_\xi^2(B_{2,r}^s)} < C\sqrt{\|u_0\|_{\widetilde{L}_\xi^2(B_{2,r}^s)}} + C\sqrt{\|v_0\|_{\widetilde{L}_\xi^2(B_{2,r}^s)}} < \rho \ll 1.$$

Therefore for any $T > 0$, we can infer from (1.24) that

$$\begin{aligned} \|\nabla_x(a^w,b^w,c^w)\|_{\widetilde{L}_T^2(B_{2,r}^{s-1})} &\lesssim \|w_0\|_{\widetilde{L}_\xi^2(B_{2,r}^s)} + (1+\epsilon)\mathcal{E}_T(w) \\ &\quad + \rho\,(\mathcal{E}_T(w) + \mathcal{D}_T(w)) + (2+\epsilon)\|\{\mathbf{I}-\mathbf{P}\}w\|_{\widetilde{L}_T^2\widetilde{L}_{\xi,\nu}^2(B_{2,r}^s)}. \end{aligned} \tag{5.2}$$

On the other hand, for any $T > 0$, by performing a spectral localization of (5.1) and taking the energy estimates in $L_{\xi,x}^2$ for each $\Delta_j w$, we obtain

$$
\begin{aligned}
&\|w\|_{\widetilde{L}^{\infty}_T\widetilde{L}^2_\xi(B^s_{2,r})}+\sqrt{2}\sqrt{\lambda_0}\|\{\mathbf{I}-\mathbf{P}\}w\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\\
&\le \|w(0)\|_{\widetilde{L}^2_\xi(B^s_{2,r})}+C\left\|2^{js}\left(\int_0^T\left|\left(\Delta_j\mathbf{\Gamma}(w,u)+\Delta_j\mathbf{\Gamma}(v,w),\Delta_j\{\mathbf{I}-\mathbf{P}\}w\right)_{\xi,x}\right|dt\right)^{\frac12}\right\|_{l^r}.
\end{aligned}
$$

For $\eta=\left(\|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})}+\|v_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})}\right)^{1/2}$, using Lemma 2.5 and (1.24) yields

$$
\begin{aligned}
&\|w\|_{\widetilde{L}^{\infty}_T\widetilde{L}^2_\xi(B^s_{2,r})}+\sqrt{2}\sqrt{\lambda_0}\|\{\mathbf{I}-\mathbf{P}\}w\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\\
&\le \|w(0)\|_{\widetilde{L}^2_\xi(B^s_{2,r})}+C\left(\sqrt{\mathcal{E}_T(u)}\sqrt{\mathcal{D}_T(w)}+\sqrt{\mathcal{D}_T(u)}\sqrt{\mathcal{E}_T(w)}\right)\sqrt{\mathcal{D}_T(w)}\\
&\quad+\left(\sqrt{\mathcal{E}_T(w)}\sqrt{\mathcal{D}_T(v)}+\sqrt{\mathcal{D}_T(w)}\sqrt{\mathcal{E}_T(v)}\right)\sqrt{\mathcal{D}_T(w)}\\
&\le \|w(0)\|_{\widetilde{L}^2_\xi(B^s_{2,r})}+C\left(\sqrt{\mathcal{E}_T(u)}+\sqrt{\mathcal{E}_T(v)}\right)\mathcal{D}_T(w)+\frac{C}{4\eta}\left[\mathcal{D}_T(u)+\mathcal{D}_T(v)\right]\mathcal{E}_T(w)+C\eta\mathcal{D}_T(w)\\
&\lesssim \|w(0)\|_{\widetilde{L}^2_\xi(B^s_{2,r})}+\rho\left(\mathcal{E}_T(w)+\mathcal{D}_T(w)\right). \qquad (5.3)
\end{aligned}
$$

If $0<\kappa_4<\min\{\frac{\sqrt{2}\sqrt{\lambda_0}}{2(2+\epsilon)},\frac{1}{1+\epsilon}\}$ and $\|u_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})}+\|v_0\|_{\widetilde{L}^2_\xi(B^s_{2,r})}$ is sufficiently small, i.e., ρ can be taken such that $\rho<\min\{\frac{1-\kappa_4(1+\epsilon)}{\kappa_4+1},\frac{\kappa_4}{\kappa_4+1}\}$, then from $\kappa_4\times$(5.2)+(5.3), we have

$$
\begin{aligned}
&\left[1-\kappa_4(1+\epsilon)-(\kappa_4+1)\rho\right]\mathcal{E}_T(w)+\kappa_4\left(\|\nabla_x(a^w,b^w,c^w)\|_{\widetilde{L}^2_T(B^{s-1}_{2,r})}+\|\{\mathbf{I}-\mathbf{P}\}w\|_{\widetilde{L}^2_T\widetilde{L}^2_{\xi,\nu}(B^s_{2,r})}\right)\\
&\qquad\le C\|w(0)\|_{\widetilde{L}^2_\xi(B^s_{2,r})}+(\kappa_4+1)\rho\mathcal{D}_T(w).
\end{aligned}
$$

Thus we obtain

$$
\left[1-\kappa_4(1+\epsilon)-(\kappa_4+1)\rho\right]\mathcal{E}_T(w)+\left[\kappa_4-(\kappa_4+1)\rho\right]\mathcal{D}_T(w)\le C\|w(0)\|_{\widetilde{L}^2_\xi(B^s_{2,r})},
$$

which implies (1.25). Hereto we complete the proof.

Acknowledgments

Hao Tang would like to express his sincere gratitude to Professor Tong Yang for his encouragement to this topic and his valuable suggestions. This work was completed when Hao Tang was visiting the School of Mathematics and Statistics at Wuhan University. He would also like to express his thanks to Professor Huijiang Zhao and his group for their kind hospitality and valuable suggestions. The authors would express their gratitude to the referees for pointing out some grammar and misspelling errors in the original manuscript, and also for their valuable suggestions and comments, which have led to a meaningful improvement of this paper.

Please cite this article in press as: Z. Liu, H. Tang, Global well-posedness for the Fokker–Planck–Boltzmann equation in Besov–Chemin–Lerner type spaces, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.02.031

References

[1] A. Arnold, P. Markowich, G. Toscani, A. Unterreiter, On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker–Planck type equations, Comm. Partial Differential Equations 26 (2001) 43–100.
[2] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, T. Yang, The Boltzmann equation without angular cutoff in the whole space: qualitative properties of solutions, Arch. Ration. Mech. Anal. 202 (2011) 599–661.
[3] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, T. Yang, Local existence with mild regularity for the Boltzmann equation, Kinet. Relat. Models 4 (2013) 1011–1041.
[4] D. Arsénio, N. Masmoudi, A new approach to velocity averaging lemmas in Besov spaces, J. Math. Pures Appl. 101 (2014) 495–551.
[5] H. Bahouri, J.-Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Springer-Verlag, Berlin, Heidelberg, 2011.
[6] M. Bisi, J.A. Carrillo, G. Toscani, Contractive metrics for a Boltzmann equation for granular gases: diffusive equilibria, J. Stat. Phys. 118 (2005) 301–331.
[7] C. Cercignani, The Boltzmann Equation and Its Applications, Appl. Math. Sci., vol. 67, Springer-Verlag, New York, 1988.
[8] C. Cercignani, R. Illner, M. Pulvirenti, The Mathematical Theory of Dilute Gases, Appl. Math. Sci., vol. 106, Springer-Verlag, New York, 1994.
[9] J.-Y. Chemin, Perfect Incompressible Fluids, Clarendon Press, Oxford, 1998.
[10] J.-Y. Chemin, N. Lerner, Flot de champs de vecteurs non lipschitziens et équations de Navier–Stokes, J. Differential Equations 121 (1995) 314–328.
[11] L. Desvillettes, About the regularizing properties of the non-cut-off Kac equation, Comm. Math. Phys. 168 (1995) 417–440.
[12] R.J. DiPerna, P.-L. Lions, On the Fokker–Planck–Boltzmann equation, Comm. Math. Phys. 120 (1988) 1–23.
[13] R.-J. Duan, Hypocoercivity of the linearized dissipative kinetic equations, Nonlinearity 24 (8) (2011) 2165–2189.
[14] R.-J. Duan, On the Cauchy problem for the Boltzmann equation in the whole space: global existence and uniform stability in $L^2(H_x^N)$, J. Differential Equations 244 (12) (2008) 3204–3234.
[15] R.-J. Duan, M. Fornasier, G. Toscani, A kinetic flocking model with diffusion, Comm. Math. Phys. 300 (2010) 95–145.
[16] R.-J. Duan, S.-Q. Liu, J. Xu, Global well-posedness in spatially critical Besov space for the Boltzmann equation, Arch. Ration. Mech. Anal., http://dx.doi.org/10.1007/s00205-015-0940-4.
[17] R.-J. Duan, R.M. Strain, Optimal time decay of the Vlasov–Poisson–Boltzmann system in $\mathbb{R}^3$, Arch. Ration. Mech. Anal. 199 (1) (2011) 291–328.
[18] N. Fournier, Finiteness of entropy for the homogeneous Boltzmann equation with measure initial condition, Ann. Appl. Probab. 25 (2) (2015) 860–897.
[19] Robert T. Glassey, The Cauchy Problem in Kinetic Theory, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.
[20] H. Grad, Asymptotic theory of the Boltzmann equation II, in: J.A. Laurmann (Ed.), Rarefied Gas Dynamics, vol. 1, Academic Press, New York, 1963, pp. 26–59.
[21] P.T. Gressman, R.M. Strain, Global classical solutions of the Boltzmann equation without angular cut-off, J. Amer. Math. Soc. 24 (3) (2011) 771–847.
[22] Y. Guo, The Boltzmann equation in the whole space, Indiana Univ. Math. J. 53 (2004) 1081–1094.
[23] Y. Guo, The Vlasov–Poisson–Boltzmann system near Maxwellians, Comm. Pure Appl. Math. 55 (9) (2002) 1104–1135.
[24] Y. Guo, The Vlasov–Maxwell–Boltzmann system near Maxwellians, Invent. Math. 153 (3) (2003) 593–630.
[25] Y. Guo, Classical solutions to the Boltzmann equation for molecules with an angular cutoff, Arch. Ration. Mech. Anal. 169 (4) (2003) 305–353.
[26] K. Hamdache, Estimations uniformes des solutions de l'équation de Boltzmann par les méthodes de viscosité artificielle et de diffusion de Fokker–Planck (Uniform estimates for solutions of the perturbed Boltzmann equation by artificial viscosity or Fokker–Planck diffusion), C. R. Acad. Sci. Paris Sér. I Math. 302 (1986) 187–190 (French).
[27] T. Kato, Quasi-linear equations of evolution with applications to partial differential equations, in: Spectral Theory and Differential Equations, in: Lecture Notes in Math., vol. 448, Springer-Verlag, Berlin, 1975, pp. 25–70.
[28] H.-L. Li, A. Matsumura, Behaviour of the Fokker–Planck–Boltzmann equation near a Maxwellian, Arch. Ration. Mech. Anal. 189 (2008) 1–44.
[29] T.-P. Liu, T. Yang, S.-H. Yu, Energy method for the Boltzmann equation, Phys. D 188 (2004) 178–192.

[30] T.-P. Liu, S.-H. Yu, Boltzmann equation: micro–macro decompositions and positivity of shock profiles, Comm. Math. Phys. 246 (2004) 133–179.

[31] S.K. Loyalka, Rarefied gas dynamic problems in environmental sciences, in: V. Boffi, C. Cercignani (Eds.), Proceedings 15th International Symposium on Rarefied Gas Dynamics, Teubner, Stuttgart, 1986.

[32] H.J. Schmeisser, H. Triebel, Topics in Fourier Analysis and Function Spaces, Wiley in Chichester, New York, 1987.

[33] V. Sohinger, R.M. Strain, The Boltzmann equation, Besov spaces, and optimal time decay rates in the whole space, Adv. Math. 261 (2014) 274–332.

[34] H. Tang, Z.-R. Liu, Continuous properties of the solution map for the Euler equations, J. Math. Phys. 55 (2014) 031504.

[35] H. Tang, Z.-R. Liu, Well-posedness of the modified Camassa–Holm equation in Besov spaces, Z. Angew. Math. Phys. 66 (2015) 1559–1580.

[36] H. Tang, S.-J. Shi, Z.-R. Liu, The dependences on initial data for the b-family equation in critical Besov space, Monatsh. Math. 177 (2015) 471–492.

[37] H. Tang, Y.-Y. Zhao, Z.-R. Liu, A note on the solution map for the periodic Camassa–Holm equation, Appl. Anal. 93 (2014) 1745–1760.

[38] H. Triebel, Theory of Function Spaces, Birkhäuser, Basel, 1983.

[39] S. Ukai, On the existence of global solutions of mixed problem for non-linear Boltzmann equation, Proc. Japan Acad. 50 (1974) 179–184.

[40] S. Ukai, Solutions of the Boltzmann equation, in: Patterns and Waves, in: Stud. Math. Appl., vol. 18, North-Holland, Amsterdam, 1986, pp. 37–96.

[41] S. Ukai, T. Yang, The Boltzmann equation in the space $L^2 \cap L^\infty_\beta$: global and time-periodic solutions, Anal. Appl. (Singap.) 4 (2006) 263–310.

[42] C. Villani, A review of mathematical topics in collisional kinetic theory, in: Handbook of Mathematical Fluid Dynamics, vol. I, North-Holland, Amsterdam, 2002, pp. 71–305.

[43] L. Xiong, T. Wang, L. Wang, Global existence and decay of solutions to the Fokker–Planck–Boltzmann equation, Kinet. Relat. Models 7 (1) (2014) 169–194.

[44] M.-Y. Zhong, H.-L. Li, Long time behavior of the Fokker–Planck–Boltzmann equation with soft potential, Quart. Appl. Math. 70 (2012) 721–742.

Please cite this article in press as: Z. Liu, H. Tang, Global well-posedness for the Fokker–Planck–Boltzmann equation in Besov–Chemin–Lerner type spaces, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.02.031