



# Large deviation principles for 3D stochastic primitive equations

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## Abstract

In this paper, we establish the Freidlin–Wentzell’s large deviations for 3D stochastic primitive equations with small noise perturbation. The weak convergence approach plays an important role.

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## 1. Introduction

The large-scale motion of the ocean can be well modeled by 3D viscous primitive equations, which are derived from the Navier–Stokes equations, with rotation, coupled with thermodynamics and salinity diffusion-transport equations, by assuming two important simplifications: Boussinesq approximation and the hydrostatic balance (see [16,17,21]). As a fundamental model in meteorology, this model has been intensively investigated because of the interests stemmed

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from physics and mathematics. For example, the mathematical study of the primitive equations originated in a series of articles by Lions, Temam and Wang in the early 1990s (see [16–19]), where they set up the mathematical framework and showed the global existence of weak solutions. Cao and Titi developed a beautiful approach to dealing with the  $L^6$ -norm of the fluctuation  $\tilde{v}$  of horizontal velocity and obtained the global well-posedness for the 3D viscous primitive equations in [3].

Along with the great successful developments of these deterministic primitive equations, the random situation has also been developed rapidly. In [15], Guo and Huang obtained the existence of universal random attractor of strong solution for the equations that the momentum equation is driven by an additive stochastic forcing and the thermodynamical equation is driven by a fixed heat source. Debussche, Glatt-Holtz, Temam and Ziane established the global well-posedness of strong solution if the primitive equations are driven by multiplicative stochastic forcing in [7]. In [9], we proved the existence of global weak solutions for 3D stochastic primitive equations driven by regular multiplicative noise, and also obtained the exponential mixing property for weak solutions which are limits of spectral Galerkin approximations.

In this paper, we are concerned with the Freidlin–Wentzell’s large deviation principle (LDP) for the stochastic primitive equations, which deals with path probability asymptotic behavior for stochastic dynamical systems with small noise. An important tool for studying the Freidlin–Wentzell’s LDP is the weak convergence approach, which is developed by Dupuis and Ellis in [10]. The key idea of this approach is to prove some variational representation formula about the Laplace transform of bounded continuous functionals, which will lead to the proof of equivalence between LDP and Laplace principle. In particular, for Brownian functionals, an elegant variational representation formula has been established by Boué and Dupuis [1], Budhiraja and Dupuis [2].

In the past two decades, there are numerous important results about LDP for stochastic partial differential equations (SPDEs). For example, Cardon-Weber [5] studied Burgers’ type SPDEs and achieved their LDP in 1999. In 2004, Cerrai and Röckner established LDP for stochastic reaction–diffusion equations with nonlinear reaction term in [6]. In 2009, under very general conditions, Liu [20] obtained the LDP for SPDEs with monotone coefficients and small multiplicative noise, which covers many models such as stochastic reaction–diffusion equations, stochastic porous media equations and fast diffusion equations, and the stochastic  $p$ -Laplace equation in Hilbert space. Gao and Sun [12] proved LDP of weak solution of the two dimensional primitive equations on the state space  $C([0, T]; H)$ .

The purpose of this paper is to establish LDP of the strong solution of 3D stochastic primitive equations by using the weak convergence method. Note that in [12] the authors considered the weak solution in two dimensional case, they used the state space  $C([0, T]; H)$  which is not suitable for our case. To deal with this problem, we change the state space to be  $C([0, T]; H^1)$  (see definition below), which leads to some higher Sobolev norm estimates and much more complicated calculation. For instance, during the procedure to prove the global well-posedness of skeleton equation, some non-trivial estimates, such as  $L^{10}$ ,  $H^1$  estimates are needed. Moreover, it’s worth mentioning that our result is obtained without adding any additional regular conditions on the noise, only those in [7] are enough. Our main result is

**Theorem 1.1.** *Suppose **Hypothesis H0** holds. Then for any  $Y_0 \in V$ ,  $\{Y^\varepsilon\}$  satisfies the large deviation principle on  $C([0, T]; V) \cap L^2([0, T]; D(A))$  with a good rate function given by (4.19).*

**Hypothesis H0** and all the above symbols will be given in the following section.

This paper is organized as follows. The mathematical framework for the stochastic primitive equations is in Sects. 2 and 3. The Freidlin–Wentzell’s large deviations and the weak convergence method are introduced in Sect. 4. The global well-posedness of skeleton solution and a priori estimates are given in Sect. 5. Finally, the large deviation principle is established in Sect. 6.

## 2. Preliminaries

Let  $D$  be a smooth bounded open domain in  $\mathbb{R}^2$ . Set  $\mathcal{O} = D \times (-1, 0)$ . Consider the 3D primitive equations on  $\mathcal{O} \times [0, T]$  driven by a stochastic forcing in a Cartesian system

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v + \theta \frac{\partial v}{\partial z} + f k \times v + \nabla P + L_1 v = \psi_1(t, v, T) \frac{dW_1}{dt}, \quad (2.1)$$

$$\partial_z P + T = 0, \quad (2.2)$$

$$\nabla \cdot v + \partial_z \theta = 0, \quad (2.3)$$

$$\frac{\partial T}{\partial t} + (v \cdot \nabla)T + \theta \frac{\partial T}{\partial z} + L_2 T = \psi_2(t, v, T) \frac{dW_2}{dt}, \quad (2.4)$$

where the horizontal velocity field  $v = (v_1, v_2)$ , the vertical velocity field  $\theta$ , the temperature  $T$  and the pressure  $P$  are all unknown functionals.  $f$  is the Coriolis parameter.  $k$  is vertical unit vector.  $W_1$  and  $W_2$  are two independent cylindrical Winner processes which will be given in Sect. 3.  $\nabla = (\partial_x, \partial_y)$ ,  $\Delta = \partial_x^2 + \partial_y^2$ . The viscosity and the heat diffusion operators  $L_1$  and  $L_2$  are given by

$$L_1 v = -A_h \Delta v - A_v \frac{\partial^2 v}{\partial z^2},$$

$$L_2 T = -K_h \Delta T - K_v \frac{\partial^2 T}{\partial z^2},$$

where  $A_h, A_v$  are positive molecular viscosities and  $K_h, K_v$  are positive conductivity constants. Without loss of generality, we assume that

$$A_h = A_v = K_h = K_v = 1.$$

We impose the same boundary conditions as [7],

$$\partial_z v = 0, \quad \theta = 0, \quad \partial_z T = 0 \quad \text{on } D \times \{0\} = \Gamma_u, \quad (2.5)$$

$$\partial_z v = 0, \quad \theta = 0, \quad \partial_z T = 0 \quad \text{on } D \times \{-1\} = \Gamma_b, \quad (2.6)$$

$$v = 0, \quad \frac{\partial T}{\partial n} = 0 \quad \text{on } \partial D \times [-1, 0] = \Gamma_l, \quad (2.7)$$

where  $n$  is the outward normal vector to  $\Gamma_l$ .

Integrating (2.3) from  $-1$  to  $z$  and using (2.5), (2.6), we have

$$\theta(t, x, y, z) := \Phi(v)(t, x, y, z) = - \int_{-1}^z \nabla \cdot v(t, x, y, z') dz', \quad (2.8)$$

moreover,

$$\int_{-1}^0 \nabla \cdot v dz = 0.$$

Integrating (2.2) from  $-1$  to  $z$ , set  $p_b$  be a certain unknown function at  $\Gamma_b$  satisfying

$$P(x, y, z, t) = p_b(x, y, t) - \int_{-1}^z T(x, y, z', t) dz'.$$

Then, (2.1)–(2.4) can be rewritten as

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v + \Phi(v) \frac{\partial v}{\partial z} + f k \times v + \nabla p_b - \int_{-1}^z \nabla T dz' + L_1 v = \psi_1(t, v, T) \frac{dW_1}{dt}, \quad (2.9)$$

$$\frac{\partial T}{\partial t} + (v \cdot \nabla)T + \Phi(v) \frac{\partial T}{\partial z} + L_2 T = \psi_2(t, v, T) \frac{dW_2}{dt}, \quad (2.10)$$

$$\int_{-1}^0 \nabla \cdot v dz = 0. \quad (2.11)$$

The boundary value conditions for (2.9)–(2.11) are given by

$$\partial_z v = 0, \quad \partial_z T = 0 \quad \text{on } \Gamma_u, \quad (2.12)$$

$$\partial_z v = 0, \quad \partial_z T = 0 \quad \text{on } \Gamma_b, \quad (2.13)$$

$$v = 0, \quad \frac{\partial T}{\partial n} = 0 \quad \text{on } \Gamma_l. \quad (2.14)$$

Denote  $Y = (v, T)$  and the initial condition

$$Y(0) = Y_0 = (v_0, T_0). \quad (2.15)$$

### 3. Formulation of the SPDE

#### 3.1. Some functional spaces

Let  $\mathcal{L}(K_1; K_2)$  (resp.  $\mathcal{L}_2(K_1; K_2)$ ) be the space of bounded (resp. Hilbert–Schmidt) linear operators from the Hilbert space  $K_1$  to  $K_2$ , whose norm is denoted by  $\|\cdot\|_{\mathcal{L}(K_1; K_2)}$  ( $\|\cdot\|_{\mathcal{L}_2(K_1; K_2)}$ ). For  $p \in \mathbb{Z}^+$ , set

$$|\phi|_p = \begin{cases} \left( \int_{\mathcal{O}} |\phi(x, y, z)|^p dx dy dz \right)^{\frac{1}{p}}, & \forall \phi \in L^p(\mathcal{O}), \\ \left( \int_D |\phi(x, y)|^p dx dy \right)^{\frac{1}{p}}, & \forall \phi \in L^p(D). \end{cases}$$

In particular,  $|\cdot|$  and  $(\cdot, \cdot)$  represent norm and inner product of  $L^2(\mathcal{O})$  (or  $L^2(D)$ ), respectively. For the classical Sobolev space  $H^m(\mathcal{O})$ ,  $m \in \mathbb{N}_+$ ,

$$\begin{cases} H^m(\mathcal{O}) = \left\{ U \mid \partial_\alpha U \in (L^2(\mathcal{O}))^3, \text{ for } |\alpha| \leq m \right\}, \\ |U|_{H^m(\mathcal{O})}^2 = \sum_{0 \leq |\alpha| \leq m} |\partial_\alpha U|^2. \end{cases}$$

It's known that  $(H^m(\mathcal{O}), |\cdot|_{H^m(\mathcal{O})})$  is a Hilbert space.  $|\cdot|_{H^p(D)}$  stands for the norm of  $H^p(D)$  for  $p \in \mathbb{Z}^+$ .

Define the working spaces for (2.9)–(2.15)

$$\begin{aligned} \mathcal{V}_1 &:= \left\{ v \in (C^\infty(\mathcal{O}))^2; \frac{\partial v}{\partial z} \Big|_{\Gamma_u, \Gamma_b} = 0, v \Big|_{\Gamma_l} = 0, \int_{-1}^0 \nabla \cdot v dz = 0 \right\}, \\ \mathcal{V}_2 &:= \left\{ T \in C^\infty(\mathcal{O}); \frac{\partial T}{\partial z} \Big|_{\Gamma_u} = 0, \frac{\partial T}{\partial z} \Big|_{\Gamma_b} = 0, \frac{\partial T}{\partial n} \Big|_{\Gamma_l} = 0 \right\}, \end{aligned}$$

$V_1$  = the closure of  $\mathcal{V}_1$  with respect to the norm  $|\cdot|_{H^1(\mathcal{O})} \times |\cdot|_{H^1(\mathcal{O})}$ ,

$V_2$  = the closure of  $\mathcal{V}_2$  with respect to the norm  $|\cdot|_{H^1(\mathcal{O})}$ ,

$H_1$  = the closure of  $\mathcal{V}_1$  with respect to the norm  $|\cdot| \times |\cdot|$ ,

$H_2$  = the closure of  $\mathcal{V}_2$  with respect to the norm  $|\cdot|$ ,

$V = V_1 \times V_2$ ,  $H = H_1 \times H_2$ .

The inner products and norms on  $V$ ,  $H$  are given by

$$\begin{aligned} (Y, \tilde{Y})_V &= (v, \tilde{v})_{V_1} + (T, \tilde{T})_{V_2}, \\ (Y, \tilde{Y}) &= (v, \tilde{v}) + (T, \tilde{T}) = (v^{(1)}, \tilde{v}^{(1)}) + (v^{(2)}, \tilde{v}^{(2)}) + (T, \tilde{T}), \\ \|Y\|_V &= (Y, Y)_V^{\frac{1}{2}} = (v, v)_{V_1}^{\frac{1}{2}} + (T, T)_{V_2}^{\frac{1}{2}}, \end{aligned}$$

where  $Y = (v, T)$ ,  $\tilde{Y} = (\tilde{v}, \tilde{T}) \in V$ , and  $(\cdot, \cdot)$  denotes the inner product in  $L^2(\mathcal{O})$ .

### 3.2. Some functionals

Define three bilinear operators  $a : V \times V \rightarrow \mathbb{R}$ ,  $a_1 : V_1 \times V_1 \rightarrow \mathbb{R}$ ,  $a_2 : V_2 \times V_2 \rightarrow \mathbb{R}$ , and their corresponding linear operators  $A : V \rightarrow V'$ ,  $A_1 : V_1 \rightarrow V_1'$ ,  $A_2 : V_2 \rightarrow V_2'$  as follows, for any  $Y = (v, T)$ ,  $\tilde{Y} = (\tilde{v}, \tilde{T}) \in V$ ,

$$a(Y, \tilde{Y}) := (AY, \tilde{Y}) = a_1(v, \tilde{v}) + a_2(T, \tilde{T}),$$

where

$$\begin{aligned} a_1(v, \tilde{v}) &:= (A_1 v, \tilde{v}) = \int_{\mathcal{O}} \left( \nabla v \cdot \nabla \tilde{v} + \frac{\partial v}{\partial z} \cdot \frac{\partial \tilde{v}}{\partial z} \right) dx dy dz \\ &= \int_{\mathcal{O}} \left( \frac{\partial v}{\partial x} \cdot \frac{\partial \tilde{v}}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{\partial \tilde{v}}{\partial y} + \frac{\partial v}{\partial z} \cdot \frac{\partial \tilde{v}}{\partial z} \right) dx dy dz \end{aligned}$$

$$= \int_{\mathcal{O}} \left( \frac{\partial v_1}{\partial x} \frac{\partial \tilde{v}_1}{\partial x} + \frac{\partial v_2}{\partial x} \frac{\partial \tilde{v}_2}{\partial x} + \frac{\partial v_1}{\partial y} \frac{\partial \tilde{v}_1}{\partial y} + \frac{\partial v_2}{\partial y} \frac{\partial \tilde{v}_2}{\partial y} + \frac{\partial v_1}{\partial z} \frac{\partial \tilde{v}_1}{\partial z} + \frac{\partial v_2}{\partial z} \frac{\partial \tilde{v}_2}{\partial z} \right) dx dy dz,$$

and

$$\begin{aligned} a_2(T, \tilde{T}) &:= (A_2 T, \tilde{T}) = \int_{\mathcal{O}} \left( \nabla T \cdot \nabla \tilde{T} + \frac{\partial T}{\partial z} \frac{\partial \tilde{T}}{\partial z} \right) dx dy dz \\ &= \int_{\mathcal{O}} \left( \frac{\partial T}{\partial x} \frac{\partial \tilde{T}}{\partial x} + \frac{\partial T}{\partial y} \frac{\partial \tilde{T}}{\partial y} + \frac{\partial T}{\partial z} \frac{\partial \tilde{T}}{\partial z} \right) dx dy dz. \end{aligned}$$

The following lemma follows from Lemma 2.4 in [17].

**Lemma 3.1.**

- (i) The operators  $a$ ,  $a_i$  ( $i = 1, 2$ ) are coercive, continuous, and therefore, the operators  $A : V \rightarrow V'$  and  $A_i : V_i \rightarrow V'_i$  ( $i = 1, 2$ ) are isomorphisms. Moreover,

$$\begin{aligned} a(Y, \tilde{Y}) &\leq C_1 \|Y\|_V \|\tilde{Y}\|_V, \\ a(Y, Y) &\geq C_2 \|Y\|_V^2, \end{aligned}$$

where  $C_1$  and  $C_2$  are two positive constants depending on boundary conditions.

- (ii) The isomorphism  $A : V \rightarrow V'$  (respectively  $A_i : V_i \rightarrow V'_i$  ( $i = 1, 2$ )) can be extended to a self-adjoint unbounded linear operator on  $H$  (respectively on  $H_i$ ,  $i = 1, 2$ ), with compact inverse  $A^{-1} : H \rightarrow H$  (respectively  $A_i^{-1} : H_i \rightarrow H_i$  ( $i = 1, 2$ )).

It's known that  $A_1$  is a self-adjoint operator with discrete spectrum in  $H_1$ . Denote by  $\{k_n\}_{n=1,2,\dots}$  the eigenbasis of  $A_1$  and suppose its associated eigenvalues  $\{v_n\}_{n=1,2,\dots}$  is increasing. Similarly,  $A_2$  is a self-adjoint operator with discrete spectrum in  $H_2$ . Let  $\{l_n\}_{n=1,2,\dots}$  be the eigenbasis of  $A_2$  with increasing corresponding eigenvalues  $\{\lambda_n\}_{n=1,2,\dots}$ . Denote  $\bar{e}_{n,0} = \begin{pmatrix} k_n \\ 0 \end{pmatrix}$

and  $\bar{e}_{0,m} = \begin{pmatrix} 0 \\ l_m \end{pmatrix}$ , it is easy to know  $\{\bar{e}_{n,0}, \bar{e}_{0,m}\}_{n,m=1,2,\dots}$  is an eigenbasis of  $(A, D(A))$ . By means of rearrangement, we can construct an eigenbasis of  $(A, D(A))$  denoted  $\{e_n\}_{n=1,2,\dots}$  such that the associated eigenvalues  $\{\mu_n\}_{n=1,2,\dots}$  is an increasing sequence.

For any  $s \in \mathbb{R}$ , the fractional power  $(A^s, D(A^s))$  of the operator  $(A, D(A))$  is defined as

$$\begin{cases} D(A^s) = \left\{ Y = \sum_{n=1}^{\infty} y_n e_n \mid \sum_{n=1}^{\infty} \mu_n^{2s} |y_n|^2 < \infty \right\}; \\ A^s Y = \sum_{n=1}^{\infty} \mu_n^s y_n e_n, \quad \text{where } Y = \sum_{n=1}^{\infty} y_n e_n. \end{cases}$$

Set

$$\|Y\|_s^A = |A^{\frac{s}{2}}Y|, \quad \mathbb{H}_s^A = D(A^{\frac{s}{2}}),$$

then  $(\mathbb{H}_0^A, \|\cdot\|_0^A) = (H, |\cdot|)$  and  $(\mathbb{H}_1^A, \|\cdot\|_1^A) = (V, \|\cdot\|_V)$ . For simplicity, denote  $\|\cdot\| = \|\cdot\|_V$ . It's obvious that  $(\mathbb{H}_s^A, \|\cdot\|_s^A)$  is a Hilbert space. Thanks to the regularity theory of the Stokes operator,  $\mathbb{H}_s^A$  is a closed subset of  $H^s(\mathcal{O})$  and  $\|\cdot\|_s^A$  is equivalent to the usual norm  $|\cdot|_{H^s(\mathcal{O})}$  for  $s \leq 2$ . Similarly, we can define  $(\mathbb{H}_s^{A_1}, \|\cdot\|_s^{A_1})$  and  $(\mathbb{H}_s^{A_2}, \|\cdot\|_s^{A_2})$ . For convenience, all of them will be denoted by  $(\mathbb{H}_s, \|\cdot\|_s)$ .

Now, we define three mappings  $b : V \times V \times V \rightarrow \mathbb{R}$ ,  $b_i : V_1 \times V_i \times V_i \rightarrow \mathbb{R}$  ( $i = 1, 2$ ) and the associated operators  $B : V \times V \rightarrow V'$ ,  $B_i : V_1 \times V_i \rightarrow V_i'$  ( $i = 1, 2$ ) by setting

$$\begin{aligned} b(Y, \tilde{Y}, \hat{Y}) &:= (B(Y, \tilde{Y}), \hat{Y}) = b_1(v, \tilde{v}, \hat{v}) + b_2(v, \tilde{T}, \hat{T}), \\ b_1(v, \tilde{v}, \hat{v}) &:= (B_1(v, \tilde{v}), \hat{v}) = \int_{\mathcal{O}} \left[ (v \cdot \nabla) \tilde{v} + \Phi(v) \frac{\partial \tilde{v}}{\partial z} \right] \cdot \hat{v} dx dy dz, \\ b_2(v, \tilde{T}, \hat{T}) &:= (B_2(v, \tilde{T}), \hat{T}) = \int_{\mathcal{O}} \left[ (v \cdot \nabla) \tilde{T} + \Phi(v) \frac{\partial \tilde{T}}{\partial z} \right] \hat{T} dx dy dz, \end{aligned}$$

for any  $Y = (v, T)$ ,  $\tilde{Y} = (\tilde{v}, \tilde{T})$ ,  $\hat{Y} = (\hat{v}, \hat{T}) \in V$ .

By integration by parts and condition (2.11), we deduce that

**Lemma 3.2.** For any  $Y, \tilde{Y} \in V$ ,

$$(B(Y, \tilde{Y}), \tilde{Y}) = b(Y, \tilde{Y}, \tilde{Y}) = b_1(v, \tilde{v}, \tilde{v}) + b_2(v, \tilde{T}, \tilde{T}) = 0.$$

Moreover, we define another mapping  $g : V \times V \rightarrow \mathbb{R}$  and the associated linear operator  $G : V \rightarrow V'$  by

$$\begin{aligned} g(Y, \tilde{Y}) &:= (G(Y), \tilde{Y}) \\ &= \int_{\mathcal{O}} \left[ f(k \times v) \cdot \tilde{v} + (\nabla p_b - \int_{-1}^z \nabla T dz') \cdot \tilde{v} \right] dx dy dz. \end{aligned}$$

By (2.11), we have

$$(v, \nabla p_b) = \left( \int_{-1}^0 v dz, \nabla p_b \right)_{L^2(D)} = - \left( p_b, \int_{-1}^0 \nabla \cdot v dz \right)_{L^2(D)} = 0.$$

Clearly,  $(v, f k \times v) = 0$ . Utilizing the Cauchy–Schwarz inequality, we obtain

**Lemma 3.3.**

(i)

$$g(Y, Y) = (G(Y), Y) = - \int_{\mathcal{O}} \left[ \left( \int_{-1}^z \nabla T dz' \right) \cdot v \right] dx dy dz.$$

(ii) *There exists a constant  $C$  independent of  $Y$  and  $\tilde{Y}$ , such that*

$$|(G(Y), Y)| \leq C(\|T\| \|v\| \vee \|T\| \|v\|), \quad (3.16)$$

$$|(G(Y), \tilde{Y})| \leq C\|v\| \|\tilde{v}\| + C(\|T\| \|\tilde{v}\| \vee \|T\| \|\tilde{v}\|). \quad (3.17)$$

Using the above functionals, we can rewrite (2.9)–(2.15) as

$$\begin{cases} dY(t) + AY(t)dt + B(Y(t), Y(t))dt + G(Y(t))dt = \psi(t, Y(t))dW(t), \\ Y(0) = Y_0, \end{cases} \quad (3.18)$$

where

$$W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \quad \psi(t, Y(t)) = \begin{pmatrix} \psi_1(t, Y(t)) & 0 \\ 0 & \psi_2(t, Y(t)) \end{pmatrix}.$$

**3.3. Some inequalities**

Let us recall some interpolation inequalities to be used later (see Sect. 4.1 in [15]).

For  $h \in H^1(D)$ ,  $D \subset \mathbb{R}^2$ ,

$$|h|_4 \leq c|h|_2^{\frac{1}{2}}|h|_{H^1(D)}^{\frac{1}{2}},$$

$$|h|_5 \leq c|h|_3^{\frac{3}{5}}|h|_{H^1(D)}^{\frac{2}{5}},$$

$$|h|_6 \leq c|h|_4^{\frac{2}{3}}|h|_{H^1(D)}^{\frac{1}{3}}.$$

For  $h \in H^1(\mathcal{O})$ ,  $\mathcal{O} = D \times (-1, 0)$ ,

$$|h|_3 \leq c|h|^{\frac{1}{2}}|h|_{H^1(\mathcal{O})}^{\frac{1}{2}},$$

$$|h|_4 \leq c|h|^{\frac{1}{4}}|h|_{H^1(\mathcal{O})}^{\frac{3}{4}},$$

$$|h|_6 \leq c|h|_{H^1(\mathcal{O})},$$

$$|h|_{\infty} \leq c|h|_{H^1(\mathcal{O})}^{\frac{1}{2}}|h|_{H^2(\mathcal{O})}^{\frac{1}{2}}.$$

Referring to page 17 in [3] and Proposition 2.2 in [4], we have



**Lemma 3.4.** Assume  $u, f, g$  be some smooth functions, then

- (i)  $|\int_{\mathcal{O}} g \cdot [(u \cdot \nabla) f] dx dy dz| \leq c |\nabla f| \|g\|_3 |u|_6 \leq c |\nabla f| \|g\|^{\frac{1}{2}} |\nabla g|^{\frac{1}{2}} |\nabla u|,$
- (ii)  $|\int_{\mathcal{O}} \Phi(u) f \cdot g dx dy dz| \leq c |\nabla u| \|g\|^{\frac{1}{2}} |\nabla g|^{\frac{1}{2}} |f|^{\frac{1}{2}} |\nabla f|^{\frac{1}{2}},$
- (iii)  $|\int_{\mathcal{O}} \Phi(u) f \cdot g dx dy dz| \leq c |f| |\nabla u|^{\frac{1}{2}} \|u\|_2^{\frac{1}{2}} |\nabla g|^{\frac{1}{2}} |g|^{\frac{1}{2}}.$

### 3.4. Definition of strong solution

For the strong solution of (3.18), we shall fix a single stochastic basis  $\mathcal{T} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P, W)$ . Here,

$$W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$$

is a cylindrical Brownian motion of the form  $W(t, \omega) = \sum_{i \geq 1} r_i w_i(t, \omega)$ , where  $\{r_i\}_{i \geq 1}$  is a complete orthonormal basis of a Hilbert space

$$U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

and  $\{w_i\}_{i \geq 1}$  is a sequence of independent standard one-dimensional real-valued Brownian motions on  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ .

Given any pair of Banach spaces  $\mathcal{X}$  and  $\mathcal{Y}$ ,  $Bnd_u(\mathcal{X}, \mathcal{Y})$  stands for the collection of all continuous mappings  $\psi : [0, \infty) \times \mathcal{X} \rightarrow \mathcal{Y}$  such that

$$\|\psi(t, x)\|_{\mathcal{Y}} \leq c(1 + \|x\|_{\mathcal{X}}), \quad x \in \mathcal{X}, \quad t \geq 0,$$

where the numerical constant  $c$  is independent of  $t$ . If, in addition,

$$\|\psi(t, x) - \psi(t, y)\|_{\mathcal{Y}} \leq c\|x - y\|_{\mathcal{X}}, \quad x, y \in \mathcal{X}, \quad t \geq 0,$$

we say  $\psi$  is in  $Lip_u(\mathcal{X}, \mathcal{Y})$ .

**Hypothesis H0** Assume  $\psi : [0, \infty) \times H \rightarrow \mathcal{L}_2(U; H)$  satisfies

$$\psi \in Lip_u(H, \mathcal{L}_2(U; H)) \cap Lip_u(V, \mathcal{L}_2(U; V)) \cap Bnd_u(V, \mathcal{L}_2(U; D(A))).$$

Now, we recall definition of strong solution to (3.18) from [7].

**Definition 3.1.** Let  $\mathcal{T} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P, W)$  be a fixed stochastic basis and suppose that  $Y_0 \in V$ .  $Y$  is called a strong solution of (3.18) if  $Y(\cdot)$  is an  $\mathcal{F}_t$ -adapted process in  $V$ , such that

$$Y(\cdot) \in L^2(\Omega; C([0, T]; V)) \bigcap L^2(\Omega; L^2([0, T]; D(A))), \quad \forall T > 0,$$

and for every  $t \geq 0$ ,

$$Y(t) + \int_0^t \left( AY + B(Y, Y) + G(Y) \right) ds = Y_0 + \int_0^t \psi(s, Y(s)) dW(s),$$

holds in  $V'$ ,  $P$ -a.s.

Referring to [7], it gives

**Theorem 3.1.** Assume *Hypothesis H0* holds. For each  $Y_0 \in V$ , there exists a unique global solution  $Y$  of (3.18) in the sense of Definition 3.1 with  $Y(0) = Y_0$ .

#### 4. Freidlin–Wentzell’s large deviations

In this section, we recall the weak convergence approach introduced by Budhiraja and Dupuis in [2]. Let us first recall some standard definitions and results from large deviation theory (see [8]).

Let  $\{Y^\varepsilon\}$  be a family random variables defined on a probability space  $(\Omega, \mathcal{F}, P)$  taking values in some Polish space  $\mathcal{E}$ .

**Definition 4.1.** (Rate Function) A function  $I : \mathcal{E} \rightarrow [0, \infty]$  is called a rate function if  $I$  is lower semicontinuous. A rate function  $I$  is called a good rate function if the level set  $\{x \in \mathcal{E} : I(x) \leq M\}$  is compact for each  $M < \infty$ .

#### Definition 4.2.

- (i) (LDP) The sequence  $\{Y^\varepsilon\}$  is said to satisfy the large deviation principle with rate function  $I$  if for each Borel subset  $A$  of  $\mathcal{E}$ ,

$$-\inf_{x \in A^\circ} I(x) \leq \liminf_{\varepsilon \rightarrow 0} \varepsilon \log P(Y^\varepsilon \in A) \leq \limsup_{\varepsilon \rightarrow 0} \varepsilon \log P(Y^\varepsilon \in A) \leq -\inf_{x \in \bar{A}} I(x),$$

where  $A^\circ$  and  $\bar{A}$  denote the interior and closure of  $A$  in  $\mathcal{E}$ , respectively.

- (ii) (Laplace principle) The sequence  $\{Y^\varepsilon\}$  is said to satisfy the Laplace principle with rate function  $I$  if for each bounded continuous real-valued function  $f$  defined on  $\mathcal{E}$ ,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \log E \left\{ \exp \left[ -\frac{1}{\varepsilon} f(Y^\varepsilon) \right] \right\} = -\inf_{x \in \mathcal{E}} \{f(x) + I(x)\}.$$

It’s well-known that the large deviation principle and the Laplace principle are equivalent if  $\mathcal{E}$  is a Polish space and the rate function is good (see [8]).

Suppose  $W(t)$  is a cylindrical Wiener process on a Hilbert space  $U$  defined on a probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, P)$  (the paths of  $W$  take values in  $C([0, T]; \mathcal{U})$ , where  $\mathcal{U}$  is another Hilbert space such that the embedding  $U \subset \mathcal{U}$  is Hilbert–Schmidt). To state the criterion obtained by Budhiraja et al. in [2], we introduce the following notions.

$$\mathcal{A} = \{ \phi : \phi \text{ is a } U\text{-valued } \{\mathcal{F}_t\}\text{-predictable process s.t. } \int_0^T |\phi(s)|_U^2 ds < \infty \text{ } P\text{-a.s.} \},$$

$$T_M = \{h \in L^2([0, T]; U) : \int_0^T |h(s)|_U^2 ds \leq M\},$$

$$\mathcal{A}_M = \{\phi \in \mathcal{A} : \phi(\omega) \in T_M, \text{ } P\text{-a.s.}\}.$$

Throughout the whole paper, we use the weak topology on  $L^2([0, T]; U)$ , under which  $T_M$  is a compact space.

Suppose  $\mathcal{G}^\varepsilon : C([0, T]; U) \rightarrow \mathcal{E}$  is a measurable mapping and  $Y^\varepsilon = \mathcal{G}^\varepsilon(W)$ . Now, we list the following sufficient conditions for the Laplace principle (equivalently, large deviation principle) of  $Y^\varepsilon$  as  $\varepsilon \rightarrow 0$ .

**Hypothesis H1** Let  $\mathcal{G}^0 : C([0, T]; U) \rightarrow \mathcal{E}$  be a measurable mapping.

- (i) For every  $M < \infty$ , let  $\{h_\varepsilon : \varepsilon > 0\} \subset \mathcal{A}_M$ . If  $h_\varepsilon$  converges to  $h$  as  $T_M$ -valued random elements in distribution, then  $\mathcal{G}^\varepsilon(W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot h_\varepsilon(s) ds)$  converges in distribution to  $\mathcal{G}^0(\int_0^\cdot h(s) ds)$ .
- (ii) For every  $M < \infty$ , the set  $K_M = \{\mathcal{G}^0(\int_0^\cdot h(s) ds) : h \in T_M\}$  is compact subset of  $\mathcal{E}$ .

The following result is due to Budhiraja et al. in [2].

**Theorem 4.1.** Suppose **Hypothesis H1** holds, then  $Y^\varepsilon$  satisfies a large deviation principle on  $\mathcal{E}$  with the good rate function  $I$  given by

$$I(f) = \inf_{\{h \in L^2([0, T]; U) : f = \mathcal{G}^0(\int_0^\cdot h(s) ds)\}} \left\{ \frac{1}{2} \int_0^T |h(s)|_U^2 ds \right\}, \quad \forall f \in \mathcal{E}. \quad (4.19)$$

By convention,  $I(\emptyset) = \infty$ .

## 5. Priori estimates

Consider the following SPDE driven by multiplicative noise

$$\begin{cases} dY^\varepsilon(t) + AY^\varepsilon(t)dt + B(Y^\varepsilon(t), Y^\varepsilon(t))dt + G(Y^\varepsilon(t))dt = \sqrt{\varepsilon}\psi(t, Y^\varepsilon)dW(t), \\ Y^\varepsilon(0) = Y_0 \in V. \end{cases} \quad (5.20)$$

Under **Hypothesis H0**, by [Theorem 3.1](#), there exists a pathwise unique strong solution of (5.20) in  $\mathfrak{R} := C([0, T]; V) \cap L^2([0, T]; D(A))$ . The norm in  $\mathfrak{R}$  is

$$\|Y\|_{\mathfrak{R}}^2 := \sup_{0 \leq t \leq T} \|Y(t)\|^2 + \int_0^T \|Y(t)\|_{D(A)}^2 dt.$$

Therefore, there exist Borel-measurable functions

$$\mathcal{G}^\varepsilon : C([0, T]; U) \rightarrow \mathfrak{R} \text{ such that } Y^\varepsilon(\cdot) = \mathcal{G}^\varepsilon(W(\cdot)). \quad (5.21)$$

In order to prove large deviation principle for  $Y^\varepsilon$ , we need to find a measurable mapping  $\mathcal{G}^0$  satisfies **Hypothesis H1**.

For  $h \in L^2([0, T]; U)$ , consider the following skeleton equation

$$\begin{cases} dY_h(t) + AY_h(t)dt + B(Y_h(t), Y_h(t))dt + G(Y_h(t))dt = \psi(t, Y_h(t))h(t)dt, \\ Y_h(0) = Y_0. \end{cases} \quad (5.22)$$

Denote

$$Y_h = (v_h, T_h)^T, \quad h = (h_1, h_2)^T,$$

then, (5.22) can be rewritten as

$$\begin{aligned} dv_h + [(v_h \cdot \nabla)v_h + \Phi(v_h)\frac{\partial v_h}{\partial z}]dt + (fk \times v_h + \nabla p_b - \int_{-1}^z \nabla T_h dz')dt + A_1 v_h dt \\ = \psi_1(t, Y_h)h_1(t)dt, \end{aligned} \quad (5.23)$$

$$dT_h + [(v_h \cdot \nabla)T_h + \Phi(v_h)\frac{\partial T_h}{\partial z}]dt + A_2 T_h dt = \psi_2(t, Y_h)h_2(t)dt. \quad (5.24)$$

### 5.1. Global well-posedness of skeleton equation

Firstly, we prove the global well-posedness of the skeleton equation (5.23)–(5.24).

**Theorem 5.1.** Assume **Hypothesis H0** holds. For any  $Y_0 = (v_0, T_0) \in V$ ,  $h \in T_M$ , (5.22) has a unique strong solution  $Y_h \in C([0, T]; V) \cap L^2([0, T]; D(A))$  on  $[0, T]$ , which depends continuously on the initial data  $Y_0$ .

Similar to [14] and [24], for initial data  $Y_0 = (v_0, T_0) \in V$ , we can prove the existence of a strong solution  $Y_h = (v_h, T_h)^T$  of (5.22) for a short interval of time, whose length depends on the initial data and the other physical parameters of the system (5.23)–(5.24). Let  $[0, \mathcal{T})$  be the maximal interval of existence of the strong solution, we will establish a priori estimates in  $H^1$  of this strong solution in this interval. In order to prove  $H^1$  norm of the strong solution, we repeat and partially refine some calculations in [3].

#### 5.1.1. A priori estimates in $H$

Taking the inner product of (5.22) with  $Y_h$  in  $L^2(\mathcal{O})$ , we obtain

$$\frac{1}{2}d|Y_h|^2 + (|\nabla Y_h|^2 + |\partial_z Y_h|^2)dt = -(B(Y_h, Y_h), Y_h)dt - (G(Y_h), Y_h)dt + (\psi(t, Y_h)h, Y_h)dt.$$

From Lemma 3.2 and Lemma 3.3, we deduce

$$\frac{1}{2}d|Y_h|^2 + (|\nabla Y_h|^2 + |\partial_z Y_h|^2)dt \leq C|Y_h||Y_h|dt + C|Y_h||\psi(t, Y_h)h|dt.$$

By the Cauchy–Schwarz inequality and the Young inequality, we have

$$\frac{1}{2}d|Y_h|^2 + (|\nabla Y_h|^2 + |\partial_z Y_h|^2)dt \leq \frac{1}{2}\|Y_h\|^2 dt + C|Y_h|^2 dt + C|\psi(t, Y_h)h|^2 dt.$$

Moreover, it follows from **Hypothesis H0** that

$$\begin{aligned} |\psi(t, Y_h)h|^2 &\leq \|\psi(t, Y_h)\|_{\mathcal{L}_2(U; H)}^2 |h|_U^2 \\ &\leq C(1 + |Y_h|^2) |h|_U^2. \end{aligned} \quad (5.25)$$

As a result, we have

$$d|Y_h|^2 + \|Y_h\|^2 dt \leq C(1 + |h|_U^2) |Y_h|^2 dt + C|h|_U^2 dt. \quad (5.26)$$

Applying Gronwall inequality to (5.26), we get

$$\sup_{t \in [0, T]} |Y_h(t)|^2 \leq C_1(|Y_0|^2, M), \quad (5.27)$$

and

$$\sup_{t \in [0, T]} |Y_h(t)|^2 + \int_0^T \|Y_h(t)\|^2 dt \leq K_1(|Y_0|^2, M), \quad (5.28)$$

where

$$\begin{aligned} C_1(|Y_0|^2, M) &= C(1 + M)e^{C(1+M)}(|Y_0|^2 + CM), \\ K_1(|Y_0|^2, M) &= C(1 + M)^2 e^{C(1+M)}(|Y_0|^2 + CM). \end{aligned}$$

### 5.1.2. Splitting

In order to obtain  $H^1$  estimates, we follow the same method as [3]. Let

$$\bar{v}_h(x, y, t) = \int_{-1}^0 v_h(x, y, z', t) dz'$$

and the fluctuation

$$\tilde{v}_h = v_h - \bar{v}_h, \quad h = (h_1, h_2)^T.$$

Referring to (32) in [3], we obtain

$$\begin{aligned} \frac{\partial \tilde{v}_h}{\partial t} - \Delta \tilde{v}_h + (\tilde{v}_h \cdot \nabla) \tilde{v}_h + [(\tilde{v}_h \cdot \nabla) \tilde{v}_h + (\nabla \cdot \tilde{v}_h) \tilde{v}_h] + \nabla p_s(x, y, t) + f k \times \tilde{v}_h \\ - \nabla \left[ \int_{-1}^0 \int_{-1}^z T_h(x, y, z', t) dz' dz \right] = \int_{-1}^0 \psi_1(t, Y_h(t)) h_1(t) dz, \end{aligned} \quad (5.29)$$

$$\begin{aligned} \nabla \cdot \tilde{v}_h &= 0, \text{ in } D, \\ \tilde{v}_h &= 0, \text{ on } \partial D. \end{aligned}$$

By subtracting (5.29) from (5.23),  $\tilde{v}_h$  satisfies

$$\begin{aligned} \frac{\partial \tilde{v}_h}{\partial t} + A_1 \tilde{v}_h + (\tilde{v}_h \cdot \nabla) \tilde{v}_h - \left( \int_{-1}^z \nabla \cdot \tilde{v}_h(x, y, z', t) dz' \right) \frac{\partial \tilde{v}}{\partial z} + (\tilde{v}_h \cdot \nabla) \tilde{v}_h + (\tilde{v}_h \cdot \nabla) \tilde{v}_h + f k \times \tilde{v}_h \\ - [(\tilde{v}_h \cdot \nabla) \tilde{v}_h + (\nabla \cdot \tilde{v}_h) \tilde{v}_h] - \nabla \left( \int_{-1}^z T_h(x, y, z', t) dz' - \int_{-1}^0 \int_{-1}^z T_h(x, y, z', t) dz' dz \right) \\ = \psi_1(t, Y_h(t)) h_1(t) - \int_{-1}^0 \psi_1(t, Y_h(t)) h_1(t) dz, \end{aligned} \quad (5.30)$$

$$\frac{\partial \tilde{v}_h}{\partial z} \Big|_{z=0} = 0, \quad \frac{\partial \tilde{v}_h}{\partial z} \Big|_{z=-1} = 0, \quad \tilde{v}_h \cdot n|_{\Gamma_l} = 0, \quad \tilde{v}_h|_{\Gamma_l} = 0.$$

### 5.1.3. Some other estimates

(1)  $L^6(\mathcal{O})$  estimates of  $\tilde{v}_h$ . Taking the inner product of (5.30) with  $|\tilde{v}_h|^4 \tilde{v}_h$  in  $L^2(\mathcal{O})$  and referring to Sect. 3.2 in [3], we obtain

$$\begin{aligned} \frac{d|\tilde{v}_h|_6^6}{dt} + 2 \int_{\mathcal{O}} (|\tilde{v}_h|^2 |\nabla \tilde{v}_h|^2 + |\tilde{v}_h|^4 |\nabla \tilde{v}_h|^2) dx dy dz \\ + 2 \int_{\mathcal{O}} (|\tilde{v}_h|^2 |\partial_z \tilde{v}_h|^2 + |\tilde{v}_h|^4 |\partial_z \tilde{v}_h|^2) dx dy dz \\ \leq C |\tilde{v}_h|^2 |\nabla \tilde{v}_h|^2 |\tilde{v}_h|^6 + C |\tilde{v}_h|^6 |\nabla \tilde{v}_h|^2 + C |\tilde{T}_h|^2 |\nabla \tilde{T}_h|^2 + C |\tilde{v}_h|^2 |\tilde{v}_h|^6 \\ + \left| \int_{\mathcal{O}} \left( \psi_1 h_1(t) - \int_{-1}^0 \psi_1 h_1(t) dz \right) \cdot |\tilde{v}_h|^4 \tilde{v}_h dx dy dz \right|. \end{aligned} \quad (5.31)$$

In this case, we only need to estimate

$$\left| \int_{\mathcal{O}} \left( \psi_1 h_1(t) - \int_{-1}^0 \psi_1 h_1(t) dz \right) \cdot |\tilde{v}_h|^4 \tilde{v}_h dx dy dz \right|.$$

Using the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \left| \int_{\mathcal{O}} \left( \psi_1 h_1(t) - \int_{-1}^0 \psi_1 h_1(t) dz \right) \cdot |\tilde{v}_h|^4 \tilde{v}_h dx dy dz \right| \\ \leq \left( \int_{\mathcal{O}} |\psi_1 h_1(t) - \int_{-1}^0 \psi_1 h_1(t) dz|^2 dx dy dz \right)^{\frac{1}{2}} \left( \int_{\mathcal{O}} |\tilde{v}_h|^{10} dx dy dz \right)^{\frac{1}{2}} \\ := I_1(t) I_2(t). \end{aligned}$$

Applying **Hypothesis H0** and (5.28), we get

$$\begin{aligned} I_1(t) &\leq C|\psi_1(t, Y_h)h_1(t)| \\ &\leq C(1 + |Y_h(t)|)|h_1(t)|_U \\ &\leq C(1 + \sup_{t \in [0, T]} |Y_h(t)|)|h_1(t)|_U \\ &\leq C|h_1(t)|_U. \end{aligned}$$

Utilizing the Sobolev inequality  $|u|_{L^{\frac{10}{3}}(\mathcal{O})} \leq C\|u\|^{\frac{3}{5}}|u|^{\frac{2}{5}}$ , we have

$$\begin{aligned} I_2^2(t) &= |\tilde{v}_h|_{10}^{10} = \|\tilde{v}_h\|^3|_{L^{\frac{10}{3}}(\mathcal{O})} \\ &\leq C\|\tilde{v}_h\|^3\|\tilde{v}_h\|^{\frac{4}{3}} \\ &\leq C\|\tilde{v}_h\|^{\frac{4}{3}}(|\tilde{v}_h|^3 + |\nabla|\tilde{v}_h|^3|^2 + |\partial_z|\tilde{v}_h|^3|^2) \\ &\leq C|\tilde{v}_h|_6^4 \left[ |\tilde{v}_h|_6^6 + \int_{\mathcal{O}} (|\tilde{v}_h|^2|\nabla|\tilde{v}_h|^2|^2 + |\tilde{v}_h|^4|\nabla\tilde{v}_h|^2) dx dy dz \right. \\ &\quad \left. + \int_{\mathcal{O}} (|\tilde{v}_h|^2|\partial_z|\tilde{v}_h|^2|^2 + |\tilde{v}_h|^4|\partial_z\tilde{v}_h|^2) dx dy dz \right] \\ &\leq C|\tilde{v}_h|_6^{10} + C|\tilde{v}_h|_6^4 \left[ \int_{\mathcal{O}} (|\tilde{v}_h|^2|\nabla|\tilde{v}_h|^2|^2 + |\tilde{v}_h|^4|\nabla\tilde{v}_h|^2) dx dy dz \right. \\ &\quad \left. + \int_{\mathcal{O}} (|\tilde{v}_h|^2|\partial_z|\tilde{v}_h|^2|^2 + |\tilde{v}_h|^4|\partial_z\tilde{v}_h|^2) dx dy dz \right]. \end{aligned}$$

As a result of the previous inequalities, we obtain

$$\begin{aligned} &\left| \int_{\mathcal{O}} \left( \psi_1 h_1(t) - \int_{-1}^0 \psi_1 h_1(t) dz \right) \cdot |\tilde{v}_h|^4 \tilde{v}_h dx dy dz \right| \\ &\leq C|h_1(t)|_U |\tilde{v}_h|_6^5 + C|h_1(t)|_U |\tilde{v}_h|_6^2 \left[ \int_{\mathcal{O}} (|\tilde{v}_h|^2|\nabla|\tilde{v}_h|^2|^2 + |\tilde{v}_h|^4|\nabla\tilde{v}_h|^2) dx dy dz \right]^{\frac{1}{2}} \\ &\quad + C|h_1(t)|_U |\tilde{v}_h|_6^2 \left[ \int_{\mathcal{O}} (|\tilde{v}_h|^2|\partial_z|\tilde{v}_h|^2|^2 + |\tilde{v}_h|^4|\partial_z\tilde{v}_h|^2) dx dy dz \right]^{\frac{1}{2}} \\ &:= I_3(t) + I_4(t) + I_5(t). \end{aligned}$$

From the simple inequality, for any  $a$ ,

$$|a|^n \leq C(1 + |a|^m), \text{ for } m \geq n, \quad (5.32)$$

we get

$$\begin{aligned} I_3(t) &\leq C(1 + |\tilde{v}_h|_6^6)(1 + |h_1(t)|_U^2) \\ &\leq C|h_1(t)|_U^2|\tilde{v}_h|_6^6 + C|\tilde{v}_h|_6^6 + C|h_1(t)|_U^2 + C. \end{aligned}$$

By the Young inequality and (5.32), we have

$$\begin{aligned} I_4(t) &\leq \frac{1}{2} \int_{\mathcal{O}} (|\tilde{v}_h|^2 |\nabla |\tilde{v}_h|^2|^2 + |\tilde{v}_h|^4 |\nabla \tilde{v}_h|^2) dx dy dz + C|h_1(t)|_U^2 |\tilde{v}_h|_6^4 \\ &\leq \frac{1}{2} \int_{\mathcal{O}} (|\tilde{v}_h|^2 |\nabla |\tilde{v}_h|^2|^2 + |\tilde{v}_h|^4 |\nabla \tilde{v}_h|^2) dx dy dz + C|h_1(t)|_U^2 |\tilde{v}_h|_6^6 + C|h_1(t)|_U^2. \end{aligned}$$

Similar to  $I_4$ , we have

$$\begin{aligned} I_5(t) &\leq \frac{1}{2} \int_{\mathcal{O}} (|\tilde{v}_h|^2 |\partial_z |\tilde{v}_h|^2|^2 + |\tilde{v}_h|^4 |\partial_z \tilde{v}_h|^2) dx dy dz + C|h_1(t)|_U^2 |\tilde{v}_h|_6^4 \\ &\leq \frac{1}{2} \int_{\mathcal{O}} (|\tilde{v}_h|^2 |\partial_z |\tilde{v}_h|^2|^2 + |\tilde{v}_h|^4 |\partial_z \tilde{v}_h|^2) dx dy dz + C|h_1(t)|_U^2 |\tilde{v}_h|_6^6 + C|h_1(t)|_U^2. \end{aligned}$$

Thus, collecting the above inequalities, we obtain

$$\begin{aligned} & \left| \int_{\mathcal{O}} (\psi_1 h_1(t) - \int_{-1}^0 \psi_1 h_1(t) dz) \cdot |\tilde{v}_h|^4 \tilde{v}_h dx dy dz \right| \\ & \leq \frac{1}{2} \int_{\mathcal{O}} (|\tilde{v}_h|^2 |\nabla |\tilde{v}_h|^2|^2 + |\tilde{v}_h|^4 |\nabla \tilde{v}_h|^2) dx dy dz + \frac{1}{2} \int_{\mathcal{O}} (|\tilde{v}_h|^2 |\partial_z |\tilde{v}_h|^2|^2 + |\tilde{v}_h|^4 |\partial_z \tilde{v}_h|^2) dx dy dz \\ & \quad + C(1 + |h_1(t)|_U^2) |\tilde{v}_h|_6^6 + C(1 + |h_1(t)|_U^2). \end{aligned} \quad (5.33)$$

Putting (5.28), (5.31) and (5.33) together, we have

$$\begin{aligned} |\tilde{v}_h(t)|_6^6 &+ \int_0^t \left[ \int_{\mathcal{O}} (|\tilde{v}_h|^2 |\nabla |\tilde{v}_h|^2|^2 + |\tilde{v}_h|^4 |\nabla \tilde{v}_h|^2) dx dy dz \right. \\ & \quad \left. + \int_{\mathcal{O}} (|\tilde{v}_h|^2 |\partial_z |\tilde{v}_h|^2|^2 + |\tilde{v}_h|^4 |\partial_z \tilde{v}_h|^2) dx dy dz \right] ds \leq K_2(t), \end{aligned} \quad (5.34)$$

where

$$K_2(t) = e^{(C(1+M)K_1^2(t))} \left[ \|v_0\|^6 + C(1+M) + K_1^2(t) \right].$$



(2)  $L^6(\mathcal{O})$  estimates of  $T_h$ . Similar to  $L^6(\mathcal{O})$  estimates of  $\tilde{v}_h$ , we obtain

$$|T_h(t)|_6^6 + \int_0^t \left( \int_{\mathcal{O}} |T_h|^4 |\nabla T_h|^2 dx dy dz + \int_{\mathcal{O}} |T_h|^4 \left| \frac{\partial T_h}{\partial z} \right|^2 dx dy dz \right) ds \leq K_3(t), \quad (5.35)$$

where

$$K_3(t) = e^{C(1+M)} \left[ \|T_0\|^6 + C(1+M) \right].$$

(3)  $|\nabla \tilde{v}_h|$  estimates. Taking the inner product of (5.29) with  $-\Delta \tilde{v}_h$  in  $L^2(D)$  and referring to Sect. 3.3.1 in [3], we have

$$\begin{aligned} & \frac{d|\nabla \tilde{v}_h|^2}{dt} + 2|\Delta \tilde{v}_h|^2 \\ & \leq C|\tilde{v}_h|^2 |\nabla \tilde{v}_h|^4 + C|\nabla \tilde{v}_h|^2 + C \int_{\mathcal{O}} |\tilde{v}_h|^4 |\nabla \tilde{v}_h|^2 dx dy dz \\ & \quad + C|\tilde{v}_h|^2 + \left| \int_D \Delta \tilde{v}_h \left( \int_{-1}^0 \psi_1(t, Y_h) h_1(t) dz' \right) dx dy \right|. \end{aligned} \quad (5.36)$$

By the Cauchy–Schwarz inequality, we get

$$\begin{aligned} \left| \int_D \Delta \tilde{v}_h \left( \int_{-1}^0 \psi_1(t, Y_h) h_1(t) dz' \right) dx dy \right| & \leq C|\Delta \tilde{v}_h| \left| \int_{-1}^0 \psi_1(t, Y_h) h_1(t) dz' \right| \\ & \leq \frac{1}{2} |\Delta \tilde{v}_h|^2 + C \left| \int_{-1}^0 \psi_1(t, Y_h) h_1(t) dz' \right|^2. \end{aligned}$$

Applying the Cauchy–Schwarz inequality, **Hypothesis H0** and by (5.28), we obtain

$$\begin{aligned} \left| \int_{-1}^0 \psi_1(t, Y_h) h_1(t) dz' \right|^2 & \leq |\psi_1(t, Y_h) h_1(t)|^2 \\ & \leq C(1 + \sup_{t \in [0, T]} |Y_h(t)|^2) |h_1(t)|_U^2 \\ & \leq C|h_1(t)|_U^2. \end{aligned} \quad (5.37)$$

Based on the above, it reaches

$$\left| \int_D \Delta \tilde{v}_h \left( \int_{-1}^0 \psi_1(t, Y_h) h_1(t) dz' \right) dx dy \right| \leq \frac{1}{2} |\Delta \tilde{v}_h|^2 + C|h_1(t)|_U^2. \quad (5.38)$$

As a result of (5.28), (5.36) and (5.38), we deduce

$$|\nabla \bar{v}_h(t)|^2 + \int_0^t |\Delta \bar{v}_h|^2 ds \leq K_4(t), \quad (5.39)$$

where

$$K_4(t) = e^{K_1^2(t)} \left[ \|v_0\|^2 + K_1(t) + K_2(t) + CM \right].$$

(4)  $\left| \frac{\partial v_h}{\partial z} \right|^2$  **estimates.** Denote  $u = \frac{\partial v_h}{\partial z}$ . Clearly,  $u$  satisfies

$$\begin{aligned} \frac{\partial u}{\partial t} + A_1 u + (v \cdot \nabla) u + \Phi(v) \frac{\partial u}{\partial z} + (u \cdot \nabla) v - (\nabla \cdot v) u + f k \times u - \nabla T \\ = \partial_z(\psi_1(t, Y_h) h_1). \end{aligned} \quad (5.40)$$

Taking the inner product of (5.40) with  $u$  in  $L^2(\mathcal{O})$  and referring to Sect. 3.3.2 in [3], we get

$$\begin{aligned} \frac{d|u|^2}{dt} + \frac{3}{2}(|\nabla u|^2 + |\partial_z u|^2) \\ \leq C(|\nabla \bar{v}_h|^4 + |\tilde{v}|_6^4)|u|^2 + C|T|^2 + \left| \int_{\mathcal{O}} \partial_z(\psi_1(t, Y_h) h_1) u dx dy dz \right|. \end{aligned}$$

By integration by parts, the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \left| \int_{\mathcal{O}} \partial_z(\psi_1(t, Y_h) h_1) u dx dy dz \right| &= \left| \int_{\mathcal{O}} \psi_1(t, Y_h) h_1 \partial_z u dx dy dz \right| \\ &\leq C |\psi_1(t, Y_h) h_1(t)| |\partial_z u| \\ &\leq \frac{1}{2} |\partial_z u|^2 + C |\psi_1(t, Y_h) h_1(t)|^2 \\ &\leq \frac{1}{2} |\partial_z u|^2 + C |h_1(t)|_{\tilde{U}}^2, \end{aligned}$$

where (5.37) is used.

As a result, we have

$$\left| \frac{\partial v_h(t)}{\partial z} \right|^2 + \int_0^t \left| \nabla \frac{\partial v_h}{\partial z} \right|^2 ds + \int_0^t \left| \frac{\partial^2 v_h}{\partial z^2} \right|^2 ds \leq K_5(t), \quad (5.41)$$

where

$$K_5(t) = e^{(K_4^2(t) + K_3^{\frac{3}{2}}(t))t} \left[ \|v_0\|^2 + K_1(t) + CM \right].$$

(5)  $|\nabla v_h|^2$  **estimates.** Taking the inner product of (5.23) with  $-\Delta v_h$  in  $L^2(\mathcal{O})$  and referring to Sect. 3.3.3 in [3], we have

$$\begin{aligned} & \frac{d|\nabla v_h|^2}{dt} + \frac{3}{2}(|\Delta v_h|^2 + |\nabla \partial_z v_h|^2) \\ & \leq C(|v|_6^4 + |\nabla v_h|^2 |\partial_z v_h|^2) |\nabla v_h|^2 + C|\nabla T_h|^2 + \left| \int_{\mathcal{O}} \psi_1(t, Y_h) h_1(t) \Delta v_h dx dy dz \right|. \end{aligned}$$

By the Cauchy–Schwarz inequality and (5.37), we get

$$\begin{aligned} \left| \int_{\mathcal{O}} \psi_1(t, Y_h) h_1(t) \Delta v_h dx dy dz \right| & \leq C |\psi_1(t, Y_h) h_1(t)| |\Delta v_h| \\ & \leq \frac{1}{2} |\Delta v_h|^2 + C |\psi_1(t, Y_h) h_1(t)|^2 \\ & \leq \frac{1}{2} |\Delta v_h|^2 + C |h_1(t)|_{\mathcal{U}}^2. \end{aligned}$$

Based on the above, we obtain

$$|\nabla v_h(t)|^2 + \int_0^t |\Delta v_h(s)|^2 ds + \int_0^t \left| \nabla \frac{\partial v_h(s)}{\partial z} \right|^2 ds \leq K_6(t), \quad (5.42)$$

where

$$K_6(t) = e^{\left(K_4^{\frac{2}{3}}(t)t + K_1(t)K_5(t)\right)} \left[ \|v_0\|^2 + K_1(t) + CM \right].$$

(6)  $\|T\|$  **estimates.** Taking the inner product of (5.24) with  $-\Delta T - T_{zz}$  in  $L^2(\mathcal{O})$  and referring to Sect. 3.3.4 in [3], we get

$$\begin{aligned} & \frac{d(|\nabla T_h|^2 + |\partial_z T_h|^2)}{dt} + \frac{3}{2}(|\Delta T_h|^2 + |\nabla \partial_z T_h|^2 + |\partial_{zz} T_h|^2) \\ & \leq C(|v_h|_6^4 + |\nabla v_h|^2 |\Delta v_h|^2) (|\nabla T_h|^2 + |\partial_z T_h|^2) + \left| \int_{\mathcal{O}} \psi_2(t, Y_h) h_2(t) (\Delta T_h + \partial_z^2 T_h) dx dy dz \right|. \end{aligned}$$

By the Cauchy–Schwarz inequality and (5.37), we have

$$\begin{aligned} & \left| \int_{\mathcal{O}} \psi_2(t, Y_h) h_2(t) (\Delta T_h + \partial_z^2 T_h) dx dy dz \right| \\ & \leq C |\psi_2(t, Y_h) h_2(t)| |\Delta T_h + \partial_z^2 T_h| \\ & \leq \frac{1}{2} (|\Delta T_h|^2 + |\partial_z^2 T_h|^2 + |\nabla \partial_z T_h|^2) + C |h_2(t)|_{\mathcal{U}}^2. \end{aligned}$$

As a result, we obtain

$$|\nabla T_h(t)|^2 + |\partial_z T_h(t)|^2 + \int_0^t \left( |\Delta T_h|^2 + |\nabla \partial_z T_h|^2 + |\partial_{zz} T_h|^2 \right) ds \leq K_7(t), \quad (5.43)$$

where

$$K_7(t) = e^{(K_4^2(t)t + K_6^2(t))} \left[ \|T_0\|^2 + CM \right].$$

#### 5.1.4. Proof of Theorem 5.1

**Proof of Theorem 5.1.** As discussed in the below of Theorem 5.1, we have indicated the existence of the strong solution of (5.22) for a short time interval using similar method as [14] and [24]. Let  $Y_h = (v_h, T_h)$  be the strong solution corresponding to the initial data  $(v_0, T_0)$  with maximal interval of existence  $[0, \mathcal{T})$ . If we assume  $\mathcal{T} < \infty$ , then it has

$$\limsup_{t \rightarrow \mathcal{T}^-} (\|v_h\| + \|T_h\|) = \infty.$$

Otherwise, the solution can be extended beyond the time  $\mathcal{T}$ . However, the above contradicts with (5.34)–(5.43). Therefore,  $\mathcal{T} = \infty$ . It remains to prove uniqueness and continuously dependence on the initial data.

Let  $Y_h^1 = (v_h^1, T_h^1, p_b^1)$ ,  $Y_h^2 = (v_h^2, T_h^2, p_b^2)$  be two strong solutions of (5.22) with initial data  $(v_0^1, T_0^1)$  and  $(v_0^2, T_0^2)$ , respectively. For convenience, we omit the index  $h$ . Denote  $r = v^1 - v^2$ ,  $\eta = T^1 - T^2$ ,  $q_b = p_b^1 - p_b^2$ , we have

$$\begin{aligned} \frac{dr}{dt} + A_1 r + (v^1 \cdot \nabla) r + (r \cdot \nabla) v^2 + \Phi(v^1) \frac{\partial r}{\partial z} + \Phi(r) \frac{\partial v^2}{\partial z} + f k \times r + \nabla q_b \\ - \int_{-1}^z \nabla \eta(x, y, z', t) dz' = \psi_1(t, Y^1(t)) h_1 - \psi_1(t, Y^2(t)) h_1, \end{aligned} \quad (5.44)$$

$$\begin{aligned} \frac{d\eta}{dt} + A_2 \eta + (v^1 \cdot \nabla) \eta + (r \cdot \nabla) T^2 + \Phi(v^1) \frac{\partial \eta}{\partial z} + \Phi(r) \frac{\partial T^2}{\partial z} \\ = \psi_2(t, Y^1(t)) h_2 - \psi_2(t, Y^2(t)) h_2, \end{aligned} \quad (5.45)$$

$$r(x, y, z, 0) = v_0^1 - v_0^2, \quad (5.46)$$

$$\eta(x, y, z, 0) = T_0^1 - T_0^2. \quad (5.47)$$

**$L^2(\mathcal{O})$  estimates of  $r$ .** Taking the inner product of (5.44) with  $r$  in  $L^2(\mathcal{O})$ , we get

$$\begin{aligned} \frac{d|r|^2}{dt} + \frac{3}{2} (|\nabla r|^2 + |\partial_z r|^2) \\ \leq C |\nabla v^2|^4 |r|^2 + C |r|^2 |\partial_z v^2|^2 \left| \nabla \frac{\partial v^2}{\partial z} \right|^2 + C |\eta|^2 \\ + \left| \int_{\mathcal{O}} \left( \psi_1(t, Y^1(t)) h_1(t) - \psi_1(t, Y^2(t)) h_1(t) \right) r dx dy dz \right|. \end{aligned}$$

By the Cauchy–Schwarz inequality, **Hypothesis H0** and (5.32), we obtain

$$\begin{aligned}
 & \left| \int_{\mathcal{O}} \left( \psi_1(t, Y^1(t)) h_1(t) - \psi_1(t, Y^2(t)) h_1(t) \right) r dx dy dz \right| \\
 & \leq |\psi_1(t, Y^1(t)) h_1(t) - \psi_1(t, Y^2(t)) h_1(t)| |r| \\
 & \leq |\psi_1(t, Y^1(t)) - \psi_1(t, Y^2(t))|_{\mathcal{L}_2(U; H)} |h_1(t)|_U |r| \\
 & \leq |Y^1(t) - Y^2(t)| |h_1(t)|_U |r| \\
 & \leq |h_1(t)|_U |r|^2 + |h_1(t)|_U |r| |\eta| \\
 & \leq C(1 + |h_1(t)|_U^2) |r|^2 + C|h_1(t)|_U^2 |\eta|^2.
 \end{aligned}$$

Then, we have

$$\begin{aligned}
 & \frac{d|r|^2}{dt} + \frac{3}{2}(|\nabla r|^2 + |\partial_z r|^2) \\
 & \leq C|\nabla v^2|^4 |r|^2 + C|r|^2 |\partial_z v^2|^2 \left| \nabla \frac{\partial v^2}{\partial z} \right|^2 + C(1 + |h_1(t)|_U^2) |r|^2 + C(1 + |h_1(t)|_U^2) |\eta|^2.
 \end{aligned} \tag{5.48}$$

Similarly, we can obtain  $L^2(\mathcal{O})$  estimates of  $\eta$ . That is

$$\begin{aligned}
 & \frac{d|\eta|^2}{dt} + \frac{3}{2}(|\nabla \eta|^2 + |\partial_z \eta|^2) \\
 & \leq C|\nabla T^2|^4 |r|^2 + C|\eta|^2 |\partial_z T^2|^2 |\nabla \partial_z T^2|^2 + \frac{1}{2} |\nabla r|^2 \\
 & \quad + \left| \int_{\mathcal{O}} \left( \psi_1(t, Y^1(t)) h_1(t) - \psi_1(t, Y^2(t)) h_2(t) \right) \eta dx dy dz \right| \\
 & \leq C|\nabla T^2|^4 |r|^2 + C|\eta|^2 |\partial_z T^2|^2 |\nabla \partial_z T^2|^2 + \frac{1}{2} |\nabla r|^2 + C(1 + |h_2(t)|_U^2) |\eta|^2 + C|h_2(t)|_U^2 |r|^2.
 \end{aligned} \tag{5.49}$$

As a result of (5.48) and (5.49), we have

$$\begin{aligned}
 & \frac{d|r|^2}{dt} + \frac{d|\eta|^2}{dt} + |\nabla r|^2 + |\partial_z r|^2 + |\nabla \eta|^2 + |\partial_z \eta|^2 \\
 & \leq C(|\nabla v^2|^4 + |\nabla T^2|^4 + |\partial_z v^2|^2 \left| \nabla \frac{\partial v^2}{\partial z} \right|^2) |r|^2 + C|\eta|^2 |\partial_z T^2|^2 |\nabla \partial_z T^2|^2 \\
 & \quad + C(1 + |h_1(t)|_U^2 + |h_2(t)|_U^2) |r|^2 + C(1 + |h_1(t)|_U^2 + |h_2(t)|_U^2) |\eta|^2.
 \end{aligned}$$

Applying Gronwall inequality, we deduce that

$$|r(t)|^2 + |\eta(t)|^2 \leq (|r(0)|^2 + |\eta(0)|^2) e^{C(K_6^2 t + K_7^2 t + K_5 K_6 + K_7^2 + C(1+M))}.$$

The above inequality proves the continuous dependence of the solutions on the initial data, and in particular, when  $r(0) = \eta(0) = 0$ , we have  $r(t) = \eta(t) = 0$ , for all  $t \geq 0$ . We complete the proof.  $\square$

From (5.41), (5.42) and (5.43), we have

**Corollary 5.2.** *Let  $Y_h$  be the unique strong solution of (5.22) with  $h \in T_M$ , then*

$$\sup_{0 \leq t \leq T} \|Y_h(t)\|^2 + \int_0^T \|Y_h(t)\|_2^2 dt \leq C(T, M, \|Y_0\|).$$

Now, we can define  $\mathcal{G}^0 : C([0, T]; \mathcal{U}) \rightarrow \Re$  by

$$\mathcal{G}^0(\tilde{h}) = \begin{cases} Y_h, & \text{if } \tilde{h} = \int_0^\cdot h(s) ds \text{ for some } h \in L^2([0, T]; U), \\ 0, & \text{otherwise.} \end{cases} \quad (5.50)$$

## 5.2. Tightness of $Y_n$

Let  $Y_n$  be the unique strong solution of (5.22) with  $h_n \in T_M$ , where  $h_n = (h_n^1, h_n^2)^T$ . In the following, we aim to prove the tightness of  $Y_n$ .

As in [11], we introduce the following space. Let  $K$  be a separable Hilbert space. Given  $p > 1$ ,  $\alpha \in (0, 1)$ , let  $W^{\alpha, p}([0, T]; K)$  be the Sobolev space of all  $u \in L^p([0, T]; K)$  such that

$$\int_0^T \int_0^T \frac{|u(t) - u(s)|_K^p}{|t - s|^{1+\alpha p}} dt ds < \infty,$$

endowed with the norm

$$|u|_{W^{\alpha, p}([0, T]; K)}^p = \int_0^T |u(t)|_K^p dt + \int_0^T \int_0^T \frac{|u(t) - u(s)|_K^p}{|t - s|^{1+\alpha p}} dt ds.$$

The following result can be found in [11].

**Lemma 5.1.** *Let  $B_0 \subset B \subset B_1$  be Banach spaces,  $B_0$  and  $B_1$  reflexive, with compact embedding of  $B_0$  in  $B$ . Let  $p \in (1, \infty)$  and  $\alpha \in (0, 1)$  be given. Let  $\Lambda$  be the space*

$$\Lambda = L^p([0, T]; B_0) \cap W^{\alpha, p}([0, T]; B_1),$$

*endowed with the natural norm. Then the embedding of  $\Lambda$  in  $L^p([0, T]; B)$  is compact.*

With the aid of Lemma 5.1, we have

**Proposition 5.3.** Under **Hypothesis H0**, the sequence  $(Y_n)_{n \in \mathbb{N}_+}$  is tight in  $L^2([0, T]; V)$ .

**Proof.** From (5.22), we have

$$\begin{aligned} Y_n(t) &= Y_0 - \int_0^t AY_n(s)ds - \int_0^t B(Y_n(s), Y_n(s))ds - \int_0^t G(Y_n(s))ds + \int_0^t \psi(s, Y_n(s))h_n(s)ds \\ &:= J_n^1 + J_n^2(t) + J_n^3(t) + J_n^4(t) + J_n^5(t). \end{aligned}$$

Referring to Sect. 4.2 in [9], we have

$$\begin{aligned} |J_n^1|^2 &\leq C_1, \\ |J_n^4|_{W^{\alpha,2}([0,T];V')}^2 &\leq C \left( \sup_{0 \leq s \leq T} |Y_n(s)|^2 \right) \leq C_{2,\alpha} \quad \alpha \in (0, \frac{1}{2}), \\ |J_n^2|_{W^{\alpha,2}([0,T];V')}^2 &\leq C \int_0^T \|Y_n(s)\|^2 ds \leq C_{3,\alpha} \quad \alpha \in (0, \frac{1}{2}), \end{aligned}$$

for suitable positive constants  $C_1, C_{2,\alpha}, C_{3,\alpha}$ . By Lemma 3.4, we obtain

$$\|B(Y, Y_1)\|_{V'} \leq C \|Y\| \|Y_1\|_2,$$

then, it gives

$$\|B(Y_n, Y_n)\|_{L^2([0,T];V')}^2 \leq C_4 \left( \sup_{0 \leq s \leq T} \|Y_n(s)\|^2 \right) \int_0^T \|Y_n(s)\|_2^2 ds. \quad (5.51)$$

As a result of Corollary 5.2, we obtain

$$|J_n^3|_{W^{\alpha,2}([0,T];V')}^2 \leq C_{5,\alpha} \quad \alpha \in (0, 1).$$

Moreover, by the Cauchy–Schwarz inequality, **Hypothesis H0** and (5.28), we have

$$\begin{aligned} \left| \int_s^t \psi(u, Y_n(u))h_n(u)du \right|^2 &\leq \int_s^t |\psi(u, Y_n(u))h_n(u)|^2 du \\ &\leq \int_s^t |h_n(u)|_U^2 (1 + |Y_n(u)|^2) du \\ &\leq \int_s^t |h_n(u)|_U^2 du + \int_s^t |h_n(u)|_U^2 |Y_n(u)|^2 du \end{aligned}$$

$$\begin{aligned}
&\leq \left(1 + \sup_{0 \leq u \leq T} |Y_n(u)|^2\right) \int_s^t |h_n(u)|_U^2 du \\
&\leq C \int_s^t |h_n(u)|_U^2 du,
\end{aligned} \tag{5.52}$$

thus, applying the Fubini theorem, we get

$$\begin{aligned}
|J_n^5|_{W^{\alpha,2}([0,T];H)}^2 &= \left| \int_0^t \psi(s, Y_n(s)) h_n(s) ds \right|_{W^{\alpha,2}([0,T];H)}^2 \\
&= \int_0^T \left| \int_0^t \psi(Y_n) h_n(s) ds \right|^2 dt + \int_0^T \int_0^T \frac{\left| \int_s^t \psi(u, Y_n(u)) h_n(u) du \right|^2}{|t-s|^{1+2\alpha}} dt ds \\
&\leq C_{6,\alpha}
\end{aligned}$$

for any  $\alpha \in (0, \frac{1}{2})$ . Collecting the previous inequalities, we obtain

$$|Y_n|_{W^{\alpha,2}([0,T];V')}^2 \leq C_{7,\alpha} \quad \forall \alpha \in (0, \frac{1}{2})$$

for some constant  $C_{7,\alpha} > 0$ .

In view of [Corollary 5.2](#),  $Y_n$  are bounded uniformly in  $n$  in the space

$$\Lambda := L^2([0, T]; D(A)) \cap W^{\alpha,2}([0, T]; V').$$

By [Lemma 5.1](#), we obtain  $(Y_n)_{n \in \mathbb{Z}^+}$  is compact in  $L^2([0, T]; V)$ .  $\square$

As a result of [Corollary 5.2](#) and [Proposition 5.3](#), we have

**Corollary 5.4.** *There exists a subsequence still denoted by  $(Y_n)_{n \in \mathbb{Z}^+}$  and  $\check{Y} \in L^\infty([0, T]; V) \cap L^2([0, T]; V) \cap L^2([0, T]; D(A))$  such that*

$$\begin{aligned}
Y_n &\rightharpoonup \check{Y} \text{ weakly star in } L^\infty([0, T]; V), \\
Y_n &\rightarrow \check{Y} \text{ strongly in } L^2([0, T]; V), \\
Y_n &\rightharpoonup \check{Y} \text{ weakly in } L^2([0, T]; D(A)).
\end{aligned}$$

### 5.3. Property of $\check{Y}$

Throughout this subsection, we fix a sequence  $(h_n)_{n \geq 0} \subset T_M$  such that  $h_n \rightharpoonup h$  weakly in  $L^2([0, T]; U)$  and denote the corresponding solution of [\(5.22\)](#) by  $Y_n$ . From [Theorem 5.1](#) and [Corollary 5.4](#), we know the weak star limit of  $Y_n$  exists, which is denoted by  $\check{Y}$ . The following proposition tells that  $\check{Y}$  is exactly the solution of [\(5.22\)](#) with  $h$ .



**Proposition 5.5.** *The above  $\check{Y}$  satisfies*

$$\begin{cases} d\check{Y}(t) + A\check{Y}(t)dt + B(\check{Y}(t), \check{Y}(t))dt + G(\check{Y}(t))dt = \psi(t, \check{Y}(t))h(t)dt, \\ \check{Y}(0) = Y_0. \end{cases} \quad (5.53)$$

Before proving, we firstly introduce a lemma on the nonlinear term.

**Lemma 5.2.** *Let  $u_v \rightarrow u$  strongly in  $L^2([0, T]; V)$  as  $v \rightarrow 0$ , then for  $w \in D(A^{\frac{3}{2}})$ ,*

$$\int_0^T (B(u_v(t), u_v(t)), w)dt \rightarrow \int_0^T (B(u(t), u(t)), w)dt \quad \text{as } v \rightarrow 0.$$

**Proof.** By the triangle inequality, we have

$$\begin{aligned} & \left| \int_0^T (B(u_v(t), u_v(t)), w)dt - \int_0^T (B(u(t), u(t)), w)dt \right| \\ & \leq \int_0^T |(B(u_v, u_v - u), w)|dt + \int_0^T |(B(u_v - u, u), w)|dt \\ & := I_1(T) + I_2(T). \end{aligned}$$

Referring to [22], we obtain

$$\|B(Y, Y_1)\|_{-3} \leq C|Y|\|Y_1\|. \quad (5.54)$$

Based on (5.54) and by the Cauchy–Schwarz inequality, we have

$$\begin{aligned} I_1(T) & \leq C \int_0^T \|u_v\| \|u_v - u\| |w|_{D(A^{\frac{3}{2}})} dt \\ & \leq C|w|_{D(A^{\frac{3}{2}})} \int_0^T \|u_v\| \|u_v - u\| dt \\ & \leq C|w|_{D(A^{\frac{3}{2}})} \left( \int_0^T \|u_v\|^2 dt \right)^{\frac{1}{2}} \left( \int_0^T \|u_v - u\|^2 dt \right)^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned}
I_2(T) &\leq C \int_0^T \|u_v - u\| |u| |w|_{D(A^{\frac{3}{2}})} dt \\
&\leq C |w|_{D(A^{\frac{3}{2}})} \int_0^T \|u_v - u\| |u| dt \\
&\leq C |w|_{D(A^{\frac{3}{2}})} \left( \int_0^T \|u_v - u\|^2 dt \right)^{\frac{1}{2}} \left( \int_0^T |u|^2 dt \right)^{\frac{1}{2}}.
\end{aligned}$$

Since  $u_v \rightarrow u$  strongly in  $L^2([0, T]; V)$ , it gives  $I_1(T) + I_2(T) \rightarrow 0$ . We complete the proof.  $\square$

**Proof of Proposition 5.5.** Denoting an orthonormal basis of  $D(A_1^{\frac{3}{2}})$  by  $\{w_j^1\}_{j \geq 1}$  and an orthonormal basis of  $D(A_2^{\frac{3}{2}})$  by  $\{w_j^2\}_{j \geq 1}$ , then we obtain an orthonormal basis of  $D(A^{\frac{3}{2}})$  denoted by  $\{w_j\}_{j \geq 1}$  by using the same method as Sect. 3. Taking a test function  $\phi(t)$  a continuously differentiable on  $[0, T]$  satisfying  $\phi(T) = 0$ . From (5.22), we have

$$\begin{aligned}
&\int_0^T \left( \frac{dY_n}{dt}, \phi(t) w_j \right) dt + \int_0^T (AY_n, \phi(t) w_j) dt + \int_0^T (B(Y_n, Y_n), \phi(t) w_j) dt \\
&\quad + \int_0^T (G(Y_n), \phi(t) w_j) dt = \int_0^T (\psi(t, Y_n(t)) h_n(t), \phi(t) w_j) dt.
\end{aligned}$$

By integration by parts, we get

$$\begin{aligned}
&-(Y_0, \phi(0) w_j) - \int_0^T (Y_n(t), \phi'(t) w_j) dt + \int_0^T (Y_n(t), \phi(t) A w_j) dt + \int_0^T (B(Y_n, Y_n), \phi(t) w_j) dt \\
&\quad + \int_0^T (G(Y_n), \phi(t) w_j) dt = \int_0^T (\psi(t, Y_n(t)) h_n(t), \phi(t) w_j) dt.
\end{aligned}$$

Denote the above equality by symbols

$$J_1 + J_2(T) + J_3(T) + J_4(T) + J_5(T) = J_6(T).$$

Since  $Y_n \rightarrow \check{Y}$  strongly in  $L^2([0, T]; V)$  and by the Cauchy–Schwarz inequality, we obtain

$$J_2(T) + J_3(T) \rightarrow - \int_0^T (\check{Y}(t), \phi'(t) w_j) dt + \int_0^T (\check{Y}(t), \phi(t) A w_j) dt. \quad (5.55)$$

It follows from [Lemma 5.2](#) that

$$J_4(T) \rightarrow \int_0^T (B(\check{Y}, \check{Y}), \phi(t)w_j)dt. \quad (5.56)$$

Since  $\check{Y} \in V$ , we can denote  $\check{Y} = (\check{v}, \check{T})$ , then

$$\begin{aligned} & \int_0^T (G(Y_n), \phi(t)w_j)dt - \int_0^T (G(\check{Y}), \phi(t)w_j)dt \\ &= \int_0^T (fk \times (v_n - \check{v}), \phi(t)w_j^1)dt + \int_0^T \left( \int_{-1}^z \nabla(T_n - \check{T})dz', \phi(t)w_j^1 \right)dt \\ &:= K_1(T) + K_2(T). \end{aligned}$$

By the Cauchy–Schwarz inequality, [Corollary 5.4](#) and [Lemma 5.2](#), we have  $K_1(T) \rightarrow 0$ . Similarly, we have

$$\begin{aligned} K_2(T) &= - \int_0^T \left( \int_{-1}^z (T_n - \check{T})dz', \phi(t)\nabla w_j^1 \right)dt \\ &\leq \int_0^T |T_n - \check{T}| |\phi(t)\nabla w_j^1| dt \rightarrow 0. \end{aligned}$$

As a result,

$$J_5(T) \rightarrow \int_0^T (G(\check{Y}), \phi(t)w_j)dt. \quad (5.57)$$

By the triangle inequality, we get

$$\begin{aligned} & \left| \int_0^T (\psi(t, Y_n)h_n(t), \phi(t)w_j)dt - \int_0^T (\psi(t, \check{Y})h(t), \phi(t)w_j)dt \right| \\ &\leq \left| \int_0^T ((\psi(t, Y_n) - \psi(t, \check{Y}))h_n(t), \phi(t)w_j)dt \right| + \left| \int_0^T (\psi(t, \check{Y})(h_n(t) - h(t)), \phi(t)w_j)dt \right|, \\ &:= K_3(T) + K_4(T). \end{aligned}$$

It follows from the Cauchy–Schwarz inequality and **Hypothesis H0** that

$$\begin{aligned}
K_3(T) &\leq \int_0^T |(\psi(t, Y_n) - \psi(t, \check{Y}))h_n(t)| |\phi(t)w_j| dt \\
&\leq \int_0^T |\psi(t, Y_n) - \psi(t, \check{Y})|_{\mathcal{L}_2(U; H)} |h_n(t)|_U |\phi(t)w_j| dt \\
&\leq C \int_0^T |Y_n - \check{Y}| |h_n(t)|_U dt \\
&\leq C \left( \int_0^T |Y_n - \check{Y}|^2 dt \right)^{\frac{1}{2}} \left( \int_0^T |h_n(t)|_U^2 dt \right)^{\frac{1}{2}}.
\end{aligned}$$

In view of Corollary 5.4, we have  $K_3(T) \rightarrow 0$ . Since  $h_n - h \rightarrow 0$  weakly in  $L^2([0, T]; U)$ , we get  $K_4(T) \rightarrow 0$ . Based on the above, we have

$$J_6(T) \rightarrow \int_0^T (\psi(t, \check{Y}))h(t), \phi(t)w_j dt. \quad (5.58)$$

From (5.55)–(5.58), for each  $j$ , we have

$$\begin{aligned}
& - \int_0^T (\check{Y}(t), \phi'(t)w_j) dt + \int_0^T (\check{Y}(t), Aw_j\phi(t)) dt + \int_0^T (B(\check{Y}, \check{Y}), \phi(t)w_j) dt \\
& \quad + \int_0^T (G(\check{Y}), \phi(t)w_j) dt \\
& = (Y_0, \phi(0)w_j) + \int_0^T (\psi(t, \check{Y}(t))h(t), \phi(t)w_j) dt.
\end{aligned} \quad (5.59)$$

Actually, (5.59) holds for any  $\zeta$ , which is a finite linear combination of  $w_j$ . That is

$$\begin{aligned}
& - \int_0^T (\check{Y}(t), \phi'(t)\zeta) dt + \int_0^T (\check{Y}(t), A\phi(t)\zeta) dt + \int_0^T (B(\check{Y}, \check{Y}), \phi(t)\zeta) dt + \int_0^T (G(\check{Y}), \phi(t)\zeta) dt \\
& = (Y_0, \phi(0)\zeta) + \int_0^T (\psi(t, \check{Y}(t))h(t), \phi(t)\zeta) dt.
\end{aligned} \quad (5.60)$$

Since  $D(A^{\frac{3}{2}})$  is dense in  $V$ , we get

$$\frac{d}{dt}(\check{Y}, \zeta) + (A\check{Y}, \zeta) + (B(\check{Y}, \check{Y}), \zeta) + (G(\check{Y}), \zeta) = (\psi(t, \check{Y}), \zeta), \quad (5.61)$$

holds as an equality in distribution in  $L^2([0, T]; V')$ , which is exactly (5.53).

Finally, it remains to prove  $\check{Y}(0) = Y_0$ . Multiplying (5.61) with the same  $\phi(t)$  as above and integrating with respect to  $t$ . By integration by parts, we have

$$\begin{aligned} - \int_0^T (\check{Y}(t), \phi'(t)\zeta) dt + \int_0^T (\check{Y}(t), A\phi(t)\zeta) dt + \int_0^T (B(\check{Y}, \check{Y}), \phi(t)\zeta) dt + \int_0^T (G(\check{Y}), \phi(t)\zeta) dt \\ = (\check{Y}(0), \phi(0)\zeta) + \int_0^T (\psi(t, \check{Y}(t))h(t), \phi(t)\zeta) dt. \end{aligned} \quad (5.62)$$

By comparison with (5.60), it gives  $(\check{Y}(0) - Y_0, \phi(0)\zeta) = 0, \forall \zeta \in D(A^{\frac{3}{2}})$ . Choosing  $\phi$  such that  $\phi(0) \neq 0$ , then

$$(\check{Y}(0) - Y_0, \zeta) = 0, \quad \forall \zeta \in D(A^{\frac{3}{2}}).$$

Since  $D(A^{\frac{3}{2}})$  is dense in  $V$ , we have  $\check{Y}(0) = Y_0$ . We complete the proof.  $\square$

In the following, we will establish the continuity of  $\check{Y}$  in  $V$ . Referring to [23], we introduce the following criterion to prove continuity.

**Lemma 5.3.** *For  $V$  and  $H$  are two Hilbert spaces ( $V'$  is the dual space of  $V$ ) with  $V \subset \subset H = H' \subset V'$ , where  $V \subset \subset H$  denotes  $V$  is compactly embedded in  $H$ . If  $u \in L^2([0, T]; V)$ ,  $\frac{du}{dt} \in L^2([0, T]; V')$ , then  $u \in C([0, T]; H)$ .*

**Proposition 5.6.** *Assume Hypothesis H0 holds, then  $\check{Y} \in C([0, T]; V)$ .*

**Proof.** In view of Lemma 5.3, we firstly need to prove  $\frac{d\check{Y}}{dt} \in L^2([0, T]; V')$ . From Proposition 5.5, we know  $\check{Y} \in L^2([0, T]; D(A)) \cap L^\infty([0, T]; V)$  and

$$\frac{d\check{Y}}{dt} = -A\check{Y} - B(\check{Y}, \check{Y}) - G(\check{Y}) + \psi(t, \check{Y})h.$$

Since  $\check{Y}$  is bounded in  $L^2([0, T]; D(A))$  and  $A$  is continuous linear operator from  $D(A)$  to  $H$ , we deduce  $A\check{Y}$  is bounded in  $L^2([0, T]; H)$ . Similar to (5.51), we have

$$\|B(\check{Y}, \check{Y})\|_{L^2([0, T]; V')} \leq C.$$

By (3.16), we have

$$\|G(\check{Y})\|_{L^2([0, T]; V')}^2 \leq C \left( \sup_{0 \leq s \leq T} |\check{Y}(s)|^2 \right) \leq C.$$

Moreover, it follows from **Hypothesis H0** that

$$\begin{aligned} \|\psi(t, \check{Y})h\|_{L^2([0,T];H)}^2 &= \int_0^T |\psi(t, \check{Y})h|^2 dt \\ &\leq \int_0^T \|\psi(t, \check{Y})\|_{\mathcal{L}_2(U;H)}^2 |h|_U^2 dt \\ &\leq \int_0^T (1 + |\check{Y}|^2) |h|_U^2 dt \\ &\leq C \sup_{t \in [0,T]} |\check{Y}(t)|^2 \int_0^T |h|_U^2 dt \leq CM. \end{aligned}$$

As a result, we get

$$\frac{d\check{Y}}{dt} \in L^2([0, T]; V').$$

From [Corollary 5.2](#), we know  $\check{Y} \in L^2([0, T]; D(A))$ , then we conclude the result by applying [Lemma 5.3](#).  $\square$

By the uniqueness of [\(5.22\)](#), we have

**Corollary 5.7.** Assume **Hypothesis H0** holds, then  $\check{Y} = Y_h$ , where  $Y_h$  is the unique strong solution of [\(5.22\)](#) with  $h$ .

Now, we can obtain

**Theorem 5.8.** Assume **Hypothesis H0** holds, then  $Y_n - \check{Y} \rightarrow 0$  in  $\mathfrak{R}$  as  $n \rightarrow \infty$ .

**Proof.** Denote  $Y_n = (v_n, T_n, p_n)$  is the solution of [\(5.22\)](#) with  $h_n = (h_n^1, h_n^2)^T$  and  $\check{Y} = (\check{v}, \check{T}, \check{p}_b)$  is the solution of [\(5.53\)](#) with  $h = (h_1, h_2)^T$ . Let  $r_n = v_n - \check{v}$ ,  $\eta_n = T_n - \check{T}$ ,  $q_n = p_n - \check{p}_b$ , we have

$$\begin{aligned} \frac{dr_n}{dt} + A_1 r_n + (v_n \cdot \nabla) r_n + (r_n \cdot \nabla) \check{v} + \Phi(v_n) \frac{\partial r_n}{\partial z} + \Phi(r_n) \frac{\partial \check{v}}{\partial z} + f k \times r_n \\ + \nabla q_n - \int_{-1}^z \nabla \eta_n(x, y, z', t) dz' = \psi_1(t, Y_n(t)) h_n^1 - \psi_1(t, \check{Y}) h_1, \end{aligned} \quad (5.63)$$

$$\begin{aligned} \frac{d\eta_n}{dt} + A_2 \eta_n + (v_n \cdot \nabla) \eta_n + (r_n \cdot \nabla) \check{T} + \Phi(v_n) \frac{\partial \eta_n}{\partial z} + \Phi(r_n) \frac{\partial \check{T}}{\partial z} \\ = \psi_2(t, Y_n) h_n^2 - \psi_2(t, \check{Y}) h_2, \end{aligned} \quad (5.64)$$

$$r_n(x, y, z, 0) = 0, \quad (5.65)$$

$$\eta_n(x, y, z, 0) = 0. \quad (5.66)$$

We need to estimate  $H^1$  estimates of  $r_n$  and  $\eta_n$ , respectively.

(1)  **$H^1$  estimates of  $r_n$ .** Taking the inner product of (5.63) with  $A_1 r_n$  in  $L^2(\mathcal{O})$  and integrating the time from 0 to  $t$ , it reaches

$$\begin{aligned} & \|r_n(t)\|^2 + 2 \int_0^t \|r_n(s)\|_2^2 ds \\ &= -2 \int_0^t \left( (v_n \cdot \nabla) r_n + \Phi(v_n) \frac{\partial r_n}{\partial z}, A_1 r_n \right) ds \\ & \quad - 2 \int_0^t \left( (r_n \cdot \nabla) \check{v} + \Phi(r_n) \frac{\partial \check{v}}{\partial z}, A_1 r_n \right) ds \\ & \quad - 2 \int_0^t \left( f k \times r_n + \nabla q_n - \int_{-1}^z \nabla \eta_n(x, y, z', t) dz', A_1 r_n \right) ds \\ & \quad + 2 \int_0^t \left( \psi_1(s, Y_n) h_n^1 - \psi_1(s, \check{Y}) h_1, A_1 r_n \right) ds \\ &:= I_1(t) + I_2(t) + I_3(t) + I_4(t). \end{aligned}$$

Applying the Cauchy–Schwarz inequality, Lemma 3.4, Corollary 5.2 and the Young inequality to  $I_1(t)$  and  $I_2(t)$ , we obtain

$$\begin{aligned} |I_1(t)| &\leq C \int_0^t \|r_n(s)\|_2 \|\nabla r_n\| |v_n|_\infty ds + C \int_0^t \|r_n(s)\|_2 \|v_n\|^{\frac{1}{2}} \|v_n\|^{\frac{1}{2}} \left| \frac{\partial r_n}{\partial z} \right|^{\frac{1}{2}} \left| \nabla \frac{\partial r_n}{\partial z} \right|^{\frac{1}{2}} ds \\ &\leq \frac{1}{4} \int_0^t \|r_n(s)\|_2^2 ds + C \int_0^t (1 + \|v_n\|^2) \|v_n\|_2^2 \|r_n\|^2 ds, \end{aligned} \quad (5.67)$$

and

$$\begin{aligned} |I_2(t)| &\leq C \int_0^t \|r_n(s)\|_2 \|\check{v}\|^{\frac{1}{2}} \|\check{v}\|^{\frac{1}{2}} \|r_n\| ds + C \int_0^t \|r_n(s)\|_2 \|r_n\|^{\frac{1}{2}} \|r_n\|^{\frac{1}{2}} \left| \frac{\partial \check{v}}{\partial z} \right|^{\frac{1}{2}} \left| \nabla \frac{\partial \check{v}}{\partial z} \right|^{\frac{1}{2}} ds \\ &\leq \frac{1}{4} \int_0^t \|r_n(s)\|_2^2 ds + C \int_0^t (1 + \|\check{v}\|^2) \|\check{v}\|_2^2 \|r_n\|^2 ds. \end{aligned} \quad (5.68)$$

Moreover, by the Cauchy–Schwarz inequality, the Young inequality and referring to Page 13 in [13], we have

$$|I_3(t)| \leq \frac{1}{4} \int_0^t \|r_n(s)\|_2^2 ds + C \int_0^t (|r_n|^2 + |\nabla \eta_n|^2) ds. \quad (5.69)$$

Finally, it's easy to deduce

$$\begin{aligned} I_4(t) &= 2 \int_0^t \left( (\psi_1(s, Y_n) - \psi_1(s, \check{Y})) h_n^1, A_1 r_n \right) ds + 2 \int_0^t \left( \psi_1(s, \check{Y})(h_n^1 - h_1), A_1 r_n \right) ds \\ &:= J_1(t) + J_2(t). \end{aligned}$$

By the Cauchy–Schwarz inequality, the Young inequality and **Hypothesis H0**, we have

$$\begin{aligned} |J_1(t)| &\leq C \int_0^t |A_1 r_n| |(\psi_1(s, Y_n) - \psi_1(s, \check{Y})) h_n^1| ds \\ &\leq C \int_0^t |A_1 r_n| \|\psi_1(s, Y_n) - \psi_1(s, \check{Y})\|_{\mathcal{L}_2(U; V)} |h_n^1|_U ds \\ &\leq C \int_0^t |A_1 r_n| \|Y_n - \check{Y}\| |h_n^1|_U ds \\ &\leq C \int_0^t |A_1 r_n| \|r_n + \eta_n\| |h_n^1|_U ds \\ &\leq \frac{1}{4} \int_0^t \|r_n(s)\|_2^2 ds + C \int_0^t (\|r_n\|^2 + \|\eta_n\|^2) |h_n^1|_U^2 ds. \end{aligned}$$

Utilizing **Hypothesis H0** and [Corollary 5.4](#), we get

$$\begin{aligned} |J_2(t)| &\leq C \int_0^t \|\psi_1(s, \check{Y})(h_n^1 - h_1)\| \|r_n\| ds \\ &\leq C \int_0^t \|\psi_1(s, \check{Y})\|_{\mathcal{L}_2(U; V)} |h_n^1 - h_1|_U \|r_n\| ds \\ &\leq C \left( \int_0^t |h_n^1 - h_1|_U^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\psi_1(s, \check{Y})\|_{\mathcal{L}_2(U; V)}^2 \|r_n\|^2 ds \right)^{\frac{1}{2}} \\ &\leq C M^{\frac{1}{2}} \left( 1 + \sup_{t \in [0, T]} \|\check{Y}\| \right) \left( \int_0^t \|r_n\|^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$



Based on the above, we obtain

$$\begin{aligned}
 |I_4(t)| &\leq \frac{1}{4} \int_0^t \|r_n(s)\|_2^2 ds + C \int_0^t (\|r_n\|^2 + \|\eta_n\|^2) |h_n^1|_U^2 ds \\
 &\quad + CM^{\frac{1}{2}} \left(1 + \sup_{t \in [0, T]} \|\check{Y}\|\right) \left(\int_0^t \|r_n\|^2 ds\right)^{\frac{1}{2}}.
 \end{aligned} \tag{5.70}$$

From (5.67)–(5.70), we have

$$\begin{aligned}
 &\|r_n(t)\|^2 + \int_0^t \|r_n(s)\|_2^2 ds \\
 &\leq C \int_0^t (1 + \|v_n\|^2) \|v_n\|_2^2 \|r_n\|^2 ds + C \int_0^t (1 + \|\check{v}\|^2) \|\check{v}\|_2^2 \|r_n\|^2 ds \\
 &\quad + C \int_0^t (|r_n|^2 + |\nabla \eta_n|^2) ds + \int_0^t (\|r_n\|^2 + \|\eta_n\|^2) |h_n^1|_U^2 ds \\
 &\quad + CM^{\frac{1}{2}} \left(\int_0^T \|r_n\|^2 ds\right)^{\frac{1}{2}}.
 \end{aligned} \tag{5.71}$$

(2)  $H^1$  estimates of  $\eta_n$ . Similar to the above, we have

$$\begin{aligned}
 &\|\eta_n(t)\|^2 + \int_0^t \|\eta_n(s)\|_2^2 ds \\
 &\leq C \int_0^t (1 + \|\check{T}\| \|\check{T}\|_2) \|r_n\|^2 ds + C \int_0^t (1 + \|v_n\|^2) \|v_n\|_2^2 \|\eta_n\|^2 ds \\
 &\quad + C \int_0^t (\|r_n\|^2 + \|\eta_n\|^2) |h_n^2|_U^2 ds + CM^{\frac{1}{2}} \left(\int_0^T \|\eta_n\|^2 ds\right)^{\frac{1}{2}}.
 \end{aligned} \tag{5.72}$$

Set  $\rho_n = (r_n, \eta_n)^T$ . From (5.71) and (5.72), we have

$$\begin{aligned}
 &\|\rho_n(t)\|^2 + \int_0^t \|\rho_n(s)\|_2^2 ds \\
 &\leq C \int_0^t (1 + \|\check{v}\|^2) \|\check{v}\|_2^2 \|r_n\|^2 ds + C \int_0^t (1 + \|\check{T}\| \|\check{T}\|_2) \|r_n\|^2 ds
 \end{aligned}$$

$$\begin{aligned}
& + C \int_0^t (1 + \|v_n\|^2) \|v_n\|_2^2 \|\rho_n\|^2 ds + C \int_0^t \|\rho_n\|^2 (1 + |h_n|_U^2) ds \\
& + CM^{\frac{1}{2}} \left[ \left( \int_0^T \|r_n\|^2 ds \right)^{\frac{1}{2}} + \left( \int_0^T \|\eta_n\|^2 ds \right)^{\frac{1}{2}} \right]. \tag{5.73}
\end{aligned}$$

Applying Gronwall inequality to (5.73), we get

$$\begin{aligned}
& \sup_{t \in [0, T]} \|\rho_n(t)\|^2 + \int_0^T \|\rho_n(s)\|_2^2 ds \\
& \leq CM^{\frac{1}{2}} \left[ \left( \int_0^T \|r_n\|^2 ds \right)^{\frac{1}{2}} + \left( \int_0^T \|\eta_n\|^2 ds \right)^{\frac{1}{2}} \right] \times \\
& \exp \left\{ C \int_0^T \left[ (1 + \|\check{v}\|^2) \|\check{v}\|_2^2 + (1 + \|\check{T}\| \|\check{T}\|_2) + (1 + \|v_n\|^2) \|v_n\|_2^2 + 1 + |h_n|_U^2 \right] ds \right\}.
\end{aligned}$$

From Corollary 5.2 and Corollary 5.4, we have

$$\lim_{n \rightarrow \infty} \left( \int_0^T \|r_n\|^2 ds + \int_0^T \|\eta_n\|^2 ds \right) = 0$$

and

$$\begin{aligned}
& \exp \left\{ C \int_0^T \left[ (1 + \|\check{v}\|^2) \|\check{v}\|_2^2 + (1 + \|\check{T}\| \|\check{T}\|_2) + (1 + \|v_n\|^2) \|v_n\|_2^2 + 1 + |h_n|_U^2 \right] ds \right\} \\
& \leq C(T, \|Y_0\|, M).
\end{aligned}$$

Therefore,

$$\|Y_n - \check{Y}\|_{\mathfrak{H}}^2 = \sup_{t \in [0, T]} \|\rho_n(t)\|^2 + \int_0^T \|\rho_n(s)\|_2^2 ds \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad \square \tag{5.74}$$

## 6. Proof of the main result

**Proof of Theorem 1.1.** Due to Theorem 4.1, it suffices to verify the two conditions in **Hypothesis H1**.

**Step 1** First, we show that the set  $K_M = \{\mathcal{G}^0(\int_0^\cdot h(s)ds) : h \in T_M\}$  is compact subset of  $\mathfrak{H}$ , where  $\mathcal{G}^0$  is defined in (5.50).

Let  $\{Y_n\}$  be a sequence in  $K_M$  where  $Y_n$  corresponds to the solution of (5.22) with  $h_n \in T_M$  in place of  $h$ . By the weak compactness of  $T_M$  in  $L^2([0, T]; U)$ , there exists a subsequence (which we still denote it by  $\{h_n\}$ ) converging to a limit  $h$  weakly in  $T_M$ . Denote by  $Y_h$  the strong solution of (5.22) with  $h$ . Utilizing Corollary 5.7, it suffices to show that  $Y_n \rightarrow Y_h$  in  $\mathfrak{H}$ . Thanks to Theorem 5.8, we complete the proof.

**Step 2** Suppose that  $\{h_\varepsilon : \varepsilon > 0\} \subset \mathcal{A}_M$  for any fixed  $M < \infty$  and  $h_\varepsilon$  converge to  $h$  as  $T_M$ -valued random elements in distribution. Recalling the definition of  $\mathcal{G}^\varepsilon$  and by Girsanov's theorem,  $\bar{Y}_{h_\varepsilon} = \mathcal{G}^\varepsilon(W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot h^\varepsilon(s)ds)$  solves the following equation

$$\begin{cases} d\bar{Y}_{h_\varepsilon}(t) + A\bar{Y}_{h_\varepsilon}(t)dt + B(\bar{Y}_{h_\varepsilon}(t), \bar{Y}_{h_\varepsilon}(t))dt + G(\bar{Y}_{h_\varepsilon}(t))dt \\ = \psi(\bar{Y}_{h_\varepsilon})h_\varepsilon dt + \sqrt{\varepsilon}\psi(t, \bar{Y}_{h_\varepsilon})dW(t), \\ \bar{Y}_{h_\varepsilon}(0) = Y_0. \end{cases} \quad (6.75)$$

In order to transform (6.75) to a deterministic model, we introduce an auxiliary process  $Z_\varepsilon = (Z_\varepsilon^1, Z_\varepsilon^2)^T$  written as

$$\begin{cases} dZ_\varepsilon(t) + AZ_\varepsilon(t)dt = \sqrt{\varepsilon}\psi(t, \bar{Y}_{h_\varepsilon})dW(t), \\ Z_\varepsilon(0) = 0. \end{cases} \quad (6.76)$$

Applying Hypothesis H0, we have

$$\lim_{\varepsilon \rightarrow 0} E \left( \sup_{0 \leq t \leq T} \|Z_\varepsilon(t)\|^2 + \int_0^T \|Z_\varepsilon(t)\|_2^2 dt \right) = 0. \quad (6.77)$$

Since  $T_M$  is a Polish space, by the Skorohod representation theorem, we can construct a stochastic basis  $(\Omega^1, \mathcal{F}^1, P^1)$  and, on this basis,  $T_M \otimes T_M \otimes C([0, T]; V) \cap L^2([0, T]; D(A))$ -valued random variables processes  $(\tilde{h}_\varepsilon, \tilde{h}, \tilde{Z}_\varepsilon)$  such that the joint distribution of  $(\tilde{h}_\varepsilon, \tilde{Z}_\varepsilon)$  is the same as  $(h_\varepsilon, Z_\varepsilon)$ ,  $\tilde{Z}_\varepsilon \rightarrow 0$   $P^1$ -a.s. in  $C([0, T]; V) \cap L^2([0, T]; D(A))$ , the distribution of  $h$  coincides with  $\tilde{h}$  and  $\tilde{h}_\varepsilon \rightarrow \tilde{h}$   $P^1$ -a.s. as  $T_M$ -valued random elements. Let  $X_{\tilde{h}_\varepsilon}(t)$  be the solution of

$$\begin{cases} dX_{\tilde{h}_\varepsilon}(t) + AX_{\tilde{h}_\varepsilon}(t)dt + B(X_{\tilde{h}_\varepsilon}(t) + \tilde{Z}_\varepsilon, X_{\tilde{h}_\varepsilon}(t) + \tilde{Z}_\varepsilon)dt + G(X_{\tilde{h}_\varepsilon}(t) + \tilde{Z}_\varepsilon)dt \\ = \psi(t, X_{\tilde{h}_\varepsilon} + \tilde{Z}_\varepsilon)\tilde{h}_\varepsilon dt, \\ X_{\tilde{h}_\varepsilon}(0) = Y_0. \end{cases} \quad (6.78)$$

The uniqueness of (6.78) implies that  $X_{\tilde{h}_\varepsilon}$  has the same distribution with  $\bar{Y}_{h_\varepsilon} - Z_\varepsilon$ . Using similar arguments as Sect. 5, we get

$$X_{\tilde{h}_\varepsilon} \rightarrow X_{\tilde{h}} \text{ in } \mathfrak{H} \quad P^1\text{-a.s.},$$

where

$$\begin{cases} dX_{\tilde{h}}(t) + AX_{\tilde{h}}(t)dt + B(X_{\tilde{h}}(t), X_{\tilde{h}}(t))dt + G(X_{\tilde{h}}(t))dt = \psi(t, X_{\tilde{h}})\tilde{h}dt, \\ X_{\tilde{h}}(0) = Y_0. \end{cases}$$

Recalling the definition of  $\mathcal{G}^0$  and in view of  $X_{\tilde{h}_\varepsilon}$  has the same distribution with  $\tilde{Y}_{h_\varepsilon} - Z_\varepsilon$ , we know  $\tilde{Y}_{h_\varepsilon} - Z_\varepsilon \rightarrow Y_h$  in  $\mathfrak{H}$ . Moreover, by (6.77), we obtain  $Z_\varepsilon \rightarrow 0$  in distribution in  $\mathfrak{H}$ . Thus,  $\tilde{Y}_{h_\varepsilon} \rightarrow Y_h$  in distribution in  $\mathfrak{H}$ . We complete the proof.  $\square$

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