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Boundedness of solutions for equations with p-Laplacian and an asymmetric nonlinear term ☆

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Abstract

For $p>1$ and $\phi_p(s):=|s|^{p-2}s$, we are concerned with the boundedness of solutions for the equation

$$(\phi_p(x'))' + \alpha\phi_p(x^+) - \beta\phi_p(x^-) = f(t,x),$$

where $x^+ = \max(x,0)$, $x^- = \max(-x,0)$ and $f(t,x)$ is 2π-periodic in t. When

$$\frac{\pi_p}{\alpha^{1/p}} + \frac{\pi_p}{\beta^{1/p}} = \frac{2\pi}{n}$$

(the "resonant" situation) and f has limits $f_\pm(t)$ as $x\to\pm\infty$, there is a function $Z(\theta)$ plays a central role for the boundedness of solutions. More precisely, if $Z(\theta)$ is of constant sign, then all solutions are bounded. Moreover, such condition also guarantees the boundedness when (α,β) near a Fučik curve.

Keywords: Boundedness of solutions; p-Laplacian; Small twist theorem

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1. Introduction

In this paper, we are concerned with the boundedness of solutions (also called "Lagrangian stability") and the existence of quasi-periodic solutions and subharmonics for the following equation

$$(\phi_p(x'))' + \alpha\phi_p(x^+) - \beta\phi_p(x^-) = f(t, x), \tag{1.1}$$

where $x^+ = \max(x, 0)$, $x^- = \max(-x, 0)$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$ and α, β are strictly positive constants. We also assume that f is smoothness and 2π-periodic in t, and has limits $f_\pm(t)$ as $x \to \pm\infty$.

When $p = 2$, $\alpha = \beta = n^2$ and $f(t, x) = p(t) - \psi(x)$, Eq. (1.1) reduces to

$$x'' + n^2x + \psi(x) = p(t). \tag{1.2}$$

Lazer–Leach [10] proved that, if

$$2(\psi(+\infty) - \psi(-\infty)) > \left| \int_0^{2\pi} p(t)e^{\mathrm{int}}\, dt \right|, \tag{1.3}$$

then Eq. (1.2) has 2π-periodic solutions. In addition, if it is assumed that $\psi(-\infty) \leqslant \psi(x) \leqslant \psi(+\infty)$, (1.3) is also a necessary condition for the existence of 2π-periodic solutions.

Motivated by the seminal works of Dancer [3], Fucik [9], Lazer–Mckenna [11], there are several authors study the existence of 2π-periodic solutions for the equation

$$x'' + \alpha x^+ - \beta x^- + \psi(x) = p(t), \tag{1.4}$$

that is, $p = 2$, $f(t, x) = p(t) - \psi(x)$ for Eq. (1.1).

When $1/\sqrt{\alpha} + 1/\sqrt{\beta} = \frac{2}{n} \in \mathbf{Q}$, the function

$$\Xi(\theta) = \frac{n}{\pi}\left(\frac{\psi(+\infty)}{\alpha} - \frac{\psi(-\infty)}{\beta}\right) - \frac{1}{2\pi}\int_0^{2\pi} p(f)S(t+\theta)\, dt \tag{1.5}$$

plays a role for the existence of 2π-periodic solutions, where S is the solution of $x'' + \alpha x^+ - \beta x^- = 0$ satisfying the initial condition $x(0) = 0$, $x'(0) = 1$. For example, Fabry–Mawhin [8] proved that if Ξ has $2s(s \neq 1)$ zeros, all being simple, then there is at least one 2π-periodic solution. Moreover, if $s \geqslant 2$, then all solutions of (1.4) with large initial conditions are unbounded. For related results, we refer [6,5,2] and references therein. More recently, Fabry–Manásevich [7] generalize the above results to (1.1). They prove that if the function

$$Z(\theta) = \int_{\{t\in[0,2\pi/n]|v(t)>0\}} f_+(t+\theta)v(t)\,dt + \int_{\{t\in[0,2\pi/n]|v(t)<0\}} f_-(t+\theta)v(t)\,dt \quad (1.6)$$

has $2s(s \neq 1)$ zeros, all being simple, then there is at least one 2π-periodic solution. Here, the function v is the solution of the equation

$$(\phi_p(x'))' + \alpha\phi_p(x^+) - \beta\phi_p(x^-) = 0$$

with the initial condition: $(v(0), v'(0)) = (0, 1)$.

Another interesting question on (1.1) is the one concerning the boundedness of all the solutions, and if the above-mentioned functions $\Xi(\theta)$ in (1.5) and $Z(\theta)$ in (1.6) play a role for the boundedness problem. For Eq. (1.2), we proved in [13] that, if p is smooth and ψ satisfies (1.3) and other reasonable conditions, then all the solutions are bounded. Wang [19] generalizes this result to Eq. (1.4). More precisely, she proves that if $\Xi(\theta)$ is of constant sign, then every solution is bounded.

It has to remark that the first result for the boundedness problem of semilinear Duffing equations is due to Ortega [16]. In that paper, he proved the boundedness of solutions for the equation

$$x'' + \alpha x^+ - \beta x^- = 1 + \varepsilon p(t)$$

if $|\varepsilon| \ll 1$. He also studies the same problem for Eq. (1.2). In [17], he proved that all the solutions are bounded when ψ is a piecewise linear function. Moreover, he gives a variant of Moser twist theorem, which becomes a basic tool for studying Lagrange stability of semilinear equations at resonance. Indeed, the proofs in [13,19] are based on Ortega's result. In the present paper, we also use his result to prove the boundedness of all solutions as well as the existence of quasi-periodic solutions for Eq. (1.1). This question are suggested by Fabry–Manásevich [7].

In order to state our main result, we first give some notations. Denoted by $F(t,x)$ with $F(t,0) = 0$ the integral of $f(t,x)$, that is,

$$F(t,x) = \int_0^x f(t,s)\,ds.$$

Let

$$\pi_p = 2(p-1)^{\frac{1}{p}} \int_0^1 (1-s^p)^{-\frac{1}{p}}\,ds \left(= \frac{\pi/p}{\sin \pi/p}\right) > 0.$$

It is not difficult to verify that every non-trivial solution of the following equation:

$$(\phi_p(x'))' + \alpha\phi_p(x^+) - \beta\phi_p(x^-) = 0$$

is T_0-periodic, where

$$T_0 = \frac{\pi_p}{\alpha^{1/p}} + \frac{\pi_p}{\beta^{1/p}}.$$

We are interested in the situation of "resonance", that is we assume that there is a integer n such that $T_0 = 2\pi/n$. This means (α, β) is at a Fučik curve.

The main result is

Theorem 1. *We suppose that the function f satisfies the following assumptions*:

(1) $f(t,x) \in C^{7,6}(\mathbf{S}^1 \times \mathbf{R})$ *and has limits*

$$\lim_{x\to\pm\infty} f(t.x) = f_\pm(t), \quad \textit{uniformly in } t$$

(2) *The following limits exist and uniformly in t*:

$$\lim_{x\to\pm\infty} x^m \frac{\partial^{n+m}}{\partial t^n \partial x^m} f(t,x) = f_{\pm,m,n}(t)$$

for $(n,m) = (0,6), (7,0)$ *and* $(7,6)$. *Moreover,* $f_{\pm,m,n}(t) \equiv 0$ *for* $m = 6,\ n = 0, 7$.

If the function $Z(\theta)$ *defined by* (1.6) *is of constant sign, then all the solutions of* (1.1) *are defined in* $\mathbf{R}$ *and for each solution x(t), we have*

$$\sup_{t\in\mathbf{R}}(|x(t)| + |x'(t)|) < +\infty.$$

Moreover, in this case, there are infinitely many subharmonic solutions and quasi-periodic solutions.

Remark. If the function $Z(\theta)$ has zeros, all being simple, then the solutions of (1.1) with large initial data are unbounded either in the future or in the past. The proof of this statement is similar to one in [1]. It is easy to construct functions f satisfying the assumptions (1) and (2) in Theorem 1. For examples, $f(t,x) = (2 + \sin t)$ arctan x and $f(t,x) = \arctan x - p(t)$.

We will also study the situation near resonance, that is equation type (1.1) with (α, β) is of the form

$$\alpha = \alpha_0 + \delta\alpha_1, \quad \beta = \beta_0 + \delta\beta_1,$$

and

$$\frac{\pi_p}{\alpha_0^{1/p}} + \frac{\pi_p}{\beta_0^{1/p}} = \frac{2\pi}{n}$$

and δ a small parameter. Denote by v_0 is the solution of the equation

$$(\phi_p(x'))' + \alpha_0\phi_p(x^+) - \beta_0\phi_p(x^-) = 0,$$

satisfying $v_0(0) = 0$, $v_0'(0) = 1$. Let

$$\begin{aligned} Z_0(\theta) = & \int_{\{t\in[0,2\pi/n]|v_0(t)>0\}} f_+(t+\theta)v_0(t)\,dt \\ & + \int_{\{t\in[0,2\pi/n]|v_0(t)<0\}} f_-(t+\theta)v_0(t)\,dt. \end{aligned} \tag{1.7}$$

In this case, the assumption on the constant sign of $Z_0(\theta)$ is also a sufficiently condition to guarantee the boundedness of solutions. However, if

$$\int_0^{2\pi} (\alpha_1|v_0^+|^p + \beta_1|v_0^-|^p)\,dt \neq 0, \tag{1.8}$$

then for almost all δ (in the sense of Lebesgue measure), the following assumption:

$$\alpha_0^{-2/p}\int_0^{2\pi} f_+(t)\,dt - \beta_0^{-2/p}\int_0^{2\pi} f_-(t)\,dt \neq 0 \tag{1.9}$$

may guarantee the boundedness of solutions for Eq. (1.1). More precisely, we have

Theorem 2. *Suppose that our assumptions on $f(t,x)$ in Theorem 1 hold. Then the following conclusions are true*:

(1) *If Z_0 in* (1.6) *is of constant sign, then there is $\delta_0 > 0$ such that for all $|\delta| \leqslant \delta_0$, every solution of Eq.* (1.1) *is bounded*.
(2) *If* (1.8), (1.9) *hold, then there is a $\delta_0 > 0$ and a set $\Lambda \subset (0, \delta_0)$ with* $\operatorname{meas}\Lambda = \delta_0$ *such that for $\delta \in \Lambda$, all solutions of Eq.* (1.1) *are bounded.*

Remark. In the following, without loss of generality and for brevity, we assume that $n = 1$, i.e.,

$$T_0 = \frac{\pi_p}{\alpha^{1/p}} + \frac{\pi_p}{\beta^{1/p}} = 2\pi.$$

Throughout this paper, we denote by $C > 1$, a universal positive constant not concerning its quantity.

2. Action-angle variables and some lemmas

In this section, we first introduce the action and angle variables (r, θ) by a symplectic transformation, and then give some technical lemmas which will be used frequently in the next sections.

Consider an auxiliary equation

$$(\phi_p(x'))' + \alpha\phi_p(x^+) - \beta\phi_p(x^-) = 0.$$

Let $v(t)$ be the solution with initial condition: $(v(0), v'(0)) = (0, 1)$. Setting $\phi_p(v') = u$, then (v, u) is a solution of the following planar system:

$$x' = \phi_q(y), \quad y' = -\alpha\phi_p(x^+) + \beta\phi_p(x^-),$$

where $q = p/(p-1) > 1$. It is not difficult to prove that

(i) $q^{-1}|u|^q + p^{-1}(\alpha|v^+|^p + \beta|v^-|^p) \equiv q^{-1}$;
(ii) $v(t)$ and $u(t)$ are 2π-periodic functions.
(iii) $v(t) > 0$ for $t \in (0, \pi_p/\alpha^{1/p})$; $v(t) < 0$ for $t \in (\pi_p/\alpha^{1/p}, 2\pi)$.

Obviously, the Eq. (1.1) is equivalent to the system

$$x' = \phi_q(y), \quad y' = -\alpha\phi_p(x^+) + \beta\phi_p(x^-) + f(t, x), \tag{2.1}$$

which is a Hamiltonian system with Hamiltonian function

$$H(x, y, t) = \frac{1}{q}|y|^q + \frac{1}{p}(\alpha|x^+|^p + \beta|x^-|^p) - F(t, x). \tag{2.2}$$

We introduce the action and angle variables via the solution $(v(t), u(t))$ as follows.

$$x = r^{\frac{1}{p}} v(\theta), \quad y = r^{\frac{1}{q}} u(\theta).$$

This transformation is called a generalized symplectic transformation as its Jacobian is q^{-1} instead of 1. Under this transformation, the system (2.1) is changed to

$$\theta' = \frac{\partial h}{\partial r}(r, \theta, t), \quad r' = \frac{\partial h}{\partial \theta}(r, \theta, t) \tag{2.3}$$

with the Hamiltonian function

$$h(r, \theta, t) = r - qF(t, r^{\frac{1}{p}} v(\theta)). \tag{2.4}$$

Note that this function is smooth in r, t and continuous in θ

In the following, we state several lemmas which will be used in the rest of this paper.

Lemma 2.1. *Under the assumptions of Theorem* 1, *we have*

$$\lim_{x\to\pm\infty} x^m \frac{\partial^{n+m}}{\partial t^n \partial x^m} f(t,x) = f_{\pm,m,n}(t) \quad \textit{uniformly in } t$$

for $0\leqslant m\leqslant 6,\ 0\leqslant n\leqslant 7$. *Moreover,*

$$f_{\pm,m,n}(t) = 0 \quad \textit{for } 1\leqslant m\leqslant 6;$$

$$f_{\pm,0,n}(t) = f_{\pm}^{(n)}(t) \quad \textit{for } 1\leqslant n\leqslant 7.$$

Proof. We prove the statement for $x \to +\infty$ only, the case of $x \to -\infty$ can be treated similarly.

First, we prove that there are 2π-periodic functions $f_{+,0,n}(\cdot)$ $(1\leqslant n\leqslant 6)$ such that

$$\lim_{x\to+\infty} \frac{\partial^n}{\partial t^n} f(t,x) = f_{+,0,n}(t) \quad \text{uniformly in } t. \tag{2.5}$$

From the equality

$$\frac{\partial^6}{\partial t^6} f(t,x) - \frac{\partial^6}{\partial t^6} f(0,x) = \int_0^t \frac{\partial^7}{\partial t^7} f(s,x)\,ds,$$

it follows that

$$\lim_{x\to+\infty}\left[\frac{\partial^6}{\partial t^6} f(t,x) - \frac{\partial^6}{\partial t^6} f(0,x)\right] = \int_0^t f_{+,0,7}(s)\,ds \quad \text{uniformly in } t.$$

Therefore

$$\lim_{x\to+\infty}\int_0^{2\pi}\left[\frac{\partial^6}{\partial t^6} f(t,x) - \frac{\partial^6}{\partial t^6} f(0,x)\right]dt = \int_0^{2\pi}\int_0^t f_{+,0,7}(s)\,ds\,dt.$$

That is

$$\lim_{x\to+\infty} \frac{\partial^6}{\partial t^6} f(0,x) = -\frac{1}{2\pi}\int_0^{2\pi}\int_0^t f_{+,0,7}(s)\,ds\,dt.$$

Let

$$f_{+,0,6}(t) = \int_0^t f_{+,0,7}(s)\,ds - \frac{1}{2\pi}\int_0^{2\pi}\int_0^t f_{+,0,7}(s)\,ds\,dt.$$

Then

$$\lim_{x\to+\infty} \frac{\partial^6}{\partial t^6} f(t,x) = f_{+,0,6}(t) \quad \text{uniformly in } t. \tag{2.6}$$

Moreover,

$$\frac{d}{dt} f_{+,0,6}(t) = f_{+,0,7}(t).$$

The periodicity of $f_{+,0,6}$ follows from the periodicity of f and (2.6).

Using the same arguments, one can prove that

$$\lim_{x\to+\infty} \frac{\partial^n}{\partial t^n} f(t,x) = f_{+,0,n}(t) \quad \text{uniformly in } t$$

and

$$\frac{d}{dt} f_{+,0,n}(t) = f_{+,0,n+1}(t)$$

for $1\leqslant n\leqslant 6$. Moreover, it is easy to check that $f_{+,0,1}(t) = f_+'(t)$.

Applying this result to the function $x^6\frac{\partial^6}{\partial x^6}f(t,x)$, we know that there are functions $f_{+,6,n}(\cdot)$, $(1\leqslant n\leqslant 6)$ such that

$$\lim_{x\to+\infty} x^6\frac{\partial^{n+6}}{\partial t^n\partial x^6} f(t,x) = f_{+,6,n}(t) \quad \text{uniformly in } t.$$

Now we turn to prove that

$$\lim_{x\to+\infty} x^m\frac{\partial^{m+n}}{\partial x^m\partial t^n} f(t,x) = 0 \quad \text{uniformly in } t \tag{2.7}$$

provided that

$$\lim_{x\to+\infty} x^{m+1}\frac{\partial^{m+n+1}}{\partial x^{m+1}\partial t^n} f(t,x) = 0 \quad \text{uniformly in } t \tag{2.8}$$

for $m\geqslant 1$.

From (2.8) we know that there is $A > 0$ such that for $x \geqslant A$,

$$\left| \frac{\partial^{m+n+1}}{\partial x^{m+1} \partial t^n} f(t, x) \right| < \frac{1}{x^{m+1}},$$

which yields that

$$\left| \frac{\partial^{m+n}}{\partial x^m \partial t^n} f(t, x_1) - \frac{\partial^{m+n}}{\partial x^m \partial t^n} f(t, x_2) \right| \leqslant \frac{1}{m} \left(\frac{1}{x_1^m} + \frac{1}{x_2^m} \right).$$

Hence, the limit of $\frac{\partial^{m+n}}{\partial x^m \partial t^n} f(t, x)$ as $x \to +\infty$ exists and uniformly in t. Moreover, from the assumption (1) of Theorem 1, we know that $f(t, x)$ is bounded, so

$$\lim_{x \to +\infty} \frac{\partial^{m+n}}{\partial x^m \partial t^n} f(t, x) = 0 \quad \text{uniformly in } t.$$

By the rule of de L'Hopital, we have

$$\lim_{x \to +\infty} x^m \frac{\partial^{m+n}}{\partial x^m \partial t^n} f(t, x) = -\frac{1}{m} \lim_{x \to +\infty} x^{m+1} \frac{\partial^{m+n+1}}{\partial x^{m+1} \partial t^n} f(t, x) = 0 \quad \text{uniformly in } t.$$

This completes the proof of this lemma. □

From this lemma, it follows that

$$\lim_{x \to \pm\infty} x^{k-1} \frac{\partial^{k+n}}{\partial t^n \partial x^k} F(t, x) = f_{\pm, k-1, n}(t) \tag{2.9}$$

uniformly in t, for $k \leqslant 7$, $n \leqslant 7$. We denote $f_{\pm,-1,n}(t) := f_{\pm,0,n}(t)$.

From (2.4), we know that

$$\frac{\partial h(r, \theta, t)}{\partial r} = 1 - \frac{q}{p} f(t, r^{\frac{1}{p}} v(\theta)) r^{\frac{1}{p}-1} v(\theta) \to 1 \quad \text{as } r \to +\infty$$

because f is bounded and $p > 1$. Hence, one can solve from (2.4) that $r = r(h, t, \theta)$ by Implicit Function Theorem for $h \gg 1$. Moreover, this function can be written in the form

$$r(h, t, \theta) = h + R(h, t, \theta), \tag{2.10}$$

where the function R is defined implicitly by

$$R(h,t,\theta) = qF(t,(h+R)^{\frac{1}{p}}v(\theta)).$$

Similar to the proof of Lemma 3.3 in [13], we know that there is a constant $C>0$ such that

$$\left| h^{k-\frac{1}{p}} \frac{\partial^{k+l}}{\partial h^k \partial t^l} R(h,t,\theta) \right| \leqslant C \tag{2.11}$$

for $0 \leqslant k \leqslant 7$, $0 \leqslant l \leqslant 7$.

The Hamiltonian system determined by the function r is

$$\frac{dh}{d\theta} = -\frac{\partial r}{\partial t}(h,t,\theta), \qquad \frac{dt}{d\theta} = \frac{\partial r}{\partial h}(h,t,\theta). \tag{2.12}$$

The relation between (2.3) and (2.12) is that if $(r(t),\theta(t))$ is a solution of (2.3) and the inverse function $t(\theta)$ of $\theta(t)$ exists, then $(h(r(t(\theta))),t(\theta))$ is a solution of (2.12) and vice versa. So in order to find quasi-periodic solutions of (2.3) and to obtain the boundedness of solutions, it is sufficient to prove the existence of quasi-periodic solutions and boundedness of solutions of (2.12).

Let $\phi_1(h,t,\theta) = (h+R)^{\frac{1}{p}} - h^{\frac{1}{p}}$. Then

$$\left| h^{k+1-\frac{2}{p}} \frac{\partial^{k+l}}{\partial h^k \partial t^l} \phi_1(h,t,\theta) \right| \leqslant C \quad \text{uniformly in } (t,\theta) \in [0,2\pi]\times[0,2\pi] \tag{2.13}$$

for $0 \leqslant k \leqslant 7$, $0 \leqslant l \leqslant 7$, where C is a positive constant. The proof follows from (2.11) and the following equality:

$$\phi_1(h,t,\theta) = \frac{1}{p}\int_0^1 (h+sR)^{\frac{1}{p}-1} R\, ds.$$

Let $\phi_2(h,t,\theta;s) = s(h+R)^{\frac{1}{p}} + (1-s)h^{\frac{1}{p}}$ for $1 \leqslant s \leqslant 1$. Then

$$\left| h^{k-\frac{1}{p}} \frac{\partial^{k+l}}{\partial h^k \partial t^l} \phi_2(h,t,\theta;s) \right| \leqslant C, \tag{2.14}$$

for $0 \leqslant k \leqslant 7$, $0 \leqslant l \leqslant 7$, where C is a positive constant. Moreover

$$\frac{1}{2}h^{\frac{1}{p}} \leqslant \phi_2(h,t,\theta;s) \leqslant 2h^{\frac{1}{p}} \quad \text{for } h \gg 1 \tag{2.15}$$

uniformly in $(s,t,\theta) \in [0,1]\times[0,2\pi]\times[0,2\pi]$.

Let

$$\Phi(h,t,\theta) = qF(t,(h+R)^{\frac{1}{p}}v(\theta)) - qF(t,h^{\frac{1}{p}}v(\theta)).$$

Combining the above inequalities, we have

Lemma 2.2. *Under the assumptions of Theorem* 1, *we have, for* $0 \leqslant k \leqslant 7$, $0 \leqslant l \leqslant 7$,

$$\left| h^{k+1-\frac{2}{p}} \frac{\partial^{k+l}}{\partial h^k \partial t^l} \Phi(h,t,\theta) \right| \leqslant C \quad uniformly\ in\ (t,\theta) \in [0,2\pi] \times [0,2\pi] \tag{2.16}$$

for some positive constant C.

Proof. From (2.14) and (2.15), it follows that, for $h \gg 1$,

$$\left| h^k \frac{\partial^{k+l}}{\partial h^k \partial t^l} \phi_2(h,t,\theta) \right| \leqslant C\phi_2(h,t,\theta) \tag{2.17}$$

for $0 \leqslant k \leqslant 7$, $0 \leqslant l \leqslant 7$, where C is a positive constant. From a direct computation, it follows that

$$\frac{\partial^{k+l}}{\partial h^k \partial t^l} f(t,\phi_2 v(\theta)) = \sum \frac{\partial^{m+n} f(t,\phi_2 v(\theta))}{\partial x^m \partial t^n} \frac{\partial^{k_1+l_1}}{\partial h^{k_1} \partial t^{l_1}} \phi_2 \cdot \cdots \cdot \frac{\partial^{k_m+l_m}}{\partial h^{k_m} \partial t^{l_m}} \phi_2 v^m(\theta),$$

where $m \leqslant k$, $n \leqslant l$, and $k_1 + \cdots + k_m = k$, $l_1 + \cdots + l_m = l - n$. From this expression, (2.17) and the assumptions on f, we have

$$\left| \frac{\partial^{k+l}}{\partial h^k \partial t^l} f(t,\phi_2(h,t,\theta)v(\theta)) \right| \leqslant C.$$

Since

$$\Phi(h,t,\theta) = \int_0^1 f(t,\phi_2(h,t,\theta)v(\theta))\phi_1(h,t,\theta)v(\theta)\,ds,$$

the proof of this lemma is completed by a direct computation combined with (2.13). $\square$

Now the Hamiltonian function r defined in (2.10) can be written in the form

$$r(h,t,\theta) = h + qF(t,h^{\frac{1}{p}}v(\theta)) + \Phi(h,t,\theta). \tag{2.18}$$

In the next section, we will give an expression of the Poincaré map of (2.12), and then prove Theorem 1 via the variant of small twist theorem [17].

3. The Proof of Theorem 1

3.1. An expression of the Poincaré map of (2.12)

Introduce a new action variable ρ and a small positive parameter ε as follows

$$h = \varepsilon^{-1}\rho, \quad \rho \in [1,2].$$

Obviously, $h \gg 1 \Leftrightarrow 0 < \varepsilon \ll 1$. The Hamiltonian system (2.12) is changed to the form

$$\frac{dt}{d\theta} = \frac{\partial H}{\partial \rho}(\rho, t, \theta; \varepsilon), \quad \frac{d\rho}{d\theta} = -\frac{\partial H}{\partial t}(\rho, t, \theta; \varepsilon), \tag{3.1}$$

where

$$H(\rho, t, \theta; \varepsilon) = \rho + q\varepsilon F(t, \varepsilon^{-\frac{1}{p}}\rho^{\frac{1}{p}}v(\theta)) + \varepsilon\Phi(\varepsilon^{-1}\rho, t, \theta). \tag{3.2}$$

Definition 3.1. We say a function $g(\rho, t, \theta; \varepsilon) \in O_k(1)$ if g is smooth in (ρ, t) and

$$\left| \frac{\partial^{k_1+k_2}}{\partial t^{k_1}\partial \rho^{k_2}} g(\rho, t, \theta; \varepsilon) \right| \leqslant C,$$

for some constant $C > 0$ which is independent of the arguments $\rho, t, \theta, \varepsilon$, where $k_1 + k_2 \leqslant k$. Similarly, we say a function $g(\rho, t, \theta; \varepsilon) \in o_k(1)$ if g is smooth in (ρ, t) and

$$\lim_{\varepsilon \to 0} \left| \frac{\partial^{k_1+k_2}}{\partial t^{k_1}\partial \rho^{k_2}} g(\rho, t, \theta; \varepsilon) \right| = 0 \quad \text{uniformly in } (\rho, t, \theta),$$

where $k_1 + k_2 \leqslant k$.

From Lemma 2.2 and (2.18), we know that the system (2.12) is of the form

$$\begin{cases} \frac{dt}{d\theta} = 1 + \frac{q}{p}\varepsilon^{\frac{1}{q}} f(t, \varepsilon^{-\frac{1}{p}}\rho^{\frac{1}{p}}v(\theta))\rho^{-\frac{1}{q}}v(\theta) + \varepsilon^{\frac{2}{q}}O_6(1), \\ \frac{d\rho}{d\theta} = -q\varepsilon F_t(t, \varepsilon^{-\frac{1}{p}}\rho^{\frac{1}{p}}v(\theta)) + \varepsilon^{\frac{2}{q}}O_6(1). \end{cases} \tag{3.3}$$

By our assumptions on f, it follows that

$$\frac{q}{p}\varepsilon^{\frac{1}{q}} f(t, \varepsilon^{-\frac{1}{p}}\rho^{\frac{1}{p}} v(\theta))\rho^{-\frac{1}{q}} v(\theta), \quad q\varepsilon F_t(t, \varepsilon^{-\frac{1}{p}}\rho^{\frac{1}{p}} v(\theta)) \in \varepsilon^{\frac{1}{q}} O_6(1),$$

which yield that, for every $(t_0, \rho_0) \in [0, 2\pi]\times[1, 2]$, the solution $(t(\theta, t_0, \rho_0), \rho(\theta, t_0, \rho_0))$ of (3.3) has the form

$$t = t_0 + \theta + \varepsilon^{\frac{1}{q}} O_6(1), \quad \rho = \rho_0 + \varepsilon^{\frac{1}{q}} O_6(1).$$

From this formula, we make the ansatz that the solution $(t(\theta, t_0, \rho_0), \rho(\theta, t_0, \rho_0))$ has the following expression:

$$t = t_0 + \theta + \varepsilon^{\frac{1}{q}} T_1(\theta, t_0, \rho_0; \varepsilon), \quad \rho = \rho_0 + \varepsilon^{\frac{1}{q}} T_2(\theta, t_0, \rho_0; \varepsilon). \tag{3.4}$$

Denote by P the Poincaré map of (3.2). Then

$$P(t_0, \rho_0) = (t_0 + 2\pi + \varepsilon^{\frac{1}{q}} T_1(2\pi, t_0, \rho_0; \varepsilon), \quad \rho_0 + \varepsilon^{\frac{1}{q}} T_2(2\pi, t_0, \rho_0; \varepsilon)).$$

From the above discussions, we know that if $\varepsilon \ll 1$, this map is well defined in the region $[\frac{1}{2}, 2] \times [0, 2\pi]$.

If one can prove that for every $\varepsilon \ll 1$ the map P has an invariant curve which is diffeomorphic to $\rho_0 = \text{const.}$, then boundedness of solutions of (1.1) follows from the standard arguments. (one may find such a discussion in [4,13,17].) In order to prove the existence of such invariant curves for every $\varepsilon \ll 1$, it suffices to verify that for every $\varepsilon \ll 1$, the Poincaré map P satisfies all the assumptions of a variant of Moser's small twist theorem which is due to Ortega [17]. In the rest of this part, we will give an expression for $(T_1(2\pi, t_0, \rho_0),\ T_2(2\pi, t_0, \rho_0))$.

From (3.2) and (3.4), the functions T_1 and T_2 satisfy

$$\begin{cases} \frac{dT_1}{d\theta} = \frac{q}{p} f(t_0 + \theta + \varepsilon^{\frac{1}{q}} T_1, \varepsilon^{-\frac{1}{p}}(\rho_0 + \varepsilon^{\frac{1}{q}} T_2)^{\frac{1}{p}} v(\theta))(\rho_0 + \varepsilon^{\frac{1}{q}} T_2)^{-\frac{1}{q}} v(\theta) + \varepsilon^{\frac{1}{q}} O_6(1), \\ \frac{dT_2}{d\theta} = -q\varepsilon^{\frac{1}{p}} F_t(t_0 + \theta + \varepsilon^{\frac{1}{q}} T_1, \varepsilon^{-\frac{1}{p}}(\rho_0 + \varepsilon^{\frac{1}{q}} T_2)^{\frac{1}{p}} v(\theta)) + \varepsilon^{\frac{1}{q}} O_6(1). \end{cases} \tag{3.5}$$

It is easy to verify that there is a constant $C > 0$ which is independent of t_0, ρ_0, θ and ε such that

$$\left|\frac{\partial^{k+l} T_1}{\partial t_0^k \partial \rho_0^l}\right|, \left|\frac{\partial^{k+l} T_2}{\partial t_0^k \partial \rho_0^l}\right| \leqslant C \quad \text{for } k + l \leqslant 6.$$

From these estimates, it follows that

$$(\rho_0 + \varepsilon^{\frac{1}{q}} T_2)^{\frac{1}{p}} v(\theta) - \rho_0^{\frac{1}{p}} v(\theta) \in \varepsilon^{\frac{1}{q}} O_6(1).$$

Similar to the proof of Lemma 2.2, we have

$$f(t_0 + \theta + \varepsilon^{\frac{1}{q}} T_1, \varepsilon^{-\frac{1}{p}} (\rho_0 + \varepsilon^{\frac{1}{q}} T_2)^{\frac{1}{p}} v(\theta)) - f(t_0 + \theta, \varepsilon^{-\frac{1}{p}} \rho_0^{\frac{1}{p}} v(\theta)) \in \varepsilon^{\frac{1}{q}} O_5(1), \quad (3.6)$$

$$F_t(t_0 + \theta + \varepsilon^{\frac{1}{q}} T_1, \varepsilon^{-\frac{1}{p}} (\rho_0 + \varepsilon^{\frac{1}{q}} T_2)^{\frac{1}{p}} v(\theta)) - F_t(t_0 + \theta, \varepsilon^{-\frac{1}{p}} \rho_0^{\frac{1}{p}} v(\theta)) \in \varepsilon^{\frac{1}{q}} O_5(1). \quad (3.7)$$

By this discussion, (3.5) can be written in the form

$$\begin{cases} \frac{dT_1}{d\theta} = \frac{q}{p} f(t_0 + \theta, \varepsilon^{-\frac{1}{p}} \rho_0^{\frac{1}{p}} v(\theta)) \rho_0^{-\frac{1}{q}} v(\theta) + \varepsilon^{\frac{1}{q}} O_5(1), \\ \frac{dT_2}{d\theta} = -q \varepsilon^{\frac{1}{p}} F_t(t_0 + \theta, \varepsilon^{-\frac{1}{p}} \rho_0^{\frac{1}{p}} v(\theta)) + \varepsilon^{\frac{1}{q}} O_5(1). \end{cases} \quad (3.8)$$

Therefore,

$$\begin{aligned} T_1(2\pi, t_0, \rho_0) &= \frac{q}{p} \int_0^{2\pi} f(t_0 + \theta, \varepsilon^{-\frac{1}{p}} \rho_0^{\frac{1}{p}} v(\theta)) \rho_0^{-\frac{1}{q}} v(\theta)\, d\theta + o_5(1), \\ T_2(2\pi, t_0, \rho_0) &= -q \int_0^{2\pi} \varepsilon^{\frac{1}{p}} F_t(t_0 + \theta, \varepsilon^{-\frac{1}{p}} \rho_0^{\frac{1}{p}} v(\theta))\, d\theta + o_5(1). \end{aligned}$$

By Lemma 2.2, (2.9) and the dominated convergence theorem, we know that

$$\begin{aligned} T_1(2\pi, t_0, \rho_0) &= \frac{q}{p} \left(\int_{\{\theta \in [0, 2\pi] v(\theta) > 0\}} f_+(t_0 + \theta) \rho_0^{-\frac{1}{q}} v(\theta)\, d\theta \right. \\ &\quad \left. + \int_{\{\theta \in [0, 2\pi] v(\theta) < 0\}} f_-(t_0 + \theta) \rho_0^{-\frac{1}{q}} v(\theta)\, d\theta \right) + o_5(1) \\ &= \frac{q}{p} Z(t_0) \rho_0^{-\frac{1}{q}} + o_5(1), \end{aligned}$$

$$T_2(2\pi, t_0, \rho_0) = -q\left(\int_{\{\theta\in[0,2\pi]v(\theta)>0\}} f'_+(t_0+\theta)\rho_0^{\frac{1}{p}} v(\theta)\,d\theta \right.$$
$$\left. + \int_{\{\theta\in[0,2\pi]v(\theta)<0\}} f'_-(t_0+\theta)\rho_0^{\frac{1}{p}} v(\theta)\,d\theta\right) + o_5(1)$$
$$= -qZ'(t_0)\rho_0^{\frac{1}{p}} + o_5(1),$$

where the function Z is defined in (1.6). So the Poincaré map of (3.3) is of the form

$$t(2\pi) = t_0 + 2\pi + \frac{q}{p}\varepsilon^{\frac{1}{q}} Z(t_0)\rho_0^{-\frac{1}{q}} + \varepsilon^{\frac{1}{q}} o_5(1),$$

$$\rho(2\pi) = \rho_0 - q\varepsilon^{\frac{1}{q}} Z'(t_0)\rho_0^{\frac{1}{p}} + \varepsilon^{\frac{1}{q}} o_5(1). \tag{3.9}$$

Since the system (3.3) is a Hamiltonian system, the Poincaré map P has the intersection property, the proof can be found in [4,17].

3.2. Proof of Theorem 1

In this part, we will prove that the Poincaré P given by (3.9) has an invariant closed curve on the cylinder $[\frac{1}{2}, 2] \times \mathbf{S}^1$ for every $\varepsilon \ll 1$. Usually, the existence of such curves is guaranteed by Moser's small twist theorem [15]. However, in the standard version, this important result is concerned with a map of the form

$$\tau_1 = \tau_0 + \omega + \delta v_0 + \cdots,$$

$$v_1 = v_0 + \cdots,$$

where ω is a fixed number, $\delta > 0$ is a small parameter and the remaining terms (indicated by dots) are of order $o(\delta)$ as $\delta \to 0$. For this reason, our map P does not meet all the conditions of Moser's theorem, it seems that one cannot apply this result to P directly. Fortunately, there is a variant of Moser's theorem [17] which allows us to prove the existence of invariant curves for P.

Under the diffeomorphism

$$\lambda = \rho^{-1}, \quad t = t,$$

the symplectic map P given by (3.9) is transformed into the form

$$Q:\begin{cases} t(2\pi)=t_0+2\pi+\varepsilon^{\frac{1}{q}}l_1(t_0,\lambda_0)+\varepsilon^{\frac{1}{q}}o_5(1),\\ \lambda(2\pi)=\lambda_0+\varepsilon^{\frac{1}{q}}l_2(t_0,\lambda_0)+\varepsilon^{\frac{1}{q}}o_5(1),\end{cases}\qquad (\lambda_0,t_0)\in[1/2,2]\times\mathbf{S}^1, \tag{3.10}$$

where

$$l_1(t_0,\lambda_0)=\frac{q}{p}Z(t_0)\lambda_0^{\frac{1}{q}},\quad l_2(t_0,\lambda_0)=qZ'(t_0)\lambda_0^{2-\frac{1}{p}}. \tag{3.11}$$

Let

$$I(t_0,\lambda_0)=\frac{1}{pZ(t_0)}\lambda_0^{\frac{1}{p}}.$$

Then it is easy to verify that the map Q satisfies all conditions of [17, Theorem 3.1]. Hence for every $\varepsilon \ll 1$, the map Q, so the map P, has an invariant closed curve diffeomorphic to $\lambda_0=$ constant. This completes the proof. $\square$

4. Proof of Theorem 2

Now we turn to prove Theorem 2. We assume that

$$\alpha=\alpha_0+\delta\alpha_1,\quad \beta=\beta_0+\delta\beta_1$$

and the constants α_0,β_0 satisfy

$$\frac{\pi_p}{\alpha_0^{1/p}}+\frac{\pi_p}{\beta_0^{1/p}}=2\pi.$$

Similar to the proof of Theorem 1, we introduce the action and angle variables via the solution $(v_0(t),u_0(t))$ as follows.

$$x=r^{\frac{1}{p}}v_0(\theta),\quad y=r^{\frac{1}{q}}u_0(\theta).$$

Under this transformation, the system (2.1) is changed to

$$\theta'=\frac{\partial h}{\partial r}(r,\theta,t),\quad r'=-\frac{\partial h}{\partial \theta}(r,\theta,t) \tag{4.1}$$

with the Hamiltonian function

$$h(r,\theta,t) = (1+\delta g(\theta))r - qF(t, r^{\frac{1}{p}} v_0(\theta)), \tag{4.2}$$

where

$$g(\theta) = \frac{1}{p}(\alpha_1|v_0^+(\theta)|^p + \beta_1|v_0^-(\theta)|^p). \tag{4.3}$$

Choose δ sufficiently small such that $1+\delta g(\theta) > \frac{1}{2}$.

Similar to the discussions in Sections 2 and 3, we may arrive at the following system

$$\frac{dt}{d\theta} = \frac{\partial H}{\partial \rho}(\rho,t,\theta;\varepsilon,\delta), \quad \frac{d\rho}{d\theta} = -\frac{\partial H}{\partial t}(\rho,t,\theta;\varepsilon,\delta), \tag{4.4}$$

where

$$\begin{aligned} H(\rho,t,\theta;\varepsilon,\delta) = {} & \frac{\rho}{1+\delta g(\theta)} + \frac{q\varepsilon}{(1+\delta g(\theta))} F\left(t, \varepsilon^{-\frac{1}{p}} \rho^{\frac{1}{p}} \frac{v_0(\theta)}{(1+\delta g(\theta))^{\frac{1}{p}}}\right) \\ & + \varepsilon\Phi(\varepsilon^{-1}\rho, t, \theta; \delta), \end{aligned} \tag{4.5}$$

where Φ satisfies (2.16).

Let

$$\omega(\delta) = \int_0^{2\pi} \frac{1}{1+\delta g(\theta)}\, d\theta,$$

$$\tilde{T}_1(t_0,\rho_0;\delta) = \frac{q}{p} Z_1(t_0;\delta)\rho_0^{-\frac{1}{q}}, \tilde{T}_2(t_0,\rho_0;\delta) = -qZ_1'(t_0;\delta)\rho_0^{\frac{1}{p}},$$

where

$$\begin{aligned} Z_1(t_0;\delta) = {} & \int_{\{\theta\in[0,2\pi]|v_0(\theta)>0\}} \frac{f_+(t_0+\theta)v_0(\theta)}{(1+\delta g(\theta))^{1+\frac{1}{p}}}\, d\theta \\ & + \int_{\{\theta\in[0,2\pi]|v_0(\theta)<0\}} \frac{f_-(t_0+\theta)v_0(\theta)}{(1+\delta g(\theta))^{1+\frac{1}{p}}}\, d\theta. \end{aligned} \tag{4.6}$$

Then the Poincaré map of (4.4) is of the form

$$\begin{cases} t(2\pi) = t_0 + 2\pi\omega(\delta) + \varepsilon^{\frac{1}{q}}\tilde{T}_1(t_0, \rho_0; \delta) + \varepsilon^{\frac{1}{q}} o_5(1), \\ \rho(2\pi) = \rho_0 + \varepsilon^{\frac{1}{q}}\tilde{T}_2(t_0, \rho_0; \delta) + \varepsilon^{\frac{1}{q}} o_5(1). \end{cases} \tag{4.7}$$

From the definition of $Z_1(t_0; \delta)$ and (1.6), we know that

$$\lim_{\delta\to 0} Z_1(t_0; \delta) = Z_0(t_0),$$

so if $Z_0 > 0$ then there is a $\delta_0 > 0$ such that $Z_1(t_0; \delta) > 0$ for $|\delta| \leqslant \delta_0$. In this case, one can prove the boundedness of solutions of (1.1) by the results of Ortega [17,18]. Indeed, if $\omega(\delta) \in \mathbf{Q}$, under the transformation $(t, \rho) \mapsto (t, \lambda = \frac{1}{\rho})$, the map defined in (4.7) is changed in the form

$$\begin{cases} t(2\pi) = t_0 + 2\pi\omega(\delta) + \frac{q}{p}\varepsilon^{\frac{1}{q}} Z_1(t_0; \delta)\lambda_0^{\frac{1}{q}} + \varepsilon^{\frac{1}{q}} o_5(1), \\ \lambda(2\pi) = \lambda_0 + q\varepsilon^{\frac{1}{q}} Z_1'(t_0; \delta)\lambda_0^{1+\frac{1}{q}} + \varepsilon^{\frac{1}{q}} o_5(1). \end{cases} \tag{4.8}$$

Let

$$I_1(t_0, \lambda_0; \delta) = \frac{1}{Z_1(t_0; \delta)}\lambda_0^{\frac{1}{p}}.$$

Then the existence of invariant closed curves of (4.8) can be implied by [17, Theorem 3.1], so all the solutions of (1.1) are bounded. If $\omega(\delta)$ is irrational, note that for δ sufficiently small, the average of Z_1 with respect to θ_0 is positive because Z_1 is positive, the existence of invariant curves as well as the boundedness of solutions follows from [18, Theorem 1]. This completes the proof of Statement (1) of Theorem 2.

Now we turn to prove the statement (2) of Theorem 2. From the definition of $\omega(\delta)$, we know that

$$\frac{d}{d\delta}\omega(\delta)|_{\delta=0} = -\int_0^{2\pi} g(\theta)\, d\theta \neq 0,$$

by the condition (1.8). So there is a $\delta_1 > 0$ such that there is a subset $\Lambda_1 \subset (0, \delta_1)$ with meas $\Lambda_1 = \delta_1$ and for each $\delta \in \Lambda_1$, $\omega(\delta)$ is irrational number. From Theorem 1 in [18], we know that the inequality (4.9) below will guarantee the existence of invariant curves for $\varepsilon \ll 1$:

$$\int_0^{2\pi} Z_1(t; \delta)\, dt_0 \neq 0. \tag{4.9}$$

From the definition of v_0, it is not difficult to see that

$$\int_{\{t_0\in[0,2\pi]|v_0(t_0)>0\}} v_0(t_0)\,dt_0 = c_0\alpha^{-\frac{2}{p}},\ \int_{\{t_0\in[0,2\pi]|v_0(t_0)<0\}} v_0(t_0)\,dt_0 = -c_0\beta^{-\frac{2}{p}},$$

where

$$c_0 = 2(p-1)^{\frac{1}{p}-\frac{1}{q}}\int_0^1 (1-s^q)^{\frac{1}{p}-\frac{1}{q}}\,ds > 0.$$

Since

$$\int_0^{2\pi}\int_{\{\theta\in[0,2\pi]|v_0(\theta)>0\}} f_+(t_0+\theta)v_0(\theta)\,d\theta\,dt_0 = c_0\alpha^{-\frac{2}{p}}\int_0^{2\pi} f_+(s)\,ds$$

and

$$\int_0^{2\pi}\int_{\{\theta\in[0,2\pi]|v_0(\theta)<0\}} f_-(t_0+\theta)v_0(\theta)\,d\theta\,dt_0 = -c_0\beta^{-\frac{2}{p}}\int_0^{2\pi} f_-(s)\,ds,$$

it follows from the condition (1.9) that

$$\int_0^{2\pi} Z_1(t_0;0)\,dt_0 \neq 0.$$

Hence, there is a $\delta_2 > 0$ such that for $|\delta|\leqslant\delta_2$, the average value of $Z_1(t_0;\delta)$ is not zero. Let $\delta_0 = \min\{\delta_1,\delta_2\}$ and $\Lambda = \Lambda\cap(0,\delta_0)$. Using the result of Ortega [18, Theorem 1] again, we know that for each $\delta\in\Lambda$, there are invariant closed curves of the map (4.7) for $\varepsilon \ll 1$. The boundedness of solutions follows from the standard arguments (see, for example, [4,13,17]). The proof of Theorem 2 is completed. $\square$

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Further reading

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