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Stability for line solitary waves of Zakharov–Kuznetsov equation

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Abstract

In this paper, we consider the stability for line solitary waves of the two dimensional Zakharov–Kuznetsov equation on $\mathbb{R} \times \mathbb{T}_L$ which is one of a high dimensional generalization of Korteweg–de Vries equation, where $\mathbb{T}_L$ is the torus with the $2\pi L$ period. The orbital and asymptotic stability of the one soliton of Korteweg–de Vries equation on the energy space was proved by Benjamin [2], Pego and Weinstein [41] and Martel and Merle [30]. We regard the one soliton of Korteweg–de Vries equation as a line solitary wave of Zakharov–Kuznetsov equation on $\mathbb{R} \times \mathbb{T}_L$. We prove the stability and the transverse instability of the line solitary waves of Zakharov–Kuznetsov equation by applying the method of Evans' function and the argument of Rousset and Tzvetkov [44]. Moreover, we prove the asymptotic stability for orbitally stable line solitary waves of Zakharov–Kuznetsov equation by using the argument of Martel and Merle [30–32] and a Liouville type theorem. If L is the critical period with respect to a line solitary wave, the line solitary wave is orbitally stable. However, since this line solitary wave is a bifurcation point of the stationary equation, the linearized operator of the stationary equation is degenerate. Because of the degeneracy of the linearized operator, we can not show the Liouville type theorem for the line solitary wave by using the usual virial type estimate. To show the Liouville type theorem for the line solitary wave, we modify a virial type estimate.

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1. Introduction

We consider the two dimensional Zakharov–Kuznetsov equation

$$u_t + \partial_x(\Delta u + u^2) = 0, \quad (t, x, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{T}_L, \tag{1.1}$$

where $\Delta = \partial_x^2 + \partial_y^2$, $u = u(t, x, y)$ is an unknown real-valued function, $\mathbb{T}_L = \mathbb{R}/2\pi L\mathbb{Z}$ and $L > 0$.

In [54], the Zakharov–Kuznetsov equation is derived to describe the propagation of ionic-acoustic waves in uniformly magnetized plasma. In [23], Lannes, Linares and Saut proved the rigorous derivation of the Zakharov–Kuznetsov equation from the Euler–Poisson system for uniformly magnetized plasmas. The Cauchy problem of the Zakharov–Kuznetsov equation has been studied for the last decade. In [8], Faminskii proved the global well-posedness of the Zakharov–Kuznetsov equation in the energy space $H^1(\mathbb{R}^2)$. This result has been pushed down to $H^s(\mathbb{R}^2)$ for $s > \frac{3}{4}$ by Linares and Pastor [24]. To study of the transverse instability of the N-soliton ϕ^N of the Korteweg–de Vries equation, Linares, Pastor and Saut [26] have proved the global well-posedness of the Zakharov–Kuznetsov equation in $\phi^N + H^1(\mathbb{R}^2)$ and $H^s(\mathbb{R} \times \mathbb{T}_L)$ for $s > \frac{3}{2}$. The result in [24] was recently improved by Grünrock and Herr [15] and Molinet and Pilod [36] who proved local well-posedness in $H^s(\mathbb{R}^2)$ for $s > \frac{1}{2}$. In [36], Molinet and Pilod showed the global well-posedness of (1.1) in $H^1(\mathbb{R} \times \mathbb{T}_L)$. Moreover, the well-posedness of the Zakharov–Kuznetsov equation in higher dimensions and the generalized Zakharov–Kuznetsov equation has been studied by [14,24,25,27,42]. The equation (1.1) has the following conservation laws:

$$M(u) = \int_{\mathbb{R}\times\mathbb{T}_L} |u|^2 dxdy, \tag{1.2}$$

$$E(u) = \int_{\mathbb{R}\times\mathbb{T}_L} \Big(\frac{1}{2}|\nabla u|^2 - \frac{1}{3}u^3\Big) dxdy, \tag{1.3}$$

where $u \in H^1(\mathbb{R} \times \mathbb{T}_L)$.

In this paper, we show the orbital stability and the asymptotic stability of line solitary waves of (1.1). By a solitary wave, we mean a non-trivial solution of (1.1) with form $u(t, x, y) = Q(x - ct, y)$, where $c > 0$ and $Q \in H^1(\mathbb{R} \times \mathbb{T}_L)$ is a solution of

$$-\Delta Q + cQ - Q^2 = 0, \quad (x, y) \in \mathbb{R} \times \mathbb{T}_L. \tag{1.4}$$

We can write the equation (1.4) as $S_c'(Q) = 0$, where

$$S_c(u) = E(u) + cM(u)$$

and S_c' is the Fréchet derivative of S_c.

The orbital stability of solitary waves is defined as follows.

Definition 1.1. We say that a solitary wave $Q(x-ct, y)$ is orbitally stable in $H^1(\mathbb{R}\times\mathbb{T}_L)$ if for any $\varepsilon>0$ there exists $\delta>0$ such that for all initial data $u_0\in H^1(\mathbb{R}\times\mathbb{T}_L)$ with $\|u_0-Q\|_{H^1}<\delta$, the solution $u(t)$ of (1.1) with $u(0)=u_0$ exists globally in positive time and satisfies

$$\sup_{t>0}\inf_{(x_0,y_0)\in\mathbb{R}\times\mathbb{T}_L}\|u(t,\cdot,\cdot)-Q(\cdot-x_0,\cdot-y_0)\|_{H^1}<\varepsilon.$$

Otherwise, we say the solitary wave $Q(x-ct, y)$ is orbitally unstable in $H^1(\mathbb{R}\times\mathbb{T}_L)$.

The orbital stability of positive solitary waves of the generalized Zakharov–Kuznetsov equation on $\mathbb{R}^N$ was showed by de Bouard [7] under the assumption of well-posedness on the energy space. In [5], Côte, Muñoz, Pilod and Simpson have proved the asymptotic stability of positive solitary waves and multi-solitary waves of the Zakharov–Kuznetsov equation on $\mathbb{R}^2$ by adapting the argument of Martel and Merle [30–32] to a multidimensional model.

The solution u to (1.1) does not depend on the variable of the transverse direction $\mathbb{T}_L$ if and only if the solution u satisfies the Korteweg–de Vries equation

$$u_t+u_{xxx}+2uu_x=0,\quad (t,x)\in\mathbb{R}\times\mathbb{R}. \tag{1.5}$$

The Korteweg–de Vries equation describes the propagation of ionic-acoustic waves in unmagnetized plasma. The equation (1.5) has the soliton solution $R_c(t,x)=Q_c(x-ct)$, where Q_c is the positive symmetric solution to

$$-\partial_x^2Q+cQ-Q^2=0,\quad Q\in H^1(\mathbb{R}). \tag{1.6}$$

Here, Q_c has the explicit form

$$Q_c(x)=\frac{3c}{2}\cosh^{-2}\Big(\frac{\sqrt{c}x}{2}\Big).$$

The orbital stability of the soliton R_c was proved by Benjamin [2]. In [41], Pego and Weinstein have showed the asymptotic stability of the soliton R_c on the exponentially weighted space by investigating a spectral property of linearized operator around Q_c. The argument of Pego and Weinstein [41] is useful to prove the asymptotic stability on the exponentially weighted space for nonintegrable equations. However, the assumption of the exponential decay of initial data yields that the solution does not have a small soliton other than the main soliton. To treat solutions including a small soliton other than the main soliton, Mizumachi [33] improved this result, using polynomial weighted spaces. In [30–32], Martel and Merle proved the asymptotic stability of the soliton for initial data on $H^1(\mathbb{R})$. To prove the asymptotic stability for initial data on $H^1(\mathbb{R})$, Martel and Merle showed the Liouville type theorem for the Korteweg–de Vries equation. The main tool to show the Liouville type theorem is the virial type estimate for solutions with some decay in space.

We regard the soliton solution R_c of (1.5) as a line solitary wave of (1.1), namely we define the line solitary wave $\tilde{R}_c$ and the solution $\tilde{Q}_c$ of (1.4) by

$$\tilde{R}_c(t,x,y)=\tilde{Q}_c(x-ct,y)=R_c(t,x)=Q_c(x-ct),\quad (t,x,y)\in\mathbb{R}\times\mathbb{R}\times\mathbb{T}_L.$$

A natural question concerning $\tilde{R}_c$ is the stability of $\tilde{R}_c$ with respect to perturbations which are periodic in the transversal direction. The stability of the line solitary wave $\tilde{R}_c$ on Kadomtsev–Petviashvili equation was studied by many papers. The stability of $\tilde{R}_c$ on KP-II was confirmed the heuristic analysis by Kadomtsev and Petviashvili [18]. In [48], Villarroel and Ablowitz showed the stability of line solitons $\tilde{R}_c$ of KP-II against decaying perturbations by the inverse scattering method. In [35] Mizumachi and Tzvetkov proved the orbital stability and the asymptotic stability of $\tilde{R}_c$ on KP-II in $L^2(\mathbb{R}\times\mathbb{T})$ by using the Bäcklund transformation. The asymptotic stability for line solitons $\tilde{R}_c$ of KP-II on $\mathbb{R}^2$ has recently been proved by Mizumachi [34]. On $\mathbb{R}^2$, because of finite speed propagations of local phase shifts along the crest of the modulating line soliton for the transverse direction, the line soliton $\tilde{R}_c$ is not orbitally stable in the usual sense. To prove the asymptotic stability, Mizumachi has showed that the local modulations of the amplitude and the phase shift of line solitons behave like a self-similar solution of the Burgers equation. For KP-I equation, Rousset and Tzvetkov proved the orbital stability and instability for line solitons $\tilde{R}_c$ of KP-I on $\mathbb{R}\times\mathbb{T}$ in [44,46] and on $\mathbb{R}^2$ in [43]. To study the relation between line solitary waves and cylindrical solitary waves, the breakup of the line solitary wave $\tilde{R}_c$ of Zakharov–Kuznetsov equation was studied numerically in [9,10,37]. The linear instability for line solitary wave $\tilde{R}_c$ of Zakharov–Kuznetsov equation on $\mathbb{R}\times\mathbb{T}_L$ with large L was showed by Bridges in [3]. The nonlinear instability for line solitary wave $\tilde{R}_c$ of Zakharov–Kuznetsov equation on $\mathbb{R}^2$ was proved by Rousset and Tzvetkov in [43]. On $\mathbb{T}_{L_1}\times\mathbb{T}_{L_2}$ with sufficiently large L_2, the linear instability of line periodic solitary waves of Zakharov–Kuznetsov equation was showed by Johnson [16] by using the method of Evan's function.

One of main results is the following:

Theorem 1.2. *Let $c>0$. Then, the following hold.*

(i) *If $0<L\le \frac{2}{\sqrt{5c}}$, then $\tilde{R}_c$ is orbitally stable.*
(ii) *If $L>\frac{2}{\sqrt{5c}}$, then $\tilde{R}_c$ is orbitally unstable.*

In Theorem 1.2, the instability for line solitary waves follows the linear instability of the linearized equation around $\tilde{R}_c$ for $L>\frac{2}{\sqrt{5c}}$. In many cases [16,17,45,51] with periodic transverse direction, the bifurcation of eigenvalues of linearized operators around line solitary waves generate the unstable mode of the linearized operator around the line solitary waves. In [1,22,29], they remarked on the relation between the symmetry breaking instability and the transverse instability. In [20,21], the symmetry breaking in ground states of nonlinear Schrödinger equations was investigated and the instability of symmetric standing waves with a large L^2-norm was proved by applying the bifurcation theory for the stationary equations. In these papers [20,21], the instability of symmetric standing waves follows the bifurcation of eigenvalues of linearized operators which yields the existence of unstable modes of the linearized operators. To prove the stability of the line solitary wave $\tilde{R}_c$ on $\mathbb{R}\times\mathbb{T}_{2/\sqrt{5c}}$, we recover the degeneracy of the linearized operator around $\tilde{Q}_c$ of the stationary equation (1.4) on $\mathbb{R}\times\mathbb{T}_{2/\sqrt{5c}}$ by using also the bifurcation theory for (1.4) and constructing the fourth order estimate of the Lyapunov function. In our case, the bifurcation of eigenvalues involves a symmetry breaking bifurcation of line solitary waves, which is proved in the following proposition.

Proposition 1.3. *Let $c_0>0$ and $L=\frac{2}{\sqrt{5c_0}}$. Then, there exist $\delta_0>0$ and $\varphi_{c_0}\in C^2((-\delta_0,\delta_0)^2,$ $H^2(\mathbb{R}\times\mathbb{T}_L))$ such that for $\vec{a}=(a_1,a_2)\in(-\delta_0,\delta_0)^2$ we have $\varphi_{c_0}(\vec{a})>0$, $\varphi_{c_0}(\vec{a})(x,y)=$*

$\varphi_{c_0}(\vec{a})(-x, y)$, $\check{c}(\vec{a}) = \check{c}(|\vec{a}|, 0)$,

$$-\Delta\varphi_{c_0}(\vec{a}) + \check{c}(\vec{a})\varphi_{c_0}(\vec{a}) - (\varphi_{c_0}(\vec{a}))^2 = 0,$$
$$\varphi_{c_0}(\vec{a}) = \tilde{Q}_{c_0} + a_1\tilde{Q}_{c_0}^{\frac{3}{2}}\cos\frac{y}{L} + a_2\tilde{Q}_{c_0}^{\frac{3}{2}}\sin\frac{y}{L} + O(|\vec{a}|^2) \quad \text{as } |\vec{a}| \to 0$$

and

$$\|\varphi_{c_0}(\vec{a})\|^2_{L^2(\mathbb{R}\times\mathbb{T}_L)} = \left\|\tilde{Q}_{c_0}\right\|^2_{L^2(\mathbb{R}\times\mathbb{T}_L)} + \frac{C_{2,c_0}}{2}|\vec{a}|^2 + o(|\vec{a}|^2) \quad \text{as } |\vec{a}| \to 0,$$

where $\check{c}(\vec{a}) = c_0 + \frac{\check{c}''(0)}{2}|\vec{a}|^2 + o(|\vec{a}|^2)$ *as* $|\vec{a}| \to 0$, $\check{c}''(0) > 0$ *and*

$$C_{2,c_0} = \frac{3\check{c}''(0)\left\|\tilde{Q}_{c_0}\right\|^2_{L^2}}{2c_0} - \frac{5\left\|\tilde{Q}_{c_0}^{\frac{3}{2}}\cos\frac{y}{L}\right\|^2_{L^2}}{2} > 0.$$

Remark 1.4. The solution $\varphi_{c_0}(\vec{a})$ is not constant in the transverse y direction. By applying the Lyapunov–Schmidt reduction and the Crandall–Rabinowitz Transversality in [21,52], we can show Proposition 1.3. In this paper, we only write the sketch of the proof Proposition 1.3 in Appendix.

We define a semi-norm $\|\cdot\|_{H^1(x>a)}$ on $H^1(\mathbb{R}\times\mathbb{T}_L)$ by

$$\|u\|^2_{H^1(x>a)} = \int_{x>a} (|\nabla u(x, y)|^2 + |u(x, y)|^2)dxdy, \quad u \in H^1(\mathbb{R}\times\mathbb{T}_L).$$

The following theorem is an main theorem for the asymptotic stability.

Theorem 1.5. *Let* $c_0 > 0$.

(i) *If* $0 < L < \frac{2}{\sqrt{5c_0}}$*, then the following holds. For any* $\beta > 0$*, there exists* $\varepsilon_{L,\beta} > 0$ *such that for* $u_0 \in H^1(\mathbb{R}\times\mathbb{T}_L)$ *with* $\left\|u_0 - \tilde{Q}_{c_0}\right\|_{H^1} < \varepsilon_{L,\beta}$*, there exist* $\rho(t) \in C^1([0, \infty), \mathbb{R})$ *and* $c_+ > 0$ *satisfying that*

$$\left\|u(t, \cdot, \cdot) - \tilde{Q}_{c+}(\cdot - \rho(t), \cdot)\right\|_{H^1(x>\beta t)} \to 0 \text{ as } t \to \infty,$$
$$\dot{\rho}(t) - c_+ \to 0 \text{ as } t \to \infty$$

and $|c_0 - c_+| \lesssim \left\|u_0 - \tilde{Q}_{c_0}\right\|_{H^1}$*, where* u *is the unique solution of* (1.1) *with* $u(0) = u_0$.

(ii) *If* $L = \frac{2}{\sqrt{5c_0}}$*, then the following holds. For any* $\beta > 0$*, there exists* $\varepsilon_\beta > 0$ *such that for* $u_0 \in H^1(\mathbb{R}\times\mathbb{T}_L)$ *with* $\left\|u_0 - \tilde{Q}_{c_0}\right\|_{H^1} < \varepsilon_\beta$*, there exist* $\rho_1(t), \rho_2(t) \in C^1([0, \infty), \mathbb{R})$*,* $c_+ > 0$ *and* $\vec{a}_+ \in \mathbb{R}^2$ *satisfying that*

$$\|u(t,\cdot,\cdot)-\Theta(\vec{a}_+,c_+)(\cdot-\rho_1(t),\cdot-\rho_2(t))\|_{H^1(x>\beta t)}\to 0 \textit{ as } t\to\infty,$$
$$\dot{\rho}_1(t)-\hat{c}_+\to 0,\ \dot{\rho}_2(t)\to 0 \textit{ as } t\to\infty,$$
$$|c_+-c_0||\vec{a}_+|=0$$

and $|c_0-c_+|+|\vec{a}_+|^2\lesssim \left\|u_0-\tilde{Q}_{c_0}\right\|_{H^1}$, *where*

$$\hat{c}_+=\begin{cases}c_+, & \vec{a}_+=(0,0),\\ \check{c}(\vec{a}_+), & c_+=c_0,\end{cases} \tag{1.7}$$
$$\Theta(\vec{a},c)(x,y)=\frac{c}{c_0}\varphi_{c_0}(\vec{a})\Big(\sqrt{\frac{c}{c_0}}x,y\Big),$$

and u is the unique solution of (1.1) with $u(0)=u_0$.

Remark 1.6. Since a neighborhood of $\tilde{Q}_{c_0}$ in $H^1(\mathbb{R}\times\mathbb{T}_L)$ contains the branch corresponding to unstable line solitary waves in the case $L=\frac{2}{\sqrt{5c_0}}$, Theorem 1.5 shows that solutions away from unstable solitary waves approach one of solitary waves in the neighborhood of $\tilde{Q}_{c_0}$ as $t\to\infty$ in the sense of the norm $H^1(x>\beta t)$.

Remark 1.7. In Theorem 1.5, the unique solution u of (1.1) with $u(0)=u_0$ means that for $T>0$ the function $u|_{[-T,T]}$ is a unique solution of (1.1) with $u(0)=u_0$ in $C([-T,T],H^1(\mathbb{R}\times\mathbb{T}_L))\cap X_T^{1,\frac{1}{2}+}$ which is defined in [36].

Remark 1.8. From Remark 1.4 and $\check{c}''(0)>0$, we obtain

$$\frac{d}{da}\|\varphi_{c_0}(\vec{a})\|^2_{L^2(\mathbb{R}\times\mathbb{T}_L)}>0$$

for small $\vec{a}$. Therefore, we can show the orbital stability of the solitary wave $\varphi_{c_0}(\vec{a})(x-\check{c}(\vec{a})t,y)$ for small $\vec{a}$ by the argument in [12] (see the proof of Theorem 1.3 in [52]). Moreover, from (ii) of Theorem 1.5, we can get the asymptotic stability of the solitary wave $\varphi_{c_0}(\vec{a})(x-\check{c}(\vec{a})t,y)$ for small $\vec{a}$.

Let us now explain the argument to prove Theorem 1.2. Since the solution $\tilde{Q}_c$ of (1.4) is not a minimizer of the functional $S_c(u)$ on $\{u\in H^1; M(u)=M(\tilde{Q}_c)\}$ for general $c>0$, we can not apply the variational argument to prove the orbital stability. Therefore, to prove the orbital stability of $\tilde{Q}_{c_0}$, we use the argument in [12,50] for $0<L<\frac{2}{\sqrt{5c_0}}$. In the case $L=\frac{2}{\sqrt{5c_0}}$, the linearized operator of (1.4) around $\tilde{Q}_{c_0}$ has an extra eigenfunction corresponding to the zero eigenvalue. Thus, we can not show the orbital stability of $\tilde{Q}_{c_0}$ by using the standard argument in [12,13,50]. Since any neighborhood of $\tilde{Q}_{c_0}$ contains the two branches which are comprised of line solitary waves $\tilde{Q}_c$ and solitary waves $\varphi_{c_0}(\vec{a})$, we can not apply the argument for the linearized operator of the evolution equation with an extra eigenfunction by Comech and Pelinovsky [6] and Maeda [28]. Because of the degeneracy of the third order term of Lyapunov functional, we can not use the argument for the instability of a standing wave on a point of interaction of two branches of standing waves in Ohta [39]. To prove the stability of $\tilde{Q}_{c_0}$, we apply the argument in [52,53].

To show the nonlinear instability of $\tilde{Q}_c$ from the existence of an unstable mode of the linearized operator around $\tilde{Q}_c$, we apply the argument by Grenier [11] and Rousset and Tzvetkov [44]. Since the simple criterion in [43,45] does not seem be applicable to the linearized operator of (1.1) around $\tilde{Q}_c$, it is difficult to get the existence of an unstable mode of the linearized operator by the implicit function theorem. For sufficiently large L, Bridges [3] showed the existence of an unstable mode by sophisticated arguments. To get the existence of an unstable mode of linearized operator for all $L > \frac{2}{\sqrt{5c_0}}$, we use the method of Evans' function by Pego and Weinstein [40] for gKdV equation.

Next we explain the main ideas and difficulties in the proof of Theorem 1.5. Since the equation (1.1) is not complete integrable, we can not use the inverse scattering method to get the asymptotic behavior of solutions. To prove the asymptotic stability, we apply the argument by Martel and Merle [30–32] and Côte et al. [5]. This argument relies on a Liouville type theorem for spatially decaying solutions around a solitary wave. From the orbital stability and the monotonicity property, solutions near by a solitary wave converge to an exponentially decaying function in $H^1(x > a)$ up to subsequence of time. Due to the Liouville type theorem, this function must be solitary waves. The main tool to prove Liouville type theorem is the virial type estimate. In the case $0 < L < \frac{2}{\sqrt{5c_0}}$, the linearized operator of (1.4) around $\tilde{Q}_{c_0}$ is coercive on $\{u \in H^1; M(u) = M(\tilde{Q}_{c_0})\}$ by modulating translation. Thus, applying the estimate of [32] we can show the virial type estimate. However, in the case $L = \frac{2}{\sqrt{5c_0}}$, the linearized operator of (1.4) around $\tilde{Q}_{c_0}$ is not coercive on the function space with the standard orthogonal condition. To get the coerciveness of linearized operator, we estimate the difference between the solution and Θ instead of the difference between the solution and solitary waves, where Θ is defined in Theorem 1.5. However, since Θ is not a solution of the stationary equation (1.4), a term including $S_c'(\Theta)$ appears in the virial type estimate. Therefore, we can not get the coerciveness of the virial type estimate by the argument in [32]. To treat the term with $S_c'(\Theta)$, we investigate the virial type estimate with a correction term $S_{\hat{c}}'(\Theta)$, where $\hat{c}$ is the suitable propagation speed of Θ. To get the coerciveness of the virial type estimate with a correction, we use the precise estimate for a quadratic form and interactions among main terms. Due to this virial type estimate with the correction, we get the Liouville type theorem around the bifurcation point $\tilde{Q}_{c_0}$.

Our plan of the present paper is as follows. In Section 2, we show the well-posedness result on the weighted space to prove the monotonicity property. The argument of this well-posedness result follows Kato in [19]. In Section 3, we prove the properties of the linearized operator of (1.1) and the estimate of the semi-group corresponding to the linearized operator. To show the linear instability of the linearized equation, we use the argument by Pego and Weinstein [40]. In Section 4, we prove (ii) of Theorem 1.2 by the argument of Rousset and Tzvetkov [44]. In Section 5, we show (i) of Theorem 1.2 by the argument of [12] and [52,53]. In Section 6, we prove the coercive type estimate of a quadratic form and the Liouville property for orbitally stable solitary waves. To get the monotonicity property, we use the Kato type local smoothing effect in Section 2. In Section 7, we prove Theorem 1.5 by applying the Liouville property and the monotonicity property in Section 6.

2. Preliminaries

In this section, we show the regularity of solutions to (1.1) on the weighted space. For that purpose, we apply the argument on KdV in [19].

From the result on well-posedness in $H^1(\mathbb{R}\times\mathbb{T}_L)$ by Molinet–Pilod [36], for initial data $u_0 \in H^1(\mathbb{R}\times\mathbb{T}_L)$ there exists the unique solution $u(t)$ of (1.1) such that $u(0)=u_0$ and for $T>0$

$$u|_{[-T,T]} \in C([-T,T], H^1(\mathbb{R}\times\mathbb{T}_L)) \cap X_T^{1,\frac{1}{2}+}.$$

Moreover, for any $T>0$, there exists a neighborhood $\mathcal{U}$ of u_0 in $H^1(\mathbb{R}\times\mathbb{T}_L)$, such that the flow map of data-solution

$$v_0 \in \mathcal{U} \mapsto v \in C([0,T), H^1(\mathbb{R}\times\mathbb{T}_L)) \cap X_T^{1,\frac{1}{2}+}$$

is smooth. Here, the function space $X^{1,\frac{1}{2}+}$ is defined in [36]. In this paper, we define H^1-solution by the solution in the function space $C([0,\infty), H^1(\mathbb{R}\times\mathbb{T}_L))$ satisfying the conservation laws $M(u(t)) = M(u(0))$ and $E(u(t)) = E(u(0))$.

Let $U_b(t) = \exp(-t(\partial_x - b)((\partial_x - b)^2 + \partial_y^2))$ for $b>0$. Then, we have, for $u \in L^2(\mathbb{R}\times\mathbb{T}_L)$ with $e^{bx}u \in L^2(\mathbb{R}\times\mathbb{T}_L)$,

$$e^{-bx}U_b(t)e^{bx}u = U_0(t)u.$$

The following lemma is concerned with decay properties of the propagator U_b.

Lemma 2.1. *Let $b>0$, $s, s' \in \mathbb{R}$, $s<s'$ and $n \in \mathbb{Z}_+ = \{k \in \mathbb{Z}; k>0\}$. Then, there exists $C = C(n,s,b) > 0$ such that for $u \in H^s(\mathbb{R}\times\mathbb{T}_L)$, $0 \le j \le n$ and $t>0$*

$$\|U_b(t)u\|_{H^{s'}} \le Ct^{-\frac{s'-s}{2}}e^{b^3t}\|u\|_{H^{s'}}, \tag{2.1}$$

$$\left\|\partial_x^j\partial_y^{n-j}U_b(t)u\right\|_{L^2} \le Ct^{-\frac{n}{2}}e^{b^3t}\|u\|_{L^2}, \tag{2.2}$$

$$\|\partial_t U_b(t)u\|_{L^2} \le Ct^{-\frac{3}{2}}e^{b^3t}\|u\|_{L^2}. \tag{2.3}$$

Proof. By the factorization we have

$$U_b(t) = \exp(tb^3)\exp(-3tb^2\partial_x)\exp(tb(3\partial_x^2+\partial_y^2))\exp(-t\partial_x\Delta).$$

Since $\exp(-3tb^2\partial_x)$ and $\exp(-t\partial_x\Delta)$ are unitary in H^s and $\exp(tb(3\partial_x^2+\partial_y^2))$ is the heat semigroup, we have the estimates (2.1)–(2.3). □

Proposition 2.2. *Let u be an H^1-solution to (1.1) with $e^{bx}u(0) \in L^2(\mathbb{R}\times\mathbb{T}_L)$ for some $b>0$. Then we have $e^{bx}u \in C([0,\infty), L^2(\mathbb{R}\times\mathbb{T}_L)) \cap C^\infty((0,\infty), H^\infty(\mathbb{R}\times\mathbb{T}_L))$ with*

$$\left\|e^{bx}u(t)\right\|_{L^2} \le e^{Kt}\left\|e^{bx}u(0)\right\|_{L^2}, \tag{2.4}$$

where K denotes various constant depending only on b and $\|u(0)\|_{L^2}$. Moreover, for any $T > 0$ and $s \geq 0$,

$$\left\| e^{bx} u(t) \right\|_{H^s} \leq K' t^{-\frac{s}{2}}, \quad 0 < t \leq T, \tag{2.5}$$

$$\left\| e^{bx} (\partial_t)^n u(t) \right\|_{H^s} \leq K' t^{-\frac{s+3n}{2}}, \quad 0 < t \leq T, n \in \mathbb{Z}_+, \tag{2.6}$$

where K' depends on s, n, T, b, $\left\| e^{bx} u(0) \right\|_{L^2}$, $M(u(0))$ and $E(u(0))$.

Proof. Let

$$q(x) = e^{bx}(1 + \varepsilon e^{2bx})^{-\frac{1}{2}}, \quad r(x) = e^{bx}(1 + \varepsilon e^{2bx})^{-1}, \quad p(x) = q(x)^2.$$

Then, we have $q, r, p \in L^\infty(\mathbb{R} \times \mathbb{T}_L)$ and

$$\partial_x p = 2br^2, \quad |\partial_x^2 p| \leq 4b^2 r^2, \quad |\partial_x^3 p| \leq 12b^3 r^2, \quad |\partial_x r| \leq br.$$

Therefore, we have

$$\frac{d}{dt}(pu, u)_{L^2} \leq -2b(3\|r\partial_x u\|_{L^2}^2 + \left\|r\partial_y u\right\|_{L^2}^2) + 12b^2\|ru\|_{L^2}^2 + \frac{8b}{3}(r^2 u, u^2)_{L^2}. \tag{2.7}$$

Then,

$$(r^2 u, u^2)_{L^2} \leq \frac{1}{2}\|r\nabla u\|_{L^2}^2 + K_0\|ru\|_{L^2}^2,$$

where K_0 depend only b and $\|u(0)\|_{L^2}$. From (2.7) and $r < q$, we obtain

$$\frac{d}{dt}\|qu\|_{L^2}^2 \leq -\frac{2b}{3}\|r\nabla u\|_{L^2}^2 + K\|ru\|_{L^2}^2 \leq -\frac{2b}{3}\|r\nabla u\|_{L^2}^2 + K\|qu\|_{L^2}^2.$$

It follows from the above inequality that $\|qu(t)\|_{L^2} \leq e^{Kt}\|qu(0)\|_{L^2}$. Applying the monotone convergence theorem, we obtain that

$$\left\| e^{bx} u(t) \right\|_{L^2} \leq e^{Kt} \left\| e^{bx} u(0) \right\|_{L^2}, \quad t \geq 0,$$

where K depends only on b and $\|u(0)\|_{L^2}$. Since $e^{bx}U_0(t) = U_b(t)e^{bx}$, by Lemma 2.1 we have for $t > 0$

$$\begin{aligned}
\left\| e^{bx} u(t) \right\|_{L^2} &\leq \left\| U_b(t) e^{bx} u(0) \right\|_{L^2} + \int_0^t \left\| U_b(t-\tau) e^{bx} \partial_x (u(\tau)^2) \right\|_{L^2} d\tau \\
&\leq C e^{b^3 t} \left\| e^{bx} u(0) \right\|_{L^2} + \int_0^t C(M(u), E(u))(t-\tau)^{-3/4} \left\| e^{bx} u(\tau) \right\|_{L^2} d\tau.
\end{aligned}$$

Here, we use $\|u(t)\|_{H^1} \leq C(M(u), E(u))$. Therefore, $e^{bx}u(t) \in C([0, \infty), L^2(\mathbb{R} \times \mathbb{T}_L))$.

Next we show (2.5). By the Sobolev embedding and the Hölder inequality, we have

$$\begin{aligned}\left\|e^{bx}\partial_x(u^2)\right\|_{H^{-\frac{1}{4}}} &\le b\left\|e^{bx}u^2\right\|_{H^{-\frac{1}{4}}}+\left\|e^{bx}u^2\right\|_{H^{\frac{3}{4}}}\\ &\le C(M(u),E(u))\left\|e^{bx}u\right\|_{H^1}.\end{aligned}$$

Thus,

$$\begin{aligned}\left\|e^{bx}u(t)\right\|_{H^{\frac{5}{4}}} &\le Ct^{-\frac{5}{8}}e^{b^3T}\left\|e^{bx}u(0)\right\|_{L^2}+K'\int_0^t(t-\tau)^{-\frac{3}{4}}\left\|e^{bx}u(\tau)\right\|_{H^1}d\tau\\ &\le Ct^{-\frac{5}{8}}e^{b^3T}\left\|e^{bx}u(0)\right\|_{L^2}+K'\int_0^t(t-\tau)^{-\frac{3}{4}}\left\|e^{bx}u(\tau)\right\|_{H^{\frac{5}{4}}}d\tau,\end{aligned}$$

where K' depends only on s, n, T, b, $\|e^{bx}u(0)\|_{L^2}$, $M(u(0))$ and $E(u(0))$. Therefore, the properties that $e^{bx}u\in C((0,\infty),H^{\frac{5}{4}}(\mathbb{R}\times\mathbb{T}_L))$ and (2.5) holds for $s=\frac{5}{4}$. By the interpolation, we obtain (2.5) for $0\le s\le\frac{5}{4}$. To prove $s>\frac{5}{4}$, we use the induction on s. Suppose (2.5) has been proved for $0\le s\le s'-\frac{1}{2}$, where $s'\ge\frac{7}{4}$. We shall show (2.5) for $0\le s\le s'$. By the Duhamel formula, we have

$$t^{\frac{s}{2}}e^{bx}u(t)=\int_0^t U_b(t-\tau)\Big(\frac{s}{2}\tau^{\frac{s}{2}-1}e^{bx}u(\tau)-\tau^{\frac{s}{2}}e^{bx}\partial_x(u(\tau)^2)\Big)d\tau.$$

Since $\|U_b(t-\tau)\|_{H^{s'-\frac{3}{2}}\to H^{s'}}\le C(t-\tau)^{-\frac{3}{4}}$,

$$t^{\frac{s'}{2}}\left\|e^{bx}u(t)\right\|_{H^{s'}}\le C\int_0^t(t-\tau)^{-\frac{3}{4}}\Big(\frac{s'}{2}\tau^{\frac{s'}{2}-1}\left\|e^{bx}u(\tau)\right\|_{H^{s'-\frac{3}{2}}}+\tau^{\frac{s'}{2}}\left\|e^{bx}\partial_x(u(\tau)^2)\right\|_{H^{s'-\frac{3}{2}}}\Big)d\tau. \tag{2.8}$$

From the assumption of the induction we have

$$\tau^{\frac{s'}{2}-1}\left\|e^{bx}u(\tau)\right\|_{H^{s'-\frac{3}{2}}}\le K'\tau^{\frac{s'}{2}-1-\frac{s'}{2}+\frac{3}{4}}=K'\tau^{-\frac{1}{4}}.$$

On the other hand, by Appendix A in [19] for $f,g\in H^s(\mathbb{R}\times\mathbb{T}_L)(s\ge\frac{5}{4})$

$$\|fg\|_{H^s}\lesssim\|f\|^{\frac{1}{2}}_{H^{\frac{3}{4}}}\|f\|^{\frac{1}{2}}_{H^{\frac{5}{4}}}\|g\|_{H^s}+\|g\|^{\frac{1}{2}}_{H^{\frac{3}{4}}}\|g\|^{\frac{1}{2}}_{H^{\frac{5}{4}}}\|f\|_{H^s}.$$

Thus, we have

$$\left\|e^{bx}\partial_x(u(\tau)^2)\right\|_{H^{s'-\frac{3}{4}}}\lesssim\left\|e^{bx}u(\tau)^2\right\|_{H^{s'-\frac{1}{2}}}\lesssim\left\|e^{\frac{bx}{2}}u(\tau)\right\|^{\frac{1}{2}}_{H^{\frac{3}{4}}}\left\|e^{\frac{bx}{2}}u(\tau)\right\|^{\frac{1}{2}}_{H^{\frac{5}{4}}}\left\|e^{\frac{bx}{2}}u(\tau)\right\|_{H^{s'-\frac{1}{2}}}.$$

From the assumption of the induction, we obtain

$$\left\| e^{bx}\partial_x(u(\tau)^2)\right\|_{H^{s'-\frac{3}{4}}} \le K'_{\frac{b}{2}}\tau^{-\frac{s'}{2}},$$

where $K'_{\frac{b}{2}}$ depends only on s, n, T, b, $\left\| e^{\frac{bx}{2}}u(0)\right\|_{L^2}$, $M(u(0))$ and $E(u(0))$. Since

$$\left\| e^{\frac{bx}{2}}u(0)\right\|_{L^2} \le (\|u(0)\|_{L^2}\left\| e^{bx}u(0)\right\|_{L^2})^{\frac{1}{2}},$$

$K'_{\frac{b}{2}}$ depends only on s, n, T, b, $\|e^{bx}u(0)\|_{L^2}$, $M(u(0))$ and $E(u(0))$. From (2.8) we obtain

$$\left\| e^{bx}u(t)\right\|_{H^{s'}} \le K''t^{-\frac{s'}{2}},$$

where K'' depends only on s, n, T, b, $\|e^{bx}u(0)\|_{L^2}$, $M(u(0))$ and $E(u(0))$. This proves (2.5) for $0 \le s \le s'$, completing the induction.

Finally we prove (2.6) by induction on n. For the case $n = 0$, it is known by (2.5). Assuming that it has been proved for all $s \ge 0$ up to a given n, we prove it for $n + 1$. By the induction hypothesis,

$$\left\| \partial_t^n\partial_x\Delta(e^{bx}u)\right\|_{H^s} \lesssim \left\| \partial_t^n(e^{bx}u)\right\|_{H^{s+3}} \le K't^{-\frac{s+3+3n}{2}}. \tag{2.9}$$

On the other hand,

$$\left\| \partial_t^n e^{bx}\partial_x(u^2)\right\|_{H^s} \lesssim \left\| \partial_t^n e^{bx}u^2\right\|_{H^{s+1}} \lesssim \sum_{j=0}^{n}\left\| e^{bx}(\partial_t^j u)(\partial_t^{n-j}u)\right\|_{H^{s+1}}.$$

By Appendix A in [19],

$$\begin{aligned}\left\| e^{bx}(\partial_t^j u)(\partial_t^{n-j}u)\right\|_{H^{s+1}} \lesssim{}& \left\| e^{\frac{bx}{2}}\partial_t^j u\right\|_{H^{s+1}}\left\| e^{\frac{bx}{2}}\partial_t^{n-j}u\right\|_{H^{\frac{3}{4}}}\left\| e^{\frac{bx}{2}}\partial_t^{n-j}u\right\|_{H^{\frac{5}{4}}}\\ &+\left\| e^{\frac{bx}{2}}\partial_t^{n-j}u\right\|_{H^{s+1}}\left\| e^{\frac{bx}{2}}\partial_t^j u\right\|_{H^{\frac{3}{4}}}\left\| e^{\frac{bx}{2}}\partial_t^j u\right\|_{H^{\frac{5}{4}}}.\end{aligned}$$

Therefore,

$$\left\| \partial_t^n e^{bx}\partial_x(u^2)\right\|_{H^s} \le K't^{-\frac{s+3+3n}{2}}. \tag{2.10}$$

From (2.9) and (2.10) we obtain (2.6) for $n + 1$, completing the induction. The property that $e^{bx}u \in C^\infty((0,\infty), H^\infty(\mathbb{R}\times\mathbb{T}_L))$ follows the estimate (2.6). $\square$

3. Linearized operator

In this section, we show the properties of the linearized operator of (1.1) around $\tilde{R}_c$. We define the linearized operator $\mathbb{L}_c$ of (1.4) around $\tilde{Q}_c$ by

$$\mathbb{L}_c = S_c''(\tilde{Q}_c) = -\Delta + c - 2\tilde{Q}_c$$

and the linearized operator $\mathcal{L}_c$ of (1.6) around Q_c by

$$\mathcal{L}_c = -\partial_x^2 + c - 2Q_c.$$

Then, the linearized operator of (1.1) around $\tilde{R}_c$ is $\partial_x \mathbb{L}_c$. From Theorem 3.4 in [4], $\mathcal{L}_c$ has the only one negative eigenvalue

$$-\lambda_c = -\frac{5c}{4}$$

and an eigenfunction $(Q_c)^{\frac{3}{2}}$ corresponding to $-\lambda_c$.

Proposition 3.1. *Let $c > 0$.*

(i) *If $0 < L \le \frac{2}{\sqrt{5c}}$, then $\partial_x \mathbb{L}_c$ has no eigenvalues with a positive real part.*

(ii) *If $0 < L < \frac{2}{\sqrt{5c}}$, then*

$$\mathrm{Ker}(\mathbb{L}_c) = \mathrm{Span}\{\partial_x \tilde{Q}_c\}.$$

(iii) *If $L = \frac{2}{\sqrt{5c}}$, then*

$$\mathrm{Ker}(\mathbb{L}_c) = \mathrm{Span}\left\{\partial_x \tilde{Q}_c, (\tilde{Q}_c)^{\frac{3}{2}} \cos\frac{y}{L}, (\tilde{Q}_c)^{\frac{3}{2}} \sin\frac{y}{L}\right\}.$$

(iv) *If $L > \frac{2}{\sqrt{5c}}$, then $\partial_x \mathbb{L}_c$ has a positive eigenvalue and the number of eigenvalues of $\partial_x \mathbb{L}_c$ with a positive real part is finite.*

Here, $\mathrm{Span}\{u_1, \dots, u_n\}$ *is the vector space spanned by vectors $u_1, \dots, u_n$.*

Proof. By the Fourier expansion, we have for $u \in H^1(\mathbb{R} \times \mathbb{T}_L)$

$$(\mathbb{L}_c u)(x, y) = \sum_{n=-\infty}^{\infty} \left(\mathcal{L}_c + \frac{n^2}{L^2}\right) u_n(x) e^{\frac{iny}{L}}, \tag{3.1}$$

where

$$u(x, y) = \sum_{n=-\infty}^{\infty} u_n(x) e^{\frac{iny}{L}}.$$

From the equation (3.1), we obtain that $\partial_x \mathbb{L}_c$ has an eigenvalue λ if and only if there exists $n \in \mathbb{Z}$ such that $\partial_x(\mathcal{L}_c + n^2/L^2)$ has an eigenvalue λ. By Theorem 3.4 in [40], the essential spectrum of $\partial_x \mathcal{L}_c$ is the imaginary axis. Moreover, from Theorem 3.1 in [40], the number of eigenvalues of $\partial_x(\mathcal{L}_c + n^2/L^2)$ with a positive real part is less than or equal to the number of negative eigenvalues of $\mathcal{L}_c + n^2/L^2$. In the case $L \le \frac{2}{\sqrt{5c}}$, since $n^2/L^2 \ge \lambda_c$ for all $n \ne 0$, $\mathcal{L}_c + n^2/L^2$ has no negative eigenvalues and (i) is verified. The kernel of $\partial_x(\mathcal{L}_c + n^2/L^2)$ is trivial if and only if the kernel of $\mathcal{L}_c + n^2/L^2$ is trivial. Therefore, for $L > \frac{2}{\sqrt{5c}}$ the kernel of $\partial_x \mathbb{L}_c$ is spanned by $\partial_x \tilde{Q}_c$. In the case $L = \frac{2}{\sqrt{5c}}$, the kernel of $\partial_x \mathbb{L}_c$ is spanned by $\partial_x \tilde{Q}_c$, $(\tilde{Q}_c)^{\frac{3}{2}} \cos \frac{y}{L}$ and $(\tilde{Q}_c)^{\frac{3}{2}} \sin \frac{y}{L}$. Thus, (ii) and (iii) are verified.

To prove (iv), we apply the method of Evans' function in [40]. We consider the following equation:

$$\partial_x \Big(\mathcal{L}_c + a \Big) u - \lambda u = 0. \tag{3.2}$$

The equation (3.2) is equivalent to the first order system

$$\partial_x \vec{u} = A(a, \lambda, x) \vec{u}, \tag{3.3}$$

where

$$\vec{u} = \begin{pmatrix} u \\ \partial_x u \\ \partial_x^2 u \end{pmatrix}, \quad A(a, \lambda, x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2\partial_x Q_c(x) - \lambda & c + a - 2Q_c(x) & 0 \end{pmatrix}.$$

First, we show that $A(a, \lambda, x)$ satisfies the assumptions **H1**, **H2**, **H3** and **H4** in Section 1 of [40]. Then, the matrix $A(a, \lambda, x)$ is analytic in λ and a for each x, so **H1** holds true. Let

$$A_\infty(a, \lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda & c + a & 0 \end{pmatrix}.$$

Then, $\lim_{|x| \to \infty} A(a, \lambda, x) = A_\infty(a, \lambda)$ and $A(a, \lambda, x)$ satisfies **H2** and **H4**. We define

$$\mu_1(a, \lambda) := \inf\{\mathrm{Re}\, \mu;\ \mu \text{ is an eigenvalue of } A_\infty(a, \lambda)\},$$

$$\mu_2(a, \lambda) := \inf\{\mathrm{Re}\, \mu;\ \mathrm{Re}\, \mu > \mathrm{Re}\, \mu_1(a, \lambda),\ \mu \text{ is an eigenvalue of } A_\infty(a, \lambda)\}.$$

Let

$$J = \{(a, \lambda) \in \mathbb{C}^2;\ A_\infty(a, \lambda) \text{ has some purely imaginary eigenvalues}\}.$$

We define J_+ be the connected component of $\mathbb{C}^2 \setminus J$ which contains $\{a \ge 0\} \times \{\lambda > 0\}$. From the perturbation theory of matrices, the number of eigenvalues counting multiplicity of $A_\infty(a, \lambda)$ having the negative real part is constant for $(a, \lambda) \in J_+$. Since the matrix $A_\infty(0, \lambda)$ has the only one simple negative eigenvalue for $\lambda > 0$, the number of eigenvalues counting multiplicity of $A_\infty(a, \lambda)$ having the negative real part is 1 for $(a, \lambda) \in J_+$. Therefore, for $(a, \lambda) \in J_+$

$$\mu_1(a,\lambda) < 0 < \mu_2(a,\lambda).$$

Moreover, for $a > -c/2$

$$\mu_1(a,0) < 0 \le \mu_2(a,0).$$

By the perturbation theory of matrices, there exists a domain $\tilde{\Omega}$ in $\mathbb{C}^2$ such that $\{a \ge 0\} \times \{\lambda \ge 0\} \subset \tilde{\Omega}$ for $(a,\lambda) \in \tilde{\Omega}$ and $A_\infty(a,\lambda)$ has the unique eigenvalue with the smallest real part $\mu_1(a,\lambda)$, which is simple and

$$\mu_1(a,\lambda) < \mu_2(a,\lambda) \tag{3.4}$$

Thus, **H3** holds true. Therefore, $A(a,\lambda,x)$ satisfies the assumptions **H1**, **H2**, **H3** and **H4** in Section 1 of [40], so we can define Evans' function $D(a,\lambda)$ for $(a,\lambda) \in \tilde{\Omega}$ by Definition 1.8 in [40]. For $(a_0,\lambda_0) \in \tilde{\Omega}$ with $\operatorname{Re}\lambda_0 > 0$, from Proposition 1.9 in [40] the kernel of the operator $\partial_x(\mathcal{L}_c + a_0) - \lambda_0$ is non-trivial if and only if $D(a_0,\lambda_0) = 0$. Since $A(a,\lambda,x)$ is analytic in a and λ for each fixed x, Evans' function $D(a,\lambda)$ is also analytic in a and λ for $(a,\lambda) \in \tilde{\Omega}$.

Let

$$\mathscr{P}(\nu) = \nu^3 - (c+a)\nu + \lambda$$

denote the characteristic polynomial of A_∞ and

$$\tilde{\mathscr{P}}(\nu) = \nu^3 + \lambda, \quad \mathscr{Q}(\nu) = -(c+a)\nu.$$

Then, the roots ν_0 of $\tilde{\mathscr{P}}(\nu) = 0$ are the cube roots of $-\lambda$, and for $|\nu - \nu_0| = o(1)$ as $|\lambda| \to \infty$ we have

$$\mathscr{Q}(\nu) = -(c+a)\nu_0(1+o(1)), \quad \frac{\partial \tilde{\mathscr{P}}}{\partial \nu}(\nu) = 3\nu_0^2(1+o(1)), \quad \left| \frac{\mathscr{Q}(\nu_0)}{\frac{\partial \tilde{\mathscr{P}}}{\partial \nu}(\nu_0)} \right| = \frac{|c+a|}{2|\lambda|^{\frac{1}{3}}}.$$

We choose $\rho(\lambda) = \rho_0 |c+a|/3|\lambda|^{\frac{1}{3}}$ for any $\rho_0 > 1$. Then, the assumption of Lemma 1.20 in [40] is satisfied and the roots of $\mathscr{P}(\nu) = 0$ are given by

$$\nu = (-\lambda)^{\frac{1}{3}} + O(|c+a||\lambda|^{-\frac{1}{3}}) \tag{3.5}$$

as $\lambda \to \infty$. From (3.5) for any labeling $\nu_1(a,\lambda)$, $\nu_2(a,\lambda)$, $\nu_3(a,\lambda)$ of roots of $A_\infty(a,\lambda)$ we have

$$\left| \frac{\nu_k}{\frac{\partial \mathscr{P}}{\partial \lambda}(\nu_j)} \right| = \frac{|\lambda|^{\frac{1}{3}}}{3|\lambda|^{\frac{2}{3}}}(1 + o(|c+a|)) = O((1+|a|)|\lambda|^{-\frac{1}{3}}),$$

as $|\lambda| \to \infty$ in $\tilde{\Omega}$. To apply Corollary 1.19 in [40], we obtain that the hypotheses of Proposition 1.17 in [40] hold. By Corollary 1.18 in [40], it follows that $D(a,\lambda) \to 1$ as $|\lambda| \to \infty$ in $\tilde{\Omega}$ for each fixed a. So for $0 \le a \le \lambda_c$,

$$D(a,\lambda) \to 1 \text{ as } \lambda \to \infty. \tag{3.6}$$

Since

$$\partial_x \mathcal{L}_c \partial_x Q_c = 0, \quad \mathcal{L}_c \partial_x Q_c = 0$$

and

$$\partial_x Q_c(x) e^{\sqrt{c}x} \to -6c^{\frac{3}{2}} \text{ as } x \to \infty, \quad Q_c(x) e^{-\sqrt{c}x} \to 6c \text{ as } x \to -\infty,$$

from (1.35) in [40] and $D(0,0) = 0$ we have

$$\begin{aligned} \frac{\partial D}{\partial a}(0,0) &= \frac{1}{\frac{\partial \mathscr{P}(\mu)}{\partial \mu}|_{(\mu,a,\lambda)=(-\sqrt{c},0,0)}} \int_{-\infty}^{\infty} \frac{Q_c}{6c} \partial_a[-\partial_x(\mathcal{L}_c + a) + \lambda]|_{(a,\lambda)=(0,0)} \frac{\partial_x Q_c}{-6c^{\frac{3}{2}}} dx \\ &= \frac{-1}{72c^{\frac{7}{2}}} \int_{-\infty}^{\infty} |\partial_x Q_c|^2 dx < 0. \end{aligned} \tag{3.7}$$

From Theorem 3.4 in [4] we have that the kernel of $\mathcal{L}_c + a$ on $L^2(\mathbb{R})$ is trivial for $0 < a < \lambda_c$. If there exists $0 < a_0 < \lambda_c$ satisfying $D(a_0, 0) = 0$, then there exists a solution u_0 of $\partial_x(\mathcal{L}_c + a_0)u = 0$ such that for all $\varepsilon > 0$ there is $C_\varepsilon > 0$ satisfying that

$$|u_0(x)| + |\partial_x u_0(x)| + |\partial_x^2 u_0(x)| \leq C_\varepsilon e^{(\mu_1+\varepsilon)x} \text{ as } x \to \infty$$

and

$$|u_0(x)| \leq C_\varepsilon e^{-\varepsilon x} \text{ as } x \to -\infty.$$

Since $((\mathcal{L}_c + a)u_0)(x) \to 0$ as $x \to \infty$, u_0 is a solution $(\mathcal{L}_c + a)u = 0$. By the property of solutions of ordinary differential equations, any solution of $(\mathcal{L}_c + a)u = 0$ decays or grows exponentially tend to $-\infty$. Thus, there are no solutions of $(\mathcal{L}_c + a)u = 0$ which grows subexponentially tend to $-\infty$ and decays exponentially tend to ∞. Hence, $u_0 \in L^2(\mathbb{R})$. This contradicts that the kernel of $\mathcal{L}_c + a$ on $L^2(\mathbb{R})$ is trivial. Thus, $D(a, 0) \neq 0$ for $0 < a < \lambda_c$. Since $D(a, \lambda)$ is real and continuous for real numbers a and λ in J_+, by (3.7) $D(a, 0)$ is negative for $0 < a < \lambda_c$. From (3.6), for a there exists $\lambda(a) > 0$ such that $D(a, \lambda(a)) = 0$. Therefore, $\partial_x(\mathcal{L}_c + a)$ has a positive eigenvalue $\lambda(a)$ for $0 < a < \lambda(a)$. Thus, $\partial_x \mathbb{L}_c$ has a positive eigenvalue for $L > \sqrt{\lambda_c}$. □

To prove the estimate of the propagator $e^{\partial_x(\mathcal{L}_c+a)t}$, we apply the following Gearhart–Greiner–Herbst–Prüss theorem, see [38].

Theorem 3.2. *Let $\mathcal{A}$ be a generator of a strongly continuous semigroup on a complex Hilbert space $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$. Then for each $t > 0$, the following spectral mapping theorem is valid*

$$\begin{aligned} \sigma(e^{\mathcal{A}}) \setminus \{0\} = \{e^{\lambda};\ &\textit{either } \mu_k := \lambda + 2\pi i k \in \sigma(\mathcal{A}) \textit{ for some } k \in \mathbb{Z} \\ &\textit{or the sequence } \{\left\|(\mu_k - \mathcal{A})^{-1}\right\|_{\mathcal{H}\to\mathcal{H}}\}_{k\in\mathbb{Z}} \textit{ is unbounded}\}. \end{aligned}$$

Proposition 3.3. *Let $a, s \geq 0$ and $s \in \mathbb{Z}$. Then, for $\varepsilon > 0$ there exists $C = C(\varepsilon, s) > 0$ for $u \in H^s(\mathbb{R})$ and $t > 0$,*

$$\left\| e^{\partial_x(\mathcal{L}_c+a)t} u \right\|_{H^s(\mathbb{R})} \leq C e^{(\mu(a)+\varepsilon)t} \|u\|_{H^s(\mathbb{R})} \tag{3.8}$$

where $\mu(a)$ is the maximum of the real pert of elements in $\sigma(\partial_x(\mathcal{L}_c + a))$.

Proof. By the compact perturbation theory the essential spectrum of $\partial_x(\mathcal{L}_c + a)$ is the essential spectrum of ∂_x^3, so the essential spectrum of $\partial_x(\mathcal{L}_c + a)$ is the imaginary axis. If we show the sequence $\{\|(\lambda + 2\pi ik - \partial_x(\mathcal{L}_c + a))^{-1}\|_{H^s \to H^s}\}_k$ is bounded for all $\operatorname{Re}\lambda > \mu(a)$, we can show the estimate (3.8) by applying Theorem 3.2 and Lemma 2 and 3 in [47] (see also the proof of Lemma 3.2 in [51]). If $s \geq 1$, we have that for $u \in H^s(\mathbb{R})$

$$\begin{aligned}
&\left\|(\lambda + 2\pi ik - \partial_x(\mathcal{L}_c + a))^{-1} u\right\|_{H^s} \\
&\lesssim \left\|(\lambda + 2\pi ik - \partial_x(\mathcal{L}_c + a))^{-1} \partial_x u\right\|_{H^{s-1}} + \left\|(\lambda + 2\pi ik - \partial_x(\mathcal{L}_c + a))^{-1} u\right\|_{H^{s-1}}.
\end{aligned}$$

Here, we use the boundedness of $(-2(Q_c)_{xx} - 2(Q_c)_x\partial_x)(\lambda + 2\pi ik - \partial_x(\mathcal{L}_c + a))^{-1}$ on $H^{s-1}(\mathbb{R})$. Therefore, the boundedness of the sequence $\{\|(\lambda + 2\pi ik - \partial_x(\mathcal{L}_c + a))^{-1}\|_{H^s \to H^s}\}_k$ follows the boundedness of the sequence $\{\|(\lambda + 2\pi ik - \partial_x(\mathcal{L}_c + a))^{-1}\|_{L^2 \to L^2}\}_k$. Thus, we prove the boundedness on $L^2(\mathbb{R})$. For $\beta \in \mathbb{C}$ we have

$$(i\beta - (i\partial_x)(\mathcal{L}_c + a))^{-1} = (I + A_\beta B)^{-1}(i\beta - (i\partial_x)((i\partial_x)^2 + c + a))^{-1},$$

where

$$\begin{aligned}
A_\beta &= 2(i\partial_x)\{i\beta - (i\partial_x)((i\partial_x)^2 + c + a)\}^{-1}\sqrt{Q_c} \\
B &= \sqrt{Q_c}.
\end{aligned}$$

Since

$$(I + A_\beta B)^{-1} = I - A_\beta(I + BA_\beta)^{-1}B,$$

for $\operatorname{Re}\lambda > \mu(a)$ the sequence $\{\|(\lambda + 2\pi ik - \partial_x(\mathcal{L}_c + a))^{-1}\|_{L^2 \to L^2}\}_k$ is bounded if and only if $\{\|(I + BA_{\lambda+2\pi ik})^{-1}\|_{L^2 \to L^2}\}_k$ is bounded. For $u \in L^2(\mathbb{R})$ we have

$$\begin{aligned}
\|BA_{\lambda+2\pi ik}u\|_{L^2} &= \left\|\sqrt{Q_c}2(i\partial_x)(i\lambda - 2\pi k - (i\partial_x)((i\partial_x)^2 + c + a))^{-1}(\sqrt{Q_c}u)\right\|_{L^2} \\
&\lesssim \left\|\eta(i\lambda - 2\pi k + \eta(\eta^2 + c + a))^{-1}\right\|_{L^1} \|u\|_{L^2}.
\end{aligned}$$

Let

$$p(\eta, k) = -\operatorname{Im}\lambda - 2\pi k + \eta(\eta^2 + c + a).$$

From (3.5) in the proof of Proposition 3.1, for $k \in \mathbb{Z}$ there exist roots $\alpha_j(k)$ $(j = 1, 2, 3)$ of $p(\eta, k) = 0$ satisfies

$$\alpha_j(k) = (2\pi k)^{\frac{1}{3}} \omega_3^j + O(|k|^{-\frac{1}{3}})$$

as $|k| \to \infty$, where ω_3 is a primitive root of $\eta^3 - 1 = 0$. Since $|\mathrm{Im}\,(\alpha_j - \alpha_3)| = \frac{\sqrt{3}}{2}(2\pi k)^{\frac{1}{3}} + O(|k|^{-\frac{1}{3}})$ as $|k| \to \infty$ for $j = 1, 2$, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{|\eta|}{|i\lambda - 2\pi k + \eta(\eta^2 + c + a)|} d\eta \\
&\le \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{|\eta|}{|\mathrm{Re}\,\lambda| + |\eta - \alpha_1(k)||\eta - \alpha_2(k)||\eta - \alpha_3(k)|} d\eta \\
&\le \frac{\sqrt{2}|k|^{-\frac{1}{3}}}{|\mathrm{Re}\,\lambda|} \sup_{-|k|^{-1} < \xi < |k|^{-1}} |\xi + \alpha_3(k) k^{-\frac{1}{3}}| \\
&\quad + \frac{|k|^{-\frac{1}{3}}}{\sqrt{2}} \int_{(-1, -|k|^{-1}) \cup (|k|^{-1}, 1)} |\xi|^{-1} d\xi \sup_{\xi \in \mathbb{R}} \frac{|\xi + \alpha_3(k) k^{-\frac{1}{3}}|}{|\xi - (\alpha_1(k) - \alpha_3(k)) k^{-\frac{1}{3}}||\xi - (\alpha_2(k) - \alpha_3(k)) k^{-\frac{1}{3}}|} \\
&\quad + \frac{|k|^{-\frac{1}{3}}}{\sqrt{2}} \int_{(-\infty, -1) \cup (1, \infty)} \frac{|\xi + \alpha_3(k) k^{-\frac{1}{3}}|}{|\xi||\xi - (\alpha_1(k) - \alpha_3(k)) k^{-\frac{1}{3}}||\xi - (\alpha_2(k) - \alpha_3(k)) k^{-\frac{1}{3}}|} d\xi \\
&\lesssim |k|^{-\frac{1}{3}} \log |k|.
\end{aligned}$$

Hence, we obtain there exists $C > 0$ such that

$$\|BA_{\lambda + 2\pi i k} u\|_{L^2} \le C|k|^{-\frac{1}{3}} (\log |k|) \|u\|_{L^2}.$$

Since $\|BA_{\lambda + 2\pi i k}\|_{L^2 \to L^2} \to 0$ as $|k| \to \infty$, $\{\left\|(I + BA_{\lambda + 2\pi i k})^{-1}\right\|_{L^2 \to L^2}\}_k$ is bounded. Thus, we obtain the conclusion. $\square$

4. Orbital instability

In this section, we prove (ii) of Theorem 1.2 by applying the argument in [44]. We assume $L > 2/\sqrt{5c}$. Let $\mu_{\max}$ be the largest eigenvalue of $\partial_x \mathbb{L}_c$. Then, there exists a positive integer k_0 such that the largest eigenvalue of $\partial_x(\mathcal{L}_c + k_0^2/L^2)$ is $\mu_{\max}$. Let χ be an eigenfunction of $\partial_x(\mathcal{L}_c + k_0^2/L^2)$ corresponding to $\mu_{\max}$. Since $\mu_{\max} > 0$, from the dichotomy for ordinary differential equations $\chi \in H^s(\mathbb{R})$ for $s > 0$. For $\delta > 0$ we define u^δ as the solution of (1.1) with initial data $\delta \chi \cos \frac{k_0 y}{L} + \tilde{Q}_c$ and we set $v^\delta(t, x, y) = u^\delta(t, x + ct, y) - \tilde{Q}_c(x)$. Then, we have $v^\delta(0, x, y) = \delta \chi(x) \cos \frac{k_0 y}{L}$ and

$$\partial_t v^\delta + \partial_x \mathbb{L}_c v^\delta + \partial_x (v^\delta)^2 = 0.$$

We define V_K^s as the function space

$$V_K^s = \Big\{u \in L^2(\mathbb{R} \times \mathbb{T}_L); u(x, y) = \sum_{j=-K}^{K} u_j(x) e^{\frac{ijk_0 y}{L}}, u_j \in H^s(\mathbb{R})\Big\},$$

and we define a norm of V_K^s as

$$\|u\|_{V_K^s} = \sup_{|j| \le K} \|u_j\|_{H^s(\mathbb{R})}, \text{ for } u = \sum_{j=-K}^{K} u_j e^{\frac{ijk_0 y}{L}} \in V_K^s.$$

To show the smallness of the high frequency part of v^δ, we consider an approximate solution

$$v_M^\delta = \sum_{l=1}^{M} \delta^l w_l, \quad w_l \in V_l^{s-l+1},$$

where w_1 is the solution of

$$\partial_t w + \partial_x \mathbb{L}_c w = 0, \quad w(0, x, y) = \chi(x) \cos \frac{k_0 y}{L},$$

and w_l is the solution of

$$\partial_t w + \partial_x \mathbb{L}_c w + \partial_x \Big(\sum_{\substack{l_1, l_2 \ge 1, \\ l_1 + l_2 = l}} w_{l_1} w_{l_2} \Big) = 0, \quad w(0, x, y) = 0.$$

Then, v_M^δ satisfies

$$\partial_t v_M^\delta + \partial_x \mathbb{L}_c v_M^\delta + \partial_x (v_M^\delta)^2 = F,$$

where

$$F = \delta^M \partial_x \Big(\sum_{\substack{1 \le l_1, l_2 \le M, \\ l_1 + l_2 > M}} \delta^{l_1 + l_2 - M} w_{l_1} w_{l_2} \Big).$$

From Proposition 3.3, we have the following lemma.

Lemma 4.1. *For $K, s, \varepsilon > 0$ there exists $C_{K,s,\varepsilon} > 0$ such that for $u \in V_K^s$*

$$\left\| e^{t \partial_x \mathbb{L}_c} u \right\|_{V_K^s} \le C_{K,s,\varepsilon} e^{(\mu_{\max} + \varepsilon) t} \|u\|_{V_K^s}.$$

Let $w^\delta = v^\delta - v_M^\delta$. Then, we have

$$\partial_t w^\delta + \partial_x \mathbb{L}_c w^\delta + 2\partial_x(w^\delta v_M^\delta) + \partial_x(w^\delta w^\delta) + F = 0. \tag{4.1}$$

Therefore,

$$\begin{aligned} \frac{d\|w^\delta\|_{L^2}^2}{dt} &= \int\limits_{\mathbb{R}\times\mathbb{T}_L} \big((w^\delta)^2 \partial_x \tilde{Q}_c - (w^\delta)^2 \partial_x v_M^\delta - F w^\delta\big) dxdy \\ &\leq \Big(1 + \|\partial_x v_M^\delta\|_{L^\infty} + \Big\|\partial_x \tilde{Q}_c\Big\|_{L^\infty}\Big)\|w^\delta\|_{L^2}^2 + \|F\|_{L^2}^2 \end{aligned} \tag{4.2}$$

From Lemma 4.1 we have that for $\varepsilon_0 > 0$ there exists $C_{M,s,\varepsilon_0} > 0$ such that

$$\|w_l(t)\|_{H^s} \leq C_{M,s,\varepsilon_0} e^{l(\mu_{\max}+\varepsilon_0)t}.$$

Therefore, there exists $C_{M,\varepsilon_0} > 0$ such that we have

$$\big\|\partial_x v_M^\delta(t)\big\|_{L^\infty} \leq C_{M,\varepsilon_0}(\delta e^{(\mu_{\max}+\varepsilon_0)t} + \delta^M e^{M(\mu_{\max}+\varepsilon_0)t}) \tag{4.3}$$

$$\|F\|_{L^2} \leq C_{M,\varepsilon_0}\delta^{M+1} e^{(M+1)(\mu_{\max}+\varepsilon_0)t}. \tag{4.4}$$

We set $T_{\delta,\varepsilon} = (\log(\varepsilon) - \log(\delta))/2\mu_{\max}$. Since $e^{(\mu_{\max}+\varepsilon_0)t} \leq \varepsilon/\delta$ for $0 < t \leq T_{\delta,\varepsilon}$, by (4.2)–(4.4) we have

$$\frac{d\|w^\delta(t)\|_{L^2}^2}{dt} \leq \Big(1 + \Big\|\partial_x \tilde{Q}_c\Big\|_{L^\infty} + 2\varepsilon C_{M,\varepsilon_0}\Big)\big\|w^\delta(t)\big\|_{L^2}^2 + C_{M,\varepsilon_0}^2 \delta^{2(M+1)} e^{2(M+1)(\mu_{\max}+\varepsilon_0)t}$$

for any $0 < \varepsilon < 1$ and $0 < t \leq T_{\delta,\varepsilon}$. Thus,

$$\begin{aligned} &\frac{d}{dt}\Big(e^{-(1+\|\partial_x \tilde{Q}_c\|_{L^\infty}+2\varepsilon C_{M,\varepsilon_0})t}\big\|w^\delta(t)\big\|_{L^2}^2\Big) \\ &\leq C_{M,\varepsilon_0}^2 \delta^{2(M+1)} e^{2(M+1)(\mu_{\max}+\varepsilon_0)t-(1+\|\partial_x \tilde{Q}_c\|_{L^\infty}+2\varepsilon C_{M,\varepsilon_0})t}. \end{aligned}$$

If we choose large M and small $\varepsilon(M)$ satisfying

$$2(M+1)(\mu_{\max}+\varepsilon_0) - (1 + \|\partial_x Q_x\|_{L^\infty} + 2\varepsilon C_{M,\varepsilon_0}) > 0,$$

then we obtain

$$\big\|w^\delta(t)\big\|_{L^2}^2 \leq C'_{M,\varepsilon_0}\delta^{2(M+1)} e^{2(M+1)(\mu_{\max}+\varepsilon_0)t}$$

for $0 < t \leq T_{\delta,\varepsilon}$. Hence, there exists $C''_{M,\varepsilon_0} > 0$ such that

$$\big\|w^\delta(T_{\delta,\varepsilon})\big\|_{L^2} \leq C''_{M,\varepsilon_0}\varepsilon^{M+1}$$

for small $\varepsilon > 0$. Let P_0 be a projection satisfying

$$(P_0 u)(x, y) = \int_{\mathbb{T}_L} u(x, z) dz, \text{ for } (x, y) \in \mathbb{R} \times \mathbb{T}_L.$$

From the definition of v_M^δ and the estimate (4.1) we have

$$\left\|(Id - P_0) v_M^\delta(t)\right\|_{L^2} \geq \sqrt{\pi} \|\chi\|_{L^2} \delta e^{\mu_{\max} t} - C_{\varepsilon_0}(\delta^2 e^{2\mu_{\max} t} + \delta^M e^{M\mu_{\max} t})$$

and

$$\begin{aligned}
\inf_{a\in\mathbb{R}} \left\| u^\delta(T_{\delta,\varepsilon}, \cdot, \cdot) - \tilde{Q}_c(\cdot + a, \cdot) \right\|_{L^2} &\geq \left\|(Id - P_0)(u^\delta(T_{\delta,\varepsilon}) - R_c(T_{\delta,\varepsilon}))\right\|_{L^2} \\
&= \left\|(Id - P_0)(v^\delta(T_{\delta,\varepsilon}))\right\|_{L^2} \\
&\geq \left\|(Id - P_0) v_M^\delta(T_{\delta,\varepsilon})\right\|_{L^2} - \left\|w^\delta(T_{\delta,\varepsilon})\right\|_{L^2} \\
&\geq \sqrt{\pi}\|\chi\|_{L^2}\varepsilon - C'''_{M,\varepsilon_0}\varepsilon^2.
\end{aligned}$$

Thus, if we choose

$$\varepsilon_1 = \frac{\sqrt{\pi}\|\chi\|_{L^2}}{2C'''_{M,\varepsilon_0}},$$

then we have for any $\delta > 0$

$$\inf_{a\in\mathbb{R}} \left\| u^\delta(T_{\delta,\varepsilon_1}, \cdot, \cdot) - \tilde{Q}_c(\cdot + a) \right\|_{L^2} \geq \frac{\sqrt{\pi}\|\chi\|_{L^2}\varepsilon_1}{2}.$$

This completes the proof of (ii) in Theorem 1.2.

5. Orbital stability

In this section, we prove (i) of Theorem 1.2 by applying the arguments in [12] and [52]. We write the outline of the proof of (i) of Theorem 1.2.

Theorem 3.3 in [12] yields the following coercive type lemma for $\mathcal{L}_{c_0}$.

Lemma 5.1. *Let $c_0 > 0$. There exists $k_0 > 0$ such that for $u \in H^1(\mathbb{R})$ with $(u, Q_{c_0})_{L^2(\mathbb{R})} = (u, \partial_x Q_{c_0})_{L^2(\mathbb{R})} = 0$,*

$$\langle \mathcal{L}_{c_0} u, u \rangle_{H^{-1}(\mathbb{R}), H^1(\mathbb{R})} \geq k_0 \|u\|_{H^1(\mathbb{R})}^2.$$

5.1. *Non-critical case* $L < \frac{2}{\sqrt{5c_0}}$

To show the orbital stability of $\tilde{R}_{c_0}$ for $L < \frac{2}{\sqrt{5c_0}}$, we apply the argument in [12,49] (see also [51,53]). Let $L < \frac{2}{\sqrt{5c_0}}$. By the Fourier expansion (3.1) we have for $u \in H^1(\mathbb{R} \times \mathbb{T}_L)$

$$\langle \mathbb{L}_{c_0} u, u\rangle_{H^{-1}(\mathbb{R}\times\mathbb{T}_L), H^1(\mathbb{R}\times\mathbb{T}_L)} = \sum_{n=-\infty}^{\infty} \left\langle \left(\mathcal{L}_{c_0} + \frac{n^2}{L^2}\right) u_n, u_n \right\rangle_{H^{-1}(\mathbb{R}), H^1(\mathbb{R})},$$

where

$$u(x, y) = \sum_{n=-\infty}^{\infty} u_n(x) e^{\frac{iny}{L}}.$$

Since $\lambda_{c_0} < L^{-2}$, $\mathcal{L}_{c_0} + n^2/L^2$ is positive for $|n| \geq 1$. From Lemma 5.1 there exists $K_0 > 0$ such that for $u \in H^1(\mathbb{R} \times \mathbb{T}_L)$ with $(u, \tilde{Q}_{c_0})_{L^2(\mathbb{R}\times\mathbb{T}_L)} = (u, \partial_x \tilde{Q}_{c_0})_{L^2(\mathbb{R}\times\mathbb{T}_L)} = 0$, we have

$$\langle \mathbb{L}_{c_0} u, u\rangle_{H^{-1}(\mathbb{R}\times\mathbb{T}_L), H^1(\mathbb{R}\times\mathbb{T}_L)} \geq K_0 \|u\|^2_{H^1(\mathbb{R}\times\mathbb{T}_L)}. \tag{5.1}$$

Combining (5.1) and the proofs of Theorem 3.4 and Theorem 3.5 in [12], we obtain the orbital stability of $\tilde{R}_{c_0}$.

5.2. *Critical case* $L = \frac{2}{\sqrt{5c_0}}$

The proof of the orbital stability of $\tilde{R}_{c_0}$ for $L = \frac{2}{\sqrt{5c_0}}$ is similar to the proof of (i) of Theorem 1.4 in [52] (see also the proof of (i) of Theorem 1.4 in [53]). Let $L = \frac{2}{\sqrt{5c_0}}$. In this case, from (iii) of Proposition 3.1 the linearized operator $\mathbb{L}_{c_0}$ has an extra eigenfunction corresponding to the zero eigenvalue. Therefore, we have to recover the degeneracy of the kernel of $\mathbb{L}_{c_0}$ from the nonlinearity of (1.1). We define the action $S_c(u)$ by $E(u) + cM(u)$.

Lemma 5.2. *There exist a neighborhood U of $(0,0)$ and a C^2 function $\gamma_c(\vec{a}) : U \to \mathbb{R}$ such that $\gamma_c(0,0) = c$ and for $\vec{a} \in U$ and $|c - c_0| < c_0/2$*

$$\begin{aligned} M\big(\Theta(\vec{a}, \gamma_c(\vec{a}))\big) &= M(\tilde{Q}_c), \\ \gamma_c(\vec{a}) - c &= -\frac{cC_{2,c_0}}{3\left\|\tilde{Q}_{c_0}\right\|^2_{L^2}} |\vec{a}|^2 + o(|\vec{a}|^2), \end{aligned} \tag{5.2}$$

where $\Theta(\vec{a}, c)(x, y) = cc_0^{-1} \varphi_{c_0}(\vec{a})(\sqrt{cc_0^{-1}}x, y)$.

Proof. Let

$$\gamma_c(\vec{a}) = c_0 \Big(\left\|\tilde{Q}_c\right\|^2_{L^2} \|\varphi_{c_0}(\vec{a})\|^{-2}_{L^2} \Big)^{\frac{2}{3}}.$$

By the definition of Θ we have

$$M\big(\Theta(\vec{a}, \gamma_c(\vec{a}))\big) = M(\tilde{Q}_c).$$

Since $\left\|\tilde{Q}_c\right\|_{L^2}^2 \left\|\tilde{Q}_{c_0}\right\|_{L^2}^{-2} = c^{\frac{3}{2}} c_0^{-\frac{3}{2}}$, we have

$$\gamma_c(\vec{a}) = c - c\frac{\left\|\varphi_{c_0}(\vec{a})\right\|_{L^2}^{\frac{4}{3}} - \left\|\tilde{Q}_{c_0}\right\|_{L^2}^{\frac{4}{3}}}{\left\|\varphi_{c_0}(\vec{a})\right\|_{L^2}^{\frac{4}{3}}} = c - \frac{cC_{2,c_0}}{3\left\|\tilde{Q}_{c_0}\right\|_{L^2}^2}|\vec{a}|^2 + o(|\vec{a}|^2). \quad \square$$

Next, we investigate the difference between Θ and $\tilde{Q}_c$ on the action S_c.

Lemma 5.3. *For $\vec{a} \in U$ and $|c - c_0| < c_0/2$,*

$$S_c\big(\Theta(\vec{a}, \gamma_c(\vec{a}))\big) - S_c(\tilde{Q}_c) = \Big(\frac{c}{c_0}\Big)^{\frac{5}{2}} \frac{5c_0 C_{2,c_0}\left\|\tilde{Q}_{c_0}^{\frac{3}{2}} \cos\frac{y}{L}\right\|_{L^2}^2}{48\left\|\tilde{Q}_{c_0}\right\|_{L^2}^2}|\vec{a}|^4 + \Big(1 - \frac{c}{c_0}\Big)\left\|\partial_y \Theta(\vec{a}, \gamma_c(\vec{a}))\right\|_{L^2}^2 + o(|\vec{a}|^4) \tag{5.3}$$

as $|\vec{a}| \to 0$.

Proof. First, we consider the case $c = c_0$. From the expansion

$$\Theta(\vec{a}, \gamma_{c_0}(\vec{a})) = \varphi_{c_0}(\vec{a})(x, y) + (\gamma_{c_0}(\vec{a}) - c_0)\partial_c Q_{c_0} + O\big((|\vec{a}| + (\gamma_{c_0}(\vec{a}) - c_0))(\gamma_{c_0}(\vec{a}) - c_0)\big), \tag{5.4}$$

we have

$$S_{c_0}\big(\Theta(\vec{a}, \gamma_{c_0}(\vec{a}))\big) - S_{c_0}(\tilde{Q}_{c_0}) = S_{\check{c}(\vec{a})}(\varphi_{c_0}(\vec{a})) - S_{c_0}(\tilde{Q}_{c_0}) + (c_0 - \check{c}(\vec{a}))M(\tilde{Q}_{c_0}) + \frac{1}{2}(\gamma_{c_0}(\vec{a}) - c_0)^2 (S_c''(\tilde{Q}_{c_0})\partial_c \tilde{Q}_{c_0}, \partial_c \tilde{Q}_{c_0})_{L^2} + o(|\vec{a}|^4),$$

where $\check{c}$ is defined in Proposition 1.3. Since $\frac{\partial \check{c}}{\partial a_1}(0, 0) = 0$ and $\frac{\partial^2 \check{c}}{\partial (a_1)^2}(0, 0) = \check{c}''(0) > 0$, there exist $\delta_1 > 0$ and the inverse function $a_1(c)$ of $\check{c}(a_1, 0)$ on from $[c_0, \check{c}(\delta_1, 0))$ to $[0, \delta_1)$. For c_1, c_2 with $c_1 \neq c_2$

$$\frac{S_{c_1}(\varphi_{c_0}(c_1)) - S_{c_2}(\varphi_{c_0}(c_2))}{c_1 - c_2} = \frac{\big(S_{c_2}''(\varphi_{c_0}(c_2))(\varphi_{c_0}(c_1) - \varphi_{c_0}(c_2)), \varphi_{c_0}(c_1) - \varphi_{c_0}(c_2)\big)_{L^2}}{2(c_1 - c_2)} + M(\varphi_{c_0}(c_1))$$

$$
\begin{aligned}
&+\frac{o((\varphi_{c_0}(c_1)-\varphi_{c_0}(c_2))^2)}{c_1-c_2}\\
&\to M(\varphi_{c_0}(c_2)) \text{ as } c_1\to c_2,
\end{aligned}
$$

where $\varphi_{c_0}(c)=\varphi_{c_0}(a_1(c),0)$. Since $S''_{c_0}(\tilde{Q}_{c_0})\partial_{a_1}\varphi_{c_0}(a_1,a_2)|_{(a_1,a_2)=(0,0)}=\mathbb{L}_{c_0}(\tilde{Q}_{c_0}^{\frac{3}{2}}\cos\frac{y}{L})=0$, for $c>c_0$

$$
\begin{aligned}
&\frac{S_c(\varphi_{c_0}(c))-S_{c_0}(\tilde{Q}_{c_0})}{c-c_0}\\
&=\frac{\left(S''_{c_0}(\tilde{Q}_{c_0})(\varphi_{c_0}(c)-\tilde{Q}_{c_0}),\varphi_{c_0}(c)-\tilde{Q}_{c_0}\right)_{L^2}}{\check{c}''(0)a_1(c)^2+o(a_1(c)^2)}+M(\varphi_{c_0}(c))+\frac{o((\varphi_{c_0}(c)-\tilde{Q}_{c_0})^2)}{\check{c}''(0)a_1(c)^2+o(a_1(c)^2)}\\
&\to M(\tilde{Q}_{c_0}) \text{ as } c\downarrow c_0.
\end{aligned}
$$

Therefore, $S_c(\varphi_{c_0}(c))$ is C^1 and $\partial_c S_c(\varphi_{c_0}(c))=M(\varphi_{c_0}(c))$. By the same way we obtain that $M(\varphi_{c_0}(c))$ is C^1 and

$$
\lim_{c\downarrow c_0}\frac{M(\varphi_{c_0}(c))-M(\tilde{Q}_{c_0})}{c-c_0}=\frac{C_{2,c_0}}{2\check{c}''(0)}.
$$

Thus, we have

$$
\begin{aligned}
&S_{\check{c}(|\vec{a}|,0)}(\varphi_{c_0}(|\vec{a}|,0))-S_{c_0}(\tilde{Q}_{c_0})+(c_0-\check{c}(|\vec{a}|,0))M(\tilde{Q}_{c_0})\\
&=\frac{C_{2,c_0}}{4\check{c}''(0)}(\check{c}(|\vec{a}|,0)-c_0)^2+o((\check{c}(|\vec{a}|,0)-c_0)^2)\\
&=\frac{C_{2,c_0}\check{c}''(0)}{16}|\vec{a}|^4+o(|\vec{a}|^4).
\end{aligned}
\tag{5.5}
$$

From Lemma 5.2 and $S''_{c_0}(\tilde{Q}_{c_0})\partial_c\tilde{Q}_{c_0}=-\tilde{Q}_{c_0}$,

$$
(\gamma_{c_0}(\vec{a})-c_0)^2\left(S''_{c_0}(\tilde{Q}_{c_0})\partial_c\tilde{Q}_{c_0},\partial_c\tilde{Q}_{c_0}\right)_{L^2}=-\frac{c_0C_{2,c_0}^2}{12\left\|\tilde{Q}_{c_0}\right\|_{L^2}^2}|\vec{a}|^4+o(|\vec{a}|^4).
\tag{5.6}
$$

Since

$$
\begin{aligned}
&S_{\check{c}(\vec{a})}(\varphi_{c_0}(\vec{a}))-S_{c_0}(\tilde{Q}_{c_0})+(c_0-\check{c}(\vec{a}))M(\tilde{Q}_{c_0})\\
&=S_{\check{c}(|\vec{a}|,0)}(\varphi_{c_0}(\vec{a}))-S_{c_0}(\tilde{Q}_{c_0})+(c_0-\check{c}(|\vec{a}|,0))M(\tilde{Q}_{c_0}),
\end{aligned}
$$

from (5.5) and (5.6) we obtain (5.3) for $c=c_0$.

Next, we consider the general cases. Since $M(\Theta(\vec{a},\gamma_c(\vec{a})))=M(\tilde{Q}_c)$, we have

$$
\begin{aligned}
&S_c\big(\Theta(\vec{a},\gamma_c(\vec{a}))\big)-S_c(\tilde{Q}_c)\\
&=\left(\frac{c}{c_0}\right)^{\frac{5}{2}}\Big(S_{c_0}\big(\Theta(\vec{a},\gamma_{c_0}(\vec{a}))\big)-S_{c_0}(\tilde{Q}_{c_0})\Big)+\Big(1-\frac{c}{c_0}\Big)\left\|\partial_y\Theta(\vec{a},\gamma_c(\vec{a}))\right\|_{L^2}^2.
\end{aligned}
$$

Therefore, we obtain (5.3) for $c>0$. $\square$

We define a distance dist_c and neighborhoods $N_{\varepsilon,c}$ and $N^l_{\varepsilon,c}$ of $\tilde{Q}_c$ by

$$\mathrm{dist}_c(u) = \inf_{x\in\mathbb{R}} \left\| u(\cdot,\cdot) - \tilde{Q}_c(\cdot - x,\cdot)\right\|_{H^1},$$
$$N_{\varepsilon,c} = \{u \in H^1(\mathbb{R}\times\mathbb{T}_L); \mathrm{dist}_c(u) < \varepsilon\},$$
$$N^l_{\varepsilon,c} = \{u \in N_{\varepsilon,c}; M(u) = M(\tilde{Q}_l)\}.$$

In the following lemma, to get a orthogonal condition we decompose functions in $N_{\varepsilon,c}$.

Lemma 5.4. *Let $\varepsilon > 0$ sufficiently small. Then, there exist $K_1 > 0$, C^2 functions $\rho : N_{\varepsilon,c_0} \to \mathbb{R}$, $c : N_{\varepsilon,c_0} \to \mathbb{R}$, $\vec{a} = (a_1, a_2) : N_{\varepsilon,c_0} \to U$ and $\eta : N_{\varepsilon,c_0} \to H^1(\mathbb{R}\times\mathbb{T}_L)$ such that for $u \in N_{\varepsilon,c_0}$*

$$u(\cdot + \rho(u), \cdot) = \Theta(\vec{a}(u), c(u))(\cdot,\cdot) + \eta(u)(\cdot,\cdot),$$
$$|c(u) - c_0| + |\vec{a}(u)| + \|\eta(u)\|_{H^1} \le K_1 \mathrm{dist}_{c_0}(u), \tag{5.7}$$

and $\big(\eta(u), \Theta(\vec{a}(u), c(u))\big)_{L^2} = \big(\eta(u), \partial_x\Theta(\vec{a}(u), c(u))\big)_{L^2} = \big(\eta(u), \partial_{a_1}\Theta(\vec{a}(u), c(u))\big)_{L^2} = \big(\eta(u), \partial_{a_2}\Theta(\vec{a}(u), c(u))\big)_{L^2} = 0.$

Proof. We define

$$G(u, c, \rho, a_1, a_2) = \begin{pmatrix} (u(\cdot+\rho,\cdot) - \Theta(\vec{a}, c), \Theta(\vec{a}, c))_{L^2} \\ (u(\cdot+\rho,\cdot) - \Theta(\vec{a}, c), \partial_x\Theta(\vec{a}, c))_{L^2} \\ (u(\cdot+\rho,\cdot) - \Theta(\vec{a}, c), \partial_{a_1}\Theta(\vec{a}, c))_{L^2} \\ (u(\cdot+\rho,\cdot) - \Theta(\vec{a}, c), \partial_{a_2}\Theta(\vec{a}, c))_{L^2} \end{pmatrix}.$$

Then, $G(\tilde{Q}_{c_0}, c_0, 0, 0, 0) = 0$. Since

$$\frac{\partial G}{\partial(c, \rho, a_1, a_2)}\Big|_{\substack{u=\tilde{Q}_{c_0}, c=c_0\\ \rho=a_1=a_2=0}}$$
$$= \begin{pmatrix} -(\partial_c\tilde{Q}_{c_0}, \tilde{Q}_{c_0})_{L^2} & 0 & 0 & 0 \\ 0 & \left\|\partial_x\tilde{Q}_{c_0}\right\|^2_{L^2} & 0 & 0 \\ 0 & 0 & -\left\|\tilde{Q}^{\frac{3}{2}}_{c_0}\cos\frac{y}{L}\right\|^2_{L^2} & 0 \\ 0 & 0 & 0 & -\left\|\tilde{Q}^{\frac{3}{2}}_{c_0}\sin\frac{y}{L}\right\|^2_{L^2} \end{pmatrix}$$

is regular, from the implicit function theorem for small $\varepsilon > 0$ there exist C^2 functions $c, \rho, a_1, a_2 : N_{\varepsilon,c_0} \to \mathbb{R}$ such that for $u \in N_{\varepsilon,c_0}$

$$G(u, c(u), \rho(u), a_1(u), a_2(u)) = 0.$$

Therefore,

$$\eta(u) = u(\cdot + \rho(u), \cdot) - \Theta(\vec{a}(u), c(u))$$

satisfies the orthogonal conditions, where $\vec{a}(u) = (a_1(u), a_2(u))$. The inequality (5.7) follows the implicit function theorem and the definition of η. $\square$

In the following lemma, we estimate $\|\Theta(\vec{a}(u), c(u)) - \Theta(\vec{a}(u), \gamma(\vec{a}(u)))\|_{H^1}$ on N^l_{ε,c_0}.

Lemma 5.5. *Let $\varepsilon > 0$ sufficiently small. There exists $C > 0$ such that for $|l - c_0| < \varepsilon^{1/2}$ and $u \in N^l_{\varepsilon,c_0}$,*

$$\begin{aligned} \big\|\Theta\big(\vec{a}(u), \gamma_l(\vec{a}(u))\big) - \Theta(\vec{a}(u), c(u))\big\|_{H^1} &\le C\|\eta(u)\|^2_{L^2}, \\ |\gamma_l(\vec{a}(u)) - c(u)| &\lesssim M(\eta(u)). \end{aligned} \tag{5.8}$$

Proof. For $u \in N^l_{\varepsilon,c_0}$,

$$\begin{aligned} M\big(\Theta(\vec{a}(u), \gamma_l(\vec{a}(u)))\big) = M(\tilde{Q}_l) &= M\big(\eta(u) + \Theta(\vec{a}(u), c(u))\big) \\ &= M(\eta(u)) + M\big(\Theta(\vec{a}(u), c(u))\big). \end{aligned}$$

For sufficiently small $\varepsilon > 0$, we have

$$|c(u) - c_0| + |\gamma_l(\vec{a}(u)) - c_0| < \frac{c_0}{2}.$$

Therefore,

$$\begin{aligned} M(\eta(u)) = M\big(\Theta(\vec{a}(u), \gamma_l(\vec{a}(u)))\big) - M\big(\Theta(\vec{a}(u), c(u))\big) &= (\gamma_l(\vec{a}(u))^{\frac{3}{2}} - c(u)^{\frac{3}{2}})M\big(\varphi_{c_0}(\vec{a}(u))\big) \\ &\gtrsim \gamma_l(\vec{a}(u)) - c(u) \ge 0. \end{aligned}$$

Since

$$\Theta\big(\vec{a}(u), \gamma_l(\vec{a}(u))\big) - \Theta(\vec{a}(u), c(u)) = \big(\gamma_l(\vec{a}(u)) - c(u)\big)\partial_c \tilde{Q}_{c_0} + o(\gamma_l(\vec{a}(u)) - c(u)),$$

we obtain

$$\big\|\Theta\big(\vec{a}(u), \gamma_l(\vec{a}(u))\big) - \Theta(\vec{a}(u), c(u))\big\|_{H^1} \lesssim \gamma_l(\vec{a}(u)) - c(u) \lesssim M(\eta(u)). \quad \square$$

Next we show the coerciveness of $S''_{c_0}(\tilde{Q}_{c_0})$ on a subspace of $H^1(\mathbb{R} \times \mathbb{T}_L)$.

Lemma 5.6. *There exist $k_2 > 0$ and $\varepsilon_0 > 0$ such that for $a_1, a_2 \in (-\varepsilon_0, \varepsilon_0)$ and $c \in (c_0 - \varepsilon_0, c_0 + \varepsilon_0)$, if $w \in H^1(\mathbb{R} \times \mathbb{T}_L)$ satisfies*

$$(w, \Theta(\vec{a}, c))_{L^2} = (w, \partial_x \Theta(\vec{a}, c))_{L^2} = (w, \partial_{a_1} \Theta(\vec{a}, c))_{L^2} = (w, \partial_{a_2} \Theta(\vec{a}, c))_{L^2} = 0,$$

then

$$\langle S''_{c_0}(\Theta(\vec{a},c))w, w\rangle_{H^{-1}(\mathbb{R}\times\mathbb{T}_L),H^1(\mathbb{R}\times\mathbb{T}_L)} \geq k_2\|w\|^2_{H^1}.$$

Proof. By the definition of S_c, $S''_{c_0}(\tilde{Q}_{c_0}) = \mathbb{L}_{c_0}$. Since $\mathcal{L}_{c_0} + n^2L^{-2}$ is positive for $|n| \geq 2$, from Lemma 5.1 we obtain that there exists $k'_2 > 0$ such that for $u \in H^1(\mathbb{R}\times\mathbb{T}_L)$ satisfying $(u, \tilde{Q}_{c_0})_{L^2} = (u, \partial_x\tilde{Q}_{c_0})_{L^2} = (u, \tilde{Q}^{\frac{3}{2}}_{c_0}\cos\frac{y}{L})_{L^2} = (u, \tilde{Q}^{\frac{3}{2}}_{c_0}\sin\frac{y}{L})_{L^2} = 0$,

$$\langle S''_{c_0}(\tilde{Q}_{c_0})u, u\rangle_{H^{-1}(\mathbb{R}\times\mathbb{T}_L),H^1(\mathbb{R}\times\mathbb{T}_L)} \geq k'_2\|u\|^2_{H^1}.$$

By a continuity argument we obtain the conclusion. □

Next, we show the orbital stability result (i) of Theorem 1.2

Proof of (i) of Theorem 1.2. Let $\varepsilon > 0$ sufficiently small. Applying Lemma 5.2–5.6, we obtain that for $u \in N^{c_0}_{\varepsilon,c_0}$

$$\begin{aligned}
&S_{c_0}(u) - S_{c_0}(\tilde{Q}_{c_0})\\
&= S_{c_0}\big(\Theta(\vec{a}(u), c(u)) + \eta(u)\big) - S_{c_0}(\tilde{Q}_{c_0})\\
&= S_{c_0}\big(\Theta(\vec{a}(u), \gamma_{c_0}(\vec{a}(u)))\big) - S_{c_0}(\tilde{Q}_{c_0})\\
&\quad + \langle S'_{c_0}\big(\Theta(\vec{a}(u), \gamma_{c_0}(\vec{a}(u)))\big), \eta(u) + \Theta(\vec{a}(u), c(u)) - \Theta\big(\vec{a}(u), \gamma_{c_0}(\vec{a}(u))\big)\rangle_{H^{-1},H^1}\\
&\quad + \frac{1}{2}\langle S''_{c_0}\big(\Theta(\vec{a}(u), \gamma_{c_0}(\vec{a}(u)))\big)\eta(u), \eta(u)\rangle_{H^{-1},H^1} + o(\|\eta(u)\|^2_{H^1})\\
&\geq \frac{5c_0C_{2,c_0}\left\|\tilde{Q}^{\frac{3}{2}}_{c_0}\cos\frac{y}{L}\right\|^2_{L^2}}{48\left\|\tilde{Q}_{c_0}\right\|^2_{L^2}}|\vec{a}(u)|^4 + k_2\|\eta(u)\|^2_{H^1}\\
&\quad + \langle S'_{c_0}\big(\Theta(\vec{a}(u), \gamma_{c_0}(\vec{a}(u)))\big), \eta(u)\rangle_{H^{-1},H^1} + o(\|\eta(u)\|^2_{H^1} + |\vec{a}(u)|^4).
\end{aligned}$$

Since $S''_{c_0}(\tilde{Q}_{c_0})\partial_c\tilde{Q}_{c_0} = -\tilde{Q}_{c_0}$ and the expansion (5.4), from Lemma 5.2 we have

$$\begin{aligned}
&\langle S'_{c_0}\big(\Theta(\vec{a}(u), \gamma_{c_0}(\vec{a}(u)))\big), \eta(u)\rangle_{H^{-1},H^1}\\
&= \langle S'_{\check{c}(\vec{a}(u))}\big(\Theta(\vec{a}(u), \gamma_{c_0}(\vec{a}(u)))\big), \eta(u)\rangle_{H^{-1},H^1}\\
&= \langle\big(S''_{\check{c}(\vec{a}(u))}(\varphi_{c_0}(\vec{a}(u))) - S''_{c_0}(\tilde{Q}_{c_0})\big)(\gamma_{c_0}(\vec{a}(u)) - c_0)\partial_c\tilde{Q}_{c_0}, \eta(u)\rangle_{H^{-1},H^1}\\
&\quad + (\gamma_{c_0}(\vec{a}(u)) - c_0)\big(\tilde{Q}_{c_0}, \eta(u)\big)_{L^2} + o(|\vec{a}(u)|^4 + \|\eta(u)\|^2_{H^1})\\
&= o(|\vec{a}(u)|^4 + \|\eta(u)\|^2_{H^1}).
\end{aligned}$$

Therefore, there exist $\varepsilon_*, k_* > 0$ such that for $u \in N^{c_0}_{\varepsilon_*,c_0}$

$$S_{c_0}(u) - S_{c_0}(\tilde{Q}_{c_0}) \geq k_*(|\vec{a}(u)|^4 + \|\eta(u)\|^2_{H^1}). \tag{5.9}$$

Now we suppose there exist $\varepsilon_0 > 0$, a sequence $\{u_n\}_n$ of solutions to (1.1) and a sequence $\{t_n\}_n$ such that $t_n > 0$, $u_n(0) \to \tilde{Q}_{c_0}$ as $n \to \infty$ in H^1 and $\mathrm{dist}_{c_0}(u_n(t_n)) > \varepsilon_0$. Let $v_n = M(\tilde{Q}_{c_0})^{\frac{1}{2}} M(u_n)^{-\frac{1}{2}} u_n(t_n)$. Then we have $M(v_n) = M(\tilde{Q}_{c_0})$, $\lim_{n\to\infty} \|v_n - u_n(t_n)\|_{H^1} = 0$ and $\lim_{n\to\infty} S_{c_0}(v_n) = S_{c_0}(\tilde{Q}_{c_0})$. Thus, by (5.9) $\lim_{n\to\infty} \vec{a}(v_n) = 0$ and $\eta(v_n) \to 0$ as $n \to \infty$ in H^1. Since $\lim_{n\to\infty} \gamma_{c_0}(\vec{a}(v_n)) = c_0$, we have $\lim_{n\to\infty} c(v_n) = c_0$. Hence, $\lim_{n\to\infty} \mathrm{dist}_{c_0}(u_n(t_n)) = 0$. This is a contradiction. We complete the proof of (i) of Theorem 1.2. □

In the following corollary, we estimate the size of the modulation parameters.

Corollary 5.7. *Let $c_0 > 0$ and $L = \frac{2}{\sqrt{5c_0}}$. Then, there exist $\delta_0, C > 0$ such that for $0 < \delta < \delta_0$ and $u_0 \in H^1(\mathbb{R} \times \mathbb{T}_L)$ with $\left\| u_0 - \tilde{Q}_{c_0} \right\|_{H^1} < \delta$, the solution u of (1.1) corresponding to the initial data u_0 satisfies*

$$|c(u(t)) - c_0| + |\vec{a}(u(t))|^2 \le C\delta, \quad t \in \mathbb{R},$$

where $c(u)$ and $\vec{a}(u)$ are defined in Lemma 5.4.

Proof. We choose $\varepsilon > 0$ which is sufficiently small. By (i) of Theorem 1.2, there exists $\delta_1 > 0$ such that for any solution u with $\left\| u(0) - \tilde{Q}_{c_0} \right\|_{H^1} = \delta < \delta_1$ satisfies $u(t) \in N_{\varepsilon,c_0}$ for $t \in \mathbb{R}$. We define $c_m > 0$ as

$$\|u_0\|_{L^2} = \left\| \tilde{Q}_{c_m} \right\|_{L^2}.$$

Applying Lemma 5.2–5.5, we obtain

$$\begin{aligned}
&S_{c_m}(u) - S_{c_m}(\tilde{Q}_{c_m}) \\
&= \frac{1}{2} \langle S''_{c_0}\big(\Theta(\vec{a}(u), \gamma_{c_m}(\vec{a}(u)))\big)\eta(u), \eta(u)\rangle_{H^{-1},H^1} \\
&\quad + \left(\frac{c_m}{c_0}\right)^{\frac{5}{2}} \frac{5c_0 C_{2,c_0} \left\| \tilde{Q}_{c_0}^{\frac{3}{2}} \cos\frac{y}{L} \right\|_{L^2}^2}{48 \left\| \tilde{Q}_{c_0} \right\|_{L^2}^2} |\vec{a}(u)|^4 + \left(1 - \frac{c_m}{c_0}\right) \left\| \partial_y \Theta\big(\vec{a}(u), \gamma_{c_m}(\vec{a}(u))\big) \right\|_{L^2}^2 \\
&\quad + o(|\vec{a}(u)|^4 + \|\eta(u_0)\|_{H^1}^2)
\end{aligned}$$

as $\delta \to 0$. Since $|c_0 - c_m| \lesssim \delta$ and

$$\begin{aligned}
\partial_y \Theta\big(\vec{a}(u), \gamma_{c_m}(\vec{a}(u))\big)(x, y) = &- \frac{a_1(u)\gamma_{c_m}(\vec{a}(u))}{c_0 L} \tilde{Q}_{c_0}^{\frac{3}{2}}\left(\sqrt{\frac{\gamma_{c_m}(\vec{a}(u))}{c_0}} x, y\right) \sin\frac{y}{L} \\
&+ \frac{a_2(u)\gamma_{c_m}(\vec{a}(u))}{c_0 L} \tilde{Q}_{c_0}^{\frac{3}{2}}\left(\sqrt{\frac{\gamma_{c_m}(\vec{a}(u))}{c_0}} x, y\right) \cos\frac{y}{L} + O(|\vec{a}(u)|^2),
\end{aligned}$$

there exist $k_3, k_4 > 0$ such that k_3 and k_4 are not depend on c_m, and

$$S_{c_m}(u) - S_{c_m}(\tilde{Q}_{c_m}) \geq k_3\|\eta(u)\|_{H^1}^2 + k_3|\vec{a}(u)|^2(|\vec{a}(u)|^2 - \delta k_4) + o(|\vec{a}(u)|^4 + \|\eta(u)\|_{H^1}^2). \tag{5.10}$$

Using the conservation laws and (5.7), we obtain

$$\begin{aligned} S_{c_m}(u) - S_{c_m}(\tilde{Q}_{c_m}) &= S_{c_m}(u_0) - S_{c_m}(\tilde{Q}_{c_m}) \\ &\lesssim \|\eta(u_0)\|_{H^1}^2 + |\vec{a}(u_0)|^2 \lesssim \delta^2. \end{aligned} \tag{5.11}$$

From (5.10) and (5.11), we have that there exist $\delta_*, k_5 > 0$ such that if $0 < \delta < \delta_*$, then

$$\|\eta(u)\|_{H^1}^2 + |\vec{a}(u)|^2(|\vec{a}(u)|^2 - \delta k_4) - k_5\delta^2 \leq 0.$$

Therefore, there exists $C(k_4, k_5) > 0$ such that

$$|\vec{a}(u)|^2 + \|\eta(u)\|_{H^1} \leq C(k_4, k_5)\delta.$$

Applying (5.8), we have

$$|c_0 - c(u)| \lesssim \|\eta(u)\|_{L^2}^2 + |\gamma_{c_m}(\vec{a}(u)) - c_m| + |c_0 - c_m| \lesssim \delta. \quad \square$$

6. Liouville property

In this section, we prove the Liouville property of (1.1). First, we show the following equation of the integration of Q_c.

Lemma 6.1. *Let $p, c > 0$. Then, we have*

$$\int_{\mathbb{R}} Q_c^{p+1} dx = \frac{3pc}{2p+1} \int_{\mathbb{R}} Q_c^p dx. \tag{6.1}$$

Proof. Since

$$-\partial_x^2 Q_c + cQ_c - Q_c^2 = 0, \tag{6.2}$$

we have

$$\begin{aligned} \int_{\mathbb{R}} Q_c^{p+1} dx &= -\int_{\mathbb{R}} Q_c^{p-1}\partial_x^2 Q_c dx + c\int_{\mathbb{R}} Q_c^p dx \\ &= (p-1)\int_{\mathbb{R}} Q_c^{p-2}(\partial_x Q_c)^2 dx + c\int_{\mathbb{R}} Q_c^p dx. \end{aligned}$$

Multiplying (6.2) by $\partial_x Q_c$ and integrating this, we obtain

$$-(\partial_x Q_c)^2 + cQ_c^2 - \frac{2}{3}Q_c^3 = 0.$$

Thus,

$$\int_{\mathbb{R}} Q_c^{p+1}dx = (p-1)\int_{\mathbb{R}} Q_c^{p-2}\Big(cQ_c^2 - \frac{2}{3}Q_c^3\Big)dx + c\int_{\mathbb{R}} Q_c^p dx \tag{6.3}$$

which implies (6.1). □

Let

$$\phi_c(x) = -\frac{\partial_x Q_c(x)}{Q_c(x)} = \sqrt{c}\tanh\frac{\sqrt{c}x}{2}.$$

Then, $\phi_c(x) \to \pm\sqrt{c}$ as $x \to \pm\infty$ and

$$\partial_x\phi_c(x) = \frac{c}{2}\cosh^{-2}\frac{\sqrt{c}x}{2} = \frac{1}{3}Q_c.$$

We introduce the following coerciveness type lemma in [32].

Lemma 6.2. *For $u \in H^1(\mathbb{R})$*

$$\begin{aligned}
-\int_{\mathbb{R}} \partial_x u\mathcal{L}_c(u\phi_c)dx &= \frac{3}{2}\int_{\mathbb{R}}\Big(\partial_x\Big(\frac{u}{Q_c}\Big)\Big)^2 Q_c^2\partial_x\phi_c dx \\
&\geq \frac{5c}{8}\Big(\int_{\mathbb{R}} 3|u|^2\partial_x\phi_c dx - \|Q_c\|_{L^3(\mathbb{R})}^{-3}\Big(\int_{\mathbb{R}} uQ_c^2 dx\Big)^2\Big).
\end{aligned}$$

Proof. Let $v = \frac{u}{Q_c}$. Since

$$\mathcal{L}_c(u\phi_c) = \mathcal{L}_c(v\partial_x Q_c) = -2\partial_x v\partial_x^2 Q_c - \partial_x^2 v\partial_x Q_c,$$

we have

$$-\int_{\mathbb{R}} \partial_x u\mathcal{L}_c(u\phi_c)dx = \int_{\mathbb{R}} \partial_x(Q_c v)\mathcal{L}_c(v\partial_x Q_c)dx = \frac{1}{2}\int_{\mathbb{R}} (\partial_x v)^2 Q_c^3 dx.$$

Let $w = vQ_c^{\frac{3}{2}}$. Using

$$\partial_x^2 Q_c = cQ_c - Q_c^2, \quad (\partial_x Q_c)^2 = cQ_c^2 - \frac{2}{3}Q_c^3,$$

we obtain that

$$\frac{1}{2}\int_{\mathbb{R}}(\partial_x v)^2 Q_c^3 dx = \frac{1}{2}\int_{\mathbb{R}} w\Big(-\partial_x^2 w + \frac{3}{2}\partial_x^2 Q_c Q_c^{-1} w + \frac{3}{4}(\partial_x Q_c)^2 Q_c^{-2} w\Big)dx$$
$$= \frac{1}{2}\int_{\mathbb{R}} w\Big(\mathcal{L}_c + \frac{5c}{4}\Big) w dx.$$

From the properties of $\mathcal{L}_c$, the operator $\mathcal{L}_c + \frac{5c}{4}$ is non-negative and the kernel of $\mathcal{L}_c + \frac{5c}{4}$ is spanned by $Q_c^{\frac{3}{2}}$. Moreover, the second eigenvalue of $\mathcal{L}_c + \frac{5c}{4}$ is $\frac{5c}{4}$. Therefore, we have

$$\int_{\mathbb{R}} w\Big(\mathcal{L}_c + \frac{5c}{4}\Big) w dx \geq \frac{5c}{4}\Big(\|w\|_{L^2}^2 - \|Q_c\|_{L^3(\mathbb{R})}^{-3}\Big(\int_{\mathbb{R}} w Q_c^{\frac{2}{3}} dx\Big)^2\Big)$$
$$= \frac{5c}{4}\Big(\int_{\mathbb{R}} 3|u|^2 \partial_x \phi_c dx - \|Q_c\|_{L^3(\mathbb{R})}^{-3}\Big(\int_{\mathbb{R}} u Q_c^2 dx\Big)^2\Big). \quad \square$$

6.1. Monotonicity properties

In this subsection, we show the monotonicity properties of (1.1). By Proposition 2.2, the equation (1.1) has the Kato type local smoothing effect. Therefore, the proof of the monotonicity properties is similar to one in [5,32]. Thus, we omit the detail of proofs in this subsection, see Section 3 in [5].

We define $\psi_R \in C^\infty(\mathbb{R}, \mathbb{R})$ by

$$\psi_R(x) = \frac{2}{\pi}\arctan(e^{x/R}), \quad x \in \mathbb{R}. \tag{6.4}$$

Then, we have $\lim_{x\to\infty}\psi_R(x) = 1$, $\lim_{x\to-\infty}\psi_R(x) = 0$,

$$\partial_x \psi_R(x) = \frac{1}{\pi R\cosh(x/R)} \text{ and } |\partial_x^3 \psi_R(x)| \leq \frac{1}{R^2}\partial_x \psi_R(x).$$

Let $\varepsilon, \beta, c_0 > 0$ and u be a solution to (1.1) satisfying that there exists $\rho \in C(\mathbb{R}, \mathbb{R})$ such that

$$\Big\| u(t,\cdot,\cdot) - \tilde{Q}_{c_0}(\cdot - \rho(t), \cdot)\Big\|_{H^1} < \varepsilon_0, \quad t \in \mathbb{R} \tag{6.5}$$

and

$$|\dot{\rho}(t) - c_0| \leq c_0/2, \quad t \in \mathbb{R}. \tag{6.6}$$

For $x_0, t_0, t \in \mathbb{R}$ we define

$$\tilde{x} = \tilde{x}(x_0, t_0, t) = x - \rho(t_0) + \frac{\beta(t_0 - t)}{2} - x_0,$$
$$\tilde{x}_- = \tilde{x}(-x_0, t, t_0,)$$

$$I_{x_0,t_0}(u(t)) = \int_{\mathbb{R}\times\mathbb{T}_L} |u(t,x,y)|^2 \psi_R(\tilde{x}(t))dxdy,$$

and

$$I^-_{x_0,t_0}(u(t)) = \int_{\mathbb{R}\times\mathbb{T}_L} |u(t,x,y)|^2 \psi_R(\tilde{x}_-(t))dxdy.$$

In the following lemma, we show the property of the parameter ρ (see Lemma 3.2 in [5]).

Lemma 6.3. *Assume that $u \in C(\mathbb{R}, H^1(\mathbb{R}\times\mathbb{T}_L))$ is a solution to (1.1) satisfying (6.5), (6.6) and that there exist $\tilde{\rho} \in C(\mathbb{R},\mathbb{R})$ and $C, \delta_0 > 0$ such that*

$$\int_{\mathbb{T}_L} |u(t, x+\tilde{\rho}(t), y)|^2 dy \le Ce^{-\delta_0|x|}, \quad (t,x) \in \mathbb{R}^2. \tag{6.7}$$

If $0 < \varepsilon_0 < \frac{1}{2}\left\| \tilde{Q}_{c_0} \right\|_{L^2(|x|\le 1)}$, then u satisfies

$$\int_{\mathbb{T}_L} |u(t, x+\rho(t), y)|^2 dy \lesssim e^{-\delta_0|x|}, \quad (t,x) \in \mathbb{R}^2, \tag{6.8}$$

where

$$\|u\|^2_{L^2(|x|\le R)} = \int_{|x|\le R} |u|^2 dxdy.$$

The following two lemmas show the L^2-monotonicity property of (1.1).

Lemma 6.4. *Let $0 < \beta < c_0/2$. Assume that $u \in C(\mathbb{R}, H^1(\mathbb{R}\times\mathbb{T}_L))$ is a solution to (1.1) satisfying (6.5) and (6.6). Then, for $x_0 > 0$, $t_0 \in \mathbb{R}$, $R \ge 2/\sqrt{\beta}$ and $t \le t_0$*

$$I_{x_0,t_0}(u(t_0)) - I_{x_0,t_0}(u(t)) \lesssim e^{-x_0/R}, \tag{6.9}$$

if $\varepsilon_0 > 0$ in (6.5) is chosen small enough. Moreover, if u satisfies the decay assumption (6.8), then

$$\int_{\mathbb{R}\times\mathbb{T}_L} |u(t_0,x,y)|^2 \psi_R(\tilde{x}(t_0))dxdy$$
$$+ \int_{-\infty}^{t_0}\int_{\mathbb{R}\times\mathbb{T}_L} (|\nabla u|^2 + |u|^2)(t,x,y)\partial_x \psi_R(\tilde{x}(t))dxdydt \lesssim e^{-x_0/R}. \tag{6.10}$$

Lemma 6.5. *Let $0 < \beta < c_0/2$. Assume that $u \in C(\mathbb{R}, H^1(\mathbb{R} \times \mathbb{T}_L))$ is a solution to (1.1) satisfying (6.5) and (6.6). Then, for $x_0 > 0$, $t_0 \in \mathbb{R}$, $R \geq 2/\sqrt{\beta}$ and $t \geq t_0$*

$$I^-_{x_0,t_0}(u(t)) - I^-_{x_0,t_0}(u(t_0)) \lesssim e^{-x_0/R}, \tag{6.11}$$

if $\varepsilon_0 > 0$ in (6.5) is chosen small enough.

The proof of Lemma 6.4 follows the proof of Lemma 3.3 in [5]. The proof of Lemma 6.5 is similar to the proof of Lemma 4.9 in [5].

We define a functional J by

$$J_{x_0,t_0}(u(t)) = \int\limits_{\mathbb{R}\times\mathbb{T}_L} \Big(|\nabla u|^2 - \frac{2}{3}u^3\Big)(t,x,y)\psi_R(\tilde{x})dxdy.$$

In the following lemma, we show the monotonicity property for J.

Lemma 6.6. *Let $0 < \beta < c_0/2$. Assume that $u \in C(\mathbb{R}, H^1(\mathbb{R} \times \mathbb{T}_L))$ is a solution to (1.1) satisfying (6.5) and (6.6). Then, for $x_0 > 0$, $t_0 \in \mathbb{R}$, $R \geq 2/\sqrt{\beta}$ and $t \leq t_0$*

$$J_{x_0,t_0}(u(t_0)) - J_{x_0,t_0}(u(t)) \lesssim e^{-x_0/R}. \tag{6.12}$$

Moreover, if u satisfies the decay assumption (6.8), then

$$\int\limits_{\mathbb{R}\times\mathbb{T}_L} |\nabla u|^2(t_0)\psi_R(\tilde{x}(t_0))dxdy \tag{6.13}$$

$$+ \int\limits_{-\infty}^{t_0}\int\limits_{\mathbb{R}\times\mathbb{T}_L} (|\nabla^2 u(t)|^2 + |\nabla u(t)|^2 + u(t)^4)(\partial_x\psi_R)(\tilde{x}(t))dxdydt \lesssim e^{-x_0/R}. \tag{6.14}$$

The proof of Lemma 6.6 is similar to the proof of Lemma 3.4 in [5].

The following proposition shows the boundedness of higher Sobolev norm of solutions satisfying the decay assumption (6.8).

Proposition 6.7. *Let $0 < \beta < c_0/2$ and $k \in \mathbb{Z}_+$. Assume that $u \in C(\mathbb{R}, H^1(\mathbb{R} \times \mathbb{T}_L))$ is a solution to (1.1) satisfying (6.5), (6.6) and the decay assumption (6.8). If $\varepsilon_0 > 0$ in (6.5) is sufficiently small, there exist $\tilde{\delta}, C = C(k) > 0$ such that*

$$\sup_{t\in\mathbb{R}} \int\limits_{\mathbb{R}\times\mathbb{T}_L} (\partial^\alpha u)^2(t, x+\rho(t), y)e^{\tilde{\delta}|x|}dxdy \leq C, \tag{6.15}$$

for $\alpha \in (\mathbb{N}_0)^2$ satisfying $|\alpha| \leq k$.

The proof of this proposition is same as the proof of Corollary 3.9 in [5].

6.2. Critical case $L = \frac{2}{\sqrt{5c_0}}$

In this section, we show the Liouville property for $L = \frac{2}{\sqrt{5c_0}}$.

Lemma 6.8. *There exist $\varepsilon_0, K_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ the following is true. For any solution $u \in C(\mathbb{R}, H^1(\mathbb{R} \times \mathbb{T}_L))$ of (1.1) satisfying*

$$\inf_{b \in \mathbb{R}} \left\| u(t, \cdot, \cdot) - Q_{c_0}(\cdot - b, \cdot) \right\|_{H^1} \leq \varepsilon$$

there exist $\vec{a} = (a_1, a_2) \in C^1(\mathbb{R}, \mathbb{R}^2)$ and $\rho, c \in C^1(\mathbb{R}, \mathbb{R})$ uniquely such that

$$\eta(t, x, y) = u(t, x + \rho(t), y) - \Theta(\vec{a}(t), c(t)) \tag{6.16}$$

satisfies for all $t \in \mathbb{R}$

$$|c(t) - c_0| + |a_1(t)| + |a_2(t)| + \|\eta(t)\|_{H^1} \leq K_0 \varepsilon, \tag{6.17}$$

$$\begin{aligned} &\int_{\mathbb{R}\times\mathbb{T}_L} \eta(t)\partial_x\Theta(\vec{a}(t), c(t))dxdy = \int_{\mathbb{R}\times\mathbb{T}_L} \eta(t)\Theta(\vec{a}(t), c(t))dxdy \\ &= \int_{\mathbb{R}\times\mathbb{T}_L} \eta(t)\Theta(\vec{a}(t), c(t))^{\frac{3}{2}} \cos\frac{y}{L}dxdy = \int_{\mathbb{R}\times\mathbb{T}_L} \eta(t)\Theta(\vec{a}(t), c(t))^{\frac{3}{2}} \sin\frac{y}{L}dxdy = 0 \end{aligned} \tag{6.18}$$

and

$$|\dot{\vec{a}}(t)| \leq K_0 \|\eta(t)\|_{L^2}, \tag{6.19}$$

$$|\dot{c}(t)| \leq \varepsilon K_0 \|\eta(t)\|_{L^2}, \tag{6.20}$$

$$|\dot{\rho}(t) - \hat{c}(t)| \leq K_0(\|\eta(t)\|_{L^2} + |c - c_0||\vec{a}|), \tag{6.21}$$

where $\hat{c}(t) = c_0^{-1} c(t) \check{c}(\vec{a}(t))$.

Proof. From Lemma 5.4, there exist C^1 mappings $\rho(t) = \rho(u(t)), c(t) = c(u(t)), \vec{a}(t) = \vec{a}(u(t)), \eta(t) = \eta(u(t))$ satisfying (6.16)–(6.18). By the calculation we have

$$\begin{aligned} \eta_t &= \partial_x(-\Delta\eta - 2\Theta\eta - \eta^2 - \Delta\Theta - \Theta^2) + \dot{\rho}\partial_x(\eta + \Theta) - \dot{\vec{a}} \cdot \partial_{\vec{a}}\Theta - \dot{c}\partial_c\Theta \\ &= \partial_x(\mathbb{L}_{c_0}\eta - \eta^2) + \partial_x S'_{c_0}(\Theta) - 2\partial_x((\Theta - \tilde{Q}_{c_0})\eta) + (\dot{\rho} - c_0)\partial_x(\eta + \Theta) - \dot{\vec{a}} \cdot \partial_{\vec{a}}\Theta - \dot{c}\partial_c\Theta, \end{aligned} \tag{6.22}$$

where $S'_{c_0}(\Theta) = -\Delta\Theta + c_0\Theta - \Theta^2$ and $\dot{\vec{a}} \cdot \partial_{\vec{a}}\Theta = \dot{a}_1\partial_{a_1}\Theta + \dot{a}_2\partial_{a_2}\Theta$. From (6.18) and $\Theta(x, y) = \Theta(-x, y)$ we obtain that

$$
\begin{aligned}
0 &= \frac{d}{dt} \int\limits_{\mathbb{R}\times\mathbb{T}_L} \eta\Theta dxdy \\
&= -\int\limits_{\mathbb{R}\times\mathbb{T}_L} \Theta\dot{\vec{a}}\cdot\partial_{\vec{a}}\Theta dxdy - \dot{c}\int\limits_{\mathbb{R}\times\mathbb{T}_L} \tilde{Q}_{c_0}\partial_c\tilde{Q}_{c_0}dxdy \\
&\quad + O\big(\|\eta\|_{L^2}(\|\eta\|_{L^2} + |\vec{a}| + |c-c_0| + |\dot{\vec{a}}| + |\dot{c}|)\big).
\end{aligned}
$$

By the expansion

$$
\Theta(\vec{a},c) = \tilde{Q}_{c_0} + O(|\vec{a}| + |c-c_0|),
$$

$$
\dot{\vec{a}}\cdot\partial_{\vec{a}}\Theta(\vec{a},c) = \dot{a}_1\tilde{Q}_{c_0}^{\frac{3}{2}}\cos\frac{y}{L} + \dot{a}_2\tilde{Q}_{c_0}^{\frac{3}{2}}\sin\frac{y}{L} + O((|\vec{a}| + |c-c_0|)|\dot{\vec{a}}|),
$$

we have

$$
\int\limits_{\mathbb{R}\times\mathbb{T}_L} \Theta\dot{\vec{a}}\cdot\partial_{\vec{a}}\Theta dxdy = O((|\vec{a}| + |c-c_0|)|\dot{\vec{a}}|).
$$

Since $\int_{\mathbb{R}\times\mathbb{T}_L} \tilde{Q}_{c_0}\partial_c\tilde{Q}_{c_0}dxdy \neq 0$,

$$
|\dot{c}| = O\big(\|\eta\|_{L^2}(\|\eta\|_{L^2} + |\vec{a}| + |c-c_0| + |\dot{\vec{a}}|) + (|\vec{a}| + |c-c_0|)|\dot{\vec{a}}|\big). \tag{6.23}
$$

From (6.18), (6.23) and $\Theta(x,y) = \Theta(-x,y)$, we obtain that

$$
\begin{aligned}
0 &= \frac{d}{dt}\int\limits_{\mathbb{R}\times\mathbb{T}_L} \eta\Theta^{\frac{3}{2}}\cos\frac{y}{L}dxdy \\
&= -\dot{a}_1\int\limits_{\mathbb{R}\times\mathbb{T}_L} \Big(\Theta^{\frac{3}{2}}\cos\frac{y}{L}\Big)\partial_{a_1}\Theta dxdy + O\big(\|\eta\|_{L^2} + (|\vec{a}| + |c-c_0|)|\dot{\vec{a}}|\big).
\end{aligned}
$$

Since

$$
\int\limits_{\mathbb{R}\times\mathbb{T}_L} \Big(\Theta^{\frac{3}{2}}\cos\frac{y}{L}\Big)\partial_{a_1}\Theta dxdy = \int\limits_{\mathbb{R}\times\mathbb{T}_L} \Big(\tilde{Q}_{c_0}^{\frac{3}{2}}\cos\frac{y}{L}\Big)^2 dxdy + O(|\vec{a}| + |c-c_0|),
$$

we obtain

$$
|\dot{a}_1| = O\big(\|\eta\|_{L^2} + (|\vec{a}| + |c-c_0|)|\dot{a}_2|\big). \tag{6.24}
$$

By the same way, from (6.23) and (6.24) we get

$$
|\dot{a}_2| = O\big(\|\eta\|_{L^2}\big). \tag{6.25}
$$

The estimates (6.19) and (6.20) follow (6.23)–(6.25). By the similar computation to (6.22),

$$\eta_t = \partial_x(\mathbb{L}_{\hat{c}}\eta - \eta^2) + \partial_x S'_{\hat{c}}(\Theta) - 2\partial_x((\Theta - \tilde{Q}_{\hat{c}})\eta) + (\dot{\rho} - \hat{c})\partial_x(\eta + \Theta) - \dot{\vec{a}} \cdot \partial_{\vec{a}}\Theta - \dot{c}\partial_c\Theta. \tag{6.26}$$

By the definition of Θ and $\hat{c}$ we have

$$\begin{aligned} S'_{\hat{c}}(\Theta) &= \frac{c^2}{c_0^2}(-\Delta\varphi_{c_0} + \check{c}\varphi_{c_0} - (\varphi_{c_0})^2) + \frac{c(c-c_0)}{c_0^2}\partial_y^2\varphi_{c_0} \\ &= \frac{c-c_0}{c_0}\partial_y^2\Theta. \end{aligned} \tag{6.27}$$

Since

$$S'_{\hat{c}}(\Theta) = \frac{c-c_0}{c_0}\partial_y^2\Theta = O(|c_0 - c||\vec{a}|), \tag{6.28}$$

from (6.18), (6.23)–(6.25), we obtain that

$$\begin{aligned} 0 &= \frac{d}{dt}\int_{\mathbb{R}\times\mathbb{T}_L} \eta(\partial_x\Theta)dxdy \\ &= (\dot{\rho} - \hat{c})\int_{\mathbb{R}\times\mathbb{T}_L} (\partial_x\tilde{Q}_{c_0})^2 dxdy + O\big(\|\eta\|_{L^2} + |c_0 - c||\vec{a}|\big). \end{aligned}$$

Thus, the estimate (6.21) holds. □

Next we prove the following Liouville type theorem.

Theorem 6.9. *Let $c_0 > 0$ and $L = \frac{2}{\sqrt{5c_0}}$. There exists $\varepsilon_0 > 0$ satisfies the following. For any solution $u \in C(\mathbb{R}, H^1(\mathbb{R}\times\mathbb{T}_L))$ to (1.1) satisfying (6.5) and (6.8), there exist $c_+ > 0$, $\vec{a}_+ = (a_{1,+}, a_{2,+})$ and ρ_0 such that*

$$\begin{gathered} u(t,x,y) = \Theta(\vec{a}_+, c_+)(x - \hat{c}_+ t + \rho_0, y), \\ |c_+ - c_0||\vec{a}_+| = 0, \end{gathered}$$

where

$$\hat{c}_+ = \begin{cases} c_+, & \vec{a}_+ = (0,0), \\ \check{c}(\vec{a}_+), & c_+ = c_0. \end{cases}$$

Proof. Let $u \in C(\mathbb{R}, H^1(\mathbb{R}\times\mathbb{T}_L))$ be solution to (1.1) satisfying (6.5) and (6.8). From Lemma 6.3 and 6.8, ρ in Lemma 6.8 satisfies (6.5) and (6.8). Let $\eta(t)$, $c(t)$, $\vec{a}(t)$, $\hat{c}(t)$ be in Lemma 6.8. We define

$$v = \mathbb{L}_{\hat{c}}\eta - \eta^2.$$

Then, v has the following almost orthogonal condition.

$$\int_{\mathbb{R}\times\mathbb{T}_L} v\partial_x \tilde{Q}_{\hat{c}} dxdy = \int_{\mathbb{R}\times\mathbb{T}_L} (\mathbb{L}_{\hat{c}}\eta - \eta^2)\partial_x \tilde{Q}_{\hat{c}} dxdy = O(\|\eta\|_{L^2}^2), \tag{6.29}$$

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{T}_L} v\partial_c \tilde{Q}_{\hat{c}} dxdy &= -\int_{\mathbb{R}\times\mathbb{T}_L} \eta \tilde{Q}_{\hat{c}} dxdy + O(\|\eta\|_{L^2}^2) \\ &= O(\|\eta\|_{L^2}(|c-c_0| + |\vec{a}| + \|\eta\|_{L^2})),\end{aligned} \tag{6.30}$$

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{T}_L} v\tilde{Q}_{\hat{c}}^{\frac{3}{2}} \cos\frac{y}{L} dxdy &= \int_{\mathbb{R}\times\mathbb{T}_L} \eta\mathbb{L}_{\hat{c}}\Big(\tilde{Q}_{\hat{c}}^{\frac{3}{2}} \cos\frac{y}{L}\Big) dxdy + O(\|\eta\|_{L^2}^2) \\ &= O(\|\eta\|_{L^2}(|c-c_0| + |\vec{a}| + \|\eta\|_{L^2})),\end{aligned} \tag{6.31}$$

$$\int_{\mathbb{R}\times\mathbb{T}_L} v\tilde{Q}_{\hat{c}}^{\frac{3}{2}} \sin\frac{y}{L} dxdy = O(\|\eta\|_{L^2}(|c-c_0| + |\vec{a}| + \|\eta\|_{L^2})). \tag{6.32}$$

From the orthogonal conditions (6.18) and Lemma 5.6, we have

$$(v,\eta)_{L^2(\mathbb{R}\times\mathbb{T}_L)} = (\mathbb{L}_{\hat{c}}\eta,\eta)_{L^2(\mathbb{R}\times\mathbb{T}_L)} - \|\eta\|_{L^3(\mathbb{R}\times\mathbb{T}_L)}^3 \geq k_2\|\eta\|_{H^1}^2 + O(\|\eta\|_{H^1}^3).$$

Therefore, if $\varepsilon_0 > 0$ is sufficiently small, then for $t \in \mathbb{R}$

$$\|\eta\|_{H^1} \lesssim \|v\|_{L^2}. \tag{6.33}$$

By (6.26), we have

$$v_t = \mathbb{L}_{\hat{c}}\eta_t + (\partial_t\mathbb{L}_{\hat{c}})\eta - 2\eta\eta_t = \mathbb{L}_{\hat{c}}\partial_x v + \mathbb{L}_{\hat{c}}\partial_x S_{\hat{c}}'(\Theta) + R(\eta,\vec{a},c), \tag{6.34}$$

where

$$\begin{aligned}R(\eta,\vec{a},c) &= -2\eta\partial_x(v + S_{\hat{c}}'(\Theta)) + (\dot{\rho}-\hat{c})\mathbb{L}_{\hat{c}}\partial_x(\eta+\Theta) - 2(\dot{\rho}-\hat{c})\eta\partial_x(\eta+\Theta) \\ &\quad + (\mathbb{L}_{\hat{c}} - 2\eta)(2\partial_x((\tilde{Q}_{\hat{c}} - \Theta)\eta) - \dot{\vec{a}}\cdot\partial_{\vec{a}}\Theta - \dot{c}\partial_c\Theta) + (\partial_t\mathbb{L}_{\hat{c}})\eta.\end{aligned}$$

Therefore,

$$\begin{aligned}&-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}\times\mathbb{T}_L} (v + S_{\hat{c}}'(\Theta))^2\phi_{\hat{c}} dxdy \\ &= -\int_{\mathbb{R}\times\mathbb{T}_L} (\mathbb{L}_{\hat{c}}\partial_x v)v\phi_{\hat{c}} dxdy - \int_{\mathbb{R}\times\mathbb{T}_L} (\mathbb{L}_{\hat{c}}\partial_x S_{\hat{c}}'(\Theta))v\phi_{\hat{c}} dxdy - \int_{\mathbb{R}\times\mathbb{T}_L} (\mathbb{L}_{\hat{c}}\partial_x v)S_{\hat{c}}'(\Theta)\phi_{\hat{c}} dxdy \\ &\quad - \int_{\mathbb{R}\times\mathbb{T}_L} (\mathbb{L}_{\hat{c}}\partial_x S_{\hat{c}}'(\Theta))S_{\hat{c}}'(\Theta)\phi_{\hat{c}} dxdy - \int_{\mathbb{R}\times\mathbb{T}_L} R(\eta,\vec{a},c)(v + S_{\hat{c}}'(\Theta))\phi_{\hat{c}} dxdy\end{aligned}$$

$$-\int_{\mathbb{R}\times\mathbb{T}_L}(\partial_t S'_{\hat{c}}(\Theta))(v+S'_{\hat{c}}(\Theta))\phi_{\hat{c}}dxdy-\frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}_L}(v+S'_{\hat{c}}(\Theta))^2\dot{\hat{c}}\partial_c\phi_{\hat{c}}dxdy. \tag{6.35}$$

We estimate each term in (6.35) separately.

(I) The estimate of $-\int_{\mathbb{R}\times\mathbb{T}_L}(\mathbb{L}_{\hat{c}}\partial_x v)v\phi_{\hat{c}}dxdy$. From the Fourier expansion $v(t,x,y)=v_0(t,x)+\sum_{n=1}^{\infty}(v_{n,1}(t,x)\cos\frac{y}{L}+v_{n,2}(t,x)\sin\frac{y}{L})$ and Lemma 6.2 we have

$$\begin{aligned}
&-\int_{\mathbb{R}\times\mathbb{T}_L}(\mathbb{L}_{\hat{c}}\partial_x v)v\phi_{\hat{c}}dxdy\\
&=-2\pi L\int_{\mathbb{R}\times\mathbb{T}_L}\Big(\mathcal{L}_{\hat{c}}\partial_x v_0\Big)v_0\phi_{\hat{c}}dx-\pi L\sum_{n\in\mathbb{Z}_+,j=1,2}\int_{\mathbb{R}}\Big(\Big(\mathcal{L}_{\hat{c}}+\frac{n^2}{L^2}\Big)\partial_x v_{n,j}\Big)v_{n,j}\phi_{\hat{c}}dx\\
&\geq\frac{5\pi L\hat{c}}{4}\Big(\int_{\mathbb{R}}|v_0|^2Q_{\hat{c}}dx-\|Q_{\hat{c}}\|_{L^3(\mathbb{R})}^{-3}\Big(\int_{\mathbb{R}}v_0Q_{\hat{c}}^2dx\Big)^2\Big)\\
&\quad+\pi L\Big(\Big(\frac{5\hat{c}}{8}+\frac{1}{6L^2}\Big)\int_{\mathbb{R}}(|v_{1,1}|^2+|v_{1,2}|^2)Q_{\hat{c}}dx-\frac{5\hat{c}}{8}\|Q_{\hat{c}}\|_{L^3(\mathbb{R})}^{-3}\Big(\int_{\mathbb{R}}v_{1,1}Q_{\hat{c}}^2dx\Big)^2\\
&\quad-\frac{5\hat{c}}{8}\|Q_{\hat{c}}\|_{L^3(\mathbb{R})}^{-3}\Big(\int_{\mathbb{R}}v_{1,2}Q_{\hat{c}}^2dx\Big)^2\Big)\\
&\quad+\pi L\sum_{n=2}^{\infty}\frac{n^2}{6L^2}\int_{\mathbb{R}}(|v_{n,1}|^2+|v_{n,2}|^2)Q_{\hat{c}}dx.
\end{aligned}\tag{6.36}$$

From the almost orthogonal condition (6.30)

$$\begin{aligned}
\Big|\int_{\mathbb{R}}v_0Q_{c_0}^2dx\Big|^2&\leq\Big|\int_{\mathbb{R}}v_0\Big(Q_{c_0}^2-\Big\|Q_{c_0}^{-\frac{1}{2}}\partial_cQ_{c_0}\Big\|_{L^2(\mathbb{R})}^{-2}\int_{\mathbb{R}}Q_{c_0}\partial_cQ_{c_0}dx'\partial_cQ_{c_0}\Big)dx\Big|^2\\
&\quad+O((|\vec{a}|+|c-c_0|+\|\eta\|_{L^2})\|\eta\|_{L^2}\|v\|_{L^2})\\
&\leq\Big\|v_0Q_{c_0}^{\frac{1}{2}}\Big\|_{L^2(\mathbb{R})}^2\Big\|Q_{c_0}^{\frac{3}{2}}-\Big\|Q_{c_0}^{-\frac{1}{2}}\partial_cQ_{c_0}\Big\|_{L^2(\mathbb{R})}^{-2}\int_{\mathbb{R}}Q_{c_0}\partial_cQ_{c_0}dx'Q_{c_0}^{-\frac{1}{2}}\partial_cQ_{c_0}\Big\|_{L^2(\mathbb{R})}^2\\
&\quad+O((|\vec{a}|+|c-c_0|+\|\eta\|_{L^2})\|\eta\|_{L^2}\|v\|_{L^2}).
\end{aligned}\tag{6.37}$$

Since

$$\Big\|Q_{c_0}^{\frac{3}{2}}-\Big\|Q_{c_0}^{-\frac{1}{2}}\partial_cQ_{c_0}\Big\|_{L^2(\mathbb{R})}^{-2}\int_{\mathbb{R}}Q_{c_0}\partial_cQ_{c_0}dx'Q_{c_0}^{-\frac{1}{2}}\partial_cQ_{c_0}\Big\|_{L^2(\mathbb{R})}^2<\|Q_{c_0}\|_{L^3(\mathbb{R})}^3,$$

from (6.37) we obtain

$$\int_{\mathbb{R}} |v_0|^2 Q_{c_0} dx - \|Q_{c_0}\|_{L^3(\mathbb{R})}^{-3} \Big(\int_{\mathbb{R}} v_0 Q_{c_0}^2 dx\Big)^2$$
$$\gtrsim \int_{\mathbb{R}} |v_0|^2 Q_{c_0} dx + O((|\vec{a}| + |c - c_0| + \|\eta\|_{L^2})\|\eta\|_{L^2}\|v\|_{L^2}). \tag{6.38}$$

On the other hand, from the almost orthogonal conditions (6.31) and (6.32)

$$\begin{aligned}\Big|\int_{\mathbb{R}} v_{1,j} Q_{c_0}^2 dx\Big|^2 &= \Big|\int_{\mathbb{R}} v_{1,j}\Big(Q_{c_0}^{\frac{3}{2}} - \|Q_{c_0}\|_{L^2(\mathbb{R})}^{-2}\int_{\mathbb{R}} Q_{c_0}^{\frac{5}{2}} dx' Q_{c_0}\Big) Q_{c_0}^{\frac{1}{2}} dx\Big|^2 \\ &\quad + O((|\vec{a}| + |c - c_0| + \|\eta\|_{L^2})\|\eta\|_{L^2}\|v\|_{L^2}) \\ &\le \Big\|Q_{c_0}^{\frac{3}{2}} - \|Q_{c_0}\|_{L^2(\mathbb{R})}^{-2}\int_{\mathbb{R}} Q_{c_0}^{\frac{5}{2}} dx' Q_{c_0}\Big\|_{L^2(\mathbb{R})}^2 \int_{\mathbb{R}} |v_{1,j}|^2 Q_{c_0} dx \\ &\quad + O((|\vec{a}| + |c - c_0| + \|\eta\|_{L^2})\|\eta\|_{L^2}\|v\|_{L^2}).\end{aligned} \tag{6.39}$$

By Lemma 6.1,

$$\int_{\mathbb{R}} Q_{c_0}^{\frac{7}{2}} dx = \frac{5c_0}{4}\int_{\mathbb{R}} Q_{c_0}^{\frac{5}{2}} dx \quad \int_{\mathbb{R}} Q_{c_0}^{\frac{5}{2}} dx = \frac{9c_0}{8}\int_{\mathbb{R}} Q_{c_0}^{\frac{3}{2}} dx. \tag{6.40}$$

From Hölder's inequality, (6.39) and (6.40), we obtain

$$\begin{aligned}&\int_{\mathbb{R}} |v_{1,j}|^2 Q_{c_0} dx - \|Q_{c_0}\|_{L^3(\mathbb{R})}^{-3}\Big(\int_{\mathbb{R}} v_{1,j} Q_{c_0}^2 dx\Big)^2 \\ &\ge \Big\|Q_{c_0}^{\frac{5}{4}}\Big\|_{L^2(\mathbb{R})}^{-2}\Big\|Q_{c_0}^{\frac{7}{4}}\Big\|_{L^2(\mathbb{R})}^{-1}\Big\|Q_{c_0}^{\frac{3}{4}}\Big\|_{L^2(\mathbb{R})}^{-1}\Big(\int_{\mathbb{R}} Q_{c_0}^{\frac{5}{2}} dx\Big)^2\int_{\mathbb{R}} |v_{1,j}|^2 Q_{c_0} dx \\ &\quad + O((|\vec{a}| + |c - c_0| + \|\eta\|_{L^2})\|\eta\|_{L^2}^2) \\ &= \sqrt{\frac{9}{10}}\int_{\mathbb{R}} |v_{1,j}|^2 Q_{c_0} dx + O((|\vec{a}| + |c - c_0| + \|\eta\|_{L^2})\|\eta\|_{L^2}\|v\|_{L^2}).\end{aligned} \tag{6.41}$$

By (6.37), (6.38) and (6.41), we obtain that there exists $k_3 > 0$ such that

$$\begin{aligned}&-\int_{\mathbb{R}\times\mathbb{T}_L} (\mathbb{L}_{\hat{c}}\partial_x v) v \phi_{\hat{c}} dx dy \\ &\ge k_3 \int_{\mathbb{R}} |v_0|^2 Q_{c_0} dx + \pi L\Big(\frac{5c_0}{8}\sqrt{\frac{9}{10}} + \frac{1}{6L^2}\Big)\int_{\mathbb{R}} (|v_{1,1}|^2 + |v_{1,2}|^2) Q_{c_0} dx\end{aligned}$$

$$+\pi L\sum_{n=2}^{\infty}\frac{n^2}{6L^2}\int_{\mathbb{R}}(|v_{n,1}|^2+|v_{n,2}|^2)Q_{c_0}dx$$
$$+O((|\vec{a}|+|c-c_0|+\|\eta\|_{L^2})\|\eta\|_{H^1}\|v\|_{L^2}). \tag{6.42}$$

(II) The estimate of $-\int_{\mathbb{R}\times\mathbb{T}_L}(\mathbb{L}_{\hat{c}}\partial_x S'_{\hat{c}}(\Theta))S'_{\hat{c}}(\Theta)\phi_{\hat{c}}dxdy$. Since

$$S'_{\hat{c}}(\Theta)=\frac{c-c_0}{c_0}\Big(a_1\tilde{Q}_{c_0}^{\frac{3}{2}}\cos\frac{y}{L}+a_2\tilde{Q}_{c_0}^{\frac{3}{2}}\sin\frac{y}{L}\Big)+O((|c-c_0|+|\vec{a}|)|c-c_0||\vec{a}|), \tag{6.43}$$

we have

$$-\int_{\mathbb{R}\times\mathbb{T}_L}(\mathbb{L}_{\hat{c}}\partial_x S'_{\hat{c}}(\Theta))S'_{\hat{c}}(\Theta)\phi_{\hat{c}}dxdy$$
$$=-\frac{\pi L(c-c_0)^2|\vec{a}|^2}{c_0^2}\int_{\mathbb{R}}\Big((\mathcal{L}_{c_0}+\frac{1}{L^2})\partial_x Q_{c_0}^{\frac{3}{2}}\Big)Q_{c_0}^{\frac{3}{2}}\phi_{c_0}dx$$
$$+O((|c-c_0|+|\vec{a}|)|c-c_0|^2|\vec{a}|^2). \tag{6.44}$$

From (6.2) and (6.3)

$$\partial_x Q_{c_0}^{\frac{3}{2}}=\frac{3}{2}Q_{c_0}^{\frac{1}{2}}\partial_x Q_{c_0}, \tag{6.45}$$
$$\partial_x^2 Q_{c_0}^{\frac{3}{2}}=\frac{9c_0}{4}Q_{c_0}^{\frac{3}{2}}-2Q_{c_0}^{\frac{5}{2}}, \tag{6.46}$$
$$\partial_x^3 Q_{c_0}^{\frac{3}{2}}=\frac{27c_0}{8}Q_{c_0}^{\frac{1}{2}}\partial_x Q_{c_0}-5Q_{c_0}^{\frac{3}{2}}\partial_x Q_{c_0}, \tag{6.47}$$
$$\partial_x^4 Q_{c_0}^{\frac{3}{2}}=\frac{3}{2}\partial_x^3(Q_{c_0}^{\frac{1}{2}}\partial_x Q_{c_0})=\frac{81c_0^2}{16}Q_{c_0}^{\frac{3}{2}}-17c_0Q_{c_0}^{\frac{5}{2}}+10Q_{c_0}^{\frac{7}{2}}. \tag{6.48}$$

From (6.45)–(6.48) we have

$$\Big(\Big(\mathcal{L}_{c_0}+\frac{1}{L^2}\Big)\partial_x Q_{c_0}^{\frac{3}{2}}\Big)Q_{c_0}^{\frac{3}{2}}\phi_{c_0}=-2c_0Q_{c_0}^4+\frac{4}{3}Q_{c_0}^5.$$

Applying Lemma 6.1, we obtain

$$-\int_{\mathbb{R}\times\mathbb{T}_L}(\mathbb{L}_{\hat{c}}\partial_x S'_{\hat{c}}(\Theta))S'_{\hat{c}}(\Theta)\phi_{\hat{c}}dxdy$$
$$=\frac{\pi L(c-c_0)^2|\vec{a}|^2}{6c_0^2}\int_{\mathbb{R}}Q_{c_0}^5dx+O((|c-c_0|+|\vec{a}|)|c-c_0|^2|\vec{a}|^2). \tag{6.49}$$

(III) The estimate of $-\int_{\mathbb{R}\times\mathbb{T}_L}(\mathbb{L}_{\hat{c}}\partial_x S'_{\hat{c}}(\Theta))v\phi_{\hat{c}}dxdy-\int_{\mathbb{R}\times\mathbb{T}_L}(\mathbb{L}_{\hat{c}}\partial_x v)S'_{\hat{c}}(\Theta)\phi_{\hat{c}}dxdy$. By the Fourier expansion and (6.43)

$$
\begin{aligned}
&- \int\limits_{\mathbb{R}\times\mathbb{T}_L} (\mathbb{L}_{\hat{c}}\partial_x S'_{\hat{c}}(\Theta))v\phi_{\hat{c}}dxdy - \int\limits_{\mathbb{R}\times\mathbb{T}_L} (\mathbb{L}_{\hat{c}}\partial_x v)S'_{\hat{c}}(\Theta)\phi_{\hat{c}}dxdy \\
&= -\frac{\pi L(c-c_0)}{c_0}\int\limits_{\mathbb{R}} \Big((\mathcal{L}_{c_0}+\frac{1}{L^2})\partial_x Q_{c_0}^{\frac{3}{2}}\Big)\Big(a_1 v_{1,1}+a_2 v_{1,2}\Big)\phi_{c_0}dx \\
&\quad + \frac{\pi L(c-c_0)}{c_0}\int\limits_{\mathbb{R}} \Big(\partial_x(\mathcal{L}_{c_0}+\frac{1}{L^2})(Q_{c_0}^{\frac{3}{2}}\phi_{c_0})\Big)\Big(a_1 v_{1,1}+a_2 v_{1,2}\Big)dx \\
&\quad + O((|c-c_0|+|\vec{a}|)|c-c_0||\vec{a}|\|v\|_{L^2}).
\end{aligned}
\tag{6.50}
$$

From (6.45)–(6.48) we have

$$
\Big(\Big(\mathcal{L}_{c_0}+\frac{1}{L^2}\Big)\partial_x Q_{c_0}^{\frac{3}{2}}\Big)\phi_{c_0} = -2c_0 Q_{c_0}^{\frac{5}{2}} + \frac{4}{3}Q_{c_0}^{\frac{7}{2}}.
$$

By the similar computation we have

$$
\partial_x(\mathcal{L}_{c_0}+\frac{1}{L^2})(Q_{c_0}^{\frac{3}{2}}\phi_{c_0}) = -\frac{10c_0}{3}Q_{c_0}^{\frac{5}{2}} + \frac{8}{3}Q_{c_0}^{\frac{7}{2}}.
$$

Therefore, applying Lemma 6.1, we obtain that

$$
\begin{aligned}
&\Big|- \int\limits_{\mathbb{R}\times\mathbb{T}_L} (\mathbb{L}_{\hat{c}}\partial_x S'_{\hat{c}}(\Theta))v\phi_{\hat{c}}dxdy - \int\limits_{\mathbb{R}\times\mathbb{T}_L} (\mathbb{L}_{\hat{c}}\partial_x v)S'_{\hat{c}}(\Theta)\phi_{\hat{c}}dxdy\Big| \\
&= \Big|\frac{\pi L(c-c_0)}{c_0}\int\limits_{\mathbb{R}} \Big(-\frac{4c_0}{3}Q_{c_0}^{\frac{5}{2}} + \frac{4}{3}Q_{c_0}^{\frac{7}{2}}\Big)\Big(a_1 v_{1,1}+a_2 v_{1,2}\Big)dx\Big| \\
&\quad + O((|c-c_0|+|\vec{a}|)|c-c_0||\vec{a}|\|v\|_{L^2}) \\
&\le \frac{3c_0\pi L}{4}\int\limits_{\mathbb{R}} (|v_{1,1}|^2+|v_{1,2}|^2)Q_{c_0}dx + \frac{20|c-c_0|^2|\vec{a}|^2\pi L}{297c_0^2}\int\limits_{\mathbb{R}\times\mathbb{T}_L} Q_{c_0}^5 dx \\
&\quad + O((|c-c_0|+|\vec{a}|)|c-c_0||\vec{a}|\|v\|_{L^2}).
\end{aligned}
\tag{6.51}
$$

(IV) The estimate of $-\int_{\mathbb{R}\times\mathbb{T}_L} R(\eta,\vec{a},c)(v+S'_{\hat{c}}(\Theta))\phi_{\hat{c}}dxdy$. Since

$$
(\partial_t\mathbb{L}_{\hat{c}})\eta = \dot{\hat{c}}\eta - 2\dot{\hat{c}}(\partial_c\tilde{Q}_{c_0})\eta
$$

and

$$
\mathbb{L}_{\hat{c}}\partial_x\eta = 2(\partial_x\tilde{Q}_{\hat{c}})\eta + v_x + \partial_x\eta^2,
$$

we have

$$
\begin{aligned}
R(\eta,\vec{a},c) &= -2\eta\partial_x(v+S'_{\hat{c}}(\Theta)) + (\dot{\rho}-\hat{c})(2(\partial_x\tilde{Q}_{\hat{c}})\eta + v_x + \partial_x\eta^2) \\
&\quad + (\dot{\rho}-\hat{c})\mathbb{L}_{\hat{c}}\partial_x\Theta - 2(\dot{\rho}-\hat{c})\eta\partial_x(\eta+\Theta)
\end{aligned}
$$

$$
\begin{aligned}
&-2(\partial_x^3(\tilde{Q}_{\hat{c}}-\Theta))\eta-6(\partial_x^2(\tilde{Q}_{\hat{c}}-\Theta))\eta_x+2(\partial_x(\tilde{Q}_{\hat{c}}-\Theta))v+2(\partial_x(\tilde{Q}_{\hat{c}}-\Theta))\eta^2\\
&-4(\partial_x(\tilde{Q}_{\hat{c}}-\Theta))\eta_{xx}+4(\tilde{Q}_{\hat{c}}-\Theta)(\partial_x\tilde{Q}_{\hat{c}})\eta+2(\tilde{Q}_{\hat{c}}-\Theta)v_x+2(\tilde{Q}_{\hat{c}}-\Theta)\partial_x\eta^2\\
&-4\eta\partial_x((\tilde{Q}_{\hat{c}}-\Theta)\eta)-(\mathbb{L}_{\hat{c}}-2\eta)(\dot{\vec{a}}\partial_{\vec{a}}\Theta+\dot{c}\partial_c\Theta)+\dot{\hat{c}}\eta-2\dot{\hat{c}}(\partial_c\tilde{Q}_{c_0})\eta. \qquad (6.52)
\end{aligned}
$$

From Lemma 6.8, integration by parts, $\mathbb{L}_{\hat{c}}\partial_x\Theta=O(|c-c_0|+|\vec{a}|)$ and $\mathbb{L}_{\hat{c}}\partial_{\vec{a}}\Theta=O(|c-c_0|+|\vec{a}|)$, we obtain

$$
-\int_{\mathbb{R}\times\mathbb{T}_L} R(\eta,\vec{a},c)(v+S'_{\hat{c}}(\Theta))\phi_{\hat{c}}\,dxdy=O((|c-c_0|+|\vec{a}|+\|\eta\|_{H^1})(|c-c_0|^2|\vec{a}|^2+\|v\|_{H^1}^2)). \tag{6.53}
$$

(V) The estimate of $-\int_{\mathbb{R}\times\mathbb{T}_L}(\partial_t S'_{\hat{c}}(\Theta))(v+S'_{\hat{c}}(\Theta))\phi_{\hat{c}}dxdy-\frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}_L}(v+S'_{\hat{c}}(\Theta))^2\dot{\hat{c}}\partial_c\phi_{\hat{c}}dxdy$. Since

$$
\partial_t S'_{\hat{c}}(\Theta))=\frac{\dot{c}}{c_0}\partial_y^2\Theta+\frac{c-c_0}{c_0}\dot{\vec{a}}\cdot\partial_{\vec{a}}\partial_y^2\Theta+\frac{c-c_0}{c_0}\dot{c}\partial_c\partial_y^2\Theta=O((|c-c_0|+|\vec{a}|)\|\eta\|_{L^2}),
$$

from Lemma 6.8 we have

$$
\begin{aligned}
&-\int_{\mathbb{R}\times\mathbb{T}_L}(\partial_t S'_{\hat{c}}(\Theta))(v+S'_{\hat{c}}(\Theta))\phi_{\hat{c}}dxdy-\frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}_L}(v+S'_{\hat{c}}(\Theta))^2\dot{\hat{c}}\partial_c\phi_{\hat{c}}dxdy\\
&=O((|c-c_0|+|\vec{a}|)(|c-c_0|^2|\vec{a}|^2+\|v\|_{L^2}^2)). \qquad (6.54)
\end{aligned}
$$

Therefore, from (I)–(V) we deduce gathering (6.42)–(6.54) that there exists $k_4>0$ such that

$$
\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}\times\mathbb{T}_L}(v+S'_{\hat{c}}(\Theta))^2\phi_{\hat{c}}dxdy\\
&\geq k_4\Big(\int_{\mathbb{R}\times\mathbb{T}_L}v^2\tilde{Q}_{c_0}dxdy+|c-c_0|^2|\vec{a}|^2\Big)\\
&\quad+O\big((|c-c_0|+|\vec{a}|+\|\eta\|_{H^1})(|c-c_0|^2|\vec{a}|^2+\|v\|_{H^1}^2)\big). \qquad (6.55)
\end{aligned}
$$

On the other hand, by (6.34), we have

$$
\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}\times\mathbb{T}_L}v^2x\,dxdy\\
&=\frac{1}{2}\int_{\mathbb{R}\times\mathbb{T}_L}(3|\partial_x v|^2+|\partial_y v|^2+\hat{c}v^2)dxdy-\int_{\mathbb{R}\times\mathbb{T}_L}v^2\partial_x(x\tilde{Q}_{\hat{c}})dxdy\\
&\quad-\int_{\mathbb{R}\times\mathbb{T}_L}(\mathbb{L}_{\hat{c}}\partial_x S'_{\hat{c}}(\Theta))vx\,dxdy-\int_{\mathbb{R}\times\mathbb{T}_L}R(\eta,\vec{a},c)vx\,dxdy. \qquad (6.56)
\end{aligned}
$$

From Proposition 6.7,

$$\Big|\int_{\mathbb{R}\times\mathbb{T}_L} \eta v_x v x dxdy\Big| \le \left\|x^2\eta\right\|_{H^1}^{\frac{1}{2}} \|\eta\|_{H^1}^{\frac{1}{2}} \|v\|_{H^1}^2 = O(\|\eta\|_{H^1}^{\frac{1}{2}} \|v\|_{H^1}^2). \tag{6.57}$$

By the similar calculation to (6.57), we have

$$\Big|\int_{\mathbb{R}\times\mathbb{T}_L} R(\eta,\vec{a},c) v x dxdy\Big| = O\big((|c-c_0|+|\vec{a}|+\|\eta\|_{H^1}^{\frac{1}{2}})\|v\|_{H^1}^2\big). \tag{6.58}$$

By the Hölder inequality and Proposition 6.7, we have

$$\Big|\int_{\mathbb{R}\times\mathbb{T}_L} v^2 \partial_x (x\tilde{Q}_{\hat{c}}) dxdy\Big| \le \Big(1+\hat{c}\left\|x^2(\partial_x\tilde{Q}_{\hat{c}})^2\tilde{Q}_{\hat{c}}^{-1}\right\|_{L^\infty}\Big)\int_{\mathbb{R}\times\mathbb{T}_L} v^2\tilde{Q}_{\hat{c}} dxdy + \frac{\hat{c}}{8}\|v\|_{L^2}^2. \tag{6.59}$$

By the Hölder inequality, we obtain there exists $C>0$ such that

$$\Big|\int_{\mathbb{R}\times\mathbb{T}_L} (\mathbb{L}_{\hat{c}}\partial_x S'_{\hat{c}}(\Theta)) v x dxdy\Big| \le \frac{\hat{c}}{8}\|v\|_{L^2}^2 + C|c-c_0||\vec{a}|. \tag{6.60}$$

We deduce gathering (6.56)–(6.60) that

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}\times\mathbb{T}_L} v^2 x dxdy \\
&\ge \frac{1}{4}\int_{\mathbb{R}\times\mathbb{T}_L} (|\nabla v|^2+\hat{c}v^2)dxdy + \Big(1+\hat{c}\left\|x^2(\partial_x\tilde{Q}_{\hat{c}})^2\tilde{Q}_{\hat{c}}^{-1}\right\|_{L^\infty}\Big)\int_{\mathbb{R}\times\mathbb{T}_L} v^2\tilde{Q}_{\hat{c}} dxdy \\
&\quad - C|c-c_0||\vec{a}| + O((|c-c_0|+|\vec{a}|+\|\eta\|_{H^1}^{\frac{1}{2}})\|v\|_{H^1}^2).
\end{aligned} \tag{6.61}$$

From (6.55) and (6.61), we obtain

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}\times\mathbb{T}_L} \Big((v+S'_{\hat{c}}(\Theta))^2\phi_{\hat{c}} + \varepsilon_+ x v^2\Big)dxdy \\
&\ge \frac{\varepsilon_+}{4}\int_{\mathbb{R}\times\mathbb{T}_L} (|\nabla v|^2 + c_0 v^2)dxdy + \frac{k_4}{2}|c-c_0|^2|\vec{a}|^2 \\
&\quad + O((|c-c_0|+|\vec{a}|+\|\eta\|_{H^1}^{\frac{1}{2}})(\|v\|_{H^1}^2 + |c-c_0|^2|\vec{a}|^2)),
\end{aligned} \tag{6.62}$$

where

$$\varepsilon_+ = \frac{k_4}{2}\Big(1+C+c_0\left\|x^2(\partial_x\tilde{Q}_{c_0})^2\tilde{Q}_{c_0}^{-1}\right\|_{L^\infty}\Big)^{-1} > 0.$$

Integrating (6.62) between t_1 and t_2, we have for sufficiently small $\varepsilon_0 > 0$

$$\int_{\mathbb{R}\times\mathbb{T}_L} \Big(\Big(v(t_1) + S'_{\hat{c}(t_1)}\big(\Theta(\vec{a}(t_1), c(t_1))\big)\Big)^2 \phi_{\hat{c}(t_1)} - \Big(v(t_2) + S'_{\hat{c}(t_2)}\big(\Theta(\vec{a}(t_2), c(t_2))\big)\Big)^2 \phi_{\hat{c}(t_2)}$$
$$+ xv(t_1)^2 - xv(t_2)^2\Big)dxdy$$
$$\geq \int_{t_1}^{t_2} \Big(\frac{\varepsilon_+}{4} \int_{\mathbb{R}\times\mathbb{T}_L} (|\nabla v(t)|^2 + c_0 v(t)^2)dxdy + k_4|c(t) - c_0|^2|\vec{a}(t)|^2\Big)dt. \tag{6.63}$$

From Proposition 6.7

$$\int_{-\infty}^{\infty} \Big(\frac{\varepsilon_+}{4} \int_{\mathbb{R}\times\mathbb{T}_L} (|\nabla v(t)|^2 + c_0 v(t)^2)dxdy + k_4|c(t) - c_0|^2|\vec{a}(t)|^2\Big)dt$$
$$\lesssim \sup_{t\in\mathbb{R}} \Big| \int_{\mathbb{R}\times\mathbb{T}_L} \Big((v + S'_{\hat{c}}(\Theta))^2\phi_{\hat{c}} + \varepsilon_+ xv^2\Big)dxdy \Big| < \infty.$$

Therefore, there exist sequences $\{t_{1,n}\}_n$ and $\{t_{2,n}\}_n$ such that

$$\lim_{n\to\infty} t_{1,n} = -\infty, \quad \lim_{n\to\infty} t_{2,n} = \infty$$

and

$$\lim_{n\to\infty} \Big| \frac{\varepsilon_+}{4} \int_{\mathbb{R}\times\mathbb{T}_L} (|\nabla v(t_{1,n})|^2 + c_0 v(t_{1,n})^2)dxdy + k_4|c(t_{1,n}) - c_0|^2|\vec{a}(t_{1,n})|^2 \Big|$$
$$= \lim_{n\to\infty} \Big| \frac{\varepsilon_+}{4} \int_{\mathbb{R}\times\mathbb{T}_L} (|\nabla v(t_{2,n})|^2 + c_0 v(t_{2,n})^2)dxdy + k_4|c(t_{2,n}) - c_0|^2|\vec{a}(t_{2,n})|^2 \Big| = 0. \tag{6.64}$$

Combining (6.63) and (6.64), we obtain that

$$\int_{-\infty}^{\infty} \Big(\frac{\varepsilon_+}{4} \int_{\mathbb{R}\times\mathbb{T}_L} (|\nabla v(t)|^2 + c_0 v(t)^2)dxdy + k_4|c(t) - c_0|^2|\vec{a}(t)|^2\Big)dt = 0$$

which implies $v \equiv 0$ and $|c - c_0||\vec{a}| \equiv 0$. By (6.33) and $v \equiv 0$, we have $\eta \equiv 0$. Therefore, we obtain the conclusion. $\square$

6.3. Non-critical case $L < \frac{2}{\sqrt{5c_0}}$

In this subsection, we show the Liouville property for $L < \frac{2}{\sqrt{5c_0}}$. Since the proof of the Liouville property for $L < \frac{2}{\sqrt{5c_0}}$ is similar to the proof of the Liouville property for $L = \frac{2}{\sqrt{5c_0}}$, we omit the detail of the proof.

Lemma 6.10. *Let $c_0 > 0$. There exist $\varepsilon_0, K > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ the following is true. For any solution $u \in C(\mathbb{R}, H^1(\mathbb{R} \times \mathbb{T}_L))$ of (1.1) satisfying*

$$\inf_{b\in\mathbb{R}} \left\| u(t,\cdot,\cdot) - Q_{c_0}(\cdot - b, \cdot) \right\|_{H^1} \le \varepsilon$$

there exist $\rho_1, c \in C^1(\mathbb{R}, \mathbb{R})$ uniquely such that

$$\eta(t,x,y) = u(t, x + \rho(t), y) - Q_{c(t)}(x)$$

satisfies for all $t \in \mathbb{R}$

$$|c(t) - c_0| + \|\eta(t)\|_{H^1} \le K_0\varepsilon,$$

$$\int\limits_{\mathbb{R}\times\mathbb{T}_L} \eta(t)\partial_x Q_{c(t)} dxdy = \int\limits_{\mathbb{R}\times\mathbb{T}_L} \eta(t) Q_{c(t)} dxdy = 0$$

and

$$|\dot{c}(t)|^{\frac{1}{2}} + |\dot{\rho}(t) - c(t)| \le K_0 \|\eta(t)\|_{L^2}.$$

The following is Liouville property in the non-critical case.

Theorem 6.11. *Let $c_0 > 0$ and $L < \frac{2}{\sqrt{5c_0}}$. There exists $\varepsilon_0 > 0$ satisfies the following. For any solution $u \in C(\mathbb{R}, H^1(\mathbb{R} \times \mathbb{T}_L))$ to (1.1) satisfying (6.5) and (6.8), there exist $c_+ > 0$ and $\rho_0 \in \mathbb{R}$ such that*

$$u(t,x,y) = Q_{c_+}(x - c_+ t + \rho_0, y).$$

Remark 6.12. The proof of Theorem 6.11 is easier then the proof of Theorem 6.9. In the case $L < \frac{2}{\sqrt{5c_0}}$, $\mathbb{L}_{c_0}$ has the following coercive type estimate. There exists $k_5 > 0$ such that for $\eta \in H^1(\mathbb{R} \times \mathbb{T}_L)$ with $(\eta, \partial_x \tilde{Q}_{c_0})_{L^2} = (\eta, \tilde{Q}_{c_0})_{L^2} = 0$,

$$\langle \mathbb{L}_{c_0}\eta, \eta\rangle_{H^{-1},H^1} \ge k_5 \|\eta\|_{H^1}^2.$$

Therefore, from Lemma 6.2 we can show a coercive type estimate for the virial identity

$$-\frac{1}{2}\frac{d}{dt} \int\limits_{\mathbb{R}\times\mathbb{T}_L} v^2(\phi_c + \varepsilon_+ x) dxdy \ge \frac{\varepsilon_+}{4} \int\limits_{\mathbb{R}\times\mathbb{T}_L} (|\nabla v|^2 + c_0 v^2) dxdy + o(\|v\|_{H^1}^2) \text{ as } \varepsilon_0 \to 0,$$

for sufficiently small $\varepsilon_+ > 0$.

7. Asymptotic stability

In this section, we prove Theorem 1.5 by applying the monotonicity property and the Liouville property in Section 6. We apply the argument by Martel and Merle [30–32] for the generalized KdV equation and Côte et al. [5] for the Zakharov–Kuznetsov equation on $\mathbb{R}^2$.

7.1. Critical case $L = \frac{2}{\sqrt{5c_0}}$

In this subsection, we consider the critical case $L = \frac{2}{\sqrt{5c_0}}$. The following proposition shows the compactness of the orbit of solutions in $H^1(x > -A)$.

Proposition 7.1. *Let $c_0 > 0$ and $L = \frac{2}{\sqrt{5c_0}}$. There exists $0 < \varepsilon_* < \varepsilon_0$ such that if $0 < \varepsilon \leq \varepsilon_*$ and $u \in C(\mathbb{R}, H^1(\mathbb{R} \times \mathbb{T}_L))$ is a solution to (1.1) satisfying $\sup_{t \in \mathbb{R}} \mathrm{dist}_{c_0}(u(t)) < \varepsilon$, then the following holds true. For any sequence $\{t_n\}_n$ with $\lim_{n\to\infty} t_n = \infty$, there exists a subsequence $\{t_{n_k}\}_k$ and $\tilde{u}_0 \in H^1(\mathbb{R} \times \mathbb{T}_L)$ such that*

$$\left\| u(t_{n_k}, \cdot + \rho(t_{n_k}), \cdot) - \tilde{u}_0 \right\|_{H^1(x > -A)} \to 0 \text{ as } k \to \infty$$

for any $A > 0$, where ρ is the function associated to u given by Lemma 6.8. Moreover, the solution $\tilde{u}$ of (1.1) with $\tilde{u}(0) = \tilde{u}_0$ satisfies

$$\left\| \tilde{u}(t, \cdot + \tilde{\rho}(t), \cdot) - \tilde{Q}_{c_0} \right\|_{H^1} \lesssim \varepsilon_0, \quad t \in \mathbb{R} \tag{7.1}$$

and

$$\int_{\mathbb{T}_L} |\tilde{u}(t, x + \tilde{\rho}(t), y)|^2 dy \lesssim e^{-\delta_1 |x|}, \quad (t, x) \in \mathbb{R}^2 \tag{7.2}$$

for some $\delta_1 > 0$, where $\tilde{\rho}$ is the function associated to $\tilde{u}$ given by Lemma 6.8 and $\tilde{\rho}(0) = 0$.

Since the proof of Proposition 7.1 is similar to the proof of Proposition 4.1 in [5], we omit the proof.

Next, we show the asymptotic stability result (ii) of Theorem 1.5.

Proof of (ii) of Theorem 1.5. Let $\beta > 0$ and u be a solution to (1.1) with $\mathrm{dist}_{c_0}(u(0)) < \varepsilon$. By Theorem 1.2 if ε is sufficiently small, then $\mathrm{dist}_{c_0}(u(t)) < \varepsilon_*$ which is defined in Proposition 7.1. Let ρ, c and $\vec{a}$ be functions associated to u given by Lemma 6.8. From Proposition 7.1, for any sequence $\{t_n\}_n$ with $\lim_{n\to\infty} t_n = \infty$, there exist a subsequence $\{t_{n_k}\}_k$, $\tilde{c}_0 > 0$, $\tilde{\vec{a}}_0 \in \mathbb{R}^2$ and $\tilde{u}_0 \in H^1(\mathbb{R} \times \mathbb{T}_L)$ such that for $A > 0$

$$u(t_{n_k}, \cdot + \rho(t_{n_k}), \cdot) \underset{k\to\infty}{\to} \tilde{u}_0 \text{ in } H^1(x > -A), \quad c(t_{n_k}) \underset{k\to\infty}{\to} \tilde{c}_0 \text{ and } \vec{a}(t_{n_k}) \underset{k\to\infty}{\to} \tilde{\vec{a}}_0.$$

Moreover, the solution $\tilde{u}$ of (1.1) with $\tilde{u}(0) = \tilde{u}_0$ satisfies (7.1) and (7.2). Let $\tilde{\rho}$, $\tilde{c}$ and $\tilde{\vec{a}}$ be functions associated to $\tilde{u}$ and given by Lemma 6.8. By the uniqueness of the decomposition in

Lemma 6.8, we have $\tilde{\rho}(0) = 0$, $\tilde{c}(0) = \tilde{c}_0$ and $\tilde{\vec{a}}(0) = \tilde{\vec{a}}_0$. Applying Theorem 6.9, we obtain that there exist $\rho_0 \in \mathbb{R}$, $c_1 > 0$ and $\vec{a}_1 \in \mathbb{R}^2$ such that $|c_1 - c_0||\vec{a}_1| = 0$ and

$$\tilde{u}(t, x, y) = \Theta(\vec{a}_1, c_1)(x - \hat{c}_1 t - \rho_0, y),$$

where

$$\hat{c}_1 = \begin{cases} c_1, & \vec{a}_1 = (0, 0), \\ \check{c}(\vec{a}_1), & c_1 = c_0. \end{cases}$$

By the uniqueness of the decomposition in Lemma 6.8, $\rho_0 = 0$, $c_1 = \tilde{c}_0$, $\vec{a}_1 = \tilde{\vec{a}}_0$ and $|\tilde{c}_0 - c_0||\tilde{\vec{a}}_0| = 0$. Since for any sequence $\{t_n\}_n$ with $\lim_{n\to\infty} t_n = \infty$ there exists a subsequence $\{t_{n_k}\}_k$ such that

$$u(t_{n_k}, \cdot + \rho(t_{n_k}), \cdot) - \Theta(\vec{a}(t_{n_k}), c(t_{n_k})) \underset{k\to\infty}{\to} 0 \text{ in } H^1(x > -A) \text{ and } |c(t_{n_k}) - c_0||\vec{a}(t_{n_k})| \underset{k\to\infty}{\to} 0,$$

we obtain

$$u(t, \cdot + \rho(t), \cdot) - \Theta(\vec{a}(t), c(t)) \underset{t\to\infty}{\to} 0 \text{ in } H^1(x > -A) \tag{7.3}$$

and

$$|c(t) - c_0||\vec{a}(t)| \underset{t\to\infty}{\to} 0. \tag{7.4}$$

Moreover, (7.3) implies that for $R > 0$ and $x_0 \in \mathbb{R}$

$$\lim_{t\to\infty} \int_{\mathbb{R}\times\mathbb{T}_L} (|\nabla\eta|^2 + |\eta|^2)(t, x - \rho(t), y)\psi_R(x - \rho(t) + x_0)dxdy = 0, \tag{7.5}$$

where $\eta(t, x, y) = u(t, x + \rho(t), y) - \Theta(\vec{a}(t), c(t))$. By (7.3), for any $\alpha > 0$ and $R > 2/\sqrt{\beta}$ there exist $x_1 \in \mathbb{R}$ and $T_1 > 0$ such that for $x_0 > x_1$ and $t > T_1$

$$\left| \int_{\mathbb{R}\times\mathbb{T}_L} |u(t, x, y)|^2 \psi_R(x - \rho(t) + x_0)dxdy - \int_{\mathbb{R}\times\mathbb{T}_L} |\Theta(\vec{a}(t), c(t))|^2 dxdy \right| < \alpha, \tag{7.6}$$

where ψ_R is defined by (6.4). From Lemma 6.5 there exists $x_2 \in \mathbb{R}$ such that for $x_0 \geq x_2$ and $t \geq t'$

$$\int_{\mathbb{R}\times\mathbb{T}_L} |u(t, x, y)|^2 \psi_R(x - \rho(t) + x_0)dxdy - \int_{\mathbb{R}\times\mathbb{T}_L} |u(t', x, y)|^2 \psi_R(x - \rho(t') + x_0)dxdy \leq \alpha. \tag{7.7}$$

By (7.6) and (7.7) we have that for $t \geq t' > T_0$

$$\int_{\mathbb{R}\times\mathbb{T}_L} |\Theta(\vec{a}(t), c(t))|^2 dxdy \leq \int_{\mathbb{R}\times\mathbb{T}_L} |\Theta(\vec{a}(t'), c(t'))|^2 dxdy + 3\alpha.$$

Since for any $\alpha > 0$

$$\limsup_{t\to\infty} \int_{\mathbb{R}\times\mathbb{T}_L} |\Theta(\vec{a}(t), c(t))|^2 dxdy \leq \liminf_{t'\to\infty} \int_{\mathbb{R}\times\mathbb{T}_L} |\Theta(\vec{a}(t'), c(t'))|^2 dxdy + 3\alpha,$$

$\int_{\mathbb{R}\times\mathbb{T}_L} |\Theta(\vec{a}(t), c(t))|^2 dxdy$ has the limit as $t \to \infty$. By the definition of Θ, we have

$$\lim_{t\to\infty} \int_{\mathbb{R}\times\mathbb{T}_L} |\Theta(\vec{a}(t), c(t))|^2 dxdy = \lim_{t\to\infty} \left(\frac{c(t)}{c_0}\right)^{\frac{3}{2}} \|\varphi_{c_0}(\vec{a}(t))\|_{L^2}^2. \tag{7.8}$$

Since $\|\varphi_{c_0}(\vec{a}(t))\|_{L^2}^2$ is strictly increasing with respect to $|\vec{a}|$, from (7.4) and (7.8) the ω-limit set of $(|\vec{a}(t)|, c(t))$ consists of at most two points. By the continuity of $\vec{a}(t)$ and $c(t)$, the ω-limit set of $(|\vec{a}(t)|, c(t))$ is the one point set which implies there exist $a_+ \geq 0$ and $c_+ > 0$ such that

$$\lim_{t\to\infty} |\vec{a}(t)| = a_+, \quad \lim_{t\to\infty} c(t) = c_+. \tag{7.9}$$

Therefore, by Corollary 5.7 we have

$$|c_+ - c_0| + |a_+|^2 \lesssim \left\| u(0) - \tilde{Q}_{c_0} \right\|_{H^1}.$$

Next, we improve the convergence of (7.17). By Lemma 6.4, for all $t_1 \leq t_2$, $x_0 > 0$ and $R > 2/\sqrt{\beta}$

$$\int_{\mathbb{R}\times\mathbb{T}_L} |u(t_2, x, y)|^2 \psi_R(\tilde{x}(t_1, t_2)) dxdy - \int_{\mathbb{R}\times\mathbb{T}_L} |u(t_1, x, y)|^2 \psi_R(\tilde{x}(t_1, t_1)) dxdy \leq Ce^{-x_0/R}, \tag{7.10}$$

where $\tilde{x}(t, \tau) = x - \rho(t) - \frac{\beta}{2}(\tau - t) + x_0$. By (6.18) if $R > \frac{1}{4c_0}$ and ε_0 is sufficiently small, then we have

$$\begin{aligned} &\left| \int_{\mathbb{R}\times\mathbb{T}_L} \eta(t, x, y)\Theta(\vec{a}(t), c(t))\psi_R(x + x_0) dxdy \right| \\ &= \left| \int_{\mathbb{R}\times\mathbb{T}_L} \eta(t, x, y)\Theta(\vec{a}(t), c(t))(1 - \psi_R(x + x_0)) dxdy \right| \\ &= \|\eta(t)\|_{L^2} \|\Theta(\vec{a}(t), c(t))(1 - \psi_R(x + x_0))\|_{L^2} \lesssim e^{-x_0/R}. \end{aligned} \tag{7.11}$$

Since

$$\big(\eta(t,x-\rho(t),y)\big)^2 = \big(u(t,x,y)\big)^2 - 2\eta(t,x-\rho(t),y)\Theta(\vec{a}(t),c(t))(x-\rho(t),y)$$
$$- \big(\Theta(\vec{a}(t),c(t))(x-\rho(t),y)\big)^2,$$

from (7.10) and (7.11) we have that there exists $C_0 > 0$ such that

$$\int_{\mathbb{R}\times\mathbb{T}_L} \big(\eta(t_2,x-\rho(t_2),y)\big)^2 \psi_R(\tilde{x}(t_1,t_2))dxdy$$
$$- \int_{\mathbb{R}\times\mathbb{T}_L} \big(\eta(t_1,x-\rho(t_1),y)\big)^2 \psi_R(\tilde{x}(t_1,t_1))dxdy$$
$$\le C_0(e^{-x_0/R} + |c(t_1)-c(t_2)| + ||\vec{a}(t_1)|^2 - |\vec{a}(t_2)|^2|).$$

For $t > 0$ large enough, we define $0 < t' < t$ such that $\rho(t') + \frac{\beta}{2}(t-t') - x_0 = \beta t$. Then, we have $t' \to \infty$ as $t \to \infty$. Combining

$$\int_{\mathbb{R}\times\mathbb{T}_L} \big(\eta(t,x-\rho(t),y)\big)^2 \psi_R(x-\beta t)dxdy$$
$$\le \int_{\mathbb{R}\times\mathbb{T}_L} \big(\eta(t',x-\rho(t'),y)\big)^2 \psi_R(x-\rho(t')+x_0)dxdy$$
$$+ C_0(e^{-x_0/R} + |c(t')-c(t)| + ||\vec{a}(t')|^2 - |\vec{a}(t)|^2|),$$

(7.5) and (7.9), we obtain for any $x_0 > 0$

$$\limsup_{t\to\infty} \int_{\mathbb{R}\times\mathbb{T}_L} \big(\eta(t,x-\rho(t),y)\big)^2 \psi_R(x-\beta t)dxdy \le C_0 e^{-x_0/R}.$$

Therefore,

$$\lim_{t\to\infty} \int_{\mathbb{R}\times\mathbb{T}_L} \big(\eta(t,x-\rho(t),y)\big)^2 \psi_R(x-\beta t)dxdy = 0. \tag{7.12}$$

From Lemma 6.6 we have for all $t_1 \le t_2$, $x_0 > 0$ and $R > 2/\sqrt{\beta}$

$$J_{x_0,t_1}(u(t_2)) - J_{x_0,t_1}(u(t_1)) \le Ce^{-x_0/R}. \tag{7.13}$$

Moreover, we have

$$\Big| \int_{\mathbb{R}\times\mathbb{T}_L} \big(u(t_2,x,y)\big)^3 \psi_R(\tilde{x}(t_1,t_2))dxdy - \int_{\mathbb{R}\times\mathbb{T}_L} \big(u(t_1,x,y)\big)^3 \psi_R(\tilde{x}(t_1,t_1))dxdy \Big|$$

$$\lesssim \Big(\int\limits_{\mathbb{R}\times\mathbb{T}_L} \big(\eta(t_1, x-\rho(t_1), y)\big)^2 \psi_R(\tilde{x}(t_1,t_2))dxdy\Big)^{\frac{1}{2}}$$
$$+ \Big(\int\limits_{\mathbb{R}\times\mathbb{T}_L} \big(\eta(t_2, x-\rho(t_2), y)\big)^2 \psi_R(\tilde{x}(t_1,t_2))dxdy\Big)^{\frac{1}{2}}$$
$$+ (e^{-x_0/R} + |c(t_1)-c(t_2)| + ||\vec{a}(t_1)| - |\vec{a}(t_2)||). \tag{7.14}$$

By (7.13) and (7.14), we get

$$\int\limits_{\mathbb{R}\times\mathbb{T}_L} \big|\nabla\eta(t_2, x-\rho(t_2), y)\big|^2 \psi_R(\tilde{x}(t_1,t_2))dxdy$$
$$- \int\limits_{\mathbb{R}\times\mathbb{T}_L} \big|\nabla\eta(t_1, x-\rho(t_1), y)\big|^2 \psi_R(\tilde{x}(t_1,t_1))dxdy$$
$$\lesssim \Big(\int\limits_{\mathbb{R}\times\mathbb{T}_L} \big(\eta(t_1, x-\rho(t_1), y)\big)^2 \psi_R(\tilde{x}(t_1,t_2))dxdy\Big)^{\frac{1}{2}}$$
$$+ \Big(\int\limits_{\mathbb{R}\times\mathbb{T}_L} \big(\eta(t_2, x-\rho(t_2), y)\big)^2 \psi_R(\tilde{x}(t_1,t_2))dxdy\Big)^{\frac{1}{2}}$$
$$+ (e^{-x_0/R} + |c(t_1)-c(t_2)| + ||\vec{a}(t_1)| - |\vec{a}(t_2)||). \tag{7.15}$$

From (7.15) with $t_1 = t'$ and $t_2 = t$ we obtain that there exists $C > 0$ such that

$$\int\limits_{\mathbb{R}\times\mathbb{T}_L} \big|\nabla\eta(t, x-\rho(t), y)\big|^2 \psi_R(x-\beta t)dxdy$$
$$\le \int\limits_{\mathbb{R}\times\mathbb{T}_L} \big|\nabla\eta(t', x-\rho(t'), y)\big|^2 \psi_R(x-\rho(t')+x_0)dxdy$$
$$+ C\Big(\int\limits_{\mathbb{R}\times\mathbb{T}_L} \big(\eta(t, x-\rho(t), y)\big)^2 \psi_R(x-\beta t)dxdy\Big)^{\frac{1}{2}}$$
$$+ C\Big(\int\limits_{\mathbb{R}\times\mathbb{T}_L} \big(\eta(t', x-\rho(t'), y)\big)^2 \psi_R(x-\rho(t')+x_0)dxdy\Big)^{\frac{1}{2}}$$
$$+ C(e^{-x_0/R} + |c(t')-c(t)| + ||\vec{a}(t')| - |\vec{a}(t)||).$$

Therefore, it follows form (7.5), (7.9) and (7.12) that for $x_0 > 0$

$$\limsup_{t\to\infty} \int\limits_{\mathbb{R}\times\mathbb{T}_L} \big|\nabla\eta(t, x-\rho(t), y)\big|^2 \psi_R(x-\beta t)dxdy \le Ce^{-x_0/R}$$

which implies

$$\lim_{t\to\infty} \int\limits_{\mathbb{R}\times\mathbb{T}_L} \left|\nabla\eta(t, x-\rho(t), y)\right|^2 \psi_R(x-\beta t)dxdy = 0. \tag{7.16}$$

Then, we define $\rho_2(t)$ by

$$\rho_2(t) = \begin{cases} \Phi^{-1}\left(\frac{\vec{a}(t)}{|\vec{a}(t)|}\right), & \text{if } |\vec{a}(t)| \neq 0 \text{ and } a_+ \neq 0, \\ 0, & \text{if otherwise,} \end{cases}$$

where $\Phi(\theta) = (\cos\theta, -\sin\theta)$ for $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Using

$$\Theta(\vec{a}(t), c(t))(x, y) = \Theta((|\vec{a}(t)|, 0), c(t))(x, y-\rho_2(t)),$$

from (7.12) and (7.16) we obtain

$$u(t, \cdot+\rho(t), y+\rho_2(t)) - \Theta((a_+, 0), c_+) \underset{t\to\infty}{\to} 0 \text{ in } H^1(x > \beta t). \tag{7.17}$$

From (6.19)–(6.21), (7.3) and (7.9), we have

$$\lim_{t\to\infty} \dot{c}(t) = \lim_{t\to\infty} |\dot{\vec{a}}(t)| = \lim_{t\to\infty} |\dot{\rho}(t) - \hat{c}_+| = 0,$$

where $\hat{c}_+$ is defined by (1.7). If $a_+ = 0$, then $\dot{\rho}_2(t) = 0$ for $t > 0$. On the other hand, if $a_+ \neq 0$, then $|\dot{\rho}_2(t)| \lesssim |\dot{\vec{a}}(t)| \to 0$ as $t \to \infty$. □

7.2. Non-critical case $L < \frac{2}{\sqrt{5c_0}}$

In this subsection, we consider (i) of Theorem 1.5. The proof of (i) of Theorem 1.5 is similar to the proof of (ii) of Theorem 1.5.

Moreover the proof of the following proposition is same as the proof of Proposition 7.1, so we omit the detail of the proof of the following proposition.

Proposition 7.2. *Let $c_0 > 0$ and $L < \frac{2}{\sqrt{5c_0}}$. There exists $0 < \varepsilon_* < \varepsilon_0$ such that if $0 < \varepsilon \le \varepsilon_*$ and $u \in C(\mathbb{R}, H^1(\mathbb{R}\times\mathbb{T}_L))$ is a solution to (1.1) satisfying $\sup_{t\in\mathbb{R}} \operatorname{dist}_{c_0}(u(t)) < \varepsilon$ then the following holds true. For any sequence $\{t_n\}_n$ with $\lim_{n\to\infty} t_n = \infty$, there exists a subsequence $\{t_{n_k}\}_k$ and $\tilde{u}_0 \in H^1(\mathbb{R}\times\mathbb{T}_L)$ such that*

$$u(t_{n_k}, \cdot+\rho(t_{n_k}), \cdot) \to \tilde{u}_0 \text{ in } H^1(x > -A) \text{ as } k \to \infty$$

for any $A > 0$, where ρ is the function associated to u given by Lemma 6.10. Moreover, the solution $\tilde{u}$ of (1.1) with $\tilde{u}(0) = \tilde{u}_0$ satisfies

$$\left\| \tilde{u}(t, \cdot+\tilde{\rho}(t), \cdot) - \tilde{Q}_{c_0} \right\|_{H^1} \lesssim \varepsilon_0, \quad t \in \mathbb{R} \tag{7.18}$$

and

$$\int_{\mathbb{T}_L} |\tilde{u}(t, x+\tilde{\rho}(t), y)|^2 dy \lesssim e^{-\delta_1|x|}, \quad (t,x) \in \mathbb{R}^2 \tag{7.19}$$

for some $\delta_1 > 0$, *where* $\tilde{\rho}$ *is the function associated to* $\tilde{u}$ *given by Lemma 6.10 and* $\tilde{\rho}(0) = 0$.

By applying Theorem 6.11 and Proposition 7.2, we obtain (i) of Theorem 1.5 from the same proof as the proof of (ii) of Theorem 1.5.

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Appendix A. Proof of Proposition 1.3

In this appendix, we show the outline of the proof of Proposition 1.3. For the completeness, see [21,52].

Let $c_0 = 4/5L^2$ and F be the function from $H^2_{sym}(\mathbb{R}\times\mathbb{T}_L) \to L^2_{sym}(\mathbb{R}\times\mathbb{T}_L)$ satisfying

$$F(\varphi, c) = -\Delta\varphi + c\varphi - \varphi^2,$$

where $L^2_{sym}(\mathbb{R}\times\mathbb{T}_L) = \{u \in L^2(\mathbb{R}\times\mathbb{T}_L) | u(x,y) = u(-x,y) = u(x,-y), (x,y) \in \mathbb{R}\times[-\pi L, \pi L]\}$, $H^2_{sym}(\mathbb{R}\times\mathbb{T}_L) = H^2(\mathbb{R}\times\mathbb{T}_L) \cap L^2_{sym}(\mathbb{R}\times\mathbb{T}_L)$ and $L^2(\mathbb{R}\times\mathbb{T}_L)$ is the set of real valued L^2-function on $\mathbb{R}\times\mathbb{T}_L$. Then, $\mathrm{Ker}(\partial_\varphi F(\tilde{Q}_{c_0}, c_0))$ is spanned by $\tilde{Q}_{c_0}^{3/2}\cos\frac{y}{L}$. Applying the Lyapunov–Schmidt decomposition, we obtain that there exists a function $h(c,a) \in H^2_{sym}(\mathbb{R}\times\mathbb{T}_L)$ such that

$$P_\perp F(\tilde{Q}_{c_0} + a\tilde{Q}_{c_0}^{3/2}\cos\frac{y}{L} + h(c,a), c) = 0,$$

where $P_\perp$ is the orthogonal projection onto $\{u \in L^2(\mathbb{R}\times\mathbb{T}_L) | \langle u, \tilde{Q}_{c_0}^{3/2}\cos\frac{y}{L}\rangle_{L^2} = 0\}$. Then, the problem $F(\tilde{Q}_{c_0} + a\tilde{Q}_{c_0}^{3/2}\cos\frac{y}{L} + h(c,a), c) = 0$ is equivalent to the problem

$$F_{||}(c,a) = \langle F(\tilde{Q}_{c_0} + a\tilde{Q}_{c_0}^{3/2}\cos\frac{y}{L} + h(c,a), c), \tilde{Q}_{c_0}^{3/2}\cos\frac{y}{L}\rangle_{L^2} = 0.$$

We apply the Crandall–Rabinowitz Transversality and we consider the problem $g(c,a) = 0$, where

$$g(c,a) = \begin{cases} \frac{F_{||}(c,a) - F_{||}(c,0)}{a}, & a \neq 0, \\ \frac{\partial F_{||}}{\partial a}(c,0), & a = 0. \end{cases}$$

Then, g is C^2 function. Here for $a \neq 0$, $F_{||}(c,a) = 0$ if and only if $g(c,a) = 0$. Since

$$\frac{\partial g}{\partial c}(c_0, 0) = -\frac{5c_0}{4}\left\| \tilde{Q}_{c_0}^{3/2} \cos\frac{y}{L}\right\|_{L^2}^2 \text{ and } \frac{\partial g}{\partial a}(c_0, 0) = 0,$$

by the implicit function theorem there exists $\check{c}(a)$ such that $g(\check{c}(a), a) = 0$. Hence, $\varphi_{c_0}(a) := \tilde{Q}_{c_0} + a\tilde{Q}_{c_0}^{3/2} \cos\frac{y}{L} + h(\check{c}(a), a)$ is a solution of $F(\varphi_{c_0}(a), \check{c}(a)) = 0$,

$$\check{c}'(0) = -\frac{\frac{\partial g}{\partial a}}{\frac{\partial g}{\partial c}}(c_0, 0) = 0$$

and

$$\check{c}''(0) = \lim_{a\to 0}\frac{\check{c}'(0)}{a} = \lim_{a\to 0}\frac{4\frac{\partial g}{\partial a}(\check{c}(a), a)}{a5c_0\left\|\tilde{Q}_{c_0}^{3/2}\cos\frac{y}{L}\right\|_{L^2}^2} = -\frac{16\left(\tilde{Q}_{c_0}^3\cos^2\frac{y}{L}, (\mathbb{L}_{c_0})^{-1}\tilde{Q}_{c_0}^3\cos^2\frac{y}{L}\right)_{L^2}}{5c_0\left\|\tilde{Q}_{c_0}^{3/2}\cos\frac{y}{L}\right\|_{L^2}^2}.$$

Using Lemma 6.1, we obtain $\check{c}''(0) > 0$ (see [52] for the detail of the calculation). Calculating $\frac{d^2}{da^2}\|\varphi_{c_0}(a)\|_{L^2}^2|_{a=0}$, we have

$$\|\varphi_{c_0}(a)\|_{L^2(\mathbb{R}\times\mathbb{T}_L)}^2 = \left\|\tilde{Q}_{c_0}\right\|_{L^2(\mathbb{R}\times\mathbb{T}_L)}^2 + \frac{C_{2,c_0}}{2}|a|^2 + o(|a|^2) \quad \text{as } |\vec{a}| \to 0,$$

where

$$C_{2,c_0} = -\frac{5c_0}{2}\left\|\tilde{Q}_{c_0}^{3/2}\cos\frac{y}{L}\right\|_{L^2(\mathbb{R}\times\mathbb{T}_L)}^2 + 2\check{c}''(0)\left(Q_{c_0}, \partial_c Q_{c_0}\right)_{L^2}.$$

The positivity of the constant C_{2,c_0} have been proved from the inequality (2.25)

$$\begin{aligned} R(p) \le & \frac{4(p+1)(p^6+18p^5-11p^4-130p^3+13p^2+16p-3)}{(5-p)(p+3)^2(5p-1)(3p+1)(p-1)} \\ & + \frac{32p^3(p+1)^4(3p-1)}{3(7p-3)(5p-1)(3p+1)(p+3)^3(p-1)} \end{aligned}$$

in [52] and the relation

$$-\frac{5c_0^2L^3C_{2,c_0}}{6\left\|\tilde{Q}_{c_0}\right\|_{L^2(\mathbb{R}\times\mathbb{T}_L)}^2} = R(2) < 0,$$

where $R(p)$ is defined in the proof of Theorem 1.3 in [52].

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