

Concentration phenomena in a diffusive aggregation model

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Abstract

We consider the drift-diffusion equation $u_t - \varepsilon \Delta u + \nabla \cdot (u \nabla K * u) = 0$ in the whole space with global-in-time bounded solutions. Mass concentration phenomena for radially symmetric solutions of this equation with small diffusivity are studied.

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1. Introduction and motivations

The nonlocal nonlinear evolution model

$$u_t - \varepsilon \Delta u + \nabla \cdot (u \nabla K * u) = 0, \quad x \in \mathbb{R}^N, \quad t > 0,$$

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describes pairwise interaction of particles with the density $u = u(x, t)$ through the convolution with a kernel $K : \mathbb{R}^N \rightarrow \mathbb{R}$. These particles are also subject to the linear diffusion represented by the Laplacian, with the diffusion coefficient $\varepsilon > 0$. Such models appear in various physical and biological settings. For instance, Astrophysics is a source of mean-field models of gravitationally attracting particles going back to the famous Chandrasekhar equation for the equilibrium of radiating stars, see [12,13]. Another source of related models is Mathematical Biology where chemotaxis (haptotaxis, angiogenesis, etc.) phenomena for populations of either cells or (micro)organisms are described by various modifications of the Keller–Segel systems, see e.g., [14].

We supplement this equation with a nonnegative, radial, bounded and integrable initial condition, and we assume that $\nabla K \in L^\infty(\mathbb{R}^N)$ which is a sufficient assumption to guarantee that the corresponding Cauchy problem has a global-in-time regular solution, uniformly bounded for $t > 0$. The main result reported in this work states that under a suitable assumption on the singularity of the kernel K and for a small diffusion coefficient $\varepsilon > 0$, one observes concentration phenomena of such global-in-time solutions, namely, ε -small neighbourhoods of the origin carry an ε -uniform portion of the total mass. Results reported in this work show what kind of concentration of solutions can be expected in “correct” mathematical models (i.e. those without finite time blowups) and how to detect them in experiments with observations accumulated over a sufficiently long time interval.

Our proofs are based on a new general methodology stemming from considerations in [15, 10,6]. Note also that our study allows us to obtain estimates of small scale quantities such as structure functions which are relevant for hydrodynamical turbulence. A similar scheme has been studied in the context of the large scale evolution of the Universe modelled by a multidimensional analogue of the classical Burgers equation $u_t - \varepsilon \Delta u + (u \cdot \nabla)u = 0$. In that context, a link to the phenomenological/formal Kolmogorov K41 theory of turbulence has been made in [8] and in the papers of the second author [9–11] culminating in the study of the asymptotic behaviour of solutions of arbitrary size in the vanishing diffusion limit in terms of various functional norms.

2. Main results

Let us now state the results of our work in detail. We consider the Cauchy problem

$$u_t - \varepsilon \Delta u + \nabla \cdot (u \nabla K * u) = 0, \quad x \in \mathbb{R}^N, \quad t > 0, \quad (2.1)$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N, \quad (2.2)$$

with an initial condition satisfying

$$u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \quad u_0 \geq 0, \quad M \equiv \int_{\mathbb{R}^N} u_0(x) \, dx > 0, \quad (2.3)$$

and with a constant diffusion coefficient $\varepsilon > 0$. Moreover, for $N = 1$ we need to assume that u_0 belongs to the Sobolev space $H^1(\mathbb{R})$.

The interaction kernel $K = K(x)$ in equation (2.1) is a radially symmetric function such that

$$K(x) = k(|x|) \quad \text{with} \quad k \in C^1(0, \infty), \quad k' \in L^\infty(0, \infty), \quad (2.4)$$

and

$$\begin{aligned}\kappa_\Lambda &\equiv - \sup_{s \in (0, \Lambda)} k'(s) \in (0, \infty) \quad \text{for each } \Lambda > 0, \\ \text{with } \kappa_0 &\equiv \lim_{\Lambda \searrow 0} \kappa_\Lambda \in (0, \infty).\end{aligned}\tag{2.5}$$

For $N = 1$, we actually need something stronger than $k' \in L^\infty(0, \infty)$, namely

$$k'' \in L^1(0, \infty).\tag{2.6}$$

The functions $K(x) = -|x|$ and $K(x) = e^{-|x|}$ are our basic examples of interaction kernels often used in applications, see e.g., [2].

Such kernels K are *mildly singular*; that is, solutions of the Cauchy problem (2.1)–(2.2) are global-in-time (see Remark 2.1 below), even though interactions are strong enough to trigger finite time blowup in the diffusion-free case $\varepsilon = 0$, see, e.g., [2,4,3] and the references therein. For more details on the positivity of solutions and the smoothing effect of the diffusion, see [18,19].

Remark 2.1. It is rather standard to show that, under assumption (2.4), (implying $\nabla K \in L^\infty(\mathbb{R}^N)$) and for each initial condition u_0 satisfying (2.3), problem (2.1)–(2.2) has a unique, nonnegative, smooth, global-in-time solution which satisfies the sign conservation property: $u(x, t) \geq 0$, and the mass conservation property

$$\int_{\mathbb{R}^N} u(x, t) \, dx = \int_{\mathbb{R}^N} u_0(x) \, dx = M \quad \text{for all } t \geq 0.\tag{2.7}$$

In fact, for $t > 0$, $m \geq 1$, and $p \in [1, \infty)$, this solution belongs to all Sobolev spaces $W^{m,p}(\mathbb{R}^N)$ and is continuously differentiable with respect to time in $W^{m,p}(\mathbb{R}^N)$, see [18,19]. In particular, integrations by parts in and time differentiation of integrals involving u in the following sections are fully justified. Also, by uniqueness, this solution is radial in x if the corresponding initial condition is. The construction of such solutions is usually performed in the framework of mild ones, and their regularity is shown afterwards up to the classical smoothness. More generally, according to [16], if $\nabla K \in L^p(\mathbb{R}^N)$ with some $p \in (N, \infty]$, then problem (2.1)–(2.2) has a unique, regular, nonnegative, radial, global-in-time solution for every nonnegative radial initial condition $u_0 \in L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ for some $q > 1$. On the other hand, it is well-known that for more singular kernels, for example when $K = -E_N$ is the fundamental solution of the Laplacian in \mathbb{R}^N with $N \geq 2$ as in the parabolic-elliptic Keller-Segel model of chemotaxis, some solutions blow up in a finite time. For a classification of kernels based on the property whether all solutions of equation (2.1) are global-in-time or they can blow up in a finite time, see [7,1,16,17].

Remark 2.2. Solutions to the Cauchy problem (2.1)–(2.2) with $\varepsilon > 0$ and under assumptions (2.3)–(2.4) are not only global-in-time but also uniformly bounded in time (but not in ε). Indeed, we prove below in Lemma 4.1 that $\sup_{t \geq 0} \|u(t)\|_p < \infty$ for each $p \in [1, \infty)$. This bound for the L^∞ -norm is obtained immediately from estimates of the Sobolev norms in Lemma 4.3 in the case $N = 1$. Such an estimate for the L^∞ -norm in higher dimensional case can be shown in an analogous way and we shall discuss it in the forthcoming paper [5].

Our goal is to study the behaviour of the family of solutions to problem (2.1)–(2.2) when $\varepsilon > 0$ is small.

Theorem 2.3 (Concentration of mass at the origin). *Let $u = u(x, t)$ be a radial, nonnegative, global-in-time solution to problem (2.1)–(2.2) with $\varepsilon > 0$, and with the interaction kernel K satisfying assumptions (2.4)–(2.6). Assume that the radial initial condition u_0 satisfies (2.3) and additionally belongs to $H^1(\mathbb{R})$ when $N = 1$. Moreover, suppose that there exists $\Lambda > 0$ such that*

$$\mu_\Lambda \equiv \int_{\mathbb{R}^N} \min\{|x|, \Lambda\} u_0(x) \, dx < \frac{\kappa_\Lambda M}{4(\kappa_\Lambda + 2\|k'\|_\infty)} \Lambda. \quad (2.8)$$

Then for some explicit numbers $\varepsilon_ > 0$, $T_* > 0$, $C_* > 0$, and $\lambda > 0$, independent of ε , the following inequality holds true*

$$\int_0^{T_*} \int_{B_{\lambda\varepsilon}} u(x, t) \, dx \, dt \geq C_* \quad \text{for all } \varepsilon \in (0, \varepsilon_*). \quad (2.9)$$

We also derive analogous estimates for L^p -norms of solutions on small balls.

Corollary 2.4. *Let $p \in [1, \infty)$. Under the assumptions of Theorem 2.3 and using the same notation, the solution u of problem (2.1)–(2.2) satisfies*

$$\int_0^{T_*} \left(\int_{B_{\lambda\varepsilon}} u(x, t)^p \, dx \right)^{1/p} dt \geq C_{**}(p) \varepsilon^{-\frac{N(p-1)}{p}}. \quad (2.10)$$

Here, the number $C_{**}(p)$ depends on the same parameters as C_* in Theorem 2.3 as well as on p .

Theorem 2.3 together with Corollary 2.4 signify that, even if the interactions described by the attractive kernel do not lead to a formation of singularities for the solution $u(t)$ neither in finite nor in the infinite time for $\varepsilon > 0$ (cf. Remarks 2.1 and 2.2), there are concentration phenomena of solutions on ε -small balls for $0 < \varepsilon \ll 1$. To the best of our knowledge, this is the first result where the small diffusion asymptotics is analyzed in a sharp way for an aggregation model. Here, we have been motivated by results of this type obtained in the specific case of the generalized Burgers equation in [9–11].

The proofs of Theorem 2.3 and of Corollary 2.4 are based on an analysis of the time evolution of the quantity $\mathcal{D}_\Lambda(u(t))$ defined below in (3.1) which is related to the mass concentration of $u(t)$ at the origin. Then, in Section 4, we prove upper bounds for the norms in L^p (and in H^1 for $N = 1$) of solutions. Using these estimates in Section 5, we derive lower bounds for the L^p -norms as stated in Theorem 2.3 and Corollary 2.4.

Remark 2.5. The order of growth $\varepsilon^{-N(p-1)/p}$ of the L^p -norms stated in inequality (2.10) is optimal because we have also an analogous (but without time average or spatial localisation) upper estimate. More precisely, we prove in Lemma 4.1 below that for each $p \in [1, \infty)$ there exists a number $C(p, u_0) > 0$ such that

$$\|u(t)\|_p \leq C(p, u_0) \varepsilon^{-N(p-1)/p} \quad \text{for all } t > 0$$

and for all sufficiently small $\varepsilon > 0$. This is a genuinely nonlinear effect since such estimates of L^p -norms for solutions of the heat equation $w_t = \varepsilon \Delta w$ are different. Indeed, it follows from the explicit form of solutions via the convolution with the Gauss–Weierstrass kernel that

$$\|w(t)\|_p \asymp (\varepsilon t)^{-N(p-1)/2p} \|w(0)\|_1.$$

Remark 2.6. Generalizations of such two-sided ε -optimal estimates to other Sobolev norms will be published in a subsequent paper [5].

Remark 2.7. If u_0 has a finite first moment, namely $\int_{\mathbb{R}^N} u_0(x)|x| dx < \infty$, then the number μ_Λ defined in (2.8) is trivially bounded from above by

$$\mu_\Lambda = \int_{\mathbb{R}^N} \min\{|x|, \Lambda\} u_0(x) dx \leq \int_{\mathbb{R}^N} u_0(x)|x| dx \quad \text{for each } \Lambda > 0.$$

Thus, assumption (2.8) is satisfied for each such initial data, if $\Lambda \kappa_\Lambda \rightarrow \infty$ as $\Lambda \rightarrow \infty$ and this property holds true e.g. in the case of the kernel $K(x) = -|x|$.

Notation. We denote by B_r the ball in \mathbb{R}^N centered at $x = 0$ with radius $r > 0$, and by $\sigma_N = 2\pi^{N/2}/\Gamma(\frac{N}{2})$ the area of the unit sphere \mathbb{S}^{N-1} in \mathbb{R}^N . For $p \in [1, \infty]$, the norms of the Lebesgue space $L^p(\mathbb{R}^N)$ are denoted by $\|\cdot\|_p$. As usual, we set $H^m = W^{m,2}$, $m > 0$, and denote the corresponding homogeneous Sobolev seminorm by $\|\cdot\|_{\dot{H}^m}$. We use the analogous notation $\|\cdot\|_{\dot{W}^{m,p}}$ for the homogeneous seminorms in $W^{m,p}$. Throughout the paper, the letter C is used for various positive numbers which may vary from line to line but depend only on the dimension N and the bounds for the kernel K : $\|\nabla K\|_\infty = \|k'\|_\infty$ and $\|k''\|_1$ for $N = 1$. The dependence upon additional parameters will be indicated explicitly.

3. Concentration of solutions

We shall describe a concentration phenomenon at the origin of solutions to problem (2.1)–(2.2) by considering the quantity

$$\mathcal{D}_\Lambda(u(t)) \equiv \begin{cases} 2u(0, t) & \text{if } N = 1, \\ (N-1) \int_{B_{3\Lambda/2}} \frac{u(x, t)}{|x|} dx & \text{if } N \geq 2, \end{cases} \quad (3.1)$$

where the scaling parameter $\Lambda > 0$ is chosen in a suitable way, according to the behaviour of the initial condition, see (2.8). The following theorem, stating that as $\varepsilon \rightarrow 0$, after time averaging, $\mathcal{D}_\Lambda(u)$ grows at least as ε^{-1} , is one of the main results of our work.

Theorem 3.1. *Let the assumptions (2.4)–(2.6) hold true, and denote by u a radial, nonnegative, global-in-time solution to problem (2.1)–(2.2) with an arbitrary $\varepsilon > 0$. Assume that the radial nonnegative initial condition u_0 satisfies (2.3), as well as condition (2.8) for some $\Lambda > 0$. Then, there exist numbers $T_\Lambda > 0$, $\mathcal{L}_\Lambda > 0$, and $\omega_\Lambda > 0$ depending only on the dimension N , mass*

M , the number μ_Λ in Assumption (2.8), and the quantities $\|k'\|_\infty$, κ_Λ in assumption (2.5) (see equations (3.14) and (3.16) below) such that

$$\int_0^{T_\Lambda} \mathcal{D}_\Lambda(u(t)) e^{-\omega_\Lambda t/\Lambda} dt \geq \frac{\Lambda \mathcal{L}_\Lambda}{\varepsilon}.$$

Proof. We define the function

$$\varphi(s) = \begin{cases} s & \text{if } 0 \leq s \leq \frac{1}{2}, \\ 1 - \frac{1}{2} \left(\frac{3}{2} - s\right)^2 & \text{if } \frac{1}{2} \leq s \leq \frac{3}{2}, \\ 1 & \text{if } s \geq \frac{3}{2}, \end{cases} \quad (3.2)$$

with the following properties

- $0 \leq \varphi(s) \leq \min\{s, 1\}$ for all $s \geq 0$,
- $0 \leq \varphi'(s) \leq 1$ for all $s \geq 0$,
- $\varphi''(s) \leq 0$ for all $s \geq 0$ such that $s \neq \frac{1}{2}$ and $s \neq \frac{3}{2}$.

For each $\Lambda > 0$, we set $\varphi_\Lambda(s) = \varphi(s/\Lambda)$ and we introduce the “truncated moment” (compared to the first moment with the function $|x|$ as weight considered in, e.g., [6])

$$\mathcal{I}_\Lambda(t) \equiv \int_{\mathbb{R}^N} \varphi_\Lambda(|x|) u(x, t) dx \quad \text{for all } t \geq 0. \quad (3.3)$$

Notice that

$$\mathcal{I}_\Lambda(t) \leq M \quad \text{for all } t \geq 0, \quad (3.4)$$

by the mass conservation property (2.7) and properties of φ_Λ . Our goal is to derive a differential inequality for \mathcal{I}_Λ , see (3.13) below. Thus, we multiply equation (2.1) by $\varphi_\Lambda(|x|)$, and integrate the resulting identity with respect to $x \in \mathbb{R}^N$.

Let us show that the contribution of the diffusive term in equation (2.1) satisfies the inequality

$$\int_{\mathbb{R}^N} \varphi_\Lambda(|x|) \Delta u(x, t) dx \leq \frac{\mathcal{D}_\Lambda(u(t))}{\Lambda} \quad \text{for all } t \geq 0. \quad (3.5)$$

Indeed, if $N \geq 2$, we integrate by parts and use the properties of φ_Λ , the positivity of u , as well as the fact that $u(x, t) = u(r, t)$ with $r = |x|$ to obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \varphi_\Lambda(|x|) \Delta u(x, t) dx &= -\frac{1}{\Lambda} \int_{\mathbb{R}^N} \varphi' \left(\frac{|x|}{\Lambda} \right) \frac{x}{|x|} \cdot \nabla u(x, t) dx \\ &= -\frac{\sigma_N}{\Lambda} \int_0^\infty \varphi' \left(\frac{r}{\Lambda} \right) r^{N-1} u_r(r, t) dr \end{aligned}$$

$$\begin{aligned}
&= \frac{\sigma_N}{\Lambda} \int_0^\infty \left[\frac{1}{\Lambda} \varphi'' \left(\frac{r}{\Lambda} \right) r^{N-1} + (N-1) \varphi' \left(\frac{r}{\Lambda} \right) r^{N-2} \right] u(r, t) \, dr \\
&\quad - \frac{\sigma_N}{\Lambda} \left[\varphi' \left(\frac{r}{\Lambda} \right) r^{N-1} u(r, t) \right]_{r=0}^{r=\infty} \\
&\leq \frac{(N-1)\sigma_N}{\Lambda} \int_0^{3\Lambda/2} u(r, t) r^{N-2} \, dr = \frac{\mathcal{D}_\Lambda(u(t))}{\Lambda}.
\end{aligned}$$

Similarly, when $N = 1$, it follows from the symmetry of u that

$$\begin{aligned}
\int_{\mathbb{R}} \varphi_\Lambda(|x|) u_{xx}(x, t) \, dx &= 2 \int_0^\infty \varphi_\Lambda(x) u_{xx}(x, t) \, dx \\
&= 2 \left[\varphi \left(\frac{x}{\Lambda} \right) u_x(x, t) \right]_{x=0}^{x=\infty} - \frac{2}{\Lambda} \int_0^\infty \varphi' \left(\frac{x}{\Lambda} \right) u_x(x, t) \, dx \\
&= -\frac{2}{\Lambda} \left[\varphi' \left(\frac{x}{\Lambda} \right) u(x, t) \right]_{x=0}^{x=\infty} + \frac{2}{\Lambda^2} \int_0^\infty \varphi'' \left(\frac{x}{\Lambda} \right) u(x, t) \, dx \\
&\leq \frac{2}{\Lambda} u(0, t) = \frac{\mathcal{D}_\Lambda(u(t))}{\Lambda},
\end{aligned}$$

which completes the proof of inequality (3.5).

Next, we estimate the contribution of the truncated moment of the nonlinear drift term

$$J_\Lambda(t) \equiv - \int_{\mathbb{R}^N} \varphi_\Lambda(|x|) \nabla \cdot (u(x, t) \nabla K * u(x, t)) \, dx.$$

Integrating by parts and using the properties of K and φ_Λ , as well as a symmetrization argument, we obtain

$$\begin{aligned}
J_\Lambda(t) &= \frac{1}{\Lambda} \int_{\mathbb{R}^N} u(x, t) \varphi' \left(\frac{|x|}{\Lambda} \right) \frac{x}{|x|} \cdot (\nabla K * u)(x, t) \, dx \\
&= \frac{1}{\Lambda} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} u(x, t) u(y, t) \varphi' \left(\frac{|x|}{\Lambda} \right) \frac{x}{|x|} \cdot \nabla K(x - y) \, dx \, dy \\
&= \frac{1}{2\Lambda} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} u(x, t) u(y, t) k'(|x - y|) \Phi_\Lambda(x, y) \, dx \, dy,
\end{aligned} \tag{3.6}$$

where

$$\Phi_{\Lambda}(x, y) \equiv \frac{x - y}{|x - y|} \cdot \left[\varphi' \left(\frac{|x|}{\Lambda} \right) \frac{x}{|x|} - \varphi' \left(\frac{|y|}{\Lambda} \right) \frac{y}{|y|} \right], \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Introducing the quantity

$$J_{\Lambda,1}(t) = \frac{1}{2\Lambda} \int_{B_{\Lambda/2}} \int_{B_{\Lambda/2}} u(x, t) u(y, t) k'(|x - y|) \Phi_{\Lambda}(x, y) \, dx \, dy,$$

we notice that, for $(x, y) \in B_{\Lambda/2} \times B_{\Lambda/2}$,

$$\Phi_{\Lambda}(x, y) = \frac{x - y}{|x - y|} \cdot \left(\frac{x}{|x|} - \frac{y}{|y|} \right) = \frac{|x| + |y|}{|x - y|} \left(1 - \frac{x \cdot y}{|x||y|} \right) \geq 1 - \frac{x \cdot y}{|x||y|} \geq 0. \quad (3.7)$$

Moreover, since $|x - y| \leq \Lambda$, by assumption (2.5), we have

$$k'(|x - y|) \leq -\kappa_{\Lambda}, \quad (x, y) \in B_{\Lambda/2} \times B_{\Lambda/2}. \quad (3.8)$$

Combining (3.7) and (3.8) we get

$$\begin{aligned} J_{\Lambda,1}(t) &\leq -\frac{\kappa_{\Lambda}}{2\Lambda} \int_{B_{\Lambda/2}} \int_{B_{\Lambda/2}} u(x, t) u(y, t) \Phi_{\Lambda}(x, y) \, dx \, dy \\ &\leq -\frac{\kappa_{\Lambda}}{2\Lambda} \int_{B_{\Lambda/2}} \int_{B_{\Lambda/2}} u(x, t) u(y, t) \left(1 - \frac{x \cdot y}{|x||y|} \right) \, dx \, dy. \end{aligned}$$

Moreover, recalling the identity

$$\int_{\mathbb{R}^N} x \psi(x) \, dx = 0,$$

valid for every radially symmetric function ψ , we end up with the inequality

$$J_{\Lambda,1}(t) \leq -\frac{\kappa_{\Lambda}}{2\Lambda} \int_{B_{\Lambda/2}} \int_{B_{\Lambda/2}} u(x, t) u(y, t) \, dx \, dy.$$

Finally, we use the mass conservation property (2.7), the inclusion

$$\left(\mathbb{R}^N \times \mathbb{R}^N \right) \setminus \left(B_{\Lambda/2} \times B_{\Lambda/2} \right) \subset \left(\mathbb{R}^N \times \left(\mathbb{R}^N \setminus B_{\Lambda/2} \right) \right) \cup \left(\left(\mathbb{R}^N \setminus B_{\Lambda/2} \right) \times \mathbb{R}^N \right), \quad (3.9)$$

and the inequality

$$\frac{1}{2} \leq \varphi_{\Lambda}(|x|) = \varphi \left(\frac{|x|}{\Lambda} \right) \quad \text{for all } x \in \mathbb{R}^N \setminus B_{\Lambda/2}, \quad (3.10)$$

to conclude that

$$\begin{aligned}
J_{\Lambda,1}(t) &\leq -\frac{\kappa_{\Lambda}}{2\Lambda}M^2 + \frac{\kappa_{\Lambda}}{2\Lambda} \int_{(\mathbb{R}^N \times \mathbb{R}^N) \setminus (B_{\Lambda/2} \times B_{\Lambda/2})} u(x,t)u(y,t) \, dx \, dy \\
&\leq -\frac{\kappa_{\Lambda}}{2\Lambda}M^2 + \frac{\kappa_{\Lambda}}{\Lambda} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus B_{\Lambda/2}} u(x,t)u(y,t) \, dx \, dy \\
&\leq -\frac{\kappa_{\Lambda}}{2\Lambda}M^2 + \frac{2\kappa_{\Lambda}}{\Lambda}M\mathcal{I}_{\Lambda}(t).
\end{aligned} \tag{3.11}$$

Next, owing to the boundedness of φ' and k' ,

$$k'(|x-y|)\Phi_{\Lambda}(x,y) \leq \|k'\|_{\infty} \left[\varphi' \left(\frac{|x|}{\Lambda} \right) + \varphi' \left(\frac{|y|}{\Lambda} \right) \right] \leq 2\|k'\|_{\infty}, \quad (x,y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

which we combine with relations (3.9) and (3.10) to estimate $J_{\Lambda}(t) - J_{\Lambda,1}(t)$, and thereby obtain

$$\begin{aligned}
J_{\Lambda}(t) - J_{\Lambda,1}(t) &= \frac{1}{2\Lambda} \int_{(\mathbb{R}^N \times \mathbb{R}^N) \setminus (B_{\Lambda/2} \times B_{\Lambda/2})} u(x,t)u(y,t)k'(|x-y|)\Phi_{\Lambda}(x,y) \, dx \, dy \\
&\leq \frac{\|k'\|_{\infty}}{\Lambda} \int_{(\mathbb{R}^N \times \mathbb{R}^N) \setminus (B_{\Lambda/2} \times B_{\Lambda/2})} u(x,t)u(y,t) \, dx \, dy \\
&\leq \frac{2\|k'\|_{\infty}}{\Lambda} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus B_{\Lambda/2}} u(x,t)u(y,t) \, dx \, dy \\
&\leq \frac{4\|k'\|_{\infty}}{\Lambda}M\mathcal{I}_{\Lambda}(t).
\end{aligned} \tag{3.12}$$

Gathering identity (3.6) and estimates (3.5), (3.11) and (3.12), we get the differential inequality

$$\Lambda \frac{d}{dt} \mathcal{I}_{\Lambda}(t) \leq \varepsilon \mathcal{D}_{\Lambda}(u(t)) - \frac{\kappa_{\Lambda}}{2}M^2 + \omega_{\Lambda} \mathcal{I}_{\Lambda}(t) \quad \text{for all } t \geq 0, \tag{3.13}$$

where

$$\omega_{\Lambda} \equiv 2M(\kappa_{\Lambda} + 2\|k'\|_{\infty}). \tag{3.14}$$

Equivalently,

$$\frac{d}{dt} [\mathcal{I}_{\Lambda}(t)e^{-\omega_{\Lambda}t/\Lambda}] \leq \left(\varepsilon \mathcal{D}_{\Lambda}(u(t)) - \frac{\kappa_{\Lambda}}{2}M^2 \right) \frac{e^{-\omega_{\Lambda}t/\Lambda}}{\Lambda} \quad \text{for all } t \geq 0.$$

After an integration with respect to time, we obtain for each $T > 0$,

$$\begin{aligned}
-\mathcal{I}_\Lambda(0) &\leq \mathcal{I}_\Lambda(T)e^{-\omega_\Lambda T/\Lambda} - \mathcal{I}_\Lambda(0) \leq \frac{1}{\Lambda} \int_0^T \left[\varepsilon \mathcal{D}_\Lambda(u(t)) - \frac{\kappa_\Lambda}{2} M^2 \right] e^{-\omega_\Lambda t/\Lambda} dt \\
&= \frac{\varepsilon}{\Lambda} \int_0^T \mathcal{D}_\Lambda(u(t)) e^{-\omega_\Lambda t/\Lambda} dt - \frac{\kappa_\Lambda}{2\omega_\Lambda} M^2 \left(1 - e^{-\omega_\Lambda T/\Lambda} \right).
\end{aligned}$$

Hence, for each $T > 0$ we have

$$\frac{\varepsilon}{\Lambda} \int_0^T \mathcal{D}_\Lambda(u(t)) e^{-\omega_\Lambda t/\Lambda} dt \geq \frac{\kappa_\Lambda}{2\omega_\Lambda} M^2 \left(1 - e^{-\omega_\Lambda T/\Lambda} \right) - \mathcal{I}_\Lambda(0). \quad (3.15)$$

By assumption (2.8) and the properties of φ , we get

$$\mathcal{I}_\Lambda(0) \leq \frac{1}{\Lambda} \int_{\mathbb{R}^N} \min\{|x|, \Lambda\} u_0(x) dx = \frac{\mu_\Lambda}{\Lambda} < \frac{\kappa_\Lambda}{2\omega_\Lambda} M^2,$$

so that choosing the numbers

$$\mathcal{L}_\Lambda \equiv \frac{1}{2} \left(\frac{\kappa_\Lambda}{2\omega_\Lambda} M^2 - \mathcal{I}_\Lambda(0) \right) > 0 \quad \text{and} \quad T_\Lambda \equiv \frac{\Lambda}{\omega_\Lambda} \log \left(\frac{\kappa_\Lambda M^2}{2\omega_\Lambda \mathcal{L}_\Lambda} \right) > 0, \quad (3.16)$$

we obtain

$$\frac{\kappa_\Lambda}{2\omega_\Lambda} M^2 \left(1 - e^{-\omega_\Lambda T/\Lambda} \right) - \mathcal{I}_\Lambda(0) \geq \mathcal{L}_\Lambda \quad \text{for each} \quad T \geq T_\Lambda, \quad (3.17)$$

which completes the proof. \square

4. Upper estimates of Lebesgue and Sobolev norms

In this section, we deal with a global-in-time, nonnegative, regular solution to problem (2.1)–(2.2) corresponding to the initial condition $0 \leq u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ (moreover, we require $u_0 \in H^1(\mathbb{R})$ for $N = 1$). For $N \geq 2$, the only assumption on the kernel is $\nabla K \in L^\infty(\mathbb{R}^N)$ and for $N = 1$ we use a more specific assumption (which can be generalized). In particular, the kernel K can be smooth and $\kappa_0 = 0$. Here, we do not require u to be radially symmetric. We will use systematically the mass conservation property (2.7).

Lemma 4.1. *Let u be a (not necessarily radial) nonnegative solution to problem (2.1)–(2.2) with $\varepsilon > 0$, with the interaction kernel satisfying $\nabla K \in L^\infty(\mathbb{R}^N)$, and with initial condition u_0 such as in (2.3). For each $p \in [1, \infty)$ there exists a constant $C_p > 0$ such that*

$$\|u(t)\|_p \leq \max \left\{ M, \|u_0\|_{\max\{2,p\}}, C_p M^{(N(p-1)+p)/p} \varepsilon^{-N(p-1)/p} \right\}$$

for all $t \geq 0$.

Proof. The case $p = 1$ is obvious by the conservation of mass (2.7).

For $p \geq 2$, we use the energy method. Integrating by parts, and then using the Hölder and the Young inequalities as well as the assumption $\nabla K \in L^\infty(\mathbb{R}^N)$, we obtain

$$\begin{aligned} \frac{1}{p(p-1)} \frac{d}{dt} \|u\|_p^p &= -\varepsilon \int_{\mathbb{R}^N} |\nabla u|^2 u^{p-2} dx + \int_{\mathbb{R}^N} u^{p-1} \nabla u \cdot (\nabla K * u) dx \\ &\leq -\frac{4\varepsilon}{p^2} \|\nabla u^{p/2}\|_2^2 + \frac{2}{p} \|u^{p/2}\|_2 \|\nabla u^{p/2}\|_2 \|\nabla K * u\|_\infty \\ &\leq \frac{4}{p^2} \|\nabla u^{p/2}\|_2 \left(-\varepsilon \|\nabla u^{p/2}\|_2 + \frac{p}{2} \|\nabla K\|_\infty \|u\|_1 \|u\|_p^{p/2} \right) \\ &\leq \frac{4}{p^2} \|\nabla u^{p/2}\|_2 \left(-\varepsilon \|\nabla u^{p/2}\|_2 + \frac{pM}{2} \|\nabla K\|_\infty \|u\|_p^{p/2} \right). \end{aligned}$$

It follows from the Hölder and Gagliardo–Nirenberg inequalities and the mass conservation (2.7) that

$$\begin{aligned} \|u\|_p^{p/2} &= \|u^{p/2}\|_2 \leq C \|\nabla u^{p/2}\|_2^{N/(N+2)} \|u^{p/2}\|_1^{2/(N+2)} \\ &\leq C \|\nabla u^{p/2}\|_2^{N/(N+2)} \left(\|u\|_p^{p(p-2)/(p-1)} M^{p/2(p-1)} \right)^{2/(N+2)} \\ &\leq C \|\nabla u^{p/2}\|_2^{N/(N+2)} M^{p/(N+2)(p-1)} \|u\|_p^{p(p-2)/(N+2)(p-1)}, \end{aligned}$$

and therefore

$$\|\nabla u^{p/2}\|_2 \geq C M^{-p/N(p-1)} \|u\|_p^{p(N(p-1)+2)/2N(p-1)}. \quad (4.1)$$

Using (4.1), we get

$$\begin{aligned} \frac{1}{p(p-1)} \frac{d}{dt} \|u\|_p^p &\leq \frac{4}{p^2} \|u\|_p^{p/2} \|\nabla u^{p/2}\|_2 \\ &\quad \times \left(-C\varepsilon M^{-p/N(p-1)} \|u\|_p^{p/N(p-1)} + \frac{pM}{2} \|\nabla K\|_\infty \right) \\ &\leq \frac{2M}{p} \|\nabla K\|_\infty \|u\|_p^{p/2} \|\nabla u^{p/2}\|_2 \\ &\quad \times \left(-\varepsilon C(p)^{-p/N(p-1)} \|u\|_p^{p/N(p-1)} M^{-(N(p-1)+p)/N(p-1)} + 1 \right) \end{aligned} \quad (4.2)$$

Let us show that the inequality (4.2) implies the estimate

$$\|u(t)\|_p \leq U_p \equiv \max \left\{ \|u_0\|_p, C(p) M^{(N(p-1)+p)/p} \varepsilon^{-N(p-1)/p} \right\} \quad \text{for all } t > 0. \quad (4.3)$$

Indeed, for $\delta > 0$, consider the set

$$A_\delta = \{t \geq 0 : \|u(t)\|_p \leq U_p + \delta\}.$$

Clearly, $0 \in A_\delta$ and the time continuity of u in $L^p(\mathbb{R}^N)$ ensures that

$$\tau_\delta := \sup\{t \geq 0 : [0, t] \subset A_\delta\} \in (0, \infty].$$

Assume now for contradiction that $\tau_\delta < \infty$. On the one hand, the definition of τ_δ implies that

$$\|u(\tau_\delta)\|_p^p = (U_p + \delta)^p \geq \|u(t)\|_p^p \quad \text{for all } t \in (0, \tau_\delta). \quad (4.4)$$

Hence,

$$\frac{d}{dt} \|u(\tau_\delta)\|_p^p \geq 0. \quad (4.5)$$

On the other hand, we infer from (4.2) and (4.4) that

$$\begin{aligned} \frac{d}{dt} \|u(\tau_\delta)\|_p^p &\leq \frac{2M}{p} \|\nabla K\|_\infty \|u(\tau_\delta)\|_p^{p/2} \|\nabla u^{p/2}(\tau_\delta)\|_2 \left(-\|u(\tau_\delta)\|_p^{p/N(p-1)} U_p^{-p/N(p-1)} + 1 \right) \\ &= \frac{2M}{p} \|\nabla K\|_\infty \|u(\tau_\delta)\|_p^{p/2} \|\nabla u^{p/2}(\tau_\delta)\|_2 \left(-\left(\frac{U_p + \delta}{U_p} \right)^{p/N(p-1)} + 1 \right) < 0, \end{aligned}$$

which contradicts (4.5). Consequently, $\tau_\delta = \infty$ and $A_\delta = [0, \infty)$ for all $\delta > 0$. Letting $\delta \rightarrow 0$ completes the proof of (4.3).

The case $1 < p < 2$ follows then from Hölder's inequality and the case $p = 2$. Indeed, by (4.3) with $p = 2$,

$$\begin{aligned} \|u(t)\|_p &\leq M^{(2-p)/p} \|u(t)\|_2^{(2p-2)/p} \\ &\leq M^{(2-p)/p} \max \left\{ \|u_0\|_2, C(2)M^{(N+2)/2} \varepsilon^{-N/2} \right\}^{2(p-1)/p} \\ &\leq \max \left\{ M^{(2-p)/p} \|u_0\|_2^{2(p-1)/2}, C(p)M^{(N(p-1)+p)/p} \varepsilon^{-N(p-1)/p} \right\} \\ &\leq \max \left\{ M, \|u_0\|_2, C(p)M^{(N(p-1)+p)/p} \varepsilon^{-N(p-1)/p} \right\}, \end{aligned}$$

as claimed. \square

In the one-dimensional case, we need an analogous estimate for a Sobolev norm. Let us first note a crucial property of the interaction kernel.

Lemma 4.2. *If $K : \mathbb{R} \rightarrow \mathbb{R}$ satisfies assumptions (2.4)–(2.6), then for each $v \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ it follows that*

$$(K' * v)_x = -2\kappa_0 v + k''(|\cdot|) * v. \quad (4.6)$$

We skip the elementary proof of this result which is related to the fact that K' has a jump of size $2\kappa_0$ at the origin.

Lemma 4.3. *Let u be a (not necessarily even) nonnegative solution to problem (2.1)–(2.2) with $N = 1$, with $\varepsilon > 0$ and with initial condition u_0 such as in (2.3). Suppose moreover that $u_0 \in H^1(\mathbb{R})$ and $K' \in L^\infty(\mathbb{R})$ has the property (4.6). Then, the following inequality holds true:*

$$\|u(t)\|_{\dot{H}^1} \leq \max \left\{ \|u_0\|_{\dot{H}^1}, C_1 M^{5/2} \varepsilon^{-3/2} \right\} \quad \text{for all } t \geq 0.$$

Proof. Integrating by parts and using Lemma 4.2 we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{\dot{H}^1}^2 &= -\varepsilon \|u\|_{\dot{H}^2}^2 - \int_{\mathbb{R}} u_x (u K' * u)_{xx} dx \\ &= -\varepsilon \|u\|_{\dot{H}^2}^2 - \int_{\mathbb{R}} u_x u_{xx} (K' * u) dx - 2 \int_{\mathbb{R}} u_x^2 (K' * u)_x dx - \int_{\mathbb{R}} u u_x (K' * u_x)_x dx \\ &= -\varepsilon \|u\|_{\dot{H}^2}^2 - \int_{\mathbb{R}} u u_x (K' * u_x)_x dx - \frac{3}{2} \int_{\mathbb{R}} u_x^2 (K' * u)_x dx \\ &= -\varepsilon \|u\|_{\dot{H}^2}^2 + \underbrace{5\kappa_0 \int_{\mathbb{R}} u u_x^2 dx}_A - \underbrace{\int_{\mathbb{R}} u u_x (k''(|\cdot|) * u_x) dx}_B - \underbrace{\frac{3}{2} \int_{\mathbb{R}} u_x^2 (k''(|\cdot|) * u) dx}_E. \end{aligned}$$

Using the Hölder and the Gagliardo–Nirenberg inequalities, we get

$$|A| \leq M \|u\|_{\dot{W}^{1,\infty}}^2 \leq C M \left(M^{1/5} \|u\|_{\dot{H}^2}^{4/5} \right)^2 \leq C M^{7/5} \|u\|_{\dot{H}^2}^{8/5},$$

and then applying, moreover, the Young inequality,

$$\begin{aligned} |B| &\leq \|u\|_2 \|u\|_{\dot{H}^1} \|k''(|\cdot|)\|_1 \|u_x\|_\infty \\ &\leq C \left(M^{4/5} \|u\|_{\dot{H}^2}^{1/5} \right) \left(M^{2/5} \|u\|_{\dot{H}^2}^{3/5} \right) \left(M^{1/5} \|u\|_{\dot{H}^2}^{4/5} \right) \\ &\leq C M^{7/5} \|u\|_{\dot{H}^2}^{8/5}, \end{aligned}$$

and

$$|E| \leq \|u\|_{\dot{H}^1}^2 \|k''(|\cdot|)\|_1 \|u\|_\infty \leq C \left(M^{2/5} \|u\|_{\dot{H}^2}^{3/5} \right)^2 \left(M^{3/5} \|u\|_{\dot{H}^2}^{2/5} \right) \leq C M^{7/5} \|u\|_{\dot{H}^2}^{8/5}.$$

Consequently, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u\|_{\dot{H}^1}^2 \leq -\varepsilon \|u\|_{\dot{H}^2}^2 + C M^{7/5} \|u\|_{\dot{H}^2}^{8/5}. \quad (4.7)$$

By the Gagliardo–Nirenberg inequality, we have $\|u\|_{\dot{H}^1} \leq C M^{2/5} \|u\|_{\dot{H}^2}^{3/5}$, which implies that

$$\|u\|_{\dot{H}^2} \geq C \|u\|_{\dot{H}^1}^{5/3} M^{-2/3}. \quad (4.8)$$

Consequently, inequalities (4.7)–(4.8) yield

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{\dot{H}^1}^2 &\leq \|u\|_{\dot{H}^2}^{8/5} \left(-C\varepsilon \|u\|_{\dot{H}^1}^{2/3} M^{-4/15} + CM^{7/5} \right) \\ &\leq C \|u\|_{\dot{H}^2}^{8/5} M^{7/5} \left(-C\varepsilon \|u\|_{\dot{H}^1}^{2/3} M^{-5/3} + 1 \right). \end{aligned}$$

Now we conclude as for the proof of Lemma 4.1. \square

5. Proofs of main results

Now we prove the results stated in Section 2

Proof of Theorem 2.3. *The case $N \geq 2$.* We recall that Λ is a number satisfying the assumption (2.8), which allows us to use Theorem 3.1. Below, we consider a parameter $\lambda > 0$; its value will be specified later.

For arbitrary $T > 0$, using twice the Hölder inequality, we get

$$\begin{aligned} \int_0^T \int_{B_{\lambda\varepsilon}} \frac{u(x,t)}{|x|} dx dt &\leq \left(\int_0^T \int_{B_{\lambda\varepsilon}} \frac{1}{|x|^{(2N-1)/2}} dx dt \right)^{2/(2N-1)} \\ &\quad \times \left(\int_0^T \int_{B_{\lambda\varepsilon}} u(x,t)^{(2N-1)/(2N-3)} dx dt \right)^{(2N-3)/(2N-1)} \\ &\leq \left(2\sigma_N T \sqrt{\lambda\varepsilon} \right)^{2/(2N-1)} \left(\int_0^T \int_{B_{\lambda\varepsilon}} u(x,t)^{(2N+1)/(2N-3)} dx dt \right)^{(2N-3)/(4N-2)} \\ &\quad \times \left(\int_0^T \int_{B_{\lambda\varepsilon}} u(x,t) dx dt \right)^{(2N-3)/(4N-2)}. \end{aligned} \tag{5.1}$$

Hence, there exists $\varepsilon_* > 0$ depending on N , M , and u_0 such that for $0 < \varepsilon \leq \varepsilon_*$, by the L^p -estimates in Lemma 4.1,

$$\begin{aligned}
& \int_0^T \int_{B_{\lambda\varepsilon}} \frac{u(x,t)}{|x|} \, dx \, dt \\
& \leq \left(\sigma_N T \sqrt{\lambda\varepsilon} \right)^{2/(2N-1)} T^{(2N-3)/(4N-2)} \left(\sup_{t \in [0,T]} \|u(t)\|_{(2N+1)/(2N-3)} \right)^{(2N+1)/(4N-2)} \\
& \quad \times \left(\int_0^T \int_{B_{\lambda\varepsilon}} u(x,t) \, dx \, dt \right)^{(2N-3)/(4N-2)} \\
& \leq CT^{(2N+1)/(4N-2)} \lambda^{1/(2N-1)} \varepsilon^{1/(2N-1)} \varepsilon^{-4N(2N+1)/(4N-2)(2N+1)} \\
& \quad \times \left(\int_0^T \int_{B_{\lambda\varepsilon}} u(x,t) \, dx \, dt \right)^{(2N-3)/(4N-2)} \\
& = CT^{(2N+1)/(4N-2)} \lambda^{1/(2N-1)} \varepsilon^{-1} \left(\int_0^T \int_{B_{\lambda\varepsilon}} u(x,t) \, dx \, dt \right)^{(2N-3)/(4N-2)}.
\end{aligned} \tag{5.2}$$

On the other hand, by the mass conservation (2.7),

$$\begin{aligned}
\int_0^T \int_{B_{3\Lambda/2} \setminus B_{\lambda\varepsilon}} \frac{u(x,t)}{|x|} \, dx \, dt & \leq \frac{1}{\lambda\varepsilon} \int_0^T \int_{B_{3\Lambda/2} \setminus B_{\lambda\varepsilon}} u(x,t) \, dx \, dt \\
& \leq \frac{1}{\lambda\varepsilon} \int_0^T \|u(t)\|_1 \, dt = \frac{MT}{\lambda\varepsilon}.
\end{aligned} \tag{5.3}$$

Recalling the definition of the quantity $\mathcal{D}_\Lambda(u)$ in (3.1), we deduce from inequalities (5.1) and (5.3) that

$$\begin{aligned}
& \frac{\varepsilon}{N-1} \int_0^T \mathcal{D}_\Lambda(u(t)) \, dt \\
& \leq CT^{(2N+1)/(4N-2)} \lambda^{1/(2N-1)} \left(\int_0^T \int_{B_{\lambda\varepsilon}} u(x,t) \, dx \, dt \right)^{(2N-3)/(4N-2)} + \frac{MT}{\lambda}.
\end{aligned} \tag{5.4}$$

Next, we infer from Theorem 3.1, the positivity of ω_Λ , and inequality (5.4) with $T = T_\Lambda$ that

$$\frac{\Lambda \mathcal{L}_\Lambda}{N-1} - \frac{MT_\Lambda}{\lambda} \leq CT_\Lambda^{(2N+1)/(4N-2)} \lambda^{1/(2N-1)} \left(\int_0^{T_\Lambda} \int_{B_{\lambda\varepsilon}} u(x, t) \, dx \, dt \right)^{(2N-3)/(4N-2)}.$$

Finally, we choose

$$\lambda \equiv \frac{2(N-1)MT_\Lambda}{\Lambda \mathcal{L}_\Lambda} \quad (5.5)$$

to complete the proof of inequality (2.9) when $N \geq 2$.

The case $N = 1$. Again, we recall that Λ is a number satisfying the assumption (2.8), which allows us to use Theorem 3.1 and we consider a parameter $\lambda > 0$ to be specified later. It follows from Theorem 3.1 and the Cauchy–Schwarz inequality that

$$\begin{aligned} \int_0^{T_\Lambda} \int_{-\lambda\varepsilon}^{\lambda\varepsilon} u(x, t) \, dx \, dt &\geq \int_0^{T_\Lambda} \int_{-\lambda\varepsilon}^{\lambda\varepsilon} \left(u(0, t) + \int_0^x u_y(y, t) \, dy \right) \, dx \, dt \\ &\geq \int_0^{T_\Lambda} \left(2\lambda\varepsilon u(0, t) - 2(\lambda\varepsilon)^{3/2} \|u_x(t)\|_2 \right) \, dt \\ &= \lambda\varepsilon \left(\int_0^{T_\Lambda} \mathcal{D}_\Lambda(u(t)) \, dt - 2T_\Lambda \sqrt{\lambda\varepsilon} \sup_{t \in [0, T_\Lambda]} \|u(t)\|_{\dot{H}^1} \right) \\ &\geq \lambda \left(\Lambda \mathcal{L}_\Lambda - 2T_\Lambda \sqrt{\lambda\varepsilon}^{3/2} \sup_{t \in [0, T_\Lambda]} \|u(t)\|_{\dot{H}^1} \right). \end{aligned}$$

We infer from Lemma 4.3 that there exists $\varepsilon_* > 0$ such that for $0 < \varepsilon \leq \varepsilon_*$, $\|u(t)\|_{\dot{H}^1} \leq C_1 M^{5/2} \varepsilon^{-3/2}$, and the above inequality leads to

$$\int_0^{T_\Lambda} \int_{-\lambda\varepsilon}^{\lambda\varepsilon} u(x, t) \, dx \, dt \geq \lambda \left(\Lambda \mathcal{L}_\Lambda - 2C_1 T_\Lambda M^{5/2} \lambda^{1/2} \right).$$

We then complete the proof by choosing $\lambda \equiv (\Lambda \mathcal{L}_\Lambda / (4C_1 M^{5/2} T_\Lambda))^2$. \square

Next, we prove the lower estimates for Lebesgue norms.

Proof of Corollary 2.4. Let $p \in [1, \infty)$. By the Hölder inequality, Theorem 2.3, and Lemma 4.1,

$$C_* \leq \int_0^{T_*} \int_{B_{\lambda\varepsilon}} u(x, t) \, dx \, dt$$

$$\leq \left(\frac{\sigma_N}{N}\right)^{\frac{p-1}{p}} (\lambda\varepsilon)^{\frac{N(p-1)}{p}} \int_0^{T_*} \left(\int_{B_{\lambda\varepsilon}} u(x, t)^p dx \right)^{1/p} dt,$$

which proves Corollary 2.4 with $C_{**}(p) = C_* \left(\frac{\sigma_N}{N}\right)^{-\frac{p-1}{p}} \lambda^{-\frac{N(p-1)}{p}}$. \square

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