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Long-time asymptotics for the nonlocal nonlinear Schrödinger equation with step-like initial data

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Abstract

We study the Cauchy problem for the integrable nonlocal nonlinear Schrödinger (NNLS) equation

$$iq_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)\bar{q}(-x,t) = 0$$

with a step-like initial data: $q(x,0) = q_0(x)$, where $q_0(x) = o(1)$ as $x \to -\infty$ and $q_0(x) = A + o(1)$ as $x \to \infty$, with an arbitrary positive constant $A > 0$. The main aim is to study the long-time behavior of the solution of this problem. We show that the asymptotics has qualitatively different form in the quarter-planes of the half-plane $-\infty < x < \infty$, $t > 0$: (i) for $x < 0$, the solution approaches a slowly decaying, modulated wave of the Zakharov-Manakov type; (ii) for $x > 0$, the solution approaches the “modulated constant”. The main tool is the representation of the solution of the Cauchy problem in terms of the solution of an associated matrix Riemann-Hilbert (RH) problem and the consequent asymptotic analysis of this RH problem.

Keywords: Nonlocal integrable equations; Riemann-Hilbert problem; Large time asymptotics; Cauchy problem with step-like initial values; Nonlinear steepest descent method

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1. Introduction

We consider the following initial value problem for the focusing nonlocal nonlinear Schrödinger (NNLS) equation with a step-like initial data:

$$iq_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)\bar{q}(-x,t) = 0, \qquad x \in \mathbb{R},\ t > 0, \tag{1.1a}$$

$$q(x,0) = q_0(x), \qquad x \in \mathbb{R}, \tag{1.1b}$$

where

$$q_0(x) \to 0 \ \text{ as } x \to -\infty \ \text{ and } \ q_0(x) \to A \text{ as } x \to \infty \tag{1.1c}$$

sufficiently fast, with some $A > 0$. Throughout the paper, $\bar{q}$ denotes the complex conjugate of q.

The nonlocal nonlinear Schrödinger equation in the form (1.1a) was introduced by M. Ablowitz and Z. Musslimani in [5]. Although this equation is just a reduction of a member of the AKNS hierarchy [3], namely, of the coupled Schrödinger equations

$$iq_t + q_{xx} + 2q^2r = 0, \tag{1.2a}$$

$$-ir_t + r_{xx} + 2r^2q = 0, \tag{1.2b}$$

corresponding to $r(x,t) = \bar{q}(-x,t)$, the NNLS equation has recently attracted much attention because of its distinctive physical and mathematical properties. Indeed, this equation is invariant under the joint transformations $x \to -x$, $t \to -t$, and complex conjugation, i.e. it is parity-time (PT) symmetric and, therefore, is related to a cutting edge research area of modern physics [8,28]. Particularly, due to the gauge-equivalence of the NNLS to the unconventional system of coupled Landau-Lifshitz (CLL) equations, this equation can find applications in the physics of nanomagnetic artificial materials [24].

Because of these features of the NNLS equation and the potential applications, other symmetry reductions of the AKNS and other hierarchies, which lead to other types of nonlocality, began to attract considerable attention. Typical examples are the reverse space-time nonlocal NLS equation and the reverse time nonlocal NLS equation, the complex/real space-time Sine-Gordon equation, the complex/real reverse space-time mKdV equation [1,6,7], the nonlocal derivative NLS equation [38], and the multidimensional nonlocal Davey-Stewartson equation [7,22].

In [6] the authors presented the Inverse Scattering Transform (IST) method to the study of the Cauchy problem for equation (1.1a), based on a variant of the Riemann-Hilbert approach, in the case of decaying initial data and obtained the one- and two-soliton solutions. In [2] and [36], a general decaying N-soliton solution of (1.1a) were found using the Hirota's direct method and the Riemann-Hilbert approach respectively (see also [37], where the N-soliton solution of the general coupled Schrödinger equations (1.2) is presented by the Riemann-Hilbert approach). The one-, two- and three-soliton solutions are obtained via the Hirota's direct method in [25] whereas in [15], the decaying one-soliton solution is obtained in terms of a double Wronskian. The soliton solutions of the focusing NNLS equation (1.1a) have some specific features: particularly, they can blow up at a finite time [2,6], and (1.1a) can simultaneously support both bright and dark soliton solutions [34].

The initial value problem for (1.1a) with the following nonzero boundary conditions:

$$q(x,t) \to q_{\pm}(t) = q_0 e^{i(\alpha t+\theta_{\pm})}, \qquad x \to \pm\infty, \tag{1.3}$$

where $q_0 > 0$, $\alpha \in \mathbb{R}$, $0 \le \theta_{\pm} < 2\pi$, is considered in [4], where the IST method is developed and the soliton solutions are constructed for certain values of the parameters $\theta_{\pm}$ (see also [2], where the general N-soliton solutions are presented).

In the present paper we assume that the solution $q(x,t)$ of problem (1.1a)–(1.1b) satisfies the following boundary conditions for all $t > 0$:

$$q(x,t) = o(1), \qquad x \to -\infty, \tag{1.4a}$$

$$q(x,t) = A + o(1), \qquad x \to +\infty \tag{1.4b}$$

(in what follows we will make the sense of $o(1)$ more precise). This choice of initial data and boundary values is inspired by the shock problems for the classical (local) NLS equation

$$iq_t(x,t) + q_{xx}(x,t) + 2|q(x,t)|^2 q(x,t) = 0, \tag{1.5}$$

which is another (local) reduction of system (1.2), with $r(x,t) = \bar{q}(x,t)$. Such problems have been considered since 1980s [9,10,13,27,30]. Particularly, in [13] the authors study the Cauchy problem for the NLS equation with the following initial condition:

$$q_0(x) = \begin{cases} 0, & x \le 0, \\ Ae^{-2iBx}, & x > 0, \end{cases} \tag{1.6}$$

assuming that the solution satisfies the boundary conditions

$$q(x,t) = o(1), \qquad x \to -\infty, \tag{1.7a}$$

$$q(x,t) = q^p(x,t) + o(1), \qquad x \to +\infty, \tag{1.7b}$$

where $q^p(x,t) = Ae^{-2iBx+2i\omega t}$ with $\omega = A^2 - 2B^2$ is a plane wave solution of the NLS equation (1.5). Notice that for the classical NLS, the both limiting functions in (1.7), i.e., $q_-(x,t) \equiv 0$ and $q_+(x,t) = q^p(x,t)$ are solutions of (1.5) whereas in the case of the NNLS equation, $q_-(x,t) \equiv 0$ is a solution, but $q_+(x,t) \equiv A$ is not. With this respect, the non-zero boundary conditions (1.4), being the simplest shock-type boundary conditions for the NNLS equation (1.1a), differ from those used for the local NLS equation.

The present paper aims at (i) the development of the Riemann-Hilbert approach to the initial value problem (1.1) with the boundary conditions (1.4) and (ii) the long-time asymptotic analysis of solutions to this problem using the nonlinear steepest-decent method [19]. The nonlinear steepest-decent method was inspired by earlier works by Manakov [32] and Its [26] (see [16] for a comprehensive historical review) and has been put into a rigorous shape by Deift and Zhou in [19], with further extensions in [17,18]. The nonlinear steepest-decent method is known to be extremely efficient for the asymptotic analysis of a wide variety of initial and initial boundary value problems for integrable systems, particularly, it has been successfully applied to many initial value problems with step-like initial data, see, e.g., [11–14,20,29,35].

The paper is organized as follows. In Section 2 we present the formalism of the IST method in the form of a multiplicative RH problem suitable for the asymptotic (as $t \to \infty$) analysis. Here we emphasize specific features of the implementation of the Riemann-Hilbert problem formalism in our case, one of them being a singularity, of particular (different for different cases of initial data) type, at the jump contour of the RH problem. The long-time asymptotic analysis of the main RH problem (and, consequently, of the solution of the Cauchy problem for the NNLS equation) is then presented in Section 3, where the main result of the paper (Theorem 1) is formulated. Two main peculiar aspects of our asymptotic results are (i) the dependence of the power-type decay parts of the asymptotics on the direction $x/t = const$ (recall that in the case of the local NLS equation (as well as for other integrable equations like the (local) Korteweg-de Vries equation, the modified Korteweg-de Vries equation, etc.), the corresponding power decay is $t^{-1/2}$ independently of the direction); (ii) the absence of a sector in the (x,t) plane, with straight boundaries $x/t = c_1$ and $x/t = c_2$, where the main term of the asymptotics is described in terms of modulated elliptic functions (which, again, is typical for local integrable nonlinear equations, with step-like initial data, including the local NLS equation [12,13]).

2. Inverse scattering transform and the Riemann-Hilbert problem

2.1. Eigenfunctions

Recall that the focusing NNLS equation (1.1a) is a compatibility condition of the following two linear equations (Lax pair) [3,4]

$$\begin{cases} \Phi_x + ik\sigma_3\Phi = U(x,t)\Phi \\ \Phi_t + 2ik^2\sigma_3\Phi = V(x,t,k)\Phi \end{cases} \tag{2.1}$$

where $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\Phi(x,t,k)$ is a 2×2 matrix-valued function, $k \in \mathbb{C}$ is an auxiliary (spectral) parameter, and the matrix coefficients $U(x,t)$ and $V(x,t,k)$ are given in terms of $q(x,t)$:

$$U(x,t) = \begin{pmatrix} 0 & q(x,t) \\ -\bar{q}(-x,t) & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \tag{2.2}$$

where $V_{11} = -V_{22} = iq(x,t)\bar{q}(-x,t)$, $V_{12} = 2kq(x,t) + iq_x(x,t)$, and $V_{21} = -2k\bar{q}(-x,t) + i(\bar{q}(-x,t))_x$.

Introduce the notations

$$U_+ = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \; U_- = \begin{pmatrix} 0 & 0 \\ -A & 0 \end{pmatrix}, \; V_+ = \begin{pmatrix} 0 & 2kA \\ 0 & 0 \end{pmatrix}, \; V_- = \begin{pmatrix} 0 & 0 \\ -2kA & 0 \end{pmatrix}. \tag{2.3}$$

Then, assuming that there exists $q(x,t)$ satisfying (1.1) and (1.4), it follows that

$$U(x,t) \to U_\pm \text{ and } V(x,t,k) \to V_\pm(k) \quad \text{as } x \to \pm\infty. \tag{2.4}$$

It is easy to see that the systems

$$\begin{cases} \Phi_x + ik\sigma_3\Phi = U_+\Phi \\ \Phi_t + 2ik^2\sigma_3\Phi = V_+(k)\Phi \end{cases}$$

and

$$\begin{cases}\Phi_x + ik\sigma_3\Phi = U_-\Phi \\ \Phi_t + 2ik^2\sigma_3\Phi = V_-(k)\Phi\end{cases}$$

are compatible (cf. (2.1)). Particularly, they are satisfied by $\Phi_\pm(x,t,k)$ defined as follows:

$$\Phi_\pm(x,t,k) = N_\pm(k)e^{-(ikx+2ik^2t)\sigma_3}, \tag{2.5}$$

where $N_+(k) = \begin{pmatrix}1 & \frac{A}{2ik}\\ 0 & 1\end{pmatrix}$ and $N_-(k) = \begin{pmatrix}1 & 0\\ \frac{A}{2ik} & 1\end{pmatrix}$. Notice that $\Phi_\pm$ are chosen in such a way that $\det\Phi_\pm \equiv 1$, which is convenient for the analysis that follows, particularly, when considering the uniqueness issue in the Riemann-Hilbert problem. On the other hand, the singularities of $N_\pm(k)$ at $k=0$ will significantly affect this analysis. Namely, the solution of the basic RH problem has a singularity as $k\to 0$, i.e. at a point on the contour of the RH problem (see (2.48) and (2.49) below).

Now define the 2×2-valued functions $\Psi_j(x,t,k)$, $j=1,2$, $-\infty<x<\infty$, $0\le t<\infty$ as the solutions of the Volterra integral equations:

$$\Psi_1(x,t,k) = N_-(k) + \int_{-\infty}^{x} G_-(x,y,t,k)\,(U(y,t)-U_-)\,\Psi_1(y,t,k)e^{ik(x-y)\sigma_3}\,dy, \tag{2.6a}$$

$$\Psi_2(x,t,k) = N_+(k) + \int_{\infty}^{x} G_+(x,y,t,k)\,(U(y,t)-U_+)\,\Psi_2(y,t,k)e^{ik(x-y)\sigma_3}\,dy, \tag{2.6b}$$

where $G_\pm(x,y,t,k) = \Phi_\pm(x,t,k)[\Phi_\pm(y,t,k)]^{-1}$. The functions $\Psi_j(x,t,k)$, $j=1,2$ are the main ingredients of the basic RH problem (see (2.29) below). The main properties of the matrices $\Psi_j(x,t,k)$ (following from the integral equations (2.6)) are summarized in Proposition 1, where we denote by $\Psi_j^{(i)}(x,t,k)$ the i-th column of $\Psi_j(x,t,k)$, $\mathbb{C}^\pm = \{k\in\mathbb{C} \mid \pm\operatorname{Im}k>0\}$, and $\overline{\mathbb{C}^\pm} = \{k\in\mathbb{C}\mid \pm\operatorname{Im}k\ge 0\}$.

Proposition 1. *The matrices $\Psi_1(x,t,k)$ and $\Psi_2(x,t,k)$ have the following properties:*

(i) *The columns $\Psi_1^{(1)}(x,t,k)$ and $\Psi_2^{(2)}(x,t,k)$ are well-defined and analytic in $k\in\mathbb{C}^+$ and continuous in $\overline{\mathbb{C}^+}\setminus\{0\}$; moreover,*

$$\Psi_1^{(1)}(x,t,k) = \begin{pmatrix}1\\0\end{pmatrix} + O(k^{-1}) \text{ and } \Psi_2^{(2)}(x,t,k) = \begin{pmatrix}0\\1\end{pmatrix} + O(k^{-1}) \quad \text{as } k\to\infty, \quad k\in\mathbb{C}^+.$$

(ii) *The columns $\Psi_1^{(2)}(x,t,k)$ and $\Psi_2^{(1)}(x,t,k)$ are well-defined and analytic in $k\in\mathbb{C}^-$ and continuous in $\overline{\mathbb{C}^-}$; moreover,*

$$\Psi_1^{(2)}(x,t,k) = \begin{pmatrix}0\\1\end{pmatrix} + O(k^{-1}) \text{ and } \Psi_2^{(1)}(x,t,k) = \begin{pmatrix}1\\0\end{pmatrix} + O(k^{-1}) \quad \text{as } k\to\infty, \quad k\in\mathbb{C}^-.$$

(iii) *The functions* $\Phi_j(x,t,k)$, $j=1,2$ *defined by*

$$\Phi_j(x,t,k)=\Psi_j(x,t,k)e^{-(ikx+2ik^2t)\sigma_3},\qquad k\in\mathbb{R}\setminus\{0\},\qquad j=1,2,\tag{2.7}$$

are the (Jost) solutions of the Lax pair equations (2.1) satisfying the boundary conditions

$$\Phi_1(x,t,k)\to\Phi_-(x,t,k),\qquad x\to-\infty,\tag{2.8a}$$

$$\Phi_2(x,t,k)\to\Phi_+(x,t,k),\qquad x\to\infty.\tag{2.8b}$$

(iv) $\det\Psi_j(x,t,k)\equiv 1$, $x\in\mathbb{R}$, $t\geq 0$, $k\in\mathbb{R}$, $\qquad j=1,2.$

(v) *The following symmetry relation holds:*

$$\Lambda\overline{\Psi_1(-x,t,-k)}\Lambda^{-1}=\Psi_2(x,t,k),\ k\in\mathbb{R}\setminus\{0\},\tag{2.9}$$

where $\Lambda=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$.

(vi) *As* $k\to 0$,

$$\Psi_1^{(1)}(x,t,k)=\frac{1}{k}\begin{pmatrix}v_1(x,t)\\v_2(x,t)\end{pmatrix}+O(1),\qquad \Psi_1^{(2)}(x,t,k)=\frac{2i}{A}\begin{pmatrix}v_1(x,t)\\v_2(x,t)\end{pmatrix}+O(k),\tag{2.10a}$$

$$\Psi_2^{(1)}(x,t,k)=-\frac{2i}{A}\begin{pmatrix}\overline{v_2}(-x,t)\\\overline{v_1}(-x,t)\end{pmatrix}+O(k),\qquad \Psi_2^{(2)}(x,t,k)=-\frac{1}{k}\begin{pmatrix}\overline{v_2}(-x,t)\\\overline{v_1}(-x,t)\end{pmatrix}+O(1),\tag{2.10b}$$

where $v_j(x,t)$, *j=1,2 solve the following system of Volterra integral equations:*

$$\begin{cases}v_1(x,t)=\int_{-\infty}^{x}q(y,t)v_2(y,t)\,dy,\\ v_2(x,t)=-i\frac{A}{2}-\int_{-\infty}^{x}\overline{q(-y,t)}v_1(y,t)\,dy.\end{cases}\tag{2.11}$$

Proof. Properties (i)-(iii) follow directly from the representation of Ψ_j in terms of the Neumann series associated with equations (2.6). The Neumann series converge provided $\int_{-\infty}^{0}|q(x,t)|dx<\infty$ and $\int_0^{\infty}|q(x,t)-A|dx<\infty$ for all $t\geq 0$ (cf. (1.4)). Item (iv) follows from the fact that U and V in (2.1) are traceless. Item (v) follows from the corresponding symmetry $\Lambda\overline{U(-x,t)}\Lambda^{-1}=U(x,t)$.

Now let us discuss Item (vi). From (2.6) and the structure of singularity of $N_\pm$ at $k=0$ it follows that, as $k\to 0$,

$$\Psi_1^{(1)}(x,t,k)=\frac{1}{k}\begin{pmatrix}v_1(x,t)\\v_2(x,t)\end{pmatrix}+O(1),\qquad \Psi_1^{(2)}(x,t,k)=\begin{pmatrix}\tilde{v}_1(x,t)\\\tilde{v}_2(x,t)\end{pmatrix}+O(k),\tag{2.12a}$$

$$\Psi_2^{(1)}(x,t,k)=\begin{pmatrix}\tilde{w}_1(x,t)\\\tilde{w}_2(x,t)\end{pmatrix}+O(k),\qquad \Psi_2^{(2)}(x,t,k)=\frac{1}{k}\begin{pmatrix}w_1(x,t)\\w_2(x,t)\end{pmatrix}+O(1)\tag{2.12b}$$

with some v_j, $\tilde{v}_j$, w_j and $\tilde{w}_j$ $(j=1,2)$. Then, the symmetry relation (2.9) implies that

$$\begin{pmatrix} w_1(x,t) \\ w_2(x,t) \end{pmatrix} = \begin{pmatrix} -\overline{v_2}(-x,t) \\ -\overline{v_1}(-x,t) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tilde{w}_1(x,t) \\ \tilde{w}_2(x,t) \end{pmatrix} = \begin{pmatrix} \overline{\tilde{v}_2}(-x,t) \\ \overline{\tilde{v}_1}(-x,t) \end{pmatrix}. \tag{2.13}$$

Further, substituting (2.12) into (2.6) we conclude that $v_j(x,t)$, $j=1,2$ satisfy (2.11) whereas $\tilde{v}_j(x,t)$, $j=1,2$ solve the following system of equations

$$\begin{cases} \tilde{v}_1(x,t) = \int_{-\infty}^{x} q(y,t)\tilde{v}_2(y,t)\,dy, \\ \tilde{v}_2(x,t) = 1 - \int_{-\infty}^{x} \overline{q(-y,t)}\tilde{v}_1(y,t)\,dy. \end{cases} \tag{2.14}$$

Comparing (2.14) with (2.11), it follows that

$$\begin{pmatrix} \tilde{v}_1(x,t) \\ \tilde{v}_2(x,t) \end{pmatrix} = \frac{2i}{A} \begin{pmatrix} v_1(x,t) \\ v_2(x,t) \end{pmatrix} \tag{2.15}$$

and thus (2.10) can be characterized in terms of two functions only, $v_1(x,t)$ and $v_2(x,t)$. $\square$

2.2. Scattering data

Since $\Phi_1(x,t,k)$ and $\Phi_2(x,t,k)$ are both well-defined for $k \in \mathbb{R} \setminus \{0\}$ and satisfy the both equations in the Lax pair (2.1), it follows that

$$\Phi_1(x,t,k) = \Phi_2(x,t,k)S(k), \;\; k \in \mathbb{R} \setminus \{0\}, \tag{2.16}$$

or, in terms of Ψ_j,

$$\Psi_1(x,t,k) = \Psi_2(x,t,k)e^{-(ikx+2ik^2t)\sigma_3}S(k)e^{(ikx+2ik^2t)\sigma_3}, \qquad k \in \mathbb{R} \setminus \{0\}, \tag{2.17}$$

where $S(k)$ is called the scattering matrix.

The symmetry relation (2.9) implies that the same relation holds for the Jost solutions $\Phi_1(x,t,k)$ and $\Phi_2(x,t,k)$:

$$\Lambda \overline{\Phi_1(-x,t,-\bar{k})}\Lambda^{-1} = \Phi_2(x,t,k), \qquad k \in \mathbb{R} \setminus \{0\}. \tag{2.18}$$

In turn, this implies that the scattering matrix $S(k)$ can be written as follows (cf. [6,33])

$$S(k) = \begin{pmatrix} a_1(k) & -\overline{b(-k)} \\ b(k) & a_2(k) \end{pmatrix}, \qquad k \in \mathbb{R} \setminus \{0\}, \tag{2.19}$$

with some $b(k)$, $a_1(k)$, and $a_2(k)$; moreover, $a_1(k)$ and $a_2(k)$ are well defined in $\overline{\mathbb{C}^+} \setminus \{0\}$ and $\overline{\mathbb{C}^-}$ respectively, where they satisfy the symmetry relations

$$\overline{a_1(-\bar{k})} = a_1(k), \qquad \overline{a_2(-\bar{k})} = a_2(k). \tag{2.20}$$

The scattering matrix $S(k)$ is uniquely determined by the initial data $q_0(x)$. Indeed, introducing the notations $\psi_1(x,k) = (\Psi_1)_{11}(x,0,k)$, $\psi_2(x,k) = (\Psi_1)_{12}(x,0,k)$, $\psi_3(x,k) =$

$(\Psi_1)_{21}(x,0,k)$ and $\psi_4(x,k)=(\Psi_1)_{22}(x,0,k)$, equations (2.6a) reduce to the systems of Volterra integral equations for ψ_1 and ψ_3:

$$\begin{cases} \psi_1(x,k)=1+\int_{-\infty}^{x} q_0(y)\psi_3(y,k)\,dy, \\ \psi_3(x,k)= \frac{A}{2ik}+\int_{-\infty}^{x} e^{2ik(x-y)}\left(A-\overline{q_0(-y)}\right)\psi_1(y,k)\,dy \\ \qquad\qquad + \frac{A}{2ik}\int_{-\infty}^{x} q_0(y)\left(1-e^{2ik(x-y)}\right)\psi_3(y,k)\,dy \end{cases} \tag{2.21}$$

and for ψ_2 and ψ_4:

$$\begin{cases} \psi_2(x,k)=\int_{-\infty}^{x} e^{-2ik(x-y)}q_0(y)\psi_4(y,k)\,dy, \\ \psi_4(x,k)= 1+\int_{-\infty}^{x}\left(A-\overline{q_0(-y)}\right)\psi_2(y,k)\,dy \\ \qquad\qquad + \frac{A}{2ik}\int_{-\infty}^{x} q_0(y)\left(e^{-2ik(x-y)}-1\right)\psi_4(y,k)\,dy. \end{cases} \tag{2.22}$$

Then the entries a_1, a_2 and b of the scattering matrix can be determined as follows:

$$a_1(k)=\lim_{x\to\infty}\left(\psi_1(x,k)-\frac{A}{2ik}\psi_3(x,k)\right),\qquad b(k)=\lim_{x\to\infty}e^{-2ikx}\psi_3(x,k), \tag{2.23}$$

and

$$a_2(k)=\lim_{x\to\infty}\psi_4(x,k). \tag{2.24}$$

Alternatively, they can be written it terms of the determinant relations:

$$a_1(k)=\det\left(\Psi_1^{(1)}(0,0,k),\,\Psi_2^{(2)}(0,0,k)\right),\qquad k\in\overline{\mathbb{C}^+}\setminus\{0\}, \tag{2.25a}$$

$$a_2(k)=\det\left(\Psi_2^{(1)}(0,0,k),\,\Psi_1^{(2)}(0,0,k)\right),\qquad k\in\overline{\mathbb{C}^-}, \tag{2.25b}$$

$$b(k)=\det\left(\Psi_2^{(1)}(0,0,k),\,\Psi_1^{(1)}(0,0,k)\right),\qquad k\in\mathbb{R}. \tag{2.25c}$$

The properties of the spectral functions, which follow from Proposition 1, are summarized in

Proposition 2. *The spectral functions $a_j(k)$, j=1,2, and $b(k)$ have the following properties*

1. *$a_1(k)$ is analytic in $k\in\mathbb{C}^+$ and continuous in $\overline{\mathbb{C}^+}\setminus\{0\}$; $a_2(k)$ is analytic in $k\in\mathbb{C}^-$ and continuous in $\overline{\mathbb{C}^-}$.*
2. *$a_j(k)=1+O\left(\frac{1}{k}\right)$, $j=1,2$ as $k\to\infty$, $k\in\overline{\mathbb{C}^{(-1)^{j+1}}}$ and $b(k)=O\left(\frac{1}{k}\right)$ as $k\to\infty$, $k\in\mathbb{R}$.*
3. *$\overline{a_1(-\bar{k})}=a_1(k)$, $k\in\overline{\mathbb{C}^+}\setminus\{0\}$; $\overline{a_2(-\bar{k})}=a_2(k)$, $k\in\overline{\mathbb{C}^-}$.*
4. *$a_1(k)a_2(k)+b(k)\overline{b(-\bar{k})}=1$, $k\in\mathbb{R}\setminus\{0\}$ (follows from* $\det S(k)=1$*).*
5. *$a_1(k)=\frac{A^2a_2(0)}{4k^2}+O(\frac{1}{k})$ as $k\to 0$, $k\in\overline{\mathbb{C}^+}$ and $b(k)=\frac{Aa_2(0)}{2ik}+O(1)$ as $k\to 0$, $k\in\mathbb{R}$.*

Remark 1. Concerning Item 5 of Proposition 2, we notice that substituting (2.10) into (2.25) yields, as $k\to 0$,

$$a_1(k) = \frac{1}{k^2}(|v_2(0,0)|^2 - |v_1(0,0)|^2) + O\left(\frac{1}{k}\right), \tag{2.26a}$$

$$a_2(k) = \frac{4}{A^2}(|v_2(0,0)|^2 - |v_1(0,0)|^2) + O(k), \tag{2.26b}$$

$$b(k) = -\frac{2i}{kA}(|v_2(0,0)|^2 - |v_1(0,0)|^2) + O(1), \tag{2.26c}$$

from which Item 5 follows. Notice that in (2.25) one can use any (x,t) instead of $(0,0)$ as arguments in the right-hand sides, which implies that $|v_2(0,0)|^2 - |v_1(0,0)|^2$ in the r.h.s. of (2.26) can be replaced by $v_2(x,t)\bar{v}_2(-x,t) - v_1(x,t)\bar{v}_1(-x,t)$, the latter being a conserved quantity (independent of x and t).

Remark 2. In the case of the pure-step initial data, i.e., when

$$q_0(x) = q_{0A}(x) := \begin{cases} 0, & x < 0, \\ A, & x > 0, \end{cases} \tag{2.27}$$

the scattering matrix $S(k)$ is as follows:

$$S(k) = [\Phi_2(0,0,k)]^{-1}\Phi_1(0,0,k) = N_+^{-1}(k)N_-(k) = \begin{pmatrix} 1+\frac{A^2}{4k^2} & -\frac{A}{2ik} \\ \frac{A}{2ik} & 1 \end{pmatrix}. \tag{2.28}$$

Particularly, in this case $a_1(k)$ has a single, simple zero (at $k = i\frac{A}{2}$) in the upper half-plane whereas $a_2(k)$ has no zeros in the lower half-plane.

2.3. The basic Riemann-Hilbert problem

The Riemann–Hilbert formalism of the IST method is based on constructing (using the Jost solutions) a piece-wise meromorphic, 2×2-valued function in the k-complex plane, whose "lack of analyticity", i.e., the jump across a contour and, if appropriate, some conditions at the singularity points, can be fully characterized in terms of the spectral data (spectral functions and a discrete set of data related to the poles) uniquely determined by the initial data.

Define the 2×2-valued function $M(x,t,k)$, piece-wise meromorphic relative to $\mathbb{R}$, as follows:

$$M(x,t,k) = \begin{cases} \left(\frac{\Psi_1^{(1)}(x,t,k)}{a_1(k)}, \Psi_2^{(2)}(x,t,k)\right), & k \in \mathbb{C}^+, \\ \left(\Psi_2^{(1)}(x,t,k), \frac{\Psi_1^{(2)}(x,t,k)}{a_2(k)}\right), & k \in \mathbb{C}^-. \end{cases} \tag{2.29}$$

Then the scattering relation (2.17) implies that the boundary values $M_\pm(x,t,k) = \lim_{k'\to k, k'\in\mathbb{C}^\pm} M(x,t,k')$, $k\in\mathbb{R}$ satisfy the multiplicative jump condition

$$M_+(x,t,k) = M_-(x,t,k)J(x,t,k), \qquad k \in \mathbb{R}\setminus\{0\}, \tag{2.30}$$

where

$$J(x,t,k) = \begin{pmatrix} 1 + r_1(k)r_2(k) & r_2(k)e^{-2ikx-4ik^2t} \\ r_1(k)e^{2ikx+4ik^2t} & 1 \end{pmatrix} \tag{2.31}$$

with the reflection coefficients defined by

$$r_1(k) := \frac{b(k)}{a_1(k)}, \quad r_2(k) := \frac{\overline{b(-k)}}{a_2(k)}. \tag{2.32}$$

Moreover, M satisfies the normalization condition

$$M(x,t,k) \to I, \qquad k \to \infty, \tag{2.33}$$

where I is the 2×2 identity matrix.

Observe that the symmetry conditions 3 in Proposition 2 imply that

$$r_1(-k)r_2(-k) = \overline{r_1(k)}\,\overline{r_2(k)}, \quad k \in \mathbb{R} \setminus \{0\}. \tag{2.34}$$

By the determinant property 4, we also have

$$1 + r_1(k)r_2(k) = \frac{1}{a_1(k)a_2(k)}, \qquad k \in \mathbb{R} \setminus \{0\}. \tag{2.35}$$

Now notice that in view of (2.26), the behavior of M as $k \to 0$ is qualitatively different in the cases $a_2(0) \neq 0$ and $a_2(0) = 0$. The former case contains the case of "pure-step initial data", see Remark 2, where $a_1(k)$ has (in $\mathbb{C}^+$) a single, simple zero located on the imaginary axis, and $a_2(k)$ has no zeros in $\mathbb{C}^-$. Since small (in the L^1 norm) perturbations of the pure-step initial data preserve these properties, we will concentrate, in the present paper, on the following two cases:

Case I: The spectral function $a_1(k)$ has one (pure imaginary) simple zero in $\overline{\mathbb{C}^+}$, say $k = ik_1$, $k_1 > 0$, and $a_2(k)$ has no zeros in $\overline{\mathbb{C}^-}$.

Case II: The spectral function $a_1(k)$ has one simple zero in $\overline{\mathbb{C}^+}$, say $k = ik_1$, $k_1 > 0$, and $a_2(k)$ has one simple zero in $\overline{\mathbb{C}^-}$ at $k = 0$. Thus we assume that $\dot{a}_2(0) \neq 0$ and, additionally, we suppose that $a_{11} := \lim\limits_{k \to 0} k a_1(k) \neq 0$.

Remark 3. Case I corresponds to the inequality $|v_2(0,0)|^2 - |v_1(0,0)|^2 \neq 0$ whereas in Case II the equality $|v_2(0,0)|^2 - |v_1(0,0)|^2 = 0$ holds, see (2.11) and (2.26). With this respect, Case I corresponds to "generic" initial conditions whereas Case II corresponds to "non-generic" ones.

Remark 4. From the symmetry relations (2.20) it follows that a_{11} is purely imaginary. Moreover, if $a_1(k)$ has one simple zero, then $\operatorname{Im} a_{11} < 0$ in Case II.

It is interesting that in contrast with the case of the local NLS, the value of k_1 can't be prescribed independently of $b(k)$.

Proposition 3. *Given $b(k)$ for $k \in \mathbb{R} \setminus \{0\}$, the zero $k = ik_1$ of $a_1(k)$ is determined as follows:*

(i) *In Case I,*

$$k_1 = \frac{A}{2}\exp\left\{-\frac{1}{2\pi i}\text{v.p.}\int_{-\infty}^{\infty}\frac{\ln\frac{\zeta^2}{\zeta^2+1}(1-b(\zeta)\bar{b}(-\zeta))}{\zeta}\,d\zeta\right\}, \tag{2.36}$$

(ii) *In Case II,*

$$k_1 = A\frac{\sqrt{(\operatorname{Re} b(0))^2+E_2^2}-\operatorname{Re} b(0)}{2E_1E_2}, \tag{2.37}$$

where

$$E_1 = \exp\left\{\frac{1}{2\pi i}\text{v.p.}\int_{-\infty}^{\infty}\frac{\ln(1-b(\zeta)\bar{b}(-\zeta))}{\zeta}\,d\zeta\right\} \quad \textit{and} \quad E_2 = \exp\left\{\frac{1}{2}\ln(1-|b(0)|^2)\right\} \tag{2.38}$$

(notice that $1-|b(0)|^2 = a_{11}\dot{a}_2(0) \neq 0$ *by assumption).*

Proof. (i) *Case I*. Define functions $\tilde{a}_1(k)$ and $\tilde{a}_2(k)$ by

$$\tilde{a}_1(k) = a_1(k)\frac{k^2}{(k-ik_1)(k+i)}, \qquad \tilde{a}_2(k) = a_2(k)\frac{k-ik_1}{k-i}.$$

Then the determinant relation (see Item 4 in Proposition 2) can be viewed as the following scalar RH problem w.r.t. $\tilde{a}_j(k)$, $j=1,2$: given $b(k)$, $k\in\mathbb{R}$, find $\tilde{a}_1(k)$ and $\tilde{a}_2(k)$ analytic and having no zeros in $\overline{\mathbb{C}^+}$ and $\overline{\mathbb{C}^-}$ respectively, satisfying the jump condition

$$\tilde{a}_1(k)\tilde{a}_2(k) = \frac{k^2}{k^2+1}(1-b(k)\bar{b}(-k)), \quad k\in\mathbb{R} \tag{2.39}$$

and the normalization conditions $\tilde{a}_j(k)\to 1$ as $k\to\infty$. The unique solution of this RH problem is given by

$$\tilde{a}_1(k) = e^{\chi(k)}, \qquad \tilde{a}_2(k) = e^{-\chi(k)},$$

where

$$\chi(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln\frac{\zeta^2}{\zeta^2+1}(1-b(\zeta)\bar{b}(-\zeta))}{\zeta-k}\,d\zeta.$$

Then $a_1(k)$ and $a_2(k)$ can be written as

$$a_1(k) = \frac{(k-ik_1)(k+i)}{k^2}e^{\chi(k)} \tag{2.40a}$$

and

$$a_2(k)=\frac{k-i}{k-ik_1}e^{-\chi(k)}, \tag{2.40b}$$

which, being evaluated at $k=0$, gives

$$a_1(k)=\frac{k_1e^{\chi(+i0)}}{k^2}(1+o(k)) \text{ and } a_2(0)=\frac{e^{-\chi(-i0)}}{k_1}. \tag{2.41}$$

On the other hand (see (2.26)),

$$a_1(k)=\frac{A^2a_2(0)}{4k^2}(1+o(k)), \qquad k\to 0. \tag{2.42}$$

Comparing (2.41) and (2.42) and taking into account that (by the Sokhotski-Plemelj formulas)

$$\chi(+i0)+\chi(-i0)=\frac{1}{\pi i}\text{v.p.}\int_{-\infty}^{\infty}\frac{\ln\frac{\zeta^2}{\zeta^2+1}(1-b(\zeta)\bar{b}(-\zeta))}{\zeta}\,d\zeta,$$

we arrive at (2.36).

(ii) *Case II.* Observe that due to the symmetry relation (2.9) and Item (vi) in Proposition 1, the behavior of $\Psi_j(x,t,k)$, $j=1,2$ as $k\to 0$ can be characterized as follows:

$$\Psi_1^{(1)}(x,t,k)=\frac{1}{k}\begin{pmatrix}v_1(x,t)\\ v_2(x,t)\end{pmatrix}+\begin{pmatrix}s_1(x,t)\\ s_2(x,t)\end{pmatrix}+O(k), \tag{2.43a}$$

$$\Psi_1^{(2)}(x,t,k)=\frac{2i}{A}\begin{pmatrix}v_1(x,t)\\ v_2(x,t)\end{pmatrix}+k\begin{pmatrix}h_1(x,t)\\ h_2(x,t)\end{pmatrix}+O(k^2), \tag{2.43b}$$

$$\Psi_2^{(1)}(x,t,k)=-\frac{2i}{A}\begin{pmatrix}\overline{v_2}(-x,t)\\ \overline{v_1}(-x,t)\end{pmatrix}-k\begin{pmatrix}\overline{h_2}(-x,t)\\ \overline{h_1}(-x,t)\end{pmatrix}+O(k^2), \tag{2.43c}$$

$$\Psi_2^{(2)}(x,t,k)=-\frac{1}{k}\begin{pmatrix}\overline{v_2}(-x,t)\\ \overline{v_1}(-x,t)\end{pmatrix}+\begin{pmatrix}\overline{s_2}(-x,t)\\ \overline{s_1}(-x,t)\end{pmatrix}+O(k), \tag{2.43d}$$

with some v_j, s_j, and h_j $(j=1,2)$. Then, using the definitions (2.25) of the spectral functions and taking into account that $|v_2(0,0)|^2-|v_1(0,0)|^2=0$ in Case II, we have as $k\to 0$:

$$a_1(k)=\frac{1}{k}(v_1\bar{s}_1-\bar{v}_1s_1-v_2\bar{s}_2+\bar{v}_2s_2)\Big|_{x,t=0}+O(1), \tag{2.44a}$$

$$a_2(k)=k\frac{2i}{A}(v_1\bar{h}_1+\bar{v}_1h_1-v_2\bar{h}_2-\bar{v}_2h_2)\Big|_{x,t=0}+O(k^2), \tag{2.44b}$$

$$b(k)=v_1\bar{h}_1-v_2\bar{h}_2+\frac{2i}{A}(\bar{v}_1s_1-\bar{v}_2s_2)\Big|_{x,t=0}+O(k). \tag{2.44c}$$

Equations (2.44) imply that

$$a_{11}=iA\,\mathrm{Re}\,b(0)-\frac{A^2}{4}\dot{a}_2(0), \tag{2.45}$$

where $a_{11}=\lim_{k\to 0}(ka_1(k))$.

On the other hand, introducing

$$\hat{a}_1(k)=a_1(k)\frac{k}{k-ik_1} \quad \text{and} \quad \hat{a}_2(k)=a_2(k)\frac{k-ik_1}{k},$$

the determinant relation can be viewed as the scalar RH problem with the jump condition

$$\hat{a}_1(k)\hat{a}_2(k)=1-b(k)\bar{b}(-k),$$

whose solution gives

$$a_1(k)=\frac{k-ik_1}{k}\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln(1-b(\zeta)\bar{b}(-\zeta))}{\zeta-k}\,d\zeta\right\}, \tag{2.46a}$$

and

$$a_2(k)=\frac{k}{k-ik_1}\exp\left\{-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln(1-b(\zeta)\bar{b}(-\zeta))}{\zeta-k}\,d\zeta\right\}. \tag{2.46b}$$

From (2.46), using the Sokhotski-Plemelj formulas, we obtain

$$a_{11}=-ik_1E_1E_2 \quad \text{and} \quad \dot{a}_2(0)=\frac{i}{k_1}E_1^{-1}E_2, \tag{2.47}$$

where E_1 and E_2 are given by (2.38), which, being compared with (2.45), uniquely determines $k_1>0$ as the solution of a quadratic equation. $\square$

Taking into account the singularities of $\Psi_j(x,t,k)$, $j=1,2$ and $a_1(k)$ at $k=0$ (see Proposition 1), the behavior of $M(x,t,k)$ at $k=0$ can be described as follows: in Case I,

$$M_+(x,t,k)=\begin{pmatrix}\frac{4}{A^2a_2(0)}v_1(x,t) & -\overline{v_2}(-x,t)\\ \frac{4}{A^2a_2(0)}v_2(x,t) & -\overline{v_1}(-x,t)\end{pmatrix}(I+O(k))\begin{pmatrix}k & 0\\ 0 & \frac{1}{k}\end{pmatrix}, \qquad k\to +i0, \tag{2.48a}$$

$$M_-(x,t,k)=\frac{2i}{A}\begin{pmatrix}-\overline{v_2}(-x,t) & \frac{v_1(x,t)}{a_2(0)}\\ -\overline{v_1}(-x,t) & \frac{v_2(x,t)}{a_2(0)}\end{pmatrix}+O(k), \qquad k\to -i0, \tag{2.48b}$$

and Case II,

$$M_+(x,t,k) = \begin{pmatrix} \frac{v_1(x,t)}{a_{11}} & -\overline{v_2}(-x,t) \\ \frac{v_2(x,t)}{a_{11}} & -\overline{v_1}(-x,t) \end{pmatrix} (I + O(k)) \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{pmatrix}, \qquad k \to +i0, \tag{2.49a}$$

$$M_-(x,t,k) = \frac{2i}{A} \begin{pmatrix} -\overline{v_2}(-x,t) & \frac{v_1(x,t)}{\dot{a}_2(0)} \\ -\overline{v_1}(-x,t) & \frac{v_2(x,t)}{\dot{a}_2(0)} \end{pmatrix} (I + O(k)) \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{pmatrix}, \qquad k \to -i0 \tag{2.49b}$$

(recall that a_{11} is determined by $a_1(k) = \frac{a_{11}}{k} + O(1)$ as $k \to 0$).

Additionally, if $a_1(ik_1) = 0$ with $k_1 > 0$ (recall that in this case we assume that this zero is simple), then $M(x,t,k)$ satisfies the residue condition

$$\operatorname*{Res}_{k=ik_1} M^{(1)}(x,t,k) = \frac{\gamma_1}{\dot{a}_1(ik_1)} e^{-2k_1 x - 4ik_1^2 t} M^{(2)}(x,t,ik_1), \quad |\gamma_1| = 1, \tag{2.50}$$

where $\Psi_1^{(1)}(0,0,ik_1) = \gamma_1 \Psi_2^{(2)}(0,0,ik_1)$. Notice that the symmetry relation (2.9) implies that $\overline{\Psi}_1^{(1)}(0,0,ik_1) = \gamma_1^{-1} \overline{\Psi}_2^{(2)}(0,0,ik_1)$ and thus $|\gamma_1| = 1$ (cf. [6]).

Notice that if $a_1(k)$ has a zero $k = \zeta_1$ that is not pure imaginary, then, due to the symmetry conditions, it also has a zero at $k = \zeta_2 = -\bar{\zeta}_1$, and the associated residue conditions have the form:

$$\operatorname*{Res}_{k=\zeta_1} M^{(1)}(x,t,k) = \frac{\eta_1}{\dot{a}_1(\zeta_1)} e^{2i\zeta_1 x + 4i\zeta_1^2 t} M^{(2)}(x,t,\zeta_1) \tag{2.51a}$$

and

$$\operatorname*{Res}_{k=\zeta_2} M^{(1)}(x,t,k) = \frac{1}{\bar{\eta}_1 \dot{a}_1(\zeta_2)} e^{2i\zeta_2 x + 4i\zeta_2^2 t} M^{(2)}(x,t,\zeta_2), \tag{2.51b}$$

where η_1 is determined by $\Psi_1^{(1)}(0,0,\zeta_1) = \eta_1 \Psi_2^{(2)}(0,0,\zeta_1)$.

Now we are at a position to formulate the Riemann-Hilbert problem, whose solution gives the solution of the initial value problem (1.1), (1.4). Let $b(k)$, $k \in \mathbb{R}$ and γ_1 with $|\gamma_1| = 1$ be the spectral data associated with the initial data $q_0(x)$ in (1.1). Then the Riemann-Hilbert problem is as follows:

Basic Riemann–Hilbert Problem: Given $b(k)$ and γ_1, find the 2×2-valued function $M(x,t,k)$, piece-wise meromorphic in k relative to $\mathbb{R}$ and satisfying the following conditions:

(i) Jump conditions. The non-tangential limits $M_\pm(x,t,k) = M(x,t,k \pm i0)$ exist a.e. for $k \in \mathbb{R}$ such that $M(x,t,\cdot) - I \in L^2(\mathbb{R} \setminus [-\varepsilon, \varepsilon])$ for any $\varepsilon > 0$ and $M_\pm(x,t,k)$ satisfy the condition

$$M_+(x,t,k) = M_-(x,t,k) J(x,t,k) \qquad \text{for a.e. } k \in \mathbb{R} \setminus \{0\}, \tag{2.52}$$

where the jump matrix $J(x,t,k)$ is given by (2.31), with r_1 and r_2 given in terms of b by (2.32) with (2.40) (Case I) or (2.46) (Case II).

(ii) Normalization at $k = \infty$:

$$M(x,t,k) = I + O(k^{-1}) \qquad \text{uniformly as } k \to \infty.$$

(iii) Residue condition (2.50) with k_1 given in terms of b using (2.36) (Case I) or (2.37) (Case II).

(iv) Singularity conditions at $k=0$: $M(x,t,k)$ satisfies (2.48) (Case I) or (2.49) (Case II), where $v_j(x,t)$, $j=1,2$ are some (not prescribed) functions.

Assume that the RH problem (i)–(iv) has a solution $M(x,t,k)$. Then the solution of the initial value problem (1.1), (1.4) is given in terms of the (12) and (21) entries of $M(x,t,k)$ as follows:

$$q(x,t)=2i\lim_{k\to\infty}kM_{12}(x,t,k), \tag{2.53}$$

and

$$q(-x,t)=-2i\lim_{k\to\infty}k\overline{M_{21}(x,t,k)}. \tag{2.54}$$

The solution of the RH problem is unique, if exists. Indeed, if M and $\tilde{M}$ are two solutions, then conditions (2.48) or (2.49) provide the boundedness of $M\tilde{M}^{-1}$ at $k=0$. Then the standard arguments based of the Liouville theorem lead to $M\tilde{M}^{-1}\equiv I$.

Remark 5. From (2.53) and (2.54) it follows that in order to present the solution of (1.1), (1.4) for all $x\in\mathbb{R}$, it is sufficient to have the solution of the RH problem for, say, $x\geq 0$ only.

Remark 6. In the general case with more zeros of $a_1(k)$ in $\mathbb{C}^+$ and/or zeros of $a_2(k)$ in $\mathbb{C}^-$, relevant residue conditions, of type (2.50) and/or (2.51), have to be specified, in terms of a prescribed set of zeros and corresponding norming constants.

Proposition 4. *The solution M of the Riemann–Hilbert problem* **(i)-(iv)** *satisfies the following symmetry condition (cf. (2.18)):*

$$M(x,t,k)=\begin{cases}\Lambda\overline{M(-x,t,-\bar{k})}\Lambda^{-1}\begin{pmatrix}\frac{1}{a_1(k)} & 0\\ 0 & a_1(k)\end{pmatrix}, & k\in\mathbb{C}^+\setminus\{0\},\\ \Lambda\overline{M(-x,t,-\bar{k})}\Lambda^{-1}\begin{pmatrix}a_2(k) & 0\\ 0 & \frac{1}{a_2(k)}\end{pmatrix}, & k\in\mathbb{C}^-\setminus\{0\}.\end{cases} \tag{2.55}$$

Proof. Follows from the symmetry of the jump matrix (2.31) in (2.52)

$$\Lambda\overline{J(-x,t,-k)}\Lambda^{-1}=\begin{pmatrix}a_2(k) & 0\\ 0 & \frac{1}{a_2(k)}\end{pmatrix}J(x,t,k)\begin{pmatrix}a_1(k) & 0\\ 0 & \frac{1}{a_1(k)}\end{pmatrix}, \qquad k\in\mathbb{R}\setminus\{0\}$$

(which, in turns, follows from (2.34) and (2.35)), and the fact that the structural conditions (2.48) and (2.49) and the residue condition (2.50) are consistent with (2.55). □

2.4. One-soliton solution

Proposition 5. *Let $a_1(k)$, $a_2(k)$, and $b(k)$ be the spectral functions (i) associated with some $q_0(x)$ and (ii) satisfying the following conditions:*

- *$b(k)=0$ for all $k\in\mathbb{R}$;*
- *$a_1(k)$ has a single, simple zero $k=ik_1$ with some $k_1>0$ in $\overline{\mathbb{C}^+}$;*
- *$a_2(k)$ has a single, simple zero $k=0$ in $\overline{\mathbb{C}^-}$.*

Also, let γ_1 be given such that $\gamma_1=e^{i\phi_1}$ with $\phi_1\in\mathbb{R}$. Then:

1. *k_1 is uniquely determined as $k_1=\frac{A}{2}$;*
2. *The Riemann–Hilbert problem (i)–(iv) has a unique solution for all (x,t) with $x\in\mathbb{R}$ and $t\geq 0$ except the set $\cup_{n\in\mathbb{Z}}\{(0,t_n)\}$ with $t_n=\frac{\phi_1}{A^2}+\frac{2\pi}{A^2}n$;*
3. *The associated exact solution $q(x,t)$ of problem (1.1), (1.4) is given by*

$$q(x,t)=\frac{A}{1-e^{-Ax-iA^2t+i\phi_1}}. \tag{2.56}$$

Proof. Since $b(0)=0$, we are in Case II, and thus Item 1 follows from Proposition 3, (ii). Moreover, (2.46) gives

$$a_1(k)=\frac{k-i\frac{A}{2}}{k},\qquad a_2(k)=\frac{k}{k-i\frac{A}{2}} \tag{2.57}$$

and thus the constants involved in (2.49) are as follows:

$$a_{11}=\frac{A}{2i},\qquad \dot{a}_2(0)=\frac{2i}{A}.$$

Now notice that since $b(k)\equiv 0$, it follows that $M(\cdot,\cdot,k)$ is a meromorphic (in $\mathbb{C}$) function with the only pole at $k=ik_1$. Then, comparing (2.49a) and (2.49b), it follows that $v_1(x,t)=-\bar{v}_2(-x,t)$ and thus the singularity conditions (2.49) reduce to a conventional residue condition:

$$\operatorname*{Res}_{k=0} M^{(2)}(x,t,k)=\frac{A}{2i}M^{(1)}(x,t,0). \tag{2.58}$$

Further, taking into account the original residue condition (2.50) and the normalization condition (ii), we arrive at the following representation for M:

$$M(x,t,k)=\begin{pmatrix}\frac{k+v_1(x,t)}{k-\frac{iA}{2}} & \frac{v_1(x,t)}{k}\\ \frac{-\bar{v}_1(-x,t)}{k-\frac{iA}{2}} & \frac{k-\bar{v}_1(-x,t)}{k}\end{pmatrix}, \tag{2.59}$$

where $v_1(x,t)$ is determined using (2.50):

$$v_1(x,t)=\frac{A}{2i}\frac{1}{1-e^{-Ax-iA^2t+i\phi_1}}. \tag{2.60}$$

Particularly, this determines the singularity set as the set of zeros of the denominator in (2.60). Finally, using (2.53) or (2.54), the soliton formula (2.56) follows. □

3. The long-time asymptotics

The shock-type long-time asymptotics for the local NLS equation with the step-like boundary conditions (1.6), (1.7) was presented in [13], where it was shown that there were always three sectors in the (x,t) half-plane ($t>0$) characterized by qualitatively different asymptotic behavior: the decaying sector (where the order of decay of q is $O(t^{-1/2})$), the sector of modulated elliptic wave, and the sector of modulated plane wave. Particularly, if $B=0$, then the modulated elliptic wave occupies the sector $0<\frac{x}{t}<8\sqrt{2}A$.

It is natural to compare this behavior with the asymptotics for the nonlocal NLS equation with the same type of the initial data (1.1b), (1.1c). This motivates us to study, in this Section, the long-time asymptotics of the solution of the initial value problem (1.1), (1.4). Our analysis is based on the adaptation of the nonlinear steepest-decent method [19] to the (oscillatory) RH problem (i)–(iv). The implementation of the method in our case has some specific features: particularly, we have to deal with a singularity on the contour, and the jump $1+r_1(k)r_2(k)$ in the scalar RH problem for $\delta(\xi,k)$ (see (3.3) below) is not, in general, real-valued.

We will show that a basic difference of the asymptotics for the nonlocal NLS equation being compared with that for the local NLS is that, while there are still the sector of decay and the sector of "modulated constant", there is no an intermediate sector between these two (although a transition zone between these sectors may exist, being characterized by a specific asymptotics along curves converging to the ray $x=0$, $t>0$).

3.1. Jump factorizations

First, notice that in view of (2.53) and (2.54), studying the RH problem for $x>0$ is sufficient for studying $q(x,t)$ for all (x,t) outside the sector $|x/t|<\varepsilon$ for any $\varepsilon>0$.

Introduce the variable $\xi:=\frac{x}{4t}$ and the phase function

$$\theta(k,\xi)=4k\xi+2k^2. \tag{3.1}$$

The jump matrix (2.31) allows, similarly to [33], two triangular factorizations:

$$J(x,t,k)=\begin{pmatrix}1 & 0\\ \frac{r_1(k)}{1+r_1(k)r_2(k)}e^{2it\theta} & 1\end{pmatrix}\begin{pmatrix}1+r_1(k)r_2(k) & 0\\ 0 & \frac{1}{1+r_1(k)r_2(k)}\end{pmatrix}\begin{pmatrix}1 & \frac{r_2(k)}{1+r_1(k)r_2(k)}e^{-2it\theta}\\ 0 & 1\end{pmatrix} \tag{3.2a}$$

and

$$J(x,t,k)=\begin{pmatrix}1 & r_2(k)e^{-2it\theta}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ r_1(k)e^{2it\theta} & 1\end{pmatrix}. \tag{3.2b}$$

Since the phase function $\theta(k,\xi)$ is the same as in the case of the local NLS, its signature table (see Fig. 1) suggests us to follow the conventional steps [19,16] involving (i) getting rid of the diagonal factor in (3.2a) and (ii) the deformation of the original RH problem (relative to the real axis) to a new one, relative to a cross, where the jump matrix converges, as $t\to\infty$, to the identity matrix uniformly away from any vicinity of the stationary phase point $k=-\xi$. But when following this scheme, we have to pay a special attention to the singularity point $k=0$.

First, introduce $\delta(\xi,k)$ as the solution of the scalar RH problem: find $\delta(\xi,k)$ analytic in $\mathbb{C}\setminus(-\infty,-\xi]$ and satisfying the conditions

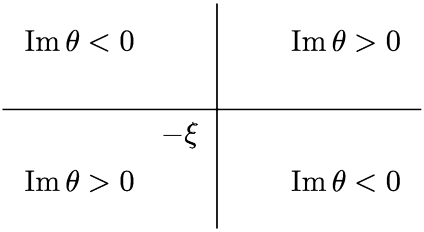

Fig. 1. Signature table.

$$\begin{cases} \delta_+(\xi,k)=\delta_-(\xi,k)(1+r_1(k)r_2(k)), & k\in(-\infty,-\xi),\\ \delta(\xi,k)\to 1, & k\to\infty. \end{cases} \tag{3.3}$$

Its solution is given by the Cauchy-type integral:

$$\delta(\xi,k)=\exp\left\{\frac{1}{2\pi i}\int\limits_{-\infty}^{-\xi}\frac{\ln(1+r_1(\zeta)r_2(\zeta))}{\zeta-k}\,d\zeta\right\} \tag{3.4}$$

(notice that since we deal with $\xi>0$, the behavior of $r_j(k)$ at $k=0$ does not affect $\delta(\xi,k)$). Then define $\tilde{M}$ with the help of δ:

$$\tilde{M}(x,t,k)=M(x,t,k)\delta^{-\sigma_3}(\xi,k). \tag{3.5}$$

Notice that in the case of the pure-step initial data (2.27), $1+r_1(k)r_2(k)=\frac{4k^2}{4k^2+A^2}$ (see Remark 2), and thus $1+r_1(k)r_2(k)$ is real-valued. However, in the general case, $1+r_1(k)r_2(k)$ can take complex values, which may cause $\delta(\xi,k)$ to be singular at $k=-\xi$ (cf. [33]).

Indeed, $\delta(\xi,k)$ can be written as

$$\delta(\xi,k)=(\xi+k)^{i\nu(-\xi)}e^{\chi(\xi,k)}, \tag{3.6}$$

where

$$\chi(\xi,k):=-\frac{1}{2\pi i}\int\limits_{-\infty}^{-\xi}\ln(k-\zeta)d_\zeta\ln(1+r_1(\zeta)r_2(\zeta)) \tag{3.7}$$

and

$$\nu(-\xi):=-\frac{1}{2\pi}\ln(1+r_1(-\xi)r_2(-\xi))=-\frac{1}{2\pi}\ln|1+r_1(-\xi)r_2(-\xi)|-\frac{i}{2\pi}\Delta(-\xi), \tag{3.8}$$

with

$$\Delta(-\xi):=\int\limits_{-\infty}^{-\xi}d\arg(1+r_1(\zeta)r_2(\zeta)).$$

In what follows we will assume that

$$\Delta(k) \in (-\pi, \pi) \qquad \text{for all } \ k \in (-\infty, 0) \tag{3.9}$$

and thus $|\operatorname{Im} \nu(k)| < \frac{1}{2}$. In this case, $\ln(1 + r_1(k)r_2(k))$ is single-valued, and the singularity of $\delta(\xi, k)$ (as well as of $\tilde{M}(x, t, k)$) at $k = -\xi$ is square integrable. More importantly, assumption (3.9) will allow us to establish correct estimates, see (3.22) in Theorem 1, i.e. the estimates with main terms dominating the error ones.

Assumption (3.9) obviously holds in the case of the pure-step initial data (2.27): in this case, $\Delta(k) \equiv 0$ for $k \in (-\infty, 0)$. With this respect, this assumption holds, particularly, if the initial data are small L^1-perturbations of $q_{0A}(x)$; we have already remarked on this aspect when formulating the conditions for Case I and Case II above.

Function $\tilde{M}(x, t, k)$ defined by (3.5) satisfies the RH problem specified by the jump, normalization, and residue conditions:

$$\tilde{M}_+(x, t, k) = \tilde{M}_-(x, t, k)\tilde{J}(x, t, k), \qquad k \in \mathbb{R} \setminus \{0\}, \tag{3.10a}$$

$$\tilde{M}(x, t, k) \to I, \qquad k \to \infty, \tag{3.10b}$$

$$\operatorname*{Res}_{k=ik_1} \tilde{M}^{(1)}(x, t, k) = \frac{\gamma_1}{\dot{a}_1(ik_1)\delta^2(\xi, ik_1)} e^{-2k_1 x - 4ik_1^2 t} \tilde{M}^{(2)}(x, t, ik_1), \quad |\gamma_1| = 1, \tag{3.10c}$$

where

$$\tilde{J}(x, t, k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{r_1(k)\delta_-^{-2}(\xi,k)}{1+r_1(k)r_2(k)} e^{2it\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{r_2(k)\delta_+^2(\xi,k)}{1+r_1(k)r_2(k)} e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & k \in (-\infty, -\xi), \\ \begin{pmatrix} 1 & r_2(k)\delta^2(\xi, k)e^{-2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r_1(k)\delta^{-2}(\xi, k)e^{2it\theta} & 1 \end{pmatrix}, & k \in (-\xi, \infty) \setminus \{0\}, \end{cases} \tag{3.10d}$$

supplemented by the singularity conditions at $k = 0$:

$$\tilde{M}_+(x, t, k) = \begin{pmatrix} \frac{4v_1(x,t)}{A^2 a_2(0)\delta(\xi,0)} & -\delta(\xi, 0)\overline{v_2}(-x, t) \\ \frac{4v_2(x,t)}{A^2 a_2(0)\delta(\xi,0)} & -\delta(\xi, 0)\overline{v_1}(-x, t) \end{pmatrix} (I + O(k)) \begin{pmatrix} k & 0 \\ 0 & \frac{1}{k} \end{pmatrix}, \qquad k \to +i0, \tag{3.10e}$$

$$\tilde{M}_-(x, t, k) = \frac{2i}{A} \begin{pmatrix} \frac{-\overline{v_2}(-x,t)}{\delta(\xi,0)} & \delta(\xi, 0)\frac{v_1(x,t)}{a_2(0)} \\ \frac{-\overline{v_1}(-x,t)}{\delta(\xi,0)} & \delta(\xi, 0)\frac{v_2(x,t)}{a_2(0)} \end{pmatrix} + O(k), \qquad k \to -i0, \tag{3.10f}$$

in Case I, and

$$\tilde{M}_+(x, t, k) = \begin{pmatrix} \frac{v_1(x,t)}{a_{11}\delta(\xi,0)} & -\delta(\xi, 0)\overline{v_2}(-x, t) \\ \frac{v_2(x,t)}{a_{11}\delta(\xi,0)} & -\delta(\xi, 0)\overline{v_1}(-x, t) \end{pmatrix} (I + O(k)) \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{pmatrix}, \qquad k \to +i0, \tag{3.10g}$$

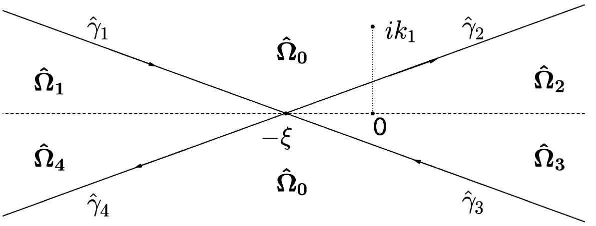

Fig. 2. Contour $\hat\Gamma=\hat\gamma_1\cup\ldots\cup\hat\gamma_4$.

$$\tilde M_-(x,t,k)=\frac{2i}{A}\begin{pmatrix}-\frac{\overline{v_2}(-x,t)}{\delta(\xi,0)} & \delta(\xi,0)\frac{v_1(x,t)}{\dot a_2(0)}\\ -\frac{\overline{v_1}(-x,t)}{\delta(\xi,0)} & \delta(\xi,0)\frac{v_2(x,t)}{\dot a_2(0)}\end{pmatrix}(I+O(k))\begin{pmatrix}1&0\\0&\frac{1}{k}\end{pmatrix},\qquad k\to -i0, \tag{3.10h}$$

in Case II.

3.2. RH problem deformations

Notice that similarly to the case of the NLS equation, assuming that $\int_{-\infty}^{0}|q_0(x)|dx<\infty$ and $\int_0^{\infty}|q_0(x)-A|dx<\infty$, the reflection coefficients $r_j(k)$, $j=1,2$, are defined, in general, for $k\in\mathbb{R}$ only (see Propositions 1 and 2). On the other hand, in the large-t analysis of $\tilde M(x,t,k)$, it is advantageous to have $r_j(k)$ continued, as meromorphic functions, into $\mathbb{C}$; then this will allow us to proceed with appropriate RH problem deformations. Otherwise $r_j(k)$ and $\frac{r_j(k)}{1+r_1(k)r_2(k)}$ have to be approximated by some rational functions with well-controlled errors (see, e.g., [16,31]).

For clarity's sake, in what follows we will assume that the initial data $q_0(x)$ are a compact perturbation of the pure step initial data $q_{0A}(x)$ (2.27), which guarantees that all $\Psi_l^m(x,0,k)$, $l,m=1,2$ (see Proposition 1) and thus $r_j(k)$ are meromorphic in $\mathbb{C}$. Then we define $\hat M(x,t,k)$ as follows (see Fig. 2):

$$\hat M(x,t,k)=\begin{cases}\tilde M(x,t,k), & k\in\hat\Omega_0,\\ \tilde M(x,t,k)\begin{pmatrix}1 & \frac{-r_2(k)\delta^2(\xi,k)}{1+r_1(k)r_2(k)}e^{-2it\theta}\\ 0 & 1\end{pmatrix}, & k\in\hat\Omega_1,\\ \tilde M(x,t,k)\begin{pmatrix}1 & 0\\ -r_1(k)\delta^{-2}(\xi,k)e^{2it\theta} & 1\end{pmatrix}, & k\in\hat\Omega_2,\\ \tilde M(x,t,k)\begin{pmatrix}1 & r_2(k)\delta^2(\xi,k)e^{-2it\theta}\\ 0 & 1\end{pmatrix}, & k\in\hat\Omega_3,\\ \tilde M(x,t,k)\begin{pmatrix}1 & 0\\ \frac{r_1(k)\delta^{-2}(\xi,k)}{1+r_1(k)r_2(k)}e^{2it\theta} & 1\end{pmatrix}, & k\in\hat\Omega_4.\end{cases} \tag{3.11}$$

Here the angles between the rays $\hat\gamma_j=\hat\gamma_j(\xi)$ and the real axis are such that the point ik_1 is located in the sector $\hat\Omega_0$. Then $\hat M(x,t,k)$ satisfies the RH problem with the jump across $\hat\Gamma=\hat\gamma_1\cup\ldots\cup\hat\gamma_4$:

$$\hat M_+(x,t,k)=\hat M_-(x,t,k)\hat J(x,t,k),\qquad k\in\hat\Gamma \tag{3.12a}$$

with

$$\hat{J}(x,t,k)=\begin{cases}\begin{pmatrix}1 & \frac{r_2(k)\delta^2(\xi,k)}{1+r_1(k)r_2(k)}e^{-2it\theta}\\ 0 & 1\end{pmatrix}, & k\in\hat{\gamma}_1,\\ \begin{pmatrix}1 & 0\\ r_1(k)\delta^{-2}(\xi,k)e^{2it\theta} & 1\end{pmatrix}, & k\in\hat{\gamma}_2,\\ \begin{pmatrix}1 & -r_2(k)\delta^2(\xi,k)e^{-2it\theta}\\ 0 & 1\end{pmatrix}, & k\in\hat{\gamma}_3,\\ \begin{pmatrix}1 & 0\\ \frac{-r_1(k)\delta^{-2}(\xi,k)}{1+r_1(k)r_2(k)}e^{2it\theta} & 1\end{pmatrix}, & k\in\hat{\gamma}_4,\end{cases} \tag{3.12b}$$

the normalization

$$\hat{M}(x,t,k)\to I, \qquad k\to\infty, \tag{3.12c}$$

and the residue condition

$$\operatorname*{Res}_{k=ik_1}\hat{M}^{(1)}(x,t,k)=c_1(x,t)\hat{M}^{(2)}(x,t,ik_1), \tag{3.12d}$$

where $c_1(x,t)=\frac{\gamma_1}{\dot{a}_1(ik_1)\delta^2(\xi,ik_1)}e^{-2k_1x-4ik_1^2t}$ with $|\gamma_1|=1$.

As for the singularity conditions at $k=0$, it is remarkable that they reduce, in the both cases, to the same residue condition having a conventional form

$$\operatorname*{Res}_{k=0}\hat{M}^{(2)}(x,t,k)=c_0(\xi)\hat{M}^{(1)}(x,t,0) \tag{3.12e}$$

with $c_0(\xi)=\frac{A\delta^2(\xi,0)}{2i}$ (cf. (2.58)).

The RH problem (3.12) involving two residue conditions (3.12d) and (3.12e) can be reduced to a regular RH problem (without residue conditions) by using the Blaschke-Potapov factors (see, e.g., [21]):

Proposition 6. *The solution $q(x,t)$ of the IV problem (1.1), (1.4) can be represented as follows:*

$$q(x,t)=-2k_1P_{12}(x,t)+2i\lim_{k\to\infty}k\hat{M}_{12}^R(x,t,k), \quad x>0, \tag{3.13a}$$

$$q(x,t)=-2k_1\overline{P_{21}(-x,t)}-2i\lim_{k\to\infty}k\overline{\hat{M}_{21}^R(-x,t,k)}, \quad x<0. \tag{3.13b}$$

Here (i) $\hat{M}^R(x,t,k)$ solves the regular Riemann-Hilbert problem:

$$\begin{cases}\hat{M}_+^R(x,t,k)=\hat{M}_-^R(x,t,k)\hat{J}^R(x,t,k), & k\in\hat{\Gamma},\\ \hat{M}^R(x,t,k)\to I, & k\to\infty,\end{cases} \tag{3.14a}$$

with

$$\hat{J}^R(x,t,k) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{k-ik_1}{k} \end{pmatrix} \hat{J}(x,t,k) \begin{pmatrix} 1 & 0 \\ 0 & \frac{k}{k-ik_1} \end{pmatrix}, \quad k \in \hat{\Gamma} \tag{3.14b}$$

and (ii) P_{12} and P_{21} are determined in terms of $\hat{M}^R$:

$$\begin{aligned} P_{12}(x,t) &= \frac{g_1(x,t)h_1(x,t)}{g_1(x,t)h_2(x,t) - g_2(x,t)h_1(x,t)}, \\ P_{21}(x,t) &= -\frac{g_2(x,t)h_2(x,t)}{g_1(x,t)h_2(x,t) - g_2(x,t)h_1(x,t)}, \end{aligned} \tag{3.15}$$

where $g(x,t) = \begin{pmatrix} g_1(x,t) \\ g_2(x,t) \end{pmatrix}$ and $h(x,t) = \begin{pmatrix} h_1(x,t) \\ h_2(x,t) \end{pmatrix}$ are given by

$$g(x,t) = ik_1 \hat{M}^{R(1)}(x,t,ik_1) - c_1(x,t)\hat{M}^{R(2)}(x,t,ik_1), \tag{3.16a}$$

$$h(x,t) = ik_1 \hat{M}^{R(2)}(x,t,0) + c_0(\xi)\hat{M}^{R(1)}(x,t,0). \tag{3.16b}$$

Proof. The solution $\hat{M}(x,t,k)$ of the Riemann-Hilbert problem (3.12) can be represented in terms of the solution $\hat{M}^R(x,t,k)$ of the regular RH problem (3.14) as follows [21]:

$$\hat{M}(x,t,k) = B(x,t,k)\hat{M}^R(x,t,k) \begin{pmatrix} 1 & 0 \\ 0 & \frac{k-ik_1}{k} \end{pmatrix}, \ k \in \mathbb{C}, \tag{3.17}$$

where the Blaschke-Potapov factor B has the form $B(x,t,k) = I + \frac{ik_1}{k-ik_1} P(x,t)$. Here $P(x,t)$ is a projection uniquely determined by the conditions

$$\ker P(x,t) = \operatorname{lin}_{\mathbb{C}} \{g(x,t)\} \quad \text{and} \quad \operatorname{Im} P(x,t) = \operatorname{lin}_{\mathbb{C}} \{h(x,t)\}, \tag{3.18}$$

where $g(x,t)$ and $h(x,t)$ are given by (3.16): this implies that the (12) and (21) elements of P are given by (3.15) whereas

$$P_{11}(x,t) = -\frac{P_{12}(x,t)g_2(x,t)}{g_1(x,t)} \quad \text{and} \quad P_{22}(x,t) = -\frac{P_{21}(x,t)g_1(x,t)}{g_2(x,t)}. \tag{3.19}$$

Finally, taking into account that

$$\hat{M}(x,t,k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 - \frac{ik_1}{k} \end{pmatrix} + \frac{ik_1}{k-ik_1} P(x,t) + \frac{\hat{M}_1^R(x,t)}{k} + O\left(\frac{1}{k^2}\right), \quad k \to \infty \tag{3.20}$$

where $\hat{M}^R(x,t,k) = I + \frac{\hat{M}_1^R(x,t)}{k} + O\left(\frac{1}{k^2}\right)$, $k \to \infty$, and using (2.53) and (2.54), the representations (3.13) follow. $\square$

Therefore, using Proposition 6, the large-t asymptotic analysis of $q(x,t)$ reduces to that for a regular RH problem (3.14). On the other hand, the latter problem has the same form as in the case of the NNLS equation on the zero background, see [33]. Consequently, one can follows the asymptotic approach, presented in [33], for obtaining the long-time asymptotics for $\hat{M}^R(x,t,k)$

at $k=ik_1$, $k=0$ (needed in (3.16)), and for large k (needed in (3.13)), which will finally lead to the long-time asymptotics of $q(x,t)$.

Before formulating detailed asymptotics, let us notice that the rough approximation $\hat{M}^R(x,t,k)\approx I$ as $t\to\infty$ with $x/t\geq\varepsilon$ for any $\varepsilon>0$ (to avoid the possible singularity of $\delta(\xi,k)$ as $\xi\to 0$), being substituted into (3.16), gives the main term of the asymptotics of $q(x,t)$ with a rough error estimate:

Proposition 7. *As $t\to\infty$,*

$$q(x,t)=A\delta^2(\xi,0)+o(1)\,\textit{for}\ x>0\quad \textit{and}\quad q(x,t)=o(1)\,\textit{for}\ x<0 \tag{3.21}$$

along any ray $\xi=\frac{x}{4t}=const>0$ or $\xi=const<0$.

Indeed, $\hat{M}^R(x,t,k)\approx I$ implies that $\begin{pmatrix} g_1(x,t)\\ g_2(x,t)\end{pmatrix}\approx\begin{pmatrix} ik_1\\ -c_1(x,t)\end{pmatrix}\approx\begin{pmatrix} ik_1\\ 0\end{pmatrix}$ and $\begin{pmatrix} h_1(x,t)\\ h_2(x,t)\end{pmatrix}\approx\begin{pmatrix} c_0(\xi)\\ ik_1\end{pmatrix}$. Accordingly, for $x>0$ we have

$$q(x,t)\approx -2k_1P_{12}(x,t)\approx -2k_1\frac{ik_1c_0(\xi)}{-k_1^2+c_0(\xi)c_1(x,t)}\approx 2ic_0(\xi)=A\delta^2(\xi,0)$$

whereas for $x<0$ we have

$$q(x,t)\approx -2k_1\overline{P_{21}(-x,t)}\approx 2k_1\frac{-\bar{c}_1(-x,t)(-ik_1)}{-k_1^2+\bar{c}_0(-\xi)\bar{c}_1(-x,t)}\approx 0.$$

Our main results make (3.21) more precise.

Theorem 1. *Consider the Cauchy problem (1.1), (1.4), where the initial data $q_0(x)$ is a compact perturbation of the pure step initial data (2.27): $q_0(x)-q_{0A}(x)=0$ for $|x|>N$ with some $N>0$. Assume that the spectral functions associated with $q_0(x)$ via (2.21)–(2.24) are such that:*

(a) *$a_1(k)$ has a single, simple zero in $\overline{\mathbb{C}^+}$ at $k=ik_1$, and $a_2(k)$ either has no zeros in $\overline{\mathbb{C}^-}$ or has a single, simple zero at $k=0$.*
(b) *$\operatorname{Im}\nu(-\xi)\in\left(-\frac{1}{2},\frac{1}{2}\right)$ for all $\xi>0$, where $\operatorname{Im}\nu(-\xi)=-\frac{1}{2\pi}\int_{-\infty}^{-\xi}d\arg(1+r_1(\zeta)r_2(\zeta))$, $r_1(k)=\frac{b(k)}{a_1(k)}$, $r_2(k)=\frac{\overline{b(-k)}}{a_2(k)}$.*

Assuming that the solution $q(x,t)$ of (1.1), (1.4) exists, its long-time asymptotics along any line $\xi=\frac{x}{4t}=const\neq 0$ is as follows:

(i) *for $x<0$:*

$$q(x,t)=t^{-\frac{1}{2}-\operatorname{Im}\nu(\xi)}\alpha_1(\xi)\exp\left\{4it\xi^2-i\operatorname{Re}\nu(\xi)\ln t\right\}+R_1(-\xi,t), \tag{3.22a}$$

(ii) *for $x>0$, three types of asymptotics are possible, depending on the value of* $\operatorname{Im}\nu(-\xi)$*:*

(a) *if* $\operatorname{Im}\nu(-\xi)\in\left(-\frac{1}{2},-\frac{1}{6}\right]$, *then*

$$q(x,t)=A\delta^2(\xi,0)+t^{-\frac{1}{2}-\operatorname{Im}\nu(-\xi)}\alpha_2(\xi)\exp\left\{-4it\xi^2+i\operatorname{Re}\nu(-\xi)\ln t\right\}+R_1(\xi,t). \tag{3.22b}$$

(b) *if* $\operatorname{Im}\nu(-\xi)\in\left(-\frac{1}{6},\frac{1}{6}\right)$, *then*

$$\begin{aligned}q(x,t)=A\delta^2(\xi,0)&+t^{-\frac{1}{2}+\operatorname{Im}\nu(-\xi)}\alpha_3(\xi)\exp\left\{4it\xi^2-i\operatorname{Re}\nu(-\xi)\ln t\right\}\\&+t^{-\frac{1}{2}-\operatorname{Im}\nu(-\xi)}\alpha_2(\xi)\exp\left\{-4it\xi^2+i\operatorname{Re}\nu(-\xi)\ln t\right\}+R_3(\xi,t).\end{aligned} \tag{3.22c}$$

(c) *if* $\operatorname{Im}\nu(-\xi)\in\left[\frac{1}{6},\frac{1}{2}\right)$, *then*

$$q(x,t)=A\delta^2(\xi,0)+t^{-\frac{1}{2}+\operatorname{Im}\nu(-\xi)}\alpha_3(\xi)\exp\left\{4it\xi^2-i\operatorname{Re}\nu(-\xi)\ln t\right\}+R_2(\xi,t). \tag{3.22d}$$

Here

$$\delta(\xi,0)=\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{-\xi}\frac{\ln(1+r_1(\zeta)r_2(\zeta))}{\zeta}\,d\zeta\right\},$$

$$\nu(-\xi)=-\frac{1}{2\pi}\ln|1+r_1(-\xi)r_2(-\xi)|-\frac{i}{2\pi}\Delta(-\xi),$$

$$\Delta(-\xi)=\int_{-\infty}^{-\xi}d\arg(1+r_1(\zeta)r_2(\zeta)),$$

$$\alpha_1(\xi)=\begin{cases}\dfrac{\sqrt{\pi}\,\exp\left\{-\frac{\pi}{2}\overline{\nu}(\xi)+\frac{\pi i}{4}-2\overline{\chi}(-\xi,\xi)-3i\overline{\nu}(\xi)\ln 2\right\}}{\overline{r_2(\xi)}\Gamma(-i\overline{\nu}(\xi))}, & r_1(-\xi)r_2(-\xi)\neq 0,\\ \frac{\overline{r_1(\xi)}e^{\frac{3\pi i}{4}}}{2\sqrt{\pi}}, & r_1(-\xi)=0, r_2(-\xi)\neq 0,\\ 0, & r_1(-\xi)\neq 0, r_2(-\xi)=0,\\ 0, & r_1(-\xi)=r_2(-\xi)=0,\end{cases}$$

$$\alpha_2(\xi)=\begin{cases}\dfrac{c_0^2(\xi)\sqrt{\pi}\,\exp\left\{-\frac{\pi}{2}\nu(-\xi)+\frac{3\pi i}{4}-2\chi(\xi,-\xi)+3i\nu(-\xi)\ln 2\right\}}{\xi^2 r_2(-\xi)\Gamma(i\nu(-\xi))}, & r_1(-\xi)r_2(-\xi)\neq 0,\\ 0, & r_1(-\xi)=0, r_2(-\xi)\neq 0,\\ \frac{c_0^2(\xi)r_1(-\xi)e^{\frac{\pi i}{4}}}{2\sqrt{\pi}\xi^2}, & r_1(-\xi)\neq 0, r_2(-\xi)=0,\\ 0, & r_1(-\xi)=r_2(-\xi)=0,\end{cases}$$

$$\alpha_3(\xi)=\begin{cases}\dfrac{\sqrt{\pi}\,\exp\left\{-\frac{\pi}{2}\nu(-\xi)+\frac{\pi i}{4}+2\chi(\xi,-\xi)-3i\nu(-\xi)\ln 2\right\}}{r_1(-\xi)\Gamma(-i\nu(-\xi))}, & r_1(-\xi)r_2(-\xi)\neq 0,\\ \frac{r_2(-\xi)e^{\frac{3\pi i}{4}}}{2\sqrt{\pi}}, & r_1(-\xi)=0, r_2(-\xi)\neq 0,\\ 0, & r_1(-\xi)\neq 0, r_2(-\xi)=0,\\ 0, & r_1(-\xi)=r_2(-\xi)=0,\end{cases}$$

with

$$\chi(\xi,k)=-\frac{1}{2\pi i}\int_{-\infty}^{-\xi}\ln(k-\zeta)d_{\zeta}\ln(1+r_1(\zeta)r_2(\zeta)),$$

where $\Gamma(\cdot)$ is the Euler Gamma-function.

The error estimates $R_1(\xi,t)$ and $R_2(\xi,t)$ are uniform in any compact subset of $\xi\in(0,\infty)$ and are as follows:

$$R_1(\xi,t)=\begin{cases}O\left(t^{-1}\right), & \operatorname{Im}\nu(-\xi)>0,\\ O\left(t^{-1}\ln t\right), & \operatorname{Im}\nu(-\xi)=0,\\ O\left(t^{-1+2|\operatorname{Im}\nu(-\xi)|}\right), & \operatorname{Im}\nu(-\xi)<0,\end{cases} \tag{3.23}$$

$$R_2(\xi,t)=\begin{cases}O\left(t^{-1+2|\operatorname{Im}\nu(-\xi)|}\right), & \operatorname{Im}\nu(-\xi)>0,\\ O\left(t^{-1}\ln t\right), & \operatorname{Im}\nu(-\xi)=0,\\ O\left(t^{-1}\right), & \operatorname{Im}\nu(-\xi)<0,\end{cases} \tag{3.24}$$

and

$$R_3(\xi,t)=R_1(\xi,t)+R_2(\xi,t)=\begin{cases}O\left(t^{-1+2|\operatorname{Im}\nu(-\xi)|}\right), & \operatorname{Im}\nu(-\xi)\neq 0,\\ O\left(t^{-1}\ln t\right), & \operatorname{Im}\nu(-\xi)=0.\end{cases}$$

Remark 7. Notice that $\delta(\xi,0)\to 1$ as $\xi\to\infty$ and thus the asymptotics (3.22b)-(3.22d) is consistent with the boundary conditions (1.4b).

Remark 8. In the case of the pure-step initial data, i.e. $q(x,0)=0$ for $x<0$ and $q(x,0)=A$ for $x\geq 0$, both assumptions of the theorem hold true. Moreover, in this case $1+r_1(k)r_2(k)=\frac{4k^2}{4k^2+A^2}$ and thus $\operatorname{Im}\nu=0$ in (3.22).

Remark 9. The problem of describing asymptotic transition between the regions $x<0$ and $x>0$ remains open. Some observations showing that this problem is far nontrivial are as follows:

1. From the point of view of the Riemann-Hilbert problem formalism, the transition region corresponds to merging the stationary phase point $k=-\xi$ and the singularity point $k=0$; to the best of our knowledge, such transition picture has not been considered in the literature.
2. The main asymptotic term for $x>0$, $A\delta^2(\xi,0)$, develops, in general, increasing oscillations as $\xi\to+0$; only in very particular cases (belonging to Case II only), where $b(0)=0$, there exists a finite limit of $\delta(\xi,0)$ $\xi\to+0$, which can be zero as well as non-zero.

3. Even in the simplest case of a soliton solution, where the asymptotics holds as $|x|$ increases, together with t, along any path in the half-planes $x > 0$ and $x < 0$, (in this case we have $\nu \equiv 0$ and thus $\delta(\xi, k) \equiv 1$), at the boundary line $x = 0$ the solution develops discrete (in t) singularities.

Sketch of proof of Theorem 1.

Here we consider the case $r_j(-\xi) \neq 0$, $j = 1, 2$ (for the cases when one of the $r_j(-\xi)$ (or the both) equals zero and thus $\nu(-\xi) = 0$, we refer to Section 1.5 of Chapter 2 in [23]). In view of (3.13), for obtaining the asymptotics (3.22) it is sufficient to estimate the solution $\hat{M}^R(x, t, k)$ of the regular RH problem (3.14) at $k = 0$, $k = ik_1$ and $k = \infty$. Noticing that this RH problem is similar to that in the case of decaying initial data [33], in what follows we will refer to [33] for the details of the relative steps in the asymptotic analysis.

First, introduce the rescaled variable z by

$$k = \frac{z}{\sqrt{8t}} - \xi, \tag{3.25}$$

so that

$$e^{2it\theta} = e^{\frac{iz^2}{2} - 4it\xi^2}.$$

Introduce the "local parametrix" $\hat{m}_0^R(x, t, k)$ as the solution of a RH problem with the jump matrix that is a "simplified $\hat{J}^R(x, t, k)$" in the sense that in its construction, $r_j(k)$, $j = 1, 2$ are replaced by the constants $r_j(-\xi)$ and $\delta(\xi, k)$ is replaced by (cf. (3.6)) $\delta \simeq \left(\frac{z}{\sqrt{8t}}\right)^{i\nu(-\xi)} e^{\chi(\xi,-\xi)}$. Such RH problem can be solved explicitly, in terms of the parabolic cylinder functions [26,33].

Indeed, $\hat{m}_0^R(x, t, k)$ can be determined by

$$\hat{m}_0^R(x, t, k) = \Delta(\xi, t) m^\Gamma(\xi, z(k)) \Delta^{-1}(\xi, t), \tag{3.26}$$

where

$$\Delta(\xi, t) = e^{(2it\xi^2 + \chi(\xi, -\xi))\sigma_3} (8t)^{-\frac{i\nu(-\xi)}{2}\sigma_3}, \tag{3.27}$$

$m^\Gamma(\xi, z)$ is determined by

$$m^\Gamma(\xi, z) = m_0(\xi, z) D_j^{-1}(\xi, z), \qquad z \in \Omega_j,\ \ j = 0, \dots, 4, \tag{3.28}$$

see Fig. 3, where γ_j corresponds to $\hat{\gamma}_j$ in accordance with (3.25). Here $D_0(\xi, z) = e^{-i\frac{z^2}{4}\sigma_3} z^{i\nu(-\xi)\sigma_3}$,

$$D_1(\xi, z) = D_0(\xi, z) \begin{pmatrix} 1 & \frac{r_2^R(-\xi)}{1 + r_1^R(-\xi) r_2^R(-\xi)} \\ 0 & 1 \end{pmatrix}, \qquad D_2(\xi, z) = D_0(\xi, z) \begin{pmatrix} 1 & 0 \\ r_1^R(-\xi) & 1 \end{pmatrix},$$

$$D_3(\xi, z) = D_0(\xi, z) \begin{pmatrix} 1 & -r_2^R(-\xi) \\ 0 & 1 \end{pmatrix}, \qquad D_4(\xi, z) = D_0(\xi, z) \begin{pmatrix} 1 & 0 \\ \frac{-r_1^R(-\xi)}{1 + r_1^R(-\xi) r_2^R(-\xi)} & 1 \end{pmatrix}$$

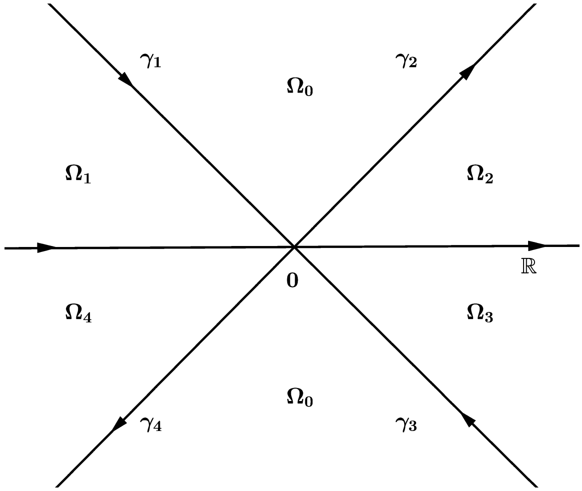

Fig. 3. Contour and domains for $m^\Gamma(\xi, z)$ in the z-plane.

with

$$r_1^R(k) = \frac{k - ik_1}{k} r_1(k), \quad r_2^R(k) = \frac{k}{k - ik_1} r_2(k),$$

and $m_0(\xi, z)$ is the solution of the following RH problem in z-plane (relative to $\mathbb{R}$, with a *constant jump matrix*):

$$\begin{cases} m_{0+}(\xi, z) = m_{0-}(\xi, z) j_0(\xi), & z \in \mathbb{R}, \\ m_0(\xi, z) = (I + O(1/z))\, e^{-i\frac{z^2}{4}\sigma_3} z^{i\nu(-\xi)\sigma_3}, & z \to \infty, \end{cases} \tag{3.29}$$

where

$$j_0(\xi) = \begin{pmatrix} 1 + r_1^R(-\xi) r_2^R(-\xi) & r_2^R(-\xi) \\ r_1^R(-\xi) & 1 \end{pmatrix}. \tag{3.30}$$

It is the RH problem for $m_0(\xi, z)$ that can be solved explicitly, in terms of the parabolic cylinder functions, see, e.g., Appendix A in [33]. Since we are interested in what happens for large t and, in view of (3.25), even finite values of k correspond to large values of z if t is large, it follows that all we actually need from $m_0(\xi, z)$ (and, correspondingly, $m^\Gamma(\xi, z)$) is its large-z asymptotics only. The latter has the form

$$m^\Gamma(\xi, z) = I + \frac{i}{z} \begin{pmatrix} 0 & \beta^R(\xi) \\ -\gamma^R(\xi) & 0 \end{pmatrix} + O(z^{-2}), \qquad z \to \infty,$$

where (cf. $\beta(\xi)$ and $\gamma(\xi)$ in [33])

$$\beta^R(\xi) = \frac{\sqrt{2\pi}\, e^{-\frac{\pi}{2}\nu(-\xi)} e^{-\frac{3\pi i}{4}}}{r_1^R(-\xi)\Gamma(-i\nu(-\xi))}, \tag{3.31a}$$

$$\gamma^R(\xi) = \frac{\sqrt{2\pi} e^{-\frac{\pi}{2}\nu(-\xi)} e^{-\frac{\pi i}{4}}}{r_2^R(-\xi)\Gamma(i\nu(-\xi))}. \tag{3.31b}$$

Now, having defined the parametrix $\hat{m}_0^R(x,t,k)$, we define $\check{M}^R(x,t,k)$ (cf. $\hat{m}(x,t,k)$ in [33]) as follows:

$$\check{M}^R(x,t,k) = \begin{cases} \hat{M}^R(x,t,k)(\hat{m}_0^R)^{-1}(x,t,k), & |k+\xi| < \varepsilon, \\ \hat{M}^R(x,t,k), & \text{otherwise}, \end{cases}$$

where ε is small enough so that $|\xi| > \varepsilon$ and $|ik_1 + \xi| > \varepsilon$. Then the sectionally analytic matrix $\check{M}^R$ has the following jumps across $\hat{\Gamma}_1 = \hat{\Gamma} \cup \{|k+\xi| = \varepsilon\}$ (the circle $|k+\xi| = \varepsilon$ is oriented counterclockwise)

$$\check{J}^R(x,t,k) = \begin{cases} \hat{m}_{0-}^R(x,t,k)\hat{J}^R(x,t,k)(\hat{m}_{0+}^R)^{-1}(x,t,k), & k \in \hat{\Gamma}, |k+\xi| < \varepsilon, \\ \left(\hat{m}_0^R\right)^{-1}(x,t,k), & |k+\xi| = \varepsilon, \\ \hat{J}^R(x,t,k), & \text{otherwise}. \end{cases} \tag{3.32}$$

The next step is the large-t evaluation of $\check{M}^R(x,t,k)$ using its representation in terms of the solution of the singular integral equation corresponding to the RH problem determined by the jump conditions (3.32) and the standard normalization condition $\check{M}^R \to I$ as $k \to \infty$. We have

$$\check{M}^R(x,t,k) = I + \frac{1}{2\pi i} \int_{\hat{\Gamma}_1} \mu(x,t,s)(\check{J}^R(x,t,s) - I)\frac{ds}{s-k}, \tag{3.33}$$

where μ solves the integral equation $\mu - C_w\mu = I$, with $w = \check{J}^R - I$. Here the Cauchy-type operator C_w is defined by $C_w f = C_-(fw)$, where $(C_-h)(k)$, $k \in \hat{\Gamma}_1$ are the right (according to the orientation of $\hat{\Gamma}_1$) non-tangential boundary values of

$$(Ch)(k') = \frac{1}{2\pi i} \int_{\hat{\Gamma}_1} \frac{h(s)}{s-k'} ds, \quad k' \in \mathbb{C} \setminus \hat{\Gamma}_1.$$

Reasoning as in [33] one can show that the main term in the large-t development of $\check{M}^R$ in (3.33) is given by the integral along the circle $|s+\xi| = \varepsilon$, which in turn gives

$$\check{M}^R(x,t,k) = I - \frac{1}{2\pi i} \int_{|s+\xi|=\varepsilon} \frac{\tilde{B}^R(\xi,t)}{(s+\xi)(s-k)} ds + R(\xi,t), \quad |k+\xi| > \varepsilon, \tag{3.34}$$

where

$$\tilde{B}^R(\xi,t) = \begin{pmatrix} 0 & i\beta^R(\xi) e^{4it\xi^2 + 2\chi(\xi,-\xi)} (8t)^{-\frac{1}{2} - i\nu(-\xi)} \\ -i\gamma^R(\xi) e^{-4it\xi^2 - 2\chi(\xi,-\xi)} (8t)^{-\frac{1}{2} + i\nu(-\xi)} & 0 \end{pmatrix} \tag{3.35}$$

and the (matrix) error estimate R has the structure $R(\xi,t)=\begin{pmatrix} R_1(\xi,t) & R_2(\xi,t)\\ R_1(\xi,t) & R_2(\xi,t)\end{pmatrix}$, with R_1 and R_2 having, in general, different orders of decay, see (3.23) and (3.24). Particularly, since $\check{M}^R=\hat{M}^R$ for all k with $|k+\xi|>\varepsilon$, we have

$$\lim_{k\to\infty} k\left(\hat{M}^R(x,t,k)-I\right)=\tilde{B}^R(\xi,t)+R(\xi,t) \tag{3.36}$$

as well as

$$\hat{M}^R(x,t,0)=I+\frac{\tilde{B}^R(\xi,t)}{\xi}+R(\xi,t), \tag{3.37a}$$

$$\hat{M}^R(x,t,ik_1)=I+\frac{\tilde{B}^R(\xi,t)}{\xi+ik_1}+R(\xi,t). \tag{3.37b}$$

Now we are at a position to evaluate $P_{12}(x,t)$ and $P_{21}(x,t)$ in (3.13). First, we evaluate $g_j(x,t)$ and $h_j(x,t)$, $j=1,2$, defined in (3.16), using (3.37) and replacing $\hat{M}^R$ by $\check{M}^R$:

$$g_1(x,t)=ik_1+R_1(\xi,t), \qquad g_2(x,t)=\frac{ik_1}{\xi+ik_1}\tilde{B}^R_{21}(\xi,t)+R_1(\xi,t),$$

$$h_1(x,t)=c_0(\xi)+\frac{ik_1}{\xi}\tilde{B}^R_{12}(\xi,t)+R_3(\xi,t), \qquad h_2(x,t)=ik_1+\frac{c_0(\xi)}{\xi}\tilde{B}^R_{21}(\xi,t)+R_3(\xi,t),$$

where $R_3(\xi,t)=R_1(\xi,t)+R_2(\xi,t)$ (we have used the standard notation for the entries of matrix $\tilde{B}^R(\xi,t)$). It follows that (we drop the arguments of the functions)

$$g_1h_1=ik_1c_0(\xi)-\frac{k_1^2}{\xi}\tilde{B}^R_{12}+R_3, \qquad g_1h_2=-k_1^2+\frac{ik_1c_0(\xi)}{\xi}\tilde{B}^R_{21}+R_3, \tag{3.38a}$$

$$g_2h_1=\frac{ik_1c_0(\xi)}{\xi+ik_1}\tilde{B}^R_{21}+R_1, \qquad g_2h_2=-\frac{k_1^2}{\xi+ik_1}\tilde{B}^R_{21}+R_1. \tag{3.38b}$$

Substituting (3.38) into (3.15), straightforward calculations give

$$P_{12}(x,t)=-\frac{ic_0(\xi)}{k_1}+\frac{\tilde{B}^R_{12}(\xi,t)}{\xi}+\frac{ic_0(\xi)^2}{\xi k_1(\xi+ik_1)}\tilde{B}^R_{21}(\xi,t)+R_3(\xi,t), \tag{3.39a}$$

$$P_{21}(x,t)=-\frac{\tilde{B}^R_{21}(\xi,t)}{\xi+ik_1}+R_1(\xi,t). \tag{3.39b}$$

Notice that formulas (3.39) involve k_1 explicitly. But using

$$\tilde{B}^R_{12}=\tilde{B}_{12}\frac{\xi}{\xi+ik_1}, \qquad \tilde{B}^R_{21}=\tilde{B}_{21}\frac{\xi+ik_1}{\xi},$$

where $\tilde{B}$ is defined similarly to $\tilde{B}^R$, see (3.31) and (3.35), with $r^R_j(-\xi)$ replaced by $r_j(-\xi)$, and substituting (3.36) and (3.39) into (3.13), it follows that the (explicit) dependence on k_1 in the resulting formulas for the main asymptotic terms vanishes, and we arrive at the asymptotic formulas (3.22).

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