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Domain decomposition methods for a class of spatially heterogeneous delayed reaction–diffusion equations ☆

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Abstract

We derive and study a class of delayed reaction–diffusion equations with spatial heterogeneity, which models the population of a single species with different habitats for mature and immature individuals. We introduce new solid cones, obtain spectral bounds of several spatial heterogeneous operators, and establish limiting non-negativeness property for the whole space and the eventual comparison principle for bounded domains. As a result, we develop new domain decomposition methods so that one can compare solutions with those to associated equations from a suitable bounded spatial domain to the whole space. Then by employing domain decomposition methods and dynamical system approaches, we obtain threshold results under the supremum norm. These results are greatly different from the existing ones of other evolution equations in unbounded domains or the whole space. The main results are applied to two examples with the Ricker birth function and with the Mackey–Glass birth function. It reveals that the size of the immature habitat can affect the reproduction and spread of the population.
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1. Introduction

Based on the interaction of intrinsic dynamics (birth and death) and the spatial diffusion in a structured population (Metz and Diekmann [27]), various spatially nonlocal and temporally delayed reaction–diffusion equation models have been derived. To be more precise, when considering the population growth of a species with age structure and diffusion in a spatial domain Λ, one faces a reaction–diffusion equation of the form,

$$\frac{\partial u}{\partial t}(t,x) = D_m \Delta u(t,x) - d_m u(t,x) + \varepsilon \int_\Lambda \Gamma(\alpha, x, y) b(u(t-\tau, y))\, dy, \ \ (t,x) \in (0,\infty) \times \Lambda. \tag{1.1}$$

Here $u(t,x)$ is the density (or number) of the mature population of a species at time t and location x; D_m and d_m are the diffusion rate and death rate of the mature population, respectively; τ is the maturation time for the species; the other two indirect parameters ε and α are defined by $\varepsilon = \exp(-\int_0^\tau d_I(a)da)$ and $\alpha = \int_0^\tau D_I(a)da$, where $D_I(a)$ and $d_I(a)$ are the age dependent diffusion rate and death rate of the immature population of the species (with $a \in [0,\tau]$), respectively; $b(u)$ is a birth function. Obviously, α measures the mobility of the immature individuals, ε measures the probability that a newborn can survive the immature period, and the kernel function $\Gamma(\alpha, x, y)$ accounts for the probability that an individual born at location y will be at location x after τ time units when it becomes mature. The kernel Γ is related to the semi-group $S(t)$ generated by solutions of the equation $\frac{\partial v}{\partial t}(t,x) = \Delta v(t,x)$ with the given boundary condition in the sense that $S(t)[\phi](x) = \int_\Lambda \Gamma(t,x,y)\phi(y)\,dy$. See [18,33,42,47] or the survey [11] for details on the derivation of such a model. In particular, for a species whose immature individuals do not move ($D_I = 0$), the kernel $\Gamma(0,x,y)$ is given by the Dirac delta function centered at x and hence (1.1) is reduced to the following spatially local equation,

$$\frac{\partial u}{\partial t}(t,x) = D_m \Delta u(t,x) - d_m u(t,x) + \varepsilon b(u(t-\tau,x)), \ \ (t,x) \in (0,\infty) \times \Lambda. \tag{1.2}$$

When the spatial domain Λ is unbounded, the existence and other qualitative properties of traveling wave solutions and the theory of asymptotic spreading of (1.1) or (1.2) are some main concerns, which have been investigated by many authors for some particular forms of the kernel function $\Gamma(\alpha, x, y)$ (to name a few, see [1,5–7,9,10,19,20,22,26,29,33,35–38,41,45,49] and the references therein).

When the spatial domain Λ is bounded, various boundary conditions were posed for (1.1) or (1.2) in Liang et al. [18] and Xu and Zhao [42], among which are the homogeneous Neumann boundary condition (or zero flux condition) and the homogeneous Dirichlet boundary condition with the former accounting for an isolated domain and the latter explaining a scenario where the boundary is hostile for the species. The global dynamics of the semiflow generated by such models subject to either the Dirichlet boundary condition or the Neumann boundary condition has been intensively and successfully studied (see, for example, [2,8,12,15,32,34,40,43,46,48]).

In the real world, the habitat (or survival region) Ω of a population's juvenile is sometimes different from that of the mature individuals. For example, tadpoles of an amphibian live in water while mature individuals can live in both land and water. Moreover, animals are typically in favor of a more comfortable environment to produce and train their offsprings, and hence adult individuals need to go to a wider region in order to get more food and living resources. The difference between habitats of mature and immature individuals, and the boundary characteristics

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of the corresponding regions can lead to different mathematical models and yield some different mathematical problems.

The purpose of this paper is to tackle one of the above-mentioned such problems under scenarios where boundaries of habitats are hostile for the species. Here the immature habitat is assumed to be bounded while the bigger mature habitat can be either bounded or unbounded. Under some standard hypotheses, the model is derived in the Appendix. In Section 2, we give some notations and provide some basic results on the boundedness, compactness, monotonicity, and smoothness of solution maps. In particular, we point out that the boundedness of the immature habitat directly makes the solution semiflow k-set contract. This, combined with some results in [14], confirms the compactness of solution trajectories and ensures the existence of a compact global attractor with the supremum norm. In Section 3, we present four lemmas to describe a new solid cone, spectral bounds, the limiting non-negative property of a class of functions, as well as the comparabilities of (limiting) solutions for (2.3) and a given function in a suitable domain. These are the keys to developing the domain decomposition methods in Section 4, which enable us to compare solutions with those of the associated equations from a suitable bounded domain to the whole space. With this and standard dynamical system theoretical arguments, we can derive threshold dynamics under the supremum norm, which are very different from the existing results of other evolution equations in unbounded domains or the whole space for which the theory of traveling waves and asymptotic spreading of (1.1) are the main concerns in [1,6,7,5,9,10,19,20, 22,26,29,33,35–38,41,45,49] and the references therein. In Section 5, we illustrate the theoretical results with two particular birth functions, the Ricker function and the Mackey–Glass function. It turns out that the habitats of mature and immature can affect the dynamics of the model. In particular, the boundedness of the immature habitat directly enables us to obtain the dynamics of the model in the sense of the supremum norm. In a future work, we shall consider the case where the immature habitat is unbounded.

2. Preliminary results

We first introduce some notations. Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$ be the sets of all nonnegative integers, real numbers, and nonnegative real numbers, respectively. For a given positive integer N, $\mathbb{R}^N$ is the N-dimensional Euclidean space. Let Ω and Λ be open domains in $\mathbb{R}^N$ with smooth boundaries such that $\Omega \subseteq \Lambda$. Let $\nu_{x,\Omega}$ (respectively, $\nu_{x,\Lambda}$) be the unit outward normal direction at $x \in \partial\Omega$ (respectively, $x \in \partial\Lambda$) of $\partial\Omega$ (respectively, $\partial\Lambda$). Let Δ denote the Laplacian operator in $\mathbb{R}^N$.

Let $X = BC(\mathbb{R}^N, \mathbb{R})$ (respectively, $Y = BUC(\mathbb{R}^N, \mathbb{R})$) be the set of all bounded and continuous functions (respectively, all bounded and uniformly continuous functions) from $\mathbb{R}^N$ to $\mathbb{R}$ equipped with the usual supremum norm $||\cdot||_X$ (respectively, $\|\cdot\|_Y$). Then both X and Y are Banach spaces. Let $X_+ = \{\phi \in X : \phi(x) \geq 0 \text{ for all } x \in \mathbb{R}^N\}$, $X_+^\circ = \{\phi \in X : \phi(x) > 0 \text{ for all } x \in \mathbb{R}^N\}$, $Y_+ = X_+ \cap Y$, and $Y_+^\circ = X_+^\circ \cap Y$. Let $C = C([-\tau, 0], X)$ be the Banach space of continuous functions from $[-\tau, 0]$ into X equipped with the supremum norm $||\cdot||_C$, $C_+ = C([-\tau, 0], X_+)$, and $C_+^\circ = C([-\tau, 0], X_+^\circ)$.

Let $X(\Lambda) = \{\phi \in X : \phi(x) = 0 \text{ for all } x \notin \Lambda\}$, $X_+(\Lambda) = X_+ \bigcap X(\Lambda)$, $X_+^\circ(\Lambda) = \{\phi \in X_+(\Lambda) : \phi(x) > 0 \text{ for all } x \in \Lambda\}$, $X_0(\Lambda; \Omega) = \{\phi \in X_+(\Lambda) : \phi|_\Omega = 0\}$, and $X_+(\Lambda; \Omega) = X_+(\Lambda) \setminus X_0(\Lambda; \Omega)$. We also introduce the following subset of $X_+(\Lambda)$,

$$Z_+(\Lambda) = \left\{\phi \in X_+^\circ(\Lambda) \cap C^1(\overline{\Lambda}) \cap C^2(\Lambda) : \frac{\partial \phi}{\partial \nu_{x,\Lambda}}(x) < 0 \text{ for all } x \in \partial\Lambda\right\}.$$

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Denote $C(\Lambda) = C([-\tau, 0], X(\Lambda))$, $C_+(\Lambda) = C_+ \bigcap C(\Lambda)$, $C^\circ_+(\Lambda) = \{\varphi \in C_+(\Lambda) : \varphi(\theta, x) > 0 \text{ for all } (\theta, x) \in [-\tau, 0] \times \Lambda\}$, $C_0(\Lambda; \Omega) = \{\varphi \in C_+(\Lambda) : \varphi|_{[-\tau,0]\times\Omega} = 0$ and $\varphi(0, \cdot)|_\Lambda = 0\}$, and $C_+(\Lambda; \Omega) = C_+(\Lambda) \setminus C_0(\Lambda; \Omega)$. It follows that X_+, C_+, $X_+(\Lambda)$, and $C_+(\Lambda)$ are closed cones in X, C, $X(\Lambda)$, and $C(\Lambda)$, respectively.

For convenience, we identify an element $\varphi \in C$ with a function from $[-\tau, 0] \times \mathbb{R}^N$ into $\mathbb{R}$. For any $a \in \mathbb{R}$, when no confusion arises, a also denotes the constant function taking the constant value a in the function spaces introduced above. For any $\phi \in X$, when no confusion arises, ϕ also denote the function taking the value $\phi(x)$ for all $(\theta, x) \in [-\tau, 0] \times \mathbb{R}^N$.

For any $\xi, \eta \in *$, we write $\xi \geq_* \eta$ if $\xi - \eta \in *_+$; $\xi >_* \eta$ if $\xi \geq_* \eta$ and $\xi \neq \eta$; $\xi \gg_* \eta$ if $\xi - \eta \in *^\circ_+$, where $*$ stands for any of X, C, $X(\Lambda)$, and $C(\Lambda)$. For simplicity of notations, when there is no confusion about the spaces involved, we just write $\geq$, $>$, $\gg$, and $||\cdot||$ for $\geq_*$, $>_*$, $\gg_*$, and $||\cdot||_*$, respectively.

For any given $r > 0$, let $X_r(\Lambda) = \{\phi \in X(\Lambda) : -r \leq \phi \leq r\}$, $X^+_r(\Lambda) = X_r(\Lambda) \cap X_+(\Lambda)$, $C_r(\Lambda) = \{\varphi \in C(\Lambda) : -r \leq \varphi \leq r\}$, $C^+_r(\Lambda) = C_r(\Lambda) \cap C_+(\Lambda)$, $B_r(\Lambda) = \{x \in \mathbb{R}^N : ||x|| \leq r \text{ or } x \notin \Lambda\}$, and $X_r = \{\phi \in X : -r \leq \phi \leq r\}$

For an interval $I \subseteq \mathbb{R}$, let $I + [-\tau, 0] = \{t + \theta : t \in I \text{ and } \theta \in [-\tau, 0]\}$. For $u : (I + [-\tau, 0]) \times \mathbb{R}^N \to \mathbb{R}$ and $t \in I$, we write $u_t(\cdot, \cdot)$ for the function defined by $u_t(\theta, x) = u(t + \theta, x)$ for $(\theta, x) \in [-\tau, 0] \times \mathbb{R}^N$.

Let $S_\Lambda(t)$ be the semigroup generated by the linear system

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = \Delta u(t, x), & (t, x) \in (0, \infty) \times \Lambda, \\ u(t, x) = 0, & (t, x) \in \mathbb{R}_+ \times (\mathbb{R}^N \setminus \Lambda), \\ u(0, x) = \phi(x), & x \in \mathbb{R}^N, \end{cases} \tag{2.1}$$

where $\phi \in X(\Lambda)$. Denote the Green function of the Dirichlet problem for the heat equation on Λ by $\Gamma^*_\Lambda : \mathbb{R}_+ \times \bar{\Lambda} \times \bar{\Lambda} \to \mathbb{R}_+$. By extending the domain of Γ^*_Λ to $\mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}^N$, we define $\Gamma_\Lambda : \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ by

$$\Gamma_\Lambda(t, x, y) = \begin{cases} \Gamma^*_\Lambda(t, x, y) & \text{for } (t, x, y) \in \mathbb{R}_+ \times \bar{\Lambda} \times \bar{\Lambda}, \\ 0 & \text{for } (t, x, y) \in \mathbb{R}_+ \times (\mathbb{R}^N \times \mathbb{R}^N \setminus \bar{\Lambda} \times \bar{\Lambda}). \end{cases}$$

Then $S_\Lambda(t)[\phi](x) = \int_{\mathbb{R}^N} \Gamma_\Lambda(t, x, y)\phi(y)\mathrm{d}y$ with $\phi \in X(\Lambda)$ solves (2.1). In particular, for $(x, \phi) \in \mathbb{R}^N \times X$, $S_{\mathbb{R}^N}(t)[\phi](x)$ satisfies

$$\begin{cases} S_{\mathbb{R}^N}(0)[\phi](x) = \phi(x) \\ S_{\mathbb{R}^N}(t)[\phi](x) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} \phi(y) \exp\left(-\frac{||x-y||^2}{4t}\right) \mathrm{d}y \quad \text{for all } t \in (0, \infty). \end{cases}$$

Note that $S_{\mathbb{R}^N}(t)|_Y$ $(t \in \mathbb{R}_+)$ is an analytic and strongly continuous semigroup on Y generated by the Y-realization Δ_Y of Δ (see, for example, Daners and Medina [3]).

In this paper, we will study asymptotic behaviors of solutions of (A.8), and thus consider the mild solutions of (A.8), which solve the following integral equation with the given initial values,

$$\begin{cases} u(t, \cdot) = e^{-d_m t} S_\Lambda(D_m t)[\varphi(0, \cdot)] \\ \qquad\qquad + \varepsilon \int_0^t e^{-d_m(t-s)} S_\Lambda(D_m(t-s))[K_\Lambda[b(u(s-\tau, \cdot))]]\mathrm{d}s, \qquad t \in \mathbb{R}_+, \\ u_0 = \varphi \in C_+(\Lambda), \end{cases} \tag{2.2}$$

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where $K_\Lambda : X(\Lambda) \to X(\Lambda)$ is defined by

$$K_\Lambda[\phi](x) = \int_\Lambda \Gamma_\Omega(\alpha, x, y)\phi(y)\mathrm{d}y = \int_{\mathbb{R}^N} \Gamma_\Omega(\alpha, \mathrm{x}, \mathrm{y})\phi(\mathrm{y})\mathrm{d}\mathrm{y} \qquad \text{for all } x \in \mathbb{R}^N.$$

According to the definitions of K_Λ and $S_\Lambda(t)$, we can easily see that the operators K_Λ and $S_\Lambda(t)$ can be extended to operators from $L^\infty(\mathbb{R}^N, \mathbb{R})$ into $L^\infty(\mathbb{R}^N, \mathbb{R})$ by using the same corresponding formulae. Note that $X(\Lambda) \subseteq X \subseteq L^\infty(\mathbb{R}^N, \mathbb{R})$.

For convenience, let $d = D_m$, $\mu = d_m$, and $f(u) = \frac{\varepsilon b(u)}{d_m}$ for all $u \in \mathbb{R}_+$. Then we consider the following rescaled form of (2.2) with the initial value $\varphi \in C_+(\Lambda)$,

$$\begin{cases} u(t,\cdot) = T_\Lambda(t)[\varphi(0,\cdot)] + \mu \int_0^t T_\Lambda(t-s)[K_\Lambda[f(u(s-\tau,\cdot))]]\mathrm{d}s, & t \in \mathbb{R}_+, \\ u_0 = \varphi \in C_+(\Lambda). \end{cases} \tag{2.3}$$

Let $k_\Lambda = ||K_\Lambda|| = \sup\left\{\frac{||K_\Lambda[\phi]||}{||\phi||} : \phi \in X_+(\Lambda) \setminus \{0\}\right\}$. Clearly, $e^{-\lambda_\Omega \alpha} \le k_\Lambda \le 1$, where λ_Ω is the *first eigenvalue* of the operator $-\Delta$ in Ω with the homogeneous Dirichlet boundary condition.

For given $\varphi \in C_+(\Lambda)$, by the method of steps and the definitions of K_Λ and $T_\Lambda(t)$, it is easy to see that (2.3) has a unique solution which exists on $\mathbb{R}_+$. Let $u^\varphi(\Lambda; t, x)$ represent the unique solution of (2.3), that is, the mild solution of (A.8) in the sense of Martin and Smith [24,25]. Then it is clear that $(u^\varphi(\Lambda;\cdot,\cdot))_t \in C_+(\Lambda)$ for all $t \in \mathbb{R}_+$ and $\varphi \in C_+(\Lambda)$. Thus solution maps of (2.3) induce a continuous semiflow on $C_+(\Lambda)$. In view of Lemma 2.3-(i) and (iii), we know that a mild solution of (A.8) is also a classical solution of (A.8) for all $t > \tau$ when f is continuously differentiable (see, for example, [24,25,39]). Moreover, it is obvious that a classical solution of (A.8) is also a mild solution of (A.8). Therefore, as far as asymptotic behaviors of solutions of (A.8) are concerned, it is sufficient to only consider solutions of (2.3).

Denote by Φ_Λ the solution semiflow of (2.3), that is, $\Phi_\Lambda : \mathbb{R}_+ \times C_+(\Lambda) \to C_+(\Lambda)$ is defined by $\Phi_\Lambda(t,\varphi) = (u^\varphi(\Lambda;\cdot,\cdot))_t$ for all $(t,\varphi) \in \mathbb{R}_+ \times C_+(\Lambda)$. We also write $\Phi_\Lambda(t,\varphi; f)$ for $\Phi_\Lambda(t,\varphi)$ to emphasize the dependence on the nonlinearity f when it is needed.

For any $0 < s < t < \infty$, define the mapping $\Phi_\Lambda(t,\cdot)|^s : C_+(\Lambda) \ni \varphi \mapsto \Phi_\Lambda(t,\cdot)|^s[\varphi] \in L^\infty([-s,0] \times \mathbb{R}^N, \mathbb{R})$, where $\Phi_\Lambda(t,\cdot)|^s[\varphi](\theta,x) = \Phi_\Lambda(t+\theta,\varphi)(0,x)$ for all $(\theta,x) \in [-s,0] \times \mathbb{R}^N$.

Now we collect some basic properties of K_Λ and $S_\Lambda(t)$ from [3,30,32].

Lemma 2.1. *The following statements are true.*

(i) $S_\Lambda(t)|_{Y \cap X(\Lambda)}$ *is an analytic and strongly continuous semigroup on* $Y \cap X(\Lambda)$.
(ii) $S_\Lambda(t)[X_+(\Lambda)] \subseteq X_+(\Lambda)$ *for all* $t \in \mathbb{R}_+$.
(iii) $S_\Lambda(t)[X_+(\Lambda) \setminus \{0\}] \subseteq Z_+(\Lambda)$ *and* $S_\Lambda(t)[X] \subseteq Y \cap X(\Lambda)$ *for* $t > 0$.
(iv) *For all* $t > 0$ *and* $\phi \in X$ *with* $\phi|_\Lambda \in C(\Lambda; \mathbb{R}_+) \setminus \{0\}$, $S_\Lambda(t)[\phi] \in X_+^\circ(\Lambda)$, $\frac{\partial S_\Lambda(t)[\phi]}{\partial \nu_{x,\Lambda}}(x) < 0$ *for all* $x \in \partial\Lambda$, *and* $S_\Lambda(t)[\phi] \le ||\phi||$.
(v) *If* $\sigma, r > 0$, *then* $\{S_\Lambda(t)[\phi]|_{B_\sigma(\Lambda)} : \phi \in X_r\}$ *is pre-compact in* $L^\infty(B_\sigma(\Lambda), \mathbb{R})$.
(vi) *There exists a constant* C_1 *such that* $||A_{S_\Lambda}[S_\Lambda(t)[\phi]]|| \le \frac{C_1}{t}||\phi||$ *for all* $(t,\phi) \in (0,\infty) \times X(\Lambda)$. *Here* A_{S_Λ} *is the generator of the strongly continuous semigroup* $S_\Lambda(\cdot)$ *and* $D(A_{S_\Lambda}) \subseteq X(\Lambda)$ *is the domain of* A_{S_Λ}.

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(vii) *For any $t_1 > t_2 > 0$, $M > 0$, and a bounded and smooth domain Ω^*, there exists $A = A(t_1, t_2, M, \Omega^*) > 0$ such that $||S_\Lambda(t)[\phi]|_{\overline{\Omega^*}}||_{C^2(\overline{\Omega^*})} \le A$ for all $(t, \phi) \in [t_1, t_2] \times X_M^+(\Lambda)$. Here $||g||_{C^2(\overline{\Omega^*})}$ represents the C^2-norm of the function g on $\overline{\Omega^*}$. Thus, for any given $\phi \in X_+(\Lambda)$, $\{S_\Lambda(t)[\phi]|_{\overline{\Omega^*}} : t \in [t_1, t_2]\}$ is compact in $C^1(\overline{\Omega^*})$.*

Lemma 2.2. *$K_\Lambda[X] \subseteq X(\Omega)$, $K_\Lambda[X_+] \subseteq X_+(\Omega)$, $K_\Lambda[X_0(\Lambda; \Omega)] = \{0\}$, and $K_\Lambda[X_+(\Lambda; \Omega)] \subseteq X_+^\circ(\Omega)$.*

By some simple computations and Lemma 2.1, we can easily obtain the following results.

Lemma 2.3. *Let $T_\Lambda(t)[\phi] = e^{-d_m t} S_\Lambda(D_m t)[\phi]$ for all $(t, \phi) \in \mathbb{R}_+ \times X$. Then the following statements are true.*

(i) *$T_\Lambda(t)|_{Y \cap X(\Lambda)}$ is an analytic and strongly continuous semigroup on $Y \cap X(\Lambda)$.*
(ii) *$T_\Lambda(t)[X_+(\Lambda)] \subseteq X_+(\Lambda)$ for all $t \in \mathbb{R}_+$.*
(iii) *$T_\Lambda(t)[X_+(\Lambda) \setminus \{0\}] \subseteq Z_+(\Lambda)$ and $T_\Lambda(t)[X] \subseteq Y \cap X(\Lambda)$ for $t > 0$.*
(iv) *If $t > 0$ and $\phi \in X$ with $\phi|_\Lambda \in C(\Lambda; \mathbb{R}_+) \setminus \{0\}$, then $T_\Lambda(t)[\phi] \in X_+^\circ(\Lambda)$, $\frac{\partial T_\Lambda(t)[\phi]}{\partial \nu_{x,\Lambda}}(x) < 0$ for all $x \in \partial\Lambda$, and $T_\Lambda(t)[\phi] \le ||\phi|| e^{-d_m t}$.*
(v) *If $\sigma, r > 0$, then $\{T_\Lambda(t)[\phi]|_{B_\sigma(\Lambda)} : \phi \in X_r\}$ is pre-compact in $L^\infty(B_\sigma(\Lambda), \mathbb{R})$.*
(vi) *There exists a constant C_1 such that $||A_{T_\Lambda}[T_\Lambda(t)[\phi]]|| \le \frac{C_1}{t}||\phi||$ for all $(t, \phi) \in (0, \infty) \times X(\Lambda)$. Here A_{T_Λ} is the generator of the strongly continuous semigroup $T_\Lambda(\cdot)$ and $D(A_{T_\Lambda}) \subseteq X(\Lambda)$ is the domain of A_{T_Λ}.*
(vii) *For any $t_1 > t_2 > 0$, $M > 0$, and a bounded and smooth domain Ω^*, there exists $A = A(t_1, t_2, M, \Omega^*) > 0$ such that $||T_\Lambda(t)[\phi]|_{\overline{\Omega^*}}||_{C^2(\overline{\Omega^*})} \le A$ for all $(t, \phi) \in [t_1, t_2] \times X_M^+(\Lambda)$. Here $||g||_{C^2(\overline{\Omega^*})}$ represents the C^2-norm of the function g on $\overline{\Omega^*}$. Thus $\{T_\Lambda(t)[\phi]|_{\overline{\Omega^*}} : t \in [t_1, t_2]$ and $\phi \in X_M^+(\Lambda)\}$ is compact in $C^1(\overline{\Omega^*})$.*

In the sequel, we always assume that Ω is bounded. We also make the following assumptions on the nonlinearity f.

(H1) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable in $\mathbb{R}_+$ with $f(0) = 0$; moreover, there exists a sequence $\{r_n\}$ in $(0, \infty)$ such that $\lim_{n\to\infty} r_n = \infty$ and $f([0, r_n]) \subseteq [0, \frac{r_n}{k_\Lambda}]$ for $n \in \mathbb{N}$.

The following result establishes the positivity, boundedness, and compactness of the solution semiflow of (2.3). We should point out that the boundedness of the immature habitat Ω directly makes the solution semiflow of (2.3) k-set contract with the supremum norm. This, combined with Theorem 2.4.1 in [14], gives the compactness of solution trajectories and the existence of a compact global attractor.

Proposition 2.1. *The following results are true.*

(i) *$\Phi_\Lambda(\mathbb{R}_+ \times C_+(\Lambda)) \subseteq C_+(\Lambda)$, $\Phi_\Lambda(\mathbb{R}_+ \times C_0(\Lambda; \Omega)) = \{0\}$, $\Phi_\Lambda(\mathbb{R}_+ \times C_+(\Lambda; \Omega)) \subseteq C_+(\Lambda; \Omega)$, $\Phi_\Lambda((\tau, \infty) \times C_+(\Lambda; \Omega)) \subseteq C_+^\circ(\Lambda)$, and $\Phi_\Lambda((\tau, \infty) \times C_+(\Lambda)) \subseteq C([-\tau, 0], Y_+)$.*
(ii) *$\Phi_\Lambda(\mathbb{R}_+ \times C_{r_n}^+(\Lambda)) \subseteq C_{r_n}^+(\Lambda)$ for all $n \in \mathbb{N}$.*

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(iii) *Let $0 < s < t$ and $k = e^{-\mu(t-s)}$. Then the mapping $\Phi_\Lambda(t,\cdot)|^s : C_+(\Lambda) \to L^\infty([-s,0]\times\mathbb{R}^N,\mathbb{R})$ is a k-set contraction mapping in the sense of $\alpha_{-s,0}[\Phi_\Lambda(t,\mathcal{A})|^s] \le k\alpha_{-\tau,0}[\mathcal{A}]$ for any bounded subset $\mathcal{A}$ in $C_+(\Lambda)$. In particular, $\Phi_\Lambda(t,\cdot)$ is a k-set contraction mapping with respect to the non-compactness measure $\alpha_{-\tau,0}$ in $C_+(\Lambda)$, where $t > \tau$ and $k = e^{-\mu(t-\tau)}$. Here $\alpha_{a,b}[\mathcal{A}]$ is the Kuratowksi non-compactness measure of the subset $\mathcal{A}$ in $L^\infty([a,b]\times\mathbb{R}^N,\mathbb{R})$ equipped with the L^∞-norm.*

Proof. (i) and (ii) follow from (2.3), Lemma 2.2, and Lemma 2.3 as well as (H1).

(iii) Fix $0 < s < t < \infty$ and $k = e^{-\mu(t-s)}$. Note that $||T_\Lambda(t)[\phi]|| \le e^{-\mu t}||\phi||$ for all $(t,\phi) \in \mathbb{R}_+ \times C(\Lambda)$. Then the mapping $\mathcal{F}_{s,t} : C(\Lambda) \ni \varphi \mapsto \mathcal{F}_{s,t}[\varphi] \in L^\infty([-s,0]\times\mathbb{R}^N,\mathbb{R})$ is a k-set contraction mapping in $C(\Lambda)$, where $\mathcal{F}_{s,t}[\varphi](\theta,x) = T_\Lambda(t+\theta)[\varphi(0,\cdot)]$ for all $(\theta,x) \in [-s,0]\times\mathbb{R}^N$.

Let us define $\mathcal{G}_t : BC([-\tau,t-\tau]\times\mathbb{R}^N,\mathbb{R}) \to BC([-t,0]\times\mathbb{R}^N,\mathbb{R})$ by $\mathcal{G}_t[\varphi](\theta,x) = \mu\int_0^{t+\theta} T_\Lambda(t+\theta-\varsigma)[K_\Lambda[f(\varphi(\varsigma-\tau,\cdot))]](x)\mathrm{d}\varsigma$ for all $\varphi \in BC([-\tau,t-\tau]\times\mathbb{R}^N,\mathbb{R})$ and $(\theta,x) \in [-t,0]\times\mathbb{R}^N$. We claim that $\mathcal{G}_t$ is a compact map for $0 < t < \infty$. In fact, for any $t, \alpha^* > 0$, $\varphi \in BC([-\tau,t-\tau]\times\mathbb{R}^N,[-\alpha^*,\alpha^*])$, and $(\theta,x) \in [-s,0]\times\mathbb{R}^N$ with $||x|| \ge 1 + 3\max\{||y|| : y \in \Omega\}$, it follows from Lemma 2.2, Lemma 2.3, and Lemma 2.1-(vi) in [44] that

$$
\begin{aligned}
|\mathcal{G}_t[\varphi](\theta,x)| &= \left|\int_0^{t+\theta} T_\Lambda(t+\theta-\varsigma)[\mu K_\Lambda[f(\varphi(\varsigma-\tau,\cdot))]](x)\mathrm{d}\varsigma\right| \\
&\le \max f([-\alpha^*,\alpha^*])\int_0^{t+\theta}\int_\Omega \mu e^{-\mu(t+\theta-\varsigma)}\Gamma_\Lambda(t+\theta-\varsigma,x,y)\mathrm{d}y\mathrm{d}\varsigma \\
&\le \max f([-\alpha^*,\alpha^*])\int_0^{t+\theta}\int_\Omega \mu e^{-\mu(t+\theta-\varsigma)}\frac{1}{(4\pi(t+\theta-\varsigma))^{\frac{N}{2}}}e^{\frac{-||x-y||^2}{4(t+\theta-\varsigma)}}\mathrm{d}y\mathrm{d}\varsigma \\
&\le \frac{\max f([-\alpha^*,\alpha^*])\mu|\Omega|}{(4\pi)^{\frac{N}{2}}}\int_0^\infty e^{-\mu\varsigma}\varsigma^{-\frac{N}{2}}e^{\frac{-||x||^2-1}{12\varsigma}}\mathrm{d}\varsigma \\
&\le \frac{\max f([-\alpha^*,\alpha^*])\mu|\Omega|}{(4\pi)^{\frac{N}{2}}}\left[\int_1^\infty e^{-\mu\varsigma}\varsigma^{-\frac{1}{2}}e^{\frac{-||x||^2-1}{12\varsigma}}\mathrm{d}\varsigma + \int_0^1 e^{-\mu\varsigma}\varsigma^{-\frac{N}{2}}e^{\frac{-||x||^2-1}{12\varsigma}}\mathrm{d}\varsigma\right] \\
&\le \frac{\max f([-\alpha^*,\alpha^*])\mu|\Omega|}{(4\pi)^{\frac{N}{2}}}\left[\int_0^\infty e^{-\mu\varsigma}\varsigma^{-\frac{1}{2}}e^{\frac{-||x||^2}{12\varsigma}}\mathrm{d}\varsigma + \int_1^\infty e^{-\frac{\mu}{\varsigma}}\varsigma^{-\frac{1}{2}}e^{\frac{-||x||^2)\varsigma}{12}}\mathrm{d}\varsigma\right] \\
&\le \frac{\max f([-\alpha^*,\alpha^*])\mu|\Omega|(6N)^{\frac{N}{2}}}{(4\pi)^{\frac{N}{2}}}\int_0^\infty e^{-\mu\varsigma}\varsigma^{-\frac{1}{2}}e^{\frac{-||x||^2}{12\varsigma}}\mathrm{d}\varsigma \\
&\le \frac{\max f([-\alpha^*,\alpha^*])|\Omega|(6N)^{\frac{N}{2}}}{(4\pi)^{\frac{N-2}{2}}}\frac{\sqrt{\mu}}{2}e^{-\frac{\sqrt{\mu}||x||}{3}}.
\end{aligned}
$$

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Then for any $\epsilon > 0$, there is $\iota > 0$ such that $|\xi(\theta, x)| < \epsilon$ for all $(\theta, x, \xi) \in [-t, 0] \times \mathbb{R}^N \setminus B_\iota \times \mathcal{G}_t[BC([-\tau, t-\tau] \times \mathbb{R}^N, [-\alpha^*, \alpha^*])]$. Again, it is easy to see that, for any $t, \alpha^*, \sigma > 0$, $\mathcal{G}_t[C([-\tau, t-\tau] \times \mathbb{R}^N, [-\alpha^*, \alpha^*])]|_{[-t,0]\times B_\sigma}$ is pre-compact in $L^\infty([-t, 0] \times B_\sigma, \mathbb{R})$ due to the Arzelà–Ascoli Theorem and Lemma 2.1-(v), Lemma 2.3-(v), where $B_\sigma = \{x \in \mathbb{R}^N : ||x|| \le \sigma\}$. It follows that there exist $\xi^1, \xi^2, \ldots, \xi^l \in \mathcal{G}_t[BC([-\tau, t-\tau] \times \mathbb{R}^N, [-\alpha^*, \alpha^*])]$ such that $\mathcal{G}_t[C([-\tau, t-\tau] \times \mathbb{R}^N, [-\alpha^*, \alpha^*])]|_{[-t,0]\times B_\iota} \subset \bigcup_{i=1}^{l} B_\epsilon(\xi^i)|_{[-t,0]\times B_\iota}$ and thus $\mathcal{G}_t[C([-\tau, t-\tau] \times \mathbb{R}^N, [-\alpha^*, \alpha^*])] \subset \bigcup_{i=1}^{l} B_\epsilon(\xi^i)$, where $B_\epsilon(\xi^i) = \{\zeta \in BC([-t, 0] \times \mathbb{R}^N, \mathbb{R}) : ||\zeta - \xi^i||_{L^\infty([-t,0]\times\mathbb{R}^N,\mathbb{R})} < \epsilon\}$. As a result, $\mathcal{G}_t[BC([-\tau, t-\tau] \times \mathbb{R}^N, [-\alpha^*, \alpha^*])]$ is precompact in $L^\infty([-t, 0] \times \mathbb{R}^N, \mathbb{R})$ and the claim is proved.

Let $\mathcal{A}$ be a bounded subset in $C_+(\Lambda)$. Then there is $\gamma \in \mathbb{R}_+$ such that $\mathcal{A} \subseteq BC([-\tau, 0] \times \mathbb{R}^N, [-\gamma, \gamma])$. By the definition of $\Phi_\Lambda(t, \cdot)|^s$, we have $\Phi_\Lambda(t, \mathcal{A})|^s \subseteq \mathcal{F}_{s,t}[\mathcal{A}] + \mathcal{G}_s[BC([-\tau, s-\tau] \times \mathbb{R}^N, [-\gamma, \gamma)]$. Thus, $\alpha_{-s,0}[\Phi_\Lambda(t, \mathcal{A})|^s] \le \alpha_{-s,0}[\mathcal{F}_{s,t}[\mathcal{A}]] + \alpha_{-s,0}[\mathcal{G}_s[BC([-\tau, s-\tau] \times \mathbb{R}^N, [-\gamma, \gamma])]] = \alpha_{-s,0}[\mathcal{F}_{s,t}[\mathcal{A}] \le k\alpha_{-\tau,0}[\mathcal{A}]$ for any bounded subset $\mathcal{A}$ in $C_+(\Lambda)$. □

The following result gives comparison principles for the solution semiflow of (2.3).

Proposition 2.2. *Let Λ_1 and Λ_2 be smooth domains with $\Omega \subseteq \Lambda_2 \subseteq \Lambda_1 \subseteq \mathbb{R}^N$ and let $g, h \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $f - g, g - h \in C(\mathbb{R}_+, \mathbb{R}_+)$ and h being nondecreasing on $\mathbb{R}_+$. Then $\Phi_{\Lambda_1}(t, \varphi; f) \ge \Phi_{\Lambda_2}(t, \varphi; g) \ge \Phi_{\Lambda_2}(t, \psi; h)$ for all $(t, \varphi, \psi) \in \mathbb{R}_+ \times C_+(\Lambda_1) \times C_+(\Lambda_2)$ with $\varphi \ge \psi$.*

Proof. Suppose that $(\varphi, \psi) \in C_+(\Lambda_1) \times C_+(\Lambda_2)$ and $\varphi \ge \psi$. It follows from Lemma 2.2, Lemma 2.3, and (2.3) that, for any $t \in [0, \tau]$,

$$
\begin{aligned}
\Phi_{\Lambda_1}(t, \varphi; f)(0, \cdot) &= T_{\Lambda_1}(t)[\varphi(0, \cdot)] + \int_0^t T_{\Lambda_1}(t - \varsigma)[K_{\Lambda_1}[f(\varphi(\varsigma - \tau, \cdot))]]\mathrm{d}\varsigma \\
&\ge T_{\Lambda_2}(t)[\varphi(0, \cdot)] + \int_0^t T_{\Lambda_2}(t - \varsigma)[K_{\Lambda_2}[g(\varphi(\varsigma - \tau, \cdot))]]\mathrm{d}\varsigma \\
&= \Phi_{\Lambda_2}(t, \varphi; g) \\
&\ge T_{\Lambda_2}(t)[\varphi(0, \cdot)] + \int_0^t T_{\Lambda_2}(t - \varsigma)[K_{\Lambda_2}[h(\varphi(\varsigma - \tau, \cdot))]]\mathrm{d}\varsigma \\
&\ge T_{\Lambda_2}(t)[\psi(0, \cdot)] + \int_0^t T_{\Lambda_2}(t - \varsigma)[K_{\Lambda_2}[h(\psi(\varsigma - \tau, \cdot))]]\mathrm{d}\varsigma \\
&= \Phi_{\Lambda_2}(t, \psi; h)(0, \cdot).
\end{aligned}
$$

This, together with the semigroup properties of Φ_Λ, implies $\Phi_{\Lambda_1}(t, \varphi; f) \ge \Phi_{\Lambda_2}(t, \varphi; g) \ge \Phi_{\Lambda_2}(t, \psi; h)$ for all $t \in \mathbb{R}_+$. The proof is complete. □

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In the following, we give several definitions.

Definition 2.1. An element $\varphi \in C_+(\Lambda)$ is called an equilibrium of Φ_Λ if $\Phi_\Lambda(t,\varphi)=\varphi$ for all $t\in\mathbb{R}_+$. A subset $\mathcal{A}$ of $C_+(\Lambda)$ is said to be positively invariant under Φ_Λ if $\Phi_\Lambda(t,\varphi)\in\mathcal{A}$ for all $\varphi\in\mathcal{A}$ and $t\in\mathbb{R}_+$.

We write $O(\Lambda;\varphi)=\{\Phi_\Lambda(t,\varphi):t\in\mathbb{R}_+\}$ for the positive semi-orbit through the point $\varphi\in C_+(\Lambda)$. The ω-limit set of $O(\Lambda;\varphi)$ is defined by $\omega(\Lambda;\varphi)=\bigcap_{t\in\mathbb{R}_+}\overline{O(\Lambda;\Phi_\Lambda(t,\varphi))}$, where $\overline{O(\Lambda;\Phi_\Lambda(t,\varphi))}$ represents the closure of $O(\Lambda;\Phi_\Lambda(t,\varphi))$. In what follows, we also write $O(f,\Lambda;\varphi)$ and $\omega(f,\Lambda;\varphi)$ for $O(\Lambda;\varphi)$ and $\omega(\Lambda;\varphi)$, respectively, to emphasize the dependence on the nonlinearity f if there is a need. By Proposition 2.1-(ii) and -(iii), we know that $\overline{O(\Lambda;\varphi)}$ is compact and hence $\omega(\Lambda;\varphi)$ is nonempty, compact, connected, and invariant. According to the invariant property of $\omega(\Lambda;\varphi)$, for $\psi\in\omega(\Lambda;\varphi)$, there is a global classical solution $u:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ of (2.3) with $u_0=\psi$ and $u_t\in\omega(\Lambda;\varphi)$ for all $t\in\mathbb{R}$.

Definition 2.2. Let ϕ^* be an equilibrium and $\mathcal{A}$ be a positively invariant set of the semiflow Φ_Λ. We say that ϕ^* is globally attractive in $\mathcal{A}$ if $\omega(\Lambda;\varphi)=\{\phi^*\}$ for all $\varphi\in\mathcal{A}$.

By using bootstrap arguments with L^p estimates, Schauder estimates, and Sobolev embedding theorems, we can obtain the following results on solutions of (2.3).

Proposition 2.3. *Let $M\in\{r_n:n\in\mathbb{N}\}$ and Ω^* be a bounded and smooth domain in Λ. Then the following results are true.*

(i) $u^\varphi(t,\cdot)\in Z_+(\Lambda)\cap C^{1,\frac12}_{loc}(\bar\Lambda)\cap C^2(\Lambda)$ *for all* $(t,\varphi)\in[3\tau,\infty)\times C_+(\Lambda;\Omega)$.

(ii) *There exists* $A=A(M,\Omega^*)>0$ *such that* $||u^\varphi(t,\cdot)||_{C^{1,1/2}(\overline{\Omega^*})}\le A$ *for all* $(t,\varphi)\in[3\tau,\infty)\times C^+_M(\Lambda)$. *Here* $||g||_{C^{1,1/2}(\overline{\Omega^*})}$ *is the sum of* $||g||_{C^1(\overline{\Omega^*})}$ *and the Hölder seminorm (with exponent* $1/2$*) of all first order partial derivatives of* g.

(iii) *Let* $\psi\in C_+(\Lambda)$. *Then* $\{\varphi(0,\cdot)|_{\overline{\Omega^*}}:\varphi\in\omega(\Lambda;\psi)\}$ *is compact in* $C^1(\overline{\Omega^*})$.

3. Four lemmas

In this section, we provide four lemmas to describe respectively a new solid cone, spectral bounds, as well as the comparabilities of (limiting) solutions for (2.3) and a given function in a suitable bounded domain/the whole space. These are the keys to developing domain decomposition methods, which are essential to establish the main results in the next section.

Firstly, we introduce a new ordered cone. Let Λ_1 be a bounded and open domain in $\mathbb{R}^N$ with smooth boundary such that $\Lambda_1\subseteq\Lambda$. Let $V(\Lambda_1)=\{\phi\in C^1(\overline{\Lambda_1},\mathbb{R}):\phi|_{\partial\Lambda_1\cap\partial\Lambda}=0\}$, $V_+(\Lambda_1)=\{\phi\in V(\Lambda_1):\phi(\Lambda_1)\subseteq\mathbb{R}_+\}$, and $V^\circ_+(\Lambda_1)=\{\phi\in V_+(\Lambda_1):\phi(\overline{\Lambda_1}\setminus(\partial\Lambda_1\cap\partial\Lambda))\subseteq(0,\infty)$ and $\frac{\partial\phi}{\partial\nu_{x,\Lambda_1}}(x)<0$ for all $x\in\partial\Lambda_1\cap\partial\Lambda\}$. Recall that ν_{x,Λ_1} represents the unit outward normal direction at $x\in\partial\Lambda_1$. Clearly, $\nu_{x,\Lambda_1}=\nu_{x,\Lambda}$ for all $x\in\partial\Lambda_1\cap\partial\Lambda$. Here, equipping V with the C^1-norm $\|\cdot\|_{C^1(\overline{\Lambda_1},\mathbb{R})}$, we can easily check that $(V(\Lambda_1),V_+(\Lambda_1))$ is an ordered Banach space, where $\|\phi\|_{C^1(\overline{\Lambda_1},\mathbb{R})}=\sup\{|\phi(x)|:x\in\overline{\Lambda_1}\}+\sup\{|\frac{\partial\phi}{\partial x_i}(x)|:x\in\overline{\Lambda_1}$ and $i\in\{1,2,\dots,N\}\}$.

The following results will be used several times later on to compare (limiting) solutions for (2.3) and a given function in a suitable bounded domain.

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Lemma 3.1. *Let $\phi \in V_+^\circ(\Lambda_1)$ and let A be a compact subset in $V_+(\Lambda_1)$. Then we have the following properties.*

(i) $T_\Lambda(t)[\phi]|_{\Lambda_1} \in V_+^\circ(\Lambda_1)$ *for all* $(t,\phi) \in (0,\infty) \times (X_+(\Lambda) \setminus \{0\})$.
(ii) $Int(V_+(\Lambda_1)) = V_+^\circ(\Lambda_1)$.
(iii) $a\phi - A \subseteq V_+^\circ(\Lambda_1) \subseteq V_+(\Lambda_1)$ *for some* $a > 0$.
(iv) $A \subseteq a\phi + V_+^\circ(\Lambda_1) \subseteq a\phi + V_+(\Lambda_1)$ *for some* $a > 0$ *whence* $A \subseteq V_+^\circ(\Lambda_1)$.

Proof. (i) follows from Lemma 2.3 and the definition of $V_+^\circ(\Lambda_1)$.

(ii) Fix $\phi^* \in V_+^\circ(\Lambda_1)$. By the smoothness of Λ_1, we know that for any $x \in \partial\Lambda_1 \cap \partial\Lambda$, there exists $\delta_x^* > 0$ such that $y - \beta\nu_{x,\Lambda_1} \in \Lambda_1$ for all $\beta \in [0, \delta_x^*)$ and $y \in B_{\delta_x^*} \cap \partial\Lambda_1$. By taking $\delta_x < \delta_x^*$, we may assume that $grad(\phi)(y - \beta\nu_{x,\Lambda_1}) \cdot \nu_{x,\Lambda_1} < 0$ for all $\beta \in [0, \delta_x)$, $y \in B_{\delta_x}(x) \cap \partial\Lambda_1$, and $\phi \in V(\Lambda_1)$ with $||\phi - \phi^*||_{C^1(\overline{\Lambda_1})} < \delta_x$. In view of the compactness of $\partial\Lambda_1 \cap \partial\Lambda$, there exist an $l \in \mathbb{N}$ and $x^0, x^1, \ldots, x^l \in \partial\Lambda_1 \cap \partial\Lambda$ such that $\partial\Lambda_1 \cap \partial\Lambda \subseteq \bigcup_{i=0}^{l} B_{\delta_{x^i}}(x^i)$. By letting $\delta = \min\{\delta_{x^i} : i = 0, 1, \ldots, l\}$ and $\mathcal{O}_i = \{y - \beta\nu_{x^i,\Lambda_1} : \beta \in [0,\delta)$ and $y \in B_{\delta_{x^i}}(x^i) \cap \partial\Lambda_1\}$, we have $grad(\phi)(y) \cdot \nu_{x^i,\Lambda_1} < 0$ for all $y \in \mathcal{O}_i$, and $\phi \in V(\Lambda_1)$ with $||\phi - \phi^*||_{C^1(\overline{\Lambda_1})} < \delta$ and $i = 0, 1, \ldots, l$. Letting $\mathcal{O} = \bigcup_{i=0}^{l} \mathcal{O}_i$, we easily see that $\overline{\Lambda_1} \setminus \mathcal{O}$ is compact and $\phi^*(\overline{\Lambda_1} \setminus \mathcal{O}) \subseteq (0,\infty)$. Therefore, $\inf \phi^*(\overline{\Lambda_1} \setminus \mathcal{O}) > 0$. Let $\delta^* = \min\{\delta, \inf\phi^*(\overline{\Lambda_1} \setminus \mathcal{O})\}$. Suppose that $\phi \in V(\Lambda_1)$ with $||\phi - \phi^*||_{C^1(\overline{\Lambda_1})} < \delta^*$. Then $\phi(\overline{\Lambda_1} \setminus \mathcal{O}) \subseteq (0,\infty)$ and $grad(\phi)(y) \cdot \nu_{x^i,\Lambda_1} < 0$ for all $y \in \mathcal{O}_i$ and $i = 0, 1, \ldots, l$. The latter, together with the definition of $V(\Lambda_1)$, implies $\phi(y) \geq 0$ for all $y \in \mathcal{O}_i \setminus (\partial\Lambda_1 \cap \partial\Lambda)$ and $i = 1, 2, \ldots, l$. According to the definition of $V_+(\Lambda_1)$, we get $\phi \in V_+(\Lambda_1)$, which implies $\phi^* \in Int(V_+(\Lambda_1))$. Consequently, by the arbitrariness of ϕ^*, we have $V_+^\circ(\Lambda_1) \subseteq Int(V_+(\Lambda_1))$.

Next, it suffices to prove $Int(V_+(\Lambda_1)) \setminus V_+^\circ(\Lambda_1) = \emptyset$. Otherwise, there exists $\phi \in Int(V_+(\Lambda_1))$ such that $\phi(\hat{x}) = 0$ for some $\hat{x} \in \overline{\Lambda_1} \setminus (\partial\Lambda_1 \cap \partial\Lambda)$ or $\frac{\partial\phi}{\partial\nu}(\tilde{x}) \geq 0$ for some $\tilde{x} \in \partial\Lambda_1 \cap \partial\Lambda$. Then there is $\phi^* \in Int(V_+(\Lambda_1))$ such that $\phi^*(\hat{x}) < 0$ or $\frac{\partial\phi}{\partial\nu}(\tilde{x}) > 0$. As a result, $\phi^*(\Lambda_1) \setminus \mathbb{R}_+ \neq \emptyset$, that is, $\phi^* \notin V_+(\Lambda_1)$, a contradiction.

(iii) By (ii) and $0 \in \phi - V_+^\circ(\Lambda_1)$, we have $a_1 A \subseteq \phi - V_+^\circ(\Lambda_1)$ for some $a_1 > 0$. In other words, $a\phi - A \subseteq V_+^\circ(\Lambda_1)$ with $a = \frac{1}{a_1} > 0$.

(iv) By (iii), we obtain that $A \subseteq \bigcup_{l=1}^{\infty}(\frac{\phi}{l} + V_+^\circ(\Lambda_1))$, which combined with the compactness of A, implies $A \subseteq a\phi + V_+^\circ(\Lambda_1)$ with $a = \frac{1}{l_0} > 0$ for some positive integer l_0. □

Denote by λ_Λ the *infimum of the real parts of spectrum* of the operator $-\Delta$ in Λ with the homogeneous Dirichlet boundary condition. Note that λ_Λ is just the *first eigenvalue* of the operator $-\Delta$ in Λ with the homogeneous Dirichlet boundary condition when Λ is bounded. Let $A_\Lambda = d\Delta - \mu Id + \mu f'(0)K_\Lambda$ and $A_{\Lambda,c} = d\Delta - \mu Id + \mu c K_\Lambda$. Obviously, $A_\Lambda = A_{\Lambda, f'(0)}$. We call A_Λ the characteristic operator of the linearized system at the zero solution of (A.8). Let us denote by $s(A_\Lambda)$ and $s(A_{\Lambda,c})$ the spectral bounds of the linear operators A_Λ and $A_{\Lambda,c}$, respectively, whence Λ is a bounded domain in $\mathbb{R}^N$. When Λ is an unbounded domain in $\mathbb{R}^N$, we shall define pseudo-spectral bounds of the linear operators A_Λ and $A_{\Lambda,c}$, still denoting by $s(A_\Lambda)$ and $s(A_{\Lambda,c})$, respectively.

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The following spectral bound theorem shows the existence of the principal eigenvalue and the corresponding eigenvector of the characteristic operator associated with the linearized system at the zero solution of (A.8), which is used to determine the threshold value.

Lemma 3.2. *For $c > 0$, let $A_{\Lambda,c}$ and $s(A_{\Lambda,c})$ be defined as above. Then the following statements hold.*

(i) *If Λ is bounded in $\mathbb{R}^N$, then $s(A_{\Lambda,c})$ is a real and simple eigenvalue of $A_{\Lambda,c}$ and there exists a unique $\xi_{\Lambda,c} \in Z_+(\Lambda)$ such that $A_{\Lambda,c}[\xi_{\Lambda,c}] = s(A_{\Lambda,c})\xi_{\Lambda,c}$, $\frac{\partial \xi_{\Lambda,c}}{\partial \nu_{x,\Lambda}}(x) < 0$ for all $x \in \partial\Lambda$, $||\xi_{\Lambda,c}|| = 1$, and $-d\lambda_\Omega - \mu + \mu c e^{-\lambda_\Omega \alpha} \le s(A_{\Lambda,c}) \le -d\lambda_\Lambda - \mu + \mu c e^{-\lambda_\Lambda \alpha} \le -\mu + \mu c$.*

(ii) *Let $0 < c_1 \le c_2$ and $\Omega \subseteq \Lambda_1 \subseteq \Lambda_2 \subseteq \mathbb{R}^N$ be bounded and smooth domains. Then $s(A_{\Lambda_1,c_1}) < s(A_{\Lambda_2,c_2})$ whence $(\Lambda_1, c_1) \ne (\Lambda_2, c_2)$.*

(iii) *If Λ is an unbounded smooth domain in $\mathbb{R}^N$, then there exist $s^* \in \mathbb{R}$ and a sequence of bounded smooth domains $\{\Lambda_l\}$ in $\mathbb{R}^N$ such that $\Omega \cup (\Lambda \cap B_l) \subseteq \Lambda_l \subseteq \Lambda_{1+l}$ for $l \in \mathbb{N}$, $\bigcup_{l\in\mathbb{N}} \Lambda_l = \Lambda$, and $s^* = \lim_{l\to\infty} s(A_{\Lambda_l,c}) \triangleq s(A_{\Lambda,c}) \in [-d\lambda_\Omega - \mu + \mu c e^{-\lambda_\Omega \alpha}, -\mu + \mu c]$ is independent of the choice of $\{\Lambda_l\}$. Moreover, if $s(A_{\Lambda,c}) > -\mu$, then there exists $\xi^* \in Z_+(\Lambda)$ such that $||\xi^*|| = 1$, $A_{\Lambda,c}[\xi^*] = s^*\xi^*$, and $\xi^* = \lim_{l\to\infty} \xi_{\Lambda_l,c} \triangleq \xi_{\Lambda,c}$ in $C^2_{loc}(\mathbb{R}^N, \mathbb{R})$. Here $B_l = \{x \in \mathbb{R}^N : ||x|| \le l\}$. Here and after, we call $s(A_\Lambda)(:= s(A_{\Lambda,f'(0)}))$ and $s(A_{\Lambda,c})$ the pseudo-spectral bounds of the linear operators A_Λ and $A_{\Lambda,c}$, respectively, whence Λ is an unbounded domain in $\mathbb{R}^N$.*

(iv) *$s(A_{\Lambda,c})$ is left continuous at $c \in (0,\infty)$. Moreover, $s(A_{\Lambda,c})$ is continuous at $c \in (0,\infty)$ when either $s(A_{\Lambda,c}) > -\mu$ or Λ is bounded.*

(v) *If $s(A_{\Lambda,c}) \le -\mu$, then there is $c^* > c$ such that $0 \ge s(A_{\Lambda,c^*}) > -\mu$.*

(vi) *If $s(A_\Lambda) > 0$, then there exist a bounded and smooth domain Λ^*, $\xi^* \in Z_+(\Lambda^*)$, and $c^* \in (0, f'(0))$ such that $||\xi^*|| = 1$, $\Omega \subseteq \Lambda^* \subseteq \Lambda$, $s(A_{\Lambda^*,c^*}) = 0$, and $A_{\Lambda^*,c^*}[\xi^*] = 0$.*

(vii) *If $s(A_\Lambda) > 0$, then there exist $c^* \in (0, f'(0))$ and $\xi \in Z_+(\Lambda)$ such that $s(A_{\Lambda,c^*}) = 0$, $||\xi|| = 1$, and $A_{\Lambda,c^*}[\xi] = 0$.*

Proof. Clearly, (i) follows from the maximum principle, the Krein–Rutman theorem in [16], and the spectral mapping theorem for semigroups in [4].

(ii) By (i), we have $A_{\Lambda_i,c_i}[\xi_{\Lambda_i,c_i}] = s(A_{\Lambda_i,c_i})\xi_{\Lambda_i,c_i}$ for $i = 1, 2$. It follows from Green's formulae, Fubini's Theorem, and the fact that $\Gamma_\Omega(\alpha, x, y) = \Gamma_\Omega(\alpha, y, x)$ for all $x, y \in \mathbb{R}^N$ and $\Gamma_\Omega(\alpha, x, y) = 0$ for all $x, y \in \mathbb{R}^N$ with $(x, y) \notin \Omega \times \Omega$ that

$$
\begin{aligned}
&(s(A_{\Lambda_1,c_1}) - s(A_{\Lambda_2,c_2})) \int_{\Lambda_1} \xi_{\Lambda_1,c_1}(x)\xi_{\Lambda_2,c_2}(x)\mathrm{d}x \\
&= \int_{\Lambda_1} \{\xi_{\Lambda_2,c_2}(x) A_{\Lambda_1,c_1}[\xi_{\Lambda_1,c_1}](x) - \xi_{\Lambda_1,c_1}(x) A_{\Lambda_2,c_2}[\xi_{\Lambda_2,c_2}](x)\}\mathrm{d}x \\
&= \int_{\Lambda_1} \Big\{\xi_{\Lambda_2,c_2}(x)[d\Delta\xi_{\Lambda_1,c_1}(x) - \mu\xi_{\Lambda_1,c_1}(x) + \mu c_1 K_{\Lambda_1}[\xi_{\Lambda_1,c_1}](x)] \\
&\quad -\xi_{\Lambda_1,c_1}(x)[d\Delta\xi_{\Lambda_2,c_2}(x) - \mu\xi_{\Lambda_2,c_2}(x) + \mu c_2 K_{\Lambda_2}[\xi_{\Lambda_2,c_2}](x)]\Big\}\mathrm{d}x \\
&= \int_{\Lambda_1} \Big\{\xi_{\Lambda_2,c_2}(x)[d\Delta\xi_{\Lambda_1,c_1}(x) + \mu c_1 K_{\Lambda_1}[\xi_{\Lambda_1,c_1}](x)]
\end{aligned}
$$

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$$
\begin{aligned}
&-\xi_{\Lambda_1,c_1}(x)[d\Delta\xi_{\Lambda_2,c_2}(x)+\mu c_2 K_{\Lambda_2}[\xi_{\Lambda_2,c_2}](x)]\Big\}\mathrm{dx}\\
&=d\int_{\Lambda_1}\{\xi_{\Lambda_2,c_2}(x)\Delta\xi_{\Lambda_1,c_1}(x)-\xi_{\Lambda_1,c_1}(x)\Delta\xi_{\Lambda_2,c_2}(x)\}\mathrm{dx}\\
&\quad+\mu(c_1-c_2)\int_{\Lambda_1}\xi_{\Lambda_2,c_2}(x)K_{\Lambda_1}[\xi_{\Lambda_1,c_1}](x)\mathrm{dx}\\
&=d\int_{\partial\Lambda_1}\xi_{\Lambda_2,c_2}(x)\frac{\partial\xi_{\Lambda_1,c_1}}{\partial\nu_{x,\Lambda_1}}(x)\mathrm{dx}+\mu(\mathrm{c}_1-\mathrm{c}_2)\int_{\Omega\Omega}\Gamma_\Omega(\alpha,\mathrm{x},\mathrm{y})\xi_{\Lambda_2,c_2}(\mathrm{x})\xi_{\Lambda_1,c_1}(\mathrm{y})\mathrm{dxdy}\\
&\le\min\left\{d\int_{\partial\Lambda_1}\xi_{\Lambda_2,c_2}(x)\,\frac{\partial\xi_{\Lambda_1,c_1}}{\partial\nu_{x,\Lambda_1}}(x)\mathrm{dx},\,\mu(\mathrm{c}_1-\mathrm{c}_2)\int_{\Omega\times\Omega}\Gamma_\Omega(\alpha,\mathrm{x},\mathrm{y})\xi_{\Lambda_2,c_2}(\mathrm{x})\xi_{\Lambda_1,c_1}(\mathrm{y})\mathrm{dxdy}\right\}.
\end{aligned}
$$

Since $c_1\le c_2$ and $\Omega\subseteq\Lambda_1\subseteq\Lambda_2\subseteq\mathbb{R}^N$, and $(\Lambda_1,c_1)\neq(\Lambda_2,c_2)$, we know that

$$
\min\left\{d\int_{\partial\Lambda_1}\xi_{\Lambda_2,c_2}(x)\frac{\partial\xi_{\Lambda_1,c_1}}{\partial\nu_{x,\Lambda_1}}(x)\mathrm{dx},\,\mu(\mathrm{c}_1-\mathrm{c}_2)\int_{\Omega\times\Omega}\Gamma_\Omega(\alpha,\mathrm{x},\mathrm{y})\xi_{\Lambda_2,c_2}(\mathrm{x})\xi_{\Lambda_1,c_1}(\mathrm{y})\mathrm{dxdy}\right\}<0.
$$

Consequently,

$$
(s(A_{\Lambda_1,c_1})-s(A_{\Lambda_2,c_2}))\int_{\Lambda_1}\xi_{\Lambda_1,c_1}(x)\xi_{\Lambda_2,c_2}(x)\mathrm{dx}<0
$$

and hence $s(A_{\Lambda_1,c_1})<s(A_{\Lambda_2,c_2})$.

(iii) Take two sequences of bounded smooth domains $\{\Lambda_l\}$ and $\{\Lambda_l^*\}$ in $\mathbb{R}^N$ with $\Omega\cup(\Lambda\cap B_l)\subseteq\Lambda_l\subseteq\Lambda_{1+l}$ and $\Omega\cup(\Lambda\cap B_l)\subseteq\Lambda_l^*\subseteq\Lambda_{1+l}^*$ for $l\in\mathbb{N}$ and $\bigcup_{l\in\mathbb{N}}\Lambda_l=\bigcup_{l\in\mathbb{N}}\Lambda_l^*=\Lambda$. By (i) and (ii), we have $-d\lambda_\Omega-\mu+\mu ce^{-\lambda_\Omega\alpha}\le s(A_{\Lambda_l,c})\le s(A_{\Lambda_{1+l},c})\le-\mu+\mu c$ and $-d\lambda_\Omega-\mu+\mu ce^{-\lambda_\Omega\alpha}\le s(A_{\Lambda_l^*,c})\le s(A_{\Lambda_{1+l}^*,c})\le-\mu+\mu c$ for all $l\in\mathbb{N}$. In view of $B_l\cap\Lambda\subseteq\Lambda_l^*$ and the boundedness of Λ_l, we can obtain that, for any $l\in\mathbb{N}$, there is $k_l\in\mathbb{N}$ such that $\Lambda_l\subseteq\Lambda_{k_l}^*$. This, together with (ii), gives $s(A_{\Lambda_l,c})\le s(A_{\Lambda^*_{k_l},c})$ for all $l\in\mathbb{N}$. It follows that $\lim_{l\to\infty}s(A_{\Lambda_l,c})\le\lim_{l\to\infty}s(A_{\Lambda_l^*,c})$. Similarly, $\lim_{l\to\infty}s(A_{\Lambda_l^*,c})\le\lim_{l\to\infty}s(A_{\Lambda_l,c})$. Therefore, $\lim_{l\to\infty}s(A_{\Lambda,c_l})=\lim_{l\to\infty}s(A_{\Lambda,c_l})\triangleq s(A_{\Lambda,c})$, also denoted by s^*. Clearly, $s^*=s(A_{\Lambda,c})\in[-d\lambda_\Omega-\mu+\mu ce^{-\lambda_\Omega\alpha},-\mu+\mu c]$.

By the standard elliptic estimates, without loss of generality, we may assume that $\xi^*=\lim_{l\to\infty}\xi_{\Lambda,c_l}\triangleq\xi_{\Lambda,c}$ in $C^2_{loc}(\mathbb{R}^N,\mathbb{R})$ with $A_{\Lambda,c}[\xi_{\Lambda,c}]=s^*\xi_{\Lambda,c}$.

Suppose that $s(A_{\Lambda,c})>-\mu$. Without loss of generality, we may assume that $s(A_{\Lambda_l,c})>-\mu$ for all $l\in\mathbb{N}$. Now it suffices to prove $||\xi_{\Lambda,c}||=1$. Otherwise, $||\xi_{\Lambda,c}||<1$. Then $\xi_{\Lambda_l,c}(x)\le\frac{1+||\xi_{\Lambda,c}||}{2}$ for all $l\in\mathbb{N}$ and $x\in\bar\Omega$. Note that $\xi_{\Lambda_l,c}$ satisfies the following equation,

$$
\begin{cases}
d\Delta v(x)=(\mu+s(A_{\Lambda_l,c}))v(x)(\ge0), & x\in\Lambda_l\setminus\bar\Omega,\\
v(x)\le\frac{1+||\xi_{\Lambda,c}||}{2}(<1), & x\in(\mathbb{R}^N\setminus\Lambda_l)\cup\bar\Omega.
\end{cases}
$$

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This, together with the maximum principle, implies $\xi_{\Lambda_l,c}(x) \le \frac{1+||\xi_{\Lambda,c}||}{2}$ for all $x \in \Lambda_l$, which yields a contradiction with $||\xi_{\Lambda_l,c}|| = 1$. Finally, we also see that $\xi_{\Lambda,c} \in Z_+(\Lambda)$ due to $\xi_{\Lambda,c} > 0$ and the standard elliptic estimates.

(iv) Firstly, we prove that $s(A_{\Lambda,c})$ is left continuous at $c \in (0,\infty)$. It suffices to prove that $s(A_{\Lambda,c}) = \lim_{l\to\infty} s(A_{\Lambda,c_l})$ whence $\{c_l\}$ is a strictly increasing sequence with $\lim_{l\to\infty} c_l = c$. Take a sequence of bounded smooth domains $\{\Lambda_l\}$ in $\mathbb{R}^N$ such that $\Omega \cup (\Lambda \cap B_l) \subseteq \Lambda_l \subseteq \Lambda_{1+l}$ for $l \in \mathbb{N}$ and $\bigcup_{l\in\mathbb{N}} \Lambda_l = \Lambda$. This, combined with (i), (iii), and the boundedness of Λ_l, implies that $s(A_{\Lambda,c_l}) = \lim_{k\to\infty} s(A_{\Lambda_k,c_l})$ and $s(A_{\Lambda_k,c}) = \lim_{l\to\infty} s(A_{\Lambda_k,c_l})$ for all k, $l \in \mathbb{N}$. Then, by contradiction, we assume that $s(A_{\Lambda,c}) > \lim_{l\to\infty} s(A_{\Lambda,c_l})$. Note that $\lim_{l\to\infty} s(A_{\Lambda,c_l}) \ge \lim_{l\to\infty} s(A_{\Lambda_k,c_l}) = s(A_{\Lambda_k,c})$ for all $k \in \mathbb{N}$. It follows that $s(A_{\Lambda,c}) > \lim_{l\to\infty} s(A_{\Lambda,c_l}) \ge \lim_{k\to\infty} s(A_{\Lambda_k,c})$, a contradiction to the definition of $s(A_{\Lambda,c})$. So $s(A_{\Lambda,c})$ is left continuous at c.

To finish the proof of (iv), it suffices to prove that $s(A_{\Lambda,\hat{c}}) = \lim_{l\to\infty} s(A_{\Lambda,c_l})$ whence $\{c_l\}$ is a strictly decreasing sequence with $\hat{c} = \lim_{l\to\infty} c_l$, where $\hat{c} \in \{c \in (0,\infty) : \Lambda \text{ is bounded or } s(A_{\Lambda,c}) > -\mu\}$. Obviously, $s(A_{\Lambda,\hat{c}}) = \lim_{l\to\infty} s(A_{\Lambda,c_l})$ follows from (i) when Λ is bounded or (iii) when $s(A_{\Lambda,\hat{c}}) > -\mu$.

(v) Suppose that $s(A_{\Lambda,c}) \le -\mu$. Without loss of generality, we may assume that Λ is unbounded since (v) follows from (iv) when Λ is bounded. By contradiction, we assume that $s(A_{\Lambda,c+\delta}) > 0$ for all $\delta > 0$. Take $\delta^* = \dfrac{1}{3\max\{\Gamma_\Omega(\alpha,x,y) : x, y \in \bar{\Omega}\}}$. According to (iii) and $s(A_{\Lambda,c+\delta^*}) > 0$, there exist $s_* \in (0, s(A_{\Lambda,c+\delta^*}))$, a bounded smooth domain Λ^* in $\mathbb{R}^N$, and $\xi^* \in Z_+(\Lambda^*)$ such that $\Omega \subseteq \Lambda^* \subseteq \Lambda$, $||\xi^*|| = 1$, $A_{\Lambda^*,c+\delta^*}[\xi^*] = s_*\xi^*$. By (iii) and $s(A_{\Lambda,c}) \le -\mu$, there exist $\omega^* \le -\mu$ and $\phi^* \in Z_+(\Lambda^*)$ such that $||\xi^*|| = 1$ and $A_{\Lambda^*,c}[\phi^*] = \omega^*\phi^*$. Similarly as in the proof of (ii),

$$
\begin{aligned}
&(s_* - \omega^*)\int_{\Lambda^*} \xi^*(x)\phi^*(x)\mathrm{d}x\\
&= \int_{\Lambda^*} \{\phi^*(x)A_{\Lambda^*,c+\delta^*}[\xi^*](x) - \xi^*(x)A_{\Lambda^*,c}[\phi^*](x)\}\mathrm{d}x\\
&= \int_{\Lambda^*} \{\phi^*(x)[d\Delta\xi^*(x) - \mu\xi^*(x) + \mu(c+\delta^*)K_{\Lambda^*}[\xi](x)]\\
&\quad -\xi^*(x)[d\Delta\phi(x) - \mu\phi(x) + \mu c K_{\Lambda^*}[\phi](x)]\}\mathrm{d}x\\
&= d\int_{\Lambda^*} \{\phi^*(x)\Delta\xi^*(x) - \xi^*(x)\Delta\phi(x)\}\mathrm{d}x\\
&\quad + \int_{\Lambda^*} \{\phi^*(x)[\mu(c+\delta^*)K_{\Lambda^*}[\xi](x)] - \xi^*(x)[\mu c K_{\Lambda^*}[\phi](x)]\}\mathrm{d}x\\
&= \int_{\Lambda^*} \{\phi^*(x)[\mu(c+\delta^*)K_{\Lambda^*}[\xi](x) - \xi^*(x)[\mu c K_{\Lambda^*}[\phi](x)]]\}\mathrm{d}x
\end{aligned}
$$

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$$
\begin{aligned}
&= \mu\delta^* \int_{\Lambda^*} \phi^*(x) \times K_{\Lambda^*}[\xi](x)\mathrm{d}x \\
&= \mu\delta^* \int_{\Omega\times\Omega} \Gamma_\Omega(\alpha, y, x)\phi^*(x)\xi(y)\mathrm{d}x\mathrm{d}y \\
&\le \frac{\mu}{3} \int_{\Omega\times\Omega} \phi^*(x)\xi(y)\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

It follows that $s_* - \omega^* \le \frac{\mu}{3}$, a contradiction to the choices of s_* and ω^*.

(vi) Take $\Lambda^* = \Lambda$ when Λ is bounded. If Λ is unbounded, then by $s(A_{\Lambda, f'(0)}) > 0$ and (iii), there exists a bounded and smooth domain Λ^* such that $\Omega \subseteq \Lambda^* \subseteq \Lambda$ and $s(A_{\Lambda^*, f'(0)}) > 0$. Statement (i) implies $\lim_{c\to 0+} s(A_{\Lambda^*,c}) \le -\mu$. This, together with $s(A_{\Lambda^*, f'(0)}) > 0$ and (iv), implies that there exist $\xi^* \in Z_+(\Lambda^*)$ and $c^* \in (0, f'(0))$ such that $||\xi^*|| = 1$, $s(A_{\Lambda^*,c^*}) = 0$, and $A_{\Lambda^*,c^*}[\xi^*] = 0$.

(vii) Note that $s(A_{\Lambda, f'(0)}) = s(A_\Lambda) > 0$ and $\lim_{c\to 0+} s(A_{\Lambda,c}) < 0$ due to (i) and (iii). Then $s(A_{\Lambda,c_1}) < 0$ for some $c_1 \in (0, f'(0))$. If Λ is bounded, then by (iv), $s(A_{\Lambda,c})$ is continuous at $c \in [c_1, f'(0)]$. Consequently, there is $c^* \in (c_1, f'(0))$ such that $s(A_{\Lambda,c^*}) = 0$. Next, suppose that Λ is unbounded. Without loss of generality, we assume that $s(A_{\Lambda,c_1}) \in (-\mu, 0]$ according to (v). Again, by (iv), $s(A_{\Lambda,c})$ is continuous at $c \in [c_1, f'(0)]$ and hence there is $c^* \in (c_1, f'(0))$ such that $s(A_{\Lambda,c^*}) = 0$. The existence of ξ follows from (i) and (iii). □

The following lemma gives the limiting non-negativeness property of a class of functions. This, combined with the boundedness of the immature habitat, is used to show that the limiting/eventual comparability of solutions for (2.3) and other related equations in a suitable bounded domain including Ω implies the comparability of the limiting comparability of solutions for (2.3) and a given function in the whole space.

Lemma 3.3. *Let $\beta > 0$ and $v \in C(\mathbb{R}_+ \times \mathbb{R}^N, [-\beta, \beta]) \cap C^{1,2}((0,\infty) \times \Lambda \setminus \bar{\Omega}, [-\beta, \beta])$. If $v(t,x)$ satisfies the following equation,*

$$
\begin{cases}
\frac{\partial v}{\partial t}(t,x) \ge d\Delta v(t,x) - \mu v(t,x), & (t,x) \in (0,\infty) \times \Lambda \setminus \bar{\Omega}, \\
v(t,x) \ge 0, & (t,x) \in \mathbb{R}_+ \times ((\mathbb{R}^N \setminus \Lambda) \cup \bar{\Omega}),
\end{cases} \tag{3.1}
$$

then $\lim_{t\to\infty}(\inf\{v(t,x) : x \in \mathbb{R}^N\}) \ge 0$.

Proof. We choose $\phi^* \in C(\mathbb{R}^N, [-\beta, \beta])$ such that $\phi^* \le v(0,\cdot)$ and $\phi^*|_{(\mathbb{R}^N\setminus\Lambda)\cup\bar{\Omega}} = 0$. Let $u(t,x)$ satisfy

$$
\begin{cases}
\frac{\partial u}{\partial t}(t,x) = d\Delta u(t,x) - \mu u(t,x), & (t,x) \in (0,\infty) \times \Lambda \setminus \bar{\Omega}, \\
u(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times ((\mathbb{R}^N \setminus \Lambda) \cup \bar{\Omega}), \\
u(0,\cdot) = \phi^*.
\end{cases}
$$

Then $-\beta e^{-\mu t} \le u(t,x) \le \beta e^{-\mu t}$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^N$ due to the Phragmén–Lindelöf type maximum principle. It follows that $\lim_{t\to\infty}(\inf\{u(t,x) : x \in \mathbb{R}^N\}) \ge 0$. Moreover, by the

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Phragmén–Lindelöf type maximum principle again, we easily see that $v(t,x) \geq u(t,x)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^N$. These imply that $\lim_{t\to\infty}(\inf\{v(t,x) : x \in \mathbb{R}^N\}) \geq 0$. □

The following lemma shows that the limiting comparability of solutions for (2.3) and a given function in a suitable bounded domain implies the eventual comparability of solutions for (2.3) and a given function in such a given bounded domain.

Lemma 3.4. *Let Λ_1 be a bounded and smooth domain and let $\psi \in C_+(\Lambda)$, $\varphi^* \in \omega(\Lambda;\psi)$, and $\xi \in V_+(\Lambda_1)$. Then the following statements are true.*

(i) *If $\varphi^*|_{[-\tau,0]\times\overline{\Lambda_1}} \geq \xi|_{\overline{\Lambda_1}}$, then, for any $\delta \in (0,1)$, there exists $t_1 = t_1(\delta,\psi,\varphi^*,\xi,\Lambda_1) > 0$ such that*

$$\Phi_\Lambda(t_1,\psi)|_{[-\tau,0]\times\overline{\Lambda_1}} \geq \delta\xi|_{\overline{\Lambda_1}}.$$

Here we write $\varphi|_{[-\tau,0]\times\overline{\Lambda_1}} \geq \xi|_{\overline{\Lambda_1}}$ when $\varphi(\theta,x) \geq \xi(x)$ for all $(\theta,x) \in [-\tau,0]\times\overline{\Lambda_1}$.

(ii) *If $0 < \varphi^*|_{[-\tau,0]\times\overline{\Lambda_1}} \leq \xi|_{\overline{\Lambda_1}}$, then, for any $\delta > 1$, there exists $t_1 = t_1(\delta,\psi,\varphi^*,\xi,\Lambda_1) > 0$ such that*

$$\Phi_\Lambda(t_1,\psi)|_{[-\tau,0]\times\overline{\Lambda_1}} \leq \delta\xi|_{\overline{\Lambda_1}}.$$

(iii) *If $\varphi|_{[-\tau,0]\times\overline{\Lambda_1}} \geq \xi|_{\overline{\Lambda_1}}$ for all $\varphi \in \omega(\Lambda;\psi)$, then, for any $\delta \in (0,1)$, there exists $t_1 = t_1(\delta,\psi,\xi,\Lambda_1) > 0$ such that*

$$\Phi_\Lambda(t,\psi)|_{[-\tau,0]\times\overline{\Lambda_1}} \geq \delta\xi|_{\overline{\Lambda_1}} \qquad \text{for all } (t,\varphi) \in [t_1,\infty)\times\omega(\Lambda;\psi).$$

(iv) *If $0 < \varphi|_{[-\tau,0]\times\overline{\Lambda_1}} \leq \xi|_{\overline{\Lambda_1}}$ for all $\varphi \in \omega(\Lambda;\psi)$, then, for any $\delta > 1$, there exists $t_1 = t_1(\delta,\psi,\xi,\Lambda_1) > 0$ such that*

$$\Phi_\Lambda(t,\psi)|_{[-\tau,0]\times\overline{\Lambda_1}} \leq \delta\xi|_{\overline{\Lambda_1}} \qquad \text{for all } (t,\varphi) \in [t_1,\infty)\times\omega(\Lambda;\psi).$$

Proof. Clearly, there is an $n^* \in \mathbb{N}$ such that $r_{n^*} \geq ||\psi||$. By Proposition 2.3-(ii) with $M = r_{n^*}$, there exists $A = A(M,\Lambda_1) > 0$ such that $||u^\varphi(t,\cdot)||_{C^{1,1/2}(\overline{\Lambda_1})} \leq A$ for all $(t,\varphi) \in [3\tau,\infty)\times C_M^+(\Lambda)$. In particular, $||u^\psi(t,\cdot)||_{C^{1,1/2}(\overline{\Lambda_1})} \leq A$ for all $t \geq 3\tau$. This and Arzelà–Ascoli Theorem imply that $O_\psi := \{u^\psi(\Lambda;t,\cdot)|_{\overline{\Lambda_1}} : t \geq 3\tau\}$ is precompact in $C^1(\overline{\Lambda_1})$. As a result, we know that $\{\varphi(\theta,\cdot)|_{\overline{\Lambda_1}} : (\theta,\varphi) \in [-\tau,0]\times\omega(\Lambda;\psi)\}$ and thus $\{\varphi^*(\theta,\cdot)|_{\overline{\Lambda_1}} : \theta \in [-\tau,0]\}$ are compact in $C^1(\overline{\Lambda_1})$. By the definition of $\omega(\Lambda;\psi)$, there is a nondecreasing sequence $\{s_k^*\}$ such that $\Phi_\Lambda(s_k^*,\psi) \to \varphi^*$ in $C_+(\Lambda)$ as $s_k^* \to \infty$.

We only prove (i) and (iv) since (ii) and (iii) can be dealt with similarity, respectively.

(i) Without loss of generality, we can assume $\xi|_{\overline{\Lambda_1}} > 0$ and $\varphi^* \in Z_+(\Lambda)$ due to Proposition 2.3-(i). In particular, $\{\varphi^*(\theta,\cdot)|_{\overline{\Lambda_1}} : \theta \in [-\tau,0]\}$ is compact in $V_+^\circ(\Lambda_1)$, which together with Lemma 3.1-(iv), implies that $\varphi^*|_{[-\tau,0]\times\overline{\Lambda_1}} \geq \delta^*\varphi^*(0,\cdot)|_{\overline{\Lambda_1}}$ for some $\delta^* > 0$.

Fix $\delta \in (0,1)$. It follows from $\varphi^*|_{[-\tau,0]\times\overline{\Lambda_1}} \geq \xi|_{\overline{\Lambda_1}}$ that $\varphi^*|_{[-\tau,0]\times\overline{\Lambda_1}} \geq (1-\delta)\delta^*\varphi^*(0,\cdot)|_{\overline{\Lambda_1}} + \delta\xi|_{\overline{\Lambda_1}}$, which implies $\{\varphi^*(\theta,\cdot)|_{\overline{\Lambda_1}} : \theta \in [-\tau,0]\} \subseteq \delta\xi|_{\overline{\Lambda_1}} + V_+^\circ(\Lambda_1)$.

To finish the proof of (i), we only need to prove that there exists $k^* \in \mathbb{N}$ such that $u^\psi(\Lambda;t,\cdot)|_{\overline{\Lambda_1}} \in \delta\xi|_{\overline{\Lambda_1}} + V_+^\circ(\Lambda_1)$ for all $t \in [s_{k^*}^* - \tau, s_{k^*}^*]$. Suppose this is not true. Then there

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exists a sequence $\{s_k\}$ with $s_k \in [s_k^* - \tau, s_k^*]$ and $\lim_{k\to\infty} s_k = \infty$ such that $u^\psi(\Lambda; s_k, \cdot)|_{\Lambda_1} \notin \delta\xi|_{\overline{\Lambda_1}} + V_+^\circ(\Lambda_1)$ for all $k \in \mathbb{N}$. But, in view of the compactness of O_ψ in $C^1(\overline{\Lambda_1})$, there exist $\theta^* \in [-\tau, 0]$ and a subsequence $\{s_k^1\}$ of $\{s_k\}$ such that $u^\psi(\Lambda; s_k^1, \cdot)|_{\Lambda_1} \to \varphi^*(\theta^*, \cdot)|_{\Lambda_1} \in \delta\xi|_{\overline{\Lambda_1}} + V_+^\circ(\Lambda_1)$ in $C^1(\overline{\Lambda_1})$ as $s_k^1 \to \infty$, and consequently, $u^\psi(\Lambda; s_k^1, \cdot)|_{\Lambda_1} \in \delta\xi|_{\overline{\Lambda_1}} + V_+^\circ(\Lambda_1)$ for all large k, a contradiction to the choices of $\{s_k^1\}$ and $\{s_k\}$.

(iv) It follows from Proposition 2.3-(i) and the fact that $0 < \varphi|_{[-\tau,0]\times\overline{\Lambda_1}} \le \xi|_{\overline{\Lambda_1}}$ for all $\varphi \in \omega(\Lambda; \psi)$ that $\omega(\Lambda; \psi) \subseteq Z_+(\Lambda)$. Hence $\xi|_{\overline{\Lambda_1}} \in V_+^\circ(\Lambda_1)$ and $\{\varphi(\theta, \cdot)|_{\overline{\Lambda_1}} : \theta \in [-\tau, 0] \text{ and } \varphi \in \omega(\Lambda; \psi)\} \subseteq V_+^\circ(\Lambda_1)$.

Suppose $\delta > 1$. It suffices to prove $\{u^\psi(\Lambda; t, \cdot)|_{\overline{\Lambda_1}} : t \ge t_1\} \subseteq \delta\xi|_{\overline{\Lambda_1}} - V_+^\circ(\Lambda_1)$ for some $t_1 > 3\tau$. Otherwise, there exists an increasing sequence $\{s_k\}$ such that $\lim_{k\to\infty} s_k = \infty$ and $u^\psi(\Lambda; s_k, \cdot)|_{\Lambda_1} \notin \delta\xi|_{\overline{\Lambda_1}} - V_+^\circ(\Lambda_1)$ for all $k \in \mathbb{N}$. But, in view of the compactness of O_ψ in $C^1(\overline{\Lambda_1})$ and the definition of $\omega(\Lambda; \psi)$, we can obtain $(\theta^*, \phi^*) \in [-\tau, 0] \times \omega(\Lambda; \psi)$ and a subsequence $\{s_k^1\}$ of $\{s_k\}$ such that $u^\psi(\Lambda; s_k^1, \cdot)|_{\Lambda_1} \to \phi^*(\theta^*, \cdot)|_{\Lambda_1} \in \delta\xi|_{\overline{\Lambda_1}} - V_+^\circ(\Lambda_1)$ in $C^1(\overline{\Lambda_1})$ as $s_k^1 \to \infty$, and consequently, $u^\psi(\Lambda; s_k^1, \cdot)|_{\Lambda_1} \in \delta\xi|_{\overline{\Lambda_1}} - V_+^\circ(\Lambda_1)$ for all large k, a contradiction. □

4. A threshold dynamics

In this section, we develop domain decomposition methods to compare solutions with those to associated equations in a suitable bounded domain related to the whole space. Then by employing domain decomposition methods and standard dynamical system theoretical arguments, we derive a threshold dynamics.

To continue the investigation on dynamics of (2.3), we assume another basic condition on the nonlinearity f.

(H2) $f(u) < f'(0)u$ for all $u \in (0, \infty)$.

We first apply Lemma 3.1 and Lemma 3.2 to obtain the existence of positive invariant sets and the persistence of nontrivial solutions to (2.3). This, together with the Schauder fixed point theorem, produces the existence of a positive equilibrium of (2.3).

For $c, \epsilon > 0$, define $h_{c,\epsilon} : \mathbb{R}_+ \to \mathbb{R}_+$ by $h_{c,\epsilon}(u) = c\min\{u, \epsilon\}$ for all $u \in \mathbb{R}_+$. Then $h_{c,\epsilon}$ is nondecreasing.

Lemma 4.1. *Assume that (H1) and (H2) hold. Suppose $s(A_\Lambda) > 0$ with c^*, Λ^*, and $\xi^* \in Z_+(\Lambda^*)$ being those in Lemma 3.2-(vi). Then, for any $M \in \{r_n : n \in \mathbb{N}\}$, there exists $\epsilon_M^* > 0$ such that $\{\varphi \in X_+ : \epsilon_M^*\xi^* \le \varphi \le M\}$ is a positively invariant set of Φ_Λ. Moreover, $\omega(\Lambda; \varphi) \subseteq \epsilon_M^*\xi^* + C_+(\Lambda)$ for any $\varphi \in C_+(\Lambda; \Omega)$ with $\varphi \le M$.*

Proof. Recall from Lemma 3.2-(vi) that $c^* \in (0, f'(0))$, $\xi^* \in Z_+(\Lambda^*)$, and Λ^* is a bounded and smooth domain with $\Omega \subseteq \Lambda^* \subseteq \Lambda$, $s(A_{\Lambda^*,c^*}) = 0$, $||\xi^*|| = 1$, and $A_{\Lambda^*,c^*}[\xi^*] = 0$.

For given $M \in \{r_n : n \in \mathbb{N}\}$, $f([0, M]) \subseteq [0, \frac{M}{k_\Lambda}]$ from (H1). Then, by (H1), there is $\epsilon_M^* > 0$ such that $f(u) > h_{c^*,\epsilon_M^*}(u)$ for all $u \in (0, M]$.

Suppose that $\varphi \in C_+(\Lambda)$ with $\epsilon_M^*\xi^* \le \varphi \le M$. Proposition 2.1-(ii) implies that $\Phi_\Lambda(t, \varphi) \le M$ for all $t \in \mathbb{R}_+$. By Proposition 2.2 and the definition of h_{c^*,ϵ_M^*}, we have $\Phi_\Lambda(\varphi, t) \ge \Phi_{\Lambda^*}(\epsilon_M^*\xi^*, t; h_{c^*,\epsilon_M^*}) = \epsilon_M^*\xi^*$ for all $t \in \mathbb{R}_+$, which yields the first conclusion.

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Next, suppose that $\varphi \in C_+(\Lambda;\Omega)$ with $\varphi \le M$. Let $u^*(t,x) = u^\varphi(\Lambda;t,x)$ for all $(t,x) \in [-\tau,\infty)\times\mathbb{R}^N$. Without loss of generality, we assume that there is $A_{\varphi,\Lambda^*} > 0$ such that $u^*(t,\cdot) \in Z_+(\Lambda)\cap C^1(\bar{\Lambda})$ and $||u^*(t,\cdot)||_{C^{1,\frac{1}{2}}(\overline{\Lambda^*})} \le A_{\varphi,\Lambda^*}$ for all $t \in [-\tau,\infty)$ according to Proposition 2.3-(i). Then $\{u^*(t,\cdot)|_{\overline{\Lambda^*}} : t \in [-\tau,0]\}$ is compact in $C^1(\overline{\Lambda^*})$. It follows from Lemma 3.1-(iv) and $supp(\xi^*) \subseteq \Lambda^*$ that there exists $\epsilon_0 \in (0,\epsilon_M^*]$ such that $u^*(t,x) \ge \epsilon_0\xi^*(x)$ for all $t \in [-\tau,0]$ and $x \in \mathbb{R}^N$. By Proposition 2.2 and the definition of h_{c^*,ϵ_M^*}, we have $\Phi_\Lambda(\varphi,t) \ge \Phi_{\Lambda^*}(\epsilon_0\xi^*,t;h_{c^*,\epsilon_M^*}) = \epsilon_0\xi^*$ and hence $u^*(t,x) \ge \epsilon_0\xi^*(x)$ for all $t \in [-\tau,\infty)$ and $x \in \mathbb{R}^N$. Therefore, by the definition of $\omega(\Lambda;\varphi)$, we have $\omega(\Lambda;\varphi) \subseteq \epsilon_0\xi^* + C_+(\Lambda)$. Let $\epsilon^* = \sup\{\epsilon \ge \epsilon_0 : \psi \ge \epsilon\xi^* \text{ for all } \psi \in \omega(\Lambda;\varphi)\}$. Obviously, $\omega(\Lambda;\varphi) \subseteq \epsilon^*\xi^* + C_+(\Lambda)$. This gives $\psi(0,\cdot) > \epsilon^*\xi^*$ for any $\psi \in \omega(\Lambda;\varphi)$ due to $supp(\xi^*) \subsetneq \bar{\Lambda}$ and Proposition 2.3-(i). It suffices to show $\epsilon^* \ge \epsilon_M^*$. Otherwise, suppose $\epsilon^* < \epsilon_M^*$. Take $\varphi^* \in \omega(\Lambda;\psi)$. By (2.3), the choice of ϵ^*, and Lemma 2.1, we know that, for all $(t,x,) \in [0,\infty)\times\mathbb{R}^N$,

$$
\begin{aligned}
&u^{\varphi^*}(\Lambda;t,x) - \epsilon^*\xi^*(x)\\
&= T_\Lambda(t)[\varphi^*(0,\cdot)](x) - T_{\Lambda^*}(t)[\epsilon^*\xi^*](x)\\
&\quad +\mu\int_0^t T_\Lambda(t-s)[K_\Lambda[f(u^{\varphi^*}(\Lambda;s-\tau,\cdot))]](x)\mathrm{d}s - \mu\int_0^t T_{\Lambda^*}(t-s)[K_{\Lambda^*}[c^*\epsilon^*\xi^*]](x)\mathrm{d}s\\
&\ge T_\Lambda(t)\varphi^*(0,\cdot) - \epsilon^*\xi^*](x) + \mu\int_0^t T_\Lambda(t-s)[K_\Lambda[f(u^{\varphi^*}(\Lambda;s-\tau,\cdot))] - K_\Lambda[c^*\epsilon^*\xi^*]](x)\mathrm{d}s\\
&\ge T_\Lambda(t)[\varphi^*(0,\cdot) - \epsilon^*\xi^*](x).
\end{aligned}
$$

Note that $\{T_\Lambda(t)[\phi(0,\cdot)-\epsilon^*\xi^*]|_{\overline{\Lambda^*}} : t \in [\tau,2\tau] \text{ and } \phi \in \omega(\Lambda;\varphi)\}$ is a compact subset in $V_+^\circ(\Lambda^*)$ by Lemma 2.3 -(iii) and (vii). Due to Lemma 3.1-(iv) and $supp(\xi^*) \subseteq \Lambda^*$, there exists $\delta^* \in (0,\epsilon_M^* - \epsilon^*)$ such that $T_\Lambda(t)[\phi(0,\cdot) - \epsilon^*\xi^*] \ge \delta^*\xi^*$ for all $t \in [\tau,2\tau]$ and $\phi \in \omega(\Lambda;\varphi)$, which gives $\Phi_\Lambda(2\tau,\varphi^*) \ge (\delta^* + \epsilon^*)\xi^*$ for all $\varphi^* \in \omega(\Lambda;\varphi)$. By the invariance of $\omega(\Lambda;\varphi)$, we have $\omega(\Lambda;\varphi;\Lambda) \subseteq (\delta^* + \epsilon^*)\xi^* + C_+(\Lambda)$, a contradiction to the choice of ϵ^*. This completes the proof. □

Now, we are ready to prove the main result of this paper, a threshold dynamics.

Theorem 4.1. *Assume that (H1) and (H2) hold. Then we have the following results.*

(i) *If $s(A_\Lambda) \le 0$, then the trivial equilibrium 0 of (2.3) attracts all solutions of (2.3) with initial values $\varphi \in C_+(\Lambda)$.*
(ii) *If $s(A_\Lambda) > 0$, then (2.3) has a positive equilibrium u_Λ^+ in $Z_+(\Lambda)$.*
(iii) *If $s(A_\Lambda) > 0$, then (2.3) is persistent in the sense that there exists $\xi \in Z_+(\Lambda)$ with the property that $||\xi|| = 1$ and, for any $M \in \{r_n : n \in \mathbb{N}\}$, there exists $\epsilon_M > 0$ such that $\{\varphi \in C_+(\Lambda) : \epsilon_M\xi \le \varphi \le M\}$ is a positively invariant set of Φ_Λ and $\omega(\Lambda;\varphi) \subseteq \epsilon_M\xi + C_+(\Lambda)$ for any $\varphi \in C_+(\Lambda;\Omega)$ with $\varphi \le M$. Therefore, $u = 0$ becomes unstable.*
(iv) *Let $s(A_\Lambda) > 0$ and let $u_+^* = ||u_\Lambda^+||$. Suppose that the nonlinear function f further satisfies the following assumption,*

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(H3) *For any closed interval* $[a,b] \neq \{1\}$ *with* $0 < a \leq b < \infty$, *either* $G((0,u_+^*] \times [a,b]) \subset (a,\infty)$ *or* $G((0,u_+^*] \times [a,b]) \subset (0,b)$, *where* $G : (0,u_+^*] \times (0,\infty) \to (0,\infty)$ *is defined by* $G(k,u) = \frac{f(ku)}{f(k)}$.

Then u_Λ^+ *attracts all solutions of (2.3) with initial values* $\varphi \in C_+(\Lambda;\Omega)$.

Proof. (i) Suppose that $s(A_\Lambda) \leq 0$. We distinguish two cases to prove statement (i).

Case 1: $\mu + s(A_\Lambda) > 0$.

In this case, by Lemma 3.2-(iii), there exists $\phi^* \in Z_+(\Lambda)$ such that $A_\Lambda[\phi^*] = s(A_\Lambda)\phi^* \leq 0$. In particular, $\phi^*|_{\bar\Omega} \in V_+^\circ(\Omega)$.

Suppose that $\varphi \in C_+(\Lambda)$. By Proposition 2.3-(iii), we can obtain that $\{\psi(0,\cdot)|_{\bar\Omega} : \psi \in \omega(\Lambda;\varphi)\}$ is compact in $V_+(\Omega)$. This, together with Lemma 3.1-(iii), implies that, $a^* \triangleq \inf\{a > 0 : \psi(0,x) \leq a\phi^*(x)$ for all $x \in \bar\Omega$ and $\psi \in \omega(\Lambda;\varphi)\} \in [0,\infty)$. Then the invariance of $\omega(\Lambda;\varphi)$ gives $\omega(\Lambda;\varphi)|_{[-\tau,0]\times\bar\Omega} \leq \phi^*|_{\bar\Omega}$. According to Lemma 3.4-(iv), for any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that $u^\varphi(\Lambda;t,x) \leq (a^* + \epsilon)\phi^*(x)$ for all $t \geq t_\epsilon$ and $x \in \bar\Omega$. Let $v(t,x) = (a^* + \epsilon)\phi^*(x) - u^\varphi(\Lambda;t + t_\epsilon,x)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^N$. Then by (H2), $v(t,x)$ satisfies (3.1) and thus $\lim_{t\to\infty}(\inf\{v(t,x) : x \in \mathbb{R}^N\}) \geq 0$ by Lemma 3.3. This, together with the definition of $\omega(\Lambda;\varphi)$ and the arbitrariness of ϵ, implies $\omega(\Lambda;\varphi) \leq a^*\phi^*$.

In the following, we show $a^* = 0$, which implies $\omega(\Lambda;\varphi) = \{0\}$. By way of contradiction, we assume $a^* > 0$.

First, we claim that $\{u^\psi(\Lambda;t,\cdot)|_{\bar\Omega}; t \in [-\tau,\tau]\} \neq \{a^*\phi^*|_{\bar\Omega}\}$ for all $\psi \in \omega(\Lambda;\varphi)$. If this is not true, then there exists $\psi \in \omega(\Lambda;\varphi)$ such that $\{u^\psi(\Lambda;t,\cdot)|_{\bar\Omega} : t \in [-\tau,\tau]\} = \{a^*\phi^*|_{\bar\Omega}\}$. Define

$$v : [-\tau,\infty) \times \mathbb{R}^N \ni (t,x) \mapsto u^\psi(\Lambda;t,x) - a^*\phi^*(x).$$

Then $v(t,x) = 0$ for any $(t,x) \in [0,\tau] \times \bar\Omega$ and by the definition of v, we know that v satisfies

$$\begin{cases} \frac{\partial v}{\partial t}(t,x) = d\Delta v(t,x) - \mu v(t,x) + K_\Lambda[f(a^*\phi) - f'(0)a^*\phi](x), & (t,x) \in (0,\tau] \times \Omega, \\ v(t,x) = 0, & (t,x) \in [0,\tau] \times \partial\Omega, \\ v(0,x) = 0, & x \in \bar\Omega. \end{cases}$$

Consequently, $K_\Lambda[f(a^*\phi) - f'(0)a^*\phi](x) = 0$ for all $(t,x) \in (0,\tau] \times \Omega$. However, by (H2), $K_\Lambda[f(a^*\phi)](x) > f'(0)a^*\phi(x)$ for all $(t,x) \in (0,\tau] \times \Omega$, a contradiction. This proves the claim.

It follows from (2.3), (H2), the above claim, Lemma 2.2, and Lemma 2.3 that, for all $\psi \in \omega(\Lambda;\varphi)$,

$$\begin{aligned} &u^\psi(\Lambda;2\tau,\cdot) \\ &= T_\Lambda(2\tau)[\psi(0,\cdot)] + \mu\int_0^{2\tau} T_\Lambda(2\tau - s)[K_\Lambda[f(u^\psi(\Lambda;s-\tau,\cdot))]]\mathrm{d}s \\ &\leq T_\Lambda(2\tau)[a^*\phi] + \mu\int_0^{2\tau} T_\Lambda(2\tau - s)[K_\Lambda[f'(0)u^\psi(\Lambda;s-\tau,\cdot)]]\mathrm{d}s \end{aligned}$$

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$$< T_\Lambda(2\tau)[a^*\phi] + \mu \int_0^{2\tau} T_\Lambda(2\tau - s)[K_\Lambda[f'(0)a^*\phi]]\mathrm{d}s$$

$$= a^*\phi.$$

This, together with the invariance of $\omega(\Lambda;\varphi)$, implies $\psi(0,\cdot) < a^*\phi^*$ for all $\psi \in \omega(\Lambda;\varphi)$. Again, by (2.3), (H2), Lemma 2.2, and Lemma 2.3, we have for all $t \in \mathbb{R}_+$ and $\psi \in \omega(\Lambda;\varphi)$, that

$$\begin{aligned}
&a^*\phi^* - u^\psi(\Lambda;t,\cdot)\\
&\geq T_\Lambda(t)[a^*\phi^* - \psi(0,\cdot)] + \mu \int_0^t T_\Lambda(t-s)[a^*K_\Lambda[f'(0)\phi^*] - K_\Lambda[f(u^\psi(\Lambda;s-\tau,\cdot))]]\mathrm{d}s\\
&\geq T_\Lambda(t)[a^*\phi^* - \psi(0,\cdot)] + \mu \int_0^t T_\Lambda(t-s)[a^*K_\Lambda[f'(0)\phi^*] - K_\Lambda[f'(0)u^\psi(\Lambda;s-\tau,\cdot)]]\mathrm{d}s\\
&= T_\Lambda(t)[a^*\phi^* - \psi(0,\cdot)] + \mu f'(0) \int_0^t T_\Lambda(t-s)[K_\Lambda[a^*\phi^* - u^\psi(\Lambda;s-\tau,\cdot)]]\mathrm{d}s\\
&\geq T_\Lambda(t)[a^*\phi^* - \psi(0,\cdot)].
\end{aligned}$$

Note that, by Lemma 2.3-(iii) and (vii), we can easily verify that $\{T_\Lambda(\tau)[a^*\phi^* - \psi(0,\cdot)]|_{\bar\Omega} : \psi \in \omega(\Lambda;\varphi)\}$ is a compact subset in $V_+^\circ(\Omega)$. Then Lemma 3.1-(iv) tells us that there exists $\delta^* \in (0, a^*)$ such that $T_\Lambda(\tau)[a^*\phi^*(x) - \psi(0,\cdot)]|_{\bar\Omega} \geq \delta^*\phi^*|_{\bar\Omega}$ and hence $u^\psi(\Lambda;\tau,\cdot)|_{\bar\Omega} < (a^* - \delta^*)\phi^*|_{\bar\Omega}$ for all $\psi \in \omega(\Lambda;\varphi)$. This, combined with the invariance of $\omega(\Lambda;\varphi)$, yields $\psi(0,\cdot)|_{\bar\Omega} \leq (a^* - \delta^*)\phi^*|_{\bar\Omega}$ for all $\psi \in \omega(\Lambda;\varphi)$, a contradiction to the choice of a^*. Therefore, $a^* = 0$.

Case 2: $\mu + s(A_\Lambda) \leq 0$.

In this case, recall that $A_{\Lambda,cf'(0)} = d\Delta - \mu Id + \mu c f'(0)K_\Lambda$ for all $c > 0$. Then $s(A_{\Lambda,f'(0)}) \leq -\mu$. Lemma 3.2-(v) implies that there is $c^* > 1$ such that $-\mu < s(A_{\Lambda,c^*f'(0)}) \leq 0$. Applying similar arguments as those in the proof of *Case 1*, we have $\omega(c^*f,\Lambda;\varphi) = \{0\}$ for all $\varphi \in C_+(\Lambda)$. Then by Proposition 2.2, $\Phi_\Lambda(t,\varphi;c^*f) \geq \Phi_\Lambda(t,\varphi)$ for all $\varphi \in C_+(\Lambda)$, which implies $0 \leq \omega(\Lambda;\varphi) \leq \omega(c^*f,\Lambda;\varphi) = \{0\}$ for all $\varphi \in C_+(\Lambda)$.

To summarize, statement (i) holds.

(ii) For given $M \in \{r_n : n \in \mathbb{N}\}$, let ξ^* and ϵ_M^* be defined as in Lemma 4.1. Denote $\mathcal{A} = \{\varphi \in C_+(\Lambda) : \epsilon_M^*\xi^* \leq \varphi \leq M\}$. By Lemma 4.1, $\Phi_\Lambda(t,\mathcal{A}) \subseteq \mathcal{A}$ for all $t \in \mathbb{R}_+$. Let $g_T(\mathcal{B}) \triangleq \overline{co}(\Phi_\Lambda(T,\mathcal{B}))$ for any $\mathcal{B} \subseteq \mathcal{A}$ and $T > 0$. Then $\Phi_\Lambda(T,(g_T)^{k-1}(\mathcal{A})) \subseteq (g_T)^k(\mathcal{A}) \subseteq (g_T)^{k-1}(\mathcal{A})$ for any positive integer k and $T > 0$.

We claim that there exists a compact convex subset $\mathcal{A}_T$ in $\mathcal{A}$ such that $\Phi_\Lambda(T,\mathcal{A}_T) \subseteq \mathcal{A}_T$ for any $T \in I \triangleq \{\frac{2\tau}{3^i} : i = 1,2,\ldots\}$. Indeed, by Proposition 2.1-(iii), $\alpha_{-\frac{T}{2},0}[g_T(\mathcal{A})|_{[-\frac{T}{2},0]\times\mathbb{R}^N}] \leq e^{-\frac{T\mu}{2}}\alpha_{-\tau,0}[\mathcal{A}]$. This, combined with the fact that $\Phi_\Lambda(t,\varphi)(\theta,x) = u^\varphi(t+\theta,x)$ for all $(t,\theta,x) \in [0,\infty)\times[-\tau,0]\times\mathbb{R}^N$, yields that, for any positive integers k and l with $l \leq k \leq 3^i$,

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$$\begin{cases} \alpha_{(\frac{1}{2}-l)T,(1-l)T}[(g_T)^k(\mathcal{A})|_{[(\frac{1}{2}-l)T,(1-l)T]\times\mathbb{R}^N}] \le e^{-\frac{T\mu}{2}}\alpha_{-\tau,0}[\mathcal{A}] & \text{for } l \le 1+\frac{3^i}{2}, \\ \alpha_{(\frac{1}{2}-l)T+\tau,(1-l)T+\tau}[(g_T)^k(\mathcal{A})|_{[(\frac{1}{2}-l)T+\tau,(1-l)T+\tau]\times\mathbb{R}^N}] \le e^{-\frac{T\mu}{2}}\alpha_{-\tau,0}[\mathcal{A}] & \text{for } l > 1+\frac{3^i}{2}. \end{cases} \tag{4.1}$$

Using (4.1) and some simple computations, we can easily obtain that $\alpha_{-\tau,0}[(g_T)^{3^i}(\mathcal{A})] \le e^{-\frac{T\mu}{2}}\alpha_{-\tau,0}[\mathcal{A}]$. Denote by $\mathcal{A}_{l,T} \triangleq (g_T)^{3^i l}(\mathcal{A})$ for any positive integer l. Then $\alpha_{-\tau,0}[\mathcal{A}_{1+l,T}] \le e^{-\frac{T\mu}{2}}\alpha_{-\tau,0}[\mathcal{A}_{l,T}] \le e^{-\frac{T\mu}{2}}\alpha_{-\tau,0}[\mathcal{A}]$. Let $\mathcal{A}_T = \bigcap_{l=1}^{\infty}\mathcal{A}_{l,T}$. Then $\mathcal{A}_T$ is a compact convex subset in $C(\Lambda)$ such that $\Phi_\Lambda(T,\mathcal{A}_T) \subseteq \mathcal{A}_T$. By the Schauder fixed point theorem, there is $\psi_T \in \mathcal{A}_T$ such that $\Phi_\Lambda(T,\psi_T) = \psi_T$. According to Proposition 2.1-(iii) and the fact that $\{\psi_T : T \in I\} = \Phi_\Lambda(2\tau,\{\psi_T : T \in I\}) \subseteq \Phi_\Lambda(2\tau,\mathcal{A})$, we know that $\alpha_{-\tau,0}[\{\psi_T : T \in I\}] = \alpha_{-\tau,0}[\Phi_\Lambda(2\tau,\{\psi_T : T \in I\})] \le e^{-\tau\mu}\alpha_{-\tau,0}[\{\psi_T : T \in I\}]$. It follows that $\{\psi_T : T \in I\}$ is precompact in $X(\Lambda)$ and hence there exist $\psi \in \mathcal{A}$ and a sequence $\{T_k\}$ in I such that $\lim_{k\to\infty} T_k = 0$ and $\lim_{k\to\infty}\psi_{T_k} = \psi$. For any $t \in (0,\infty)$, there exist $s_k \in [0,T_k)$ and a nonnegative integer N_k such that $t = N_kT_k + s_k$. Obviously, $\lim_{k\to\infty} s_k = 0$. It follows that $\Phi_\Lambda(t,\psi) = \lim_{k\to\infty}\Phi_\Lambda(t,\psi_{T_k}) = \lim_{k\to\infty}\Phi_\Lambda(s_k,\psi_{T_k}) = \psi$, which implies that $u^+_\Lambda \triangleq \psi$ is a positive equilibrium of (2.3) located in $Z_+(\Lambda)$.

(iii) It suffices to obtain the persistence of (2.3) since the instability of $u = 0$ follows from the persistence of (2.3).

Note that $s(A_{\Lambda,f'(0)}) > 0$ due to $s(A_\Lambda) > 0$. Then Lemma 3.2 implies that there exists $c^* \in (0,f'(0))$ such that $s(A_{\Lambda,c^*}) = 0$. Hence there exists $\xi \in Z_+(\Lambda)$ such that $||\xi|| = 1$ and $A_{\Lambda,c^*}[\xi] = 0$.

For given $M \in \{r_n : n \in \mathbb{N}\}$, $f([0,M]) \subseteq [0,\frac{M}{k_\Lambda}]$ from (H1). Then, by (H2), there is $\epsilon_M > 0$ such that $f(u) > h_{c^*,\epsilon_M}(u)$ for all $u \in (0,M]$. Suppose that $\varphi \in C_+(\Lambda)$ with $\epsilon_M\xi \le \varphi \le M$. Proposition 2.1-(ii) implies that $\Phi_\Lambda(t,\varphi) \le M$ for all $t \in \mathbb{R}_+$. By Proposition 2.2 and the choice of h_{c^*,ϵ_M}, we have $\Phi_\Lambda(\varphi,t) \ge \Phi_\Lambda(\epsilon_M\xi,t;h_{c^*,\epsilon_M}(\cdot)) = \epsilon_M\xi$ for all $t \in \mathbb{R}_+$, which yields the first conclusion.

Next, suppose that $\varphi \in C_+(\Lambda;\Omega)$ with $\varphi \le M$. Let $u(t,x) = u^\varphi(\Lambda;t,x)$ for all $(t,x) \in [-\tau,\infty)\times\mathbb{R}^N$. Without loss of generality, we assume that $u(t,\cdot) \in \omega(\Lambda;\varphi)\cap Z_+(\Lambda)\cap C^{1,\frac{1}{2}}_{loc}(\bar{\Lambda})$ for all $t \in [-\tau,\infty)$ according to Proposition 2.3-(i). From the proof of Lemma 4.1, there exist a bounded and smooth domain Λ^* and $\xi^* \in Z_+(\Lambda^*)$ such that $\Omega \subseteq \Lambda^* \subseteq \Lambda$, $||\xi^*|| = 1$, and $\omega(\Lambda;\varphi) \subseteq \epsilon_M\xi^* + C_+(\Lambda)$. By expanding Λ^* properly, we may assume that $\partial\Omega\cap\partial(\Lambda^*) = \partial\Omega\cap\partial\Lambda$. It follows from Lemma 3.4-(iii) and the compactness of $supp(\xi^*)$ that there is $t_1 > 0$ such that $u(t,\cdot) \ge \frac{\epsilon_M}{2}\xi^*$ for all $t \ge t_1$. Note that $\xi^*|_{\bar{\Omega}}, \xi|_{\bar{\Omega}}$, and $u(t,\cdot)|_{\bar{\Omega}} \in V^\circ_+(\Omega)$ for all $t \in [-\tau,t_1]$ due to the choice of Λ^*. According to Proposition 2.3-(ii), we know that $\{u(t,\cdot)|_{\overline{\Lambda^*}} : t \in [-\tau,t_1]\}$ is compact in $C^1(\overline{\Omega})$. Therefore, by Lemma 3.1-(iv), there is $\epsilon_1 > 0$ such that $\xi^*|_{\bar{\Omega}} \ge \epsilon_1\xi|_{\bar{\Omega}}$ and $u(t,\cdot)|_{\bar{\Omega}} \ge \epsilon_1\xi|_{\bar{\Omega}}$ for all $t \in [-\tau,t_1]$. Let $\epsilon_0 = \min\{\epsilon_1,\frac{\epsilon_1\epsilon_M}{2}\}$. Then we have $u(t,x) \ge \epsilon_0\xi(x)$ for all $t \in [-\tau,\infty)$ and $x \in \bar{\Omega}$.

Let $v(t,x) = u(t,x) - \epsilon_0\xi(x)$ for all $t \in [-\tau,\infty)$ and $x \in \mathbb{R}^N$. Then v satisfies (3.1) and hence $\omega(\Lambda;\varphi) \subseteq \epsilon_0\xi + C_+(\Lambda)$ by Lemma 3.3 and the definition of $\omega(\Lambda;\varphi)$.

Let $\epsilon^* = \sup\{\epsilon \ge \epsilon_0 : \psi \ge \epsilon\xi \text{ for all } \psi \in \omega(\Lambda;\varphi)\}$. Obviously, $\omega(\Lambda;\varphi) \subseteq \epsilon^*\xi + C_+(\Lambda)$. It suffices to show $\epsilon^* \ge \epsilon_M$. We achieve this by with contradiction as follows.

Suppose $\epsilon^* < \epsilon_M$. Clearly, for any $\psi \in \omega(\Lambda;\varphi)$, there exists $\theta_\psi \in [0,2\tau)$ such that $u^\psi(\theta_\psi,\cdot) > \epsilon^*\xi$ due to choices of ξ and ϵ^*. By (2.3), the choice of ϵ^*, and Lemma 2.1, we know that, for any $\psi \in \omega(\Lambda;\varphi)$,

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$$\begin{aligned} u^{\psi}(\Lambda;2\tau,\cdot) &= \epsilon^*\xi + \Phi_\Lambda(\theta_\psi,\psi)(2\tau-\theta_\psi,\cdot) - \epsilon^*\xi \\ &= \epsilon^*\xi + T_\Lambda(2\tau-\theta_\psi)[u^{\psi}(\Lambda;\theta_\psi,\cdot) - \epsilon^*\xi] \\ &\quad + \mu\int_0^{2\tau-\theta_\psi} T_\Lambda(2\tau-\theta_\psi-s)[K_\Lambda[f(u^{\psi}(\Lambda;s+\theta_\psi-\tau,\cdot))] - \epsilon^* K_\Lambda[c^*\xi]]\mathrm{d}s \\ &\geq \epsilon^*\xi + T_\Lambda(2\tau-\theta_\psi)[u^{\psi}(\Lambda;\theta_\psi,\cdot) - \epsilon^*\xi] \\ &> \epsilon^*\xi. \end{aligned}$$

This, combined with the invariance of $\omega(\Lambda;\varphi)$, implies $\psi(0,\cdot) > \epsilon^*\xi$ for all $\psi \in \omega(\Lambda;\varphi)$. It follows from (2.3), the choice of ϵ^*, and Lemma 2.1 that for any $\psi \in \omega(\Lambda;\varphi)$,

$$\begin{aligned} u^{\psi}(\Lambda;\tau,\cdot) &= \epsilon^*\xi + u^{\psi}(\Lambda;\tau,\cdot) - \epsilon^*\xi \\ &= \epsilon^*\xi + T_\Lambda(\tau)[\psi(0,\cdot) - \epsilon^*\xi] \\ &\quad + \mu\int_0^{\tau} T_\Lambda(\tau-s)[K_\Lambda[f(u^{\psi}(\Lambda;s-\tau,\cdot))] - \epsilon^* K_\Lambda[c^*\xi]]\mathrm{d}s \\ &\geq \epsilon^*\xi + T_\Lambda(\tau)[\psi(0,\cdot) - \epsilon^*\xi]. \end{aligned}$$

Note that, by Lemma 2.3-(iii) and (vii), $\{T_\Lambda(\tau)[\psi(0,\cdot) - \epsilon^*\xi^*]|_{\bar\Omega} : \psi \in \omega(\Lambda;\varphi)\}$ is a compact subset in $V_+^\circ(\Omega)$. By Lemma 3.1-(iv), there exists $\delta^* \in (0, \epsilon_M^* - \epsilon^*)$ such that $T_\Lambda(\tau)[\psi(0,\cdot) - \epsilon^*\xi]|_{\bar\Omega} \geq \delta^*\xi|_{\bar\Omega}$ for all $\psi \in \omega(\Lambda;\varphi)$. This, together with the invariance of $\omega(\Lambda;\varphi)$, implies $\psi|_{[-\tau,0]\times\bar\Omega} \geq (\delta^* + \epsilon^*)\xi|_{\bar\Omega}$ for all $\psi \in \omega(\Lambda;\varphi)$. It follows from Lemma 3.4-(iii) that there is $t_2 > 0$ such that $u(t,\cdot) \geq \frac{\delta^*+2\epsilon^*}{2}\xi$ for all $t \geq t_2$. Let $\tilde v(t,x) = u(t+t_2+\tau,x) - \frac{\delta^*+2\epsilon^*}{2}\xi(x)$ for all $t \in [-\tau,\infty)$ and $x \in \mathbb{R}^N$. Then $\tilde v$ satisfies (3.1) and consequently, by Lemma 3.3 and the definition of $\omega(\Lambda;\varphi)$, we easily see $\omega(\Lambda;\varphi) \subseteq \frac{\delta^*+2\epsilon^*}{2}\xi + C_+(\Lambda)$, a contradiction to the choice of ϵ^*. This proves $\epsilon^* \geq \epsilon_M$.

(iv) Suppose $\varphi \in C_+(\Lambda;\Omega)$. By statements (ii, iii) and Proposition 2.3-(i, iii), $\{\psi(0,\cdot)|_{\bar\Omega} : \psi \in \omega(\Lambda;\varphi)\}$ is a compact subset in $V_+^\circ(\Omega)$. Therefore, Lemma 3.1 implies $0 < a^{**} \leq b^{**} < \infty$, where $a^{**} = \sup\{a \geq 0 : \psi(0,x) \geq a u_+(x) \text{ for all } (\psi,x) \in \omega(\Lambda;\varphi)\times\overline{\Omega}\}$ and $b^{**} = \inf\{b \geq 0 : \psi(0,x) \leq b u_+(x) \text{ for all } (\psi,x) \in \omega(\Lambda;\varphi)\times\overline{\Omega}\}$. By the choices of a^{**} and b^{**}, and Lemma 3.3, we have $0 < a^{**}u_+ \leq \psi \leq b^{**}u_+$ for all $\psi \in \omega(\Lambda;\varphi)$. To prove statement (iv), it suffices to prove that $a^{**} = b^{**} = 1$. Otherwise, $a^{**} \neq 1$ or $b^{**} \neq 1$. By assumption (H3) with $[a,b] = [a^{**},b^{**}]$, we know that either (I) $f(ku) > a^{**}f(u)$ for all $(k,u) \in [a^{**},b^{**}]\times(0,u_+^*]$ or (II) $f(ku) < b^{**}f(u)$ for all $(k,u) \in [a^{**},b^{**}]\times(0,u_+^*]$.

In the following, we only give the detail to get a contradiction for (I) since (II) can be dealt with similarity.

It follows from (2.3), Lemma 2.2, and Lemma 2.3 that, for any $\psi \in \omega(\Lambda;\varphi)$,

$$\begin{aligned} u^{\psi}(\Lambda;\tau,\cdot) &= a^{**}u_+ + u^{\psi}(\Lambda;\tau,\cdot) - a^{**}u_+ \\ &= a^{**}u_+ + T_\Lambda(\tau)[\psi(0,\cdot)] + \mu\int_0^{\tau} T_\Lambda(\tau-s)[K_\Lambda[f(u^{\psi}(\Lambda;s-\tau,\cdot))]]\mathrm{ds} - \mathrm{a}^{**}\mathrm{u}_+ \\ &= a^{**}u_+ + T_\Lambda(\tau)[\psi(0,\cdot) - a^{**}u_+] \end{aligned}$$

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$$+\mu\int_0^\tau T_\Lambda(\tau-s)[K_\Lambda[f(u^\psi(\Lambda;s-\tau,\cdot))-a^{**}f(u_+)]](x)\mathrm{d}s$$
$$>a^{**}u_+ + T_\Lambda(\tau)[\psi(0,\cdot)-a^{**}u_+]$$
$$\geq a^{**}u_+.$$

This, together with the invariance of $\omega(\Lambda;\varphi)$, implies $\psi(0,\cdot)>a^{**}u_+$ for all $\psi\in\omega(\Lambda;\varphi)$. Then by Lemma 2.3-(iii, vii) and Lemma 3.1, there exists $\delta^*>0$ such that $T_\Lambda(\tau)[\psi(0,\cdot)-a^{**}u_+]|_{\bar\Omega}\geq\delta^*u_+|_{\bar\Omega}$ for all $\psi\in\omega(\Lambda;\varphi)$. As a result, we easily see that $u^\psi(\Lambda;\tau,\cdot)|_{\bar\Omega}\geq(a^{**}+\delta^*)u_+|_{\bar\Omega}$ for all $\psi\in\omega(\Lambda;\varphi)$. It follows from the invariance of $\omega(\Lambda;\varphi)$ that $\psi(\theta,\cdot)|_{\bar\Omega}\geq(a^{**}+\delta^*)u_+|_{\bar\Omega}$ for all $(\theta,\psi)\in[-\tau,0]\times\omega(\Lambda;\varphi)$, a contradiction to the choice of a^{**}.

This completes the proof. □

The condition (H3) in Theorem 4.1 requires information about $u_+^*=||u_\Lambda^+||$. However, we only know the existence of the positive state steady u_Λ^+ but do not know, in general, the details about u_Λ^+ (hence u_+^*). This makes Theorem 4.1 less feasible. Motivated by [46,47], we find that the following condition depending only on the property of the nonlinear function f can serve as an alternative, which can be much easier to verify that (H3):

(H4) There exists $u^*>0$ such that $f(u^*)=u^*$. Moreover, for any closed interval $[a,b]\neq\{1\}$ with $0<a\leq b<\infty$, either $G((0,f^*]\times[a,b])\subset(a,\infty)$ or $G((0,f^*]\times[a,b])\subset(0,b)$, where $f^*=\sup_{k\in[0,u^*]}f(k)$ and $G:(0,f^*]\times(0,\infty)\to(0,\infty)$ is defined by $G(k,u)=\frac{f(ku)}{f(k)}$.

Theorem 4.2. *Assume that (H1) and (H2) hold with* $[0,\frac{r_n}{k_\Lambda}]$ *being replaced by* $[0,r_n]$. *Then we have the following results.*

(i) *If* $s(A_\Lambda)\leq-d\lambda_\Lambda-\mu+\mu f'(0)e^{-\lambda_\Lambda\alpha}\leq0$, *then the trivial equilibrium* 0 *of (2.3) attracts all solutions of (2.3) with initial values* $\varphi\in C_+(\Lambda)$.

(ii) *If* $s(A_\Lambda)\geq-d\lambda_\Omega-\mu+\mu f'(0)e^{-\lambda_\Omega\alpha}>0$, *then* $u=0$ *becomes unstable and (2.3) has a positive equilibrium* $u_+\in Z_+(\Lambda)$. *Moreover, (2.3) is persistent in the sense that there exists* $\xi\in Z_+(\Lambda)$ *with the property that* $||\xi||=1$ *and for any* $M\in\{r_n:n\in\mathbb{N}\}$, *there exists* $\epsilon_M>0$ *such that* $\{\varphi\in C_+(\Lambda):\epsilon_M\xi\leq\varphi\leq M\}$ *is a positively invariant set of* Φ_Λ *and* $\omega(\Lambda;\varphi)\subseteq\epsilon_M\xi+C_+(\Lambda)$ *for any* $\varphi\in C_+(\Lambda;\Omega)$ *with* $\varphi\leq M$.

(iii) *Let* $-d\lambda_\Omega-\mu+\mu f'(0)e^{-\lambda_\Omega\alpha}>0$. *Suppose that the nonlinear function* f *also satisfies assumption (H4). Then* u_+ *attracts all solutions of (2.3) with initial values* $\varphi\in C_+(\Lambda;\Omega)$.

Proof. (i) and (ii) follow from (i) and (ii) in Theorem 4.1 due to Lemma 3.2-(ii).

According to the proof of Theorem 2.5 [46], we can easily verify (H3) under (H4). Therefore, (iii) is a direct consequence of Theorem 4.1-(iii) and Lemma 3.2-(ii). □

5. Examples

In this section, we illustrate Theorem 4.2 by considering equation (2.3) with specific birth functions, namely, the Ricker function and the Mackey–Glass function. We will leave the applications of Theorem 4.1 to readers.

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Example 5.1. Consider the equation with the Ricker birth function,

$$\begin{cases} \frac{\partial u}{\partial t} = D_m \Delta u - d_m u \\ \qquad\quad +\varepsilon \int_\Lambda \Gamma_\Omega(\alpha, x, y) p u(t-\tau, y) e^{-u(t-\tau,y)} \mathrm{d}y, & (t,x) \in (0,\infty)\times\Lambda, \\ u(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times \partial\Lambda, \\ u(t,x) = \varphi(t,x), & (t,x) \in [-\tau, 0]\times\bar{\Lambda}, \end{cases} \tag{5.1}$$

where $p > 0$ and $\varphi \in C_+(\Lambda)$. We recall that D_m and d_m are the diffusion and death rates for the mature individuals, respectively, while $\alpha = \int_0^\tau D_I(a)\mathrm{da}$ and $\varepsilon = \exp(-\int_0^\tau d_I(a)\mathrm{da})$ with $D_I(a)$ and $d_I(a)$ being the diffusion and death rates of the immature, respectively. Note that the Ricker birth function, $b(u) = pue^{-u}$ for all $u \in \mathbb{R}_+$, has been widely used in models for fish and for blowfly (see, for example, [13,5,21,19,28,31–33,46]).

The following lemma follows from Lemma 3.7 and Lemma 4.2 in [47].

Lemma 5.1. *Let $f(u) = \frac{\varepsilon p}{d_m} u e^{-u}$ for all $u \in \mathbb{R}_+$. Then the following statements are true.*

(i) *The assumptions (H1) and (H2) with $r_n = p + n$ hold.*
(ii) *If $2ed_m \geq \varepsilon p > d_m$, then assumption (H4) holds.*

By applying Theorem 4.2 and Lemma 5.1, we then obtain the following results for (5.1).

Theorem 5.1. *The following statements hold.*

(i) *If $-D_m\lambda_\Lambda - d_m + \varepsilon p e^{-\lambda_\Lambda \alpha} \leq 0$, then the trivial equilibrium 0 of (5.1) attracts all solutions with initial values $\varphi \in C_+(\Lambda)$.*
(ii) *If $-D_m\lambda_\Omega - d_m + \varepsilon p e^{-\lambda_\Omega \alpha} > 0$, then $u = 0$ becomes unstable and (5.1) has a positive equilibrium $u_+ \in Z_+(\Lambda)$. Furthermore, (5.1) is persistent in the sense that there exists $\xi \in Z_+(\Lambda)$ with the property that $||\xi|| = 1$, and, for any $M \in \{p + n : n \in \mathbb{N}\}$, there exists $\epsilon_M > 0$ such that $\{\varphi \in C_+(\Lambda) : \epsilon_M \xi \leq \varphi \leq M\}$ is a positively invariant set of Φ_Λ and $\omega(\Lambda;\varphi) \subseteq \epsilon_M \xi + C_+(\Lambda)$ for any $\varphi \in C_+(\Lambda;\Omega)$ with $\varphi \leq M$.*
(iii) *Let $-D_m\lambda_\Omega - d_m + \varepsilon p e^{-\lambda_\Omega \alpha} > 0$ and $\varepsilon p \leq 2ed_m$. Then u_+ attracts all solutions of (5.1) with initial values $\varphi \in C_+(\Lambda;\Omega)$.*

Example 5.2. Consider the equation with the Mackey–Glass birth function,

$$\begin{cases} \frac{\partial u}{\partial t} = D_m \Delta u - d_m u + \varepsilon \int_\Lambda \Gamma_\Omega(\alpha, x, y) \frac{p u(t-\tau,y)}{1+(u(t-\tau,y))^n} \mathrm{d}y, & (t,x) \in (0,\infty)\times\Lambda, \\ u(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times \partial\Lambda, \\ u(t,x) = \varphi(t,x), & (t,x) \in [-\tau, 0]\times\bar{\Lambda}, \end{cases} \tag{5.2}$$

where n, $p > 0$, and $\varphi \in C_+(\Lambda)$. In this example, the birth function $b(u) = \frac{pu}{1+u^n}$ for all $u \in \mathbb{R}_+$ was originally used by Mackey and Glass [23] to model the blood cell production in an ordinary differential equation model. Since then modified models have been studied by many researchers.

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One of the topics on these models is the stability of a positive equilibrium, accounting for a long term stable blood concentration level. See, for example, [17] and the references therein.

The following lemma follows from Lemma 3.7 and Lemma 4.1 in [47].

Lemma 5.2. *Let* $f(u) = \frac{\varepsilon p}{d_m}\frac{u}{1+u^n}$ *for all* $u \in \mathbb{R}_+$. *Then the following statements are true.*

(i) *The assumptions (H1) and (H2) with* $r_n = n + \frac{\varepsilon p}{d_m}$ *hold.*

(ii) *If* $\varepsilon p > d_m$ *and* $n \le \max\{2\frac{(\varepsilon p)^n(n-1)^{n-1}+n^n d_m^n}{(\varepsilon p)^n(n-1)^{n-1}}, \frac{\varepsilon p}{\varepsilon p - d_m}\}$, *then assumption (H4) holds.*

By applying Theorem 4.2 and Lemma 5.2, we then obtain the following results for (5.2).

Theorem 5.2. *The following statements hold.*

(i) *If* $-D_m\lambda_\Lambda - d_m + \varepsilon p e^{-\lambda_\Lambda \alpha} \le 0$, *then the trivial equilibrium* 0 *of* (5.2) *attracts all solutions with initial values* $\varphi \in C_+(\Lambda)$.

(ii) *If* $-D_m\lambda_\Omega - d_m + \varepsilon p e^{-\lambda_\Omega \alpha} > 0$, *then* $u = 0$ *becomes unstable and* (5.2) *has a positive equilibrium* $u_+ \in Z_+(\Lambda)$. *Moreover,* (5.2) *is persistent in the sense that there exists* $\xi \in Z_+(\Lambda)$ *with the property that* $||\xi|| = 1$ *and for any* $M \in \{n + \frac{\varepsilon p}{d_m} : n \in \mathbb{N}\}$, *there exists* $\epsilon_M > 0$ *such that* $\{\varphi \in C_+(\Lambda) : \epsilon_M \xi \le \varphi \le M\}$ *is a positively invariant set of* Φ_Λ *and* $\omega(\Lambda;\varphi) \subseteq \epsilon_M \xi + C_+(\Lambda)$ *for any* $\varphi \in C_+(\Lambda;\Omega)$ *with* $\varphi \le M$.

(iii) *Let* $-D_m\lambda_\Omega - d_m + \varepsilon p e^{-\lambda_\Omega \alpha} > 0$. *Suppose that* $n \le \max\{2\frac{(\varepsilon p)^n(n-1)^{n-1}+n^n d_m^n}{(\varepsilon p)^n(n-1)^{n-1}}, \frac{\varepsilon p}{\varepsilon p - d_m}\}$. *Then* u_+ *attracts all solutions of* (5.2) *with initial values* $\varphi \in C_+(\Lambda;\Omega)$.

Theorem 5.1 and Theorem 5.2 reveal that the dynamics of equation (2.3) depends not only on the death rates, the diffusion rates, and the birth functions, but also on the choice of the mature-immature habitats. Specifically, (I) the (pseudo-)spectral bound $s(A_\Lambda)$ of the linear operator A_Λ depends on the habitats and is increasing with the increase of the sizes of both Ω and Λ (see Lemma 3.2); (II) the boundedness of the immature habitat Ω directly enables us to obtain the threshold dynamics about the global attractivity of the trivial equilibrium, the existence and global attractivity of the positive equilibrium, and the persistence of other solutions in the sense of the supremum norm regardless of whether the mature habitat Λ is bounded or not.

We point out that if the immature habitat Ω is unbounded, then the problem is more complex. Generally, we can't get the compactness of solution trajectories and the existence of a compact global attractor in the sense of the supremum norm. However, some dynamics with respect to the compact open topology and asymptotic propagation properties have been obtained when $\Omega = \Lambda = \mathbb{R}^N$ or $\Omega = \Lambda = \mathbb{R}_+$ in [5,47]. In the future, we shall consider the case where Ω is unbounded and $\Omega \subsetneq \Lambda$.

Appendix A. Derivation of the model

The purpose of this appendix is to derive a population growth model for a single species with age structure and diffusion. It is assumed that the immature individuals live in a bounded habitat which is part of that of the mature ones. The maturation time is taken as a constant τ (> 0). We assume that the boundaries of the habitats are hostile to the species, that is, we impose a Dirichlet boundary condition.

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Consider a single species whose mature individuals live in Λ while immature ones live in Ω. Let $W(t,a,x)$ be the population density of the species at time $t \in \mathbb{R}_+$ and location $x \in \Lambda$ of age $a \in \mathbb{R}_+$. Denote by $D(a)$ and $d(a)$ the diffusion rate and the death rate, respectively, at age a. Then $W(t,a,x)$ satisfies the following evolution equation (see, for instance, Metz–Diekmann [27]),

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial a} = D(a)\Delta W - d(a)W, \qquad t > 0, (a,x) \in ([0,\tau] \times \Omega) \cup ((\tau,\infty) \times \Lambda). \tag{A.1}$$

As mentioned earlier, we consider the homogeneous Dirichlet boundary condition on the pair $(\Lambda; \Omega)$,

$$W(t,a,x) = 0, \qquad t \in \mathbb{R}_+ \text{ and } (a,x) \in ([0,\tau] \times (\bar{\Lambda} \setminus \Omega)) \cup ((\tau,\infty) \times \partial\Lambda), \tag{A.2}$$

which accounts for the scenario that the region $\mathbb{R}^N \setminus \Omega$ is hostile to the immature individuals while the region $\mathbb{R}^N \setminus \Lambda$ is hostile to the mature ones. Then the density of the mature individuals at time t and location x, $u(t,x)$, is given by

$$u(t,x) = \int_\tau^\infty W(t,a,x)\mathrm{da}. \tag{A.3}$$

By the biological meaning of $W(t,a,x)$, one should have

$$W(t,\infty,x) = 0, \qquad (t,x) \in \mathbb{R}_+ \times \Lambda \tag{A.4}$$

and

$$W(t,0,x) = b(u(t,x)), \qquad (t,x) \in \mathbb{R}_+ \times \Omega,$$

where $b : \mathbb{R}_+ \to \mathbb{R}_+$ is the birth function.

Differentiating (A.3) with respect to t and making use of (A.1) and (A.4), we get

$$\frac{\partial u(t,x)}{\partial t} = W(t,\tau,x) + D_m \Delta u - d_m u, \qquad t > 0, x \in \Lambda. \tag{A.5}$$

Here, for simplicity of presentation, we have assumed that the diffusion rate and the death rate for the mature individuals are age-independent, that is, $D(a) = D_m$ and $d(a) = d_m$ for $a \in (\tau,\infty)$. We need to determine $W(t,\tau,x)$ in terms of $u(t,x)$. To this end, for any $s \in \mathbb{R}_+$, let $V^s(t,x) = W(t,t-s,x)$ for $(t,x) \in [s,s+\tau] \times \bar{\Omega}$. Then by (A.1), (A.2), and (A.4), we have

$$\begin{cases} \frac{\partial V^s}{\partial t}(t,x) = D(t-s)\Delta V^s(t,x) - d(t-s)V^s(t,x), & (t,x) \in (s,s+\tau] \times \Omega, \\ V^s(t,x) = 0, & (t,x) \in [s,s+\tau] \times (\bar{\Lambda} \setminus \Omega), \\ V^s(s,x) = b(u(s,x)), & x \in \bar{\Omega}. \end{cases} \tag{A.6}$$

One easily checks that

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$$V^s(t,x) = e^{-\int_0^{t-s} d(\varsigma)\mathrm{d}\varsigma} S_\Omega\left(\int_0^{t-s} D(\varsigma)\mathrm{d}\varsigma\right)[b(u(s,\cdot))](x)$$

$$= e^{-\int_0^{t-s} d(\varsigma)\mathrm{d}\varsigma} \int_\Omega \Gamma_\Omega\left(\int_0^{t-s} D(\varsigma)\mathrm{d}\varsigma, x, y\right) b(u(s,y))\mathrm{d}y$$

satisfies (A.6). Therefore, for $(t,x) \in \mathbb{R}_+ \times \bar{\Lambda}$,

$$W(t,\tau,x) = \varepsilon S_\Omega(\alpha)[b(u(t-\tau,\cdot))](x) = \varepsilon \int_\Lambda \Gamma_\Omega(\alpha,x,y) b(u(t-\tau,y))\mathrm{d}y, \tag{A.7}$$

where $\alpha = \int_0^\tau D_I(a)\mathrm{da}$ and $\varepsilon = \exp(-\int_0^\tau d_I(a)\mathrm{da})$. Here we have used $D_I(a)$ and $d_I(a)$ to denote the diffusion and death rates of the immature individuals, respectively, that is, $D_I = D|_{[0,\tau]}$ and $d_I = d|_{[0,\tau]}$.

In summary, by (A.2), (A.5), and (A.7) as well as the initial value condition, we have obtained the following initial-boundary value problem of a delayed reaction–diffusion equation with spatial heterogeneity,

$$\begin{cases} \frac{\partial u}{\partial t} = D_m \Delta u - d_m u + \varepsilon \int_\Lambda \Gamma_\Omega(\alpha,x,y) b(u(t-\tau,y))\mathrm{d}y, & (t,x) \in (0,\infty) \times \Lambda, \\ u(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times (\mathbb{R}^N \setminus \Lambda), \\ u(t,x) = \varphi(t,x), & (t,x) \in [-\tau,0] \times \mathbb{R}^N, \end{cases} \tag{A.8}$$

where $\varphi \in C_+(\Lambda)$.

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