

# Partial differential equations with differential constraints

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## Abstract

A geometric setting for constrained exterior differential systems on fibered manifolds with  $n$ -dimensional bases is proposed. Constraints given as submanifolds of jet bundles (locally defined by systems of first-order partial differential equations) are shown to carry a natural geometric structure, called the *canonical distribution*. Systems of second-order partial differential equations subjected to differential constraints are modeled as exterior differential systems defined on constraint submanifolds. As an important particular case, *Lagrangian systems* subjected to first-order differential constraints are considered. Different kinds of constraints are introduced and investigated (Lagrangian constraints, constraints adapted to the fibered structure, constraints arising from a (co)distribution, semi-holonomic constraints, holonomic constraints).

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## 1. Introduction

Within the classical calculus of variations and optimal control theory, equations subjected to different kinds of constraints are investigated, providing mathematical models for motion of various systems appearing in mechanics and engineering. Recently, namely constraints given by systems of ordinary differential equations have been intensively studied with the help of methods of differential geometry and global analysis, and a general theory of “non-holonomic systems” in fibered manifolds was founded. This concerns a geometric version of Chetaev equations [6] and its generalization to constraints given by higher-order ODEs, a geometric model for constrained ODEs as differential systems defined directly on constraint submanifolds, a theory covering non-Lagrangian systems as well as higher-order ODEs with higher-order differential constraints, study of symmetries of constrained Lagrangian systems, Hamiltonian constrained systems, and many other questions (see e.g. [5,8,13,22,23,25,29–34,36]). All the above-mentioned results, however, have been achieved for systems of *ordinary differential equations*; *partial differential equations*, except of a pioneer work [3], have not yet been studied.

In this paper we propose a generalization of the theory of non-holonomic systems to second-order partial differential equations subjected to constraints given by first-order PDEs. Our task is to transfer to this case main ideas from [22,25]. The exposition consists of the following four parts:

In Section 2 we present a geometric setting for systems of second-order partial differential equations

$$E_{\sigma} \left( x^i, \gamma^v, \frac{\partial \gamma^v}{\partial x^p}, \frac{\partial^2 \gamma^v}{\partial x^p \partial x^q} \right) = 0, \quad 1 \leq \sigma \leq m, \quad (1.1)$$

for mappings  $(x^i) \rightarrow (\gamma^v(x^i))$ ,  $1 \leq i \leq n$ ,  $1 \leq v \leq m$ , between smooth manifolds. Eqs. (1.1) are modeled by a *dynamical form* and its *Lepage class* on a jet prolongation of a fibered manifold  $\pi : Y \rightarrow X$ , where  $\dim X = n$  and  $\dim Y = n + m$ , and solutions are interpreted as integral sections of a corresponding *exterior differential system* generated by  $n$ -forms. This approach relates the global theory of differential equations to the calculus of variations in fibered manifolds [21]: it enables, on one hand, easily to consider variational equations as a special case, and on the other hand, to enlarge and generalize to the “non-variational” case some methods which have been developed to investigate exclusively variational equations.

In Section 3 we study systems of first-order PDEs which have the meaning of differential *constraints* in fibered manifolds, i.e., which are fibered submanifolds of  $J^1 Y \rightarrow Y$ . In this paper we focus on a significant class of constraints, which we call *regular constraints* (characterized by rank condition (3.3)). As a key-result it is shown that every regular constraint is endowed with a natural geometric structure, namely, a subbundle of the tangent bundle, which we call the *canonical distribution*. This subdistribution of the Cartan distribution has an analogy in non-holonomic mechanics, where it plays a role of “generalized virtual displacements”. Thus, we can say that regular constraints comply with a generalized *D’Alembert principle*.

Section 4 deals with *constrained PDEs*. First of all, we associate with unconstrained equations new equations, defined on the constraint submanifold. The geometric model for (unconstrained) PDEs, together with the canonical distribution of the constraint gives the constrained equations represented by an *exterior differential system on the constraint submanifold*. In particular, we are interested in constrained *variational equations*, and we find a constrained Euler–Lagrange operator. While in the (unconstrained) calculus of variations on fibered manifolds a Lagrangian is a differential form which can be locally represented by a function,  $L$ , it turns out that a “constrained Lagrangian” is a differential form which *cannot* be represented by a single function. Next, we study constrained PDEs as local *deformations* of unconstrained PDEs, and we obtain equations which generalize to the case of PDEs *Chetaev equations*, known from non-holonomic mechanics. We also show that generalized Chetaev equations and constrained equations are equivalent.

Section 5 is devoted to a detail study of different kinds of constraints, which are covered by our setting. It turns out that for partial differential equations one has more interesting constraints than for ODEs. In particular, there appear constraints which we call *Lagrangian*, and  $\pi$ -*adapted*. Besides, one has, similarly as in mechanics, *constraints defined by a distribution on  $Y$* , *semi-holonomic*, and *holonomic* constraints. We study properties of these constraints and their relations.

## 2. Dynamical forms in jet bundles

### 2.1. Fibered manifolds and their prolongations

Throughout this paper, we assume all manifolds and maps be smooth, and use standard notations:  $T$  and  $J^r$  denotes the tangent and the  $r$ -jet prolongation functor, respectively,  $d$  the exterior derivative,  $*$  the pull-back,  $i_\xi$  the contraction by a vector field  $\xi$ , etc. The summation convention is used unless otherwise explicitly stated.

Let us briefly recall main concepts from the theory of fibered manifolds and the corresponding calculus. For more details we refer to [19,21,35] (see also [4,14]).

We consider a *fibered manifold*  $\pi : Y \rightarrow X$  with a base  $X$  of dimension  $n$ , and total space  $Y$ ,  $\dim Y = m + n$ , and its *jet prolongations*  $\pi_r : J^r Y \rightarrow X$ ; for simplicity of notations, we also write  $J^0 Y = Y$  and  $\pi_0 = \pi$ . There are naturally induced fibered manifolds  $\pi_{r,s} : J^r Y \rightarrow J^s Y$ , where  $r > s \geq 0$ . In this paper we shall mainly work with the first and second jet prolongation of  $\pi$ , i.e., with fibered manifolds  $\pi_1, \pi_2$ . Local *fibered coordinates* on  $Y$  are denoted by  $(x^i, y^\sigma)$ , where  $1 \leq i \leq n$ ,  $1 \leq \sigma \leq m$ , and the associated coordinates on  $J^1 Y$  and  $J^2 Y$  by  $(x^i, y^\sigma, y_j^\sigma)$  and  $(x^i, y^\sigma, y_j^\sigma, y_{jk}^\sigma)$ , where  $1 \leq j \leq k \leq n$ , respectively. In formulas, we use summation over all values of indices (not only over non-decreasing sequences). In calculations we use on  $J^1 Y$  (resp.  $J^2 Y$ ), either a *canonical basis* of one forms,  $(dx^i, dy^\sigma, dy_j^\sigma)$  (resp.  $(dx^i, dy^\sigma, dy_j^\sigma, dy_{jk}^\sigma)$ ), or a *basis adapted to the contact structure*,  $(dx^i, \omega^\sigma, dy_j^\sigma)$ , (resp.  $(dx^i, \omega^\sigma, \omega_j^\sigma, dy_{jk}^\sigma)$ ), where

$$\omega^\sigma = dy^\sigma - y_k^\sigma dx^k, \quad \omega_j^\sigma = dy_j^\sigma - y_{jk}^\sigma dx^k. \quad (2.1)$$

Next, we denote

$$\begin{aligned}\omega_0 &= dx^1 \wedge \cdots \wedge dx^n, \quad \omega_{j_1} = i_{\partial/\partial x^{j_1}} \omega_0, \\ \omega_{j_1 j_2} &= i_{\partial/\partial x^{j_2}} \omega_{j_1}, \quad \dots, \quad \omega_{j_1 \dots j_n} = i_{\partial/\partial x^{j_n}} \omega_{j_1 \dots j_{n-1}}.\end{aligned}\quad (2.2)$$

By a *section*  $\gamma$  of  $\pi$  we mean a mapping  $\gamma : U \rightarrow Y$ , defined on an open subset  $U$  of  $X$ , such that  $\pi \circ \gamma = \text{id}_U$ . In fibered coordinates, components of a section  $\gamma$  of  $\pi$  take the form  $(x^i, \gamma^\sigma)$ , where the  $\gamma^\sigma$ 's are functions of the  $x^i$ 's. Components of the first jet prolongation  $J^1\gamma$  (which is a section of  $\pi_1$ ) take the form  $(x^i, \gamma^\sigma, \partial\gamma^\sigma/\partial x^j)$ . Similarly, components of the second jet prolongation  $J^2\gamma$  of  $\gamma$  become  $(x^i, \gamma^\sigma, \partial\gamma^\sigma/\partial x^j, \partial^2\gamma^\sigma/\partial x^j \partial x^k)$ . A section of  $\pi_r$  is called *holonomic* if it is the  $r$ -jet prolongation of a section of  $\pi$ .

A vector field  $\xi$  on  $J^r Y$ ,  $r \geq 0$ , is called  $\pi_r$ -*projectable* if there exists a vector field  $\xi_0$  on  $X$  such that  $T\pi_r.\xi = \xi_0 \circ \pi_r$ , and  $\pi_r$ -*vertical* if it projects onto a zero vector field on  $X$ , i.e.,  $T\pi_r.\xi = 0$ . Quite similarly one can define a  $\pi_{r,s}$ -*projectable* or a  $\pi_{r,s}$ -*vertical* vector field on  $J^r Y$ , where  $r > s$ . A differential  $k$ -form  $\eta$  on  $J^r Y$  is called  $\pi_r$ -*horizontal* (resp.  $\pi_{r,s}$ -*horizontal*) if  $i_\xi \eta = 0$  for every  $\pi_r$ -vertical (resp.  $\pi_{r,s}$ -vertical) vector field  $\xi$  on  $J^r Y$ . Note that  $\pi_r$ -horizontal forms are those which in fibered coordinates contain wedge products of the base differentials  $dx^i$  only (with components dependent upon all the fibered coordinates). Similarly,  $\pi_{r,0}$ -horizontal forms contain wedge products of the total space differentials  $dx^i$ 's and  $dy^\sigma$ 's only, etc. To every  $k$ -form  $\eta$  on  $J^r Y$  one can assign a unique horizontal  $k$ -form on  $J^{r+1}Y$ , denoted by  $h\eta$  and called the *horizontal part of*  $\eta$ . The mapping  $h$  is defined to be an  $R$ -linear wedge product preserving mapping such that for every function  $f$  on  $J^r Y$ ,  $hf = f \circ \pi_{r+1,r}$ , and

$$hdx^i = dx^i, \quad hdy^\sigma = y_k^\sigma dx^k, \quad hdy_j^\sigma = y_{jk}^\sigma dx^k, \quad \text{etc.} \quad (2.3)$$

In particular,  $hdf = d_i f dx^i$ , where  $d_i$  is the *ith total derivative operator* which for a first-order function  $f$  takes the form

$$d_i \equiv \frac{d}{dx^i} = \frac{\partial}{\partial x^i} + y_i^\sigma \frac{\partial}{\partial y^\sigma} + y_{ki}^\sigma \frac{\partial}{\partial y_k^\sigma}. \quad (2.4)$$

By definition of  $h$ , for any form  $\eta$  of degree  $k > n$ ,  $h\eta = 0$ . A  $k$ -form  $\eta$  on  $J^r Y$  ( $r \geq 1$ ) is called *contact* if for every section  $\gamma$  of  $\pi$ ,  $J^r \gamma^* \eta = 0$ . A contact  $k$ -form is called *1-contact* (resp. *q-contact*,  $q \geq 2$ ) if for every vertical vector field  $\xi$ , the  $(k-1)$ -form  $i_\xi \eta$  is horizontal (resp.  $(q-1)$ -contact).

**Theorem 2.1** (Krupka [19]). *Every  $k$ -form  $\eta$  on  $J^r Y$  has a canonical decomposition*

$$\pi_{r+1,r}^* \eta = h\eta + p_1 \eta + p_2 \eta + \cdots + p_k \eta, \quad (2.5)$$

where  $h\eta$  is a unique horizontal form, and  $p_q\eta$ ,  $q = 1, 2, \dots, k$ , are unique  $q$ -contact forms.

The forms  $h\eta$  and  $p_q\eta$ ,  $q = 1, 2, \dots, k$ , above are called *the horizontal part of  $\eta$* , and *the  $q$ -contact part of  $\eta$* , respectively.

## 2.2. Differential equations modeled by dynamical forms

**Definition 2.2.** By a second-order dynamical form on a fibered manifold  $\pi : Y \rightarrow X$  we understand a differential  $(n+1)$ -form on  $J^2Y$  which is 1-contact, and horizontal with respect to the projection onto  $Y$ .

In fibered coordinates one gets

$$E = E_\sigma \omega^\sigma \wedge \omega_0, \quad (2.6)$$

where  $E_\sigma$  are functions depending upon  $(x^i, y^\rho, y^\rho_p, y^\rho_{pq})$ . A section  $\gamma$  of  $\pi$  defined on an open set  $U \subset X$  is called a *path* of  $E$  if

$$E \circ J^2\gamma = 0. \quad (2.7)$$

The above equation, called *equation for paths* of a dynamical form  $E$ , takes in fibered coordinates the form of a system of  $m$  second-order partial differential equations,

$$E_\sigma \circ J^2\gamma = 0, \quad 1 \leq \sigma \leq m, \quad (2.8)$$

or, more explicitly,

$$E_\sigma \left( x^i, \gamma^\rho, \frac{\partial \gamma^\rho}{\partial x^p}, \frac{\partial^2 \gamma^\rho}{\partial x^p \partial x^q} \right) = 0, \quad 1 \leq \sigma \leq m, \quad (2.9)$$

where  $m = \dim Y - \dim X$  is the fiber dimension. Note that equations for paths of a dynamical form on a fibered manifold (with the base dimension  $n$  and fiber dimension  $m$ ) can be regarded as a *global characterization of (local) differential equations* (2.9) for graphs of mappings  $R^n \rightarrow R^m$ . Dynamical forms represent quite a wide class of systems of differential equations: in particular, they contain *all second-order variational PDEs*.

**Definition 2.3.** Let  $E$  be a dynamical form on  $J^2Y$ . We say that a  $(n+1)$ -form  $\alpha$  on  $J^rY$  is related to  $E$  if  $p_1\alpha = E$ . Taking into account Theorem 2.1, we can see that  $\alpha$  is related to  $E$  if and only if

$$\pi_{r+1,r}^* \alpha = E + F, \quad (2.10)$$

where  $F$  is an at least 2-contact form. The family of all to  $E$  related  $(n+1)$ -forms, defined possibly on open subsets of  $J^r Y$ , will be called the Lepage class of  $E$  of order  $r$ , and denoted by  $[\alpha]^r$ .

We can see that every second-order dynamical form has related Lepage classes of order  $r$  for every  $r \geq 2$ . Lepage classes are used for a geometrical description of equations (2.7) (i.e., (2.9)) by means of *exterior differential systems* as follows:

**Proposition 2.4.** *Let be a dynamical form on  $J^2 Y$ ,  $[\alpha]^r$  its Lepage class of order  $r$ . For  $\alpha \in [\alpha]^r$  consider the ideal  $\mathcal{H}_\alpha$  in the exterior algebra on  $J^r Y$ , generated by the system of  $n$ -forms*

$$i_\xi \alpha, \quad \text{where } \xi \text{ runs over all } \pi_r\text{-vertical vector fields on } J^r Y. \quad (2.11)$$

The following conditions are equivalent:

- (1) A section  $\gamma$  of  $\pi$  is a path of  $E$ , i.e.,  $E \circ J^2 \gamma = 0$ .
- (2) For every  $\alpha \in [\alpha]^r$ ,  $J^r \gamma$  is an integral section of the ideal  $\mathcal{H}_\alpha$ , i.e.,

$$J^r \gamma^* i_\xi \alpha = 0, \quad \text{for every } \pi_r\text{-vertical vector field } \xi \text{ on } J^r Y. \quad (2.12)$$

- (3) For every  $\alpha \in [\alpha]^r$ ,

$$J^{r+1} \gamma^* i_{\hat{\xi}} \hat{\alpha} = 0, \quad \text{for every } \pi_{r+1}\text{-vertical vector field } \hat{\xi} \text{ on } J^{r+1} Y, \quad (2.13)$$

where  $\hat{\alpha}$  is the at most 2-contact part of  $\alpha$ .

**Proof.** Suppose (1). Then  $E_\sigma \circ J^2 \gamma = 0$ ,  $1 \leq \sigma \leq m$ . This means that for every  $\pi_2$ -vertical vector field  $\zeta$  on  $J^2 Y$ ,

$$\zeta = \zeta^\sigma \frac{\partial}{\partial y^\sigma} + \zeta_i^\sigma \frac{\partial}{\partial y_i^\sigma} + \zeta_{ij}^\sigma \frac{\partial}{\partial y_{ij}^\sigma}, \quad (2.14)$$

we have

$$J^2 \gamma^* i_\zeta E = J^2 \gamma^* ((E_\sigma \zeta^\sigma) \omega_0) = \left( (E_\sigma \zeta^\sigma) \circ J^2 \gamma \right) \omega_0 = 0. \quad (2.15)$$

If  $\zeta$  is a vertical vector field on  $J^r Y$ , denote by  $\hat{\zeta}$  a vector field on  $J^{r+1} Y$  which projects onto  $\zeta$ . Since Eq. (2.15) depends only upon the total space components of vector fields on  $J^2 Y$ , we can see that also for every  $\pi_r$ -vertical vector field  $\zeta$  on  $J^r Y$  where  $r > 2$ ,

$$J^{r+1} \gamma^* i_{\hat{\zeta}} \pi_{r+1,2}^* E = 0. \quad (2.16)$$

Hence, for every  $\alpha \in [\alpha]^r$ , and every vertical  $\xi$ ,

$$J^r \gamma^* i_\xi \alpha = J^{r+1} \gamma^* i_\xi \pi_{r+1,r}^* \alpha = J^{r+1} \gamma^* i_\xi p_1 \alpha = J^{r+1} \gamma^* i_\xi \pi_{r+1,2}^* E = 0. \quad (2.17)$$

Conversely, suppose that  $\gamma$  satisfies Eqs. (2.12). Taking any  $\alpha \in [\alpha]^r$ , and using that (possibly up to a projection)  $E = p_1 \alpha$ , we get from (2.17) by similar arguments as above that  $E \circ J^2 \gamma = 0$ .

Assertions (2) and (3) are equivalent, as seen immediately from (2.17).  $\square$

**Definition 2.5.** We shall call the ideals  $\mathcal{H}_\alpha$  and  $\mathcal{H}_{\hat{\alpha}}$  introduced in Proposition 2.4 Hamiltonian ideal and principal Hamiltonian ideal of the  $(n+1)$ -form  $\alpha$ , respectively, and the form  $\hat{\alpha}$  the principal part of  $\alpha$ .

Proposition 2.4 says that all Hamiltonian ideals and principal Hamiltonian ideals associated with  $E$  have the same *holonomic integral sections*, and these *coincide with prolonged paths of  $E$*  (i.e., with *solutions of Eqs. (2.7)*, respectively, (2.9)).

**Remark 2.6.** The terminology for  $\mathcal{H}_\alpha$  reflects that one used in the calculus of variations. If Eqs. (2.9) (i.e., (2.7)) are variational (i.e., are Euler–Lagrange equations), then related Eqs. (2.12) are called Hamilton Eqs. (see [10,20,24], also [7,9], etc.).

### 2.3. Equations polynomial in the second derivatives

We shall study second-order PDEs which admit a *first-order Lepage class*. In view of the above considerations this means that equations of this kind are described by means of *exterior differential systems on  $J^1 Y$* .

**Proposition 2.7.** Let  $E = E_\sigma \omega^\sigma \wedge \omega_0$  be a dynamical form on  $J^2 Y$ .  $E$  has a Lepage class of order 1 if and only if the functions  $E_\sigma$ ,  $1 \leq \sigma \leq m$ , are polynomials of degree  $\leq n$  in the variables  $y_{ji}^v$ , i.e.,

$$E_\sigma = A_\sigma + B_{\sigma v_1}^{j_1 i_1} y_{j_1 i_1}^{v_1} + B_{\sigma v_1 v_2}^{j_1 i_1 j_2 i_2} y_{j_1 i_1}^{v_1} y_{j_2 i_2}^{v_2} + \cdots + B_{\sigma v_1 \dots v_n}^{j_1 i_1 \dots j_n i_n} y_{j_1 i_1}^{v_1} \cdots y_{j_n i_n}^{v_n}, \quad (2.18)$$

and the coefficients  $B_{\sigma v_1 \dots v_k}^{j_1 i_1 \dots j_k i_k}$  (where  $2 \leq k \leq n$ ) are completely antisymmetric in the indices  $i_1, \dots, i_k$ .

**Proof.** In a basis adapted to the contact structure, every  $(n+1)$ -form  $\alpha$  on  $J^1 Y$  takes the form “polynomial” in  $dy_j^v$ , i.e.,

$$\begin{aligned} \alpha = & \beta_0 + \beta_{v_1}^{j_1 i_1} \wedge dy_{j_1}^{v_1} + \beta_{v_1 v_2}^{j_1 i_1 j_2 i_2} \wedge dy_{j_1}^{v_1} \wedge dy_{j_2}^{v_2} + \beta_{v_1 v_2 v_3}^{j_1 i_1 j_2 i_2 j_3 i_3} \wedge dy_{j_1}^{v_1} \wedge dy_{j_2}^{v_2} \wedge dy_{j_3}^{v_3} \\ & + \cdots + \beta_{v_1 \dots v_n}^{j_1 i_1 \dots j_n i_n} \wedge dy_{j_1}^{v_1} \wedge \cdots \wedge dy_{j_n}^{v_n} + \beta_{v_1 \dots v_{n+1}}^{j_1 i_1 \dots j_{n+1} i_{n+1}} dy_{j_1}^{v_1} \wedge \cdots \wedge dy_{j_{n+1}}^{v_{n+1}}, \end{aligned} \quad (2.19)$$

where the  $\beta$ 's are  $p$ -forms ( $n+1 \geq p \geq 0$ ) expressed by means of wedge products of the  $dx^i$ 's and  $\omega^\rho$ 's only. Substituting  $dy_j^v = \omega_j^v + y_{ji}^v dx^i$ , we obtain the lift  $\pi_{2,1}^* \alpha$  of  $\alpha$  expressed as a sum of contact parts, and we can see that all components of  $\pi_{2,1}^* \alpha$  are polynomials in the variables  $y_{ji}^v$ . In particular, this concerns the first term, i.e., the 1-contact part  $E = p_1 \alpha$ , which is by assumption a dynamical form. Taking into account that the term  $\omega_{j_1}^{v_1} \wedge \cdots \wedge \omega_{j_{n+1}}^{v_{n+1}}$  gives no contribution to  $E$  (indeed,  $y_{j_1 i_1}^{v_1} dx^{i_1} \wedge y_{j_2 i_2}^{v_2} dx^{i_2} \wedge \cdots \wedge y_{j_{n+1} i_{n+1}}^{v_{n+1}} dx^{i_{n+1}} = 0$ ), we conclude that the components of  $E$  are polynomials of degree at most  $n$ . Finally, the antisymmetry condition for the  $B$ 's appears, since  $B_{\sigma v_1 \dots v_p}^{j_1 i_1 \dots j_p i_p}$  arise as components at  $\omega^\sigma \wedge y_{j_1 k_1}^{v_1} dx^{k_1} \wedge y_{j_2 k_2}^{v_2} dx^{k_2} \wedge \cdots \wedge y_{j_p k_p}^{v_p} dx^{k_p} \wedge \omega_{i_1 \dots i_p}$ , which are completely antisymmetric in the indices  $i_1 \dots i_p$ .

Conversely, assume that fiber chart components  $E_\sigma$ ,  $1 \leq \sigma \leq m$ , of  $E$  are polynomials characterized by the proposition. Put

$$\begin{aligned} \alpha_0 = & A_\sigma \omega^\sigma \wedge \omega_0 + B_{\sigma v_1}^{j_1 i_1} \omega^\sigma \wedge dy_{j_1}^{v_1} \wedge \omega_{i_1} + \frac{1}{2} B_{\sigma v_1 v_2}^{j_1 i_1 j_2 i_2} \omega^\sigma \wedge dy_{j_1}^{v_1} \wedge dy_{j_2}^{v_2} \wedge \omega_{i_1 i_2} \\ & + \frac{1}{3!} B_{\sigma v_1 v_2 v_3}^{j_1 i_1 j_2 i_2 j_3 i_3} \omega^\sigma \wedge dy_{j_1}^{v_1} \wedge dy_{j_2}^{v_2} \wedge dy_{j_3}^{v_3} \wedge \omega_{i_1 i_2 i_3} \\ & + \cdots + \frac{1}{n!} B_{\sigma v_1 \dots v_n}^{j_1 i_1 \dots j_n i_n} \omega^\sigma \wedge dy_{j_1}^{v_1} \wedge \cdots \wedge dy_{j_n}^{v_n} \wedge \omega_{i_1 \dots i_n}. \end{aligned} \quad (2.20)$$

Then  $\alpha_0$  is a local form on  $J^1 Y$  such that  $p_1 \alpha_0 = E$ , i.e., it generates a first-order Lepage class of  $E$ .  $\square$

In accordance with [11], we say that a dynamical form  $E$  on  $J^2 Y$  is  $J^1 Y$ -pertinent if it possesses a first-order Lepage class, i.e., its components  $E_\sigma$  take the form described by Proposition 2.7. In what follows, we denote a first-order Lepage class  $[\alpha]$  simply by  $[\alpha]$ , and we write

$$\alpha_1 \sim \alpha_2 \quad \text{for} \quad \alpha_1, \alpha_2 \in [\alpha]. \quad (2.21)$$

The  $(n+1)$ -form  $\alpha_0$  given by (2.20) is a local first-order form related with  $E$ , which is “minimal” in the sense that it does not contain any free terms. All the first-order related  $(n+1)$ -forms are then characterized as follows:

**Corollary 2.8.** *The first-order Lepage class  $[\alpha]$  of a  $J^1 Y$ -pertinent dynamical form  $E$  consists of all (local) forms*

$$\alpha = \alpha_0 + F, \quad (2.22)$$

where  $\alpha_0$  is given by (2.20), and  $F$  is an at least 2-contact form (defined on an open subset of  $J^1 Y$ ). The class  $[\alpha]$  contains a subclass of forms belonging to the ideal generated by the 1-contact forms  $\omega^\sigma$ ,  $1 \leq \sigma \leq m$ ; in particular, one can even consider invariant representatives such that  $F$  is  $\pi_{1,0}$ -horizontal (contains no  $dy_j^v$ ).



For dynamical forms whose components are *affine in the second derivatives* (i.e., for *quasilinear second-order PDE*) the situation further simplifies:

**Corollary 2.9.** *Every dynamical form  $E$  on  $J^2Y$  with components affine in the second derivatives, i.e., such that*

$$E_\sigma = A_\sigma + B_{\sigma\nu}^{ji} y_{ji}^\nu, \quad (2.23)$$

*is  $J^1Y$ -pertinent, and its first-order Lepage class  $[\alpha]$  consists of the following forms:*

$$\begin{aligned} \alpha &= A_\sigma \omega^\sigma \wedge \omega_0 + B_{\sigma\nu}^{ji} \omega^\sigma \wedge dy_{(j}^\nu \wedge \omega_{i)} + F \\ &= \left( E_\sigma - \frac{\partial E_\sigma}{\partial y_{ij}^\nu} y_{ij}^\nu \right) \omega^\sigma \wedge \omega_0 + \frac{\partial E_\sigma}{\partial y_{ij}^\nu} \omega^\sigma \wedge dy_j^\nu \wedge \omega_i + F, \end{aligned} \quad (2.24)$$

where  $F$  is at least 2-contact, and  $(j, i)$  denotes symmetrization in the indicated indices.

#### 2.4. Variational equations

Among equations we have considered up to now, there is an important family of *variational* equations, having many specific properties. We briefly recall without proofs basic concepts from the calculus of variations on fibered manifolds in order to put variational equations into the above general scheme. The exposition follows [15,16,19,21], where more results and proofs can be found.

A horizontal  $n$ -form  $\lambda$  on  $J^1Y$  (where  $n = \dim X$ ) is called a *first-order Lagrangian*. A form  $\rho$  such that  $h\rho = \lambda$ , and  $p_1 d\rho$  is  $\pi_{1,0}$ -horizontal is called *Lepagean equivalent* of  $\lambda$  [15]. Lepagean equivalents of a first-order Lagrangian  $\lambda = L\omega_0$  take the form

$$\rho = \Theta_\lambda + \mu = L\omega_0 + \frac{\partial L}{\partial y_j^\sigma} \omega^\sigma \wedge \omega_j + \mu, \quad (2.25)$$

where  $\Theta_\lambda$  is the *Poincaré–Cartan form*, and  $\mu$  is an arbitrary at least 2-contact form. Family (2.25) of Lepagean equivalents of  $\lambda$  contains the following  $n$ -form:

$$\rho_\lambda = L\omega_0 + \sum_{k=1}^n \frac{1}{(k!)^2} \frac{\partial^k L}{\partial y_{j_1}^{\sigma_1} \cdots \partial y_{j_k}^{\sigma_k}} \omega^{\sigma_1} \wedge \cdots \wedge \omega^{\sigma_k} \wedge \omega_{j_1 \cdots j_k}, \quad (2.26)$$

called *Krupka form* (see [17,2]). If  $\rho$  is a Lepagean equivalent of  $\lambda$  then the action functions of  $\rho$  and  $\lambda$  are the same, and the *paths of the dynamical form*

$$E_\lambda = p_1 d\rho \quad (2.27)$$

are extremals of the Lagrangian  $\lambda$ .  $E_\lambda$  is called the *Euler–Lagrange form* of  $\lambda$ , its components *Euler–Lagrange expressions*, and equations for paths of  $E$  are called *Euler–Lagrange equations*. It holds  $E_\lambda = \varepsilon_\sigma(L)\omega^\sigma \wedge \omega_0$ , where

$$\varepsilon_\sigma(L) = \frac{\partial L}{\partial y^\sigma} - \frac{d}{dx^j} \frac{\partial L}{\partial y_j^\sigma}, \quad 1 \leq \sigma \leq m. \quad (2.28)$$

Since  $\lambda$  is a first-order Lagrangian, the Euler–Lagrange expressions (2.28) are *affine in the second derivatives*. Keeping notations of (2.18) we have

$$E_\sigma = A_\sigma + B_{\sigma v}^{ij} y_{ij}^v = A_\sigma + B_{\sigma v}^{(ij)} y_{ij}^v, \quad (2.29)$$

where

$$E_\sigma = \varepsilon_\sigma(L), \quad A_\sigma = \varepsilon'_\sigma(L) \equiv \frac{\partial L}{\partial y^\sigma} - \frac{d'}{dx^j} \frac{\partial L}{\partial y_j^\sigma},$$

$$B_{\sigma v}^{ij} = B_{\sigma v}^{ij}(L) \equiv -\frac{\partial^2 L}{\partial y_i^\sigma \partial y_j^v}, \quad (2.30)$$

and  $B_{\sigma v}^{(ij)} = \frac{1}{2} (B_{\sigma v}^{ij} + B_{\sigma v}^{ji})$ . Above,  $d'/dx^j$  is the *cut total derivative*

$$\frac{d'}{dx^j} = \frac{d}{dx^j} - y_{ji}^v \frac{\partial}{\partial y_i^v} = \frac{\partial}{\partial x^j} + y_j^v \frac{\partial}{\partial y^v}. \quad (2.31)$$

*Euler–Lagrange equations* take one of the following equivalent *intrinsic* forms:

$$J^1 \gamma^* i_\xi d\rho = 0 \quad \text{for every vertical vector field } \xi \text{ on } J^1 Y.$$

$$E_\lambda \circ J^2 \gamma = 0. \quad (2.32)$$

The first equation comes from the first variation formula for the Lagrangian  $\lambda$ , the second one reflects the fact that the Euler–Lagrange form  $E_\lambda$  is a dynamical form.

By definition of  $E_\lambda$ , formula (2.29), and Corollary 2.9 we immediately get:

**Proposition 2.10.** *Let  $\lambda$  be a first-order Lagrangian. Then its Euler–Lagrange form  $E_\lambda$  has a Lepage class defined on  $J^1 Y$ . Moreover, in the first-order Lepage class of  $E_\lambda$  there are the following distinguished representatives:*

$$d\Theta_\lambda \sim d\rho_\lambda \sim \alpha_{0\lambda} \sim \alpha_\lambda, \quad (2.33)$$

where

$$\begin{aligned}\alpha_{0\lambda} &= \varepsilon'_\sigma(L) \omega^\sigma \wedge \omega_0 + B_{\sigma\nu}^{(ji)}(L) \omega^\sigma \wedge dy_j^\nu \wedge \omega_i, \\ \alpha_\lambda &= \varepsilon'_\sigma(L) \omega^\sigma \wedge \omega_0 + B_{\sigma\nu}^{ji}(L) \omega^\sigma \wedge dy_j^\nu \wedge \omega_i.\end{aligned}\quad (2.34)$$

Euler–Lagrange equations (2.32) then read

$$J^1\gamma^*i_\xi\alpha = 0 \quad \text{for every vertical vector field } \xi \text{ on } J^1Y, \quad (2.35)$$

where  $\alpha$  is any element belonging to the first-order Lepage class of  $E_\lambda$ , and they are equations for holonomic integral sections of the ideal  $\mathcal{H}_\alpha$  (2.11).

**Remark 2.11.** It is known how to recognize whether a dynamical form  $E$  coincides (at least locally) with the Euler–Lagrange form of a Lagrangian (see [12] for second-order ODEs, [1,18] for PDEs of any order). Necessary and sufficient conditions for variationality of second-order dynamical forms take the following form of conditions on the “left-hand sides” of the corresponding equations:

$$\begin{aligned}\frac{\partial E_\sigma}{\partial y_{ij}^\nu} - \frac{\partial E_\nu}{\partial y_{ij}^\sigma} &= 0, \quad \frac{\partial E_\sigma}{\partial y_j^\nu} + \frac{\partial E_\nu}{\partial y_j^\sigma} - 2d_i \frac{\partial E_\nu}{\partial y_{ij}^\sigma} = 0, \\ \frac{\partial E_\sigma}{\partial y^\nu} - \frac{\partial E_\nu}{\partial y^\sigma} + d_i \frac{\partial E_\nu}{\partial y_i^\sigma} - d_i d_j \frac{\partial E_\nu}{\partial y_{ij}^\sigma} &= 0.\end{aligned}\quad (2.36)$$

A (local) Lagrangian then can be computed using the Tonti–Vainberg formula

$$L = y^\sigma \int_0^1 E_\sigma \left( x^i, uy^v, uy_j^v, uy_{jk}^v \right) du. \quad (2.37)$$

Next, it is known that a dynamical form  $E$  is (locally) variational if and only if the Lepage class  $[\alpha]^2$  of  $E$  contains a closed representative (i.e., there exists  $\alpha \in [\alpha]^2$  such that  $d\alpha = 0$  [18,22,11]).

### 3. Constraint structure in $J^1Y$

**Definition 3.1.** By a non-holonomic constraint in  $J^1Y$  we shall mean a fibered submanifold  $Q$  of  $\pi_{1,0} : J^1Y \rightarrow Y$ ,  $\text{codim } Q = \kappa$ , where  $1 \leq \kappa \leq mn - 1$ . This means that in any fibered chart a constraint  $Q$  can be expressed by equations

$$f^\alpha \left( x^i, y^\sigma, y_j^\sigma \right) = 0, \quad 1 \leq \alpha \leq \kappa, \quad (3.1)$$

such that

$$\text{rank} \left( \frac{\partial f^\alpha}{\partial y_j^\sigma} \right) = \kappa, \quad \text{where } \alpha \text{ labels rows and } \sigma, j \text{ label columns.} \quad (3.2)$$

If moreover

$$\text{rank} \left( \frac{\partial f^\alpha}{\partial y_j^\sigma} \right) = k, \quad \text{where } \alpha, j \text{ label rows and } \sigma \text{ labels columns,} \quad (3.3)$$

for some  $k$ ,  $1 \leq k \leq m-1$ , we say that  $Q$  is a regular (non-holonomic) constraint of corank  $(\kappa, k)$ .

**Remark 3.2.** Notice that condition (3.3) is invariant. Indeed, with obvious notations we have

$$\bar{F}_\sigma^{\alpha j} = \frac{\partial f^\alpha}{\partial \bar{y}_j^\sigma} = \frac{\partial f^\alpha}{\partial y_l^v} \frac{\partial y_l^v}{\partial \bar{y}_j^\sigma} = \frac{\partial f^\alpha}{\partial y_l^v} \frac{\partial y_l^v}{\partial \bar{y}^\sigma} \frac{\partial \bar{x}^j}{\partial x^l} = F_v^{\alpha l} B_\sigma^v A_l^j,$$

i.e., in matrix notation,

$$\begin{pmatrix} \bar{F}^1 \\ \bar{F}^2 \\ \vdots \\ \bar{F}^\kappa \end{pmatrix} = \begin{pmatrix} A F^1 \\ A F^2 \\ \vdots \\ A F^\kappa \end{pmatrix} \cdot B = \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & & A \end{pmatrix} \cdot \begin{pmatrix} F^1 \\ F^2 \\ \vdots \\ F^\kappa \end{pmatrix} \cdot B.$$

Since the matrices  $A, B$  are regular, we get

$$\text{rank} \left( \frac{\partial f^\alpha}{\partial \bar{y}_j^\sigma} \right) = \text{rank} \begin{pmatrix} \bar{F}^1 \\ \bar{F}^2 \\ \vdots \\ \bar{F}^\kappa \end{pmatrix} = \text{rank} \begin{pmatrix} F^1 \\ F^2 \\ \vdots \\ F^\kappa \end{pmatrix} = \text{rank} \left( \frac{\partial f^\alpha}{\partial y_j^\sigma} \right),$$

as desired.

Let  $(V, \psi)$  be a fibered chart on  $Y$ ,  $(V_1, \psi_1)$  the associated chart on  $J^1Y$ ,  $U \subset V_1$  an open set. A regular constraint  $Q$  in  $J^1Y$  of corank  $(\kappa, k)$  naturally gives rise to the following distributions, defined on  $U$ :

- (1)  $\mathcal{D}_U$ , annihilated by the 1-forms  $df^\alpha$ ,  $1 \leq \alpha \leq \kappa$ . The rank condition (3.2) guarantees that  $\mathcal{D}_U$  has a constant corank equal to  $\kappa$  on  $U$ , i.e., its rank is  $n + m + nm - \kappa$ .

(2)  $\tilde{C}'_U$ , annihilated by the following 1-forms,

$$\phi^{\alpha j} = f^\alpha dx^j + \frac{1}{n} \frac{\partial f^\alpha}{\partial y_j^\sigma} \omega^\sigma, \quad 1 \leq \alpha \leq \kappa, \quad 1 \leq j \leq n. \quad (3.4)$$

These 1-forms are not independent, however, due to the rank condition (3.3), there exist functions  $c_{\alpha j}^a$ ,  $1 \leq a \leq k$ ,  $1 \leq \alpha \leq \kappa$ ,  $1 \leq j \leq n$ , on  $U$ , such that the  $(k \times m)$ -matrix

$$M = (M_\sigma^a), \quad \text{where} \quad M_\sigma^a = \frac{1}{n} c_{\alpha j}^a \frac{\partial f^\alpha}{\partial y_j^\sigma}, \quad (3.5)$$

has maximal rank (equal to  $k$ ). This means that

$$\phi^a = c_{\alpha j}^a \phi^{\alpha j} = c_{\alpha j}^a f^\alpha dx^j + \frac{1}{n} c_{\alpha j}^a \frac{\partial f^\alpha}{\partial y_j^\sigma} \omega^\sigma = c_{\alpha j}^a f^\alpha dx^j + M_\sigma^a \omega^\sigma, \quad 1 \leq a \leq k, \quad (3.6)$$

are independent at each point of  $U$ . Hence, the distribution

$$\tilde{C}_U = \text{annih}\{\phi^a, 1 \leq a \leq k\}, \quad (3.7)$$

has a constant corank  $k$ , i.e.,  $\text{rank } \tilde{C}_U = n + m + nm - k$ . 1-forms annihilating the distribution  $\tilde{C}_U$  will be called *canonical constraint 1-forms* of the constraint  $Q$ .

(3)  $C_U$ , annihilated by  $k + \kappa$  independent 1-forms  $\phi^a, df^\alpha$ ,  $1 \leq a \leq k$ ,  $1 \leq \alpha \leq \kappa$ .

Immediately from the above constructions we can see that the following assertions hold:

**Proposition 3.3.**  $Q \cap U$  is an integral submanifold of  $\mathcal{D}_U$ . Hence, for every  $x \in Q$ , the forms  $df^\alpha(x)$ ,  $1 \leq \alpha \leq \kappa$ , annihilate the tangent space  $T_x Q$  to the manifold  $Q$  at  $x$ , i.e., along  $Q$ ,  $\mathcal{D} = \text{annih}\{df^\alpha, 1 \leq \alpha \leq \kappa\} = TQ$ .

**Corollary 3.4.** Let  $Q$  be a constraint of codimension  $\kappa$  in  $J^1 Y$ , and let  $f^\alpha = 0$  and  $f'^\alpha = 0$ , where  $1 \leq \alpha \leq \kappa$ , be two sets of equations of  $Q$  on an open set  $U \subset V_1 \subset J^1 Y$ . Then there are functions  $\gamma_\beta^\alpha$  on  $U$  such that at each point of  $U$ ,  $(\gamma_\beta^\alpha)$  is a regular matrix, and  $df'^\alpha = \gamma_\beta^\alpha df^\beta$ . In particular, at each point  $x \in Q \cap U$ ,

$$\frac{\partial f'^\alpha}{\partial y_j^\sigma} = \gamma_\beta^\alpha \frac{\partial f^\beta}{\partial y_j^\sigma}. \quad (3.8)$$

**Proposition 3.5.**  $\mathcal{C}_U$  is a subdistribution of both  $\tilde{\mathcal{C}}_U$  and  $\mathcal{D}_U$ . At the points of  $Q \cap U$ , the distributions  $\mathcal{C}_U$  and  $\tilde{\mathcal{C}}_U \cap \mathcal{D}$  coincide, and define a distribution of corank  $k$  on  $Q \cap U$ .

Now, we shall show that the above local distributions on the constraint  $Q$  unite into a global distribution on  $Q$ .

**Theorem 3.6.** Let  $Q$  be a regular constraint in  $J^1Y$  of corank  $(\kappa, k)$ , let  $\iota: Q \rightarrow J^1Y$  be the canonical embedding of the submanifold  $Q$  into  $J^1Y$ . If  $\phi^a$ ,  $1 \leq a \leq k$ , are independent local canonical constraint 1-forms, put

$$\varphi^a = \iota^* \phi^a = (M_\sigma^a \circ \iota) \iota^* \omega^\sigma, \quad 1 \leq a \leq k. \quad (3.9)$$

Then

$$\mathcal{C} = \text{annih}\{\varphi^a, 1 \leq a \leq k\} \quad (3.10)$$

is a distribution of corank  $k$  on  $Q$ .

**Proof.** Taking into account Propositions 3.3 and 3.5, it is sufficient to show that if  $\phi^{\alpha j}$  defined on  $U_1$  and  $\bar{\phi}^{\alpha j}$  defined on  $U_2$  such that  $U_1 \cap U_2 \cap Q \neq \emptyset$  are constraint 1-forms of  $Q$  annihilating the distribution  $\tilde{\mathcal{C}}'_{U_1}$  and  $\tilde{\mathcal{C}}'_{U_2}$ , respectively, then on  $U_1 \cap U_2 \cap Q$ ,

$$\bar{\phi}^{\alpha j} = c_{\beta l}^{\alpha j} \phi^{\beta l} \quad (3.11)$$

for some functions  $c_{\beta l}^{\alpha j}$ , meaning that  $\tilde{\mathcal{C}}_{U_1} = \tilde{\mathcal{C}}_{U_2}$  at the points of  $U_1 \cap U_2 \cap Q$ .

Denote  $(x^i, y^\sigma, y_j^\sigma)$  and  $(\bar{x}^i, \bar{y}^\sigma, \bar{y}_j^\sigma)$  associated fibered coordinates on  $U_1$  and  $U_2$ , respectively, and assume that the constraint  $Q$  is given by the equations  $f^\alpha(x^i, y^\sigma, y_j^\sigma) = 0$  on  $U_1$ , and  $f'^\alpha(\bar{x}^i, \bar{y}^\sigma, \bar{y}_j^\sigma) = 0$  on  $U_2$ , where  $1 \leq \alpha \leq \kappa$ . We have

$$\phi^{\alpha j} = f^\alpha dx^j + \frac{1}{n} \frac{\partial f^\alpha}{\partial y_j^\sigma} \omega^\sigma, \quad \bar{\phi}^{\alpha j} = f'^\alpha d\bar{x}^j + \frac{1}{n} \frac{\partial f'^\alpha}{\partial \bar{y}_j^\sigma} \bar{\omega}^\sigma. \quad (3.12)$$

Now, by transformation rules and by (3.8), we get

$$\begin{aligned} n\bar{\phi}^{\alpha j} &= \iota^* \left( \frac{\partial f'^\alpha}{\partial \bar{y}_j^\sigma} \bar{\omega}^\sigma \right) = \iota^* \left( \frac{\partial f'^\alpha}{\partial y_l^\nu} \frac{\partial y_l^\nu}{\partial \bar{y}_j^\sigma} \frac{\partial \bar{y}^\sigma}{\partial y^\rho} \omega^\rho \right) = \iota^* \left( \gamma_\beta^\alpha \frac{\partial f^\beta}{\partial y_l^\nu} \frac{\partial \bar{x}^j}{\partial x^l} \frac{\partial y^\nu}{\partial \bar{y}^\sigma} \frac{\partial \bar{y}^\sigma}{\partial y^\rho} \omega^\rho \right) \\ &= \iota^* \left( \gamma_\beta^\alpha \frac{\partial \bar{x}^j}{\partial x^l} \frac{\partial f^\beta}{\partial y_l^\nu} \omega^\nu \right) = c_{\beta l}^{\alpha j} \iota^* \phi^{\beta l} = c_{\beta l}^{\alpha j} n\phi^{\beta l}, \end{aligned} \quad (3.13)$$

proving our assertion.  $\square$

**Definition 3.7.** The distribution  $\mathcal{C}$  on  $Q$  defined in Theorem 3.6 will be called canonical distribution. 1-forms belonging to the annihilator,  $\mathcal{C}^0$ , of  $\mathcal{C}$ , will be called canonical constraint 1-forms. The ideal in the exterior algebra of differential forms on  $Q$  generated by  $\mathcal{C}^0$  will be called canonical constraint ideal, and denoted by  $\mathcal{I}(\mathcal{C}^0)$ . Elements of  $\mathcal{I}(\mathcal{C}^0)$  will then be called canonical constraint forms.

Note that, by definition,  $\mathcal{C}$  is the *characteristic distribution* of the ideal  $\mathcal{I}(\mathcal{C}^0)$ .

Let us find vector fields belonging to the canonical distribution.

**Theorem 3.8.** *The canonical distribution  $\mathcal{C}$  on  $Q$  is locally spanned by the following vector fields:*

$$\begin{aligned}\frac{\partial_c}{\partial x^i} &\equiv \frac{\partial}{\partial x^i} + \sum_{a=1}^k (F_i^a \circ \iota) \frac{\partial}{\partial y^{m-k+a}}, \quad 1 \leq i \leq n, \\ \frac{\partial_c}{\partial y^s} &\equiv \frac{\partial}{\partial y^s} + \sum_{a=1}^k (G_s^a \circ \iota) \frac{\partial}{\partial y^{m-k+a}}, \quad 1 \leq s \leq m-k, \\ \frac{\partial}{\partial z^J}, \quad &1 \leq J \leq nm - \kappa,\end{aligned}\tag{3.14}$$

where  $(x^i, y^\sigma, z^J, f^\alpha)$ ,  $1 \leq i \leq n$ ,  $1 \leq \sigma \leq m$ ,  $1 \leq J \leq nm - \kappa$ ,  $1 \leq \alpha \leq \kappa$ , denote fibered coordinates adapted to the submanifold  $\iota : Q \rightarrow J^1Y$ , the functions  $G_s^a$  represent (at each point) a fundamental system of solutions of the system of independent homogeneous algebraic equations for  $m$  unknowns  $\Xi^\sigma$ ,  $1 \leq \sigma \leq m$ ,

$$M_\sigma^a \Xi^\sigma = 0, \quad 1 \leq a \leq k,\tag{3.15}$$

and, for every  $i = 1, 2, \dots, n$ , the  $F_i^a$  are solutions of the equations

$$M_\sigma^a F_i^\sigma = M_\sigma^a y_i^\sigma - f^\alpha c_{\alpha i}^a, \quad 1 \leq a \leq k,\tag{3.16}$$

(where  $y_j^\sigma$  are considered as functions of  $z^J, f^\beta$ ) corresponding to the choice of all the parameters equal to zero.

**Proof.** The rank condition (3.2) guarantees that in a neighborhood of every point in  $Q$  one can find coordinates  $(x^i, y^\sigma, z^J, f^\alpha)$ , where  $1 \leq i \leq n$ ,  $1 \leq \sigma \leq m$ , and  $1 \leq J \leq nm - \kappa$ ,  $1 \leq \alpha \leq \kappa$ . Consider the distribution  $\tilde{\mathcal{C}}_U$  on  $U \subset J^1Y$  such that  $U \cap Q \neq \emptyset$ . For a vector field  $\xi$  on  $U$  denote

$$\xi = \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma} + \tilde{\Xi}^J \frac{\partial}{\partial z^J} + \tilde{\Xi}^\alpha \frac{\partial}{\partial f^\alpha}.\tag{3.17}$$

The condition  $i_{\xi}\phi^a = 0$  for all  $a$ , gives us the following system of equations for the components of  $\xi$ :

$$c_{\alpha j}^a f^{\alpha} \xi^j + M_{\sigma}^a \left( \Xi^{\sigma} - y_l^{\sigma} \xi^l \right) = 0, \quad 1 \leq a \leq k, \quad (3.18)$$

i.e.,

$$M_{\sigma}^a \Xi^{\sigma} = \left( M_{\sigma}^a y_l^{\sigma} - c_{\alpha l}^a f^{\alpha} \right) \xi^l, \quad 1 \leq a \leq k, \quad (3.19)$$

where  $y_j^{\nu}$  are functions of the adapted coordinates  $(x^i, y^{\sigma}, z^J, f^{\beta})$ . By assumption,  $\text{rank } M = k$ . This means that one can express  $k$  of the functions  $\Xi^{\sigma}$ , e.g. (without loss of generality)  $\Xi^{m-k+a}$ , where  $1 \leq a \leq k$ , by means of ‘parameters’  $\xi^i, \Xi^s, 1 \leq i \leq n, 1 \leq s \leq m-k$ . Hence, the general solution of  $i_{\xi}\phi^a = 0, 1 \leq a \leq k$ , is

$$\begin{aligned} \xi = & \xi^i \frac{\partial}{\partial x^i} + \sum_{s=1}^{m-k} \Xi^s \frac{\partial}{\partial y^s} + \sum_{a=1}^k \Xi^{m-k+a} \left( \xi^1, \dots, \xi^n, \Xi^1, \dots, \Xi^{m-k} \right) \frac{\partial}{\partial y^{m-k+a}} \\ & + \Xi^J \frac{\partial}{\partial z^J} + \Xi^{\alpha} \frac{\partial}{\partial f^{\alpha}}, \end{aligned} \quad (3.20)$$

where  $\xi^i, \Xi^s, \Xi^J$  and  $\Xi^{\alpha}$  are arbitrary functions and  $\Xi^{m-k+a}$  are solutions of Eqs. (3.19). Hence, one can take independent vector fields on  $U$  spanning the distribution  $\tilde{\mathcal{C}}_U$  as follows:

$$\begin{aligned} & \frac{\partial}{\partial x^i} + \sum_{a=1}^k F_i^a \frac{\partial}{\partial y^{m-k+a}}, \quad 1 \leq i \leq n, \\ & \frac{\partial}{\partial y^s} + \sum_{a=1}^k G_s^a \frac{\partial}{\partial y^{m-k+a}}, \quad 1 \leq s \leq m-k, \\ & \frac{\partial}{\partial z^J}, \quad \frac{\partial}{\partial f^{\alpha}}, \quad 1 \leq J \leq nm - \kappa, \quad 1 \leq \alpha \leq \kappa, \end{aligned} \quad (3.21)$$

where  $G_s^a, 1 \leq s \leq m-k$ , is a fundamental system of solutions of (3.15) (i.e., (3.19) with  $\xi^1 = \dots = \xi^n = 0$ ), and  $F_i^a, 1 \leq i \leq n$ , are solutions of (3.19) for  $\Xi^1 = \dots = \Xi^{m-k} = 0$  (here the subscript  $i$  corresponds to the choice  $\xi^i = 1, \xi^j = 0$  for  $j \neq i$ ). Since  $\mathcal{C} = \tilde{\mathcal{C}}_U \cap TQ$ , we finally get

$$\mathcal{C} = \text{span} \left\{ \frac{\partial}{\partial x^i} + \sum_{a=1}^k \bar{F}_i^a \frac{\partial}{\partial y^{m-k+a}}, \frac{\partial}{\partial y^s} + \sum_{a=1}^k \bar{G}_s^a \frac{\partial}{\partial y^{m-k+a}}, \frac{\partial}{\partial z^J} \right\}, \quad (3.22)$$



where, as above,  $1 \leq i \leq n$ ,  $1 \leq s \leq m - k$ ,  $1 \leq J \leq nm - \kappa$ , and  $\bar{F}_i^a = F_i^a \circ \iota$ ,  $\bar{G}_s^a = G_s^a \circ \iota$ .  $\square$

The canonical distribution  $\mathcal{C}$  on  $Q$  is a subbundle of the tangent bundle  $TQ \rightarrow Q$ . In general, however, it *need not be completely integrable*. We shall study conditions for the complete integrability of  $\mathcal{C}$  in Section 5.

**Remark 3.9** (Notations adapted to the constraint structure). (i) The following conventions concerning notation of indices will be used, and summation over repeated indices will be understood (if not otherwise explicitly stated):

$$\begin{aligned} 1 \leq i, j, l \leq n, \quad 1 \leq \alpha, \beta, \gamma \leq \kappa, \quad 1 \leq J \leq nm - \kappa, \\ 1 \leq \sigma, \nu, \rho \leq m, \quad 1 \leq a, b, c \leq k, \quad 1 \leq p, r, s \leq m - k. \end{aligned} \quad (3.23)$$

(ii) Taking into account that the matrix (3.5) in (3.6) has maximal rank,  $k$ , one can express  $k$  of the contact 1-forms  $\omega^\sigma$  by means of the constraint forms  $\phi^a$ ,  $1 \leq a \leq m$ , and the remaining  $\omega^\nu$ 's. Without loss of generality we may suppose that this concerns the forms  $\omega^{m-k+a}$ , where  $1 \leq a \leq k$ . In an adapted basis  $(x^i, y^\sigma, z^J, f^\alpha)$  and in the notations of the above theorem it holds

$$\begin{aligned} \omega^{m-k+a} &= \mu_b^a \left( \phi^b - M_s^b \omega^s - c_{\alpha j}^b f^\alpha dx^j \right) \\ &= \mu_b^a \phi^b + G_s^a \omega^s + \left( F_j^a + G_s^a y_j^s - y_j^{m-k+a} \right) dx^j, \end{aligned} \quad (3.24)$$

where  $(\mu_b^a)$  is an appropriate regular matrix. Here and in what follows,  $y_j^\sigma$  are considered as functions of the adapted coordinates  $(x^i, y^\sigma, z^J, f^\alpha)$ . Similarly, the rank condition (3.2) guarantees that one can express the forms  $dz^\alpha$  by means of  $(df^\beta, dx^i, dy^\sigma, dz^J)$ . Thus, we have on  $J^1Y$  the following bases of 1-forms, adapted to the constraint structure:

$$\left( dx^i, dy^s, \phi^a, dz^J, df^A \right), \quad \text{or} \quad \left( dx^i, \omega^s, \phi^a, dz^J, df^A \right); \quad (3.25)$$

Consequently, with obvious notations we may write

$$\bar{\omega}^{m-k+a} \equiv \iota^* \omega^{m-k+a} = \bar{\varphi}^a + \bar{G}_s^a \bar{\omega}^s, \quad (3.26)$$

where  $\bar{\omega}^s = \iota^* \omega^s$ , and  $\bar{\varphi}^a = \iota^*(\mu_b^a \phi^b) = (\mu_b^a \circ \iota) \phi^b$ . We can see that, on  $Q$ , instead of a canonical basis  $(dx^i, dy^\sigma, dz^J)$ , or a basis  $(dx^i, \bar{\omega}^\sigma, dz^J)$  adapted to the induced contact structure, it is worth to work with bases adapted to the constraint structure,

where the canonical constraint 1-forms appear:

$$(dx^i, dy^s, \bar{\varphi}^a, dz^J), \quad (dx^i, \bar{\omega}^s, \bar{\varphi}^a, dz^J). \quad (3.27)$$

(iii) Keeping the above notations we can express the functions  $G_s^a$  and  $F_j^a$  appearing in (3.21) and (3.14) as follows:

$$G_s^a = \mu_b^a M_s^b, \quad F_j^a = y_j^{m-k+a} - G_s^a y_j^s - \mu_b^a c_{\alpha j}^b f^\alpha. \quad (3.28)$$

We also put

$$y_j^\sigma \circ \iota = g_j^\sigma. \quad (3.29)$$

With this notation,

$$\bar{G}_s^a = (\mu_b^a M_s^b) \circ \iota, \quad \bar{F}_j^a = g_j^{m-k+a} - \bar{G}_s^a g_j^s. \quad (3.30)$$

(iv) The vector fields  $\partial_c/\partial x^i$  and  $\partial_c/\partial y^s$  on  $Q$  defined by (3.14) will be called *constraint partial derivative operators*. We put

$$\begin{aligned} \frac{d'_c}{dx^i} &= \frac{\partial_c}{\partial x^i} + g_i^s \frac{\partial_c}{\partial y^s}, \\ \frac{d_c}{dx^i} &= \frac{\partial_c}{\partial x^i} + g_i^s \frac{\partial_c}{\partial y^s} + z_i^J \frac{\partial}{\partial z^J} = \frac{d'_c}{dx^i} + z_i^J \frac{\partial}{\partial z^J}, \end{aligned} \quad (3.31)$$

and call the above operators the *ith cut constraint* and *constraint total derivative operator*, respectively. We note that the operators  $d_c/dx^i$  act on functions on  $Q$ , giving rise to functions on  $\tilde{Q}$ , the lift of  $Q$  in  $J^2Y$  (defined as the manifold of all points  $J_x^2\gamma \in J^2Y$  such that  $J_x^1\gamma \in Q$ ).

(v) The exterior derivative of a function  $f$  on  $Q$  can be expressed as follows:

$$\begin{aligned} df &= \left( \frac{\partial f}{\partial x^j} + \frac{\partial f}{\partial y^\sigma} g_j^\sigma \right) dx^j + \frac{\partial f}{\partial y^s} \bar{\omega}^s + \frac{\partial f}{\partial y^{m-k+a}} \bar{\omega}^{m-k+a} + \frac{\partial f}{\partial z^J} dz^J \\ &= \left( \frac{\partial f}{\partial x^j} + \frac{\partial f}{\partial y^s} g_j^s + \frac{\partial f}{\partial y^{m-k+a}} (\bar{F}_j^a + \bar{G}_s^a g_j^s) \right) dx^j \\ &\quad + \left( \frac{\partial f}{\partial y^s} + \bar{G}_s^a \frac{\partial f}{\partial y^{m-k+a}} \right) \bar{\omega}^s + \frac{\partial f}{\partial y^{m-k+a}} \bar{\varphi}^a + \frac{\partial f}{\partial z^J} dz^J \\ &= \frac{d'_c f}{dx^j} dx^j + \frac{\partial_c f}{\partial y^s} \bar{\omega}^s + \frac{\partial f}{\partial y^{m-k+a}} \bar{\varphi}^a + \frac{\partial f}{\partial z^J} dz^J, \end{aligned} \quad (3.32)$$

since by (3.30)

$$g_j^{m-k+a} = \bar{F}_j^a + \bar{G}_s^a g_j^s. \quad (3.33)$$

(vi) Let us compute the explicit expression for  $d\bar{\varphi}^a$  which will often be used later

$$\begin{aligned} d\bar{\varphi}^a &= d\bar{\omega}^{m-k+a} - d\bar{G}_s^a \wedge \bar{\omega}^s - \bar{G}_s^a d\bar{\omega}^s \\ &= -dg_j^{m-k+a} \wedge dx^j - d\bar{G}_s^a \wedge \bar{\omega}^s + \bar{G}_s^a dg_j^s \wedge dx^j \\ &= \left( \bar{G}_s^a \frac{d'_c g_j^s}{dx^i} - \frac{d'_c g_j^{m-k+a}}{dx^i} \right) dx^i \wedge dx^j \\ &\quad + \left( \bar{G}_s^a \frac{\partial_c g_j^s}{\partial y^r} + \frac{d'_c \bar{G}_r^a}{dx^j} - \frac{\partial_c g_j^{m-k+a}}{\partial y^r} \right) \bar{\omega}^r \wedge dx^j \\ &\quad + \left( \bar{G}_s^a \frac{\partial g_j^s}{\partial z^J} - \frac{\partial g_j^{m-k+a}}{\partial z^J} \right) dz^J \wedge dx^j - \frac{\partial_c \bar{G}_s^a}{\partial y^r} \bar{\omega}^r \wedge \bar{\omega}^s - \frac{\partial \bar{G}_s^a}{\partial z^J} dz^J \wedge \bar{\omega}^s \\ &\quad + \left( \bar{G}_s^a \frac{\partial g_j^s}{\partial y^{m-k+b}} - \frac{\partial g_j^{m-k+a}}{\partial y^{m-k+b}} \right) \bar{\varphi}^b \wedge dx^j - \frac{\partial \bar{G}_s^a}{\partial y^{m-k+b}} \bar{\varphi}^b \wedge \bar{\omega}^s, \end{aligned} \quad (3.34)$$

and denote

$$C_{Jj}^a = \bar{G}_s^a \frac{\partial g_j^s}{\partial z^J} - \frac{\partial g_j^{m-k+a}}{\partial z^J} = -\frac{\partial \bar{F}_j^a}{\partial z^J} - \frac{\partial \bar{G}_s^a}{\partial z^J} g_j^s, \quad C_{js}^{ai} = C_{Jj}^a \frac{\partial z^J}{\partial y_i^s}. \quad (3.35)$$

## 4. Constrained systems

### 4.1. Constrained PDEs

Let  $Q$  be a regular constraint in  $J^1Y$ ,  $\mathcal{I}(\mathcal{C}^0)$  the associated canonical constraint ideal. Since for every  $q$ -contact form  $\eta$  on  $J^1Y$   $i^*\eta$  is a  $q$ -contact form on  $Q$ , we have the following equivalence relation on  $(n+1)$ -forms on  $Q$ :

$$\alpha_1 \approx \alpha_2 \quad \text{if} \quad \alpha_1 - \alpha_2 = \bar{F} + \varphi, \quad (4.1)$$

where  $\bar{F}$  is an at least 2-contact  $(n+1)$ -form on  $Q$ , and  $\varphi$  is a constraint  $(n+1)$ -form. We denote by  $[[\alpha]]$  the class of  $\alpha$ .

If  $[\alpha]$  is a Lepage class on  $J^1Y$  then obviously for any of its two elements,

$$\alpha_1 \sim \alpha_2 \Rightarrow i^*\alpha_1 \approx i^*\alpha_2. \quad (4.2)$$

This enables us to associate with a given system of second-order PDEs, polynomial in the second derivatives, a system of equations defined on the constraint  $Q$ . Recall that the equations we consider are characterized by a dynamical form with components (“left-hand sides” of the equations) given by Proposition 2.7 (formula (2.18)).

**Definition 4.1.** Let  $E$  be a  $J^1Y$ -pertinent dynamical form on  $Y$ ,  $[\alpha]$  its Lepage class on  $J^1Y$ . By the constrained system associated with  $E$  and the constraint  $Q$  we shall mean the equivalence class  $[[\iota^*\alpha]]$ .

A general element of the class  $[[\iota^*\alpha]]$  is of the form

$$\bar{\alpha} = \iota^*\alpha + \bar{F} + \varphi, \quad (4.3)$$

where  $\alpha \in [\alpha]$  is any  $(n+1)$ -form related with  $E$ ,  $\bar{F}$  is at least 2-contact, and  $\varphi \in \mathcal{I}(C^0)$ . In particular, we have a distinguished representative  $\bar{\alpha}_0 = \iota^*\alpha_0 \in [[\iota^*\alpha]]$  (cf. (2.20)), as well as those  $\bar{\alpha} \approx \bar{\alpha}_0$  which belong to the ideal generated by the forms  $\bar{\omega}^\sigma = \iota^*\omega^\sigma$ .

**Proposition 4.2.** In adapted fibered coordinates  $(x^i, y^\sigma, z^J)$  on  $Q$ ,

$$\begin{aligned} \bar{\alpha}_0 \approx & \mathcal{A}_s \bar{\omega}^s \wedge \omega_0 + \mathcal{B}_{sJ_1}^{i_1} \bar{\omega}^s \wedge dz^{J_1} \wedge \omega_{i_1} + \frac{1}{2} \mathcal{B}_{sJ_1J_2}^{i_1i_2} \bar{\omega}^s \wedge dz^{J_1} \wedge dz^{J_2} \wedge \omega_{i_1i_2} \\ & + \cdots + \frac{1}{n!} \mathcal{B}_{sJ_1 \dots J_n}^{i_1 \dots i_n} \bar{\omega}^s \wedge dz^{J_1} \wedge \cdots \wedge dz^{J_n} \wedge \omega_{i_1 \dots i_n}, \end{aligned} \quad (4.4)$$

where

$$\begin{aligned} \mathcal{A}_s = & \bar{A}_s + \bar{A}_{m-k+a} \bar{G}_s^a + \left( \bar{B}_{sv_1}^{j_1i_1} + \bar{B}_{m-k+a v_1}^{j_1i_1} \bar{G}_s^a \right) \frac{d'_c g_{j_1}^{v_1}}{dx^{i_1}} \\ & + \left( \bar{B}_{sv_1v_2}^{j_1i_1j_2i_2} + \bar{B}_{m-k+av_1v_2}^{j_1i_1j_2i_2} \bar{G}_s^a \right) \frac{d'_c g_{j_1}^{v_1}}{dx^{i_1}} \frac{d'_c g_{j_2}^{v_2}}{dx^{i_2}} \\ & + \cdots + \left( \bar{B}_{sv_1 \dots v_n}^{j_1i_1 \dots j_ni_n} + \bar{B}_{m-k+av_1 \dots v_n}^{j_1i_1 \dots j_ni_n} \bar{G}_s^a \right) \frac{d'_c g_{j_1}^{v_1}}{dx^{i_1}} \cdots \frac{d'_c g_{j_n}^{v_n}}{dx^{i_n}}, \\ \mathcal{B}_{sJ_1}^{i_1} = & \left( \bar{B}_{sv_1}^{j_1i_1} + \bar{B}_{m-k+av_1}^{j_1i_1} \bar{G}_s^a \right) \frac{\partial g_{j_1}^{v_1}}{\partial z^{J_1}} \\ & + 2 \left( \bar{B}_{sv_1v_2}^{j_1i_1j_2i_2} + \bar{B}_{m-k+av_1v_2}^{j_1i_1j_2i_2} \bar{G}_s^a \right) \frac{\partial g_{j_1}^{v_1}}{\partial z^{J_1}} \frac{d'_c g_{j_2}^{v_2}}{dx^{i_2}} \\ & + \cdots + n \left( \bar{B}_{sv_1 \dots v_n}^{j_1i_1 \dots j_ni_n} + \bar{B}_{m-k+av_1 \dots v_n}^{j_1i_1 \dots j_ni_n} \bar{G}_s^a \right) \frac{\partial g_{j_1}^{v_1}}{\partial z^{J_1}} \frac{d'_c g_{j_2}^{v_2}}{dx^{i_2}} \cdots \frac{d'_c g_{j_n}^{v_n}}{dx^{i_n}}, \end{aligned}$$

$$\begin{aligned}
\mathcal{B}_{sJ_1J_2}^{i_1i_2} &= \left( \bar{B}_{sv_1v_2}^{j_1i_1j_2i_2} + \bar{B}_{m-k+av_1v_2}^{j_1i_1j_2i_2} \bar{G}_s^a \right) \frac{\partial g_{j_1}^{v_1}}{\partial z^{J_1}} \frac{\partial g_{j_2}^{v_2}}{\partial z^{J_2}} \\
&\quad + \cdots + \frac{n(n-1)}{2} \left( \bar{B}_{sv_1 \dots v_n}^{j_1i_1 \dots j_ni_n} + \bar{B}_{m-k+av_1 \dots v_n}^{j_1i_1 \dots j_ni_n} \bar{G}_s^a \right) \\
&\quad \times \frac{\partial g_{j_1}^{v_1}}{\partial z^{J_1}} \frac{\partial g_{j_2}^{v_2}}{\partial z^{J_2}} \frac{d'_c g_{j_3}^{v_3}}{dx^{i_3}} \cdots \frac{d'_c g_{j_n}^{v_n}}{dx^{i_n}}, \\
&\quad \vdots \\
\mathcal{B}_{sJ_1 \dots J_n}^{i_1 \dots i_n} &= \left( \bar{B}_{sv_1 \dots v_n}^{j_1i_1 \dots j_ni_n} + \bar{B}_{m-k+av_1 \dots v_n}^{j_1i_1 \dots j_ni_n} \bar{G}_s^a \right) \frac{\partial g_{j_1}^{v_1}}{\partial z^{J_1}} \cdots \frac{\partial g_{j_n}^{v_n}}{\partial z^{J_n}}, \tag{4.5}
\end{aligned}$$

and  $\bar{A}_\sigma = A_\sigma \circ \iota$ ,  $\bar{B}_{\sigma v_1}^{j_1 i_1} = B_{\sigma v_1}^{j_1 i_1} \circ \iota$ , etc.

**Proof.** By (2.20) and in the notations of Remark 3.9 we have

$$\begin{aligned}
\bar{\alpha}_0 &= \iota^* \alpha_0 = \bar{A}_\sigma \bar{\omega}^\sigma \wedge \omega_0 + \bar{B}_{\sigma v_1}^{j_1 i_1} \bar{\omega}^\sigma \wedge dg_{j_1}^{v_1} \wedge \omega_{i_1} \\
&\quad + \frac{1}{2} \bar{B}_{\sigma v_1 v_2}^{j_1 i_1 j_2 i_2} \bar{\omega}^\sigma \wedge dg_{j_1}^{v_1} \wedge dg_{j_2}^{v_2} \wedge \omega_{i_1 i_2} \\
&\quad + \cdots + \frac{1}{n!} \bar{B}_{\sigma v_1 \dots v_n}^{j_1 i_1 \dots j_n i_n} \bar{\omega}^\sigma \wedge dg_{j_1}^{v_1} \wedge \cdots \wedge dg_{j_n}^{v_n} \wedge \omega_{i_1 \dots i_n} \\
&\approx (\bar{A}_s + \bar{A}_{m-k+a} \bar{G}_s^a) \bar{\omega}^s \wedge \omega_0 \\
&\quad + \left( \bar{B}_{sv_1}^{j_1 i_1} + \bar{B}_{m-k+a v_1}^{j_1 i_1} \bar{G}_s^a \right) \bar{\omega}^s \wedge \left( \frac{d'_c g_{j_1}^{v_1}}{dx^{l_1}} dx^{l_1} + \frac{\partial g_{j_1}^{v_1}}{\partial z^{J_1}} dz^{J_1} \right) \wedge \omega_{i_1} \\
&\quad + \frac{1}{2} \left( \bar{B}_{sv_1 v_2}^{j_1 i_1 j_2 i_2} + \bar{B}_{m-k+av_1 v_2}^{j_1 i_1 j_2 i_2} \bar{G}_s^a \right) \bar{\omega}^s \\
&\quad \wedge \left( \frac{d'_c g_{j_1}^{v_1}}{dx^{l_1}} dx^{l_1} + \frac{\partial g_{j_1}^{v_1}}{\partial z^{J_1}} dz^{J_1} \right) \wedge \left( \frac{d'_c g_{j_2}^{v_2}}{dx^{l_2}} dx^{l_2} + \frac{\partial g_{j_2}^{v_2}}{\partial z^{J_2}} dz^{J_2} \right) \wedge \omega_{i_1 i_2} \\
&\quad + \cdots + \frac{1}{n!} \left( \bar{B}_{sv_1 \dots v_n}^{j_1 i_1 \dots j_n i_n} + \bar{B}_{m-k+av_1 \dots v_n}^{j_1 i_1 \dots j_n i_n} \bar{G}_s^a \right) \bar{\omega}^s \\
&\quad \wedge \left( \frac{d'_c g_{j_1}^{v_1}}{dx^{l_1}} dx^{l_1} + \frac{\partial g_{j_1}^{v_1}}{\partial z^{J_1}} dz^{J_1} \right) \wedge \cdots \wedge \left( \frac{d'_c g_{j_n}^{v_n}}{dx^{l_n}} dx^{l_n} + \frac{\partial g_{j_n}^{v_n}}{\partial z^{J_n}} dz^{J_n} \right) \wedge \omega_{i_1 \dots i_n}, \tag{4.6}
\end{aligned}$$

from which formulas (4.4), (4.5) easily follow.  $\square$

**Corollary 4.3.** If  $E_\sigma$  are affine in the second derivatives (i.e., represent quasilinear second-order PDEs) then

$$\bar{\alpha}_0 \approx \mathcal{A}_s \bar{\omega}^s \wedge \omega_0 + \mathcal{B}_{sJ_1}^{i_1} \bar{\omega}^s \wedge dz^{J_1} \wedge \omega_{i_1}, \tag{4.7}$$

where

$$\begin{aligned} \mathcal{A}_s &= \bar{A}_s + \bar{A}_{m-k+a} \bar{G}_s^a + \left( \bar{B}_{sv_1}^{j_1 i_1} + \bar{B}_{m-k+a v_1}^{j_1 i_1} \bar{G}_s^a \right) \frac{d'_c g_{j_1}^{v_1}}{dx^{i_1}}, \\ \mathcal{B}_{sJ_1}^{i_1} &= \left( \bar{B}_{sv_1}^{j_1 i_1} + \bar{B}_{m-k+a v_1}^{j_1 i_1} \bar{G}_s^a \right) \frac{\partial g_{j_1}^{v_1}}{\partial z^{J_1}}. \end{aligned} \quad (4.8)$$

Recall that unconstrained equations were PDEs for sections  $\gamma: W \rightarrow Y$ ,  $W \subset X$ , of the fibered manifold  $\pi: Y \rightarrow X$ . Solutions of constrained equations have to obey the constraint condition

$$J^1\gamma(W) \subset Q, \quad (4.9)$$

i.e., have to satisfy the system of  $\kappa$  first-order PDE defining the constraint  $Q$ ,

$$f^\kappa(x^i, y^\sigma, y_j^\sigma) \circ J^1\gamma = 0. \quad (4.10)$$

Now, in accordance with the understanding of unconstrained equations as equations for holonomic integral sections of (any) Hamiltonian ideal  $\mathcal{H}_\alpha$  related with a Lepage class  $[\alpha]$ , we can consider constrained equations as equations for holonomic integral sections of an appropriate ideal in the exterior algebra on  $Q$ :

**Definition 4.4.** Let  $Q$  be a constraint in  $J^1Y$  with the canonical distribution  $\mathcal{C}$ ,  $E$  a dynamical form on  $J^2Y$ , and  $[[I^*\alpha]]$  its corresponding constrained system. For every  $\bar{\alpha} \in [[I^*\alpha]]$  consider the ideal  $\bar{\mathcal{H}}_{\bar{\alpha}}$  on  $Q$ , generated by the system of  $n$ -forms

$$i_\xi \bar{\alpha}, \quad \text{where } \xi \text{ runs over all vertical vector fields on } Q \text{ belonging to } \mathcal{C}. \quad (4.11)$$

We shall call  $\bar{\mathcal{H}}_{\bar{\alpha}}$  constraint Hamiltonian ideal. Sections  $\gamma: W \rightarrow Y$  of  $\pi$  such that  $J^1\gamma(W) \subset Q$ , which are integral sections of  $\bar{\mathcal{H}}_{\bar{\alpha}}$ , i.e., satisfy

$$J^1\gamma^* i_\xi \bar{\alpha} = 0, \quad \text{for every vertical vector field } \xi \in \mathcal{C}, \quad (4.12)$$

will be called constrained paths of  $E$ . Eqs. (4.12) will be called constrained equations associated with  $E$  and the constraint  $Q$ .

In adapted fibered coordinates, Eqs. (4.12) represent a system of  $m-k$  second-order PDE for the components of  $\gamma$ , polynomial in the second derivatives:

$$\left( \mathcal{A}_s + \mathcal{B}_{sJ_1}^{i_1} z_{i_1}^{J_1} + \mathcal{B}_{sJ_1J_2}^{i_1i_2} z_{i_1}^{J_1} z_{i_2}^{J_2} + \cdots + \mathcal{B}_{sJ_1 \dots J_n}^{i_1 \dots i_n} z_{i_1}^{J_1} \cdots z_{i_n}^{J_n} \right) \circ J^2\gamma = 0. \quad (4.13)$$

In fact, due to the following proposition, in equations (4.13) only  $m-k$  unknown functions  $\gamma^1(x^i), \dots, \gamma^{m-k}(x^i)$  appear. Therefore we shall also refer to them as to *reduced equations* for the constrained system  $[[\iota^*\alpha]]$ .

**Proposition 4.5.** *A section  $\gamma$  of  $\pi$  satisfies the constraint condition (4.9) (i.e., (4.10)) if and only if  $J^1\gamma$  is an integral section of the canonical distribution  $\mathcal{C}$ . This means that for every  $a$ ,*

$$J^1\gamma^*\bar{\varphi}^a = 0. \quad (4.14)$$

In coordinates,

$$\frac{\partial \gamma^{m-k+a}}{\partial x^j} = g_j^{m-k+a} \circ J^1\gamma. \quad (4.15)$$

**Proof.** Let  $\gamma$  be a section of  $\pi$  satisfying (4.9). This means that  $\delta = J^1\gamma$  is a *holonomic* section of the fibered manifold  $Q \rightarrow X$ , meaning that  $\delta$  is an integral section of the induced contact distribution on  $Q$ . However, this distribution is annihilated by the 1-forms  $\iota^*\omega^\sigma = \bar{\omega}^\sigma$ . Now, from (3.26) we can see that  $\delta$  is an integral section of  $\mathcal{C}$ . The converse is trivial.  $\square$

We can conclude that constrained paths can be locally obtained by solving the system of simultaneous  $kn$  first-order PDE (4.15) and  $m-k$  second-order PDE (4.13). Notice that complete integrability of the distribution  $\mathcal{C}$  is not so essential, since we are looking for integral *sections* (which are locally  $n$ -dimensional submanifolds of  $Q$ ), not for integral manifolds of  $\mathcal{C}$ . In fact, in analogy with non-holonomic mechanics (ordinary differential equations) one can expect that namely the situations where  $\mathcal{C}$  is *not* completely integrable will be of interest in the theory and applications of PDEs with differential constraints.

#### 4.2. Constrained Lagrangian systems

If the unconstrained equations are equations for extremals of a first-order Lagrangian  $\lambda$ , i.e., if  $E = E_\lambda$ , we have in the Lepage class  $[\alpha]$  of  $E_\lambda$  distinguished representatives, which we can use for construction of the corresponding constrained system (see Proposition 2.10). In the (unconstrained) calculus of variations one usually takes the form  $d\Theta_\lambda$  (see e.g. [10,9]), however, in many situations the form  $d\rho_\lambda$  may be more useful [17,2,11], or one can even utilize a general Lepagean  $(n+1)$ -form  $d\rho$  [7,20,24,26]. As we have seen above, in the constrained situation, the *constrained Lagrangian system* is the *equivalence class*  $[[\iota^*d\Theta_\lambda]]$ , and for study of constrained equations any of its representatives is appropriate. Of course, the work with the most simple ones,  $\bar{\alpha}_0$  or  $\bar{\alpha}_{0\lambda}$ , or with the most simple closed one,  $d\Theta_\lambda$ , can be most convenient.

**Definition 4.6.** Let  $[[\iota^*d\Theta_\lambda]]$  be a constrained Lagrangian system. Each of the forms  $\iota^*d\Theta_\lambda + \varphi$ , where  $\varphi \in \mathcal{I}(\mathcal{C}^0)$ , will be called a constrained Poincaré–Cartan  $(n+1)$ -form of  $\lambda$ . Similarly, each of the forms  $\iota^*d\rho_\lambda + \varphi$ , where  $\varphi \in \mathcal{I}(\mathcal{C}^0)$ , will be called a constrained Krupka  $(n+1)$ -form of  $\lambda$ . Paths of a constrained system will be called constrained extremals, and Eqs. (4.12) where  $\bar{\alpha}$  is any element of the class  $[[\iota^*d\Theta_\lambda]]$  will be called constrained Euler–Lagrange equations of  $\lambda$ .

For  $\lambda = L\omega_0$  denote

$$\bar{L} = L \circ \iota, \quad \bar{L}_a^j = \frac{\partial L}{\partial y_j^{m-k+a}} \circ \iota, \quad (4.16)$$

where the above are functions of adapted fibered coordinates,  $(x^i, y^\sigma, z^J)$ , on  $Q$ , and

$$\Theta_{\iota^*\lambda} = \bar{L}\omega_0 + \frac{\partial \bar{L}}{\partial y_j^s} \bar{\omega}^s \wedge \omega_j = \bar{L}\omega_0 + \frac{\partial \bar{L}}{\partial z^J} \frac{\partial z^J}{\partial y_j^s} \bar{\omega}^s \wedge \omega_j. \quad (4.17)$$

In keeping with notations introduced in Remark 3.9 we can easily find the following relation:

**Proposition 4.7.**

$$\iota^*\Theta_\lambda = \Theta_{\iota^*\lambda} + \bar{L}_a^i C_{is}^{aj} \bar{\omega}^s \wedge \omega_j + \bar{L}_a^j \bar{\varphi}^a \wedge \omega_j. \quad (4.18)$$

**Proof.** We have

$$\begin{aligned} \iota^*\Theta_\lambda &= \bar{L}\omega_0 + \left( \frac{\partial L}{\partial y_j^s} \circ \iota \right) \bar{\omega}^s \wedge \omega_j + \left( \frac{\partial L}{\partial y_j^{m-k+a}} \circ \iota \right) \bar{\omega}^{m-k+a} \wedge \omega_j \\ &= \bar{L}\omega_0 + \left( \left( \frac{\partial L}{\partial y_j^s} \circ \iota \right) + \bar{L}_a^j \bar{G}_s^a \right) \bar{\omega}^s \wedge \omega_j + \bar{L}_a^j \bar{\varphi}^a \wedge \omega_j. \end{aligned} \quad (4.19)$$

On the other hand,

$$\begin{aligned} \Theta_{\iota^*\lambda} &= \bar{L}\omega_0 + \frac{\partial \bar{L}}{\partial z^J} \frac{\partial z^J}{\partial y_j^s} \bar{\omega}^s \wedge \omega_j \\ &= \bar{L}\omega_0 + \left( \left( \frac{\partial L}{\partial y_j^s} \circ \iota \right) + \bar{L}_a^j \bar{G}_s^a \right) \bar{\omega}^s \wedge \omega_j - \bar{L}_a^i C_{is}^{aj} \bar{\omega}^s \wedge \omega_j, \end{aligned} \quad (4.20)$$



since from  $\iota^*dL = d\bar{L}$  one gets

$$\begin{aligned} \frac{\partial \bar{L}}{\partial z^J} &= \left( \frac{\partial L}{\partial y_i^\sigma} \circ \iota \right) \frac{\partial g_i^\sigma}{\partial z^J} = \left( \frac{\partial L}{\partial y_i^r} \circ \iota \right) \frac{\partial g_i^r}{\partial z^J} + \left( \frac{\partial L}{\partial y_i^{m-k+a}} \circ \iota \right) \frac{\partial g_i^{m-k+a}}{\partial z^J} \\ &= \left( \left( \frac{\partial L}{\partial y_i^r} \circ \iota \right) + \bar{L}_a^i \bar{G}_r^a \right) \frac{\partial g_i^r}{\partial z^J} - \bar{L}_a^i C_{Ji}^a. \end{aligned} \quad (4.21)$$

Comparing (4.19) and (4.20) we obtain the desired formula.  $\square$

For convenience of notations let us introduce the  $\mathcal{C}$ -modified Euler–Lagrange operator and cut  $\mathcal{C}$ -modified Euler–Lagrange operator, respectively:

$$\begin{aligned} \mu_s &= \frac{\partial_c}{\partial y^s} - \frac{d_c}{dx^i} \left( \frac{\partial}{\partial y_i^s} \right) - \frac{\partial_c g_j^r}{\partial y^s} \frac{\partial}{\partial y_j^r} = \frac{\partial_c}{\partial y^s} - \frac{d_c}{dx^i} \left( \frac{\partial z^J}{\partial y_i^s} \frac{\partial}{\partial z^J} \right) - \frac{\partial_c g_j^r}{\partial y^s} \frac{\partial z^J}{\partial y_j^r} \frac{\partial}{\partial z^J}, \\ \mu'_s &= \frac{\partial_c}{\partial y^s} - \frac{d'_c}{dx^i} \left( \frac{\partial}{\partial y_i^s} \right) - \frac{\partial_c g_j^r}{\partial y^s} \frac{\partial}{\partial y_j^r} = \frac{\partial_c}{\partial y^s} - \frac{d'_c}{dx^i} \left( \frac{\partial z^J}{\partial y_i^s} \frac{\partial}{\partial z^J} \right) - \frac{\partial_c g_j^r}{\partial y^s} \frac{\partial z^J}{\partial y_j^r} \frac{\partial}{\partial z^J}. \end{aligned} \quad (4.22)$$

**Theorem 4.8.** *Let  $\lambda$  be a Lagrangian in  $J^1Y$ ,  $Q \subset J^1Y$  a regular constraint. Let  $\gamma : W \rightarrow Y$  be a section of the fibered manifold  $\pi : Y \rightarrow X$  such that  $J^1\gamma(W) \subset Q$ . In adapted fibered coordinates, the constrained Euler–Lagrange equations take one of the following equivalent forms:*

(1) By means of  $L$ ,

$$\left( \mathcal{A}_s + \mathcal{B}_{sJ}^i z_i^J \right) \circ J^2\gamma = 0, \quad (4.23)$$

where  $\mathcal{A}_s, \mathcal{B}_{sJ}^i$  are given by (4.8), where (cf. (2.30))

$$\bar{A}_\sigma = e'_\sigma(L) \circ \iota, \quad \bar{B}_{\sigma\nu}^{ij} = - \left( \frac{\partial^2 L}{\partial y_i^\sigma \partial y_j^\nu} \right) \circ \iota. \quad (4.24)$$

(2) By means of  $\bar{L}$  and  $\bar{L}_a^j$ ,

$$\left( \mu_s(\bar{L}) - \bar{L}_a^j \mu_s(g_j^{m-k+a}) - C_{js}^{ai} \frac{d_c \bar{L}_a^j}{dx^i} \right) \circ J^2\gamma = 0, \quad (4.25)$$

meaning that the functions  $\mathcal{A}_s, \mathcal{B}_{sJ}^i$  are equivalently expressed as follows:

$$\begin{aligned}\mathcal{A}_s &= \mu'_s(\bar{L}) - \bar{L}_a^j \mu'_s \left( g_j^{m-k+a} \right) - \mathcal{C}_{js}^{ai} \frac{d'_c \bar{L}_a^j}{dx^i}, \\ \mathcal{B}_{sJ}^i &= -\frac{\partial}{\partial z^J} \left( \frac{\partial \bar{L}}{\partial z^K} \frac{\partial z^K}{\partial y_i^s} \right) + \bar{L}_a^i \frac{\partial}{\partial z^J} \left( \bar{G}_s^a \delta_j^i - \mathcal{C}_{is}^{aj} \right) - \mathcal{C}_{js}^{ai} \frac{\partial \bar{L}_a^j}{\partial z^J} \\ &= -\frac{\partial}{\partial z^J} \left( \frac{\partial \bar{L}}{\partial z^K} \frac{\partial z^K}{\partial y_i^s} \right) + \bar{L}_a^j \frac{\partial}{\partial z^J} \left( \frac{\partial g_j^{m-k+a}}{\partial z^K} \frac{\partial z^K}{\partial y_i^s} \right) - \mathcal{C}_{js}^{ai} \frac{\partial \bar{L}_a^j}{\partial z^J}. \quad (4.26)\end{aligned}$$

**Proof.** The first part of the theorem is clear. Let us prove the second one. By Proposition 4.7 and with notations of Remark 3.9 we obtain:

$$\begin{aligned}i^* d\Theta_\lambda &\approx d\Theta_{i^*\lambda} + d \left( \bar{L}_a^i \mathcal{C}_{is}^{aj} \right) \wedge \bar{\omega}^s \wedge \omega_j - \bar{L}_a^i \mathcal{C}_{is}^{aj} dg_j^s \wedge \omega_0 + \bar{L}_a^j d\bar{\varphi}^a \wedge \omega_j \\ &\approx d\Theta_{i^*\lambda} - \mathcal{C}_{js}^{ai} \frac{d'_c \bar{L}_a^j}{dx^i} \bar{\omega}^s \wedge \omega_0 \\ &\quad - \bar{L}_a^j \left( \frac{d'_c \mathcal{C}_{js}^{ai}}{dx^i} + \mathcal{C}_{jr}^{ai} \frac{\partial_c g_i^r}{\partial y^s} + \frac{\partial_c g_j^{m-k+a}}{\partial y^s} - \frac{d'_c \bar{G}_s^a}{dx^j} - \bar{G}_r^a \frac{\partial_c g_j^r}{\partial y^s} \right) \bar{\omega}^s \wedge \omega_0 \\ &\quad - \left( \frac{\partial(\bar{L}_a^j \mathcal{C}_{js}^{ai})}{\partial z^J} - \bar{L}_i^a \frac{\partial \bar{G}_s^a}{\partial z^J} \right) \bar{\omega}^s \wedge dz^J \wedge \omega_i. \quad (4.27)\end{aligned}$$

However,

$$\begin{aligned}\mu'_s \left( g_j^{m-k+a} \right) &= \frac{\partial_c g_j^{m-k+a}}{\partial y^s} - \frac{d'_c}{dx^i} \left( \frac{\partial z^J}{\partial y_i^s} \frac{\partial g_j^{m-k+a}}{\partial z^J} \right) - \frac{\partial_c g_i^r}{\partial y^s} \frac{\partial z^J}{\partial y_i^r} \frac{\partial g_j^{m-k+a}}{\partial z^J} \\ &= \frac{\partial_c g_j^{m-k+a}}{\partial y^s} - \frac{d'_c}{dx^i} \left( \bar{G}_s^a \delta_j^i - \mathcal{C}_{js}^{ai} \right) - \frac{\partial_c g_i^r}{\partial y^s} \left( \bar{G}_r^a \delta_j^i - \mathcal{C}_{jr}^{ai} \right) \\ &= \frac{\partial_c g_j^{m-k+a}}{\partial y^s} - \frac{d'_c \bar{G}_s^a}{dx^j} + \frac{d'_c \mathcal{C}_{js}^{ai}}{dx^i} - \frac{\partial_c g_j^r}{\partial y^s} \bar{G}_r^a + \frac{\partial_c g_i^r}{\partial y^s} \mathcal{C}_{jr}^{ai}, \quad (4.28)\end{aligned}$$

and

$$\begin{aligned}\mu_s \left( g_j^{m-k+a} \right) - \mu'_s \left( g_j^{m-k+a} \right) &= -\frac{\partial}{\partial z^J} \left( \frac{\partial g_j^{m-k+a}}{\partial z^K} \frac{\partial z^K}{\partial y_i^s} \right) z_i^J \\ &= -\frac{\partial}{\partial z^J} \left( \bar{G}_s^a \delta_j^i - \mathcal{C}_{js}^{ai} \right) z_i^J. \quad (4.29)\end{aligned}$$

Substituting into (4.27) we get

$$\begin{aligned} \iota^* d\Theta_\lambda &\approx d\Theta_{\iota^*\lambda} - \left( \bar{L}_a^j \mu'_s (g_j^{m-k+a}) + C_{js}^{ai} \frac{d'_c \bar{L}_a^j}{dx^i} \right) \bar{\omega}^s \wedge \omega_0 \\ &\quad + \left( \bar{L}_a^j \frac{\partial}{\partial z^J} \left( \frac{\partial g_j^{m-k+a}}{\partial z^K} \frac{\partial z^K}{\partial y_i^s} \right) - C_{js}^{ai} \frac{\partial \bar{L}_a^j}{\partial z^J} \right) \bar{\omega}^s \wedge dz^J \wedge \omega_i. \end{aligned} \quad (4.30)$$

Finally, expressing  $d\Theta_{\iota^*\lambda}$  we obtain

$$\begin{aligned} d\Theta_{\iota^*\lambda} &\approx \left( \frac{\partial_c \bar{L}}{\partial y^s} - \frac{d'_c}{dx^i} \left( \frac{\partial \bar{L}}{\partial z^J} \frac{\partial z^J}{\partial y_i^s} \right) - \frac{\partial \bar{L}}{\partial z^J} \frac{\partial z^J}{\partial y_j^r} \frac{\partial_c g_j^r}{\partial y^s} \right) \bar{\omega}^s \wedge \omega_0 \\ &\quad - \frac{\partial}{\partial z^J} \left( \frac{\partial \bar{L}}{\partial z^K} \frac{\partial z^K}{\partial y_i^s} \right) \bar{\omega}^s \wedge dz^J \wedge \omega_i \\ &= \mu'_s(\bar{L}) \bar{\omega}^s \wedge \omega_0 - \frac{\partial}{\partial z^J} \left( \frac{\partial \bar{L}}{\partial z^K} \frac{\partial z^K}{\partial y_i^s} \right) \bar{\omega}^s \wedge dz^J \wedge \omega_i. \end{aligned} \quad (4.31)$$

Formulas (4.30) and (4.31) give us the representative  $\bar{\alpha}_0 \approx d\Theta_{\iota^*\lambda}$  (the components of which determine the corresponding constrained equations),

$$\begin{aligned} \bar{\alpha}_0 &= \mathcal{A}_s \bar{\omega}^s \wedge \omega_0 + \mathcal{B}_{sJ}^i \bar{\omega}^s \wedge dz^J \wedge \omega_i \\ &= \left( \mu'_s(\bar{L}) - \bar{L}_a^j \mu'_s (g_j^{m-k+a}) + C_{js}^{ai} \frac{d'_c \bar{L}_a^j}{dx^i} \right) \bar{\omega}^s \wedge \omega_0 \\ &\quad - \left( \frac{\partial}{\partial z^J} \left( \frac{\partial \bar{L}}{\partial z^K} \frac{\partial z^K}{\partial y_i^s} \right) - \bar{L}_a^j \frac{\partial}{\partial z^J} \left( \frac{\partial g_j^{m-k+a}}{\partial z^K} \frac{\partial z^K}{\partial y_i^s} \right) + C_{js}^{ai} \frac{\partial \bar{L}_a^j}{\partial z^J} \right) \bar{\omega}^s \wedge dz^J \wedge \omega_i, \end{aligned} \quad (4.32)$$

as desired.  $\square$

**Remark 4.9.** Proposition 4.7 and Theorem 4.8 show that for general constraints the  $n$ -form  $\Theta_{\iota^*\lambda}$  is not a Lepagean form for the constrained equations. This means that  $\iota^*\lambda$  has not the meaning of a “constrained Lagrangian”. A proper Lepagean form is, however,  $\iota^*\Theta_\lambda$  (or  $\iota^*\rho$ , where  $\rho$  is any Lepagean equivalent of  $\lambda$ ), since  $d\iota^*\Theta_\lambda = \iota^*d\Theta_\lambda$  gives rise to the constrained Euler–Lagrange equations. In this way, the role of a constrained Lagrangian is played by the (local)  $n$ -form

$$\lambda_C = \bar{L}\omega_0 + \bar{L}_a^j \bar{\varphi}^a \wedge \omega_j. \quad (4.33)$$

Consequently, a constrained Lagrangian system typically cannot be locally determined by a single function defined on the constraint, but is determined rather by  $1 + nk$  “constraint Lagrange functions”,  $\bar{L}, \bar{L}_a^j$  on  $Q$ .

**Definition 4.10.** The operator defined by (4.25), i.e.,

$$\mathcal{E}_s^C(\bar{L}, \bar{L}_a^j) = \mu_s(\bar{L}) - \bar{L}_a^j \mu_s \left( g_j^{m-k+a} \right) - c_{js}^{ai} \frac{d_c \bar{L}_a^j}{dx^i} \quad (4.34)$$

will be called the constraint Euler–Lagrange operator.

We can define the concept of a *constraint-horizontal form* on  $Q$  as a form annihilated by vertical vector fields belonging to the canonical distribution  $\mathcal{C}$  (cf. [27]). Then  $\lambda_{\mathcal{C}}$  is constraint-horizontal, and  $\mathcal{E}^C$  is a map acting on constraint-horizontal  $n$ -forms on  $Q$ , assigning them classes of *dynamical forms* on  $\tilde{Q} \subset J^2Y$  ( $\tilde{Q}$  is a natural prolongation of  $Q$ ). Indeed,  $\mathcal{E}^C(\lambda_{\mathcal{C}})$  is determined up to a dynamical form  $\Phi \in \mathcal{I}(\mathcal{C}^0)$ ; in coordinates,

$$\mathcal{E}^C(\lambda_{\mathcal{C}}) = \mathcal{E}_s^C(\bar{L}, \bar{L}_a^j) \bar{\omega}^s \wedge \omega_0 + \Phi_a \bar{\varphi}^a \wedge \omega_0. \quad (4.35)$$

#### 4.3. Chetaev equations

We have introduced differential equations with constraints as geometric objects defined directly on constraint manifolds. Another (but equivalent) model for constrained equations arises from their understanding as deformations of the original (unconstrained) equations, defined on  $J^1Y$ , in a neighborhood of the constraint. We adopt this idea from [22,23] where it has been proposed for the case of second and higher-order ODEs.

Let  $Q \subset J^1Y$  be a regular constraint. To a point  $x \in Q$  consider an appropriate open set  $U \subset J^1Y$  (open in  $J^1Y$ ) where  $Q$  is given by equations  $f^\alpha = 0$ , and the corresponding distribution  $\tilde{\mathcal{C}}_U$  defined on  $U$ . Recall that by (3.7)  $\tilde{\mathcal{C}}_U$  is annihilated by  $k$  linearly independent 1-forms defined on  $U$ ,

$$\phi^a = c_{\alpha j}^a f^\alpha dx^j + M_\sigma^a \omega^\sigma, \quad \text{where} \quad M_\sigma^a = \frac{1}{n} c_{\alpha j}^a \frac{\partial f^\alpha}{\partial y_j^\sigma}. \quad (4.36)$$

Denote by  $\mathcal{I}_U$  the ideal on  $U$  generated by (4.36).

Let  $E$  be a  $J^1Y$ -pertinent dynamical form on  $J^2Y$ ,  $[\alpha]$  its Lepage class. (Recall that  $E$  is characterized by Proposition 2.7). If  $\Phi \in \mathcal{I}_U$  is a *dynamical form*, put

$$E_\Phi = E - \pi_{2,1}^* \Phi. \quad (4.37)$$

$E_\Phi$  is a  $J^1Y$ -pertinent dynamical form on  $\pi_{2,1}^{-1}(U)$ , hence has a Lepage class  $[\alpha_\Phi]$  defined on  $U$ . Moreover, we can easily see that  $\alpha_1 \sim \alpha_2 \Rightarrow \alpha_{1\Phi} \sim \alpha_{2\Phi}$ .

**Definition 4.11.** We shall call  $E_\Phi$  a deformation of  $E$  induced by the constraint  $Q$ . Similarly, the Lepage class  $[\alpha_\Phi]$  will be called a deformation of  $[\alpha]$  induced by  $Q$ . Equations for paths of  $E_\Phi$  will be called deformed equations. A corresponding dynamical form  $\Phi$  will be called energy-momentum form of the constraint  $Q$ .

Note that by definition,  $\Phi = \phi^a \wedge \eta_a$ , where  $\eta_a$  are *horizontal  $n$ -forms* defined on  $U$ ; in fibered coordinates,  $\eta_a = h_a \omega_0$ . With help of (4.36) we write

$$\Phi = \phi^a \wedge \eta_a = h_a M_\sigma^a \omega^\sigma \wedge \omega_0 = \frac{1}{n} h_a c_{\alpha j}^a \frac{\partial f^\alpha}{\partial y_j^\sigma} \omega^\sigma \wedge \omega_0 = \lambda_{\alpha j} \frac{\partial f^\alpha}{\partial y_j^\sigma} \omega^\sigma \wedge \omega_0, \quad (4.38)$$

and call the functions

$$\lambda_{\alpha j} = \frac{1}{n} h_a c_{\alpha j}^a \quad (4.39)$$

*Lagrange multipliers.* Hence, energy-momentum forms of the constraint  $Q$  read

$$\Phi = \Phi_\sigma \omega^\sigma \wedge \omega_0, \quad \text{where} \quad \Phi_\sigma = \lambda_{\alpha j} \frac{\partial f^\alpha}{\partial y_j^\sigma}, \quad (4.40)$$

and we can see that they are determined by the constraint up to Lagrange multipliers.

Obviously, the concept of energy-momentum form of the constraint does not depend upon a choice of local generators of the distribution  $\tilde{C}_U$ . Indeed, if  $\psi^a$  are other independent 1-forms annihilating  $\tilde{C}_U$ , it holds  $\psi^a = A_b^a \phi^b$  for a regular matrix  $(A_b^a)$  on  $U$ , and we get  $\Phi = \eta_a \wedge \phi^a = A_a^b \eta_a \wedge \psi^b = \kappa_a \wedge \psi^a$ .

**Remark 4.12.** The definition of energy-momentum form of the constraint  $Q$  gives a local  $(n+1)$ -form on every appropriate open set  $U$ . However, one can obtain a global form  $\Phi$  with help of a partition of unity subordinate to a cover  $\{U_i\}$  of  $Q$ . Moreover, as an immediate consequence of Corollary 3.4 and Proposition 3.5 it turns out that any two energy-momentum forms along the constraint  $Q$  coincide up to Lagrange multipliers.

We shall be interested in *constrained paths* of the deformed equations, i.e., those paths that pass in the constraint manifold  $(J^1\gamma(W) \subset Q \cap U)$ . Immediately from the definitions we get:

**Proposition 4.13.** *The following conditions are equivalent:*

- (1) A section  $\gamma: W \rightarrow Y$  of  $\pi$  is a constrained path of  $E_\Phi$ .
- (2) For any  $\alpha_\Phi \in [\alpha_\Phi]$ ,  $J^1\gamma$  is an integral section of the Hamiltonian ideal  $\mathcal{H}_{\alpha_\Phi}$ , and  $J^1\gamma(W) \subset Q \cap U$ .

(3)  $\gamma$  satisfies the following system of second-order PDE:

$$A_\sigma + B_{\sigma v_1}^{j_1 i_1} y_{j_1 i_1}^{v_1} + B_{\sigma v_1 v_2}^{j_1 i_1 j_2 i_2} y_{j_1 i_1}^{v_1} y_{j_2 i_2}^{v_2} + \cdots + B_{\sigma v_1 \dots v_n}^{j_1 i_1 \dots j_n i_n} y_{j_1 i_1}^{v_1} \cdots y_{j_n i_n}^{v_n} = \lambda_{\alpha j} \frac{\partial f^\alpha}{\partial y_j^\sigma}, \quad (4.41)$$

together with the equations of the constraint,  $f^\alpha = 0$ .

If, in particular  $E = E_\lambda$  (i.e., the unconstrained equations are Euler–Lagrange equations of a Lagrangian  $\lambda = L\omega_0$ ), then the corresponding deformed equations (4.41) take the form

$$\frac{\partial L}{\partial y^\sigma} - \frac{d}{dx^j} \frac{\partial L}{\partial y_j^\sigma} = \lambda_{\alpha j} \frac{\partial f^\alpha}{\partial y_j^\sigma}. \quad (4.42)$$

**Remark 4.14.** Eqs. (4.42) were obtained also in [3]. These equations generalize to (variational) partial differential equations the Chetaev equations, proposed by Chetaev in 1930 to describe motion of mechanical Lagrangian systems subjected to constraints involving time, positions and velocities of particles [6] (so-called non-holonomic mechanics). The right-hand sides of Chetaev’s equations in mechanics are interpreted as components of a force, called constraint (or Chetaev) force; it is determined up to Lagrange multipliers, which have to be evaluated with help of deformed equations. As we can see from (4.41), for partial differential equations the meaning of Lagrange multipliers and deformed equations is analogous.

Let us clarify the relation between the deformed and reduced equations.

**Theorem 4.15.**

(1) For every  $U$  and every dynamical form  $\Phi \in \mathcal{I}(U)$ , the constrained system associated with  $E_\Phi$  coincides with the constrained system associated with  $E$ , i.e.,

$$[[i^* \alpha_\Phi]] = [[i^* \alpha]]. \quad (4.43)$$

(2) For sections  $\gamma : W \rightarrow Y$  of  $\pi$  such that  $J^1 \gamma(W) \subset Q$ , deformed equations and reduced equations are equivalent.

**Proof.** By definition of  $[\alpha_\Phi]$ , every element of the class is of the form

$$\alpha_\Phi = \alpha - \Phi + F, \quad (4.44)$$

where  $F$  is an at least 2-contact form on  $U$ . Hence

$$i^* \alpha_\Phi = i^* \alpha + i^* \Phi + i^* F \approx i^* \alpha, \quad (4.45)$$

since  $i^*\Phi$  is a constraint form on  $Q$ , and  $i^*F$  is an at least 2-contact form on  $Q$ . This means that  $i^*\alpha_\Phi \in [[i^*\alpha]]$ . Conversely, for every fixed  $\Phi$ ,

$$i^*\alpha \approx i^*\alpha - i^*\Phi = i^*\alpha_\Phi, \quad (4.46)$$

i.e.,  $i^*\alpha \in [[i^*\alpha_\Phi]]$ .

The second part of Theorem 4.15 is a direct consequence of the first part.  $\square$

## 5. Particular cases of regular constraints in $J^1Y$

In this section we introduce some particular cases of constraints, such as *Lagrangian constraints*,  *$\pi$ -adapted constraints*, *constraints defined by a distribution on  $Y$* , *semi-holonomic constraints*, and *holonomic constraints*.

### 5.1. Lagrangian constraints

Let  $Q$  be a regular constraint in  $J^1Y$ ,  $\mathcal{C} = \text{annih}\{\bar{\varphi}^a\}$  its canonical distribution,  $\mathcal{I}$  the constraint ideal. For an open subset  $U \subset J^1Y$  where  $Q$  is given by equations  $f^a = 0$  consider the related constraint distribution  $\tilde{\mathcal{C}}_U = \text{annih}\{\phi^a\}$  and the constraint ideal  $\mathcal{I}_U$  on  $U$ .

**Definition 5.1.** A constraint  $Q$  is called *Lagrangian* in  $U$  if for all  $a$ , the forms  $p_1 d\phi^a$  are horizontal with respect to the projection onto  $Y$ .  $Q$  is called a *Lagrangian constraint* if it is Lagrangian in an open neighborhood of the submanifold  $Q$ .

We note that the definition of a Lagrangian constraint does not depend upon a choice of forms annihilating the distribution  $\tilde{\mathcal{C}}_U$ . Indeed, if  $\psi^a$  is another system of independent 1-forms annihilating  $\tilde{\mathcal{C}}_U$ , one has  $\psi^a = A_b^a \phi^b$ , where  $(A_b^a)$  is a regular matrix on  $U$ . Hence

$$p_1 d\psi^a = p_1 (A_b^a d\phi^b) + p_1 (dA_b^a \wedge \phi^b) = A_b^a p_1 d\phi^b + (hdA_b^a) \wedge \phi^b \quad (5.1)$$

which is  $\pi_{1,0}$ -horizontal, since all  $\phi^a$  are  $\pi_{1,0}$ -horizontal, as can be seen from their definition.

Next, note that the definition of a Lagrangian constraint means that for all  $a$  and  $i$ , the 1-contact part of  $d(\phi^a \wedge \omega_i) = d(c_{xi}^a f^a \omega_0 + M_\sigma^a \omega^\sigma \wedge \omega_i)$  is an *Euler–Lagrange form*. This means, however, that the  $n$ -forms

$$\phi_i^a = c_{xi}^a f^a \omega_0 + M_\sigma^a \omega^\sigma \wedge \omega_i = c_{xi}^a f^a \omega_0 + M_\sigma^a \delta_i^j \omega^\sigma \wedge \omega_j \quad (5.2)$$

are *Lepagean*, and

$$\Lambda_i^a = c_{xi}^a f^a \omega_0 \quad (5.3)$$

are local *Lagrangians* for the constraint  $Q$ .

**Theorem 5.2.** (1) A constraint  $Q$  is Lagrangian on  $U$  if and only if

$$\frac{\partial (c_{\alpha j}^a f^\alpha)}{\partial y_l^\sigma} = 0, \quad j \neq l,$$

$$\frac{\partial (c_{\alpha 1}^a f^\alpha)}{\partial y_1^\sigma} = \frac{\partial (c_{\alpha 2}^a f^\alpha)}{\partial y_2^\sigma} = \dots = \frac{\partial (c_{\alpha n}^a f^\alpha)}{\partial y_n^\sigma} = \frac{1}{n} c_{\alpha i}^a \frac{\partial f^\alpha}{\partial y_i^\sigma}. \quad (5.4)$$

(2) Consider a covering of  $Q$  by adapted fibered charts  $(x^i, y^\sigma, z^J)$ . If the constraint  $Q$  is Lagrangian then

$$C_{Jj}^a = 0, \quad (5.5)$$

where the functions  $C_{Jj}^a$  are given by (3.35).

**Proof.** (1) Computing  $d\phi^a$  where  $\phi^a$  are given by (4.36) and taking the 1-contact part we get

$$\begin{aligned} p_1 d\phi^a &= p_1 \left( d(c_{\alpha j}^a f^\alpha) \wedge dx^j + \frac{1}{n} d \left( c_{\alpha j}^a \frac{\partial f^\alpha}{\partial y_j^\sigma} \right) \wedge \omega^\sigma - \frac{1}{n} c_{\alpha j}^a \frac{\partial f^\alpha}{\partial y_j^\sigma} dy_l^\sigma \wedge dx^l \right) \\ &= \left( \frac{\partial (c_{\alpha j}^a f^\alpha)}{\partial y^\sigma} - \frac{1}{n} \frac{d}{dx^j} \left( c_{\alpha i}^a \frac{\partial f^\alpha}{\partial y_i^\sigma} \right) \right) \omega^\sigma \wedge dx^j \\ &\quad + \left( \frac{\partial (c_{\alpha j}^a f^\alpha)}{\partial y_l^\sigma} - \frac{1}{n} c_{\alpha i}^a \frac{\partial f^\alpha}{\partial y_i^\sigma} \delta_j^l \right) \omega_l^\sigma \wedge dx^j. \end{aligned} \quad (5.6)$$

This means that  $Q$  is Lagrangian on  $U$  iff

$$\frac{\partial (c_{\alpha j}^a f^\alpha)}{\partial y_l^\sigma} - \frac{1}{n} c_{\alpha i}^a \frac{\partial f^\alpha}{\partial y_i^\sigma} \delta_j^l = 0. \quad (5.7)$$

Formula (5.7) gives for  $j \neq l$  the first of (5.4), and for  $j = l$

$$\frac{\partial (c_{\alpha j}^a f^\alpha)}{\partial y_j^\sigma} = \frac{1}{n} \sum_{i=1}^n c_{\alpha i}^a \frac{\partial f^\alpha}{\partial y_i^\sigma}, \quad \text{for every fixed } j \text{ (no summation over } j), \quad (5.8)$$

which is the second relation of (5.4).



(2) Expressing  $\phi^a$  in fibered coordinates  $(x^i, y^\sigma, z^J, f^\alpha)$  adapted to the constraint, we can see that if  $Q$  is Lagrangian then for all 1-forms  $\bar{\varphi}^a$  annihilating the canonical distribution  $\mathcal{C}$ ,  $p_1 d\bar{\varphi}^a = 0$ . By (3.34) we obtain (5.4) as desired.  $\square$

Notice that indeed, (5.7) is the condition for  $\phi_i^a$  (5.2) be a *Lepagean  $n$ -form*.

**Remark 5.3.** For  $\dim X = 1$  (ordinary differential equations/non-holonomic mechanics) every non-holonomic constraint  $Q$  in  $J^1Y$  is Lagrangian. Indeed, in this case (in the notation  $(t, q^\sigma, \dot{q}^\sigma)$  for fibered coordinates on  $U \subset J^1Y$ )

$$\phi^\alpha = f^\alpha dt + \frac{\partial f^\alpha}{\partial \dot{q}^\sigma} \omega^\sigma, \quad 1 \leq \alpha \leq k = \text{codim } Q, \quad (5.9)$$

meaning that all  $\phi^a$  are Lepagean 1-forms, i.e.,  $\Lambda^\alpha = f^\alpha dt$  are local Lagrangians for the constraint, and  $E_{\Lambda^\alpha} = p_1 d\phi^\alpha$  are the corresponding Euler–Lagrange forms.

## 5.2. $\pi$ -adapted constraints

**Definition 5.4.** A regular constraint  $Q \in J^1Y$  of corank  $(\kappa, k)$  is called  $\pi$ -adapted if  $\kappa = kn$ , and  $Q$  can be locally defined by a system of  $kn$  first-order partial differential equations in normal form,

$$f_j^a \equiv y_j^{m-k+a} - g_j^a(x^i, y^\sigma, y_l^s) = 0, \quad 1 \leq a \leq k < m, \quad 1 \leq j \leq n, \quad (5.10)$$

where the functions  $g_j^a$  above depend upon  $x^i$ ,  $1 \leq i \leq n$ ,  $y^\sigma$ ,  $1 \leq \sigma \leq m$ , and  $y_l^s$ ,  $1 \leq s \leq m - k$ ,  $1 \leq l \leq n$ .

Note that:

- The submanifold  $Q \subset J^1Y$  has corank  $kn$ .
- The rank condition (3.2) is a consequence of (3.3). Indeed, (3.3) becomes

$$\text{rank} \left( \frac{\partial f_j^a}{\partial y_i^\sigma} \right) = k, \quad \text{where } (a, j, i) \text{ label rows and } \sigma \text{ label columns.} \quad (5.11)$$

However, by (5.10),

$$\frac{\partial f_j^a}{\partial y_i^{m-k+b}} = \delta_b^a \delta_j^i, \quad \frac{\partial f_j^a}{\partial y_i^s} = -\frac{\partial g_j^a}{\partial y_i^s}, \quad (5.12)$$

hence the matrix in (3.2) takes the form

$$\left( \frac{\partial f_j^a}{\partial y_i^\sigma} \right) = \begin{pmatrix} -\frac{\partial g_j^a}{\partial y_i^s} & \delta_b^a \delta_j^i \end{pmatrix}, \quad \text{where } (a, j) \text{ label rows}$$

and  $(\sigma, i) = (s, b, i)$  label columns, (5.13)

i.e., it is a  $(kn \times mn)$ -matrix with the  $(kn \times kn)$  unit submatrix. Consequently, its rank is maximal and equal to  $kn = \kappa$ , meaning that the rank condition (3.2) holds.

Summarizing, we have an *equivalent definition* of a  $\pi$ -adapted constraint as a submanifold  $Q \subset J^1Y$  of corank  $kn$ , which can be locally expressed by Eqs. (5.10), and satisfies the rank condition (5.11).

For a  $\pi$ -adapted constraint we have the extended local constraint distribution  $\tilde{C}_U$  annihilated by  $kn^2$  (non-independent) 1-forms

$$\phi_j^{ai} = f_j^a dx^i + \frac{1}{n} \frac{\partial f_j^a}{\partial y_i^\sigma} \omega^\sigma = \left( y_j^{m-k+a} - g_j^a \right) dx^i - \frac{1}{n} \left( \frac{\partial g_j^a}{\partial y_i^s} \omega^s + \delta_j^i \omega^{m-k+a} \right), \quad (5.14)$$

or, equivalently, by  $k$  independent 1-forms

$$\begin{aligned} \phi^a &= \phi_j^{ai} \delta_i^j = f_i^a dx^i + \frac{1}{n} \frac{\partial f_i^a}{\partial y_i^\sigma} \omega^\sigma \\ &= \left( y_i^{m-k+a} - g_i^a \right) dx^i - \frac{1}{n} \frac{\partial g_i^a}{\partial y_i^s} \omega^s + \omega^{m-k+a}. \end{aligned} \quad (5.15)$$

Rewriting the formulas for the canonical distribution, we obtain:

**Proposition 5.5.** *The canonical distribution  $\mathcal{C}$  of a  $\pi$ -adapted constraint  $Q$  is locally annihilated by  $(k$  linearly independent) 1-forms*

$$\varphi^a = -\frac{1}{n} \frac{\partial g_i^a}{\partial y_i^s} \omega^s + \bar{\omega}^{m-k+a}. \quad (5.16)$$

*Equivalently,  $\mathcal{C}$  is locally spanned by the following  $n + m - k + n(m - k)$  independent vector fields:*

$$\begin{aligned} \frac{\partial_c}{\partial x^j} &= \frac{\partial}{\partial x^j} + \left( g_j^a - \frac{1}{n} \frac{\partial g_i^a}{\partial y_i^s} y_j^s \right) \frac{\partial}{\partial y^{m-k+a}}, \\ \frac{\partial_c}{\partial y^s} &= \frac{\partial}{\partial y^s} + \frac{1}{n} \frac{\partial g_i^a}{\partial y_i^s} \frac{\partial}{\partial y^{m-k+a}}, \quad \frac{\partial}{\partial y_j^s}. \end{aligned} \quad (5.17)$$

Next, in keeping with notations introduced in Section 3 we obtain:

$$g_j^s \equiv y_j^s \circ \iota = y_j^s, \quad g_j^{m-k+a} \equiv y_j^{m-k+a} \circ \iota = g_j^a = F_j^a + G_s^a y_j^s, \\ \bar{\omega}^{m-k+a} = dy^{m-k+a} - g_j^a dx^j, \quad G_s^a = \frac{1}{n} \frac{\partial g_i^a}{\partial y_i^s}, \quad F_j^a = g_j^a - \frac{1}{n} \frac{\partial g_i^a}{\partial y_i^s} y_j^s, \quad (5.18)$$

and since the functions  $F_j^a$  and  $G_s^a$  do not depend upon the  $y^{m-k+b}$ , we can use the same notation for  $\bar{F}$  and  $F$ , resp. for  $\bar{G}$  and  $G$ . Adapted coordinates on  $Q$  simply become  $(x^i, y^\sigma, y_j^s)$ , and a corresponding adapted basis of 1-forms is  $(dx^i, dy^s, \varphi^a, dy_j^s)$ . Finally, the *cut C-modified Euler-Lagrange operator* simplifies to:

$$\mu'_s = \frac{\partial_c}{\partial y^s} - \frac{d'_c}{dx^i} \left( \frac{\partial}{\partial y_i^s} \right). \quad (5.19)$$

**Theorem 5.6.** *Every  $\pi$ -adapted constraint is Lagrangian. Equivalently, it can be defined by a system of  $kn$  separable equations, i.e., (5.10) where*

$$\frac{\partial g_j^a}{\partial y_i^s} = 0, \quad i \neq j, \quad (5.20)$$

and such that

$$\frac{\partial g_1^a}{\partial y_1^s} = \frac{\partial g_2^a}{\partial y_2^s} = \dots = \frac{\partial g_n^a}{\partial y_n^s} = h_s^a(x^i, y^v). \quad (5.21)$$

**Proof.** With help of (5.12), the rank condition (5.11) implies (5.20) and (5.21). However, these conditions are equivalent with

$$C_{sj}^{ap} \equiv \frac{1}{n} \frac{\partial g_i^a}{\partial y_i^s} \delta_j^p - \frac{\partial g_j^a}{\partial y_p^s} = 0, \quad (5.22)$$

meaning that the constraint is Lagrangian. For detail arguments and computations we refer to [28].  $\square$

$\pi$ -adapted constraints and corresponding Lagrangian and Hamiltonian constrained systems are studied in [28].

### 5.3. Constraints defined by a (co)distribution on $Y$

We shall show that every weakly horizontal distribution (or, equivalently, a codistribution) of a constant rank on  $Y$  gives rise to a non-holonomic constraint structure in  $J^1Y$ .

Let  $1 \leq k < m$ . Consider a *weakly horizontal* distribution  $\mathcal{D}$  on  $Y$  of corank  $k$  (hence of rank  $m + n - k$ ). Recall [21] that this means that  $\mathcal{D}$  has a vertical subdistribution of rank  $m - k$  (sections of  $\pi$  are among admissible integral mappings). Equivalently, if  $\mathcal{D}$  is locally annihilated by a system of  $k$  linearly independent 1-forms  $\eta^a$ ,  $1 \leq a \leq k$ , the weak-horizontality condition means that the related distribution on  $J^1Y$ , annihilated by the contact forms  $p\eta^a$ ,  $1 \leq a \leq k$ , has the same corank  $k$ . In fibered coordinates, where

$$\eta^a = A_i^a dx^i + B_\sigma^a dy^\sigma, \quad (5.23)$$

we have

$$p\eta^a = B_\sigma^a \omega^\sigma, \quad (5.24)$$

and the weak-horizontality condition reads

$$\text{rank}(B_\sigma^a) = \max = k. \quad (5.25)$$

The distribution  $\mathcal{D}$  gives rise to distribution on  $J^1Y$ , annihilated by the pull-backs of (5.23), i.e., by the 1-forms

$$\pi_{1,0}^* \eta^a = h\eta^a + p\eta^a = (A_i^a - B_\sigma^a y_i^\sigma) dx^i + B_\sigma^a \omega^\sigma. \quad (5.26)$$

Putting

$$f_i^a = A_i^a - B_\sigma^a y_i^\sigma, \quad (5.27)$$

and realizing that

$$\frac{\partial f_j^a}{\partial y_i^\sigma} = B_\sigma^a \delta_j^i, \quad \frac{\partial f_i^a}{\partial y_i^\sigma} = n B_\sigma^a, \quad (5.28)$$

we can see that  $Q \subset J^1Y$  defined by the equations

$$f_i^a \equiv A_i^a + B_\sigma^a y_i^\sigma = 0, \quad 1 \leq a \leq k, \quad 1 \leq j \leq n, \quad (5.29)$$

(which are affine in the first derivatives) or, equivalently,

$$h\eta^a = 0, \quad 1 \leq a \leq k, \quad (5.30)$$

is a  $\pi$ -adapted constraint of codimension  $kn$  in  $J^1Y$ . The extended constraint distribution of  $Q$  is defined on  $J^1Y$ , and is annihilated by the 1-forms

$$\phi^a = f_i^a dx^i + \frac{1}{n} \frac{\partial f_i^a}{\partial y_i^\sigma} \omega^\sigma = (A_i^a - B_\sigma^a y_i^\sigma) dx^i + B_\sigma^a \omega^\sigma = \pi_{1,0}^* \eta^a. \quad (5.31)$$

Hence the canonical distribution of  $Q$  becomes

$$\mathcal{C} = \text{annih} \{ \varphi^a = \iota^* \phi^a = \iota^* \pi_{1,0}^* \eta^a = \iota^* p\eta^a, \quad 1 \leq a \leq k \}. \quad (5.32)$$

Summarizing, we have obtained:

**Proposition 5.7.** *A weakly horizontal distribution  $\mathcal{D} = \text{annih}\{\eta^a, 1 \leq a \leq k\}$  on  $Y$  defines a constraint structure  $(Q, \mathcal{C})$  in  $J^1Y$  by  $\iota : Q \rightarrow J^1Y : h\eta^a = 0, 1 \leq a \leq k$ , and  $\mathcal{C} = \text{annih}\{\iota^* p\eta^a, 1 \leq a \leq k\}$ . This constraint is  $\pi$ -adapted (hence Lagrangian), and projectable.*

From (5.32) we immediately get

**Proposition 5.8.** *Let  $(Q, \mathcal{C})$  be a constraint structure in  $J^1Y$  defined by a weakly horizontal distribution  $\mathcal{D}$  on  $Y$  (alternatively, a codistribution  $\mathcal{D}^0$ ) of a constant rank. If  $\mathcal{D}$  is completely integrable then the canonical distribution  $\mathcal{C}$  is completely integrable.*

**Proof.** One only has to take into account that a completely integrable distribution is locally annihilated by exact forms, and that the exterior derivative and the pull-back commute.  $\square$

Note that if  $\mathcal{D}$  is completely integrable then  $\mathcal{D} = \text{annih}\{du^a, 1 \leq a \leq k\}$ . This means that  $Q$  is given by equations

$$f_j^a \equiv \frac{du^a}{dx^j} = 0, \quad (5.33)$$

it holds

$$B_\sigma^a = \frac{\partial u^a}{\partial y^\sigma}, \quad \text{i.e.,} \quad \text{rank} \left( \frac{\partial u^a}{\partial y^\sigma} \right) = k, \quad (5.34)$$

and the canonical distribution  $\mathcal{C}$  is annihilated by 1-forms  $\varphi^a = \iota^* d\tilde{u}^a = d\iota^* \tilde{u}^a$ , where  $\tilde{u}^a$  denotes the lift of  $u^a$  to  $J^1Y$ ,  $\tilde{u}^a = u^a \circ \pi_{1,0}$ .

#### 5.4. Semi-holonomic constraints

**Definition 5.9.** Let  $Q$  be a regular constraint in  $J^1Y$ . We shall call  $Q$  semi-holonomic if the canonical distribution  $\mathcal{C}$  of  $Q$  is completely integrable.

**Proposition 5.10.** A constraint in  $J^1Y$  defined by a completely integrable distribution  $\mathcal{D}$  on  $Y$  is semi-holonomic.

**Proposition 5.11.** Any semi-holonomic constraint  $Q$  comes from a distribution  $\mathcal{D}$  on  $Y$ . Consequently, every semi-holonomic constraint is  $\pi$ -adapted, and can be locally given by separable equations, affine in the first derivatives.

**Proof.** By assumption, the canonical distribution  $\mathcal{C}$  is completely integrable. Hence, locally there exist  $k$  linearly independent exact 1-forms  $du^a$  on  $Q$ ,  $1 \leq a \leq k$ , annihilating  $\mathcal{C}$ , i.e., such that  $du^a = c_b^a \varphi^b$  for some functions  $c_b^a$ ,  $1 \leq a, b \leq k$ . Since by (3.33)

$$\varphi^a = -F_j^a dx^j - G_s^a dy^s + dy^{m-k+a} \quad (5.35)$$

and

$$\begin{aligned} du^a &= c_b^a \varphi^b = -c_b^a F_j^b dx^j - G_s^b dy^s + c_b^a dy^{m-k+b} \\ &= \frac{\partial u^a}{\partial x^i} dx^i + \frac{\partial u^a}{\partial y^s} dy^s + \frac{\partial u^a}{\partial y^{m-k+b}} dy^{m-k+b} + \frac{\partial u^a}{\partial z^J} dz^J, \end{aligned} \quad (5.36)$$

we get  $\partial u^a / \partial z^J = 0$ . Since  $u^a$  are functions on  $Q$ , i.e., in adapted coordinates  $\partial u^a / \partial f^\alpha = 0$ , we conclude that

$$\frac{\partial u^a}{\partial y_j^\sigma} = 0, \quad (5.37)$$

meaning that the  $u^a$  are functions on an open subset of  $Y$ . Consequently,  $Q$  comes from the distribution  $\mathcal{D} = \text{annih}\{du^a\}$  which is defined on  $Y$ , and  $f_j^a \equiv du^a / dx^j = 0$  are equations of  $Q$  which are affine in the  $y_i^y$ 's and separable.  $\square$

More precisely, we have the following equivalent characterizations of semi-holonomic constraints:

**Theorem 5.12.** Let  $Q$  be a  $\pi$ -adapted constraint in  $J^1Y$ . The following conditions are equivalent:

- (1)  $Q$  is semi-holonomic.
- (2) The constraint ideal  $\mathcal{I}$  is closed.
- (3) For every  $a$ ,  $d\varphi^a \approx 0$ .

(4)  $Q$  can be locally given by separable equations in normal form

$$f_j^a \equiv y_j^{m-k+a} - g_j^a(x^i, y^\sigma, y_j^s) = 0, \quad 1 \leq a \leq k < m, \quad 1 \leq j \leq n, \quad (5.38)$$

such that the functions  $g_j^a$  satisfy the relations

$$\begin{aligned} \frac{\partial^2 g_i^a}{\partial y_p^s \partial y_l^r} &= 0, & \mu_s(g_j^a) &= \frac{\partial_c g_j^a}{\partial y^s} - \frac{d_c}{dx^i} \frac{\partial g_j^a}{\partial y_i^s} = 0, \\ \frac{\partial g_1^a}{\partial y_1^s} &= \frac{\partial g_2^a}{\partial y_2^s} = \dots = \frac{\partial g_n^a}{\partial y_n^s}, & \frac{d_c g_i^a}{dx^j} &= \frac{d_c g_j^a}{dx^i}. \end{aligned} \quad (5.39)$$

**Proof.** Equivalence of (1) and (2) is obvious, since  $\mathcal{C}$  is a generating distribution for the ideal  $\mathcal{I}$ . Equivalence of (2) and (3) comes from the definition of the relation  $\approx$ . It remains to show equivalence of (3) and (4). From (3.34)) we can see that the condition  $d\varphi^a \approx 0$  means that the second and the last condition of (5.39) hold with the corresponding cut operators  $\mu'_s$  and  $d'_c/dx^l$ , and the functions  $g_i^a$  satisfy

$$\frac{\partial^2 g_i^a}{\partial y_p^s \partial y_l^r} = 0. \quad (5.40)$$

Since  $Q$  is a Lagrangian constraint by Theorem 5.6, the third of the relations in (5.39) holds, and we conclude that the  $g_i^a$  are affine in the first derivatives. Consequently, we can write  $\mu_s$  and  $d_c/dx^j$  instead of the cut operators, and we obtain (5.39) as desired. Conversely, computing  $d\varphi^a$  we can see that (5.38) and (5.39) guarantee that  $d\varphi^a \approx 0$ .  $\square$

### 5.5. Holonomic constraints

By a *holonomic constraint* in  $Y$  one means a fibered submanifold  $Q_0 \rightarrow X$  of the fibered manifold  $\pi : Y \rightarrow X$ . Hence, a holonomic constraint  $Q_0$  of codimension  $k$ , where  $1 \leq k < m$ , can be locally given by a system of *algebraic* equations

$$u^a(x^i, y^\sigma) = 0, \quad 1 \leq a \leq k, \quad (5.41)$$

where

$$\text{rank} \left( \frac{\partial u^a}{\partial y^\sigma} \right) = k. \quad (5.42)$$

The first jet prolongation  $J^1Q_0$  of a holonomic constraint is a submanifold of  $J^1Y$  locally given by equations

$$u^a = 0, \quad \frac{du^a}{dx^j} = 0, \quad (5.43)$$

and it is fibered both over  $X$  and  $Y$ . Now,  $Q \subset J^1Y$  with equations

$$f_j^a \equiv \frac{du^a}{dx^j} = 0 \quad (5.44)$$

is a  $\pi$ -adapted constraint such that  $J^1Q_0 \subset Q$ . This means that holonomic constraints can be considered as a special case of non-holonomic constraints not only formally but also from the geometrical point of view. It is important to notice the following key property of holonomic constraints, which explains the essence of differences between holonomic and (nontrivially) non-holonomic constraint structures:

**Theorem 5.13.** *Let  $Q_0$  be a holonomic constraint in  $Y$ ,  $Q \subset J^1Y$  the related  $\pi$ -adapted constraint. Then the canonical distribution  $\mathcal{C}$  of  $Q$  is at each point  $x \in J^1Q_0$  equal to  $T_x J^1Q_0$ . Consequently,  $\mathcal{C}(J^1Q_0)$  is projectable, and projects onto the tangent distribution  $TQ_0$  of  $Q_0$ .*

**Proof.** By definition,  $\mathcal{C} = \text{annih}\{l^*du^a\} = \text{annih}\{dl^*u^a\}$ . Hence, along  $J^1Q_0$ , where moreover  $u^a = 0$ , we get  $\mathcal{C} = \text{annih}\{0\} = TJ^1Q_0$ .  $\square$

We can see that in the holonomic case the (restricted) canonical distribution is simply the tangent distribution, i.e., it means *no restrictions on the tangent space of the constraint manifold* (this is nothing but a geometric understanding of the classical *D'Alembert's principle* known from classical mechanics).

Now, it is easy to realize that *holonomic constrained equations* are simply *restrictions to the constraint manifold* (arise by *pull-back* from the unconstrained ones). Precisely, we have the following:

**Corollary 5.14.** *Let  $i_0 : Q_0 \rightarrow Y$  be a holonomic constraint,  $E$  a  $J^1Y$ -pertinent dynamical form on  $J^2Y$ ,  $[\alpha]$  its first-order Lepage class. Then the constrained system on  $J^1Q_0$  takes the form*

$$[[J^1i_0^*\alpha]] = [J^1i_0^*\alpha] = J^1i_0^*\alpha \bmod \{\text{at least 2-contact forms on } J^1Q_0\}. \quad (5.45)$$

*This means that the constrained equations are equations for paths of the  $J^1Q_0$ -pertinent dynamical form*

$$E_C = J^2i_0^*E, \quad (5.46)$$

*defined on  $J^2Q_0$ .*



If, in particular,  $E$  is variational, and  $\lambda$  is a local Lagrangian for  $E$  defined on  $J^1Y$ , then

$$J^1i_0^* \Theta_\lambda = \Theta_{J^1i_0^* \lambda}, \quad (5.47)$$

and  $E_C$  is the Euler–Lagrange form of the Lagrangian

$$\lambda_C = J^1i_0^* \lambda, \quad (5.48)$$

defined on an open subset of  $J^1Q_0$ .

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