



# Global gradient estimates for the borderline case of double phase problems with BMO coefficients in nonsmooth domains <sup>☆</sup>

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## Abstract

We consider a double phase problem with BMO coefficient in divergence form on a bounded nonsmooth domain. The problem under consideration is characterized by the fact that both ellipticity and growth switch between a type of polynomial and a type of logarithm according to the position, which describes a feature of strongly anisotropic materials. We obtain the global Calderón–Zygmund type estimates for the distributional solution in the case that the associated nonlinearity has a small BMO and the boundary of the domain is sufficiently flat in the Reifenberg sense.

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## 1. Introduction

This paper is concerned with the borderline case of double phase problems in divergence form. The problem under consideration is given by

$$\begin{aligned} \operatorname{div} \left( \beta(x) \left[ |Du|^{p-2} Du + a(x) |Du|^{p-2} \log(e + |Du|) Du \right] \right) \\ = \operatorname{div} \left( |F|^{p-2} F + a(x) |F|^{p-2} \log(e + |F|) F \right) \quad \text{in } \Omega, \end{aligned} \quad (1.1)$$

where  $1 < p < \infty$  is a fixed number,  $\Omega \subset \mathbb{R}^n$  is a bounded open domain with  $n \geq 2$ , and  $F : \Omega \rightarrow \mathbb{R}^n$  is a given vector field. Here the coefficient function  $\beta : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies  $\nu \leq \beta(\cdot) \leq L$  for some fixed constants  $0 < \nu \leq L < \infty$ , and the function  $a : \mathbb{R}^n \rightarrow \mathbb{R}$  is always assumed to be non-negative and bounded.

We remark that the equation (1.1) is closely related to the functional

$$v \in W^{1,1}(\Omega) \mapsto \mathcal{H}(v, \Omega) - \int_{\Omega} (|F|^{p-2} F + a(x) |F|^{p-2} \log(e + |F|) F, Dv) dx, \quad (1.2)$$

where the energy functional  $\mathcal{H}$  is given by

$$\mathcal{H}(v, \Omega) := \int_{\Omega} \beta(x) [|Dv|^p + a(x) |Dv|^p \log(e + |Dv|)] dx. \quad (1.3)$$

It is worth pointing out that this energy functional  $\mathcal{H}$  can be regarded as a borderline case of  $(p, q)$ -energy functionals

$$\mathcal{H}_{p,q}(v, \Omega) := \int_{\Omega} \beta(x) [|Dv|^p + a(x) |Dv|^q] dx, \quad 1 < p < q. \quad (1.4)$$

The main object of this paper is to investigate optimal conditions on the coefficient functions  $\beta(\cdot)$ ,  $a(\cdot)$  and a minimal geometric assumption on  $\partial\Omega$  under which the natural Calderón-Zygmund type relation

$$|F|^p + a(x) |F|^p \log(e + |F|) \in L^\gamma(\Omega) \implies |Du|^p + a(x) |Du|^p \log(e + |Du|) \in L^\gamma(\Omega) \quad (1.5)$$

and the corresponding global gradient estimate

$$\begin{aligned} \left( \int_{\Omega} [|Du|^p + a(x) |Du|^p \log(e + |Du|)]^\gamma dx \right)^{\frac{1}{\gamma}} \\ \leq c \left( \int_{\Omega} [|F|^p + a(x) |F|^p \log(e + |F|)]^\gamma dx \right)^{\frac{1}{\gamma}} \end{aligned} \quad (1.6)$$

hold true for every  $\gamma \in [1, \infty)$ .

In recent years, there has been an increasing interest in the functionals (1.3) and (1.4). A  $(p, q)$ -energy functional was first considered by Zhikov [45,48] in order to describe a feature of strongly anisotropic materials with hardening exponents  $p$  and  $q$ . In the functional  $\mathcal{H}_{p,q}$ , the modulating coefficient  $a(\cdot)$  determines the composition of the mixture of two different materials. The functional  $\mathcal{H}_{p,q}$  has both  $p$ - and  $q$ -growth terms in the gradient variable when  $a(x) > 0$ , while it has only  $p$ -growth term on the zero set  $\{a(x) = 0\}$ . In this respect, functionals  $\mathcal{H}_{p,q}$  and  $\mathcal{H}$  are called double phase functionals. There is a wide literature on the regularity theory for the double phase problems, see for instance [4,5,11–13].

In particular, the regularity of the modulating coefficient  $a(\cdot)$  is closely related to how to control the size of the phase transition. For the functional  $\mathcal{H}_{p,q}$ , it is essential to assume that  $a(\cdot) \in C^{0,\alpha}$  for some  $\alpha > \alpha_0$ , where the number  $\alpha_0 > 0$  depends on the gap between  $p$  and  $q$ , as shown in [18,22]. On the other hand, in the borderline case,  $a(\cdot)$  is not necessary to be Hölder continuous, but it seems that the correct condition is to be log-Hölder continuous. Here, the point is that the log-Hölder continuity of  $a(\cdot)$  is exactly dual to the size of the phase transition in  $\mathcal{H}$ .

We also emphasize that the Euler–Lagrange equations of the energy functionals (1.3) and (1.4) involve non-uniformly elliptic operators. This means that if we let

$$A_{\mathcal{H}}(x, \xi) := \beta(x) \left[ |\xi|^{p-2} \xi + a(x) |\xi|^{p-2} \log(e + |\xi|) \xi \right] \quad (1.7)$$

and if we denote by  $D_{\xi} A_{\mathcal{H}}(x, \xi)$  the Jacobian matrix of  $A_{\mathcal{H}}(x, \xi)$  with respect to the variable  $\xi$ , then the ratio between the highest and lowest eigenvalue of  $D_{\xi} A_{\mathcal{H}}(x, \xi)$  is comparable to  $1 + a(x) \log(e + |\xi|)$ , and hence it is unbounded with respect to the gradient variable  $\xi$ . There have been a lot of research activities regarding non-uniformly elliptic operators, see [16,17,27,39,41,42,46] and references therein.

In this paper we consider discontinuous coefficients of bounded mean oscillations (BMO) and nonsmooth domains which are not necessarily Lipschitz continuous. To deal with logarithmic terms in the functional (1.2) and conduct a careful analysis near the nonsmooth boundary, we utilize the framework of Orlicz spaces and Musielak–Orlicz spaces. In addition, we adopt the so-called maximal function free technique which was first introduced in [2] and later utilized in [7–9,15,43,44]. The advantage of using the maximal function free technique lies in the fact that it can be applied to the problems with a lack of homogeneity properties.

This paper is organized as follows. In the next section we introduce some backgrounds to state the main results. In Section 3 we set up notation and terminology. We also present some preliminaries and discuss covering arguments. Section 4 is devoted to the comparison estimates. Finally, in the last section we establish global gradient estimates for the borderline case of double phase problems in nonsmooth domains.

## 2. Background and main results

We deal with a distributional solution of the Dirichlet problem

$$\begin{cases} \operatorname{div} A(x, Du) = \operatorname{div} G(x, F) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

where  $\partial\Omega$  is the boundary of  $\Omega$ .

The above problem is a generic one whose model is given by (1.1). Throughout this paper, the nonlinearity  $A : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is assumed to be measurable with respect to  $x$ , differentiable with respect to  $\xi \neq 0$  and to satisfy the following structural conditions:

$$|A(x, \xi)| + |\xi| |D_\xi A(x, \xi)| \leq L \left[ |\xi|^{p-1} + a(x) |\xi|^{p-1} \log(e + |\xi|) \right], \quad (2.2)$$

$$\langle D_\xi A(x, \xi) \eta, \eta \rangle \geq \nu \left[ |\xi|^{p-2} + a(x) |\xi|^{p-2} \log(e + |\xi|) \right] |\eta|^2, \quad (2.3)$$

for almost every  $x \in \mathbb{R}$  and for all  $\xi, \eta \in \mathbb{R}^n$ , where  $0 < \nu \leq L < +\infty$  are fixed constants and  $a : \mathbb{R}^n \rightarrow \mathbb{R}$  is a non-negative and bounded function.

We define the auxiliary vector fields  $V_p, V_{\log} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by

$$V_p(\xi) := |\xi|^{\frac{p-2}{2}} \xi \quad \text{and} \quad V_{\log}(\xi) := \left( |\xi|^{p-2} \log(e + |\xi|) + \frac{|\xi|^{p-1}}{p(e + |\xi|)} \right)^{\frac{1}{2}} \xi,$$

whenever  $\xi \in \mathbb{R}^n$ . Then the structural condition (2.3) implies the following monotonicity property:

$$\langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle \geq \tilde{\nu} \left[ |V_p(\xi) - V_p(\eta)|^2 + a(x) |V_{\log}(\xi) - V_{\log}(\eta)|^2 \right], \quad (2.4)$$

where  $\tilde{\nu}$  is a positive constant depending only on  $n, p$  and  $\nu$ . In particular, for the case  $p \geq 2$ , the above monotonicity property can be reduced to

$$\langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle \geq \tilde{\nu} \left[ |\xi - \eta|^p + a(x) |\xi - \eta|^p \log(e + |\xi - \eta|) \right]. \quad (2.5)$$

In addition, we remark that  $|V_p(\xi)|^2 = |\xi|^p$  and that  $|V_{\log}(\xi)|^2$  is comparable to  $|\xi|^p \log(e + |\xi|)$ , more precisely,

$$|\xi|^p \log(e + |\xi|) \leq |V_{\log}(\xi)|^2 \leq 2|\xi|^p \log(e + |\xi|), \quad \forall \xi \in \mathbb{R}^n. \quad (2.6)$$

On the other hand, a vector field  $G : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  in the nonhomogeneous term of (2.1) is assumed to be a Carathéodory function, namely, measurable in  $x$  and continuous in  $\xi$ , satisfying the following growth condition:

$$|G(x, \xi)| \leq L \left[ |\xi|^{p-1} + a(x) |\xi|^{p-1} \log(e + |\xi|) \right]. \quad (2.7)$$

We now introduce a regularity assumption on the modulating coefficient function  $a(\cdot)$ . We note that the energy functional  $\mathcal{H}$  in (1.3) is a borderline case of  $(p, q)$ -energy functionals  $\mathcal{H}_{p,q}$  in (1.4). On the double phase problems with  $(p, q)$ -growth, it is necessary to assume that  $a(\cdot)$  is Hölder continuous, see for instance [18, 22]. However, in the borderline case, this assumption should be superfluous, as the phase transition between  $|Du|^p$  and  $|Du|^p \log(e + |Du|)$  is less dramatic. Here, we assume that  $a(\cdot)$  is only log-Hölder continuous. Specifically,  $a(\cdot)$  is assumed to be a continuous map satisfying

$$|a(x_1) - a(x_2)| \leq \omega(|x_1 - x_2|) \quad (2.8)$$

whenever  $x_1, x_2 \in \mathbb{R}$ , where  $\omega : [0, \infty) \rightarrow [0, 1]$  is a non-decreasing and concave modulus of continuity with a decay of logarithmic type as follows:

$$\lim_{r \rightarrow 0} \omega(r) \log \left( \frac{1}{r} \right) = 0. \quad (2.9)$$

We next discuss a regularity assumption on the nonlinearity with respect to  $x$ -variable. Let us denote by  $(f)_U$  to mean the integral average of a locally integrable function  $f = f(x)$  over a bounded domain  $U$  in  $\mathbb{R}^n$ , that is,

$$(f)_U = \int_U f(x) dx = \frac{1}{|U|} \int_U f(x) dx.$$

With this notation, we define

$$\Theta(A, B_r(y))(x) := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \left| \frac{A(x, \xi)}{|\xi|^{p-1} + a(x)|\xi|^{p-1} \log(e + |\xi|)} - \left( \frac{A(\cdot, \xi)}{|\xi|^{p-1} + a(\cdot)|\xi|^{p-1} \log(e + |\xi|)} \right)_{B_r(y)} \right|. \quad (2.10)$$

Throughout this paper,  $0 < \delta < \frac{1}{8}$  is a small constant to be determined later. On the other hand,  $R_0$  can be any positive number. Our main assumption on the nonlinearity is the following.

**Definition 2.1.** We say that  $A(x, \xi)$  is  $(\delta, R_0)$ -vanishing if

$$\sup_{0 < r \leq R_0} \sup_{y \in \mathbb{R}^n} \int_{B_r(y)} \Theta(A, B_r(y))(x) dx \leq \delta.$$

We remark that it follows from the growth condition (2.2) that for any  $\kappa > 1$ ,

$$\sup_{0 < r \leq R_0} \sup_{y \in \mathbb{R}^n} \int_{B_r(y)} \Theta^\kappa dx \leq (2L)^{\kappa-1} \delta. \quad (2.11)$$

We now introduce a geometric assumption on the boundary of the domain.

**Definition 2.2.** We say that  $\Omega$  is  $(\delta, R_0)$ -Reifenberg flat if for each  $x \in \partial\Omega$  and for each  $r \in (0, R_0]$ , there exists a coordinate system  $\{y^1, \dots, y^n\}$  such that  $x = 0$  in this coordinate system and that

$$B_r(0) \cap \{y^n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y^n > -\delta r\}.$$

We remark that a Reifenberg flat domain can go beyond Lipschitz category, not necessarily given by graphs. Nevertheless, it satisfies the following measure density conditions:

$$\sup_{0 < r \leq R_0} \sup_{x \in \Omega} \frac{|B_r(x)|}{|B_r(x) \cap \Omega|} \leq \left( \frac{2}{1-\delta} \right)^n \leq \left( \frac{16}{7} \right)^n, \quad (2.12)$$

$$\inf_{0 < r \leq R_0} \inf_{x \in \Omega} \frac{|B_r(x) \cap \Omega^c|}{|B_r(x)|} \geq \left( \frac{1-\delta}{2} \right)^n \geq \left( \frac{7}{16} \right)^n. \quad (2.13)$$

We refer to [10,24,28,29,32,33,36,37,40] and references therein regarding its properties and applications in analysis and geometry.

We next return to a distributional solution to (2.1) under consideration.

**Definition 2.3.** We say that  $u \in W_0^{1,1}(\Omega)$  is a distributional solution to (2.1) if it satisfies

$$\int_{\Omega} \langle A(x, Du), D\varphi \rangle dx = \int_{\Omega} \langle G(x, F), D\varphi \rangle dx, \quad (2.14)$$

for every test function  $\varphi \in C_0^\infty(\Omega)$ .

We clearly point out that if  $u \in W_0^{1,1}(\Omega)$  is a distributional solution to (2.1), with the natural integrability assumption  $H(x, Du), H(x, F) \in L^1(\Omega)$ , then (2.14) still holds for every function  $\varphi \in W_0^{1,1}(\Omega)$  with  $H(x, D\varphi) \in L^1(\Omega)$ , see Corollary 3.5 in the next section. We will also present necessary issues of solutions including existence, uniqueness and standard energy estimate in the next section.

In the rest of the paper we shall use the notation

$$H(x, \xi) := |\xi|^p + a(x)|\xi|^p \log(e + |\xi|), \quad (2.15)$$

for  $x \in \Omega$  and  $\xi \in \mathbb{R}^n$ . According to the above remark (2.6), we have

$$\frac{1}{2} \left[ |V_p(\xi)|^2 + a(x)|V_{\log}(\xi)|^2 \right] \leq H(x, \xi) \leq |V_p(\xi)|^2 + a(x)|V_{\log}(\xi)|^2. \quad (2.16)$$

We now state the main result in this paper.

**Theorem 2.4.** Let  $u \in W_0^{1,1}(\Omega)$  be the distributional solution to (2.1), with

$$H(x, Du), H(x, F) \in L^1(\Omega). \quad (2.17)$$

Suppose that

$$H(x, F) \in L^\gamma(\Omega) \quad \text{for some } \gamma \in (1, \infty). \quad (2.18)$$

Then there exists a constant  $\delta = \delta(n, p, v, L, \|a\|_{L^\infty(\Omega)}, \|H(\cdot, F)\|_{L^1(\Omega)}, \gamma) > 0$  such that if  $A$  is  $(\delta, R_0)$ -vanishing and  $\Omega$  is  $(\delta, R_0)$ -Reifenberg flat for some  $R_0 > 0$ , then we have

$$H(x, Du) \in L^\gamma(\Omega)$$

with the estimate

$$\left( \int_{\Omega} [H(x, Du)]^{\gamma} dx \right)^{\frac{1}{\gamma}} \leq c \left( \int_{\Omega} [H(x, F)]^{\gamma} dx \right)^{\frac{1}{\gamma}}, \quad (2.19)$$

where  $c = c(n, p, \nu, L, \omega(\cdot), \|a\|_{L^{\infty}(\Omega)}, \|H(\cdot, F)\|_{L^1(\Omega)}, \gamma, R_0, \Omega)$  is a positive constant.

**Remark 2.5.** Some research activities have been carried out on double phase problems in the borderline case, see for instance [4,5]. However, there are few results regarding Calderón–Zygmund type estimate for those problems, as far as we are concerned. The above Calderón–Zygmund estimate (2.19) is global, and we also obtain a local estimate in the proof of Theorem 2.4, see (5.20) in Section 5.

**Remark 2.6.** It is worth pointing out that there is a close relationship between the double phase problem in the borderline case and the  $p(x)$ -Laplacian type problem whose energy functional is given by

$$v \in W^{1,1}(\Omega) \mapsto \int_{\Omega} |Dv|^{p(x)} dx, \quad 1 < p(x) < \infty. \quad (2.20)$$

As shown in [1,4,5,8], the regularity assumption on the modulating coefficient  $a(x)$  is exactly the same as that on the variable exponent  $p(x)$ . In fact, the variable exponent  $p(x)$  in the functional (2.20) yields a logarithmic perturbation in the gradient when the point  $x$  varies, which is linked to a double phase functional in the borderline case (1.3).

### 3. Notations and preliminaries

From now on, for the sake of convenience, we employ the letter  $c$  to denote any universal constants which can be explicitly computed in terms of known quantities such as  $n, p, \nu, L, \gamma, \|a\|_{L^{\infty}(\Omega)}, \|H(\cdot, F)\|_{L^1(\Omega)}$  and  $\omega(\cdot)$ . Thus the exact value denoted by  $c$  might be different from line to line. We start with some standard notation.

- (1) For a point  $y \in \mathbb{R}^n$  and for  $r > 0$ ,  $B_r(y) := \{x \in \mathbb{R}^n : |x - y| < r\}$ ,  $\Omega_r(y) := B_r(y) \cap \Omega$ ,  $\partial_w \Omega_r(y) := B_r(y) \cap \partial\Omega$ . If the center is clear in the context, we shall omit denoting it as follows:  $B_r \equiv B_r(y)$ ,  $\Omega_r \equiv \Omega_r(y)$ .
- (2)  $B_r^+ = B_r(0) \cap \{x^n > 0\}$ ,  $T_r = B_r(0) \cap \{x^n = 0\}$ .
- (3) The number  $e$  is the Euler's number and  $\log t$  is the natural logarithm of  $t > 0$ .
- (4) For  $\alpha > 0$ ,  $\log^{\alpha}(e + t)$  denotes the quantity  $[\log(e + t)]^{\alpha}$ .

We will use the following logarithmic inequalities later:

$$\log(e + st) \leq \log(e + s) + \log(e + t), \quad (3.1)$$

$$\log(e + t) \leq c(\alpha) \log(e + t^{\alpha}), \quad (3.2)$$

for all  $s, t \geq 0$  and  $\alpha \geq 1$ .

### 3.1. Orlicz spaces and Musielak–Orlicz spaces

We first recall some definitions and basic properties on the Orlicz spaces. A Young function  $g : [0, \infty) \rightarrow [0, \infty)$  is a nondecreasing convex function such that

$$g(0) = 0, \lim_{t \rightarrow \infty} g(t) = \infty, \lim_{t \rightarrow 0+} \frac{g(t)}{t} = 0, \lim_{t \rightarrow \infty} \frac{g(t)}{t} = \infty.$$

For a given Young function  $g$ , the complementary Young function  $g^*$  to  $g$  is given by

$$g^*(t) = \sup\{ts - g(s) : s \geq 0\}, \quad t \geq 0.$$

We remark that this  $g^*$  satisfies all the conditions to be a Young function and that  $(g^*)^* = g$ .

We say that  $g$  satisfies the  $\Delta_2$ -condition, denoted by  $g \in \Delta_2$ , if there exists  $\mu_1 > 1$  such that  $g(2t) \leq \mu_1 g(t)$  for all  $t \geq 0$ . We denote by  $\Delta_2(g)$  the smallest constant  $\mu_1$ . Also we say that  $g$  satisfies the  $\nabla_2$ -condition, denoted by  $g \in \nabla_2$ , if there exists  $\mu_2 > 1$  such that  $g(t) \leq \frac{1}{2\mu_2} g(\mu_2 t)$  for all  $t \geq 0$ . We note that  $g \in \nabla_2$  if and only if  $g^* \in \Delta_2$ .

If  $g \in \Delta_2$ , then there exist two constants  $\kappa_1 = \kappa_1(\Delta_2(g))$  and  $\kappa_2 = \kappa_2(\Delta_2(g))$  with  $1 < \kappa_1 \leq \kappa_2 < \infty$  such that

$$c^{-1} \min\{\theta^{\kappa_1}, \theta^{\kappa_2}\} g(t) \leq g(\theta t) \leq c \max\{\theta^{\kappa_1}, \theta^{\kappa_2}\} g(t) \quad \text{for all } t, \theta \geq 0, \quad (3.3)$$

where the constant  $c \geq 1$  is independent of  $\theta$  and  $t$  (see [26]).

If  $g \in \Delta_2 \cap \nabla_2$ , then for any  $\varepsilon \in (0, 1]$ , there exists a positive constant  $c$  depending on  $\varepsilon$ ,  $\Delta_2(g)$  and  $\Delta_2(g^*)$  such that

$$st \leq \varepsilon g(s) + cg^*(t) \quad \text{for all } s, t \geq 0. \quad (3.4)$$

This inequality is called Young's inequality. Furthermore, from the following property of the complementary Young function (see for instance [3])

$$g^*\left(\frac{g(t)}{t}\right) \leq g(t), \quad (3.5)$$

we obtain a modified form of Young's inequality:

$$s \frac{g(t)}{t} \leq \varepsilon g(s) + cg(t) \quad \text{for all } s \geq 0 \text{ and } t > 0. \quad (3.6)$$

In particular, when  $g(t) = t^p \log(e + t)$  with  $p > 1$ , we have

$$st^{p-1} \log(e + t) \leq \varepsilon s^p \log(e + s) + ct^p \log(e + t) \quad \text{for all } s, t \geq 0. \quad (3.7)$$

For a Young function  $g$ , the Orlicz class  $K^g(\Omega)$  consists of all measurable functions  $v : \Omega \rightarrow \mathbb{R}$  satisfying

$$\int_{\Omega} g(|v(x)|) dx < +\infty.$$



The Orlicz space  $L^g(\Omega)$  is the vector space generated by  $K^g(\Omega)$ . If  $g \in \Delta_2$ , then  $K^g(\Omega) = L^g(\Omega)$  and this space is a Banach space under the Luxemburg norm

$$\|v\|_{L^g(\Omega)} = \inf \left\{ \sigma > 0 : \int_{\Omega} g\left(\frac{|v(x)|}{\sigma}\right) dx \leq 1 \right\}.$$

If  $g \in \Delta_2 \cap \nabla_2$ , then for any  $v \in L^g(\Omega)$  and  $w \in L^{g^*}(\Omega)$ ,

$$\int_{\Omega} |vw| dx \leq 2 \|v\|_{L^g(\Omega)} \|w\|_{L^{g^*}(\Omega)}. \quad (3.8)$$

The followings are some examples of Young functions and the corresponding Orlicz spaces:

$$g_1(t) = t^p, \quad p \geq 1,$$

which generates the classical Lebesgue space  $L^p(\Omega)$ , and

$$g_2(t) = t^p \log^{\alpha}(e + t), \quad p \geq 1, \quad \alpha > 0,$$

generates the Orlicz–Zygmund space  $L^p \log^{\alpha} L(\Omega)$ . Above all,  $L \log^{\alpha} L$  space includes any  $L^p$  spaces for  $p > 1$  with the following estimate:

$$\|v\|_{L \log^{\alpha} L(\Omega)} \leq c(p) \left( \int_{\Omega} |v|^p dx \right)^{\frac{1}{p}}.$$

Moreover, for every  $p > 1$  and  $v \in L^p(\Omega)$ , we have that (see [1,25] for more details)

$$\int_{\Omega} |v| \log^{\alpha} \left( e + \frac{|v|}{(|v|)_{\Omega}} \right) dx \leq c(p, \alpha) \left( \int_{\Omega} |v|^p dx \right)^{\frac{1}{p}}. \quad (3.9)$$

We now introduce the Musielak–Orlicz spaces which generalize the Orlicz spaces. Let  $g : \Omega \times [0, \infty) \rightarrow [0, \infty)$  be a function satisfying the following conditions:

- (a)  $g(x, \cdot)$  is a Young function for every  $x \in \Omega$ ,
- (b)  $g(\cdot, t)$  is a measurable function for every  $t \geq 0$ .

Such a function  $g(x, t)$  is called a Musielak–Orlicz function. We will denote by  $g(\cdot)$  a Musielak–Orlicz function to emphasize the dependence on  $x$ . As before, we say that  $g(\cdot)$  satisfies the  $\Delta_2$ -condition, denoted by  $g(\cdot) \in \Delta_2$ , if there exists  $\mu > 1$  such that  $g(x, 2t) \leq \mu g(x, t)$  for all  $x \in \Omega$ ,  $t \geq 0$ .

The Musielak–Orlicz class  $K^{g(\cdot)}(\Omega)$  consists of all measurable functions  $v : \Omega \rightarrow \mathbb{R}$  satisfying

$$\int_{\Omega} g(x, |v(x)|) dx < +\infty,$$

and the Musielak–Orlicz space  $L^{g(\cdot)}(\Omega)$  is the vector space generated by  $K^{g(\cdot)}(\Omega)$ . If  $g(\cdot) \in \Delta_2$ , then  $K^{g(\cdot)}(\Omega) = L^{g(\cdot)}(\Omega)$  and this space is a Banach space under the Luxemburg type norm

$$\|v\|_{L^{g(\cdot)}(\Omega)} = \inf \left\{ \sigma > 0 : \int_{\Omega} g\left(x, \frac{|v(x)|}{\sigma}\right) dx \leq 1 \right\}.$$

The Musielak–Orlicz–Sobolev space  $W^{1,g(\cdot)}(\Omega)$  is the function space of all measurable functions  $v \in L^{g(\cdot)}(\Omega)$  such that its distributional gradient vector  $Dv$  belongs to  $L^{g(\cdot)}(\Omega; \mathbb{R}^n)$ . If  $v \in W^{1,g(\cdot)}(\Omega)$ , we define its norm to be

$$\|v\|_{W^{1,g(\cdot)}(\Omega)} = \|v\|_{L^{g(\cdot)}(\Omega)} + \|Dv\|_{L^{g(\cdot)}(\Omega; \mathbb{R}^n)}.$$

The space  $W_0^{1,g(\cdot)}(\Omega)$  is defined as the closure of  $C_0^\infty(\Omega)$  in  $W^{1,g(\cdot)}(\Omega)$ . For a deeper discussion of the Musielak–Orlicz space and the associated Sobolev space, we refer the reader to [6,14,20,21,34,38] and references therein.

### 3.2. Extension lemmas

Here we introduce the zero extension lemma and the McShane extension lemma.

**Lemma 3.1.** [3] *Let  $U$  be a bounded domain in  $\mathbb{R}^n$ , and let  $v \in W_0^{1,p}(U)$  for some  $p \geq 1$ . Let  $\tilde{v}$  denote the zero extension of  $v$  and  $\tilde{Dv}$  denote the zero extension of  $Dv$ , that is,*

$$\tilde{v}(x) := \begin{cases} v(x) & \text{if } x \in U, \\ 0 & \text{if } x \in U^c, \end{cases} \quad \tilde{Dv}(x) := \begin{cases} Dv(x) & \text{if } x \in U, \\ 0 & \text{if } x \in U^c. \end{cases}$$

*Then  $D\tilde{v} = \tilde{Dv}$  in the weak sense, and hence  $\tilde{v} \in W^{1,p}(\mathbb{R}^n)$ .*

**Lemma 3.2.** [31] *Assume  $U \subset \mathbb{R}^n$ , and let  $b : U \rightarrow \mathbb{R}$  be a continuous function which has a modulus of continuity  $\omega$  satisfying (2.9). Then there exists an extension  $\tilde{b} : \mathbb{R}^n \rightarrow \mathbb{R}$  of  $b$  such that  $\tilde{b}$  admits  $\omega$  as a modulus of continuity and  $\|\tilde{b}\|_{L^\infty(\mathbb{R}^n)} = \|b\|_{L^\infty(U)}$ .*

**Remark 3.3.** Lemma 3.2 shows that it does not matter whether the coefficient function  $a(\cdot)$  is defined in  $\Omega \subset \mathbb{R}^n$  or in the whole domain  $\mathbb{R}^n$ . Hence we can assume that the coefficient function  $a(\cdot)$  is defined in  $\mathbb{R}^n$  as in the previous section. In addition, it follows from Lemma 3.1 that if  $H(x, Du) \in L^1(\Omega)$  for  $u \in W_0^{1,1}(\Omega)$ , then we have  $H(x, D\tilde{u}) \in L^1(\mathbb{R}^n)$ , where  $\tilde{u}$  is the zero extension of  $u$ .

### 3.3. Testing and solvability

We next discuss distributional solutions. The following proposition provides a criterion for admissible test functions.

**Proposition 3.4.** *Let  $B \subset \Omega$  be a ball and let  $W : B \rightarrow \mathbb{R}^n$  be a measurable vector field such that  $H(x, W) \in L^1(B)$  and which is a distributional solution to the equation*

$$\operatorname{div} S(x, W) = 0 \quad \text{in } B. \quad (3.10)$$

*Here we assume that the vector field  $S : B \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfies the growth condition*

$$|S(x, \xi)| \leq L \left[ |\xi|^{p-1} + a(x) |\xi|^{p-1} \log(e + |\xi|) \right] \quad (3.11)$$

*for almost every  $x \in B$  and for all  $\xi \in \mathbb{R}^n$ , where  $a(\cdot)$  has a modulus of continuity with a decay of logarithmic type (2.9). Then every function  $\varphi \in W_0^{1,1}(B)$  with  $H(x, D\varphi) \in L^1(B)$  satisfies*

$$\int_B \langle S(x, W), D\varphi \rangle dx = 0. \quad (3.12)$$

**Proof.** We only need to consider the case  $B = B_1(0) = B_1$  by dilation and translation. Let us begin by constructing a sequence of functions  $\{\varphi_k\} \subset C_0^\infty(B_1)$  such that

$$D\varphi_k \longrightarrow D\varphi \quad \text{a.e.} \quad \text{and} \quad H(x, D\varphi_k) \longrightarrow H(x, D\varphi) \quad \text{in } L^1(B_1). \quad (3.13)$$

The following construction is adapted from [13,18,47]. By Lemma 3.2, the continuous function  $a(\cdot)$  can be extended on  $\mathbb{R}^n$ , with the same modulus of continuity. For simplicity of notation, we continue to write  $a(\cdot)$  for the extension. Also we can take the zero extension of  $\varphi$  by Lemma 3.1, and hence we can assume that  $\varphi \in W^{1,1}(\mathbb{R}^n)$  with  $H(x, D\varphi) \in L^1(\mathbb{R}^n)$ . Let  $\psi \in C_0^\infty(B_1)$  be a mollifier with  $\psi \geq 0$ ,  $\int_{\mathbb{R}^n} \psi dx = 1$ , and set

$$\psi_r(x) := \frac{1}{r^n} \psi\left(\frac{x}{r}\right)$$

for  $x \in B_r$  with  $r > 0$ . Then it is clear that  $\psi_r \in C_0^\infty(B_r)$ ,  $\int_{\mathbb{R}^n} \psi_r dx = 1$  and  $0 \leq \psi_r \leq c(n)r^{-n}$ . Now we define, for  $0 < r < \frac{1}{4}$ ,

$$\widehat{a}_r(x) := a\left(\frac{x}{1-2r}\right), \quad \widehat{\varphi}_r(x) := \varphi\left(\frac{x}{1-2r}\right), \quad \varphi_r := \widehat{\varphi}_r * \psi_r \in C_0^\infty(B_{1-r}).$$

We also define

$$a_r(x) := \inf_{y \in B_r(x)} \widehat{a}_r(y), \quad H_r(x, \xi) := |\xi|^p + a_r(x) |\xi|^p \log(e + |\xi|)$$

for  $x \in B_1$  and  $\xi \in \mathbb{R}^n$ . By Hölder's inequality, we have

$$|D\varphi_r(x)| = |D\widehat{\varphi}_r * \psi_r(x)| \leq c \left( \int_{B_1} |D\widehat{\varphi}_r|^p dy \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^n} \psi_r^{\frac{p}{p-1}} dy \right)^{\frac{p-1}{p}} \leq cr^{-\frac{n}{p}}$$

for every  $x \in B_1$ . Then it follows from the definitions of  $a_r(\cdot)$  and  $\widehat{a}_r(\cdot)$  that

$$\begin{aligned} H(x, D\varphi_r(x)) &\leq |a(x) - a_r(x)| |D\varphi_r(x)|^p \log(e + |D\varphi_r(x)|) + H_r(x, D\varphi_r(x)) \\ &\leq c\omega(r) \log(e + cr^{-\frac{n}{p}}) |D\varphi_r(x)|^p + H_r(x, D\varphi_r(x)). \end{aligned}$$

Here we note that

$$\begin{aligned} \log(e + cr^{-\frac{n}{p}}) &\leq c \log(e + r^{-\frac{n}{p}}) \\ &\leq c \log(2r^{-\frac{n}{p}}) = c \left[ \log 2 + \frac{n}{p} \log \left( \frac{1}{r} \right) \right] \\ &\leq c \left[ 1 + \log \left( \frac{1}{r} \right) \right] \leq 2c \log \left( \frac{1}{r} \right), \end{aligned}$$

if  $r > 0$  is sufficiently small. Therefore, we see that

$$\begin{aligned} H(x, D\varphi_r(x)) &\leq c\omega(r) \log \left( \frac{1}{r} \right) |D\varphi_r(x)|^p + H_r(x, D\varphi_r(x)) \\ &\leq cH_r(x, D\varphi_r(x)), \end{aligned} \quad (3.14)$$

since  $a(\cdot)$  has a modulus of continuity with a decay of logarithmic type (2.9). By Jensen's inequality, we now estimate the right-hand side of (3.14) as follows:

$$\begin{aligned} H_r(x, D\varphi_r(x)) &\leq \int_{B_r(x)} H_r(x, D\widehat{\varphi}_r(y)) \psi_r(x-y) dy \\ &= \int_{B_r(x)} H_r \left( x, \frac{1}{1-2r} D\varphi \left( \frac{y}{1-2r} \right) \right) \psi_r(x-y) dy \\ &\leq \frac{1}{(1-2r)^{p+1}} \int_{B_r(x)} H_r \left( x, D\varphi \left( \frac{y}{1-2r} \right) \right) \psi_r(x-y) dy \\ &\leq 2^{p+1} \int_{B_r(x)} H \left( \frac{y}{1-2r}, D\varphi \left( \frac{y}{1-2r} \right) \right) \psi_r(x-y) dy \\ &= 2^{p+1} \left[ H \left( \frac{\cdot}{1-2r}, D\varphi \left( \frac{\cdot}{1-2r} \right) \right) * \psi_r \right] (x). \end{aligned} \quad (3.15)$$

Combining (3.14) and (3.15), we deduce that

$$H(x, D\varphi_r(x)) \leq c \left[ H \left( \frac{\cdot}{1-2r}, D\varphi \left( \frac{\cdot}{1-2r} \right) \right) * \psi_r \right] (x). \quad (3.16)$$

Using the fact that

$$H \left( \frac{\cdot}{1-2r}, D\varphi \left( \frac{\cdot}{1-2r} \right) \right) * \psi_r \longrightarrow H(\cdot, D\varphi(\cdot)) \quad \text{in } L^1(B_1)$$

as  $r \rightarrow 0$ , we can apply a generalized version of Lebesgue dominated convergence theorem to obtain a sequence of functions  $\{\varphi_k\} := \{\varphi_{r_k}\} \subset C_0^\infty(B_1)$  satisfying (3.13), for a suitable sequence  $\{r_k\}$  converging to zero.

We next show that

$$\int_{B_1} \langle S(x, W), D\varphi_k \rangle dx \longrightarrow \int_{B_1} \langle S(x, W), D\varphi \rangle dx, \quad (3.17)$$

as  $k \rightarrow \infty$ . From the growth condition (3.11) and Young's inequality (3.7) with  $\varepsilon = 1$ , we have

$$\begin{aligned} |\langle S(x, W), D\varphi_k \rangle| &\leq L \left[ |W|^{p-1} + a(x)|W|^{p-1} \log(e + |W|) \right] |D\varphi_k| \\ &\leq c [H(x, W) + H(x, D\varphi_k)], \end{aligned}$$

where  $c = c(p, L)$  is a positive constant. The generalized version of Lebesgue dominated convergence theorem now yields (3.17). Furthermore, from the fact that

$$\int_{B_1} \langle S(x, W), D\varphi_k \rangle dx = 0, \quad \forall k \in \mathbb{N},$$

we have the desired conclusion (3.12).  $\square$

Combining the previous proposition with Remark 3.3, we have the following result.

**Corollary 3.5.** *Let  $u \in W_0^{1,1}(\Omega)$  be a distributional solution to (2.1) with (2.17), under the assumptions (2.9) on the modulus of continuity  $\omega(\cdot)$  of the function  $a(\cdot)$ . Then we have*

$$\int_{\Omega} \langle A(x, Du), D\varphi \rangle dx = \int_{\Omega} \langle G(x, F), D\varphi \rangle dx,$$

for every function  $\varphi \in W_0^{1,1}(\Omega)$  with  $H(x, D\varphi) \in L^1(\Omega)$ .

Now we prove an existence result for the Dirichlet problem (2.1).

**Theorem 3.6.** *Suppose that  $H(x, F) \in L^1(\Omega)$ . Then there exists a unique distributional solution  $u \in W_0^{1,1}(\Omega)$  to (2.1) such that  $H(x, Du) \in L^1(\Omega)$ . Furthermore, the standard energy estimate*

$$\int_{\Omega} H(x, Du) dx \leq c \int_{\Omega} H(x, F) dx, \quad (3.18)$$

holds for a positive constant  $c = c(n, p, v, L)$ .

**Proof.** By abuse of notation, we continue to write  $H(x, \xi)$  also when  $\xi \in \mathbb{R}$ . Our proof starts with the observation that the function  $H : \Omega \times [0, \infty) \rightarrow [0, \infty)$  under consideration is a Musielak–Orlicz function. In addition,  $H(\cdot)$  satisfies the  $\Delta_2$ -condition, as

$$\begin{aligned} H(x, 2t) &= (2t)^p + a(x)(2t)^p \log(e + 2t) \\ &\leq 2^{p+1} [t^p + a(x)t^p \log(e + t)] = 2^{p+1} H(x, t), \end{aligned}$$

for all  $x \in \Omega$  and  $t \geq 0$ . Hence  $L^{H(\cdot)}(\Omega)$  is a Banach space. By the absence of Lavrentiev phenomenon discussed in [18] and the result of [38], there exists a distributional solution  $u \in W_0^{1, H(\cdot)}(\Omega)$  to the problem (2.1). It follows immediately that  $u \in W_0^{1,1}(\Omega)$  with  $H(x, Du) \in L^1(\Omega)$ . Furthermore, from Corollary 3.5, we can choose  $u$  as a test function, that is, we have

$$\int_{\Omega} \langle A(x, Du), Du \rangle dx = \int_{\Omega} \langle G(x, F), Du \rangle dx.$$

By using the monotonicity property (2.4) with  $\eta = 0$ , the growth condition (2.7) of  $G$  and Young's inequality (3.7) with  $\tau \in (0, 1)$ , we obtain

$$\begin{aligned} \tilde{v} \int_{\Omega} H(x, Du) dx &\leq \int_{\Omega} \langle A(x, Du), Du \rangle dx = \int_{\Omega} \langle G(x, F), Du \rangle dx \\ &\leq L \int_{\Omega} [ |F|^{p-1} + a(x)|F|^{p-1} \log(e + |F|) ] |Du| dx \\ &\leq L \left[ \tau \int_{\Omega} H(x, Du) dx + c(\tau, p) \int_{\Omega} H(x, F) dx \right]. \end{aligned}$$

The standard energy estimate (3.18) now follows by taking  $\tau = \frac{\tilde{v}}{2L}$ .

We next show the uniqueness of solutions. Suppose that  $u_1, u_2 \in W_0^{1,1}(\Omega)$  are distributional solutions to (2.1) with  $H(x, Du_1), H(x, Du_2) \in L^1(\Omega)$ . Then we can take  $\varphi = u_1 - u_2 \in W_0^{1,1}(\Omega)$  as a test function by Corollary 3.5, and hence we obtain

$$\int_{\Omega} \langle A(x, Du_1) - A(x, Du_2), Du_1 - Du_2 \rangle dx = 0 \quad (3.19)$$

Combining (3.19) with the monotonicity property (2.4) yields  $Du_1 \equiv Du_2$  in  $\Omega$ . Since  $u_1 = 0 = u_2$  on  $\partial\Omega$ , we conclude that  $u_1 \equiv u_2$  in  $\Omega$ .  $\square$

### 3.4. Regularity results for reference equations

We first discuss higher integrability results for reference equations in which the nonhomogeneous term equals to zero.

**Lemma 3.7.** *Let  $h \in W^{1,1}(B_{2r})$  be a distributional solution to*

$$\operatorname{div} A(x, Dh) = 0 \quad \text{in } B_{2r}, \quad (3.20)$$

with  $H(x, Dh) \in L^1(B_{2r})$  and  $B_{2r} \subset \Omega$ . Then there exists a universal constant  $\sigma = \sigma(n, p, v, L, \omega(\cdot), \|Dh\|_{L^p(B_{2r})}) > 0$  such that

$$\int_{B_r} [H(x, Dh)]^{1+\sigma} dx \leq c \left( \int_{B_{2r}} H(x, Dh) dx \right)^{1+\sigma},$$

where  $c = c(n, p, v, L, \omega(\cdot), \|Dh\|_{L^p(B_{2r})})$  is a positive constant.

**Proof.** If we prove that the following Caccioppoli type inequality

$$\int_{B_\rho} H(x, Dh) dx \leq c \int_{B_{2\rho}} H\left(x, \frac{|h - (h)_{B_{2\rho}}|}{\rho}\right) dx \quad (3.21)$$

holds whenever  $B_{2\rho} \equiv B_{2\rho}(y) \subset B_{2r}$  for some  $c = c(n, p, v, L) > 0$ , then the lemma follows by the same method as in [5, Section 4.1].

To show (3.21), we first define the complementary Musielak–Young function of  $H(\cdot)$  by for each  $x \in \Omega$ ,

$$H^*(x, t) := \sup\{ts - H(x, s) : s \geq 0\}.$$

Then  $H^* : \Omega \times [0, \infty) \rightarrow [0, \infty)$  is also a Musielak–Young function, and it is clear that  $H(\cdot) \in \Delta_2 \cap \nabla_2$  and  $H^*(\cdot) \in \Delta_2 \cap \nabla_2$ . Thus we have the following Young’s inequality: for any  $\tau \in (0, 1]$ , there exists a positive constant  $c = c(\tau, p)$  such that for  $s, t \geq 0$  and  $x \in \Omega$ ,

$$st \leq \tau H^*(x, s) + cH(x, t).$$

It also follows from [35, Lemma 2.2] that there exist  $\kappa \in (1, \infty)$  and  $c \geq 1$ , both depending only on  $p$ , such that

$$H^*(x, \theta t) \leq c\theta^\kappa H^*(x, t) \quad (3.22)$$

for all  $x \in B_{2r}$ ,  $t \geq 0$  and all  $\theta \in [0, 1]$ .

Let  $q := \frac{\kappa}{\kappa-1} > 1$ , and let  $\zeta \in C_0^\infty(B_{2\rho})$  be a cut-off function such that  $0 \leq \zeta \leq 1$ ,  $\zeta \equiv 1$  in  $B_\rho$  and  $|D\zeta| \leq \frac{2}{\rho}$ . Then we note from (3.22) and [35, Lemma 2.2] that

$$H^*\left(x, \frac{H(x, t)}{t} \zeta^{q-1}\right) \leq c \zeta^q H^*\left(x, \frac{H(x, t)}{t}\right) \leq c \zeta^q H(x, t) \quad (3.23)$$

for  $x \in B_{2\rho}$  and  $t > 0$ . We now take  $\varphi = \zeta^q (h - (h)_{B_{2\rho}})$  as a test function in (3.20). Indeed, Proposition 3.4 ensures that the above choice of  $\varphi$  is valid from the fact that  $H(x, Dh) \in L^1(B_{2r})$ . We thus get

$$\int_{B_{2\rho}} \langle A(x, Dh), Dh \rangle \zeta^q dx = -q \int_{B_{2\rho}} \langle A(x, Dh), D\zeta \rangle \zeta^{q-1} (h - (h)_{B_{2\rho}}) dx.$$

Then it follows from the monotonicity property (2.4) with  $\eta = 0$ , the growth condition (2.2), Young's inequality and (3.23) that

$$\begin{aligned}
 & \widetilde{v} \int_{B_{2\rho}} H(x, Dh) \zeta^q dx \\
 & \leq \int_{B_{2\rho}} \langle A(x, Dh), Dh \rangle \zeta^q dx = -q \int_{B_{2\rho}} \langle A(x, Dh), D\zeta \rangle \zeta^{q-1} (h - (h)_{B_{2\rho}}) dx \\
 & \leq 2qL \int_{B_{2\rho}} \frac{H(x, Dh)}{|Dh|} \zeta^{q-1} \frac{|h - (h)_{B_{2\rho}}|}{\rho} dx \\
 & \leq \tau \int_{B_{2\rho}} H^* \left( x, \frac{H(x, Dh)}{|Dh|} \zeta^{q-1} \right) dx + c(\tau) \int_{B_{2\rho}} H \left( x, \frac{|h - (h)_{B_{2\rho}}|}{\rho} \right) dx \\
 & \leq c\tau \int_{B_{2\rho}} H(x, Dh) \zeta^q dx + c(\tau) \int_{B_{2\rho}} H \left( x, \frac{|h - (h)_{B_{2\rho}}|}{\rho} \right) dx.
 \end{aligned}$$

Consequently, we obtain the Caccioppoli inequality (3.21) by taking  $\tau = \frac{\widetilde{v}}{2c}$ , and the lemma follows.  $\square$

We remark that the dependence on  $\|Dh\|_{L^p(B_{2r})}$  of the constants  $c$  and  $\sigma$  in Lemma 3.7 is natural when dealing with non-uniformly elliptic problems. Moreover, the constant  $c$  is a non-decreasing function of  $\|Dh\|_{L^p(B_{2r})}$ , as discussed in [5]. Moreover, as in the proof of [35, Theorem 3.9], one can find the boundary version of Lemma 3.7 as follows:

**Lemma 3.8.** *Suppose that  $\Omega$  is  $(\delta, R_0)$ -Reifenberg flat. Let  $h \in W^{1,1}(\Omega_{2r})$  be a distributional solution to*

$$\begin{cases} \operatorname{div} A(x, Dh) = 0 & \text{in } \Omega_{2r}, \\ h = 0 & \text{on } \partial_w \Omega_{2r}, \end{cases}$$

with  $H(x, Dh) \in L^1(\Omega_{2r})$ ,  $0 < 2r \leq R_0$  and

$$B_{2r}^+ \subset B_{2r} \cap \Omega \subset B_{2r} \cap \{x^n > -4\delta r\}.$$

Then there exists a universal constant  $\sigma = \sigma(n, p, v, L, \omega(\cdot), \|Dh\|_{L^p(\Omega_{2r})}) > 0$  such that

$$\int_{\Omega_r} [H(x, Dh)]^{1+\sigma} dx \leq c \left( \int_{\Omega_{2r}} H(x, Dh) dx \right)^{1+\sigma},$$

where  $c = c(n, p, v, L, \omega(\cdot), \|Dh\|_{L^p(\Omega_{2r})})$  is a positive constant.



We next discuss Lipschitz estimates for reference problems in which the associated nonlinearity has no  $x$ -dependence. To be specific, consider a vector-valued function  $A_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfying the following structural conditions:

$$|A_0(\xi)| + |\xi| |D_\xi A_0(\xi)| \leq L \left[ |\xi|^{p-1} + a_0 |\xi|^{p-1} \log(e + |\xi|) \right], \quad (3.24)$$

$$\langle D_\xi A_0(\xi) \eta, \eta \rangle \geq \nu \left[ |\xi|^{p-2} + a_0 |\xi|^{p-2} \log(e + |\xi|) \right] |\eta|^2, \quad (3.25)$$

for every  $\xi, \eta \in \mathbb{R}^n$ , where  $0 < \nu \leq L < +\infty$  and  $a_0 \geq 0$  are fixed constants. In accordance with (2.15), we use the notation

$$H_0(\xi) := |\xi|^p + a_0 |\xi|^p \log(e + |\xi|), \quad (3.26)$$

for  $\xi \in \mathbb{R}^n$ .

Then we have the following interior Lipschitz estimate.

**Lemma 3.9.** [19,30] *Let  $v \in W^{1,1}(B_{2r})$  be a distributional solution to*

$$\operatorname{div} A_0(Dv) = 0 \quad \text{in } B_{2r},$$

*with  $H_0(Dv) \in L^1(B_{2r})$ . Then we have  $Dv \in L^\infty(B_r)$  with the estimate*

$$\sup_{B_r} H_0(Dv) \leq c \int_{B_{2r}} H_0(Dv) dx,$$

*where  $c = (n, p, \nu, L, a_0)$  is a universal constant.*

We next present a boundary version of the above lemma. Note that the mapping  $t \mapsto t^p + a_0 t^p \log(e + t)$  is invertible on  $[0, \infty)$ , to obtain the following result in much the same way as [13, Theorem 2.2].

**Lemma 3.10.** *Let  $v \in W^{1,1}(B_{2r}^+)$  be a distributional solution to*

$$\begin{cases} \operatorname{div} A_0(Dv) = 0 & \text{in } B_{2r}^+, \\ v = 0 & \text{on } T_{2r}, \end{cases}$$

*with  $H_0(Dv) \in L^1(B_{2r}^+)$ . Then we have  $Dv \in L^\infty(B_r^+)$  with the estimate*

$$\sup_{B_r^+} H_0(Dv) \leq c \int_{B_{2r}^+} H_0(Dv) dx,$$

*where  $c = (n, p, \nu, L, a_0)$  is a universal constant.*

### 3.5. Covering argument

To establish a desired global estimate, we investigate a covering argument. We first assume that

$$0 < \tilde{R} \leq \min \left\{ R_0, \frac{1}{e} \right\}, \quad (3.27)$$

where  $R_0$  is the number given in [Theorem 2.4](#). In this subsection and the next section, we fix a point  $x_0 \in \Omega$  and concentrate on  $\Omega_R = \Omega_R(x_0) = B_R(x_0) \cap \Omega$  with  $R \leq \tilde{R}$ .

Let  $u \in W_0^{1,1}(\Omega)$  be the distributional solution to [\(2.1\)](#) with [\(2.17\)](#). We first select radii  $r_1, r_2$  such that  $\frac{R}{2} \leq r_1 < r_2 \leq R$  and consider the upper level sets

$$E(\lambda, s) := \{x \in \Omega_s : H(x, Du(x)) > \lambda\}, \quad \frac{R}{2} \leq s \leq R, \quad \lambda > 0. \quad (3.28)$$

For each  $y \in E(\lambda, r_1)$ , we define a continuous function  $\Phi_y : (0, r_2 - r_1] \rightarrow [0, \infty)$  by

$$\Phi_y(\rho) := \int_{\Omega_\rho(y)} \left[ H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx, \quad (3.29)$$

where  $\delta > 0$  is to be determined later. From the Lebesgue differentiation theorem and [\(3.28\)](#), it follows that for almost every  $y \in E(\lambda, r_1)$ ,

$$\begin{aligned} \lim_{\rho \rightarrow 0} \Phi_y(\rho) &= H(y, Du(y)) + \frac{1}{\delta} H(y, F(y)) \\ &\geq H(y, Du(y)) > \lambda. \end{aligned} \quad (3.30)$$

On the other hand, using [\(2.12\)](#), for any  $\rho \in [\frac{r_2 - r_1}{156}, r_2 - r_1]$ , we obtain

$$\begin{aligned} \Phi_y(\rho) &= \int_{\Omega_\rho(y)} \left[ H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx \\ &\leq \frac{|\Omega_{r_2}|}{|\Omega_\rho(y)|} \int_{\Omega_{r_2}} \left[ H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx \\ &\leq \frac{|B_{r_2}|}{|B_\rho(y)|} \frac{|B_\rho(y)|}{|\Omega_\rho(y)|} \int_{\Omega_{r_2}} \left[ H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx \\ &\leq \left( \frac{r_2}{\frac{r_2 - r_1}{156}} \right)^n \left( \frac{16}{7} \right)^n \int_{\Omega_{r_2}} \left[ H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx \\ &\leq \frac{400^n r_2^n}{(r_2 - r_1)^n} \int_{\Omega_{r_2}} \left[ H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx =: \lambda_0. \end{aligned} \quad (3.31)$$

From now on, we consider positive numbers  $\lambda$  satisfying

$$\lambda > \lambda_0. \quad (3.32)$$

Since  $\Phi_y$  is a continuous function, it follows from (3.30)–(3.32) that for almost every  $y \in E(\lambda, r_1)$ , there exists an exit time radius  $\rho_y \in (0, \frac{r_2-r_1}{156})$  such that

$$\Phi_y(\rho_y) = \lambda \quad \text{and} \quad \Phi_y(\rho) < \lambda \quad \text{for all } \rho \in (\rho_y, r_2 - r_1]. \quad (3.33)$$

Note that the family  $\{\Omega_{\rho_y}(y)\} = \{B_{\rho_y}(y) \cap \Omega\}$  covers  $E(\lambda, r_1)$  up to a negligible set. Applying Vitali's covering lemma, there exists a countable family of disjoint sets  $\{\Omega_{\rho_i}(y_i)\}_{i=1}^\infty$  with  $y_i \in E(\lambda, r_1)$  and  $\rho_i = \rho_{y_i} \in (0, \frac{r_2-r_1}{156})$  such that

$$E(\lambda, r_1) \subset \bigcup_{i \geq 1} \Omega_{5\rho_i}(y_i) \cup \text{negligible set}, \quad (3.34)$$

and

$$\Phi_{y_i}(\rho_i) = \lambda \quad \text{and} \quad \Phi_{y_i}(\rho) < \lambda \quad \text{for all } \rho \in (\rho_i, r_2 - r_1]. \quad (3.35)$$

We first discuss the interior case  $B_{20\rho_i}(y_i) \subset \Omega$ . Let us consider the concentric balls

$$B_i^0 \equiv B_{\rho_i}(y_i) = \Omega_{\rho_i}(y_i), \quad B_i^j \equiv B_{5^j\rho_i}(y_i) = \Omega_{5^j\rho_i}(y_i) \quad \text{for } j = 1, 2, 3, 4. \quad (3.36)$$

We recall that

$$0 < \rho_i < 5^j\rho_i \leq 20\rho_i < 156\rho_i < r_2 - r_1 \leq \frac{R}{2} \leq \frac{\tilde{R}}{2} < \tilde{R} \leq \min \left\{ R_0, \frac{1}{e} \right\}. \quad (3.37)$$

Then (3.29) and (3.35) yield that, for  $j = 1, 2, 3, 4$ ,

$$\int_{B_i^j} \left[ H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx < \lambda. \quad (3.38)$$

For the boundary case  $B_{20\rho_i}(y_i) \not\subset \Omega$ , we fix a boundary point  $\widehat{y}_i \in B_{20\rho_i}(y_i) \cap \partial\Omega$ . It is clear that

$$\Omega_{5\rho_i}(y_i) \subset \Omega_{25\rho_i}(\widehat{y}_i). \quad (3.39)$$

Since the domain  $\Omega$  is  $(\delta, R_0)$ -Reifenberg flat, there exists a coordinate system  $\{z^1, \dots, z^n\}$ , after a proper rotation and translation, such that in this new coordinate system,

$$\begin{cases} y_i = z_i, & \widehat{y}_i + 156\delta\rho_i(0, \dots, 0, 1) \text{ is the origin,} \\ B_{156\rho_i}^+ \subset \Omega_{156\rho_i} \subset B_{156\rho_i} \cap \{z^n > -312\delta\rho_i\}. \end{cases} \quad (3.40)$$

We assume that

$$0 < \delta \leq \frac{1}{312}. \quad (3.41)$$

From (3.39)–(3.41), we see that

$$\Omega_{5\rho_i}(z_i) \subset \Omega_{26\rho_i}(0) \quad (3.42)$$

in the coordinate system. Now we set

$$\Omega_i^0 \equiv \Omega_{\rho_i}(z_i), \quad \Omega_i^j \equiv \Omega_{26^j\rho_i} = \Omega_{26^j\rho_i}(0) \quad \text{for } j = 1, 2, 3, 4, 5. \quad (3.43)$$

Then it follows from (3.39)–(3.43) that

$$\Omega_i^j \subset \Omega_{130\rho_i} \subset \Omega_{156\rho_i}(z_i), \quad B_i^{j+} \equiv B_{26^j\rho_i}^+ \subset \Omega_i^j \subset B_{26^j\rho_i} \cap \{z^n > -312\delta\rho_i\}. \quad (3.44)$$

Hence we deduce that, for  $j = 1, 2, 3, 4, 5$ ,

$$\begin{aligned} & \oint_{\Omega_i^j} \left[ H(z, Du) + \frac{1}{\delta} H(z, F) \right] dz \\ & \leq \frac{|\Omega_{156\rho_i}(z_i)|}{|\Omega_{26\rho_i}|} \oint_{\Omega_{156\rho_i}(z_i)} \left[ H(z, Du) + \frac{1}{\delta} H(z, F) \right] dz \\ & \leq \frac{|B_{156\rho_i}(z_i)|}{|B_{26\rho_i}^+|} \oint_{\Omega_{156\rho_i}(z_i)} \left[ H(z, Du) + \frac{1}{\delta} H(z, F) \right] dz \\ & \leq 2 \cdot 6^n \oint_{\Omega_{156\rho_i}(z_i)} \left[ H(z, Du) + \frac{1}{\delta} H(z, F) \right] dz \\ & < 2 \cdot 6^n \lambda. \end{aligned} \quad (3.45)$$

Consequently, we have

$$\oint_{\Omega_i^5} H(x, Du) dx \leq c\lambda \quad \text{and} \quad \oint_{\Omega_i^5} H(x, F) dx \leq c\delta\lambda. \quad (3.46)$$

#### 4. Comparison estimates

The comparison estimates will be divided into the interior case and the boundary case. In this section, we mainly deal with the boundary case since similar results for the interior case can be proved in much a simpler way. In this case, we adopt the new coordinate system (3.40) and (3.42)–(3.44), as the related quantities are invariant under such translation and rotation.

Let  $u \in W_0^{1,1}(\Omega)$  be the distributional solution to (2.1) with (2.17), and let  $h_i \in W^{1,1}(\Omega_i^5)$  be the distributional solution to the homogeneous problem

$$\begin{cases} \operatorname{div} A(x, Dh_i) = 0 & \text{in } \Omega_i^5, \\ h_i = u & \text{on } \partial\Omega_i^5, \end{cases} \quad (4.1)$$

with  $H(x, Dh_i) \in L^1(\Omega_i^5)$ . It follows from the standard energy estimate for (4.1) and the uniform bound (3.46) that

$$\int_{\Omega_i^5} H(x, Dh_i) dx \leq c \int_{\Omega_i^5} H(x, Du) dx \leq c\lambda, \quad (4.2)$$

for a universal constant  $c = c(n, p, v, L)$ .

**Lemma 4.1.** *Let  $u \in W_0^{1,1}(\Omega)$  be the distributional solution to (2.1) with (2.17). Then for any  $0 < \varepsilon < 1$ , there exists a constant  $\delta = \delta(n, p, v, L, \varepsilon) > 0$  such that if (3.46) holds and if  $h_i \in W^{1,1}(\Omega_i^5)$  is the distributional solution to (4.1) with  $H(x, Dh_i) \in L^1(\Omega_i^5)$ , then we have*

$$\int_{\Omega_i^5} \left( |V_p(Du) - V_p(Dh_i)|^2 + a(x) |V_{\log}(Du) - V_{\log}(Dh_i)|^2 \right) dx \leq \varepsilon\lambda. \quad (4.3)$$

**Proof.** We take  $\varphi = u - h_i \in W_0^{1,1}(\Omega_i^5)$  as a test function in (2.1) and (4.1) to find that

$$\int_{\Omega_i^5} \langle A(x, Du) - A(x, Dh_i), D(u - h_i) \rangle dx = \int_{\Omega_i^5} \langle G(x, F), D(u - h_i) \rangle dx.$$

Indeed, Corollary 3.5 ensures that the above choice of  $\varphi$  is valid from (4.2). Using the monotonicity property (2.4), the growth condition (2.7), Young's inequality (3.7) with  $\tau \in (0, 1)$ , the uniform bounds (3.46) and (4.2), we obtain

$$\begin{aligned} & \int_{\Omega_i^5} \left( |V_p(Du) - V_p(Dh_i)|^2 + a(x) |V_{\log}(Du) - V_{\log}(Dh_i)|^2 \right) dx \\ & \leq c \int_{\Omega_i^5} \langle A(x, Du) - A(x, Dh_i), D(u - h_i) \rangle dx \\ & = c \int_{\Omega_i^5} \langle G(x, F), D(u - h_i) \rangle dx \\ & \leq c \int_{\Omega_i^5} \left[ |F|^{p-1} + a(x) |F|^{p-1} \log(e + |F|) \right] [|Du| + |Dh_i|] dx \end{aligned}$$

$$\begin{aligned} &\leq \tau \int_{\Omega_i^5} H(x, Du) dx + \tau \int_{\Omega_i^5} H(x, Dh_i) dx + c(\tau) \int_{\Omega_i^5} H(x, F) dx \\ &\leq c\tau\lambda + c(\tau)\delta\lambda. \end{aligned}$$

The conclusion (4.3) now follows by taking  $\tau = \frac{\varepsilon}{2c}$  and  $\delta = \frac{\varepsilon}{2c(\tau)}$ .  $\square$

We next consider a vector field  $\tilde{A} : \Omega_i^4 \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  defined by

$$\tilde{A}(x, \xi) := \left[ \frac{1 + a(x_{i,M}) \log(e + |\xi|)}{1 + a(x) \log(e + |\xi|)} \right] A(x, \xi), \quad (4.4)$$

where  $x_{i,M} \in \overline{\Omega_i^4}$  is a point such that

$$a(x_{i,M}) = \sup_{x \in \Omega_i^4} a(x). \quad (4.5)$$

This vector field  $\tilde{A}$  actually depends on  $i$ , but we omit the index  $i$  for simplicity. We now claim that the nonlinearity  $\tilde{A}$  satisfies the following structural conditions: for almost every  $x \in \Omega_i^4$  and for all  $\xi, \eta \in \mathbb{R}^n$ ,

$$|\tilde{A}(x, \xi)| + |\xi| |D_\xi \tilde{A}(x, \xi)| \leq 2L \left[ |\xi|^{p-1} + a(x_{i,M}) |\xi|^{p-1} \log(e + |\xi|) \right], \quad (4.6)$$

$$\langle D_\xi \tilde{A}(x, \xi) \eta, \eta \rangle \geq \frac{\nu}{2} \left[ |\xi|^{p-2} + a(x_{i,M}) |\xi|^{p-2} \log(e + |\xi|) \right] |\eta|^2, \quad (4.7)$$

where  $L$  and  $\nu$  are the positive constants presented in (2.2) and (2.3), respectively. To see this, we first compute  $D_\xi \tilde{A}(x, \xi)$  as follows:

$$\begin{aligned} D_\xi \tilde{A}(x, \xi) &= \left[ \frac{1 + a(x_{i,M}) \log(e + |\xi|)}{1 + a(x) \log(e + |\xi|)} \right] D_\xi A(x, \xi) \\ &\quad + D_\xi \left[ \frac{1 + a(x_{i,M}) \log(e + |\xi|)}{1 + a(x) \log(e + |\xi|)} \right] \otimes A(x, \xi) \\ &= \frac{1 + a(x_{i,M}) \log(e + |\xi|)}{1 + a(x) \log(e + |\xi|)} D_\xi A(x, \xi) \\ &\quad + \left[ \frac{a(x_{i,M}) - a(x)}{(1 + a(x) \log(e + |\xi|))^2 (e + |\xi|) |\xi|} \right] \xi \otimes A(x, \xi). \end{aligned} \quad (4.8)$$

Then it follows from (2.2), (4.4) and (4.8) that for almost every  $x \in \Omega_i^4$  and for all  $\xi \in \mathbb{R}^n$ ,

$$\begin{aligned} |\tilde{A}(x, \xi)| + |\xi| |D_\xi \tilde{A}(x, \xi)| &\leq \left[ \frac{1 + a(x_{i,M}) \log(e + |\xi|)}{1 + a(x) \log(e + |\xi|)} \right] (|A(x, \xi)| + |\xi| |D_\xi A(x, \xi)|) \\ &\quad + \left[ \frac{a(x_{i,M}) - a(x)}{(1 + a(x) \log(e + |\xi|))^2} \cdot \frac{|\xi|}{e + |\xi|} \right] |A(x, \xi)| \end{aligned}$$

$$\begin{aligned}
 &\leq \left[ \frac{1 + a(x_{i,M}) \log(e + |\xi|)}{1 + a(x) \log(e + |\xi|)} \right] (|A(x, \xi)| + |\xi| |D_\xi A(x, \xi)|) \\
 &\quad + \left[ \frac{1 + a(x_{i,M}) \log(e + |\xi|)}{1 + a(x) \log(e + |\xi|)} \right] |A(x, \xi)| \\
 &\leq 2 \left[ \frac{1 + a(x_{i,M}) \log(e + |\xi|)}{1 + a(x) \log(e + |\xi|)} \right] (|A(x, \xi)| + |\xi| |D_\xi A(x, \xi)|) \\
 &\leq 2L \left[ |\xi|^{p-1} + a(x_{i,M}) |\xi|^{p-1} \log(e + |\xi|) \right].
 \end{aligned}$$

It also follows from (2.3) and (4.8) that

$$\begin{aligned}
 \langle D_\xi \tilde{A}(x, \xi) \eta, \eta \rangle &= \left[ \frac{1 + a(x_{i,M}) \log(e + |\xi|)}{1 + a(x) \log(e + |\xi|)} \right] \langle D_\xi A(x, \xi) \eta, \eta \rangle \\
 &\quad + \left[ \frac{a(x_{i,M}) - a(x)}{(1 + a(x) \log(e + |\xi|))^2 (e + |\xi|) |\xi|} \right] \langle \xi \otimes A(x, \xi) \eta, \eta \rangle \\
 &\geq \nu \left[ |\xi|^{p-2} + a(x_{i,M}) |\xi|^{p-2} \log(e + |\xi|) \right] |\eta|^2 \\
 &\quad - \left[ \frac{a(x_{i,M}) - a(x)}{(1 + a(x) \log(e + |\xi|))^2 (e + |\xi|)} \right] |A(x, \xi)| |\eta|^2 \\
 &\geq \nu \left[ |\xi|^{p-2} + a(x_{i,M}) |\xi|^{p-2} \log(e + |\xi|) \right] |\eta|^2 \\
 &\quad - L \left[ a(x_{i,M}) - a(x) \right] \left[ \frac{|\xi|^{p-1} + a(x) |\xi|^{p-1} \log(e + |\xi|)}{(1 + a(x) \log(e + |\xi|))^2 (e + |\xi|)} \right] |\eta|^2 \\
 &\geq \nu \left[ |\xi|^{p-2} + a(x_{i,M}) |\xi|^{p-2} \log(e + |\xi|) \right] |\eta|^2 \\
 &\quad - L \left[ a(x_{i,M}) - a(x) \right] |\xi|^{p-2} |\eta|^2 \\
 &\geq (\nu - L [a(x_{i,M}) - a(x)]) \left[ |\xi|^{p-2} + a(x_{i,M}) |\xi|^{p-2} \log(e + |\xi|) \right] |\eta|^2.
 \end{aligned}$$

We now choose  $\tilde{R} > 0$  so small that

$$0 \leq a(x_{i,M}) - a(x) \leq \omega(208\rho_i) \leq \omega(\tilde{R}) \leq \frac{\nu}{2L}. \quad (4.9)$$

Then the structural condition (4.7) follows.

We remark that the structural condition (4.7) implies the following monotonicity property: for almost every  $x \in \Omega_i^4$  and for all  $\xi, \eta \in \mathbb{R}^n$ ,

$$\langle \tilde{A}(x, \xi) - \tilde{A}(x, \eta), \xi - \eta \rangle \geq \frac{\tilde{\nu}}{2} \left[ |V_p(\xi) - V_p(\eta)|^2 + a(x_{i,M}) |V_{\log}(\xi) - V_{\log}(\eta)|^2 \right]. \quad (4.10)$$

With the solution  $h_i$  to (4.1), we next let  $w_i$  be the distributional solution to

$$\begin{cases} \operatorname{div} \tilde{A}(x, Dw_i) = 0 & \text{in } \Omega_i^4, \\ w_i = h_i & \text{on } \partial\Omega_i^4. \end{cases} \quad (4.11)$$

According to [Lemma 3.8](#), we see that  $Dh_i \in L^{p(1+\sigma)}(\Omega_i^4)$  for some positive constant  $\sigma = \sigma(n, p, v, L, \omega(\cdot), \|Dh_i\|_{L^p(\Omega_i^4)}) > 0$ . From the fact  $L^{p(1+\sigma)}(\Omega_i^4) \subset L^p \log L(\Omega_i^4)$ , we have

$$\int_{\Omega_i^4} H(x_{i,M}, Dh_i) dx = \int_{\Omega_i^4} |Dh_i|^p dx + a(x_{i,M}) \int_{\Omega_i^4} |Dh_i|^p \log(e + |Dh_i|) dx < +\infty.$$

In addition, choosing  $w_i - h_i$  as a test function in [\(4.11\)](#) and using the structural conditions [\(4.6\)–\(4.7\)](#), we have the standard energy estimate for [\(4.11\)](#) as follows:

$$\int_{\Omega_i^4} H(x_{i,M}, Dw_i) dx \leq c \int_{\Omega_i^4} H(x_{i,M}, Dh_i) dx < +\infty, \quad (4.12)$$

for a universal constant  $c = c(n, p, v, L)$ .

**Lemma 4.2.** *Under the assumptions and conclusions in [Lemma 4.1](#), there exists  $\tilde{R} = \tilde{R}(n, p, v, L, \omega(\cdot), \|H(\cdot, F)\|_{L^1(\Omega)}, \varepsilon) > 0$  such that if  $w_i \in W^{1,1}(\Omega_i^4)$  is the distributional solution to [\(4.11\)](#) with  $H(x_{i,M}, Dw_i) \in L^1(\Omega_i^4)$ , then we have*

$$\int_{\Omega_i^4} \left( |V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M}) |V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2 \right) dx \leq \varepsilon \lambda. \quad (4.13)$$

Furthermore, we have

$$\int_{\Omega_i^4} H(x_{i,M}, Dh_i) dx \leq c \lambda, \quad (4.14)$$

where  $c = c(n, p, v, L, \|H(\cdot, F)\|_{L^1(\Omega)})$  is a positive constant.

**Proof.** We take  $\varphi = h_i - w_i \in W_0^{1,1}(\Omega_i^4)$  as a test function in [\(4.1\)](#) and [\(4.11\)](#). Indeed, [Proposition 3.4](#) and [Remark 3.3](#) ensure that the above choice of  $\varphi$  is valid from [\(4.5\)](#) and [\(4.12\)](#), and hence we have

$$\int_{\Omega_i^4} \langle \tilde{A}(x, Dh_i) - \tilde{A}(x, Dw_i), D(h_i - w_i) \rangle dx = \int_{\Omega_i^4} \langle \tilde{A}(x, Dh_i) - A(x, Dh_i), D(h_i - w_i) \rangle dx. \quad (4.15)$$

It follows from [\(4.10\)](#) and [\(4.15\)](#) that

$$\int_{\Omega_i^4} \left( |V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M}) |V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2 \right) dx$$



$$\begin{aligned}
 &\leq c \int_{\Omega_i^4} \langle \tilde{A}(x, Dh_i) - \tilde{A}(x, Dw_i), D(h_i - w_i) \rangle dx \\
 &= c \int_{\Omega_i^4} \langle \tilde{A}(x, Dh_i) - A(x, Dh_i), D(h_i - w_i) \rangle dx \\
 &\leq c \int_{\Omega_i^4} |\tilde{A}(x, Dh_i) - A(x, Dh_i)| |Dh_i - Dw_i| dx.
 \end{aligned} \tag{4.16}$$

By the definition of  $\tilde{A}$  and the growth condition of  $A$ , we have

$$\begin{aligned}
 |\tilde{A}(x, Dh_i) - A(x, Dh_i)| &= \left[ \frac{1 + a(x_{i,M}) \log(e + |Dh_i|)}{1 + a(x) \log(e + |Dh_i|)} - 1 \right] |A(x, Dh_i)| \\
 &= \left[ \frac{(a(x_{i,M}) - a(x)) \log(e + |Dh_i|)}{1 + a(x) \log(e + |Dh_i|)} \right] |A(x, Dh_i)| \\
 &\leq L(a(x_{i,M}) - a(x)) |Dh_i|^{p-1} \log(e + |Dh_i|) \\
 &\leq L\omega(208\rho_i) |Dh_i|^{p-1} \log(e + |Dh_i|).
 \end{aligned} \tag{4.17}$$

Note that  $\omega(c\rho_i) \leq c\omega(\rho_i)$  for  $c \geq 1$ , by the concavity of  $\omega(\cdot)$ . Then it follows from (4.16), (4.17) and Young's inequality with  $\tau \in (0, 1)$  that

$$\begin{aligned}
 &\int_{\Omega_i^4} \left( |V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M}) |V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2 \right) dx \\
 &\leq c \int_{\Omega_i^4} \omega(\rho_i) |Dh_i|^{p-1} \log(e + |Dh_i|) (|Dh_i| + |Dw_i|) dx \\
 &\leq c\tau^{-\frac{1}{p-1}} \int_{\Omega_i^4} \omega(\rho_i)^{\frac{p}{p-1}} |Dh_i|^p \log^{\frac{p}{p-1}}(e + |Dh_i|) dx + c\tau \int_{\Omega_i^4} (|Dh_i| + |Dw_i|)^p dx \\
 &\leq c_1 \tau^{-\frac{1}{p-1}} \omega(\rho_i)^{\frac{p}{p-1}} \int_{\Omega_i^4} |Dh_i|^p \log^{\frac{p}{p-1}}(e + |Dh_i|) dx \\
 &\quad + c_1 \tau \int_{\Omega_i^4} (|Dh_i|^p + |Dw_i|^p) dx \\
 &=: \text{I} + \text{II},
 \end{aligned} \tag{4.18}$$

for some positive constant  $c_1 = c_1(n, p, v, L)$ .

We first estimate I. Using (3.1), (3.2), (3.37) and Lemma 3.8, we derive

$$\begin{aligned}
 \text{I} &= c_1 \tau^{-\frac{1}{p-1}} \omega(\rho_i)^{\frac{p}{p-1}} \int_{\Omega_i^4} |Dh_i|^p \log^{\frac{p}{p-1}}(e + |Dh_i|) dx \\
 &\leq c \tau^{-\frac{1}{p-1}} \omega(\rho_i)^{\frac{p}{p-1}} \int_{\Omega_i^4} |Dh_i|^p \log^{\frac{p}{p-1}} \left( e + \frac{|Dh_i|^p}{(|Dh_i|^p)_{\Omega_i^4}} \right) dx \\
 &\quad + c \tau^{-\frac{1}{p-1}} \omega(\rho_i)^{\frac{p}{p-1}} \log^{\frac{p}{p-1}} \left( e + (|Dh_i|^p)_{\Omega_i^4} \right) \int_{\Omega_i^4} |Dh_i|^p dx \\
 &\leq c \tau^{-\frac{1}{p-1}} \omega(\rho_i)^{\frac{p}{p-1}} \left( \int_{\Omega_i^4} |Dh_i|^{p(1+\sigma)} dx \right)^{\frac{1}{1+\sigma}} \\
 &\quad + c \tau^{-\frac{1}{p-1}} \omega(\rho_i)^{\frac{p}{p-1}} \log^{\frac{p}{p-1}} \left( e + \frac{c}{\rho_i^n} \int_{\Omega_i^4} H(x, Dh_i) dx \right) \int_{\Omega_i^4} |Dh_i|^p dx \\
 &\leq c \tau^{-\frac{1}{p-1}} \omega(\rho_i)^{\frac{p}{p-1}} \left( \int_{\Omega_i^4} [H(x, Dh_i)]^{1+\sigma} dx \right)^{\frac{1}{1+\sigma}} \\
 &\quad + c \tau^{-\frac{1}{p-1}} \omega(\rho_i)^{\frac{p}{p-1}} \log^{\frac{p}{p-1}} \left( e + \frac{c}{\rho_i^n} \|H(\cdot, F)\|_{L^1(\Omega)} \right) \int_{\Omega_i^4} H(x, Dh_i) dx \\
 &\leq c \tau^{-\frac{1}{p-1}} \omega(\rho_i)^{\frac{p}{p-1}} \int_{\Omega_i^5} H(x, Dh_i) dx \\
 &\quad + c \tau^{-\frac{1}{p-1}} \omega(\rho_i)^{\frac{p}{p-1}} \log^{\frac{p}{p-1}} \left( \frac{1}{\rho_i} \right) \int_{\Omega_i^4} H(x, Dh_i) dx \\
 &\leq c \tau^{-\frac{1}{p-1}} \omega(\rho_i)^{\frac{p}{p-1}} \log^{\frac{p}{p-1}} \left( \frac{1}{\rho_i} \right) \int_{\Omega_i^5} H(x, Dh_i) dx \\
 &\leq c \tau^{-\frac{1}{p-1}} \left( \omega(\rho_i) \log \left( \frac{1}{\rho_i} \right) \right)^{\frac{p}{p-1}} \lambda, \tag{4.19}
 \end{aligned}$$

for a positive constant  $c = c(n, p, v, L, \|H(\cdot, F)\|_{L^1(\Omega)})$ . In the same manner we see that

$$\omega(\rho_i) \int_{\Omega_i^4} |Dh_i|^p \log(e + |Dh_i|) dx \leq c\omega(\rho_i) \log\left(\frac{1}{\rho_i}\right) \int_{\Omega_i^5} H(x, Dh_i) dx,$$

and hence

$$\begin{aligned} \int_{\Omega_i^4} H(x_{i,M}, Dh_i) dx &= \int_{\Omega_i^4} H(x, Dh_i) dx \\ &\quad + \int_{\Omega_i^4} (a(x_{i,M}) - a(x)) |Dh_i|^p \log(e + |Dh_i|) dx \\ &\leq \int_{\Omega_i^4} H(x, Dh_i) dx + c\omega(\rho_i) \int_{\Omega_i^4} |Dh_i|^p \log(e + |Dh_i|) dx \\ &\leq c \left( 1 + \omega(\rho_i) \log\left(\frac{1}{\rho_i}\right) \right) \int_{\Omega_i^5} H(x, Dh_i) dx \\ &\leq c \int_{\Omega_i^5} H(x, Dh_i) dx. \end{aligned} \quad (4.20)$$

Now the conclusion (4.14) follows from (4.20) and (4.2).

We next estimate II. Using (4.12), (4.20) and (4.2), we have

$$\begin{aligned} \text{II} &= c_1 \tau \int_{\Omega_i^4} (|Dh_i|^p + |Dw_i|^p) dx \\ &\leq c\tau \int_{\Omega_i^4} [H(x_{i,M}, Dh_i) + H(x_{i,M}, Dw_i)] dx \\ &\leq c\tau \int_{\Omega_i^4} H(x_{i,M}, Dh_i) dx \leq c\tau \int_{\Omega_i^5} H(x, Dh_i) dx \leq c\tau\lambda. \end{aligned} \quad (4.21)$$

Combining (4.18) with (4.19) and (4.21) yields

$$\begin{aligned} \int_{\Omega_i^4} \left( |V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M}) |V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2 \right) dx \\ \leq c \left[ \tau^{-\frac{1}{p-1}} \left( \omega(\rho_i) \log\left(\frac{1}{\rho_i}\right) \right)^{\frac{p}{p-1}} + \tau \right] \lambda. \end{aligned}$$

Taking  $\tau = \omega(\rho_i) \log\left(\frac{1}{\rho_i}\right)$  in the above inequality, we get

$$\oint_{\Omega_i^4} \left( |V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M}) |V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2 \right) dx \leq c_2 \omega(\rho_i) \log \left( \frac{1}{\rho_i} \right) \lambda, \quad (4.22)$$

for some positive constant  $c_2 = c_2(n, p, v, L, \|H(\cdot, F)\|_{L^1(\Omega)})$ . Now, from (2.9), we take so small  $\tilde{R} > 0$  that for any  $r \leq \tilde{R}$ ,

$$c_2 \omega(r) \log \left( \frac{1}{r} \right) \leq \varepsilon. \quad (4.23)$$

Then the conclusion (4.13) follows from (3.37), (4.22) and (4.23).  $\square$

Let  $\tilde{A}_0(\xi)$  denote the integral average of  $\tilde{A}(\cdot, \xi)$  over  $B_i^{3+}$  with respect to  $x$ -variable for each fixed  $\xi \in \mathbb{R}^n$ , that is,

$$\tilde{A}_0(\xi) = \oint_{B_i^{3+}} \tilde{A}(x, \xi) dx = \frac{1}{|B_i^{3+}|} \int_{B_i^{3+}} \tilde{A}(x, \xi) dx. \quad (4.24)$$

We note that  $\tilde{A}_0(\xi)$  satisfies the structural conditions (4.6) and (4.7). We now let  $v_i$  be the unique distributional solution to

$$\begin{cases} \operatorname{div} \tilde{A}_0(Dv_i) = 0 & \text{in } \Omega_i^3, \\ v_i = w_i & \text{on } \partial\Omega_i^3. \end{cases} \quad (4.25)$$

with  $H(x_{i,M}, Dv_i) \in L^1(\Omega_i^3)$ . It follows from (4.12), (4.14) and the standard energy estimate for (4.25) with the choice of  $a(\cdot) \equiv a(x_{i,M})$  that

$$\oint_{\Omega_i^3} H(x_{i,M}, Dv_i) dx \leq c \oint_{\Omega_i^3} H(x_{i,M}, Dw_i) dx \leq c\lambda, \quad (4.26)$$

for a positive constant  $c = c(n, p, v, L, \|H(\cdot, F)\|_{L^1(\Omega)})$ .

**Lemma 4.3.** *Let  $u \in W_0^{1,1}(\Omega)$  be the distributional solution to (2.1) with (2.17). Then for any  $0 < \varepsilon < 1$ , there exists  $\delta = \delta(n, p, v, L, \|H(\cdot, F)\|_{L^1(\Omega)}, \varepsilon) > 0$  such that if (3.46) holds and if  $h_i \in W^{1,1}(\Omega_i^5)$  is the distributional solution to (4.1) with  $H(x, Dh_i) \in L^1(\Omega_i^5)$ ,  $w_i \in W^{1,1}(\Omega_i^4)$  is the distributional solution to (4.11) with  $H(x_{i,M}, Dw_i) \in L^1(\Omega_i^4)$  and  $v_i \in W^{1,1}(\Omega_i^3)$  is the distributional solution to (4.25) with  $H(x_{i,M}, Dv_i) \in L^1(\Omega_i^3)$ , then we have*

$$\oint_{\Omega_i^3} \left( |V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M}) |V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2 \right) dx \leq \varepsilon\lambda. \quad (4.27)$$

**Proof.** We take  $\varphi = w_i - v_i \in W_0^{1,1}(\Omega_i^3)$  as a test function in (4.11) and (4.25) to see that

$$\oint_{\Omega_i^3} \langle \tilde{A}_0(Dw_i) - \tilde{A}_0(Dv_i), D(w_i - v_i) \rangle dx = - \oint_{\Omega_i^3} \langle \tilde{A}(x, Dw_i) - \tilde{A}_0(Dw_i), D(w_i - v_i) \rangle dx. \quad (4.28)$$

It follows from (4.10) and (4.28) that

$$\begin{aligned} & \oint_{\Omega_i^3} \left( |V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M}) |V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2 \right) dx \\ & \leq c \oint_{\Omega_i^3} \langle \tilde{A}_0(Dw_i) - \tilde{A}_0(Dv_i), D(w_i - v_i) \rangle dx \\ & = -c \oint_{\Omega_i^3} \langle \tilde{A}(x, Dw_i) - \tilde{A}_0(Dw_i), D(w_i - v_i) \rangle dx \\ & \leq c \oint_{\Omega_i^3} |\tilde{A}(x, Dw_i) - \tilde{A}_0(Dw_i)| |Dw_i - Dv_i| dx. \end{aligned} \quad (4.29)$$

From a direct calculation, we obtain

$$\sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{|\tilde{A}(x, \xi) - \tilde{A}_0(\xi)|}{|\xi|^{p-1} + a(x_{i,M}) |\xi|^{p-1} \log(e + |\xi|)} = \Theta(A, B_i^{3+})(x). \quad (4.30)$$

Indeed, it follows from the definition of  $\tilde{A}(x, \xi)$  that

$$\begin{aligned} & \frac{|\tilde{A}(x, \xi) - \tilde{A}_0(\xi)|}{|\xi|^{p-1} + a(x_{i,M}) |\xi|^{p-1} \log(e + |\xi|)} \\ & = \left| \frac{\tilde{A}(x, \xi)}{|\xi|^{p-1} + a(x_{i,M}) |\xi|^{p-1} \log(e + |\xi|)} - \frac{\int_{B_i^{3+}} \tilde{A}(x, \xi) dx}{|\xi|^{p-1} + a(x_{i,M}) |\xi|^{p-1} \log(e + |\xi|)} \right| \\ & = \left| \frac{A(x, \xi)}{|\xi|^{p-1} + a(x) |\xi|^{p-1} \log(e + |\xi|)} - \int_{B_i^{3+}} \frac{A(x, \xi)}{|\xi|^{p-1} + a(x) |\xi|^{p-1} \log(e + |\xi|)} dx \right| \\ & = \left| \frac{A(x, \xi)}{|\xi|^{p-1} + a(x) |\xi|^{p-1} \log(e + |\xi|)} - \left( \frac{A(\cdot, \xi)}{|\xi|^{p-1} + a(\cdot) |\xi|^{p-1} \log(e + |\xi|)} \right)_{B_i^{3+}} \right|, \end{aligned}$$

and hence the equality (4.30) follows from (2.10). Substituting (4.30) into (4.29), we have

$$\begin{aligned}
 & \int_{\Omega_i^3} (|V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M})|V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2) dx \\
 & \leq c \int_{\Omega_i^3} \Theta(A, B_i^{3+}) \left[ |Dw_i|^{p-1} + a(x_{i,M})|Dw_i|^{p-1} \log(e + |Dw_i|) \right] |Dw_i - Dv_i| dx \\
 & \leq c_0 \int_{\Omega_i^3} \Theta(A, B_i^{3+}) \left[ |Dw_i|^p + a(x_{i,M})|Dw_i|^p \log(e + |Dw_i|) \right] dx \\
 & \quad + c_0 \int_{\Omega_i^3} \Theta(A, B_i^{3+}) |Dw_i|^{p-1} |Dv_i| dx \\
 & \quad + c_0 \int_{\Omega_i^3} \Theta(A, B_i^{3+}) a(x_{i,M}) |Dw_i|^{p-1} \log(e + |Dw_i|) |Dv_i| dx \\
 & =: I_1 + I_2 + I_3,
 \end{aligned} \tag{4.31}$$

for some positive constant  $c_0 = c_0(n, p, \nu, L)$ .

We first estimate  $I_1$ . Using Hölder's inequality and the higher integrability for  $Dw_i$  with the choice of  $a(\cdot) \equiv a(x_{i,M})$ , we get

$$\begin{aligned}
 I_1 &= c_0 \int_{\Omega_i^3} \Theta(A, B_i^{3+}) H(x_{i,M}, Dw_i) dx \\
 &\leq c \left( \int_{\Omega_i^3} \Theta^{\frac{1+\sigma}{\sigma}} dx \right)^{\frac{\sigma}{1+\sigma}} \left( \int_{\Omega_i^3} [H(x_{i,M}, Dw_i)]^{1+\sigma} dx \right)^{\frac{1}{1+\sigma}} \\
 &\leq c \left( \int_{\Omega_i^3} \Theta^{\frac{1+\sigma}{\sigma}} dx \right)^{\frac{\sigma}{1+\sigma}} \int_{\Omega_i^4} H(x_{i,M}, Dw_i) dx.
 \end{aligned}$$

We note from (2.11) and (3.44) that

$$\begin{aligned}
 \int_{\Omega_i^3} \Theta^{\frac{1+\sigma}{\sigma}} dx &\leq \frac{1}{|B_i^{3+}|} \left( \int_{B_i^{3+}} \Theta^{\frac{1+\sigma}{\sigma}} dx + \int_{\Omega_i^3 \setminus B_i^{3+}} \Theta^{\frac{1+\sigma}{\sigma}} dx \right) \\
 &\leq \int_{B_i^{3+}} \Theta^{\frac{1+\sigma}{\sigma}} dx + \frac{|\Omega_i^3 \setminus B_i^{3+}|}{|B_i^{3+}|} (2L)^{\frac{1+\sigma}{\sigma}} \\
 &\leq c\delta.
 \end{aligned} \tag{4.32}$$

Using the energy estimates (4.12) and (4.14), we get

$$I_1 \leq c\delta^{\frac{\sigma}{1+\sigma}} \int_{\Omega_i^4} H(x_{i,M}, Dw_i) dx \leq c\delta^{\frac{\sigma}{1+\sigma}} \lambda. \quad (4.33)$$

We next estimate  $I_2$ . Using Young's inequality with  $\tau \in (0, 1)$ , Hölder's inequality and the higher integrability for  $Dw_i$ , we derive

$$\begin{aligned} I_2 &= c_0 \int_{\Omega_i^3} \Theta(A, B_i^{3+}) |Dw_i|^{p-1} |Dv_i| dx \\ &\leq c\tau^{-\frac{1}{p-1}} \int_{\Omega_i^3} \Theta^{\frac{p}{p-1}} |Dw_i|^p dx + c\tau \int_{\Omega_i^3} |Dv_i|^p dx \\ &\leq c\tau^{-\frac{1}{p-1}} \int_{\Omega_i^3} \Theta^{\frac{p}{p-1}} H(x_{i,M}, Dw_i) dx + c\tau \int_{\Omega_i^3} H(x_{i,M}, Dv_i) dx \\ &\leq c\tau^{-\frac{1}{p-1}} \left( \int_{\Omega_i^3} \Theta^{\frac{p}{p-1} \frac{1+\sigma}{\sigma}} dx \right)^{\frac{\sigma}{1+\sigma}} \left( \int_{\Omega_i^3} [H(x_{i,M}, Dw_i)]^{1+\sigma} dx \right)^{\frac{1}{1+\sigma}} \\ &\quad + c\tau \int_{\Omega_i^3} H(x_{i,M}, Dv_i) dx \\ &\leq c\tau^{-\frac{1}{p-1}} \left( \int_{\Omega_i^3} \Theta^{\frac{p}{p-1} \frac{1+\sigma}{\sigma}} dx \right)^{\frac{\sigma}{1+\sigma}} \int_{\Omega_i^4} H(x_{i,M}, Dw_i) dx + c\tau \int_{\Omega_i^3} H(x_{i,M}, Dv_i) dx. \end{aligned}$$

Likewise as in (4.32), one see that

$$\int_{\Omega_i^3} \Theta^{\frac{p}{p-1} \frac{1+\sigma}{\sigma}} dx \leq c\delta.$$

Utilizing the energy estimates (4.12), (4.14) and (4.26), we have

$$\begin{aligned} I_2 &\leq c\tau^{-\frac{1}{p-1}} \delta^{\frac{\sigma}{1+\sigma}} \int_{\Omega_i^4} H(x_{i,M}, Dw_i) dx + c\tau \int_{\Omega_i^3} H(x_{i,M}, Dv_i) dx \\ &\leq c\tau^{-\frac{1}{p-1}} \delta^{\frac{\sigma}{1+\sigma}} \lambda + c\tau \lambda. \end{aligned} \quad (4.34)$$

We now estimate  $I_3$ . To do this, we set  $g(t) := t^p \log(e+t)$  for  $t \geq 0$ . Note that  $g \in \Delta_2 \cap \nabla_2$ . Using Young's inequality (3.4) with  $\tau \in (0, 1)$ , (3.3) and (3.5), we have

$$\begin{aligned}
I_3 &= c_0 a(x_{i,M}) \int_{\Omega_i^3} \Theta(A, B_i^{3+}) |Dw_i|^{p-1} \log(e + |Dw_i|) |Dv_i| dx \\
&\leq c(\tau) a(x_{i,M}) \int_{\Omega_i^3} g^* \left( \Theta \frac{g(|Dw_i|)}{|Dw_i|} \right) dx + c\tau a(x_{i,M}) \int_{\Omega_i^3} g(|Dv_i|) dx \\
&\leq c(\tau) a(x_{i,M}) \int_{\Omega_i^3} \max\{\Theta^{\kappa_1}, \Theta^{\kappa_2}\} g^* \left( \frac{g(|Dw_i|)}{|Dw_i|} \right) dx + c\tau a(x_{i,M}) \int_{\Omega_i^3} g(|Dv_i|) dx \\
&\leq c(\tau) \int_{\Omega_i^3} (\Theta^{\kappa_1} + \Theta^{\kappa_2}) a(x_{i,M}) g(|Dw_i|) dx + c\tau \int_{\Omega_i^3} a(x_{i,M}) g(|Dv_i|) dx \\
&\leq c(\tau) \int_{\Omega_i^3} (\Theta^{\kappa_1} + \Theta^{\kappa_2}) H(x_{i,M}, Dw_i) dx + c\tau \int_{\Omega_i^3} H(x_{i,M}, Dv_i) dx,
\end{aligned}$$

for some positive constants  $\kappa_1$  and  $\kappa_2$  depending only on  $p$ . Then it follows from the higher integrability for  $Dw_i$  that

$$\begin{aligned}
I_3 &\leq c(\tau) \left( \int_{\Omega_i^3} \Theta^{\kappa_1 \cdot \frac{1+\sigma}{\sigma}} dx \right)^{\frac{\sigma}{1+\sigma}} \left( \int_{\Omega_i^3} [H(x_{i,M}, Dw_i)]^{1+\sigma} dx \right)^{\frac{1}{1+\sigma}} \\
&\quad + c(\tau) \left( \int_{\Omega_i^3} \Theta^{\kappa_2 \cdot \frac{1+\sigma}{\sigma}} dx \right)^{\frac{\sigma}{1+\sigma}} \left( \int_{\Omega_i^3} [H(x_{i,M}, Dw_i)]^{1+\sigma} dx \right)^{\frac{1}{1+\sigma}} \\
&\quad + c\tau \int_{\Omega_i^3} H(x_{i,M}, Dv_i) dx \\
&\leq c(\tau) \left( \int_{\Omega_i^3} \Theta^{\kappa_1 \cdot \frac{1+\sigma}{\sigma}} dx \right)^{\frac{\sigma}{1+\sigma}} \int_{\Omega_i^4} H(x_{i,M}, Dw_i) dx \\
&\quad + c(\tau) \left( \int_{\Omega_i^3} \Theta^{\kappa_2 \cdot \frac{1+\sigma}{\sigma}} dx \right)^{\frac{\sigma}{1+\sigma}} \int_{\Omega_i^4} H(x_{i,M}, Dw_i) dx + c\tau \int_{\Omega_i^3} H(x_{i,M}, Dv_i) dx.
\end{aligned}$$

By a similar argument, we obtain

$$I_3 \leq c(\tau) \delta^{\frac{\sigma}{1+\sigma}} \lambda + c\tau \lambda. \quad (4.35)$$



Combining (4.31) with (4.33)–(4.35) yields

$$\begin{aligned} & \int_{\Omega_i^3} \left( |V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M}) |V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2 \right) dx \\ & \leq c \delta^{\frac{\sigma}{1+\sigma}} \lambda + c \tau^{-\frac{1}{p-1}} \delta^{\frac{\sigma}{1+\sigma}} \lambda + c(\tau) \delta^{\frac{\sigma}{1+\sigma}} \lambda + c \tau \lambda \\ & \leq c(\tau) \delta^{\frac{\sigma}{1+\sigma}} \lambda + c \tau \lambda. \end{aligned}$$

The conclusion (4.27) now follows by taking  $\tau = \frac{\varepsilon}{2c}$  and  $\delta = \left( \frac{\varepsilon}{2c(\tau)} \right)^{\frac{1+\sigma}{\sigma}}$ .  $\square$

Since the domain under consideration can go beyond Lipschitz category, a boundary issue arises and we should deal with it more carefully. The following compactness lemma is the key to get a desired comparison estimate for the boundary case.

**Lemma 4.4.** *Let  $A_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a vector-valued function satisfying the structural conditions (3.24) and (3.25). Then for any  $\varepsilon > 0$ , there exists a small  $\delta = \delta(n, p, L, a_0, \varepsilon) > 0$  such that if*

$$B_{3r}^+ \subset \Omega_{3r} \subset B_{3r} \cap \{x^n > -6\delta r\}, \quad (4.36)$$

for some fixed  $r > 0$ , and if  $v \in W^{1,1}(\Omega_{3r})$  is a distributional solution to

$$\begin{cases} \operatorname{div} A_0(Dv) = 0 & \text{in } \Omega_{3r}, \\ v = 0 & \text{on } \partial_w \Omega_{3r}, \end{cases} \quad (4.37)$$

with  $H_0(Dv) \in L^1(\Omega_{3r})$  and

$$\int_{\Omega_{3r}} H_0(Dv) dx \leq \tilde{c} \quad (4.38)$$

for some constant  $\tilde{c} > 1$ , then there exists a distributional solution  $\bar{v} \in W^{1,1}(B_{2r}^+)$  to

$$\begin{cases} \operatorname{div} A_0(D\bar{v}) = 0 & \text{in } B_{2r}^+, \\ \bar{v} = 0 & \text{on } T_{2r}, \end{cases} \quad (4.39)$$

with

$$\int_{B_{2r}^+} H_0(D\bar{v}) dx \leq \left( \frac{3}{2} \right)^n \tilde{c}, \quad (4.40)$$

such that

$$\int_{\Omega_r} \left( |V_p(Dv) - V_p(D\bar{v})|^2 + a_0 |V_{\log}(Dv) - V_{\log}(D\bar{v})|^2 \right) dx \leq \tilde{c} \varepsilon, \quad (4.41)$$

where  $\bar{v}$  is extended by zero from  $B_r^+$  to  $B_r \supset \Omega_r$ .

**Proof.** We first note that it suffices to prove the lemma only for the case  $r = 1$  by scaling. Indeed, if we set  $\widehat{v}(x) := \frac{1}{r}v(rx)$  for  $x \in \widehat{\Omega}_3$ , where  $\widehat{\Omega}_3 := \left\{ \frac{1}{r}x : x \in \Omega_{3r} \right\}$ , then for any  $x \in \widehat{\Omega}_3$ , we see that  $D\widehat{v}(x) = Dv(rx)$ ,

$$\begin{aligned} H_0(D\widehat{v})(x) &= |D\widehat{v}(x)|^p + a_0|D\widehat{v}(x)|^p \log(e + |D\widehat{v}(x)|) \\ &= |Dv(rx)|^p + a_0|Dv(rx)|^p \log(e + |Dv(rx)|) = H_0(Dv)(rx), \end{aligned}$$

$$B_3^+ \subset \widehat{\Omega}_3 \subset B_3 \cap \{x^n > -6\delta\}$$

and  $\widehat{v} \in W^{1,1}(\widehat{\Omega}_3)$  is a distributional solution to

$$\begin{cases} \operatorname{div} A_0(D\widehat{v}) = 0 & \text{in } \widehat{\Omega}_3, \\ \widehat{v} = 0 & \text{on } \partial_w \widehat{\Omega}_3, \end{cases}$$

with

$$\int_{\widehat{\Omega}_3} H_0(D\widehat{v}) dx \leq \widetilde{c}.$$

In the same manner, one can regain (4.39)–(4.41).

We prove this lemma by contradiction via a compactness argument. If not, then we could find  $\varepsilon_0 > 0$ ,  $\{\Omega_3^k\}_{k=1}^\infty$  with

$$B_3^+ \subset \Omega_3^k \subset B_3 \cap \left\{ x^n > -\frac{6}{k} \right\}, \quad (4.42)$$

and distributional solutions  $v_k \in W^{1,1}(\Omega_3^k)$  to

$$\begin{cases} \operatorname{div} A_0(Dv_k) = 0 & \text{in } \Omega_3^k, \\ v_k = 0 & \text{on } \partial_w \Omega_3^k, \end{cases} \quad (4.43)$$

with

$$\int_{\Omega_3^k} H_0(Dv_k) dx \leq \widetilde{c} \quad (4.44)$$

such that

$$\int_{\Omega_1^k} \left( |V_p(Dv_k) - V_p(D\overline{v})|^2 + a_0|V_{\log}(Dv_k) - V_{\log}(D\overline{v})|^2 \right) dx > \widetilde{c}\varepsilon_0, \quad (4.45)$$

for any distributional solution  $\overline{v} \in W^{1,1}(B_2^+)$  to (4.39) with (4.40) when  $r = 1$ .

By abuse of notation, we shall continue to write  $H_0(\xi)$  also when  $\xi \in \mathbb{R}$ . Then the function  $H_0(t) = t^p + a_0 t^p \log(e + t)$ ,  $t \geq 0$  is a Young function with  $H_0 \in \Delta_2 \cap \nabla_2$ . Let us consider the Orlicz space  $L^{H_0}(\Omega)$ . We remark that the space  $L^{H_0}(\Omega)$  is nothing but the Lebesgue space  $L^p(\Omega)$  if  $a_0 = 0$ , and the Orlicz space  $L^p \log L(\Omega)$  if  $a_0 > 0$ . Therefore, the space  $L^{H_0}(\Omega)$  and the associated Sobolev space  $W^{1,H_0}(\Omega)$  are reflexive Banach spaces.

Let  $\tilde{v}_k$  be the zero extension of  $v_k$  from  $\Omega_3^k$  to  $B_3$ . Then by Lemma 3.1 and (4.44),  $\{H_0(D\tilde{v}_k)\}_{k=1}^\infty$  is uniformly bounded in  $L^1(B_3)$ . Hence  $\{\tilde{v}_k\}_{k=1}^\infty$  is uniformly bounded in  $W^{1,H_0}(B_3)$  and also  $\{v_k\}_{k=1}^\infty$  is uniformly bounded in  $W^{1,H_0}(B_3^+)$ . From the above remark, there exist subsequences, which we still denote by  $\{\tilde{v}_k\}_{k=1}^\infty$  and  $\{v_k\}_{k=1}^\infty$ ,  $v_0 \in W^{1,H_0}(B_3)$  and  $v_\infty \in W^{1,H_0}(B_3^+)$  such that

$$\tilde{v}_k \rightharpoonup v_0 \text{ weakly in } W^{1,H_0}(B_3), \quad \tilde{v}_k \rightarrow v_0 \text{ strongly in } L^{H_0}(B_3), \quad (4.46)$$

and

$$v_k \rightharpoonup v_\infty \text{ weakly in } W^{1,H_0}(B_3^+), \quad v_k \rightarrow v_\infty \text{ strongly in } L^{H_0}(B_3^+). \quad (4.47)$$

We note from (4.42), (4.43), (4.44) and (4.47) that  $v_\infty$  is a solution to (4.39) when  $r = 1$ , see [10]. Furthermore, it follows from the weakly lower semicontinuity, (4.42), (4.44) and (4.47) that

$$\begin{aligned} \int_{B_2^+} H_0(Dv_\infty) dx &\leq \left(\frac{3}{2}\right)^n \int_{B_3^+} H_0(Dv_\infty) dx \\ &\leq \left(\frac{3}{2}\right)^n \liminf_{k \rightarrow \infty} \int_{B_3^+} H_0(Dv_k) dx \\ &= \left(\frac{3}{2}\right)^n \liminf_{k \rightarrow \infty} \frac{|\Omega_3^k|}{|B_3^+|} \int_{\Omega_3^k} H_0(Dv_k) dx \leq \left(\frac{3}{2}\right)^n \tilde{c}. \end{aligned}$$

For the zero extension  $\tilde{v}_\infty$  of  $v_\infty$  from  $B_3^+$  to  $B_3$ , we note that

$$\tilde{v}_\infty = v_0 \quad \text{and then} \quad D\tilde{v}_\infty = Dv_0 \quad \text{a.e. in } B_3, \quad (4.48)$$

since  $v_k \rightarrow v_\infty$  a.e. in  $B_3^+$  and  $\tilde{v}_k \rightarrow v_0$  a.e. in  $B_3$ .

We now choose a cut-off function  $\zeta \in C_0^\infty(B_2)$  such that  $0 \leq \zeta \leq 1$ ,  $\zeta \equiv 1$  in  $B_1$  and  $|D\zeta| \leq 2$ . Then we find that

$$\begin{aligned} &\int_{B_1} \langle A_0(D\tilde{v}_k) - A_0(Dv_0), D\tilde{v}_k - Dv_0 \rangle dx \\ &\leq \int_{B_2} \zeta \langle A_0(D\tilde{v}_k) - A_0(Dv_0), D\tilde{v}_k - Dv_0 \rangle dx \end{aligned}$$

$$\begin{aligned}
&= \int_{B_2} \zeta \langle A_0(D\tilde{v}_k), D\tilde{v}_k - Dv_0 \rangle dx + \int_{B_2} \zeta \langle A_0(Dv_0), D\tilde{v}_k - Dv_0 \rangle dx \\
&=: I_1 + I_2.
\end{aligned} \tag{4.49}$$

We first estimate  $I_1$  as follows:

$$\begin{aligned}
I_1 &= \int_{B_2} \zeta \langle A_0(D\tilde{v}_k), D\tilde{v}_k - Dv_0 \rangle dx \\
&= \int_{B_2} \langle A_0(D\tilde{v}_k), D(\zeta(\tilde{v}_k - v_0)) \rangle dx - \int_{B_2} \langle A_0(D\tilde{v}_k), D\zeta \rangle (\tilde{v}_k - v_0) dx \\
&= \int_{\Omega_2^k} \langle A_0(Dv_k), D(\zeta(\tilde{v}_k - v_0)) \rangle dx - \int_{B_2} \langle A_0(D\tilde{v}_k), D\zeta \rangle (\tilde{v}_k - v_0) dx \\
&= - \int_{B_2} \langle A_0(D\tilde{v}_k), D\zeta \rangle (\tilde{v}_k - v_0) dx,
\end{aligned}$$

since one can take  $\zeta(\tilde{v}_k - v_0) \in W_0^{1,p}(\Omega_2^k)$  as a test function in the problem (4.43). From (3.24), (3.3) and (3.5), we see that

$$\begin{aligned}
H_0^*(|A_0(D\tilde{v}_k)|) &\leq H_0^*\left(L \frac{H_0(|D\tilde{v}_k|)}{|D\tilde{v}_k|}\right) \\
&\leq c(L^{\kappa_1^*} + L^{\kappa_2^*}) H_0^*\left(\frac{H_0(|D\tilde{v}_k|)}{|D\tilde{v}_k|}\right) \\
&\leq c(L^{\kappa_1^*} + L^{\kappa_2^*}) H_0(D\tilde{v}_k),
\end{aligned} \tag{4.50}$$

for some positive constants  $\kappa_1^*$  and  $\kappa_2^*$  depending only on  $p$  and  $a_0$ . Then it follows from the Luxemburg norm property, (4.50) and (4.44) that

$$\begin{aligned}
\|A_0(D\tilde{v}_k)\|_{L^{H_0^*}(B_2)} &\leq 1 + \int_{B_2} H_0^*(|A_0(D\tilde{v}_k)|) dx \\
&\leq 1 + c \int_{B_3} H_0^*(|A_0(D\tilde{v}_k)|) dx \\
&\leq 1 + c \int_{B_3} H_0(D\tilde{v}_k) dx \leq 1 + c\tilde{c},
\end{aligned} \tag{4.51}$$

where  $c$  is a constant depending on  $n$ ,  $p$ ,  $L$  and  $a_0$ . Hence we obtain

$$\begin{aligned}
 |I_1| &= \left| \int_{B_2} \langle A_0(D\tilde{v}_k), D\zeta \rangle (\tilde{v}_k - v_0) dx \right| \\
 &\leq 2 \int_{B_2} |A_0(D\tilde{v}_k)| |\tilde{v}_k - v_0| dx \\
 &\leq 4|B_2| \|A_0(D\tilde{v}_k)\|_{L^{H_0^*}(B_2)} \|\tilde{v}_k - v_0\|_{L^{H_0}(B_2)} \\
 &\leq 4|B_2|(1 + c\tilde{c}) \|\tilde{v}_k - v_0\|_{L^{H_0}(B_2)} \longrightarrow 0 \quad \text{as } k \rightarrow \infty,
 \end{aligned} \tag{4.52}$$

by (3.8), (4.46) and (4.51).

On the other hand, we deduce from (4.46) that

$$I_2 = \int_{B_2} \langle \zeta A_0(Dv_0), D\tilde{v}_k - Dv_0 \rangle dx \longrightarrow 0 \quad \text{as } k \rightarrow \infty. \tag{4.53}$$

Combining (4.49) with (4.52) and (4.53) yields

$$\lim_{k \rightarrow \infty} \int_{B_1} \langle A_0(D\tilde{v}_k) - A_0(Dv_0), D\tilde{v}_k - Dv_0 \rangle dx = 0.$$

We note from (2.4) with  $a(\cdot) = a_0 = 0$  that

$$\langle A_0(\xi) - A_0(\eta), \xi - \eta \rangle \geq \tilde{v} \left[ |V_p(\xi) - V_p(\eta)|^2 + a_0 |V_{\log}(\xi) - V_{\log}(\eta)|^2 \right]$$

for every  $\xi, \eta \in \mathbb{R}^n$ , where  $\tilde{v}$  is a positive constant depending on  $n$ ,  $p$  and  $v$ . Therefore, we conclude that

$$\lim_{k \rightarrow \infty} \int_{B_1} \left( |V_p(Dv_k) - V_p(Dv_0)|^2 + a_0 |V_{\log}(Dv_k) - V_{\log}(Dv_0)|^2 \right) dx = 0,$$

which is contrary to (4.45). This finishes the proof.  $\square$

The following lemma is a direct consequence of Lemma 3.10 and Lemma 4.4.

**Lemma 4.5.** *Let  $u \in W_0^{1,1}(\Omega)$  be the distributional solution to (2.1) with (2.17). Then for any  $\varepsilon > 0$ , there exists a constant  $\delta = \delta(n, p, L, \|a\|_{L^\infty(\Omega)}, \varepsilon) \in (0, \frac{1}{312})$  such that if (3.46) holds and if  $h_i \in W^{1,1}(\Omega_i^5)$  is the distributional solution to (4.1) with  $H(x, Dh_i) \in L^1(\Omega_i^5)$ ,  $w_i \in W^{1,1}(\Omega_i^4)$  is the distributional solution to (4.11) with  $H(x_{i,M}, Dw_i) \in L^1(\Omega_i^4)$  and  $v_i \in W^{1,1}(\Omega_i^3)$  is the distributional solution to (4.25) with  $H(x_{i,M}, Dv_i) \in L^1(\Omega_i^3)$ , then there exists a distributional solution  $\bar{v}_i \in W^{1,1}(B_i^{2+})$  to*

$$\begin{cases} \operatorname{div} \tilde{A}_0(D\bar{v}_i) = 0 & \text{in } B_i^{2+}, \\ \bar{v}_i = 0 & \text{on } T_i^2, \end{cases} \tag{4.54}$$

such that

$$\int_{\Omega_i^1} \left( |V_p(Dv_i) - V_p(D\bar{v}_i)|^2 + a(x_{i,M}) |V_{\log}(Dv_i) - V_{\log}(D\bar{v}_i)|^2 \right) dx \leq \varepsilon \lambda \quad (4.55)$$

and

$$\sup_{\Omega_i^1} H(x_{i,M}, D\bar{v}_i) \leq c\lambda \quad (4.56)$$

for some positive constant  $c = c(n, p, \nu, L, \|H(\cdot, F)\|_{L^1(\Omega)})$ , where  $\bar{v}_i$  is extended by zero from  $B_i^{2+}$  to  $B_i^2 \supset \Omega_i^2$ .

**Remark 4.6.** For the interior case, Lemma 4.1 and Lemma 4.2 with  $\Omega_i^j$  replaced by  $B_i^{j-1}$ , respectively, still hold. Furthermore, we have the interior Lipschitz regularity for the solution  $v_i$  to the reference problem (4.11). In fact, it follows from Lemma 3.9 and (4.26) that

$$\sup_{B_i^1} H(x_{i,M}, Dv_i) \leq c\lambda, \quad (4.57)$$

where  $x_{i,M} \in \overline{B_i^3}$  is a point such that

$$a(x_{i,M}) = \sup_{x \in B_i^3} a(x) \quad (4.58)$$

and  $c = c(n, p, \nu, L, \|H(\cdot, F)\|_{L^1(\Omega)})$  is a positive constant.

## 5. Global gradient estimates

We start with the following technical lemma, see [23, Lemma 4.3].

**Lemma 5.1.** Let  $\phi : [R_1, R_2] \rightarrow [0, \infty)$  be a bounded function. Suppose that for any  $s_1$  and  $s_2$  with  $0 < R_1 \leq s_1 < s_2 \leq R_2$ ,

$$\phi(s_1) \leq \vartheta \phi(s_2) + \frac{P}{(s_2 - s_1)^\kappa} + Q,$$

where  $P, Q \geq 0$ ,  $\kappa > 0$  and  $\vartheta \in [0, 1)$ . Then there holds

$$\phi(R_1) \leq c(\vartheta, \kappa) \left[ \frac{P}{(R_2 - R_1)^\kappa} + Q \right].$$

In this section we establish a global gradient estimate for the distributional solution to (2.1) with  $H(x, Du) \in L^1(\Omega)$ . We use the comparison results which we have shown in the previous section and apply the covering arguments which we have discussed in Section 3 to prove the main result as follows.

**Proof of Theorem 2.4.** We first choose a universal constant

$$\tilde{R} = \tilde{R}(n, p, v, L, \omega(\cdot), \|H(\cdot, F)\|_{L^1(\Omega)}, R_0, \varepsilon) > 0 \quad (5.1)$$

so that the conditions (3.27), (4.9) and (4.23) hold true. According to the covering argument presented in Subsection 3.5, there exists a countable family of disjoint sets  $\{\Omega_{\rho_i}(y_i)\}_{i=1}^{\infty}$  satisfying (3.34) and (3.35). For the interior case, we know from (2.16), Lemma 4.1, Lemma 4.2, Lemma 4.3 and Remark 4.6 that for any  $0 < \varepsilon < 1$ , there exists  $\delta = \delta(n, p, v, L, \|H(\cdot, F)\|_{L^1(\Omega)}, \varepsilon) > 0$  so that one can find  $h_i \in W^{1,1}(B_i^4)$ ,  $w_i \in W^{1,1}(B_i^3)$  and  $v_i \in W^{1,1}(B_i^2)$  satisfying

$$\begin{aligned} & \int_{B_i^4} \left( |V_p(Du) - V_p(Dh_i)|^2 + a(x) |V_{\log}(Du) - V_{\log}(Dh_i)|^2 \right) dx \\ & + \int_{B_i^3} \left( |V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M}) |V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2 \right) dx \\ & + \int_{B_i^2} \left( |V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M}) |V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2 \right) dx \\ & \leq \varepsilon \lambda \end{aligned} \quad (5.2)$$

and

$$\sup_{B_i^1} \left( |V_p(Dv_i)|^2 + a(x_{i,M}) |V_{\log}(Dv_i)|^2 \right) \leq c_1 \lambda, \quad (5.3)$$

where  $c_1 = c_1(n, p, v, L, \|H(\cdot, F)\|_{L^1(\Omega)})$  is a positive constant and  $x_{i,M} \in \overline{B_i^3}$  is a point such that  $a(x_{i,M}) = \sup_{x \in B_i^3} a(x)$ .

On the other hand, for the boundary case, we see from (2.16), Lemma 4.1, Lemma 4.2, Lemma 4.3, Lemma 4.4 and Lemma 4.5 that for any  $0 < \varepsilon < 1$ , there exists  $\delta = \delta(n, p, v, L, \|a\|_{L^\infty(\Omega)}, \|H(\cdot, F)\|_{L^1(\Omega)}, \varepsilon) > 0$  so that one can find  $h_i \in W^{1,1}(\Omega_i^5)$ ,  $w_i \in W^{1,1}(\Omega_i^4)$ ,  $v_i \in W^{1,1}(\Omega_i^3)$  and  $\bar{v}_i \in W^{1,1}(\Omega_i^2)$  satisfying

$$\begin{aligned} & \int_{\Omega_i^5} \left( |V_p(Du) - V_p(Dh_i)|^2 + a(x) |V_{\log}(Du) - V_{\log}(Dh_i)|^2 \right) dx \\ & + \int_{\Omega_i^4} \left( |V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M}) |V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2 \right) dx \\ & + \int_{\Omega_i^3} \left( |V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M}) |V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2 \right) dx \end{aligned}$$

$$\begin{aligned}
& + \int_{\Omega_i^1} \left( |V_p(Dv_i) - V_p(D\bar{v}_i)|^2 + a(x_{i,M}) |V_{\log}(Dv_i) - V_{\log}(D\bar{v}_i)|^2 \right) dx \\
& \leq \varepsilon \lambda
\end{aligned} \tag{5.4}$$

and

$$\sup_{\Omega_i^1} \left( |V_p(D\bar{v}_i)|^2 + a(x_{i,M}) |V_{\log}(D\bar{v}_i)|^2 \right) \leq c_2 \lambda, \tag{5.5}$$

where  $c_2 = c_2(n, p, v, L, \|H(\cdot, F)\|_{L^1(\Omega)})$  is a positive constant and  $x_{i,M} \in \overline{\Omega_i^4}$  is a point such that  $a(x_{i,M}) = \sup_{x \in \Omega_i^4} a(x)$ .

Consequently, for any  $0 < \varepsilon < 1$ , there exists a small constant

$$\delta = \delta(n, p, v, L, \|a\|_{L^\infty(\Omega)}, \|H(\cdot, F)\|_{L^1(\Omega)}, \varepsilon) > 0 \tag{5.6}$$

such that (5.2)–(5.5) hold true.

Let us now estimate  $|\Omega_{\rho_i}(y_i)|$ . It follows from (3.35) that

$$\begin{aligned}
\lambda |\Omega_{\rho_i}(y_i)| &= \int_{\Omega_{\rho_i}(y_i)} \left[ H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx \\
&\leq \int_{\{x \in \Omega_{\rho_i}(y_i) : H(x, Du(x)) > \frac{\lambda}{4}\}} H(x, Du(x)) dx + \frac{\lambda}{4} |\Omega_{\rho_i}(y_i)| \\
&\quad + \frac{1}{\delta} \int_{\{x \in \Omega_{\rho_i}(y_i) : H(x, F(x)) > \frac{\delta \lambda}{4}\}} H(x, F(x)) dx + \frac{\lambda}{4} |\Omega_{\rho_i}(y_i)|,
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{\lambda}{2} |\Omega_{\rho_i}(y_i)| &\leq \int_{\{x \in \Omega_{\rho_i}(y_i) : H(x, Du) > \frac{\lambda}{4}\}} H(x, Du) dx \\
&\quad + \frac{1}{\delta} \int_{\{x \in \Omega_{\rho_i}(y_i) : H(x, F) > \frac{\delta \lambda}{4}\}} H(x, F) dx.
\end{aligned} \tag{5.7}$$

For the interior case, using (2.16), (4.58) and (5.3), we have

$$\begin{aligned}
& 8c_1 \lambda |\{x \in B_{5\rho_i}(y_i) : H(x, Du) > 8c_1 \lambda\}| + \int_{\{x \in B_{5\rho_i}(y_i) : H(x, Du) > 8c_1 \lambda\}} H(x, Du) dx \\
& \leq 2 \int_{\{x \in B_{5\rho_i}(y_i) : H(x, Du) > 8c_1 \lambda\}} H(x, Du) dx
\end{aligned}$$



$$\begin{aligned}
 &\leq 2 \int_{\{x \in B_{5\rho_i}(y_i) : H(x, Du) > 8c_1\lambda\}} \left( |V_p(Du)|^2 + a(x)|V_{\log}(Du)|^2 \right) dx \\
 &\leq 8 \left[ \int_{B_{5\rho_i}(y_i)} \left( |V_p(Du) - V_p(Dh_i)|^2 + a(x)|V_{\log}(Du) - V_{\log}(Dh_i)|^2 \right) dx \right. \\
 &\quad + \int_{B_{5\rho_i}(y_i)} \left( |V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M})|V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2 \right) dx \\
 &\quad + \int_{B_{5\rho_i}(y_i)} \left( |V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M})|V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2 \right) dx \\
 &\quad \left. + \int_{\{x \in B_{5\rho_i}(y_i) : H(x, Du) > 8c_1\lambda\}} \left( |V_p(Dv_i)|^2 + a(x_{i,M})|V_{\log}(Dv_i)|^2 \right) dx \right] \\
 &\leq 8 \left[ \int_{B_i^4} \left( |V_p(Du) - V_p(Dh_i)|^2 + a(x)|V_{\log}(Du) - V_{\log}(Dh_i)|^2 \right) dx \right. \\
 &\quad + \int_{B_i^3} \left( |V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M})|V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2 \right) dx \\
 &\quad \left. + \int_{B_i^2} \left( |V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M})|V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2 \right) dx \right] \\
 &\quad + 8c_1\lambda |\{x \in B_{5\rho_i}(y_i) : H(x, Du) > 8c_1\lambda\}|, \tag{5.8}
 \end{aligned}$$

where we have used the following elementary inequality:

$$(t_1 + t_2 + \cdots + t_N)^2 \leq N(t_1^2 + t_2^2 + \cdots + t_N^2), \tag{5.9}$$

for any  $N \in \mathbb{N}$  and  $t_1, t_2, \dots, t_N \in \mathbb{R}$ . Then it follows from (5.8) and (5.2) that

$$\begin{aligned}
 &\int_{\{x \in B_{5\rho_i}(y_i) : H(x, Du) > 8c_1\lambda\}} H(x, Du) dx \\
 &\leq 8 \left[ \int_{B_i^4} \left( |V_p(Du) - V_p(Dh_i)|^2 + a(x)|V_{\log}(Du) - V_{\log}(Dh_i)|^2 \right) dx \right.
 \end{aligned}$$

$$\begin{aligned}
& + \int_{B_i^3} \left( |V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M}) |V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2 \right) dx \\
& + \int_{B_i^2} \left( |V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M}) |V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2 \right) dx \Bigg] \\
& \leq 8|B_i^4| \left[ \int_{B_i^4} \left( |V_p(Du) - V_p(Dh_i)|^2 + a(x) |V_{\log}(Du) - V_{\log}(Dh_i)|^2 \right) dx \right. \\
& \quad + \int_{B_i^3} \left( |V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M}) |V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2 \right) dx \\
& \quad \left. + \int_{B_i^2} \left( |V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M}) |V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2 \right) dx \right] \\
& \leq 8|B_i^4| \varepsilon \lambda \\
& = 8 \cdot 20^n |B_i^0| \varepsilon \lambda. \tag{5.10}
\end{aligned}$$

Now for the boundary case, by a similar argument, we deduce from (2.16), (4.5), (5.5) and (5.9) that

$$\begin{aligned}
& 10c_2\lambda |\{x \in \Omega_{5\rho_i}(y_i) : H(x, Du) > 10c_2\lambda\}| + \int_{\{x \in \Omega_{5\rho_i}(y_i) : H(x, Du) > 10c_2\lambda\}} H(x, Du) dx \\
& \leq 2 \int_{\{x \in \Omega_{5\rho_i}(y_i) : H(x, Du) > 10c_2\lambda\}} H(x, Du) dx \\
& \leq 2 \int_{\{x \in \Omega_{5\rho_i}(y_i) : H(x, Du) > 10c_2\lambda\}} \left( |V_p(Du)|^2 + a(x) |V_{\log}(Du)|^2 \right) dx \\
& \leq 10 \left[ \int_{\Omega_i^5} \left( |V_p(Du) - V_p(Dh_i)|^2 + a(x) |V_{\log}(Du) - V_{\log}(Dh_i)|^2 \right) dx \right. \\
& \quad + \int_{\Omega_i^4} \left( |V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M}) |V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2 \right) dx \\
& \quad \left. + \int_{\Omega_i^3} \left( |V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M}) |V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2 \right) dx \right]
\end{aligned}$$

$$\begin{aligned}
 & + \int_{\Omega_i^1} \left( |V_p(Dv_i) - V_p(D\bar{v}_i)|^2 + a(x_{i,M}) |V_{\log}(Dv_i) - V_{\log}(D\bar{v}_i)|^2 \right) dx \Bigg] \\
 & + 10c_2\lambda |\{x \in \Omega_{5\rho_i}(y_i) : H(x, Du) > 10c_2\lambda\}|. \tag{5.11}
 \end{aligned}$$

Then we obtain from (5.11) and (5.4) that

$$\begin{aligned}
 & \int_{\{x \in \Omega_{5\rho_i}(y_i) : H(x, Du) > 10c_2\lambda\}} H(x, Du) dx \\
 & \leq 10 \left[ \int_{\Omega_i^5} \left( |V_p(Du) - V_p(Dh_i)|^2 + a(x) |V_{\log}(Du) - V_{\log}(Dh_i)|^2 \right) dx \right. \\
 & \quad + \int_{\Omega_i^4} \left( |V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M}) |V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2 \right) dx \\
 & \quad + \int_{\Omega_i^3} \left( |V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M}) |V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2 \right) dx \\
 & \quad \left. + \int_{\Omega_i^1} \left( |V_p(Dv_i) - V_p(D\bar{v}_i)|^2 + a(x_{i,M}) |V_{\log}(Dv_i) - V_{\log}(D\bar{v}_i)|^2 \right) dx \right] \\
 & \leq 10|\Omega_i^5| \left[ \int_{\Omega_i^5} \left( |V_p(Du) - V_p(Dh_i)|^2 + a(x) |V_{\log}(Du) - V_{\log}(Dh_i)|^2 \right) dx \right. \\
 & \quad + \int_{\Omega_i^4} \left( |V_p(Dh_i) - V_p(Dw_i)|^2 + a(x_{i,M}) |V_{\log}(Dh_i) - V_{\log}(Dw_i)|^2 \right) dx \\
 & \quad + \int_{\Omega_i^3} \left( |V_p(Dw_i) - V_p(Dv_i)|^2 + a(x_{i,M}) |V_{\log}(Dw_i) - V_{\log}(Dv_i)|^2 \right) dx \\
 & \quad \left. + \int_{\Omega_i^1} \left( |V_p(Dv_i) - V_p(D\bar{v}_i)|^2 + a(x_{i,M}) |V_{\log}(Dv_i) - V_{\log}(D\bar{v}_i)|^2 \right) dx \right] \\
 & \leq 10|\Omega_i^5| \varepsilon \lambda \\
 & \leq 10 \cdot 300^n |\Omega_i^0| \varepsilon \lambda, \tag{5.12}
 \end{aligned}$$

where for the last inequality we have used the fact that

$$|\Omega_i^5| \leq |B_{130\rho_i}(y_i)| = 130^n |B_{\rho_i}(y_i)| \leq 130^n \left(\frac{16}{7}\right)^n |\Omega_{\rho_i}(y_i)| \leq 300^n |\Omega_i^0|,$$

which follows from (3.44) and the measure density condition (2.12).

Consequently, in either case, we deduce from (5.10) and (5.12) that

$$\int_{\{x \in \Omega_{5\rho_i}(y_i) : H(x, Du) > c_0\lambda\}} H(x, Du) dx \leq 10 \cdot 300^n \varepsilon \lambda |\Omega_{\rho_i}(y_i)|, \quad (5.13)$$

where  $c_0 = \max\{8c_1, 10c_2\}$  is a universal constant. Combining (5.7) and (5.13) gives

$$\begin{aligned} & \int_{\{x \in \Omega_{5\rho_i}(y_i) : H(x, Du) > c_0\lambda\}} H(x, Du) dx \\ & \leq 20 \cdot 300^n \varepsilon \left[ \int_{\{x \in \Omega_{\rho_i}(y_i) : H(x, Du) > \frac{\lambda}{4}\}} H(x, Du) dx \right. \\ & \quad \left. + \frac{1}{\delta} \int_{\{x \in \Omega_{\rho_i}(y_i) : H(x, F) > \frac{\delta\lambda}{4}\}} H(x, F) dx \right]. \quad (5.14) \end{aligned}$$

For the sake of simplicity as in (3.28), let us denote the upper level set of  $H(x, F)$  by

$$\Xi(\lambda, s) := \{x \in \Omega_s : H(x, F(x)) > \lambda\}, \quad \frac{R}{2} \leq s \leq R, \quad \lambda > 0. \quad (5.15)$$

Since the family  $\{\Omega_{\rho_i}(y_i)\}_{i=1}^\infty$  is disjoint, it follows from (3.34), (3.37) and (5.14) that

$$\begin{aligned} & \int_{E(c_0\lambda, r_1)} H(x, Du) dx \leq \sum_{i \geq 1} \int_{\{x \in \Omega_{5\rho_i}(y_i) : H(x, Du) > c_0\lambda\}} H(x, Du) dx \\ & \leq 300^{n+1} \varepsilon \left[ \sum_{i \geq 1} \int_{\{x \in \Omega_{\rho_i}(y_i) : H(x, Du) > \frac{\lambda}{4}\}} H(x, Du) dx \right. \\ & \quad \left. + \frac{1}{\delta} \sum_{i \geq 1} \int_{\{x \in \Omega_{\rho_i}(y_i) : H(x, F) > \frac{\delta\lambda}{4}\}} H(x, F) dx \right] \end{aligned}$$

$$\begin{aligned}
 &= 300^{n+1} \varepsilon \left[ \int_{\bigcup_{i \geq 1} \{x \in \Omega_{\rho_i}(y_i) : H(x, Du) > \frac{\lambda}{4}\}} H(x, Du) dx \right. \\
 &\quad \left. + \frac{1}{\delta} \int_{\bigcup_{i \geq 1} \{x \in \Omega_{\rho_i}(y_i) : H(x, F) > \frac{\delta \lambda}{4}\}} H(x, F) dx \right] \\
 &\leq 300^{n+1} \varepsilon \left[ \int_{E\left(\frac{\lambda}{4}, r_2\right)} H(x, Du) dx + \frac{1}{\delta} \int_{\Xi\left(\frac{\delta \lambda}{4}, r_2\right)} H(x, F) dx \right].
 \end{aligned}$$

After a change of variable with respect to  $\lambda$ , we conclude that

$$\int_{E(\lambda, r_1)} H(x, Du) dx \leq 300^{n+1} \varepsilon \left[ \int_{E\left(\frac{\lambda}{4c_0}, r_2\right)} H(x, Du) dx + \frac{1}{\delta} \int_{\Xi\left(\frac{\delta \lambda}{4c_0}, r_2\right)} H(x, F) dx \right], \quad (5.16)$$

whenever  $\lambda > c_0 \lambda_0$ , where  $\lambda_0$  has been defined in (3.31).

To estimate  $H(x, Du)^\gamma$ , we recall that Fubini's theorem yields

$$(\gamma - 1) \int_0^M \lambda^{\gamma-2} \int_{E(\lambda, r_1)} H(x, Du) dx d\lambda = \int_{\Omega_{r_1}} H(x, Du) [H(x, Du)_M]^{\gamma-1} dx$$

for any  $M > 0$ , where  $H(x, Du)_M := \min\{H(x, Du), M\}$  is the truncated function of  $H(x, Du)$ . Here, we note that the right-hand side of the above identity is finite, as  $H(x, Du) \in L^1(\Omega)$  and the truncated function  $H(x, Du)_M$  is bounded. Then, for  $M > c_0 \lambda_0$ , it follows from (5.16) that

$$\begin{aligned}
 &\int_{\Omega_{r_1}} H(x, Du) [H(x, Du)_M]^{\gamma-1} dx \\
 &= (\gamma - 1) \int_0^{c_0 \lambda_0} \lambda^{\gamma-2} \int_{E(\lambda, r_1)} H(x, Du) dx d\lambda \\
 &\quad + (\gamma - 1) \int_{c_0 \lambda_0}^M \lambda^{\gamma-2} \int_{E(\lambda, r_1)} H(x, Du) dx d\lambda \\
 &\leq (\gamma - 1) \int_0^{c_0 \lambda_0} \lambda^{\gamma-2} d\lambda \int_{\Omega_R} H(x, Du) dx
 \end{aligned}$$

$$\begin{aligned}
& + 300^{n+1} \varepsilon \left[ (\gamma - 1) \int_{c_0 \lambda_0}^M \lambda^{\gamma-2} \int_{E\left(\frac{\lambda}{4c_0}, r_2\right)} H(x, Du) dx d\lambda \right. \\
& \quad \left. + \frac{1}{\delta} \cdot (\gamma - 1) \int_{c_0 \lambda_0}^M \lambda^{\gamma-2} \int_{\Xi\left(\frac{\delta \lambda}{4c_0}, r_2\right)} H(x, F) dx d\lambda \right] \\
& \leq (c_0 \lambda_0)^{\gamma-1} \int_{\Omega_R} H(x, Du) dx \\
& \quad + 300^{n+1} \varepsilon \left[ (\gamma - 1) \int_0^M \lambda^{\gamma-2} \int_{E\left(\frac{\lambda}{4c_0}, r_2\right)} H(x, Du) dx d\lambda \right. \\
& \quad \left. + \frac{1}{\delta} \cdot (\gamma - 1) \int_0^\infty \lambda^{\gamma-2} \int_{\Xi\left(\frac{\delta \lambda}{4c_0}, r_2\right)} H(x, F) dx d\lambda \right]. \quad (5.17)
\end{aligned}$$

Utilizing a change of variable and Fubini's theorem, we can calculate the last double integrals in the above display as follows:

$$\begin{aligned}
& (\gamma - 1) \int_0^M \lambda^{\gamma-2} \int_{E\left(\frac{\lambda}{4c_0}, r_2\right)} H(x, Du) dx d\lambda \\
& = (4c_0)^{\gamma-1} (\gamma - 1) \int_0^{\frac{M}{4c_0}} \lambda^{\gamma-2} \int_{E(\lambda, r_2)} H(x, Du) dx d\lambda \\
& \leq (4c_0)^{\gamma-1} (\gamma - 1) \int_0^M \lambda^{\gamma-2} \int_{E(\lambda, r_2)} H(x, Du) dx d\lambda \\
& = (4c_0)^{\gamma-1} \int_{\Omega_{r_2}} H(x, Du) [H(x, Du)_M]^{\gamma-1} dx, \quad (5.18)
\end{aligned}$$

and

$$\begin{aligned}
 & \frac{1}{\delta} \cdot (\gamma - 1) \int_0^\infty \lambda^{\gamma-2} \int_{\Xi\left(\frac{\delta\lambda}{4c_3}, r_2\right)} H(x, F) dx d\lambda \\
 &= \frac{1}{\delta} \left(\frac{4c_0}{\delta}\right)^{\gamma-1} (\gamma - 1) \int_0^\infty \lambda^{\gamma-2} \int_{\Xi(\lambda, r_2)} H(x, F) dx d\lambda \\
 &= \frac{1}{\delta} \left(\frac{4c_0}{\delta}\right)^{\gamma-1} \int_{\Omega_{r_2}} [H(x, F)]^\gamma dx.
 \end{aligned} \tag{5.19}$$

Combining (5.17) with (5.18) and (5.19) gives

$$\begin{aligned}
 & \int_{\Omega_{r_1}} H(x, Du) [H(x, Du)_M]^{\gamma-1} dx \\
 & \leq (c_0 \lambda_0)^{\gamma-1} |\Omega_R| \int_{\Omega_R} H(x, Du) dx \\
 & \quad + 300^{n+1} (4c_0)^{\gamma-1} \varepsilon \left[ \int_{\Omega_{r_2}} H(x, Du) [H(x, Du)_M]^{\gamma-1} dx + \frac{1}{\delta^\gamma} \int_{\Omega_{r_2}} [H(x, F)]^\gamma dx \right].
 \end{aligned}$$

We now take  $\varepsilon = \varepsilon(n, p, v, L, \|H(\cdot, F)\|_{L^1(\Omega)}, \gamma) \in (0, 1)$  small enough in order to obtain

$$\begin{aligned}
 & \int_{\Omega_{r_1}} H(x, Du) [H(x, Du)_M]^{\gamma-1} dx \\
 & \leq \frac{1}{2} \int_{\Omega_{r_2}} H(x, Du) [H(x, Du)_M]^{\gamma-1} dx \\
 & \quad + (c_0 \lambda_0)^{\gamma-1} |\Omega_R| \int_{\Omega_R} H(x, Du) dx + \frac{1}{\delta^\gamma} \int_{\Omega_R} [H(x, F)]^\gamma dx.
 \end{aligned}$$

Note that once  $\varepsilon \equiv \varepsilon(n, p, v, L, \|H(\cdot, F)\|_{L^1(\Omega)}, \gamma) \in (0, 1)$  is chosen, one can find the corresponding constants  $\tilde{R} \equiv \tilde{R}(n, p, v, L, \omega(\cdot), \|H(\cdot, F)\|_{L^1(\Omega)}, \gamma, R_0) > 0$  and  $\delta \equiv \delta(n, p, v, L, \|a\|_{L^\infty(\Omega)}, \|H(\cdot, F)\|_{L^1(\Omega)}, \gamma) > 0$ , see (5.1) and (5.6) respectively. Recalling the definition of  $\lambda_0$  in (3.31), we see that

$$\begin{aligned}
 \lambda_0 &= \frac{400^n r_2^n}{(r_2 - r_1)^n} \int_{\Omega_{r_2}} \left[ H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx \\
 &= \frac{400^n r_2^n}{(r_2 - r_1)^n} \frac{|B_{r_2}|}{|\Omega_{r_2}|} \frac{|B_R|}{|B_{r_2}|} \frac{|\Omega_R|}{|B_R|} \int_{\Omega_R} \left[ H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{400^n r_2^n}{(r_2 - r_1)^n} \left(\frac{16}{7}\right)^n \frac{R^n}{r_2^n} \int_{\Omega_R} \left[ H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx \\
&\leq \frac{10^{3n} R^n}{(r_2 - r_1)^n} \int_{\Omega_R} \left[ H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\int_{\Omega_{r_1}} H(x, Du) [H(x, Du)_M]^{\gamma-1} dx \\
&\leq \frac{1}{2} \int_{\Omega_{r_2}} H(x, Du) [H(x, Du)_M]^{\gamma-1} dx \\
&\quad + \frac{(10^{3n} c_0)^{\gamma-1} R^{n(\gamma-1)} |\Omega_R|}{(r_2 - r_1)^{n(\gamma-1)}} \left( \int_{\Omega_R} \left[ H(x, Du) + \frac{1}{\delta} H(x, F) \right] dx \right)^\gamma \\
&\quad + \frac{1}{\delta^\gamma} \int_{\Omega_R} [H(x, F)]^\gamma dx.
\end{aligned}$$

We now apply [Lemma 5.1](#) when

$$\phi(s) = \int_{\Omega_s} H(x, Du) [H(x, Du)_M]^{\gamma-1} dx, \quad \kappa = n(\gamma - 1) > 0 \quad \text{and} \quad \vartheta = \frac{1}{2},$$

to discover that

$$\begin{aligned}
\int_{\Omega_{R/2}} H(x, Du) [H(x, Du)_M]^{\gamma-1} dx &\leq c |\Omega_R| \left( \int_{\Omega_R} [H(x, Du) + H(x, F)] dx \right)^\gamma \\
&\quad + c \int_{\Omega_R} [H(x, F)]^\gamma dx.
\end{aligned}$$

Hence we have

$$\int_{\Omega_{R/2}} H(x, Du) [H(x, Du)_M]^{\gamma-1} dx \leq c \left( \int_{\Omega_R} H(x, Du) dx \right)^\gamma + c \int_{\Omega_R} [H(x, F)]^\gamma dx,$$

for some positive constant  $c = c(n, p, \nu, L, \|a\|_{L^\infty(\Omega)}, \|H(\cdot, F)\|_{L^1(\Omega)}, \gamma)$ . Using Fatou's lemma, we obtain a local estimate up to the boundary as follows:



$$\int_{\Omega_{R/2}} [H(x, Du)]^\gamma dx \leq c \left( \int_{\Omega_R} H(x, Du) dx \right)^\gamma + c \int_{\Omega_R} [H(x, F)]^\gamma dx, \quad (5.20)$$

which holds for every  $0 < R \leq \tilde{R}$ .

Since  $\Omega$  is bounded in  $\mathbb{R}^n$ , there exists a finite family of balls  $\{B_{\tilde{R}/2}(x_j)\}_{j=1}^N$  with  $x_j \in \Omega$  for  $j = 1, \dots, N$  which covers  $\Omega$ . This clearly forces

$$\int_{\Omega} [H(x, Du)]^\gamma dx \leq \sum_{j=1}^N \int_{\Omega_{\tilde{R}/2}(x_j)} [H(x, Du)]^\gamma dx. \quad (5.21)$$

Using the local estimate (5.20) with  $R = \tilde{R}$ , the standard energy estimate (3.18), Hölder's inequality and the measure density condition (2.12), we deduce that for each  $j = 1, \dots, N$ ,

$$\begin{aligned} \int_{\Omega_{\tilde{R}/2}(x_j)} [H(x, Du)]^\gamma dx &\leq c |\Omega_{\tilde{R}}(x_j)|^{1-\gamma} \left( \int_{\Omega_{\tilde{R}}(x_j)} H(x, Du) dx \right)^\gamma + c \int_{\Omega_{\tilde{R}}(x_j)} [H(x, F)]^\gamma dx \\ &\leq c |\Omega_{\tilde{R}}(x_j)|^{1-\gamma} \left( \int_{\Omega} H(x, Du) dx \right)^\gamma + c \int_{\Omega} [H(x, F)]^\gamma dx \\ &\leq c |\Omega_{\tilde{R}}(x_j)|^{1-\gamma} \left( \int_{\Omega} H(x, F) dx \right)^\gamma + c \int_{\Omega} [H(x, F)]^\gamma dx \\ &\leq c (|\Omega_{\tilde{R}}(x_j)|^{1-\gamma} |\Omega|^{\gamma-1} + 1) \int_{\Omega} [H(x, F)]^\gamma dx \\ &\leq c (|B_{\tilde{R}}(x_j)|^{1-\gamma} |\Omega|^{\gamma-1} + 1) \int_{\Omega} [H(x, F)]^\gamma dx. \end{aligned} \quad (5.22)$$

We note that the constant  $c \equiv c(n, p, \nu, L, \|a\|_{L^\infty(\Omega)}, \|H(\cdot, F)\|_{L^1(\Omega)}, \gamma)$  in the above display is independent of  $j$ . Then it follows from (5.21) and (5.22) that

$$\begin{aligned} \int_{\Omega} [H(x, Du)]^\gamma dx &\leq \sum_{j=1}^N c (|B_{\tilde{R}}(x_j)|^{1-\gamma} |\Omega|^{\gamma-1} + 1) \int_{\Omega} [H(x, F)]^\gamma dx \\ &= cN (|B_{\tilde{R}}|^{1-\gamma} |\Omega|^{\gamma-1} + 1) \int_{\Omega} [H(x, F)]^\gamma dx \\ &\leq c \int_{\Omega} [H(x, F)]^\gamma dx, \end{aligned}$$

for some positive constant  $c = c(n, p, \nu, L, \omega(\cdot), \|a\|_{L^\infty(\Omega)}, \|H(\cdot, F)\|_{L^1(\Omega)}, \gamma, R_0, \Omega)$ . This is the desired conclusion (2.19).  $\square$

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