



Convexity of level sets of minimal graph on space form with nonnegative curvature

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Abstract

For the minimal graph defined on a convex ring in the space form with nonnegative curvature, we obtain the regularity and the strict convexity about its level sets by the continuity method.

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1. Introduction

The Geometry especially the convexity of level sets of the solutions to elliptic partial differential equations has been interesting to us for a long time. For instance, Alfhors [1] concluded that level curves of Green function on simply connected convex domain in the plane are the convex Jordan curves. Shiffman [31] studied the minimal annulus in R^3 whose boundary consists of two closed convex curves in parallel planes P_1 , P_2 , he derived that the intersection of the surface with any parallel plane P , between P_1 and P_2 , is a convex Jordan curve. In 1957, Gabriel [11] proved that the level sets of the Green function on a 3-dimensional bounded convex domain are strictly convex and Lewis [21] extended Gabriel's result to p -harmonic functions in higher dimensions. Makar-Limanov [24] and Brascamp–Lieb [4] got the results on the Poisson equation

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and first eigenvalue equation with Dirichlet boundary value problem on bounded convex domain. Caffarelli–Spruck [7] generalized Lewis’s results [21] to a class of semilinear elliptic partial differential equations. Motivated by the result of Caffarelli–Friedman [5], Korevaar [19] gave a new proof on the results of Gabriel and Lewis [11,21] using the deformation process and the constant rank theorem of the second fundamental form of convex level sets of p -harmonic function. Moreover, he also concluded in his paper [19] that level sets of minimal graph defined on convex rings are strictly convex. Kawohl [18] gave a survey of this subject. For more recent related extensions, please see the papers by Bianchin–Longinetti–Salani [3], Xu [38] and Bian–Guan–Ma–Xu [2].

On the curvature estimates of the level sets, Ortel–Schneider [29], Longinetti [22,23] proved that the curvature of level curves attains its minimum on the boundary (see also Talenti [33] for related results) for 2-dimensional harmonic function with convex level curves. Furthermore, Longinetti studied the precise relation between the curvature of the convex level lines and the height of minimal graph in [23]. The curvature estimate of the level sets of the solution to partial differential equations then had no new progress until recently, Ma–Ou–Zhang [25] got the Gaussian curvature estimates of the convex level sets of harmonic functions which depend on the Gaussian curvature of the boundary and the norm of the gradient on the boundary in $\mathbb{R}^n$. Furthermore, in [26] the concavity of the Gaussian curvature of the convex level sets of p -harmonic functions with respect to the height was derived to describe the variation of the curvature along the height of the function. In [16], the lower bound of the principal curvature of the convex level sets of the solution to a kind of fully nonlinear elliptic equations was derived. For Poisson equations and a class of semilinear elliptic partial differential equations, Caffarelli–Spruck [7] concluded that the level sets of their solutions are all convex with respect to the gradient direction, the curvature estimate of the level sets has been got by Wang–Zhang [37], and in the same paper they also described the geometrical properties of the level sets of the minimal graph. In the sequel, following the technique in [26], Wang [34] got the precise relation between the curvature of the convex level sets and the height of minimal graph of general dimensions which generalized the previous results of Longinetti [23]. In [35] and [36], similar results about the geometrical properties of the solution to some partial differential equations were derived.

For the Riemannian manifold case, Papadimitrakakis [30] concluded the convexity of the level curves of harmonic functions on convex rings in the hyperbolic plane via one complex variable tools. Ma–Zhang [27] generalized Papadimitrakakis’s results to space form of general dimensions. Partial results in [27] can be stated as follows.

Theorem 1.1. ([27]) *Let (M^n, g) be a space form with constant sectional curvature 1 or -1 , and Ω_0 and Ω_1 be bounded smooth strictly convex domains in M^n , $n \geq 2$ and $\bar{\Omega}_1 \subseteq \Omega_0$. Let ω satisfy*

$$\begin{cases} \Delta \omega = 0 & \text{in } \Omega = \Omega_0 \setminus \bar{\Omega}_1, \\ \omega = 0 & \text{on } \partial\Omega_0, \\ \omega = 1 & \text{on } \partial\Omega_1. \end{cases}$$

Then $\nabla \omega \neq 0$ is valid everywhere in Ω and all the level sets of ω are strictly convex with respect to $\nabla \omega$.

Based on the above strict convexity of the level sets of harmonic functions defined on the convex ring in space forms, following the technique in [19], we come in this paper to consider another important geometrical object, the minimal graph defined on the convex ring in space form with nonnegative curvature. We mainly get the following theorem.

Theorem 1.2. *Let (M^n, g) be a space form with constant sectional curvature $\epsilon \geq 0$, and Ω_0 and Ω_1 be bounded smooth strictly convex domains in M^n , $n \geq 2$ and $\bar{\Omega}_1 \subseteq \Omega_0$. Assume that the following minimal graph equation defined on $\Omega = \Omega_0 \setminus \bar{\Omega}_1$*

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 & \text{in } \Omega = \Omega_0 \setminus \bar{\Omega}_1, \\ u = 0 & \text{on } \partial\Omega_0, \\ u = 1 & \text{on } \partial\Omega_1 \end{cases} \quad (1.1)$$

has a smooth solution u . Then $\nabla u \neq 0$ is valid everywhere in Ω and all the level sets of u are strictly convex with respect to ∇u .

Following this theorem, we immediately get a geometrical property of the minimal graph defined on the convex ring.

Corollary 1.3. *Let (M^n, g) be a space form with constant sectional curvature $\epsilon \geq 0$, and Ω_0 and Ω_1 be bounded smooth strictly convex domains in M^n , $n \geq 2$ and $\bar{\Omega}_1 \subseteq \Omega_0$. Suppose that u is the solution to (1.1). Then $|\nabla u|$ increases strictly along the gradient direction.*

Remark 1.4. Remark that the case of $\epsilon = 0$ has been already concluded in [19]. And it is a pity that we can not derive similar information or counterexample to the case $\epsilon < 0$. For the similar and important problem, the convexity of the level sets of the first eigenvalue on the domain in space forms, we know that the case of $\epsilon = 0$ is due to Brascamp–Lieb [4] and the positive curvature case of $\epsilon = 1$ belongs to Lee–Wang [20] while in [39] a counterexample was constructed to show that the level sets of the first eigenfunction of a domain with negative curvature are not convex.

The rest of the paper is organized as follows: in section 2, we list the notations and the preliminaries being used during the process of the proof. In section 3, we give the existence of the solution of minimal graph defined on convex rings in space form with smaller height than that in (1.1) and deduce the $C^{2,\alpha}$ continuity of the solution with respect to the boundary value. In section 4, we prove a constant rank theorem of the second fundamental form of the level sets of minimal graph. In section 5, we prove the regularity and strict convexity of the level sets of minimal graph.

2. Notations and preliminaries

In this section, we introduce some notations and preliminaries for the main results.

Firstly, we give the derivative commutation formula in Riemannian geometry.

Lemma 2.1. *Let u be a smooth function defined on Riemannian manifold M^n with constant sectional curvature and denote by R_{ijkl} the curvature tensor of M^n , then*

$$u_{ijk} = u_{ikj} + u_l R_{lijk}$$

and

$$u_{ijkl} = u_{klij} - u_{\xi j} R_{\xi k li} - u_{i \xi} R_{\xi k lj} + u_{\xi l} R_{\xi i j k} + u_{k \xi} R_{\xi i j l}.$$

Remark that Einstein summation convention here is adopted. $\square$

In the sequel, we list out a linear algebra formula [8,9,17] used frequently in our proof.

Proposition 2.2. *Let u be a smooth function and (D^2u) be its Hessian matrix. Assume that*

$$(D^2u) = \begin{pmatrix} u_{ij} & u_{in} \\ u_{nj} & u_{nn} \end{pmatrix}_{n \times n}.$$

Denote by $\sigma_k(A)$ the k -th elementary symmetric function of the eigenvalue of the matrix A . Then we have

$$\begin{aligned} \sigma_{l+1}(D^2u) &= \sigma_{l+1}(u_{ij}) + u_{nn}\sigma_l(u_{ij}) - \sum_i u_{ni}u_{in}\sigma_{l-1}(u_{pq}|i) \\ &\quad + \sum_{i \neq j} u_{ni}u_{jn}u_{ij}\sigma_{l-2}(u_{pq}|ij) - \sum_{i \neq j, j \neq k, k \neq i} u_{ni}u_{jn}u_{ik}u_{kj}\sigma_{l-3}(u_{pq}|ijk) + T, \end{aligned} \quad (2.1)$$

where we denote by $(u_{pq}|i)$ the symmetric matrix obtained from (u_{pq}) by deleting the i -row and i -column and by $(u_{pq}|ij)$ the symmetric matrix obtained from (u_{pq}) when deleting the i, j -rows and i, j -columns, and similarly we define $(u_{pq}|ijk)$, and also every term in the polynomial T includes at least 3 factors like u_{rs} with $r \neq s$. So if the matrix (u_{pq}) is diagonal at a fixed point, we then have at this point that

$$T = 0, \quad DT = 0, \quad \text{and} \quad D^2T = 0.$$

Remark that all the Latin indices will vary from 1 to $n - 1$. $\square$

For the calculation principle of $\sigma_k(A)$, we do not mention much more here and one can refer to [38] for details.

For a smooth function u defined on a Riemannian manifold M^n , its graph can be considered as a hypersurface in $M^n \times \mathbb{R}$ with canonical product Riemannian metric. In [32], the mean curvature of this hypersurface has been already deduced.

Proposition 2.3. ([32]) *Let $u : M^n \rightarrow \mathbb{R}$ be a smooth function defined on a Riemannian manifold M^n . Consider the graph of u , denoted by $\Sigma_u = F(M^n)$, where $F : M^n \rightarrow M^n \times \mathbb{R}$ is defined to be $F(p) = (p, u(p))$ and $M^n \times \mathbb{R}$ is equipped with the canonical product Riemannian metric. Then the mean curvature of Σ_u is*

$$H = -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right),$$

here, div is the divergence operator on M^n . $\square$

The following corollary is obvious.

Corollary 2.4. *Let u be a smooth function defined on M^n , then the graph of u , Σ_u , is minimal in $M^n \times \mathbb{R}$ if and only if $\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0$ holds on M^n . $\square$*

We remark that in [10] the equation of minimal graph defined on manifolds is also deduced.

Let us define a notation as follows in the whole paper: we denoted by $\Sigma^c(u)$ the level set $\{x \in \Omega \mid u(x) = c\}$ for a smooth function u and a regular value c of u .

The following lemma is essential to express the geometric quantity of the level sets in terms of the function itself. We omit the proof and one can refer to Proposition 2.1 in [27].

Lemma 2.5. *Let u be a smooth function defined on M^n with nonzero gradient everywhere. Assume that the level sets of u are convex with respect to the normal direction. Let $\{e_\alpha, \alpha = 1, 2, \dots, n\}$ be a local orthogonal frame on M^n . Then the k -th curvature of the level set $\Sigma^c(u)$ is*

$$\sigma_k[\Sigma^c(u)] = (-1)^k \sum_{\alpha, \beta=1}^n \frac{\partial \sigma_{k+1}(D^2u)}{\partial u_{\alpha\beta}} u_{\alpha} u_{\beta} |\nabla u|^{-(k+2)}, \quad (2.2)$$

where $1 \leq k \leq n-1$.

Furthermore, if we take $e_n = \frac{\nabla u}{|\nabla u|}$ as the unit normal direction, then the second fundamental form of the level set of u is

$$h_{ij} = -\frac{u_{ij}}{|\nabla u|}. \quad (2.3)$$

3. The solution with relatively smaller height

In this section, based on the condition that the equation (1.1) is solvable, we settle the existence of the solution to minimal graph equation with height less than the one in the equation (1.1). And then we deduce some properties of the solution. Firstly, we deduce a prior estimate of the minimal graph defined on the convex ring. More precisely, we have

Proposition 3.1. *Let $u: \Omega \rightarrow \mathbb{R}$ be a minimal graph defined on Ω in an n -dimensional space form M with nonnegative curvature. Then we have*

$$\sup_{x \in \Omega} |\nabla u|(x) \leq \sup_{x \in \partial \Omega} |\nabla u|(x). \quad (3.1)$$

Proof. Let $\phi = \frac{1}{2}|\nabla u|^2$. We choose the frame $\{e_\alpha\}$ on the manifold such that the Riemannian curvature tensor takes the form $R_{\alpha\xi\beta\eta} = \epsilon(\delta_{\alpha\beta}\delta_{\xi\eta} - \delta_{\alpha\eta}\delta_{\beta\xi})$, where $\epsilon \geq 0$ and $\delta_{\alpha\beta}$ is the Kronecker symbol. It follows that

$$\begin{aligned} & \sum_{\alpha, \beta} [(1 + |\nabla u|^2)\delta_{\alpha\beta} - u_{\alpha}u_{\beta}]\phi_{\alpha\beta} \\ &= \sum_{\alpha, \beta, \gamma} [(1 + |\nabla u|^2)\delta_{\alpha\beta} - u_{\alpha}u_{\beta}](u_{\gamma}u_{\gamma\alpha\beta} + u_{\gamma\alpha}u_{\gamma\beta}) \\ &= \sum_{\alpha, \beta, \gamma} u_{\gamma}[(1 + |\nabla u|^2)\delta_{\alpha\beta} - u_{\alpha}u_{\beta}]u_{\gamma\alpha\beta} + (1 + |\nabla u|^2) \sum_{\alpha, \beta} u_{\alpha\beta}^2 - \sum_{\gamma} \phi_{\gamma}^2, \end{aligned} \quad (3.2)$$

by Lemma 2.1 we have

$$\begin{aligned}
 & \sum_{\alpha, \beta} [(1 + |\nabla u|^2) \delta_{\alpha\beta} - u_{\alpha} u_{\beta}] \phi_{\alpha\beta} \\
 & \geq \sum_{\alpha, \beta, \gamma} u_{\gamma} [(1 + |\nabla u|^2) \delta_{\alpha\beta} - u_{\alpha} u_{\beta}] (u_{\alpha\beta\gamma} + u_{\xi} R_{\xi\alpha\gamma\beta}) - \sum_{\gamma} \phi_{\gamma}^2 \\
 & \geq - \sum_{\alpha, \beta, \gamma} u_{\gamma} [(1 + |\nabla u|^2) \delta_{\alpha\beta} - u_{\alpha} u_{\beta}]_{\gamma} u_{\alpha\beta} - \sum_{\gamma} \phi_{\gamma}^2 \\
 & = -2\Delta u \sum_{\gamma} u_{\gamma} \phi_{\gamma} + 2 \sum_{\gamma} \phi_{\gamma}^2 - \sum_{\gamma} \phi_{\gamma}^2 \\
 & = \sum_{\gamma} \phi_{\gamma}^2 - 2\Delta u \sum_{\gamma} u_{\gamma} \phi_{\gamma}.
 \end{aligned} \tag{3.3}$$

Thus by the weak maximum principle we reach the conclusion. $\square$

Now, we can derive a result of the minimal graph on the convex ring in space forms with nonnegative curvature which is fundamental to the whole paper.

Theorem 3.2. *Let (M^n, g) be a space form with constant sectional curvature $\epsilon \geq 0$, and Ω_0 and Ω_1 be bounded smooth strictly convex domains in M^n , $n \geq 2$ and $\bar{\Omega}_1 \subseteq \Omega_0$. Assume the equation (1.1) is solvable, then the following minimal graph equation defined on Ω for $0 < \tau < 1$*

$$\begin{cases} \operatorname{div}(\frac{\nabla u^{\tau}}{\sqrt{1+|\nabla u^{\tau}|^2}}) = 0 & \text{in } \Omega = \Omega_0 \setminus \bar{\Omega}_1, \\ u^{\tau} = 0 & \text{on } \partial\Omega_0, \\ u^{\tau} = \tau & \text{on } \partial\Omega_1 \end{cases} \tag{3.4}$$

has a unique smooth solution u^{τ} . Moreover, there exists a positive constant $C_1 > 0$ such that

$$||u^t - u^{\tau}||_{C^{2,\alpha}(\bar{\Omega})} \leq C_1 |t - \tau|$$

for some $\alpha \in (0, 1)$ and any $t \in (0, 1]$. Especially, setting $t \rightarrow 0$, we get that

$$||u^{\tau}||_{C^{2,\alpha}(\bar{\Omega})} \leq C_1 \tau.$$

Proof. We denote by u^1 the solution of (1.1). According to the comparison principle of quasi-linear partial differential equation, we derive that $0 \leq u^{\tau} < u^1$ and $u^{\tau} = u^1 = 0$ on $\partial\Omega_0$ for $\tau \in (0, 1)$, thus we have $|\nabla u^{\tau}| \leq |\nabla u^1|$ on $\partial\Omega_0$. On the other hand, $u^1 - 1 + \tau$ is the unique solution of

$$\begin{cases} \operatorname{div}(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = 0 & \text{in } \Omega = \Omega_0 \setminus \bar{\Omega}_1, \\ u = \tau - 1 & \text{on } \partial\Omega_0, \\ u = \tau & \text{on } \partial\Omega_1 \end{cases} \tag{3.5}$$

and thus $u^1 - 1 + \tau < u^{\tau} \leq \tau$ in Ω and $u^1 - 1 + \tau = u^{\tau} = \tau$ on $\partial\Omega_1$, so by comparison principle we deduce that $|\nabla u^{\tau}| \leq |\nabla(u^1 - 1 + \tau)| = |\nabla u^1|$ on $\partial\Omega_1$.

According to [Proposition 3.1](#) and the discussions above, we then deduce that the solution to the minimal graph equation (3.4) for $\tau \in (0, 1)$ has a priori gradient estimate as follows

$$\sup_{x \in \bar{\Omega}} |\nabla u^\tau|(x) \leq \sup_{x \in \partial\Omega} |\nabla u^\tau|(x) \leq \sup_{x \in \partial\Omega} |\nabla u^1|(x). \quad (3.6)$$

Now, by the classical theory of quasilinear elliptic partial differential equations [\[12\]](#) we then get the existence conclusion.

Uniqueness and smoothness of the solution are obvious.

We pause for a while to recall some remarks of ω^τ , the harmonic function defined on the same convex rings and with the same boundary value, i.e.

$$\begin{cases} \Delta \omega^\tau = 0 & \text{in } \Omega = \Omega_0 \setminus \bar{\Omega}_1, \\ \omega^\tau = 0 & \text{on } \partial\Omega_0, \\ \omega^\tau = \tau & \text{on } \partial\Omega_1. \end{cases} \quad (3.7)$$

It is obvious that $\omega^\tau = \tau \omega^1$ and it follows that

$$\|\omega^\tau\|_{C^{2,\alpha}(\bar{\Omega})} = \tau \|\omega^1\|_{C^{2,\alpha}(\bar{\Omega})} \leq C_2 \tau \quad (3.8)$$

for some $\alpha \in (0, 1)$ and $C_2 > 0$.

Additionally, by [\[27\]](#) we know that $|\nabla \omega^1| \neq 0$ holds on $\bar{\Omega}$, we then can take a positive constant $C_0 > 0$ such that for $x \in \Omega$,

$$|\nabla \omega^\tau|(x) \geq C_0 \tau. \quad (3.9)$$

Now, we come to prove the $C^{2,\alpha}$ continuity of the solution u^τ with respect to the boundary value. Without loss of generality we assume that $t > \tau$. Let $h = u^t - u^\tau$, then h would satisfy a linear divergence type equation. In fact, h has boundary value 0, $t - \tau$ on the boundary $\partial\Omega_0$, $\partial\Omega_1$, respectively. Furthermore, if we rewrite the minimal graph equation to the form $\operatorname{div}(A(\nabla u)) = 0$, we have by following the discussion in [\[28\]](#) or the theory in [\[12\]](#) that

$$\begin{aligned} 0 &= \operatorname{div}(A(\nabla u^t)) - \operatorname{div}(A(\nabla u^\tau)) = \sum_{\alpha} D_{\alpha} (A^{\alpha}(\nabla u^t) - A^{\alpha}(\nabla u^\tau)) \\ &= \sum_{\alpha} D_{\alpha} \left(A^{\alpha}(s \nabla u^t + (1-s) \nabla u^\tau) \Big|_0^1 \right) \\ &= \sum_{\alpha} D_{\alpha} (m_{\alpha\beta}(x) D_{\beta} h), \end{aligned} \quad (3.10)$$

where,

$$m_{\alpha\beta}(x) = \int_0^1 \frac{\partial A^{\alpha}}{\partial p_{\beta}} (s \nabla u^t + (1-s) \nabla u^\tau) ds.$$

Anyway, $h(x)$ satisfies the following uniformly elliptic differential equation for the uniform gradient bound we have deduced,

$$\begin{cases} D_\alpha(m_{\alpha\beta}(x)D_\beta h) = 0 & \text{in } \Omega = \Omega_0 \setminus \bar{\Omega}_1, \\ h = 0 & \text{on } \partial\Omega_0, \\ h = t - \tau & \text{on } \partial\Omega_1. \end{cases}$$

In order to give the $C^{2,\alpha}$ estimate of h , according to Theorem 8.33 of [12] or Theorem 8.33' in [28], we need to extend the boundary value of h to a smooth function defined on the whole domain $\bar{\Omega}$. Fortunately, the harmonic function $\omega^{t-\tau}$ is just suitable for this target. Thus we have by the maximum principle and (3.8),

$$\|h\|_{C^{2,\alpha}(\bar{\Omega})} \leq C \left(\|h\|_{L^\infty(\Omega)} + \|\omega^{t-\tau}\|_{C^{2,\alpha}(\bar{\Omega})} \right) \leq C_1(t - \tau). \quad (3.11)$$

This completes the whole proof of the theorem. $\square$

4. A constant rank theorem

The constant rank theorem joint with deformation process is usually applied to prove the convexity of the solution or the convexity of the level sets of the solution. Based on this essential tool, lots of important results concerning to convexity appeared [6,13–15,27] recently. In this section, we will show a constant rank theorem as follows.

Theorem 4.1. *Let Ω be a smooth bounded connected domain on the space form M^n with constant curvature $\epsilon \geq 0$. Let $u \in C^4(\Omega) \cap C^2(\bar{\Omega})$ be the solution to the prescribed mean curvature equation*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = H(x),$$

where $H(x) \geq 0$ satisfies the structure condition

$$3H_\alpha H_\beta + 4\epsilon H^2 \delta_{\alpha\beta} \leq 2H H_{\alpha\beta}. \quad (4.1)$$

Assume $|\nabla u| \neq 0$ in Ω and the level sets of u are all convex with respect to the normal ∇u , then the second fundamental form of the level sets of u must have the same rank at all points in Ω .

Proof. We firstly represent the equation into the following

$$\sum_{\alpha,\beta} \left[(1 + |\nabla u|^2) \delta_{\alpha\beta} - u_\alpha u_\beta \right] u_{\alpha\beta} = H(x)(1 + |\nabla u|^2)^{\frac{3}{2}} \triangleq f(x, \nabla u) \text{ in } \Omega.$$

The theorem is obviously of local feature, so we may assume that the level set $\Sigma^c(u)$ is connected for each c in a neighborhood of some $c_0 = u(x_0)$. We will compute in a neighborhood of some point $x_0 \in \Omega$ such that the minimal rank l of the second fundamental form of $\Sigma^{u(x_0)}(u)$ is attained at x_0 . Without loss of generality, we can assume that $0 \leq l \leq n - 2$, otherwise we then have got the constant rank $n - 1$.

Let U be a small open neighborhood of x_0 such that for each $x \in U$, there are l “good” eigenvalues of the second fundamental form of the level sets which are bounded from below by a positive constant, and the other $n - 1 - l$ eigenvalues of the second fundamental form of the level sets are very small and we name them “bad” eigenvalues. In the following, we will denote by G, B the index set of “good” eigenvalues and “bad” eigenvalues respectively.

For any $x \in U$ fixed, we can choose a frame $e_1, e_2, \dots, e_{n-1}, e_n$ such that $|\nabla u|(x) = u_n(x) > 0$ and the matrix (u_{ij}) ($i, j = 1, 2, \dots, n-1$) is diagonal at x .

By (2.3), the second fundamental form of Σ^c then is also diagonal at x . Without loss of generality we could assume $h_{11} \leq h_{22} \leq \dots \leq h_{n-1n-1}$ and there exists a positive constant $C > 0$ depending only on $\|u\|_{C^4}$ and the domain U such that $h_{n-1n-1} \geq C$ for all $x \in U$. For convenience, we let $G = \{n-l, n-l+1, \dots, n-1\}$ and $B = \{1, 2, \dots, n-l-1\}$ be the “good” and “bad” sets of indices, respectively. We also denote

$$B = \{h_{11}, h_{22}, \dots, h_{n-l-1n-l-1}\} \text{ and } G = \{h_{n-ln-l}, \dots, h_{n-1n-1}\}.$$

Note that for a fixed $\delta > 0$, we can choose the neighborhood U small enough such that $h_{jj} < \delta$ for all $j \in B$ and $x \in U$.

Denote by λ_i , $i = 1, 2, \dots, n-1$ the principal curvature of level sets of u and $\lambda = (\lambda_1, \dots, \lambda_{n-1})$, it is obvious that $\lambda_i = h_{ii}$ under the assumption above. We set $\mu_i = u_{ii}$, $\mu = (\mu_1, \dots, \mu_{n-1})$ during the whole proof, then $\mu_i = -u_n \lambda_i$ for $i = 1, 2, \dots, n-1$ at x .

Generally, the following notations are usually necessary to conclude the constant rank theorems.

For two functions defined on Ω , h and k , for $y \in \Omega$, we denote by $h(y) \leq k(y)$ if and only if

$$(h-k)(y) \leq (C_3\phi + C_4|\nabla\phi|)(y).$$

Also, we call that $h(y) \sim k(y)$ if and only if $h(y) \leq k(y)$ and $k(y) \leq h(y)$. Generally, we say that $h \sim k$ is valid if and only if $h(y) \sim k(y)$ for any $y \in \Omega$ holds.

During the whole proof, the Greek indices such as α, β etc. will vary from 1 to n while the Latin indices such as i, j, k etc. will vary from 1 to $n-1$.

For each c , set

$$\varphi = |\nabla u|^{l+3} \sigma_{l+1}(\lambda) = (-1)^{l+1} \sum_{\alpha, \beta} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{\alpha\beta}} u_{\alpha} u_{\beta}. \quad (4.2)$$

From now on, all the calculations will be done at x with the above frame. In order to simplify the process of the proof, we introduce the notation [27] $b_{ij,\xi}$ as follows,

$$u_n u_{ij\xi} = -u_n^2 b_{ij,\xi} + u_{ni} u_{j\xi} + u_{nj} u_{i\xi} + u_{n\xi} u_{ij}. \quad (4.3)$$

Easy to get for $j \in B$,

$$\lambda_j \sim 0.$$

Now we come to compute the first order derivative of φ .

$$\begin{aligned} \varphi_{\xi} &= (-1)^{l+1} \left(\sum_{\alpha, \beta} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{\alpha\beta}} u_{\alpha} u_{\beta} \right)_{\xi} \\ &= (-1)^{l+1} \sum_{\alpha, \beta, \gamma, \delta} \frac{\partial^2 \sigma_{l+2}(D^2 u)}{\partial u_{\alpha\beta} \partial u_{\gamma\delta}} u_{\alpha} u_{\beta} u_{\gamma\delta\xi} + (-1)^{l+1} \sum_{\alpha, \beta} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{\alpha\beta}} (u_{\alpha\xi} u_{\beta} + u_{\alpha} u_{\beta\xi}) \\ &= O_1 + O_2; \end{aligned}$$

For the term O_1 , by (2.1),

$$\begin{aligned} O_1 &= (-1)^{l+1} \sum_{\alpha, \beta, \gamma, \delta} \frac{\partial^2 \sigma_{l+2}(D^2 u)}{\partial u_{\alpha\beta} \partial u_{\gamma\delta}} u_{\alpha} u_{\beta} u_{\gamma\delta\xi} = (-1)^{l+1} u_n^2 \sum_{\gamma, \delta} \frac{\partial^2 \sigma_{l+2}(D^2 u)}{\partial u_{nn} \partial u_{\gamma\delta}} u_{\gamma\delta\xi} \\ &= (-1)^{l+1} u_n^2 \sum_i \frac{\partial \sigma_{l+1}(u_{pq})}{\partial u_{ii}} u_{ii\xi} = (-1)^{l+1} u_n^2 \sum_i \sigma_l(\mu | i) u_{ii\xi} \\ &\sim (-1)^{l+1} u_n^2 \sum_{i \in B} \sigma_l(\mu | i) u_{ii\xi} = -u_n^{l+2} \sum_{i \in B} \sigma_l(\lambda | i) u_{ii\xi} \sim -u_n^{l+2} \sigma_l(G) \sum_{j \in B} u_{jj\xi}; \end{aligned} \quad (4.4)$$

For the term O_2 ,

$$\begin{aligned} O_2 &= (-1)^{l+1} \sum_{\alpha, \beta} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{\alpha\beta}} (u_{\alpha\xi} u_{\beta} + u_{\alpha} u_{\beta\xi}) = 2(-1)^{l+1} u_n \sum_{\beta} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{n\beta}} u_{\beta\xi} \\ &= 2(-1)^{l+1} u_n \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{nn}} u_{n\xi} + 2(-1)^{l+1} u_n \sum_i \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{ni}} u_{i\xi} \\ &= 2(-1)^{l+1} u_n \sigma_{l+1}(\mu) u_{n\xi} + 2(-1)^l u_n \sum_i \sigma_l(\mu | i) u_{in} u_{i\xi} \\ &\sim 2u_n^{l+1} \sum_{i \in B} \sigma_l(\lambda | i) u_{in} u_{i\xi} \sim 2u_n^{l+1} \sigma_l(G) \sum_{j \in B} u_{jn} u_{j\xi}; \end{aligned} \quad (4.5)$$

Therefore, by (4.3), (4.4) and (4.5), we have

$$\begin{aligned} \varphi_{\xi} &\sim -u_n^{l+2} \sigma_l(G) \sum_{j \in B} u_{jj\xi} + 2u_n^{l+1} \sigma_l(G) \sum_{j \in B} u_{jn} u_{j\xi} \\ &= -u_n^{l+1} \sigma_l(G) \left(u_n \sum_{j \in B} u_{jj\xi} - 2 \sum_{j \in B} u_{jn} u_{j\xi} \right) \\ &\sim u_n^{l+2} \sigma_l(G) \sum_{j \in B} b_{jj, \xi}. \end{aligned} \quad (4.6)$$

So we deduce that

$$\sum_{j \in B} b_{jj, \xi} \sim 0, \quad \forall 1 \leq \xi \leq n. \quad (4.7)$$

Now we set out to compute the second order derivative of φ .

$$\begin{aligned} \varphi_{\xi\eta} &= (-1)^{l+1} \sum_{\alpha, \beta} \left(\frac{\partial^2 \sigma_{l+2}(D^2 u)}{\partial u_{\alpha\beta} \partial u_{\gamma\delta}} u_{\alpha} u_{\beta} u_{\gamma\delta\xi} \right)_{\eta} \\ &\quad + (-1)^{l+1} \sum_{\alpha, \beta} \left(\frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{\alpha\beta}} (u_{\alpha\xi} u_{\beta} + u_{\alpha} u_{\beta\xi}) \right)_{\eta} \end{aligned}$$

Denote by $a^{\xi\eta} = (1 + |\nabla u|^2)\delta_{\xi\eta} - u_{\xi}u_{\eta}$. Direct calculations show that

$$\sum_{\xi,\eta} a^{\xi\eta} \varphi_{\xi\eta} = I + II + III + IV + V, \quad (4.8)$$

where

$$\begin{aligned} I &= (-1)^{l+1} \sum_{\alpha,\beta,\gamma,\delta,\tau,\theta,\xi,\eta} a^{\xi\eta} \frac{\partial^3 \sigma_{l+2}(D^2 u)}{\partial u_{\alpha\beta} \partial u_{\gamma\delta} \partial u_{\tau\theta}} u_{\alpha} u_{\beta} u_{\gamma\delta\xi} u_{\tau\theta\eta}; \\ II &= 4(-1)^{l+1} \sum_{\alpha,\beta,\gamma,\delta,\xi,\eta} a^{\xi\eta} \frac{\partial^2 \sigma_{l+2}(D^2 u)}{\partial u_{\alpha\beta} \partial u_{\gamma\delta}} u_{\alpha\eta} u_{\beta} u_{\gamma\delta\xi}; \\ III &= (-1)^{l+1} \sum_{\alpha,\beta,\gamma,\delta,\xi,\eta} a^{\xi\eta} \frac{\partial^2 \sigma_{l+2}(D^2 u)}{\partial u_{\alpha\beta} \partial u_{\gamma\delta}} u_{\alpha} u_{\beta} u_{\gamma\delta\xi\eta}; \\ IV &= 2(-1)^{l+1} \sum_{\alpha,\beta,\xi,\eta} a^{\xi\eta} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{\alpha\beta}} u_{\alpha\xi\eta} u_{\beta}; \\ V &= 2(-1)^{l+1} \sum_{\alpha,\beta,\xi,\eta} a^{\xi\eta} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{\alpha\beta}} u_{\alpha\xi} u_{\beta\eta}. \end{aligned} \quad (4.9)$$

In the following, we come to deal with the above five terms one by one.

For I , we compute step by step and will arrive at (4.20).

$$\begin{aligned} I &= (-1)^{l+1} \sum_{\alpha,\beta,\gamma,\delta,\tau,\theta,\xi,\eta} a^{\xi\eta} \frac{\partial^3 \sigma_{l+2}(D^2 u)}{\partial u_{\alpha\beta} \partial u_{\gamma\delta} \partial u_{\tau\theta}} u_{\alpha} u_{\beta} u_{\gamma\delta\xi} u_{\tau\theta\eta} \\ &= (-1)^{l+1} u_n^2 \sum_{\xi} a^{\xi\xi} \sum_{i,j,k,l} \frac{\partial^2 \sigma_{l+1}(u_{pq})}{\partial u_{ij} \partial u_{kl}} u_{ij\xi} u_{kl\xi} \\ &= (-1)^{l+1} u_n^2 \sum_{\xi} a^{\xi\xi} \sum_{i \neq j} \sigma_{l-1}(\mu |ij) \left(u_{ii\xi} u_{jj\xi} - u_{ij\xi}^2 \right) \\ &= u_n^{l+1} \sum_{\xi} a^{\xi\xi} \sum_{i \neq j} \sigma_{l-1}(\lambda |ij) \left(u_{ii\xi} u_{jj\xi} - u_{ij\xi}^2 \right) \\ &= u_n^{l+1} \sum_{\xi} a^{\xi\xi} \left(\sum_{i,j \in G, i \neq j} + \sum_{i \in G, j \in B} + \sum_{j \in G, i \in B} + \sum_{i,j \in B, i \neq j} \right) \sigma_{l-1}(\lambda |ij) \left(u_{ii\xi} u_{jj\xi} - u_{ij\xi}^2 \right) \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (4.10)$$

It is easy to conclude that

$$I_1 \sim 0. \quad (4.11)$$

For the term $I_2 + I_3$, it follows that

$$\begin{aligned}
 I_2 + I_3 &= 2u_n^{l+1} \sum_{\xi} a^{\xi\xi} \sum_{i \in G, j \in B} \sigma_{l-1}(\lambda |ij) \left(u_{ii\xi} u_{jj\xi} - u_{ij\xi}^2 \right) \\
 &\sim 2u_n^{l-1} \sum_{\xi} a^{\xi\xi} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) \left((u_n u_{ii\xi}) (u_n u_{jj\xi}) - (u_n u_{ij\xi})^2 \right) \\
 &= 2u_n^{l-1} \sum_{\xi} a^{\xi\xi} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) \left(-u_n^2 b_{ii,\xi} + 2u_{ni} u_{i\xi} \right. \\
 &\quad \left. + u_{n\xi} u_{ii} \right) \left(-u_n^2 b_{jj,\xi} + 2u_{nj} u_{j\xi} \right) \\
 &\quad - 2u_n^{l-1} \sum_{\xi} a^{\xi\xi} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) \left(-u_n^2 b_{ij,\xi} + u_{ni} u_{j\xi} + u_{nj} u_{i\xi} \right)^2 \\
 &= I_{21} + I_{22}.
 \end{aligned} \tag{4.12}$$

By (4.7), we have

$$\begin{aligned}
 I_{21} &= 4u_n^{l-1} \sum_{\xi} a^{\xi\xi} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) \left(-u_n^2 b_{ii,\xi} + 2u_{ni} u_{i\xi} + u_{n\xi} u_{ii} \right) u_{nj} u_{j\xi} \\
 &= 4u_n^{l-1} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) \left(-u_n^2 b_{ii,n} + 2u_{ni}^2 + u_{nn} u_{ii} \right) u_{nj}^2 \\
 &= -4u_n^{l+1} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) b_{ii,n} u_{nj}^2 + 8u_n^{l-1} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{ni}^2 u_{nj}^2 \\
 &\quad - 4lu_n^l u_{nn} \sigma_l(G) \sum_{j \in B} u_{nj}^2,
 \end{aligned} \tag{4.13}$$

and for the second term I_{22} , we get

$$\begin{aligned}
 I_{22} &= -2u_n^{l+3} \sum_{\xi} a^{\xi\xi} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) b_{ij,\xi}^2 \\
 &\quad + 8u_n^{l+1} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{ni} u_{nj} b_{ij,n} + 4u_n^{l+1} \left(1 + u_n^2 \right) \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{nj} u_{ii} b_{ij,i} \\
 &\quad - 8u_n^{l-1} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{ni}^2 u_{nj}^2 - 2u_n^{l-1} \left(1 + u_n^2 \right) \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{ii}^2 u_{nj}^2 \\
 &= -2u_n^{l+3} \sum_{\xi} a^{\xi\xi} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) b_{ij,\xi}^2 \\
 &\quad + 8u_n^{l+1} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{ni} u_{nj} b_{ij,n} - 4u_n^{l+2} \left(1 + u_n^2 \right) \sigma_l(G) \sum_{i \in G, j \in B} u_{nj} b_{ij,i} \\
 &\quad - 8u_n^{l-1} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{ni}^2 u_{nj}^2 - 2u_n^{l+1} \left(1 + u_n^2 \right) \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2.
 \end{aligned} \tag{4.14}$$

Therefore, by (4.12), (4.13) and (4.14) we get

$$\begin{aligned}
I_2 + I_3 = & -2u_n^{l+3} \sum_{\xi} a^{\xi\xi} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) b_{ij,\xi}^2 - 4u_n^{l+1} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{nj}^2 b_{ii,n} \\
& + 8u_n^{l+1} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{ni} u_{nj} b_{ij,n} - 4u_n^{l+2} (1 + u_n^2) \sigma_l(G) \sum_{i \in G, j \in B} u_{nj} b_{ii,j} \\
& - 2u_n^{l+1} (1 + u_n^2) \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2 - 4lu_n^l u_{nn} \sigma_l(G) \sum_{j \in B} u_{nj}^2. \quad (4.15)
\end{aligned}$$

For the term I_4 ,

$$\begin{aligned}
I_4 \sim & u_n^{l-1} \sigma_{l-1}(G) \sum_{\xi} a^{\xi\xi} \sum_{i, j \in B, i \neq j} \left(-u_n^2 b_{ii,\xi} + 2u_{ni} u_{i\xi} \right) \left(-u_n^2 b_{jj,\xi} + 2u_{nj} u_{j\xi} \right) \\
& - u_n^{l-1} \sigma_{l-1}(G) \sum_{\xi} a^{\xi\xi} \sum_{i, j \in B, i \neq j} \left(-u_n^2 b_{ij,\xi} + u_{ni} u_{j\xi} + u_{nj} u_{i\xi} \right)^2 \quad (4.16) \\
= & I_{41} + I_{42}.
\end{aligned}$$

According to (4.7) we derive

$$\begin{aligned}
I_{41} = & u_n^{l-1} \sigma_{l-1}(G) \sum_{\xi} a^{\xi\xi} \sum_{i, j \in B, i \neq j} \left(-u_n^2 b_{ii,\xi} + 2u_{ni} u_{i\xi} \right) \left(-u_n^2 b_{jj,\xi} + 2u_{nj} u_{j\xi} \right) \\
= & u_n^{l-1} \sigma_{l-1}(G) \sum_{\xi} a^{\xi\xi} \sum_{i, j \in B, i \neq j} \left[u_n^4 b_{ii,\xi} b_{jj,\xi} - 4u_n^2 u_{nj} u_{j\xi} b_{ii,\xi} + 4u_{ni} u_{nj} u_{i\xi} u_{j\xi} \right] \quad (4.17) \\
= & -u_n^{l+3} \sigma_{l-1}(G) \sum_{i \in B} b_{ii,n}^2 - u_n^{l+3} (1 + u_n^2) \sigma_{l-1}(G) \sum_{j=1}^{n-1} \sum_{i \in B} b_{ii,j}^2 \\
& + 4u_n^{l+1} \sigma_{l-1}(G) \sum_{j \in B} u_{nj}^2 b_{jj,n} + 4u_n^{l-1} \sigma_{l-1}(G) \sum_{i, j \in B, i \neq j} u_{ni}^2 u_{nj}^2
\end{aligned}$$

and

$$\begin{aligned}
I_{42} = & -u_n^{l-1} \sum_{\xi} a^{\xi\xi} \sigma_{l-1}(G) \sum_{i, j \in B, i \neq j} \left(-u_n^2 b_{ij,\xi} + u_{ni} u_{j\xi} + u_{nj} u_{i\xi} \right)^2 \\
= & -u_n^{l-1} \sigma_{l-1}(G) \sum_{i, j \in B, i \neq j} \left(-u_n^2 b_{ij,n} + 2u_{ni} u_{nj} \right)^2 \\
& - u_n^{l+3} (1 + u_n^2) \sigma_{l-1}(G) \sum_{k=1}^{n-1} \sum_{i, j \in B, i \neq j} b_{ij,k}^2 \quad (4.18) \\
= & -u_n^{l+3} \sigma_{l-1}(G) \sum_{i, j \in B, i \neq j} b_{ij,n}^2 + 4u_n^{l+1} \sigma_{l-1}(G) \sum_{i, j \in B, i \neq j} u_{ni} u_{nj} b_{ij,n} \\
& - 4u_n^{l-1} \sigma_{l-1}(G) \sum_{i, j \in B, i \neq j} u_{ni}^2 u_{nj}^2 - u_n^{l+3} (1 + u_n^2) \sigma_{l-1}(G) \sum_{k=1}^{n-1} \sum_{i, j \in B, i \neq j} b_{ij,k}^2.
\end{aligned}$$

Then we can conclude from (4.16), (4.17) and (4.18) that

$$\begin{aligned}
 I_4 = & -u_n^{l+3} \sigma_{l-1}(G) \sum_{i,j \in B} b_{ij,n}^2 \\
 & -u_n^{l+3} \left(1 + u_n^2\right) \sigma_{l-1}(G) \sum_{k=1}^{n-1} \sum_{i,j \in B} b_{ij,k}^2 + 4u_n^{l+1} \sigma_{l-1}(G) \sum_{i,j \in B} u_{ni} u_{nj} b_{ij,n}.
 \end{aligned} \tag{4.19}$$

Now, together with (4.10), (4.11), (4.15) and (4.19) we have

$$\begin{aligned}
 I \sim & -2u_n^{l+3} \sum_{\xi} a^{\xi\xi} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) b_{ij,\xi}^2 - 4u_n^{l+1} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{nj}^2 b_{ii,n} \\
 & + 8u_n^{l+1} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{ni} u_{nj} b_{ij,n} - 4u_n^{l+2} \left(1 + u_n^2\right) \sigma_l(G) \sum_{i \in G, j \in B} u_{nj} b_{ii,j} \\
 & - 2u_n^{l+1} \left(1 + u_n^2\right) \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2 - 4u_n^l u_{nn} \sigma_l(G) \sum_{j \in B} u_{nj}^2 \\
 & - u_n^{l+3} \sigma_{l-1}(G) \sum_{i,j \in B} b_{ij,n}^2 - u_n^{l+3} \left(1 + u_n^2\right) \sigma_{l-1}(G) \sum_{k=1}^{n-1} \sum_{i,j \in B} b_{ij,k}^2 \\
 & + 4u_n^{l+1} \sigma_{l-1}(G) \sum_{i,j \in B} u_{ni} u_{nj} b_{ij,n}.
 \end{aligned} \tag{4.20}$$

For the term II , we will arrive at (4.44).

$$\begin{aligned}
 II = & 4(-1)^{l+1} \sum_{\xi} a^{\xi\xi} \sum_{\alpha, \beta, \gamma, \delta} \frac{\partial^2 \sigma_{l+2}(D^2 u)}{\partial u_{\alpha\beta} \partial u_{\gamma\delta}} u_{\alpha\xi} u_{\beta} u_{\gamma\delta\xi} \\
 = & 4(-1)^{l+1} u_n \sum_{\xi} a^{\xi\xi} \sum_{\gamma, \delta} \frac{\partial^2 \sigma_{l+2}(D^2 u)}{\partial u_{nn} \partial u_{\gamma\delta}} u_{n\xi} u_{\gamma\delta\xi} \\
 & + 4(-1)^{l+1} u_n \sum_{\xi} a^{\xi\xi} \sum_i \sum_{\gamma, \delta} \frac{\partial^2 \sigma_{l+2}(D^2 u)}{\partial u_{in} \partial u_{\gamma\delta}} u_{i\xi} u_{\gamma\delta\xi} = II_1 + II_2.
 \end{aligned} \tag{4.21}$$

For the first term II_1 we have by (4.3),

$$\begin{aligned}
 II_1 = & 4(-1)^{l+1} u_n \sum_{\xi} a^{\xi\xi} \sum_{\gamma, \delta} \frac{\partial^2 \sigma_{l+2}(D^2 u)}{\partial u_{nn} \partial u_{\gamma\delta}} u_{n\xi} u_{\gamma\delta\xi} \\
 = & 4(-1)^{l+1} u_n \sum_{\xi} a^{\xi\xi} \sum_i \sigma_l(\mu|i) u_{n\xi} u_{ii\xi} \\
 \sim & -4u_n^{l+1} \sum_{\xi} a^{\xi\xi} \sum_{i \in B} \sigma_l(G) u_{n\xi} u_{ii\xi}
 \end{aligned} \tag{4.22}$$

$$\begin{aligned}
&= -4u_n^l \sigma_l(G) \sum_{\xi} a^{\xi\xi} \sum_{i \in B} u_{n\xi} (-u_n^2 b_{ii,\xi} + 2u_{ni} u_{i\xi}) \\
&= -8f u_n^l \sigma_l(G) \sum_{j \in B} u_{nj}^2 - 8u_n^{l+1} (1 + u_n^2) \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2.
\end{aligned}$$

In the following, we come to settle the second term II_2 .

$$\begin{aligned}
II_2 &= 4(-1)^{l+1} u_n \sum_{\xi} a^{\xi\xi} \frac{\partial^2 \sigma_{l+2}(D^2 u)}{\partial u_{in} \partial u_{\gamma\delta}} u_{i\xi} u_{\gamma\delta\xi} \\
&= 4(-1)^{l+1} u_n \sum_{\xi} a^{\xi\xi} \sum_{i=1}^{n-1} \frac{\partial}{\partial u_{\gamma\delta}} \left(-u_{ni} \sigma_l(u_{pq} | i) + \sum_{j \neq i} u_{nj} u_{ij} \sigma_{l-1}(u_{pq} | ij) \right) u_{i\xi} u_{\gamma\delta\xi} \\
&= II_{21} + II_{22}.
\end{aligned} \tag{4.23}$$

It follows that

$$\begin{aligned}
II_{21} &= 4(-1)^{l+1} u_n \sum_{\xi} a^{\xi\xi} \sum_{i=1}^{n-1} \frac{\partial}{\partial u_{\gamma\delta}} (-u_{ni} \sigma_l(u_{pq} | i)) u_{i\xi} u_{\gamma\delta\xi} \\
&= 4(-1)^{l+1} u_n \sum_{\xi} a^{\xi\xi} \sum_{i=1}^{n-1} (-\sigma_l(u_{pq} | i)) u_{i\xi} u_{ni\xi} \\
&\quad - 4(-1)^{l+1} u_n \sum_{\xi} a^{\xi\xi} \sum_{i \neq j} \sigma_{l-1}(u_{pq} | ij) u_{ni} u_{i\xi} u_{jj\xi} = II_{211} + II_{212};
\end{aligned} \tag{4.24}$$

For II_{211} , we derive that

$$\begin{aligned}
II_{211} &= 4(-1)^{l+1} u_n \sum_{\xi} a^{\xi\xi} \sum_{i=1}^{n-1} (-\sigma_l(u_{pq} | i)) u_{i\xi} u_{ni\xi} \\
&\sim 4u_n^{l+1} \sigma_l(G) \sum_{\xi} a^{\xi\xi} \sum_{i \in B} u_{i\xi} u_{ni\xi} \\
&\sim 4u_n^{l+1} \sigma_l(G) \sum_{i \in B} u_{in} u_{nin} = 4u_n^{l+1} \sigma_l(G) \sum_{i \in B} u_{in} u_{nni}.
\end{aligned} \tag{4.25}$$

We now pause for a while to derive a observation for the proceed of the whole proof. Differentiate both sides of the equation $a^{\alpha\beta} u_{\alpha\beta} = f$ with respect to e_i ($i \in B$),

$$\sum_{\alpha, \beta} a^{\alpha\beta} ,_i u_{\alpha\beta} + \sum_{\alpha, \beta} a^{\alpha\beta} u_{\alpha\beta i} = f_i.$$

Easy to get that for $i \in B$,

$$\sum_{\alpha, \beta} a^{\alpha\beta} {}_{,i} u_{\alpha\beta} = 2u_n u_{ni} \Delta u - 2u_n \sum_{\alpha} u_{n\alpha} u_{i\alpha} = -2u_n^2 u_{ni} \sigma_1(G). \quad (4.26)$$

Therefore, by (4.3) the term u_{nni} now becomes

$$u_{nni} = f_i + 2u_n^2 u_{ni} \sigma_1(G) - u_n^{-1} (1 + u_n^2) \sum_{j \in G} (-u_n^2 b_{jj,i} + u_{ni} u_{jj}). \quad (4.27)$$

Plug this expression into (4.25), we have

$$\begin{aligned} II_{211} &= 4u_n^{l+1} \sigma_l(G) \sum_{j \in B} u_{nj} f_j + 4u_n^{l+1} \sigma_l(G) \sigma_1(G) (1 + 3u_n^2) \sum_{j \in B} u_{nj}^2 \\ &\quad + 4u_n^{l+2} \sigma_l(G) (1 + u_n^2) \sum_{i \in G, j \in B} u_{nj} b_{ii,j}. \end{aligned} \quad (4.28)$$

For the term II_{212} ,

$$\begin{aligned} II_{212} &= -4(-1)^{l+1} u_n \sum_{\xi} a^{\xi\xi} \sum_{i \neq j} \sigma_{l-1}(u_{pq} |ij) u_{ni} u_{i\xi} u_{jj\xi} \\ &= 4(-1)^l u_n \sum_{\xi} a^{\xi\xi} \left(\sum_{i,j \in G, i \neq j} + \sum_{i \in G, j \in B} + \sum_{j \in G, i \in B} + \sum_{i,j \in B, i \neq j} \right) \sigma_{l-1}(\mu |ij) u_{ni} u_{i\xi} u_{jj\xi} \\ &= II_{2121} + II_{2122} + II_{2123} + II_{2124}. \end{aligned} \quad (4.29)$$

Obvious to find that

$$II_{2121} \sim 0. \quad (4.30)$$

By (4.3) and (4.7), we get

$$\begin{aligned} II_{2122} &= 4(-1)^l \sum_{\xi} a^{\xi\xi} \sum_{i \in G, j \in B} \sigma_{l-1}(\mu |ij) u_{ni} u_{i\xi} (-u_n^2 b_{jj,\xi} + 2u_{nj} u_{j\xi} + u_{n\xi} u_{jj}) \\ &\sim -4u_n^{l-1} \sum_{\xi} a^{\xi\xi} \sum_{i \in G, j \in B} \sigma_{l-1}(\lambda |i) u_{ni} u_{i\xi} (-u_n^2 b_{jj,\xi} + 2u_{nj} u_{j\xi}) \\ &= -8u_n^{l-1} \sum_{i \in G, j \in B} \sigma_{l-1}(G |i) u_{ni}^2 u_{nj}^2, \end{aligned} \quad (4.31)$$

similarly,

$$\begin{aligned}
II_{2123} &= 4(-1)^l \sum_{\xi} a^{\xi\xi} \sum_{j \in G, i \in B} \sigma_{l-1}(\mu |ij) u_{ni} u_{i\xi} \left(-u_n^2 b_{jj,\xi} + 2u_{nj} u_{j\xi} + u_{n\xi} u_{jj} \right) \\
&\sim 4u_n^{l+1} \sum_{i \in G, j \in B} \sigma_{l-1}(\lambda |i) u_{nj}^2 b_{ii,n} - 8u_n^{l-1} \sum_{i \in G, j \in B} \sigma_{l-1}(\lambda |i) u_{ni}^2 u_{nj}^2 \\
&\quad + 4lu_n^l u_{nn} \sigma_l(G) \sum_{j \in B} u_{nj}^2,
\end{aligned} \tag{4.32}$$

and

$$\begin{aligned}
II_{2124} &= 4(-1)^l \sum_{\xi} a^{\xi\xi} \sum_{i,j \in B, i \neq j} \sigma_{l-1}(\mu |ij) u_{ni} u_{i\xi} \left(-u_n^2 b_{jj,\xi} + 2u_{nj} u_{j\xi} + u_{n\xi} u_{jj} \right) \\
&= -4u_n^{l-1} \sum_{i,j \in B, i \neq j} \sigma_{l-1}(\lambda |ij) u_{in}^2 \left(-u_n^2 b_{jj,n} + 2u_{nj}^2 \right) \\
&\sim 4u_n^{l+1} \sigma_{l-1}(G) \sum_{i,j \in B, i \neq j} u_{in}^2 b_{jj,n} - 8u_n^{l-1} \sigma_{l-1}(G) \sum_{i,j \in B, i \neq j} u_{in}^2 u_{nj}^2 \\
&\sim -4u_n^{l+1} \sigma_{l-1}(G) \sum_{j \in B} u_{nj}^2 b_{jj,n} - 8u_n^{l-1} \sigma_{l-1}(G) \sum_{i,j \in B, i \neq j} u_{ni}^2 u_{nj}^2.
\end{aligned} \tag{4.33}$$

Therefore, according to (4.30), (4.31), (4.32) and (4.33) we have

$$\begin{aligned}
II_{212} &\sim 4u_n^{l+1} \sum_{i \in G, j \in B} \sigma_{l-1}(G |i) u_{nj}^2 b_{ii,n} - 4u_n^{l+1} \sigma_{l-1}(G) \sum_{j \in B} u_{nj}^2 b_{jj,n} \\
&\quad - 16u_n^{l-1} \sum_{i \in G, j \in B} \sigma_{l-1}(G |i) u_{ni}^2 u_{nj}^2 - 8u_n^{l-1} \sigma_{l-1}(G) \sum_{i,j \in B, i \neq j} u_{ni}^2 u_{nj}^2 \\
&\quad + 4lu_n^l u_{nn} \sigma_l(G) \sum_{j \in B} u_{nj}^2.
\end{aligned} \tag{4.34}$$

By (4.28) and (4.34), we get that

$$\begin{aligned}
II_{21} &\sim 4u_n^{l+1} \sigma_l(G) \sum_{j \in B} u_{nj} f_j + 4u_n^{l+1} \sigma_l(G) \sigma_1(G) \left(1 + 3u_n^2 \right) \sum_{j \in B} u_{nj}^2 \\
&\quad + 4u_n^{l+2} \sigma_l(G) \left(1 + u_n^2 \right) \sum_{i \in G, j \in B} u_{nj} b_{ii,j} + 4u_n^{l+1} \sum_{i \in G, j \in B} \sigma_{l-1}(G |i) u_{nj}^2 b_{ii,n} \\
&\quad - 4u_n^{l+1} \sigma_{l-1}(G) \sum_{j \in B} u_{nj}^2 b_{jj,n} - 16u_n^{l-1} \sum_{i \in G, j \in B} \sigma_{l-1}(G |i) u_{ni}^2 u_{nj}^2 \\
&\quad + 4lu_n^l u_{nn} \sigma_l(G) \sum_{j \in B} u_{nj}^2 - 8u_n^{l-1} \sigma_{l-1}(G) \sum_{i,j \in B, i \neq j} u_{ni}^2 u_{nj}^2.
\end{aligned} \tag{4.35}$$

In the sequel, we come to settle the term II_{22} .

$$\begin{aligned}
 II_{22} &= 4(-1)^{l+1} u_n \sum_{\xi=1}^n a^{\xi\xi} \sum_{i \neq j} \sum_{\gamma, \delta} \frac{\partial}{\partial u_{\gamma\delta}} [u_{nj} u_{ji} \sigma_{l-1}(u_{pq} | ij)] u_{i\xi} u_{\gamma\delta\xi} \\
 &= 4u_n^l \sum_{\xi=1}^n a^{\xi\xi} \left(\sum_{i,j \in G, i \neq j} + \sum_{i \in G, j \in B} + \sum_{j \in G, i \in B} + \sum_{i,j \in B, i \neq j} \right) u_{nj} \sigma_{l-1}(\lambda | ij) u_{i\xi} u_{ji\xi} \\
 &= II_{221} + II_{222} + II_{223} + II_{224}.
 \end{aligned} \tag{4.36}$$

Step by step, we will give the final expression of II_{22} . Firstly,

$$II_{221} \sim 0. \tag{4.37}$$

Secondly, by (4.3) and Lemma 2.1,

$$\begin{aligned}
 II_{222} &\sim 4u_n^l \sum_{\xi=1}^n a^{\xi\xi} \sum_{i \in G, j \in B} u_{nj} \sigma_{l-1}(G | i) u_{i\xi} u_{ij\xi} \\
 &= 4u_n^l \sum_{i \in G, j \in B} \sigma_{l-1}(G | i) u_{nj} u_{ni} u_{ijn} + 4u_n^l (1 + u_n^2) \sum_{i \in G, j \in B} \sigma_{l-1}(G | i) u_{nj} u_{ii} u_{iji} \\
 &= 4u_n^l \sum_{i \in G, j \in B} \sigma_{l-1}(G | i) u_{nj} u_{ni} u_{ijn} \\
 &\quad - 4u_n^l (1 + u_n^2) \sum_{i \in G, j \in B} \sigma_l(G) u_{nj} [-u_n^2 b_{ii,j} + u_{nj} u_{ii}] \\
 &= 4u_n^l \sum_{i \in G, j \in B} \sigma_{l-1}(G | i) u_{nj} u_{ni} u_{ijn} + 4u_n^{l+2} (1 + u_n^2) \sigma_l(G) \sum_{i \in G, j \in B} u_{nj} b_{ii,j} \\
 &\quad + 4u_n^{l+1} (1 + u_n^2) \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2.
 \end{aligned} \tag{4.38}$$

A similar process shows that

$$II_{223} \sim 4u_n^l \sum_{i \in G, j \in B} \sigma_{l-1}(G | i) u_{nj} u_{ni} u_{ijn}, \tag{4.39}$$

and

$$II_{224} \sim 4u_n^l \sigma_{l-1}(G) \sum_{i, j \in B, i \neq j} u_{nj} u_{ni} u_{ijn}. \tag{4.40}$$

Combining (4.37)–(4.40), we have

$$\begin{aligned}
II_{22} \sim & 8u_n^l \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{nj} u_{ni} u_{ijn} + 4u_n^l \sigma_{l-1}(G) \sum_{i, j \in B, i \neq j} u_{nj} u_{ni} u_{ijn} \\
& + 4u_n^{l+2} (1 + u_n^2) \sigma_l(G) \sum_{i \in G, j \in B} u_{nj} b_{ii,j} + 4u_n^{l+1} (1 + u_n^2) \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2.
\end{aligned} \tag{4.41}$$

Once again by (4.3), we revise the form of II_{22} into the following

$$\begin{aligned}
II_{22} \sim & -8u_n^{l+1} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{nj} u_{ni} b_{ij,n} - 4u_n^{l+1} \sigma_{l-1}(G) \sum_{i, j \in B, i \neq j} u_{nj} u_{ni} b_{ij,n} \\
& + 16u_n^{l-1} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{ni}^2 u_{nj}^2 + 8u_n^{l-1} \sigma_{l-1}(G) \sum_{i, j \in B, i \neq j} u_{ni}^2 u_{nj}^2 \\
& + 4u_n^{l+2} (1 + u_n^2) \sigma_l(G) \sum_{i \in G, j \in B} u_{nj} b_{ii,j} + 4u_n^{l+1} (1 + u_n^2) \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2.
\end{aligned} \tag{4.42}$$

Combining (4.23), (4.35) and (4.42), we then have

$$\begin{aligned}
II_2 \sim & 4u_n^{l+1} \sigma_l(G) \sum_{j \in B} u_{nj} f_j + 8u_n^{l+1} \sigma_l(G) \sigma_1(G) (1 + 2u_n^2) \sum_{j \in B} u_{nj}^2 \\
& + 8u_n^{l+2} \sigma_l(G) (1 + u_n^2) \sum_{i \in G, j \in B} u_{nj} b_{ii,j} + 4u_n^{l+1} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{nj}^2 b_{ii,n} \\
& + 4lu_n^l u_{nn} \sigma_l(G) \sum_{j \in B} u_{nj}^2 - 8u_n^{l+1} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{nj} u_{ni} b_{ij,n} \\
& - 4u_n^{l+1} \sigma_{l-1}(G) \sum_{i, j \in B} u_{nj} u_{ni} b_{ij,n}.
\end{aligned} \tag{4.43}$$

Therefore, by (4.21), (4.22) and (4.43),

$$\begin{aligned}
II \sim & -8fu_n^l \sigma_l(G) \sum_{j \in B} u_{nj}^2 + 4u_n^{l+1} \sigma_l(G) \sum_{j \in B} u_{nj} f_j + 8u_n^{l+3} \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2 \\
& + 8u_n^{l+2} \sigma_l(G) (1 + u_n^2) \sum_{i \in G, j \in B} u_{nj} b_{ii,j} + 4u_n^{l+1} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{nj}^2 b_{ii,n} \\
& + 4lu_n^l u_{nn} \sigma_l(G) \sum_{j \in B} u_{nj}^2 - 8u_n^{l+1} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{nj} u_{ni} b_{ij,n} \\
& - 4u_n^{l+1} \sigma_{l-1}(G) \sum_{i, j \in B} u_{nj} u_{ni} b_{ij,n}.
\end{aligned} \tag{4.44}$$

So, by (4.20) and (4.44), it follows that

$$\begin{aligned}
 I + II &\sim -2u_n^{l+3} \sum_{\xi} a^{\xi\xi} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) b_{ij,\xi}^2 - u_n^{l+3} \sigma_{l-1}(G) \sum_{i,j \in B} b_{ij,n}^2 \\
 &\quad - u_n^{l+3} \left(1 + u_n^2\right) \sigma_{l-1}(G) \sum_{k=1}^{n-1} \sum_{i,j \in B} b_{ij,k}^2 - 8f u_n^l \sigma_l(G) \sum_{j \in B} u_{nj}^2 \\
 &\quad - 2u_n^{l+1} \left(1 - 3u_n^2\right) \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2 + 4u_n^{l+1} \sigma_l(G) \sum_{j \in B} u_{jn} f_j \\
 &\quad + 4u_n^{l+2} \left(1 + u_n^2\right) \sigma_l(G) \sum_{i \in G, j \in B} u_{nj} b_{ii,j} .
 \end{aligned} \tag{4.45}$$

Now, it's time for us to compute the third term III .

By [Lemma 2.1](#), we then have

$$\begin{aligned}
 III &= (-1)^{l+1} \sum_{\alpha,\beta,\gamma,\delta,\xi,\eta} a^{\xi\eta} \frac{\partial^2 \sigma_{l+2}(D^2 u)}{\partial u_{\alpha\beta} \partial u_{\gamma\delta}} u_{\alpha} u_{\beta} u_{\gamma\delta\xi\eta} \\
 &= (-1)^{l+1} u_n^2 \sum_{\alpha,\beta,\gamma,\delta} a^{\alpha\beta} \frac{\partial \sigma_{l+1}(u_{pq})}{\partial u_{\gamma\delta}} u_{\gamma\delta\alpha\beta} \\
 &\sim -u_n^{l+2} \sigma_l(G) \sum_{\alpha,\beta} \sum_{i \in B} a^{\alpha\beta} u_{ii\alpha\beta} \\
 &= -u_n^{l+2} \sigma_l(G) \sum_{\alpha,\beta} \sum_{i \in B} a^{\alpha\beta} \left(u_{\alpha\beta ii} + 2 \sum_{\xi} u_{\xi i} R_{\xi\alpha i \beta} - 2 \sum_{\xi} u_{\alpha\xi} R_{\xi i \beta i} \right) \\
 &= III_1 + III_2 + III_3 .
 \end{aligned} \tag{4.46}$$

For the last two terms III_2 and III_3 , it is obvious to deduce that

$$III_2 \sim 0, \quad III_3 \sim 2(n-l-1) \epsilon f u_n^{l+2} \sigma_l(G) . \tag{4.47}$$

Remark that ϵ is just the sectional curvature of the space form.

In the following, we come to compute the more complicated term III_1 .

$$\begin{aligned}
 III_1 &= -u_n^{l+2} \sigma_l(G) \sum_{\alpha,\beta} \sum_{i \in B} a^{\alpha\beta} u_{\alpha\beta ii} \\
 &= -u_n^{l+2} \sigma_l(G) \sum_{\alpha,\beta} \sum_{i \in B} \left[(a^{\alpha\beta} u_{\alpha\beta})_{ii} - (a^{\alpha\beta})_{ii} u_{\alpha\beta} - 2(a^{\alpha\beta})_i u_{\alpha\beta i} \right] \\
 &= -u_n^{l+2} \sigma_l(G) \sum_{j \in B} f_{jj} + III_{11} + III_{12} .
 \end{aligned} \tag{4.48}$$

For III_{11} , we have

$$\begin{aligned}
III_{11} &= u_n^{l+2} \sigma_l(G) \sum_{\alpha, \beta} \sum_{j \in B} (a^{\alpha\beta})_{jj} u_{\alpha\beta} \\
&= u_n^{l+2} \sigma_l(G) \sum_{\alpha, \beta} \sum_{j \in B} \left(2 \sum_{\xi} u_{\xi} u_{\xi j} \delta_{\alpha\beta} - 2 u_{\alpha j} u_{\beta} \right)_j u_{\alpha\beta} \\
&= 2 u_n^{l+2} \sigma_l(G) \sum_{j \in B} \sum_{\alpha} u_{\alpha j}^2 \Delta u - 2 u_n^{l+2} \sigma_l(G) \sum_{\alpha, \beta} \sum_{j \in B} u_{\alpha j} u_{\beta j} u_{\alpha\beta} \\
&\quad + 2 u_n^{l+3} \sigma_l(G) \sum_{j \in B} u_{njj} \Delta u - 2 u_n^{l+3} \sigma_l(G) \sum_{\alpha} \sum_{j \in B} u_{\alpha jj} u_{n\alpha} \\
&= III_{111} + III_{112},
\end{aligned} \tag{4.49}$$

where,

$$\begin{aligned}
III_{111} &= 2 u_n^{l+2} \sigma_l(G) \sum_{j \in B} \sum_{\alpha} u_{\alpha j}^2 \Delta u - 2 u_n^{l+2} \sigma_l(G) \sum_{\alpha, \beta} \sum_{j \in B} u_{\alpha j} u_{\beta j} u_{\alpha\beta} \\
&= 2 u_n^{l+2} \sigma_l(G) \sum_{j \in B} u_{nj}^2 (\Delta u - u_{nn}) = -2 u_n^{l+3} \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2
\end{aligned} \tag{4.50}$$

and by [Lemma 2.1](#) and [\(4.3\)](#),

$$\begin{aligned}
III_{112} &= 2 u_n^{l+3} \sigma_l(G) \sum_{j \in B} u_{njj} (\Delta u - u_{nn}) - 2 u_n^{l+3} \sigma_l(G) \sum_{i=1}^{n-1} \sum_{j \in B} u_{ni} u_{ijj} \\
&= -2 u_n^{l+4} \sigma_l(G) \sigma_1(G) \sum_{j \in B} (u_{jjn} + \epsilon u_n) - 2 u_n^{l+3} \sigma_l(G) \sum_{i=1}^{n-1} \sum_{j \in B} u_{ni} u_{jji} \\
&= -4 u_n^{l+3} \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2 - 2(n-l-1) \epsilon u_n^{l+5} \sigma_l(G) \sigma_1(G).
\end{aligned} \tag{4.51}$$

Therefore, combining [\(4.49\)](#), [\(4.50\)](#) and [\(4.51\)](#) we get

$$III_{11} = -6 u_n^{l+3} \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2 - 2(n-l-1) \epsilon u_n^{l+5} \sigma_l(G) \sigma_1(G). \tag{4.52}$$

For the term III_{12} , we have by [\(4.3\)](#) that

$$\begin{aligned}
III_{12} &= 2 u_n^{l+2} \sigma_l(G) \sum_{\alpha, \beta} \sum_{j \in B} (a^{\alpha\beta})_j u_{\alpha\beta j} \\
&= 4 u_n^{l+3} \sigma_l(G) \sum_{\alpha} \sum_{j \in B} u_{nj} u_{\alpha\alpha j} - 4 u_n^{l+3} \sigma_l(G) \sum_{\alpha} \sum_{j \in B} u_{\alpha j} u_{\alpha nj}
\end{aligned} \tag{4.53}$$

$$\begin{aligned}
 &= 4u_n^{l+3} \sigma_l(G) \sum_{i=1}^{n-1} \sum_{j \in B} u_{nj} u_{ii,j} \\
 &= -4u_n^{l+4} \sigma_l(G) \sum_{i \in G, j \in B} u_{nj} b_{ii,j} - 4u_n^{l+3} \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2.
 \end{aligned}$$

Therefore, by (4.48), (4.52) and (4.53) we have

$$\begin{aligned}
 III_1 &= -u_n^{l+2} \sigma_l(G) \sum_{j \in B} f_{jj} - 4u_n^{l+4} \sigma_l(G) \sum_{i \in G, j \in B} u_{nj} b_{ii,j} \\
 &\quad - 10u_n^{l+3} \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2 - 2(n-l-1) \epsilon u_n^{l+5} \sigma_l(G) \sigma_1(G). \quad (4.54)
 \end{aligned}$$

So, by (4.46), (4.47) and (4.54), it follows that

$$\begin{aligned}
 III &\sim -u_n^{l+2} \sigma_l(G) \sum_{j \in B} f_{jj} - 4u_n^{l+4} \sigma_l(G) \sum_{i \in G, j \in B} u_{nj} b_{ii,j} \\
 &\quad - 10u_n^{l+3} \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2 - 2(n-l-1) \epsilon u_n^{l+5} \sigma_l(G) \sigma_1(G) \quad (4.55) \\
 &\quad + 2(n-l-1) \epsilon f u_n^{l+2} \sigma_l(G).
 \end{aligned}$$

For the term IV , we show by Lemma 2.1 and (4.26) that

$$\begin{aligned}
 IV &= 2(-1)^{l+1} \sum_{\alpha, \beta, \xi, \eta} a^{\xi \eta} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{\alpha \beta}} u_{\alpha \xi \eta} u_{\beta} \\
 &= 2(-1)^{l+1} u_n \sum_{\xi, \eta} a^{\xi \eta} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{nn}} u_{n \xi \eta} + 2(-1)^{l+1} u_n \sum_{\xi, \eta} a^{\xi \eta} \sum_{i=1}^{n-1} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{in}} u_{i \xi \eta} \\
 &= 2(-1)^{l+1} u_n \sum_{\xi, \eta} a^{\xi \eta} \sigma_{l+1}(u_{pq}) u_{n \xi \eta} + 2(-1)^l u_n \sum_{\xi, \eta} a^{\xi \eta} \sum_{i=1}^{n-1} \sigma_l(u_{pq} | i) u_{ni} u_{i \xi \eta} \\
 &\sim 2u_n^{l+1} \sigma_l(G) \sum_{\xi, \eta} a^{\xi \eta} \sum_{j \in B} u_{nj} u_{j \xi \eta} \\
 &= 2u_n^{l+1} \sigma_l(G) \sum_{j \in B} u_{nj} \sum_{\xi, \eta} a^{\xi \eta} u_{\xi \eta j} \\
 &= 2u_n^{l+1} \sigma_l(G) \sum_{j \in B} u_{nj} f_j + 4u_n^{l+3} \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2. \quad (4.56)
 \end{aligned}$$

For the term V , we have

$$\begin{aligned}
V &= 2(-1)^{l+1} \sum_{\alpha, \beta, \xi, \eta} a^{\xi \eta} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{\alpha \beta}} u_{\alpha \xi} u_{\beta \eta} \\
&= 2(-1)^{l+1} \sum_{\xi, \eta} a^{\xi \eta} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{nn}} u_{n \xi} u_{n \eta} + 2(-1)^{l+1} \sum_{\xi, \eta} a^{\xi \eta} \sum_{i, j=1}^{n-1} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{ij}} u_{i \xi} u_{j \eta} \\
&\quad + 4(-1)^{l+1} \sum_{\xi, \eta} a^{\xi \eta} \sum_{j=1}^{n-1} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{nj}} u_{n \xi} u_{j \eta} \\
&= V_1 + V_2 + V_3.
\end{aligned} \tag{4.57}$$

It is obvious to see that

$$V_1 \sim 0, \tag{4.58}$$

and

$$\begin{aligned}
V_3 &= 4(-1)^{l+1} \sum_{\xi, \eta} a^{\xi \eta} \sum_{j=1}^{n-1} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{nj}} u_{n \xi} u_{j \eta} \\
&\sim 4(-1)^l \sum_{\xi} a^{\xi \xi} \sum_{j \in B} \sigma_l(u_{pq} | j) u_{jn} u_{n \xi} u_{j \xi} \\
&\sim 4u_n^l u_{nn} \sigma_l(G) \sum_{j \in B} u_{jn}^2.
\end{aligned} \tag{4.59}$$

For the term V_2 , we have

$$\begin{aligned}
V_2 &= 2(-1)^{l+1} \sum_{\xi, \eta} a^{\xi \eta} \sum_{i, j=1}^{n-1} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{ij}} u_{i \xi} u_{j \eta} \\
&= 2(-1)^{l+1} \sum_{\xi} a^{\xi \xi} \sum_{i=1}^{n-1} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{ii}} u_{i \xi}^2 + 2(-1)^{l+1} \sum_{\xi} a^{\xi \xi} \sum_{i, j=1, i \neq j}^{n-1} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{ij}} u_{i \xi} u_{j \xi} \\
&= V_{21} + V_{22}.
\end{aligned} \tag{4.60}$$

For the term V_{21} , by (2.1),

$$\begin{aligned}
V_{21} &= 2(-1)^{l+1} \sum_{\xi} a^{\xi \xi} \sum_{i=1}^{n-1} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{ii}} u_{i \xi}^2 \\
&= 2(-1)^{l+1} \sum_{\xi} a^{\xi \xi} \sum_{i=1}^{n-1} \left(u_{nn} \sigma_l(u_{pq} | i) - \sum_{j \neq i} u_{nj}^2 \sigma_{l-1}(u_{pq} | ij) \right) u_{i \xi}^2 \\
&\sim -2u_n^l u_{nn} \sigma_l(G) \sum_{j \in B} u_{nj}^2 - 2u_n^{l-1} \sum_{\xi} a^{\xi \xi} \sum_{i=1}^{n-1} \sum_{j \neq i} \sigma_{l-1}(\lambda | ij) u_{nj}^2 u_{i \xi}^2.
\end{aligned} \tag{4.61}$$

As the calculations before, we divide the term including $i \neq j$ into four parts and then derive

$$\begin{aligned}
 V_{21} &\sim -2u_n^l u_{nn} \sigma_l(G) \sum_{j \in B} u_{nj}^2 - 2u_n^{l-1} \sum_{\xi} a^{\xi\xi} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{nj}^2 u_{i\xi}^2 \\
 &\quad - 2u_n^{l-1} \sum_{i \in B, j \in G} \sigma_{l-1}(G|j) u_{nj}^2 u_{ni}^2 - 2u_n^{l-1} \sigma_{l-1}(G) \sum_{i, j \in B, i \neq j} u_{nj}^2 u_{ni}^2 \\
 &\sim -2u_n^l u_{nn} \sigma_l(G) \sum_{j \in B} u_{nj}^2 - 4u_n^{l-1} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{nj}^2 u_{ni}^2 \\
 &\quad - 2u_n^{l+1} (1 + u_n^2) \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2 - 2u_n^{l-1} \sigma_{l-1}(G) \sum_{i, j \in B, i \neq j} u_{nj}^2 u_{ni}^2.
 \end{aligned} \tag{4.62}$$

For the term V_{22} , it follows that

$$\begin{aligned}
 V_{22} &= 2(-1)^{l+1} \sum_{\xi} a^{\xi\xi} \sum_{i \neq j} \frac{\partial \sigma_{l+2}(D^2 u)}{\partial u_{ij}} u_{i\xi} u_{j\xi} \\
 &= 2(-1)^{l+1} \sum_{\xi} a^{\xi\xi} \sum_{i \neq j} u_{ni} u_{nj} \sigma_{l-1}(u_{pq}|ij) u_{i\xi} u_{j\xi} \\
 &\sim 4u_n^{l-1} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) u_{ni}^2 u_{nj}^2 + 2u_n^{l-1} \sigma_{l-1}(G) \sum_{i, j \in B, i \neq j} u_{ni}^2 u_{nj}^2.
 \end{aligned} \tag{4.63}$$

By (4.60), (4.62) and (4.63),

$$V_2 \sim -2u_n^l u_{nn} \sigma_l(G) \sum_{j \in B} u_{nj}^2 - 2u_n^{l+1} (1 + u_n^2) \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2. \tag{4.64}$$

Therefore, by (4.57), (4.58), (4.59) and (4.59) we derive that

$$\begin{aligned}
 V &\sim 2u_n^l u_{nn} \sigma_l(G) \sum_{j \in B} u_{nj}^2 - 2u_n^{l+1} (1 + u_n^2) \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2 \\
 &= 2u_n^l f \sigma_l(G) \sum_{j \in B} u_{nj}^2.
 \end{aligned} \tag{4.65}$$

Now, combine (4.8), (4.45), (4.55), (4.56) and (4.65) we derive finally that

$$\begin{aligned}
 \sum_{\alpha, \beta} a^{\alpha\beta} \varphi_{\alpha\beta} &\sim -2u_n^{l+3} \sum_{\xi} a^{\xi\xi} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) b_{ij, \xi}^2 - u_n^{l+3} \sigma_{l-1}(G) \sum_{i, j \in B} b_{ij, n}^2 \\
 &\quad - u_n^{l+3} (1 + u_n^2) \sigma_{l-1}(G) \sum_{k=1}^{n-1} \sum_{i, j \in B} b_{ij, k}^2 + 4u_n^{l+2} \sigma_l(G) \sum_{i \in G, j \in B} u_{nj} b_{ii, j} \\
 &\quad - 2u_n^{l+1} \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2 - 2(n-l-1) \epsilon u_n^{l+5} \sigma_l(G) \sigma_1(G)
 \end{aligned} \tag{4.66}$$

$$\begin{aligned}
& -u_n^{l+2}\sigma_l(G)\sum_{j\in B}f_{jj}-6fu_n^l\sigma_l(G)\sum_{j\in B}u_{nj}^2+6u_n^{l+1}\sigma_l(G)\sum_{j\in B}u_{jn}f_j \\
& +2(n-l-1)\epsilon fu_n^{l+2}\sigma_l(G).
\end{aligned}$$

Since $a^{\xi\xi} \geq 1$ for any $\xi = 1, 2, \dots, n$, $b_{ij,i} = b_{ii,j}$ for $i \in G$, $j \in B$ and $\epsilon \geq 0$, we then have

$$\begin{aligned}
\sum_{\alpha,\beta} a^{\alpha\beta} \varphi_{\alpha\beta} & \leq -2u_n^{l+3} \sum_{i \in G, j \in B} \sigma_{l-1}(G|i) b_{ii,j}^2 + 4u_n^{l+2} \sigma_l(G) \sum_{i \in G, j \in B} u_{nj} b_{ii,j} \\
& - 2u_n^{l+1} \sigma_l(G) \sigma_1(G) \sum_{j \in B} u_{nj}^2 \\
& - u_n^{l+2} \sigma_l(G) \sum_{j \in B} f_{jj} - 6fu_n^l \sigma_l(G) \sum_{j \in B} u_{nj}^2 + 6u_n^{l+1} \sigma_l(G) \sum_{j \in B} u_{jn} f_j \\
& + 2(n-l-1)\epsilon fu_n^{l+2} \sigma_l(G).
\end{aligned} \tag{4.67}$$

After a observation, we can simplify the above expression into the following form

$$\begin{aligned}
\sum_{\alpha,\beta} a^{\alpha\beta} \varphi_{\alpha\beta} & \leq -2u_n^{l+1} \sum_{i \in G, j \in B} \left(u_n \sqrt{\sigma_{l-1}(G|i)} b_{ii,j} - \sqrt{\sigma_l(G) \lambda_i} u_{nj} \right)^2 \\
& - u_n^l \sigma_l(G) \sum_{j \in B} \left[6fu_{nj}^2 - 6u_n f_j u_{nj} + u_n^2 (f_{jj} - 2\epsilon f) \right].
\end{aligned} \tag{4.68}$$

Now, we come to compute the term involving f . Note that $f(x, \nabla u) = H(x)(1 + |\nabla u|^2)^{\frac{3}{2}}$, then we get for $j \in B$,

$$\begin{aligned}
f(x, \nabla u) & = H(1 + u_n^2)^{\frac{3}{2}}, \\
[f(x, \nabla u)]_j & = H_j(1 + u_n^2)^{\frac{3}{2}} + 3Hu_n(1 + u_n^2)^{\frac{1}{2}}u_{nj},
\end{aligned} \tag{4.69}$$

and

$$\begin{aligned}
[f(x, \nabla u)]_{jj} & = H_{jj}(1 + u_n^2)^{\frac{3}{2}} + 6H_j u_n(1 + u_n^2)^{\frac{1}{2}}u_{nj} + 3Hu_n^2(1 + u_n^2)^{-\frac{1}{2}}u_{nj}^2 \\
& + 3H(1 + u_n^2)^{\frac{1}{2}}u_{nj}^2 + 3H(1 + u_n^2)^{\frac{1}{2}}u_n u_{njj}.
\end{aligned} \tag{4.70}$$

By [Lemma 2.1](#), [\(4.3\)](#) and the fact that $\epsilon \geq 0$, we have for $j \in B$,

$$\begin{aligned}
[f(x, \nabla u)]_{jj} & = H_{jj}(1 + u_n^2)^{\frac{3}{2}} + 6H_j u_n(1 + u_n^2)^{\frac{1}{2}}u_{nj} + 3Hu_n^2(1 + u_n^2)^{-\frac{1}{2}}u_{nj}^2 \\
& + 9H(1 + u_n^2)^{\frac{1}{2}}u_{nj}^2 + 3Hu_n^2(1 + u_n^2)^{\frac{1}{2}}\epsilon - 3Hu_n^2(1 + u_n^2)^{\frac{1}{2}}b_{jj,n} \\
& \geq H_{jj}(1 + u_n^2)^{\frac{3}{2}} + 6H_j u_n(1 + u_n^2)^{\frac{1}{2}}u_{nj} + 3Hu_n^2(1 + u_n^2)^{-\frac{1}{2}}u_{nj}^2 \\
& + 9H(1 + u_n^2)^{\frac{1}{2}}u_{nj}^2 - 3Hu_n^2(1 + u_n^2)^{\frac{1}{2}}b_{jj,n}.
\end{aligned} \tag{4.71}$$

Therefore, by [\(4.7\)](#) we derive

$$\sum_{j \in B} \left[6f u_{nj}^2 - 6u_n f_j u_{nj} + u_n^2 (f_{jj} - 2\epsilon f) \right] \geq \sum_{j \in B} \left(A u_{nj}^2 + B u_{nj} + C \right), \quad (4.72)$$

where

$$\begin{aligned} A &= 3H(2 + u_n^2)(1 + u_n^2)^{-\frac{1}{2}} \geq 6H(1 + u_n^2)^{-\frac{1}{2}}, \\ B &= -6u_n H_j (1 + u_n^2)^{\frac{1}{2}}, \\ C &= u_n^2 (H_{jj} - 2\epsilon H)(1 + u_n^2)^{\frac{3}{2}}. \end{aligned} \quad (4.73)$$

Under the structure condition (4.1), we easily get that for $j \in B$,

$$A u_{nj}^2 + B u_{nj} + C \geq 0. \quad (4.74)$$

Thus,

$$\sum_{\alpha, \beta} a^{\alpha\beta} \varphi_{\alpha\beta} \leq 0. \quad (4.75)$$

And this finishes the proof of the constant rank theorem. $\square$

5. Strict convexity of level sets

In this section, we follow the idea of [19] to prove Theorem 1.2 by the continuity method and then to prove Corollary 1.3.

Remark that the equation we concentrate on is the following

$$\begin{cases} \operatorname{div} \left(\frac{\nabla u^\tau}{\sqrt{1 + |\nabla u^\tau|^2}} \right) = 0 & \text{in } \Omega = \Omega_0 \setminus \bar{\Omega}_1, \\ u^\tau = 0 & \text{on } \partial\Omega_0, \\ u^\tau = \tau & \text{on } \partial\Omega_1 \end{cases} \quad (5.1)$$

for $0 < \tau \leq 1$. Remark that the existence of the solution for $0 < \tau < 1$ is already clear to us by Theorem 3.2.

Firstly, when the height of the minimal graph, τ , is small enough, the level sets must be regular and strictly convex. In fact, if τ is small, by Theorem 3.2 we know that the norm of the gradient will be sufficiently small which would force the minimal graph to be looked as something like the graph of a harmonic function since the similarity of the equations they satisfy. Precisely, we have

Lemma 5.1. *Let (M^n, g) be a space form with constant sectional curvature $\epsilon \geq 0$, and Ω_0 and Ω_1 be bounded smooth strictly convex domains in M^n , $n \geq 2$ and $\bar{\Omega}_1 \subseteq \Omega_0$. Then there exists $\delta_0 > 0$ such that for any $0 < \tau \leq \delta_0$, the solution to the minimal graph equation (5.1) satisfies that $\nabla u^\tau \neq 0$ and all level sets of u^τ are strictly convex with respect to ∇u^τ .*

Proof. Let ω^τ be the harmonic function defined in (3.7). Denote by $\rho = u^\tau - \omega^\tau$ and it is easy to see that ρ satisfies the following Poisson type equation

$$\begin{cases} \Delta \rho = \sum_{\xi, \eta} u_{\xi\eta}^{\tau} u_{\xi}^{\tau} u_{\eta}^{\tau} - |\nabla u^{\tau}|^2 \Delta u^{\tau} & \text{in } \Omega = \Omega_0 \setminus \bar{\Omega}_1, \\ \rho = 0 & \text{on } \partial\Omega_0, \\ \rho = 0 & \text{on } \partial\Omega_1. \end{cases} \quad (5.2)$$

Denoted by $f = \sum_{\xi, \eta} u_{\xi\eta}^{\tau} u_{\xi}^{\tau} u_{\eta}^{\tau} - |\nabla u^{\tau}|^2 \Delta u^{\tau}$, according to [Theorem 3.2](#) we know that

$$|f|_{C^{\alpha}(\bar{\Omega})} \leq C\tau^2. \quad (5.3)$$

The Schauder theory then gives that

$$\begin{aligned} \|u^{\tau} - \omega^{\tau}\|_{C^{2,\alpha}(\bar{\Omega})} &\leq C \left(\|\rho\|_{L^{\infty}(\Omega)} + |f|_{C^{\alpha}(\bar{\Omega})} \right) \\ &\leq C \left(\|f\|_{L^{\infty}(\bar{\Omega})} + |f|_{C^{\alpha}(\bar{\Omega})} \right) \leq C_5\tau^2. \end{aligned} \quad (5.4)$$

Especially, we get for any $x \in \Omega$,

$$|\nabla u^{\tau} - \nabla \omega^{\tau}|(x) \leq C_5\tau^2. \quad (5.5)$$

It follows by [\(3.9\)](#) that for any $x \in \Omega$,

$$|\nabla u^{\tau}|(x) \geq |\nabla \omega^{\tau}|(x) - C_5\tau^2 \geq C_0\tau - C_5\tau^2, \quad (5.6)$$

thus there exists some $\delta_0 > 0$ such that for any $\tau \in (0, \delta_0]$ and for any $x \in \Omega$,

$$|\nabla u^{\tau}|(x) > 0, \quad (5.7)$$

which guarantees the regularity of the level sets in this situation.

Next, we come to consider the strict convexity of the level sets.

Notations as before, according to [\[27\]](#) we know that the level sets of ω^1 are strictly convex, then

$$\lambda_{\min}(\omega^1) \geq C_6 \quad (5.8)$$

for some positive constant C_6 , where $\lambda_{\min}(\psi)$ is denoted to be the minimal eigenvalue of the second fundamental forms of the level sets of ψ with respect to the gradient direction. Remark that $\lambda_{\min}(\psi)$ is an expression via the derivatives up to 2-order of ψ .

Since the relation of the level sets between ω^{τ} and ω^1 is that for $s \in [0, \tau]$,

$$\Sigma^s(\omega^{\tau}) = \Sigma^{\frac{s}{\tau}}(\omega^1),$$

we deduce that for any $0 < \tau \leq 1$

$$\lambda_{\min}(\omega^{\tau}) \geq C_6. \quad (5.9)$$

By [\(5.4\)](#), u^{τ} and ω^{τ} is $C^{2,\alpha}$ close enough if τ is small enough, so by adjusting suitably δ_0 above, we can get that for $\tau \in (0, \delta_0]$,

$$\lambda_{\min}(u^\tau) \geq C_7 > 0.$$

This gives the strict convexity of the level sets of minimal graph with a small height. $\square$

Proof of Theorem 1.2. We now begin to conclude the theorem by deformation process.

Set

$$T = \{\tau \in (0, 1] \mid \forall s \in (0, \tau], u^s \text{ has nonzero gradient and strictly convex level sets} \}.$$

By Lemma 5.1, we know that $(0, \delta_0] \subset T$.

For the openness, we assume firstly that $\tau_0 \in T$, that is to say, for all $x \in \Omega$ we have $|\nabla u^{\tau_0}|(x) > 0$ and the level sets of u^{τ_0} have positive second fundamental form with respect to its gradient. Namely, there is a $C_8 > 0$ such that

$$|\nabla u^{\tau_0}| \geq C_8, \quad \lambda_{\min}(u^{\tau_0}) \geq C_8. \quad (5.10)$$

Then Theorem 3.2 and almost the same procedure as Lemma 5.1 produce that there exists some $\delta > 0$ such that for all $\tau \in [\tau_0 - \delta, \tau_0 + \delta]$, u^τ has nonsingular gradient and strictly convex level sets, this concludes the openness.

For the closedness, we assume without loss of generality that

$$\tau_i \in T, \quad \tau_\infty \in (0, 1], \quad \delta_0 \leq \tau_i < \tau_\infty, \quad \text{and} \quad \lim_{i \rightarrow \infty} \tau_i = \tau_\infty.$$

Obviously, u^{τ_∞} is the unique solution to (5.1) at the time τ_∞ . In the sequel, we will prove that $\tau_\infty \in T$, that is to say, u^{τ_∞} has nonzero gradient and strictly convex level sets.

For each $\tau_i \in T$, according to the assumption we have $|\nabla u^{\tau_i}| > 0$ and the level sets of u^{τ_i} are strictly convex with respect to ∇u^{τ_i} . Discussion as before, taking the frame such that $e_n = \frac{\nabla u^{\tau_i}}{|\nabla u^{\tau_i}|}$, we can derive by (2.3) that $u_{ll}^{\tau_i} < 0$ for $l = 1, 2, \dots, n-1$. Thus we deduce from the equation of minimal graph that

$$\frac{\sum_{\alpha, \beta=1}^n u_{\alpha\beta}^{\tau_i} u_{\alpha}^{\tau_i} u_{\beta}^{\tau_i}}{|\nabla u^{\tau_i}|^2} = u_{nn}^{\tau_i} = -(1 + u_n^2) \sum_{l=1}^{n-1} u_{ll}^{\tau_i} > 0. \quad (5.11)$$

Therefore, the norm of the gradient of u^{τ_i} will increase along the gradient direction, in fact,

$$\sum_{\alpha=1}^n u_{\alpha}^{\tau_i} |\nabla u^{\tau_i}|_{\alpha}^2 = 2 \sum_{\alpha, \beta=1}^n u_{\alpha\beta}^{\tau_i} u_{\alpha}^{\tau_i} u_{\beta}^{\tau_i} > 0. \quad (5.12)$$

Thus the minimum of $|\nabla u^{\tau_i}|$ has to be attained on the exterior boundary $\partial\Omega_0$.

On the other hand, it is obvious from comparison principle that for any i ,

$$u^{\delta_0} \leq u^{\tau_i} \quad \text{and} \quad u^{\delta_0} = u^{\tau_i} = 0 \quad \text{on} \quad \partial\Omega_0,$$

therefore according to Hopf strong maximum principle, a positive constant C_{10} depending upon u^{δ_0} can be taken to assure that for all $x \in \partial\Omega_0$,

$$|\nabla u^{\tau_i}|(x) > |\nabla u^{\delta_0}|(x) \geq C_{10} > 0. \quad (5.13)$$

Thus,

$$\forall x \in \Omega, \quad |\nabla u^{\tau_i}|(x) \geq C_{10} > 0. \quad (5.14)$$

Once again taking advantage of [Theorem 3.2](#) and the fact that $\tau_i \rightarrow \tau_\infty$, we then have

$$|\nabla u^{\tau_\infty}| > 0. \quad (5.15)$$

Finally, for the strict convexity of the level sets of each u^{τ_i} and [Theorem 3.2](#), the limit u^{τ_∞} could only have convex level sets. Then the constant rank [Theorem 4.1](#) and the strict convexity of $\Sigma^0(u^{\tau_\infty}) = \partial\Omega_0$ ensure that the level sets of u^{τ_∞} must be all strictly convex. This is to say that $\tau_\infty \in T$ then T is relatively closed in $(0, 1]$.

In summary, we have by the connection of $(0, 1]$ that $T = (0, 1]$, and this finally concludes the whole proof of [Theorem 1.2](#). $\square$

Proof of Corollary 1.3. Based on [Theorem 1.2](#), we know that the level sets of u are regular and strictly convex. Then the corollary is immediately derived by the same discussion as [\(5.11\)](#) and [\(5.12\)](#). $\square$

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References

- [1] L.V. Ahlfors, *Conformal Invariants: Topics in Geometric Function Theory*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York–Düsseldorf–Johannesburg, 1973, pp. 5–6.
- [2] B.J. Bian, P.F. Guan, X.N. Ma, L. Xu, A constant rank theorem for quasiconcave solutions of fully nonlinear partial differential equations, *Indiana Univ. Math. J.* 60 (2011) 101–120.
- [3] C. Bianchini, M. Longinetti, P. Salani, Quasiconcave solutions to elliptic problems in convex rings, *Indiana Univ. Math. J.* 58 (2009) 1565–1590.
- [4] H. Brascamp, E. Lieb, On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation, *J. Funct. Anal.* 22 (4) (1976) 366–389.
- [5] L. Caffarelli, A. Friedman, Convexity of solutions of some semilinear elliptic equations, *Duke Math. J.* 52 (1985) 431–455.
- [6] L. Caffarelli, P. Guan, X.N. Ma, A constant rank theorem for solutions of fully nonlinear elliptic equations, *Comm. Pure Appl. Math.* 60 (12) (2007) 1769–1791.
- [7] L. Caffarelli, J. Spruck, Convexity properties of solutions to some classical variational problems, *Comm. Partial Differential Equations* 7 (1982) 1337–1379.
- [8] C.Q. Chen, B.W. Hu, A microscopic convexity principle for spacetime convex solutions of fully nonlinear parabolic equations, *Acta Math. Sin. (Engl. Ser.)* 4 (2013) 651–674.
- [9] C.Q. Chen, X.N. Ma, P. Salani, On space–time quasiconcave solutions of the heat equation, [arXiv:1405.6373 \[math.AP\]](#).

- [10] T.H. Colding, W.P. Minicozzi II, *Minimal Surfaces*, Courant Lecture Notes in Mathematics, vol. 4, New York University, Courant Institute of Mathematical Sciences, New York, 1999.
- [11] R. Gabriel, A result concerning convex level surfaces of 3-dimensional harmonic functions, *J. Lond. Math. Soc.* 32 (1957) 286–294.
- [12] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, second edition, Springer-Verlag, Berlin, 1983.
- [13] P. Guan, C.S. Lin, X.N. Ma, The Christoffel–Minkowski problem II: Weingarten curvature equations, *Chin. Ann. Math. Ser. B* 27 (6) (2006) 595–614.
- [14] P. Guan, X.N. Ma, The Christoffel–Minkowski problem I: convexity of solutions of a Hessian equation, *Invent. Math.* 151 (3) (2003) 553–577.
- [15] P. Guan, X.N. Ma, F. Zhou, The Christoffel–Minkowski problem III: existence and convexity of admissible solutions, *Comm. Pure Appl. Math.* 59 (9) (2006) 1352–1376.
- [16] P. Guan, L. Xu, Convexity estimates for level sets of quasiconcave solutions to fully nonlinear elliptic equations, *J. Reine Angew. Math.* 680 (2013) 41–67.
- [17] B.W. Hu, X.N. Ma, Constant rank theorem of the spacetime convex solution of heat equation, *Manuscripta Math.* 138 (1–2) (2012) 89–118.
- [18] B. Kawohl, *Rearrangements and Convexity of Level Sets in PDE*, Lecture Notes in Math., vol. 1150, Springer-Verlag, Berlin, 1985.
- [19] N.J. Korevaar, Convexity of level sets for solutions to elliptic ring problems, *Comm. Partial Differential Equations* 15 (4) (1990) 541–556.
- [20] Y.I. Lee, A.N. Wang, Estimate of $\lambda_2 - \lambda_1$ on spheres, *Chinese J. Math.* 15 (2) (1987) 95–97.
- [21] J.L. Lewis, Capacitary functions in convex rings, *Arch. Ration. Mech. Anal.* 66 (1977) 201–224.
- [22] M. Longinetti, Convexity of the level lines of harmonic functions, *Boll. Unione Mat. Ital. A* 6 (1983) 71–75 (in Italian).
- [23] M. Longinetti, On minimal surfaces bounded by two convex curves in parallel planes, *J. Differential Equations* 67 (1987) 344–358.
- [24] L.G. Makar-Limanov, Solution of Dirichlet’s problem for the equation $\Delta u = -1$ on a convex region, *Math. Notes Acad. Sci. USSR* 9 (1971) 52–53.
- [25] X.N. Ma, Q.Z. Ou, W. Zhang, Gaussian curvature estimates for the convex level sets of p -harmonic functions, *Comm. Pure Appl. Math.* 63 (2010) 0935–0971.
- [26] X. Ma, W. Zhang, The concavity of the Gaussian curvature of the convex level sets of p -harmonic functions with respect to the height, *Commun. Math. Stat.* 1 (4) (2013) 465–489.
- [27] X.N. Ma, Y.B. Zhang, The convexity and the Gaussian curvature estimates for the level sets of harmonic functions on convex rings in space forms, *J. Geom. Anal.* 24 (1) (2014) 337–374.
- [28] J. McCuan, Continua of H -graphs: convexity and isoperimetric stability, *Calc. Var. Partial Differential Equations* 9 (1999) 297–325.
- [29] M. Ortel, W. Schneider, Curvature of level curves of harmonic functions, *Canad. Math. Bull.* 26 (4) (1983) 399–405.
- [30] M. Papadimitrakis, On convexity of level curves of harmonic functions in the hyperbolic plane, *Proc. Amer. Math. Soc.* 114 (3) (1992) 695–698.
- [31] M. Shiffman, On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes, *Ann. of Math.* 63 (1956) 77–90.
- [32] J. Spruck, Interior gradient estimates and existence theorems for constant mean curvature graphs in $M^n \times \mathbb{R}$, *Pure Appl. Math. Q.* 3 (3) (2007) 785–800.
- [33] G. Talenti, On functions, whose lines of steepest descent bend proportionally to level lines, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* 10 (4) (1983) 587–605.
- [34] P.H. Wang, The concavity of the Gaussian curvature of the convex level sets of minimal surfaces with respect to the height, *Pacific J. Math.* 267 (2) (2014) 489–509.
- [35] P.H. Wang, X.J. Wang, The geometric properties of harmonic functions on 2-dimensional Riemannian manifolds, *Nonlinear Anal.* 103 (2014) 2–8.
- [36] P.H. Wang, L.L. Zhao, Some geometrical properties of convex level sets of minimal graph on 2-dimensional Riemannian manifolds, *Nonlinear Anal.* 130 (2016) 1–17.
- [37] P.H. Wang, W. Zhang, Gaussian curvature estimates for the convex level sets of solutions for some nonlinear elliptic partial differential equations, *J. Partial Differ. Equ.* 25 (3) (2012) 239–275.

- [38] L. Xu, A microscopic convexity theorem of level sets for solutions to elliptic equations, *Calc. Var. Partial Differential Equations* 40 (1) (2011) 51–63.
- [39] Ying Shih, A counterexample to the convexity property of the first eigenfunction on a convex domain of negative curvature, *Comm. Partial Differential Equations* 14 (7) (1989) 867–876.