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On heat convection equations in a half space with non-decaying data and Stokes semi-group on Besov spaces based on L^∞

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ABSTRACT

The local existence theorem for the Boussinesq equations in the half space $\mathbb{R}^{n+1}_+ = \{x = (\tilde{x}, x_n) \in \mathbb{R}^{n+1};\ \tilde{x} \in \mathbb{R}^n,\ x_{n+1} > 0\}$ with non-decaying data is shown. To this end, the Stokes operator is shown to generate an analytic semi-group on the Besov spaces of negative order based on L^∞.

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1. Introduction

In the present paper, the heat convection in a viscous incompressible fluid in the $n+1$-dimensional half space $\mathbb{R}^{n+1}_+ = \{x = (\tilde{x}, x_{n+1}) \in \mathbb{R}^{n+1};\ \tilde{x} \in \mathbb{R}^n,\ x_{n+1} > 0\}$ is considered. It is described by the Boussinesq equations:

$$\text{(B)}\quad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = g\theta, \quad t > 0,\ x = (\tilde{x}, x_{n+1}) \in \mathbb{R}^{n+1}_+, \\ \partial_t \theta - \Delta \theta + u \cdot \nabla \theta = 0, \quad t > 0,\ x = (\tilde{x}, x_{n+1}) \in \mathbb{R}^{n+1}_+, \\ \operatorname{div} u = 0, \quad t > 0,\ x = (\tilde{x}, x_{n+1}) \in \mathbb{R}^{n+1}_+, \\ u|_{\partial \mathbb{R}^{n+1}_+} = 0, \quad \theta|_{\partial \mathbb{R}^{n+1}_+} = S(\tilde{x}, t), \\ u|_{t=0} = u_0, \quad \theta|_{t=0} = \theta_0, \end{cases}$$

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where $u = (u^1(x,t), u^2(x,t), \dots, u^{n+1}(x,t))$, $\theta = \theta(x,t)$, and $p = p(x,t)$ denote the velocity vector, the temperature, and the pressure, respectively, of the fluid at the point $(x,t) \in \mathbb{R}^n \times (0,\infty)$. Here, u_0 and θ_0 are given initial data, $S = S(\tilde{x},t)$ is a given heat source on the boundary of the half space, and g is the given constant vector that denotes the acceleration of gravity. Throughout the present paper, g is assumed to be a vertical constant vector, that is, $g = (0,0,\dots,0,|g|)$.

Heat convection has been investigated in various domains [4,9,10,12,13,19,20,29,30,32,33,36]. In these studies, the qth integrability condition was imposed on the initial data $u_0, \theta_0 \in L^q$ for some $q < \infty$. This condition implies that $u_0(x)$ and $\theta_0(x)$ decay at infinity in some sense, if the domain is the whole space $\mathbb{R}^n$, the half space $\mathbb{R}^{n+1}_+$, or an exterior domain.

On the other hand, Giga, Inui and Matsui [15], Cannon and Knightly [5], and Cannone [6] showed the local existence of solutions to the Navier–Stokes equations in $\mathbb{R}^n$:

$$\text{(N–S)} \quad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in } t > 0 \text{ and } x \in \mathbb{R}^n, \\ \operatorname{div} u = 0 & \text{in } t > 0 \text{ and } x \in \mathbb{R}^n, \\ u|_{t=0} = u_0 & \text{in } x \in \mathbb{R}^n \end{cases}$$

with *non-decaying* initial data $u_0 \in L^\infty(\mathbb{R}^n)$. The solution of the integral equation is referred to as a *mild solution*. Cannone [6] proved the existence of mild solutions to (N–S) for more general initial data in Besov-type spaces. Giga, Inui and Matsui [15] showed that the mild solutions in $L^\infty(\mathbb{R}^n)$ actually satisfy (N–S) in a strong sense. Moreover, they proved the uniqueness of solutions to (N–S) under certain conditions. (See also Giga, Inui, J. Kato and Matsui [16] and J. Kato [21].) In addition, Koch and Tataru [23] showed the existence of mild solutions to (N–S) in $BMO^{-1}(\mathbb{R}^n)$. (See also Sawada [34] and Maekawa and Terasawa [31].) Concerning time-global solvability in two dimensions, Giga, Matsui and Sawada [17] proved the global existence of solutions to (N–S) with *non-decaying* initial data $u_0 \in L^\infty(\mathbb{R}^2)$.

For the case in which the domain is the half space $\mathbb{R}^{n+1}_+$ $(n \geqslant 1)$, Solonnikov [37] proved the local unique existence of solutions to the Navier–Stokes equations with *non-decaying* initial data $u_0 \in L^\infty(\mathbb{R}^{n+1}_+)$. The proof of Solonnikov [37] is based on estimates for the entries of the Green matrix for the Stokes problem. On the other hand, Desch, Hieber and Prüss [8] showed that the Stokes operator generates an analytic semi-group on $L^\infty_\sigma(\mathbb{R}^{n+1}_+)$.

Concerning heat convection, Sawada and the present author [35] proved the unique local existence theorem for the Boussinesq equations in the whole space with *non-decaying* initial data. More precisely, if $u_0 \in L^\infty$ with $\operatorname{div} u = 0$ in $\mathcal{D}'$ and $\theta_0 \in \dot{B}^0_{\infty,1}$, then there exists a unique local solution $(u, \nabla p)$ to the Boussinesq equations. Note that the space $\dot{B}^0_{\infty,1}$ includes several non-decaying functions, such as $\sin(a \cdot x) + (1 + x_1^2)^{-1}$. Recently, in a previous paper [39], the global existence of the solution to the 2-D Boussinesq equations with *non-decaying* initial data $(u_0, \theta_0) \in L^\infty(\mathbb{R}^2) \times \dot{B}^0_{\infty,1}(\mathbb{R}^2)$ was shown. However, it is more interesting and physically reasonable to consider the heat convection in a domain with the boundary heated by a heat source than to consider that in the whole space. In the present paper, fluid in the half space $\mathbb{R}^{n+1}_+$ $(n \geqslant 2)$ with the boundary heated by heat source S is considered.

In order to solve (B), the inhomogeneous boundary condition in (B) is replaced by the homogeneous boundary condition, as follows. Let h be a smooth function of x_{n+1} such that

$$h(x_{n+1}) = \begin{cases} 1 & (0 \leqslant x_{n+1} < 1) \\ 0 & (x_{n+1} > 2) \end{cases} \in C^\infty\big([0,\infty)\big).$$

The function $S(\tilde{x},t)$ on $\partial\mathbb{R}^{n+1}_+ \times [0,\infty)$ is extended into the function $\bar{S}(\tilde{x}, x_{n+1}, t)$ in $\mathbb{R}^{n+1}_+ \times [0,\infty)$, as follows:

$$\bar{S}(x,t) = S(\tilde{x},t)h(x_{n+1}).$$

Let $\bar{\theta} = \theta - \bar{S}$ and $\bar{\theta}_0(x) = \theta_0(x) - \bar{S}(x,0)$. Then, (B) is transformed into

$$(\mathrm{B}') \quad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = g(\bar{\theta} + \bar{S}), \\ \partial_t \bar{\theta} - \Delta \bar{\theta} + u \cdot \nabla \bar{\theta} = -\partial_t \bar{S} + \Delta \bar{S} - u \cdot \nabla \bar{S}, \\ \operatorname{div} u = 0, \quad t > 0, \ x = (\tilde{x}, x_{n+1}) \in \mathbb{R}^{n+1}_+, \\ u|_{\partial \mathbb{R}^{n+1}_+} = 0, \quad \bar{\theta}|_{\partial \mathbb{R}^{n+1}_+} = 0, \\ u|_{t=0} = u_0, \quad \bar{\theta}|_{t=0} = \bar{\theta}_0. \end{cases}$$

Rather than the problem (B), the problem (B′) is considered hereinafter.

Next, the Helmholtz operator in $\mathbb{R}^{n+1}_+$ is defined according to Giga, Inui, Mahalov, Matsui and Saal [14]. Let f be a (scalar-, vector-, or tensor-valued) function defined on $\mathbb{R}^{n+1}_+$. The 0-extension $\mathbf{e}^0 f$, the even extension $\mathbf{e}^+ f$, and the odd extension $\mathbf{e}^- f$ of f are defined as follows:

$$\mathbf{e}^0 f(\tilde{x}, x_{n+1}) := \begin{cases} f(\tilde{x}, x_{n+1}) & (x_{n+1} \geqslant 0), \\ 0 & (x_{n+1} < 0), \end{cases}$$

$$\mathbf{e}^{\pm} f(\tilde{x}, x_{n+1}) := \begin{cases} f(\tilde{x}, x_{n+1}) & (x_{n+1} \geqslant 0), \\ \pm f(\tilde{x}, -x_{n+1}) & (x_{n+1} < 0). \end{cases}$$

For a vector field $u = (u^1(x), u^2(x), \ldots, u^{n+1}(x))$ in $\mathbb{R}^{n+1}_+$, let Eu be the extension by means of the reflection with respect to $\partial \mathbb{R}^{n+1}$, i.e.,

$$(Eu)^j := \mathbf{e}^+ u^j \quad \text{for } 1 \leqslant j \leqslant n, \qquad (Eu)^{n+1} := \mathbf{e}^- u^{n+1}.$$

Recall the Helmholtz operator P in the whole space $\mathbb{R}^{n+1}$:

$$P = (P_{ij})_{1 \leqslant i,j \leqslant n+1} = (\delta_{i,j} + R_i R_j)_{1 \leqslant i,j \leqslant n+1},$$

where R_j is the Riesz transform defined by $R_j = \partial_j (-\Delta)^{-1/2}$. Using the above operator, the Helmholtz operator P_+ in the half space $\mathbb{R}^{n+1}_+$ can be defined by

$$P_+ = PE. \tag{1.1}$$

This formula (1.1) was given by [14], and this formula plays an important role in proving the main theorems of the present study. Formally applying P_+ to the first equation of (B′) reveals that (B′) is transformed into abstract evolution equations

$$(\mathrm{AB}) \quad \begin{cases} u_t + Au + P_+(u \cdot \nabla u) = P_+\big(g(\bar{\theta} + \bar{S})\big), \\ \bar{\theta}_t - \Delta \bar{\theta} + u \cdot \nabla \bar{\theta} = -\partial_t \bar{S} + \Delta \bar{S} - u \cdot \nabla \bar{S}, \\ u|_{\partial \mathbb{R}^{n+1}_+} = 0, \quad \bar{\theta}|_{\partial \mathbb{R}^{n+1}_+} = 0, \\ u|_{t=0} = u_0, \quad \bar{\theta}|_{t=0} = \bar{\theta}_0, \end{cases}$$

where $A = -P_+ \Delta$ is called the Stokes operator. It is unclear as to whether $P_+(g\bar{S}) = P(0, \ldots, 0, -|g| S \mathbf{e}^- h)$ belongs to L^∞. Therefore, the strong L^∞-solution to (AB) cannot be constructed. Hence, in the present paper, instead of L^∞, the Besov space $B^{-s}_{\infty,q}$ $(0 < s < 1)$, which is wider than L^∞, is used. In order to solve (AB) in $B^{-s}_{\infty,q}$, it is shown that $-A$ generates an analytic semi-group on $B^{-s}_{\infty,q}(\mathbb{R}^{n+1}_+)$ by using the formula of resolvent $(\lambda + A)^{-1}$ given by Desch, Hieber and Prüss [8].[1]

[1] In [8], $P_+ \Delta$ is denoted by A. In the present paper, $-P_+ \Delta$ is denoted by A.

In the present paper, the Besov spaces $B^{-s}_{\infty,q}$ are restricted to the case $0 < s < 1$. This restriction comes from the condition (2.9), which guarantees the boundedness of the zero-extension operator from $B^{-s}_{\infty,q}(\mathbb{R}^{n+1}_+)$ into $B^{-s}_{\infty,q}(\mathbb{R}^{n+1})$. (See (2.11).) This boundedness plays important role in proving the main theorems of the present study. There is another reason for the restriction $s < 1$. In order to estimate the nonlinear term $u \cdot \nabla u = \nabla \cdot (u \otimes u)$ in $B^{-s}_{\infty,q}$, the inequality (5.10) is used. This requires the condition $s < 1$.

Note that the Stokes problem and the (global or local) solvability of the Navier–Stokes equations in various Besov spaces on domains with nonempty boundaries have been investigated by, e.g., Amann [1], Cannone, Planchon and Schonbek [7] and Fujiwara and Yamazaki [11]. (Abstract Besov spaces were used by Giga and Sohr [18] and Kobayashi and Muramatu [22].) However, in these studies, the Besov spaces $B^{s}_{p,q}$ were restricted to the case in which $p < \infty$. Hence, these results are not comparable to those of the present paper.

It is also notable that Giga, Inui, Mahalov, Matsui and Saal [14] recently proved the unique local existence of mild solutions in $\dot{B}^0_{\infty,1}(\mathbb{R}^2; L^p(\mathbb{R}_+))$ $(1 < p < \infty)$ for the 3-D rotating Navier–Stokes equations with the Coriolis term $\omega e_3 \times u$ in the half space $\mathbb{R}^3_+$. Since $\dot{B}^0_{\infty,1}(\mathbb{R}^2; L^p(\mathbb{R}_+))$ is strictly smaller than $B^{-s}_{\infty,q}(\mathbb{R}^3_+)$ for $\frac{1}{p} < s < 1$ and since in the present paper the Coriolis term is not considered, there is no inclusion between their result and the result of the present study. After finishing the present paper, we noticed that Maremonti [27] proved the unique existence of classical solutions to the Stokes equations in the half space with initial data increasing at infinity. He also showed the unique local existence of bounded solutions to Navier–Stokes equations in the half space.

The remainder of the present paper is organized as follows. In Section 2, the Besov space is introduced, and the main results are described. In Section 3, the formula of the resolvent of $-A$ given in [8] is presented. In Sections 4 and 5, the proofs of the main results are presented.

In the present paper, a point in $\mathbb{R}^n$ is denoted by $\tilde{x} = (x_1, x_2, \ldots, x_n)$, and a point in $\mathbb{R}^{n+1}_+$ is denoted by $x = (\tilde{x}, x_{n+1})$, or sometimes $x = (\tilde{x}, y)$.

2. Preliminaries and main results

2.1. Function spaces

Before presenting the results of the present study, the definition of the Besov space (cf. [2]) is presented. Let ψ and φ_j be the Littlewood–Paley decomposition satisfying $\mathcal{F}\varphi_j(\xi) = (\mathcal{F}\varphi_0)(2^{-j}\xi) \in C^\infty_0(\mathbb{R}^{n+1})$, $\operatorname{supp} \mathcal{F}\varphi_0 \subset \{1/2 < |\xi| < 2\}$,

$$\sum_{j=-\infty}^{\infty} \mathcal{F}\varphi_j(\xi) = 1 \quad (\xi \neq 0) \quad \text{and} \quad 1 = \mathcal{F}\psi(\xi) + \sum_{j=0}^{\infty} \mathcal{F}\varphi_j(\xi) \quad \bigl(\xi \in \mathbb{R}^{n+1}\bigr).$$

Here, $\mathcal{F}f$ denotes the Fourier transform. Let $\mathcal{S}'(\mathbb{R}^{n+1})$ be the space of all tempered distributions, and let $\mathcal{P}$ be the set of all polynomials.

Definition 1 *(Besov space on $\mathbb{R}^{n+1}$).* (Cf. [2,40].) Let $\alpha \in \mathbb{R}$, $1 \leqslant p, q \leqslant \infty$. The Besov spaces are defined as follows:

$$B^\alpha_{p,q} = B^\alpha_{p,q}\bigl(\mathbb{R}^{n+1}\bigr) := \bigl\{ f \in \mathcal{S}';\ \|f\|_{B^\alpha_{p,q}} < \infty \bigr\},$$
$$\dot{B}^\alpha_{p,q} = \dot{B}^\alpha_{p,q}\bigl(\mathbb{R}^{n+1}\bigr) := \bigl\{ f \in \mathcal{S}'/\mathcal{P};\ \|f\|_{\dot{B}^\alpha_{p,q}} < \infty \bigr\},$$

where

$$\|f\|_{B^\alpha_{p,q}} := \|\psi * f\|_p + \left\{ \sum_{j=0}^{\infty} \bigl(2^{j\alpha}\|\varphi_j * f\|_p\bigr)^q \right\}^{1/q} \quad (q < \infty),$$

$$\|f\|_{B^\alpha_{p,\infty}} := \|\psi * f\|_p + \sup_{j=0,1,\ldots} 2^{j\alpha}\|\varphi_j * f\|_p,$$

$$\|f\|_{\dot{B}^{\alpha}_{p,q}} := \left\{ \sum_{j=-\infty}^{\infty} \left(2^{j\alpha}\|\varphi_j * f\|_p\right)^q \right\}^{1/q} \quad (q<\infty),$$

$$\|f\|_{\dot{B}^{\alpha}_{p,\infty}} := \sup_{-\infty<j<\infty} 2^{j\alpha}\|\varphi_j * f\|_p.$$

Recall that

$$\left\{ f\in\mathcal{S}';\ \|f\|_{\dot{B}^{\alpha}_{p,q}}<\infty,\ f=\sum_{j=-\infty}^{\infty}\phi_j * f \text{ in } \mathcal{S}' \right\} \cong \dot{B}^{\alpha}_{p,q} \tag{2.1}$$

holds, if

$$\alpha < n/p, \quad \text{or} \quad (\alpha,q)=(n/p,1), \tag{2.2}$$

for details, see, e.g., [26]. Hence, when α, p and q satisfy (2.2), the definition of homogeneous Besov space may be modified as

$$\dot{B}^{\alpha}_{p,q} := \left\{ f\in\mathcal{S}';\ \|f\|_{\dot{B}^{\alpha}_{p,q}}<\infty,\ f=\sum_{j=-\infty}^{\infty}\phi_j * f \text{ in } \mathcal{S}' \right\}. \tag{2.3}$$

Hereinafter, (2.3) is used as the definition of $\dot{B}^{\alpha}_{p,q}$ when α, p and q satisfy (2.2). Then, if α, p and q satisfy (2.2), $\dot{B}^{\alpha}_{p,q}$ is a subspace of $\mathcal{S}'$ and

$$\|f\|_{\dot{B}^{\alpha}_{p,q}}=0 \quad \text{if and only if} \quad f=0 \quad \text{in } \mathcal{S}'.$$

Note that

$$\begin{aligned} &\dot{B}^0_{\infty,1}\subset B^0_{\infty,1}\subset C\big(\mathbb{R}^{n+1}\big)\subset L^\infty\subset \dot{B}^0_{\infty,\infty},\\ &\|f\|_{\dot{B}^{\alpha}_{p,q}}\leqslant C(\alpha)\|f\|_{B^{\alpha}_{p,q}} \quad \text{if } \alpha>0,\\ &\|f\|_{B^{\alpha}_{p,q}}\leqslant C(\alpha)\|f\|_{\dot{B}^{\alpha}_{p,q}} \quad \text{if } \alpha<0. \end{aligned} \tag{2.4}$$

Next, the Besov spaces on the half space $\mathbb{R}^{n+1}_+$ are defined as restrictions of $B^{\alpha}_{p,q}(\mathbb{R}^{n+1})$. For simplicity, $B^{\alpha}_{p,q}(\mathbb{R}^{n+1}_+)$ is sometimes denoted as $B^{\alpha}_{p,q,+}$.

Definition 2 *(Besov space on $\mathbb{R}^{n+1}_+$).* (Cf. [40].) Let $\alpha\in\mathbb{R}$, $1\leqslant p,q\leqslant\infty$. The Besov spaces on the half space are defined as follows:

$$B^{\alpha}_{p,q,+}=B^{\alpha}_{p,q}\big(\mathbb{R}^{n+1}_+\big) := \big\{ f;\ \exists g\in B^{\alpha}_{p,q}\big(\mathbb{R}^{n+1}\big) \text{ s.t. } f=g \text{ in } \mathcal{D}'\big(\mathbb{R}^{n+1}_+\big)\big\}, \tag{2.5}$$

$$\|f\|_{B^{\alpha}_{p,q,+}} := \inf\|g\|_{B^{\alpha}_{p,q}}, \tag{2.6}$$

where the infimum is taken over all $g\in B^{\alpha}_{p,q}(\mathbb{R}^{n+1})$ in the sense of (2.5),

$$\dot{B}^{\alpha}_{p,q,+}=\dot{B}^{\alpha}_{p,q}\big(\mathbb{R}^{n+1}_+\big) := \big\{ f;\ \exists g\in \dot{B}^{\alpha}_{p,q}\big(\mathbb{R}^{n+1}\big) \text{ s.t. } f=g \text{ in } \mathcal{D}'\big(\mathbb{R}^{n+1}_+\big)\big\}, \tag{2.7}$$

$$\|f\|_{\dot{B}^{\alpha}_{p,q,+}} := \inf\|g\|_{\dot{B}^{\alpha}_{p,q}}, \tag{2.8}$$

where the infimum is taken over all $g\in \dot{B}^{\alpha}_{p,q}(\mathbb{R}^{n+1})$ in the sense of (2.7).

Next, the zero-extension of distributions $f \in B^{\alpha}_{p,q,+}$ shall be defined. Let $\chi(x)$ be the characteristic function of the half space $\mathbb{R}^{n+1}_{+}$, i.e., $\chi(x) = 1_{(0,\infty)}(x_{n+1})$. If

$$1 \leqslant p \leqslant \infty, \qquad \frac{1}{p} - 1 < \alpha < \frac{1}{p}, \tag{2.9}$$

then χ is a *pointwise multiplier* for $B^{\alpha}_{p,q}$, and the following holds:

$$\|\chi g\|_{B^{\alpha}_{p,q}} \leqslant C\|g\|_{B^{\alpha}_{p,q}} \quad \text{for all } g \in B^{\alpha}_{p,q}, \tag{2.10}$$

where the constant C is independent of g. For the definitions of χg and the "pointwise multiplier" and for the proof of (2.10), refer to Triebel [40, pp. 140, 158]. If $\alpha > 0$, the definition of χg coincides a multiplication of $\chi \in L^{\infty}$ and $g \in L^{p}$ in the usual sense. For the case in which $\alpha \leqslant 0$, if (α, p, q) satisfies (2.9), then χg coincides the following definition:

$$(\chi g, \phi) := (g, \chi\phi) \quad \text{for all } \phi \in \mathcal{S}\big(\mathbb{R}^{n+1}\big),$$

where $(\cdot,\cdot)$ denotes the duality pairing. Based on (2.10), if (α, p, q) satisfies (2.9), there exists a constant $C = C(n, \alpha, p, q) > 0$, such that

$$\frac{1}{C}\big\|\mathbf{e}^0 f\big\|_{B^{\alpha}_{p,q}} \leqslant \|f\|_{B^{\alpha}_{p,q,+}} \leqslant C\big\|\mathbf{e}^0 f\big\|_{B^{\alpha}_{p,q}} \quad \text{for all } f \in B^{\alpha}_{p,q,+}, \tag{2.11}$$

where the definition of the zero-extension $\mathbf{e}^0 f$ for $f \in B^{\alpha}_{p,q,+} = B^{\alpha}_{p,q}(\mathbb{R}^{n+1}_{+})$ is as follows:

$$\mathbf{e}^0 f := \chi g \quad \text{for some } g \in B^{\alpha}_{p,q} \text{ with } f = g \in \mathcal{D}'\big(\mathbb{R}^{n+1}_{+}\big).$$

This definition is independent of the choice of g. The above definition of $\mathbf{e}^0 f$ is equivalent to the following:

$$\big(\mathbf{e}^0 f, \phi\big) := (g, \chi\phi) \quad \text{for all } \phi \in \mathcal{S}\big(\mathbb{R}^{n+1}\big).$$

Similarly, if (α, p, q) satisfies (2.9), for $f \in B^{\alpha}_{p,q,+}$, $\mathbf{e}^- f$, $\mathbf{e}^+ f$ and Ef are defined as follows:

$$\begin{aligned}
(\mathbf{e}^- f, \phi) &:= \big(g, \chi\big\{\phi(\tilde{x}, x_{n+1}) - \phi(\tilde{x}, -x_{n+1})\big\}\big) \quad \text{for all } \phi \in \mathcal{S}\big(\mathbb{R}^{n+1}\big),\\
(\mathbf{e}^+ f, \phi) &:= \big(g, \chi\big\{\phi(\tilde{x}, x_{n+1}) + \phi(\tilde{x}, -x_{n+1})\big\}\big) \quad \text{for all } \phi \in \mathcal{S}\big(\mathbb{R}^{n+1}\big),\\
Ef &:= \big(\mathbf{e}^+ f^1, \mathbf{e}^+ f^2, \ldots, \mathbf{e}^+ f^n, \mathbf{e}^- f^{n+1}\big),
\end{aligned}$$

where g is some distribution with $f = g$ in $\mathcal{D}'(\mathbb{R}^{n+1}_{+})$. If (α, p, q) satisfies (2.9), then

$$\|\mathbf{e}^{\pm} f\|_{B^{\alpha}_{p,q}} \cong \big\|\mathbf{e}^0 f\big\|_{B^{\alpha}_{p,q}} \cong \|f\|_{B^{\alpha}_{p,q,+}}.$$

For distributions g defined in the whole space $\mathbb{R}^{n+1}$, the restriction of g on $\mathbb{R}^{n+1}_{+}$ is denoted by $g|_{\mathbb{R}^{n+1}_{+}}$. For the sake of simplicity, $\mathbf{e}^0(g|_{\mathbb{R}^{n+1}_{+}})$, $\mathbf{e}^-(g|_{\mathbb{R}^{n+1}_{+}})$, $\mathbf{e}^+(g|_{\mathbb{R}^{n+1}_{+}})$, and $E(g|_{\mathbb{R}^{n+1}_{+}})$ are denoted by

$$\mathbf{e}^0 g, \ \mathbf{e}^- g, \ \mathbf{e}^+ g, \ \text{and } Eg,$$

respectively.

Next, the spaces of solenoidal vector fields in $B^{-s}_{\infty,q,+} = B^{-s}_{\infty,q}(\mathbb{R}^{n+1}_{+})$ $(0 < s < 1)$ are defined.

Definition 3 *(Spaces of solenoidal vector fields).* Let $0 < s < 1$, $1 \leqslant q \leqslant \infty$. Set

$$B^{-s}_{\infty,q,+,\sigma} := \{u \in (B^{-s}_{\infty,q,+})^{n+1};\ \operatorname{div} Eu = 0 \text{ in } \mathcal{S}'\},$$
$$B_\sigma := \{u \in (BUC(\bar{\mathbb{R}}^{n+1}_+))^{n+1};\ \operatorname{div} u = 0,\ u^{n+1}|_{\partial\mathbb{R}^{n+1}_+} = 0\},$$
$$C^m_\sigma(\mathbb{R}^{n+1}_+) := B_\sigma \cap C^m(\mathbb{R}^{n+1}_+) \quad (m = 1, 2, \ldots),$$

where *BUC* is the set of bounded uniformly continuous functions and C^m is the set of all m-times continuously differentiable functions f with $\sum_{|\alpha|\leqslant m} \sup_{x\in\mathbb{R}^{n+1}_+} |\partial^\alpha f(x)| < \infty$. Here, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ denotes a multi-index with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_{n+1}$ and ∂^α denotes $\partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_{n+1}}_{x_{n+1}}$.

Remark 1. (a) Note that for $u \in B^{-s}_{\infty,q,+,\sigma}$ $(0 < s < 1 < q < \infty)$, $(Eu(x_{n+1}))^{n+1} = \mathbf{e}^- u^{n+1}(x_{n+1}) \in C((-\infty,\infty); B^{-s}_{\infty,q}(\mathbb{R}^n))$, and hence $u^{n+1}|_{\partial\mathbb{R}^{n+1}_+} = 0$. (See Appendix A.) Note also that if $u \in W^{1,\infty}(\mathbb{R}^{n+1}_+)$ with $\operatorname{div} u = 0$ in $\mathbb{R}^{n+1}_+$ and $u^{n+1}|_{\partial\mathbb{R}^{n+1}_+} = 0$, then $\operatorname{div} Eu = 0$ in $\mathcal{S}'(\mathbb{R}^{n+1})$.

(b) For $f \in B^{-s}_{\infty,q,+,\sigma}$, let

$$f_N = \psi_N * Ef,$$

where $\psi_k = 2^{(n+1)k}\psi(2^k\cdot)$. Since f^{n+1}_N is a continuous odd function with respect to x_{n+1}, $f^{n+1}_N|_{\partial\mathbb{R}^{n+1}_+} = 0$ and $f_N \in B_\sigma$. Then, if $1 \leqslant q < \infty$, then B_σ is dense in $B^{-s}_{\infty,q,+,\sigma}$, because $\|Ef - f_N\|_{B^{-s}_{\infty,q}} \to 0$ as $N \to \infty$. Note also that $C^m_\sigma(\mathbb{R}^{n+1}_+)$ is also dense in $B^{-s}_{\infty,q,+,\sigma}$, because $f_N \in C^m(\mathbb{R}^{n+1}_+)$.

(c) $B^{-s}_{\infty,q}(\mathbb{R}^N)$ is wider than the uniformly local Lebesgue spaces $L^p_{uloc}(\mathbb{R}^N)$ for $\frac{N}{p} < s < 1$. Hence, for example, $\sum_{k\in\mathbb{Z}^N} \frac{1}{|x-k|^{s-\epsilon}} e^{-|x-k|}$ belongs to $B^{-s}_{\infty,q}(\mathbb{R}^N)$.

2.2. Main theorems

The following theorem is essentially due to the explicit formula of the resolvent of the Stokes operator given by Desch, Hieber and Prüss [8]. It is known that the Stokes operator $A_r = -P_+\Delta_r$ is a closed operator on

$$L^r_\sigma(\mathbb{R}^{n+1}_+) := \{v \in (L^r(\mathbb{R}^{n+1}_+))^{n+1};\ \operatorname{div} v = 0 \text{ in } \mathcal{D}'(\mathbb{R}^{n+1}_+),\ v|_{\partial\mathbb{R}^{n+1}_+} = 0\}$$

and $-A_r$ generates an analytic semi-group on $L^r_\sigma(\mathbb{R}^{n+1}_+)$ for $1 < r < \infty$. Desch, Hieber and Prüss [8] gave the explicit formula of the resolvent $R(\lambda) = (\lambda + A_r)^{-1}$ and showed that there exists a closed operator A_∞ in L^∞_σ such that $R(\lambda) = (\lambda + A_\infty)^{-1}$, $\lambda > 0$, where

$$L^\infty_\sigma(\mathbb{R}^{n+1}_+) = \left\{ f \in (L^n(\mathbb{R}^{n+1}_+))^n;\ \int_{\mathbb{R}^{n+1}_+} f \cdot \nabla\phi\, dx = 0 \text{ for all } \phi \in L^1_{loc}(\mathbb{R}^{n+1}_+) \text{ with } \nabla\phi \in L^1(\mathbb{R}^{n+1}_+) \right\}.$$

Moreover, they showed that $-A_\infty$ generates an analytic semi-group on L^∞_σ. In the present paper, this fact is proven to also hold on $B^{-s}_{\infty,q,+,\sigma}$. In Section 4 of the present paper, using the explicit formula of $R(\lambda)$ of Desch, Hieber and Prüss, it is shown that $R(\lambda)$ is a bounded operator on $B^{-s}_{\infty,q,+,\sigma}$ and that there exists a closed densely defined operator $A_{s,q,+}$ in $B^{-s}_{\infty,q,+,\sigma}$ such that

$$R(\lambda) = (\lambda + A_{s,q,+})^{-1}, \quad \lambda > 0.$$

Here, the domain $D(A_{s,q,+})$ of $A_{s,q,+}$ is defined by

$$D(A_{s,q,+}) := R(1)B^{-s}_{\infty,q,+,\sigma} = \bigl\{u \in B^{-s}_{\infty,q,+,\sigma};\ u = R(1)v,\ v \in B^{-s}_{\infty,q,+,\sigma}\bigr\}. \tag{2.12}$$

Hereinafter, for the sake of simplicity, $A_{s,q,+}$ is denoted by A. It is now possible to state the main theorems of the present study.

Theorem 1. *Let $0 < s < 1$, $1 \leqslant q < \infty$.*

(1) *$-A$ generates an analytic continuous semi-group $\{e^{-tA}\}_{t\geqslant 0}$ on $B^{-s}_{\infty,q,+,\sigma}$. (For the definition of $A = A_{s,q,+}$, see (4.29).)*
(2) *Let $f \in B^{-s}_{\infty,q,+,\sigma}$, $-s < m < 2 - s$, $0 \leqslant \alpha \leqslant 1$ and $0 < T < \infty$. Then, the following hold*:
   (a) $\|e^{-tA}f\|_{B^{-s}_{\infty,q,+,\sigma}} \leqslant C(s)\|f\|_{B^{-s}_{\infty,q,+,\sigma}}$ $(t > 0)$,
   (b) $\|\mathbf{e}^- e^{-tA}f\|_{B^m_{\infty,1}} \leqslant C(s,m)(1+\frac{1}{t})^{(m+s)/2}\|f\|_{B^{-s}_{\infty,q,+,\sigma}}$ $(t > 0)$,
   (c) $\|\nabla e^{-tA}f\|_{B^{-s}_{\infty,1,+}} \leqslant C\|\nabla \mathbf{e}^- e^{-tA}f\|_{B^{-s}_{\infty,1}} \leqslant C(s,m)(1+\frac{1}{t})^{1/2}\|f\|_{B^{-s}_{\infty,q,+,\sigma}}$ $(t > 0)$,
   (d) $\|(e^{-tA} - e^{-\tau A})f\|_{B^{-s}_{\infty,q,+,\sigma}} \leqslant C(s,T)(t-\tau)^{\alpha}\tau^{-\alpha}\|f\|_{B^{-s}_{\infty,q,+,\sigma}}$ $(0 < \tau < t \leqslant T)$,
   (e) $\|\nabla(e^{-tA} - e^{-\tau A})f\|_{B^{-s}_{\infty,1,+}} \leqslant C(s,T)(t-\tau)^{\alpha}\tau^{-\alpha-1/2}\|f\|_{B^{-s}_{\infty,q,+,\sigma}}$ $(0 < \tau < t \leqslant T)$.

Remark 2. (i) Although $u(t) = e^{-tA_r}a$ satisfies the Stokes system

$$\partial_t u - \Delta u + \nabla p = 0 \quad \text{in } \mathbb{R}^{n+1}_+, \qquad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^{n+1}_+,$$
$$u|_{\partial\mathbb{R}^{n+1}_+} = 0, \qquad u|_{t=0} = a \tag{2.13}$$

for $a \in L^r_\sigma$ $(1 < r < \infty)$, it is unclear whether $u = e^{-tA}a$ satisfies the Stokes system for non-decaying initial data $a \in B^{-s}_{\infty,q,+,\sigma}$, because $L^r \cap B^{-s}_{\infty,q,+,\sigma}$ is not dense in $B^{-s}_{\infty,q,+,\sigma}$. In Appendix A, it is proven that $u = e^{-tA}a$ satisfies the Stokes system for $a \in B^{-s}_{\infty,q,+,\sigma}$. Hence, $A = A_{s,q,+}$ can be referred to as the Stokes operator on $B^{-s}_{\infty,q,+,\sigma}$.

(ii) Letting $t = 1$, for $0 < m < 2 - s$ and $0 < s' < s < 1$, Theorem 1(b) yields the following:

$$\bigl\|\mathbf{e}^- e^{-A}f\bigr\|_{\dot B^m_{\infty,1}} \leqslant C\|f\|_{B^{-s}_{\infty,q,+,\sigma}} \leqslant C\|f\|_\infty \quad \text{and} \quad \bigl\|\mathbf{e}^- e^{-A}f\bigr\|_{\dot B^m_{\infty,1}} \leqslant C\|f\|_{B^{-s}_{\infty,q,+}} \leqslant C\|f\|_{\dot B^{-s'}_{\infty,\infty,+}}.$$

These inequalities and the basic scaling argument yield the following decay estimates for Stokes equations:

$$\bigl\|\mathbf{e}^- e^{-tA}f\bigr\|_{\dot B^m_{\infty,1}} \leqslant Ct^{-m/2}\|f\|_\infty \quad \text{and} \quad \bigl\|\mathbf{e}^- e^{-tA}f\bigr\|_{\dot B^m_{\infty,1}} \leqslant Ct^{-(m+s')/2}\|f\|_{\dot B^{-s'}_{\infty,\infty,+}}.$$

(iii) For the case in which the domain is the whole space $\mathbb{R}^N$ $(N \geqslant 1)$, it is well known that for $\lambda \in \mathbb{C} - \{0\}$ with $\arg\lambda < \theta (< \pi)$, $\|(\lambda - \Delta)^{-1}f\|_\infty \leqslant C(N,\theta)\|f\|_\infty/|\lambda|$ holds, and hence the heat semi-group $e^{t\Delta}$ is an analytic semi-group on $B^\alpha_{\infty,q}(\mathbb{R}^N)$ and $\dot B^\alpha_{\infty,q}(\mathbb{R}^N)$ for all $\alpha \in \mathbb{R}$, $1 \leqslant q < \infty$.

Next, the local existence theorem for the Boussinesq equations is stated.

Theorem 2. *Let $0 < s < 1$, $1 \leqslant q < \infty$, $u_0 \in B^{-s}_{\infty,q,+,\sigma}$, $\mathbf{e}^-\bar\theta_0 \in \dot B^0_{\infty,1}(\mathbb{R}^{n+1})$, and $S, \partial_{\tilde x}^2 S, \partial_t S \in C^\alpha([0,\infty); L^\infty(\mathbb{R}^n_{\tilde x}))$ for some $\alpha > 0$. Then, there exist $T > 0$ and a solution $(u, \bar\theta)$ to (AB) on $[0, T)$ such that*

$$u \in C\big([0,T];B^{-s}_{\infty,q,+,\sigma}\big)\cap C^1\big((0,T];B^{-s}_{\infty,q,+,\sigma}\big)\cap C\big((0,T];D(A)\big), \tag{2.14}$$

$$\mathbf{e}^{-}\bar\theta \in C\big([0,T];\dot B^0_{\infty,1}\big)\cap C^1\big((0,T];L^\infty\big) \quad \text{and} \quad \Delta\mathbf{e}^{-}\bar\theta \in C\big((0,T];L^\infty\big), \tag{2.15}$$

$$t^{1/2}\nabla u(t) \in L^\infty\big(0,T;B^{-s}_{\infty,1,+}\big), \tag{2.16}$$

$$t^{1/2}\nabla \mathbf{e}^{-}\bar\theta(t) \in L^\infty\big(0,T;\dot B^0_{\infty,1}\big). \tag{2.17}$$

As for uniqueness, $(u,\bar\theta)$ is the only solution of (AB) *in the class* (2.14)–(2.15)*.*

Here, $D(A)=R(1)B^{-s}_{\infty,q,+,\sigma}\subset B^{-s}_{\infty,q,+,\sigma}\cap B^{-s+2-\epsilon}_{\infty,q,+}$ for all $\epsilon>0$. (See Section 4, (4.13).) Hence, $D(A)\subset W^{1,\infty}$. Moreover, Section 4 reveals that $v|_{\partial\mathbb{R}^{n+1}_+}=0$ for $v\in D(A)$. Therefore, the solution u given in Theorem 2 satisfies the boundary condition $u(t)|_{\partial\mathbb{R}^{n+1}_+}=0$ for $t>0$. Since $\dot B^0_{\infty,1}\subset BUC(\mathbb{R}^{n+1})$ and $\mathbf{e}^{-}\bar\theta(\tilde x,x_{n+1})$ is an odd function with respect to the x_{n+1}-variable, (2.15) implies $\mathbf{e}^{-}\bar\theta|_{\partial\mathbb{R}^{n+1}_+}=0$. In addition, the solution $(u,\bar\theta)$ to (AB) given in Theorem 2 satisfies the Boussinesq equations (B). (See Remark 2(i) and Appendix A.4.)

Before closing this section, a number of lemmas are stated.

Lemma 2.1. (a) *Let* $\sup_{\tilde x\in\mathbb{R}^n}|f(\tilde x,x_{n+1})|\in L^p(0,\infty)$ $(1\leqslant p\leqslant\infty)$*. Then,* $\mathbf{e}^{+}f,\mathbf{e}^{-}f$ *and* $\mathbf{e}^0 f$ *belong to the homogeneous Besov space* $\dot B^{-\frac{1}{p}}_{\infty,\infty}$*, and it holds that*

$$\|\mathbf{e}^{\pm}f\|_{\dot B^{-\frac{1}{p}}_{\infty,\infty}} \cong \big\|\mathbf{e}^0 f\big\|_{\dot B^{-\frac{1}{p}}_{\infty,\infty}} \leqslant C\|f\|_{L^p(0,\infty;L^\infty(\mathbb{R}^n_{\tilde x}))}. \tag{2.18}$$

(b) *Let* $0<s<1$, $1\leqslant q\leqslant\infty$*. Then,*

$$\big\|\mathbf{e}^0 f\big\|_{\dot B^{-s}_{\infty,q}} \leqslant C\|f\|_{L^1_w(0,\infty;L^\infty(\mathbb{R}^n_{\tilde x}))\cap L^\infty(\mathbb{R}^{n+1}_+)}.$$

Here,

$$\begin{aligned}
\|f\|_{L^1_w(0,\infty;L^\infty(\mathbb{R}^n_{\tilde x}))\cap L^\infty(\mathbb{R}^{n+1}_+)} &= \|f\|_{L^1_w(0,\infty;L^\infty(\mathbb{R}^n_{\tilde x}))}+\|f\|_{L^\infty(\mathbb{R}^{n+1}_+)},\\
\|f\|_{L^1_w(0,\infty;L^\infty(\mathbb{R}^n_{\tilde x}))} &= \Big\|\sup_{\tilde x\in\mathbb{R}^n}\big|f(\tilde x,x_{n+1})\big|\Big\|_{L^1_w(0,\infty)}\\
&= \sup_s s\mu\Big\{x_{n+1}\in(0,\infty);\ \sup_{\tilde x}\big|f(\tilde x,x_{n+1})\big|>s\Big\},
\end{aligned}$$

where μ denotes the Lebesgue measure.

Concerning the proof of Lemma 2.1, recall that for $1\leqslant a\leqslant r$

$$\|u\|_{\dot B^{-N/r}_{\infty,\infty}} \leqslant C\|u\|_{\mathcal{M}^r_a(\mathbb{R}^N)} \tag{2.19}$$

(see Kozono and Yamazaki [26, (2.3)]), where $\mathcal{M}^r_a(\mathbb{R}^N)$ is the Morrey space defined by

$$\|u\|_{\mathcal{M}^r_a(\mathbb{R}^N)} := \sup_{x_0\in\mathbb{R}^N}\sup_{R>0} R^{N/r-N/a}\left\{\int_{|x-x_0|<R}|u|^a(x)\,dx\right\}^{1/a}.$$

Since

$$\|u\|_{\mathcal{M}^{pN}_{p}(\mathbb{R}^N)} \leqslant C\|u\|_{L^p(-\infty,\infty;L^\infty(\mathbb{R}^{N-1}))},$$

the estimate (2.18) by (2.19) with $N = n+1$, $a = p$, and $r = pN = p(n+1)$ is obtained. Here, (b) follows immediately from (a).

Proposition 2.1. *Let $y, h, q, \gamma > 0$ and $d \in \mathbb{R}$. Then, it holds that*

$$\begin{aligned}&\int_0^\infty dy' \int_{\mathbb{R}^n} d\tilde{x} \int_0^\infty dr\, r^{n-2} \int_0^\infty e^{-q(r+s)(|\tilde{x}|+y'+y)-qy'} \frac{(r+s)^\gamma (1+r+s)^d}{(1+y+r+s)^h}\, ds \\ &\quad = C\int_0^\infty e^{-qsy} \frac{s^{\gamma-1}(1+s)^{d-1}}{(1+y+s)^h}\, ds, \end{aligned} \tag{2.20}$$

where C is a constant depending only on q and n.

The above proposition is proved by elementary calculation, as follows:

$$\begin{aligned} F(\tilde{x}, y') &= \int_0^\infty dr\, r^{n-2} \int_0^\infty e^{-q(r+s)(|\tilde{x}|+y'+y)-qy'} \frac{(r+s)^\gamma (1+r+s)^d}{(1+y+r+s)^h}\, ds \\ &= \int_0^\infty dr\, r^{n-2} \int_r^\infty e^{-qs(|\tilde{x}|+y'+y)-qy'} \frac{s^\gamma (1+s)^d}{(1+y+s)^h}\, ds \\ &= C\int_0^\infty \frac{s^{n+\gamma-1}(1+s)^d}{(1+y+s)^h} e^{-qs(|\tilde{x}|+y'+y)-qy'}\, ds, \end{aligned}$$

$$\begin{aligned} \int_{\mathbb{R}^n} F(\tilde{x}, y')\, d\tilde{x} &= C\int_0^\infty \left(\int_0^\infty \rho^{n-1} e^{-q\rho s}\, d\rho\right) e^{-q(y+y')s-qy'} \frac{s^{n+\gamma-1}(1+s)^d}{(1+y+s)^h}\, ds \\ &= C\int_0^\infty e^{-q(y+y')s-qy'} \frac{s^{\gamma-1}(1+s)^d}{(1+y+s)^h}\, ds, \end{aligned}$$

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^n} F(\tilde{x}, y')\, d\tilde{x}\, dy' &= C\int_0^\infty \left(\int_0^\infty e^{-q(1+s)y'}\, dy'\right) e^{-qys} \frac{s^{\gamma-1}(1+s)^d}{(1+y+s)^h}\, ds \\ &= C\int_0^\infty e^{-qys} \frac{s^{\gamma-1}(1+s)^{d-1}}{(1+y+s)^h}\, ds. \end{aligned}$$

Proposition 2.2. (1) *Let $\beta > 0$ and $q > 0$ be constants. Then, it holds that*

$$\int_0^\infty \frac{e^{-q\beta s} s^\gamma}{(1+s)(1+\beta+s)}\, ds \leqslant \begin{cases} \frac{C(q,\gamma)}{(1+\beta)^2} & \text{if } \gamma = 0, \\ \frac{C(q)\log(e+1/\beta)}{(1+\beta)^3} & \text{if } \gamma = 1, \\ \frac{C(q,\gamma)}{\beta(1+\beta)^3} & \text{if } \gamma = 2, \\ \frac{C(q,\gamma)}{\beta^2(1+\beta)^3} & \text{if } \gamma = 3, \end{cases} \tag{2.21}$$

where $C(q,\gamma)$ is a constant that is independent of β.

(2) *Let $\beta > 0$ and $q > 0$ be constants. Then, it holds that*

$$\int_0^\infty \frac{e^{-q\beta s} s^\gamma}{(1+\beta+s)}\,ds \leqslant \begin{cases} \frac{C(q)\log(e+1/\beta)}{(1+\beta)^2} & \text{if } \gamma = 0, \\ \frac{C(q,\gamma)}{\beta^\gamma(1+\beta)^2} & \text{if } \gamma = 1,2,3, \end{cases} \tag{2.22}$$

where $C(q,\gamma)$ is a constant that is independent of β.

(3) *Let $\beta > 0$ and $q > 0$ be constants. Then, it holds that*

$$\int_0^\infty \frac{e^{-q\beta s}}{(1+s)}\,ds \leqslant C(q)\frac{\log(e+1/\beta)}{1+\beta}, \tag{2.23}$$

where $C(q)$ is a constant that is independent of β.

The proof of Proposition 2.2 is elementary and so is omitted here.

Proposition 2.3. *For all $f \in \dot{B}^\gamma_{\infty,\infty}$, the following holds*:

$$\left\| e^{t\Delta} f \right\|_{\dot{B}^m_{\infty,1}} \leqslant C(n,m,\gamma) t^{-(m-\gamma)/2} \|f\|_{\dot{B}^\gamma_{\infty,\infty}} \quad \text{for } t > 0 \text{ and } m > \gamma. \tag{2.24}$$

The proof of Proposition 2.3 is simple. Letting $2N > m - \gamma$ and $t = 1$, we have

$$\begin{aligned} \left\| e^{\Delta} f \right\|_{\dot{B}^m_{\infty,1}} &= \sum_{j\leqslant 0} 2^{(m-\gamma)j} 2^{\gamma j} \left\| \varphi_j * e^{\Delta} f \right\|_\infty + \sum_{j\geqslant 1} 2^{(m-\gamma)j} 2^{\gamma j} \left\| \varphi_j * e^{\Delta} f \right\|_\infty \\ &\leqslant \sum_{j\leqslant 0} 2^{(m-\gamma)j} \|f\|_{\dot{B}^\gamma_{\infty,\infty}} + \sum_{j\geqslant 1} 2^{(m-\gamma)j} 2^{\gamma j} \left\| (-\Delta)^{-N} \tilde{\varphi}_j * (-\Delta)^N e^{\Delta} \varphi_j * f \right\|_\infty \\ &\leqslant C\|f\|_{\dot{B}^\gamma_{\infty,\infty}} + C \sum_{j\geqslant 1} 2^{(m-\gamma-2N)j} \|f\|_{\dot{B}^\gamma_{\infty,\infty}} \leqslant C\|f\|_{\dot{B}^\gamma_{\infty,\infty}}, \end{aligned}$$

where $\tilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$, and we used $\|(-\Delta)^{-N}\tilde{\varphi}_j\|_1 = C2^{-2Nj}$. Combining this inequality with the scaling argument yields (2.24).

3. Explicit formula of the resolvent of $P_+\Delta$ given by D–H–P

In this section, the formula of the resolvent of $-A$ given by Desch, Hieber and Prüss [8] is recalled. Let $\lambda \in \Sigma_\pi := \{z \in \mathbb{C} - \{0\};\ |\arg z| < \pi\}$ and $f \in (L^r(\mathbb{R}^{n+1}_+))^{n+1}$ with $\operatorname{div} f = 0$. The resolvent problem for the Stokes problem is as follows:

$$\begin{aligned} \lambda u - \Delta u + \nabla p &= f \quad \text{in } \mathbb{R}^{n+1}_+, \\ \operatorname{div} u &= 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\ u|_{\partial\mathbb{R}^{n+1}_+} &= 0. \end{aligned} \tag{3.1}$$

For convenience, notations similar to those in [8] are used. (Whereas in [8], the operator A denotes $P_+\Delta$, in the present paper, A denotes $-P_+\Delta$.) For $(\tilde{x}, y) \in \mathbb{R}^{n+1}_+ \equiv \{(\tilde{x}, y) \in \mathbb{R}^{n+1};\ \tilde{x} \in \mathbb{R}^n,\ y > 0\}$, $u = u(\tilde{x}, y) = (v, w)^T$ is written with $v = v(\tilde{x}, y) = (u^1, u^2, \ldots, u^n)^T$, $w = w(\tilde{x}, y) = u^{n+1}$, and $f = (f_v, f_w)^T$ is written with $f_v = (f^1, \ldots, f^n)^T$. Assume that $f_w(\tilde{x}, 0) = 0$ for all $\tilde{x} \in \mathbb{R}^n$. Applying the Fourier transform with respect to $\tilde{x}$, we have

$$\big(\lambda+|\xi|^2\big)\hat{v}(\xi,y)-\partial_y^2\hat{v}(\xi,y)=\hat{f}_v(\xi,y)-i\xi\cdot\hat{p}(\xi,y),\quad \xi\in\mathbb{R}^n,\ y>0, \tag{3.2}$$

$$\big(\lambda+|\xi|^2\big)\hat{w}(\xi,y)-\partial_y^2\hat{w}(\xi,y)=\hat{f}_w(\xi,y)-\partial_y\hat{p}(\xi,y),\quad \xi\in\mathbb{R}^n,\ y>0, \tag{3.3}$$

$$i\xi\cdot\hat{v}(\xi,y)+\partial_y\hat{w}(\xi,y)=0,\quad \xi\in\mathbb{R}^n,\ y>0, \tag{3.4}$$

$$\hat{v}(\xi,0)=0,\quad \xi\in\mathbb{R}^n, \tag{3.5}$$

$$\hat{w}(\xi,0)=0,\quad \xi\in\mathbb{R}^n. \tag{3.6}$$

Following [8], we have

$$\begin{aligned}\hat{v}&=\hat{v}_1+\hat{v}_2,\\ \hat{w}&=\hat{w}_1+\hat{w}_2,\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}\omega(\xi)&=\big(\lambda+|\xi|^2\big)^{1/2},\\ \hat{v}_1(\xi,y)&=\frac{1}{2\omega(\xi)}\int_0^\infty\big[e^{-\omega(\xi)|y-s|}-e^{-\omega(\xi)(y+s)}\big]\hat{f}_v(\xi,s)\,ds,\\ \hat{v}_2(\xi,y)&=\frac{\xi(e^{-|\xi|y}-e^{-\omega(\xi)y})}{|\xi|(\omega(\xi)-|\xi|)\omega(\xi)}\int_0^\infty e^{-\omega(\xi)s}\xi\cdot\hat{f}_v(\xi,s)\,ds,\\ \hat{w}_1(\xi,y)&=\frac{1}{2\omega(\xi)}\int_0^\infty\big[e^{-\omega(\xi)|y-s|}-e^{-\omega(\xi)(y+s)}\big]\hat{f}_w(\xi,s)\,ds,\\ \hat{w}_2(\xi,y)&=\frac{i}{\omega(\xi)}\frac{e^{-|\xi|y}-e^{-\omega(\xi)y}}{\omega(\xi)-|\xi|}\int_0^\infty e^{-\omega(\xi)s}\xi\cdot\hat{f}_v(\xi,s)\,ds.\end{aligned} \tag{3.8}$$

For details, see [8, Section 2].

In addition, v_1 and w_1 are solutions to the following equations:

$$\begin{cases}\lambda v_1-\Delta v_1=f_v & \text{in } \mathbb{R}^{n+1}_+,\\ v_1|_{\partial\mathbb{R}^{n+1}_+}=0,\end{cases}\qquad \begin{cases}\lambda w_1-\Delta w_1=f_w & \text{in } \mathbb{R}^{n+1}_+,\\ w_1|_{\partial\mathbb{R}^{n+1}_+}=0\end{cases}$$

and

$$v_1=(\lambda-\Delta)^{-1}\mathbf{e}^-f_v,\qquad w_1=(\lambda-\Delta)^{-1}\mathbf{e}^-f_w, \tag{3.9}$$

where $\mathbf{e}^-$ is defined by $\mathbf{e}^-u:=(\mathbf{e}^-u^1,\mathbf{e}^-u^2,\dots,\mathbf{e}^-u^{n+1})$ for vector fields u in $\mathbb{R}^{n+1}_+$ and $(\lambda-\Delta)^{-1}$ is the resolvent of the Laplacian $\Delta=\partial_{x_1}^2+\cdots+\partial_{x_n}^2+\partial_y^2$ in the whole space $\mathbb{R}^{n+1}=\mathbb{R}^n_x\times\mathbb{R}_y$. Then, the resolvent $R(\lambda)$ of $-A_r=P_+\Delta$ has the following formula:

$$R(\lambda)f=(v,w)=(\lambda-\Delta)^{-1}\mathbf{e}^-f+(v_2,w_2). \tag{3.10}$$

Moreover, v_2 and w_2 has the following representations:

$$v_2(\tilde{x}, y) = \int_0^\infty dy' \int_{\mathbb{R}^n} r_v(\tilde{x} - \tilde{x}', y, y', \lambda) f_v(\tilde{x}', y')\, d\tilde{x}', \tag{3.11}$$

$$w_2(\tilde{x}, y) = \int_0^\infty dy' \int_{\mathbb{R}^n} r_w(\tilde{x} - \tilde{x}', y, y', \lambda) f_v(\tilde{x}', y')\, d\tilde{x}', \tag{3.12}$$

$$r_v(\tilde{x}, y, y', \lambda) = \frac{1}{(2\pi)^n} \int e^{i\tilde{x}\cdot\xi}\ ye^{-|\xi|y} e^{-\omega(\xi)y'} \frac{\xi^T\xi}{|\xi|\omega(\xi)} \phi\big(y\big(\omega(\xi) - |\xi|\big)\big)\, d\xi, \tag{3.13}$$

$$r_w(\tilde{x}, y, y', \lambda) = \frac{1}{(2\pi)^n} \int e^{i\tilde{x}\cdot\xi} ye^{-|\xi|y} e^{-\omega(\xi)y'} \frac{i\xi}{\omega(\xi)} \phi\big(y\big(\omega(\xi) - |\xi|\big)\big)\, d\xi, \tag{3.14}$$

$$\phi(z) = \frac{1 - e^{-z}}{z} = \sum_{k=1}^\infty \frac{(-z)^{k-1}}{k!} \quad (z \in \mathbb{C}), \tag{3.15}$$

where

$$\xi^T = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} \quad \text{and} \quad \xi^T\xi = (\xi_i\xi_j)_{i,j=1,2,\dots,n}.$$

Note that

$$\partial_{y'}^k \partial_{\tilde{x}}^\alpha r_v(\tilde{x}, y, y', \lambda) = \frac{1}{(2\pi)^n} \int e^{i\tilde{x}\cdot\xi} ye^{-|\xi|y} e^{-\omega(\xi)y'} \big(-\omega(\xi)\big)^k \frac{(i\xi)^\alpha \xi^T\xi}{|\xi|\omega(\xi)} \phi\big(y\big(\omega(\xi) - |\xi|\big)\big)\, d\xi, \tag{3.16}$$

$$\partial_{y'}^k \partial_{\tilde{x}}^\alpha r_w(\tilde{x}, y, y', \lambda) = \frac{1}{(2\pi)^n} \int e^{i\tilde{x}\cdot\xi} ye^{-|\xi|y} e^{-\omega(\xi)y'} \big(-\omega(\xi)\big)^k \frac{(i\xi)^\alpha i\xi}{\omega(\xi)} \phi\big(y\big(\omega(\xi) - |\xi|\big)\big)\, d\xi. \tag{3.17}$$

The following lemma plays an important role in proving Theorem 1.

Lemma 3.1. (1) *Let $k = 0, 1$, and let α be a multi-index with $|\alpha| \leqslant 3$. Assume that $0 < \theta < \pi$. Then, there exists a constant C that depends only on n, k, θ and α such that*

$$\int_0^\infty dy' \int_{\mathbb{R}^n} \big|\partial_{y'}^k \partial_{\tilde{x}}^\alpha r_v(\tilde{x}, y, y', \lambda)\big|\, d\tilde{x} + \int_0^\infty dy' \int_{\mathbb{R}^n} \big|\partial_{y'}^k \partial_{\tilde{x}}^\alpha r_w(\tilde{x}, y, y', \lambda)\big|\, d\tilde{x}$$
$$\leqslant \begin{cases} \dfrac{Cy}{|\lambda|^{\frac{1}{2}}(1+|\lambda|^{\frac{1}{2}}y)^2} & \text{if } (k, |\alpha|) = (0, 0), \\ \dfrac{Cy\log(e+1/(|\lambda|^{\frac{1}{2}}y))}{(1+|\lambda|^{\frac{1}{2}}y)^3} & \text{if } (k, |\alpha|) = (0, 1), \\ \dfrac{C}{(1+|\lambda|^{\frac{1}{2}}y)^3} & \text{if } (k, |\alpha|) = (0, 2), \\ \dfrac{C}{y(1+|\lambda|^{\frac{1}{2}}y)^3} & \text{if } (k, |\alpha|) = (0, 3), \\ \dfrac{Cy\log(e+1/(|\lambda|^{\frac{1}{2}}y))}{(1+|\lambda|^{\frac{1}{2}}y)^2} & \text{if } (k, |\alpha|) = (1, 0), \\ \dfrac{C}{y^{|\alpha|-1}(1+|\lambda|^{\frac{1}{2}}y)^2} & \text{if } (k, |\alpha|) = (1, 1), (1, 2), (1, 3) \end{cases} \tag{3.18}$$

for all $\lambda \in \Sigma_\theta$. Here, $\Sigma_\theta = \{z \in \mathbb{C} - \{0\};\ |\arg z| < \theta\}$.

(2) *Let α be a multi-index with $|\alpha| \leqslant 1$. Assume $0 < \theta < \pi$. Then, there exists a constant C depending only on n, θ and α such that*

$$\int_0^\infty dy' \int_{\mathbb{R}^n} \left|\partial_{\tilde{x}}^\alpha \partial_y^2 r_v(\tilde{x}, y, y', \lambda)\right| d\tilde{x} + \int_0^\infty dy' \int_{\mathbb{R}^n} \left|\partial_{\tilde{x}}^\alpha \partial_y^2 r_w(\tilde{x}, y, y', \lambda)\right| d\tilde{x} \leqslant \begin{cases} \frac{C}{1+|\lambda|^{\frac{1}{2}} y} \log(e + \frac{1}{|\lambda|^{\frac{1}{2}} y}) & \text{if } |\alpha| = 0, \\ \frac{C}{y(1+|\lambda|^{\frac{1}{2}} y)} & \text{if } |\alpha| = 1 \end{cases} \tag{3.19}$$

and

$$\int_0^\infty dy' \int_{\mathbb{R}^n} \left|\partial_{y'} \partial_y^2 r_v(\tilde{x}, y, y', \lambda)\right| d\tilde{x} + \int_0^\infty dy' \int_{\mathbb{R}^n} \left|\partial_{y'} \partial_y^2 r_w(\tilde{x}, y, y', \lambda)\right| d\tilde{x} \leqslant \frac{C}{y} \tag{3.20}$$

for all $\lambda \in \Sigma_\theta$.

Proof. Lemma 3.1 is proven in the same manner as Lemma 1 given by [8]. First, (3.18) is shown. From (3.13) and (3.14), we have $r_v(\tilde{x}, y, y', \lambda) = |\lambda|^{\frac{n-1}{2}} r_v(|\lambda|^{\frac{1}{2}}\tilde{x}, |\lambda|^{\frac{1}{2}} y, |\lambda|^{\frac{1}{2}} y', \frac{\lambda}{|\lambda|})$ and $r_w(\tilde{x}, y, y', \lambda) = |\lambda|^{\frac{n-1}{2}} r_w(|\lambda|^{\frac{1}{2}}\tilde{x}, |\lambda|^{\frac{1}{2}} y, |\lambda|^{\frac{1}{2}} y', \frac{\lambda}{|\lambda|})$. Hence, it may be assumed that $\lambda \in \Sigma_\theta$ with $|\lambda| = 1$. Let $Q_{\tilde{x}} = (q_{ij})_{i,j=1,2,\dots,n}$ be a rotation matrix such that

$$\tilde{x} Q_{\tilde{x}} = \left(x^1, x^2, \dots, x^n\right) Q_{\tilde{x}} = \left(|\tilde{x}|, 0, 0, \dots, 0\right),$$

and let $\xi Q_{\tilde{x}} = (a, rb)$, $a \in \mathbb{R}$, $r > 0$, $b \in S^{n-1}$. Then, (3.16) and (3.17) yield

$$\begin{aligned} \partial_{y'}^k \partial_{\tilde{x}}^\alpha r_v(\tilde{x}, y, y', \lambda) = c_n \int_0^\infty dr\, r^{n-2} \int_{S^{n-1}} db \int_{-\infty}^\infty e^{i|\tilde{x}|a} \frac{y(-\sqrt{\lambda + r^2 + a^2})^k}{\sqrt{\lambda + r^2 + a^2}\sqrt{r^2 + a^2}} e^{-y\sqrt{r^2+a^2}} \\ \times e^{-y'\sqrt{\lambda+r^2+a^2}} \phi\left(y\left(\sqrt{\lambda + r^2 + a^2} - \sqrt{r^2 + a^2}\right)\right)\left(i(a, rb) Q_{\tilde{x}}^T\right)^\alpha \\ \times Q_{\tilde{x}} \begin{pmatrix} a \\ rb^T \end{pmatrix} (a, rb) Q_{\tilde{x}}^T \, da, \\ \partial_{y'}^k \partial_{\tilde{x}}^\alpha r_w(\tilde{x}, y, y', \lambda) = c_n \int_0^\infty dr\, r^{n-2} \int_{S^{n-1}} db \int_{-\infty}^\infty e^{i|\tilde{x}|a} \frac{y(-\sqrt{\lambda + r^2 + a^2})^k}{\sqrt{\lambda + r^2 + a^2}} e^{-y\sqrt{r^2+a^2}} \\ \times e^{-y'\sqrt{\lambda+r^2+a^2}} \phi\left(y\left(\sqrt{\lambda + r^2 + a^2} - \sqrt{r^2 + a^2}\right)\right)\left(i(a, rb) Q_{\tilde{x}}^T\right)^\alpha i(a, rb) Q_{\tilde{x}}^T \, da. \end{aligned}$$

Let $\Re z$ and $\Im z$ denote the real and imaginary parts of $z \in \mathbb{C}$, respectively. There exists a sufficiently small number $\varepsilon_0 = \varepsilon_0(\theta) > 0$ such that the functions $\sqrt{r^2 + z^2}$, $\sqrt{\lambda + r^2 + z^2}$, $\frac{1}{\sqrt{r^2+z^2}}$, and $\frac{1}{\sqrt{\lambda+r^2+z^2}}$ are analytic in the domain $\{z \in \mathbb{C};\ 0 \leqslant \Im z < \varepsilon_0(r + |\Re z|)\}$ and the following properties hold: if $a = s + i\varepsilon(r + |s|)$, $\varepsilon \in (0, \varepsilon_0)$, $s \in \mathbb{R}$, $r > 0$, then there exists a constant $c = c(\theta, \varepsilon) > 0$, such that

$$c\left|r^2 + a^2\right| \leqslant \left(r + |s|\right)^2 \leqslant \frac{1}{c}\left|r^2 + a^2\right|, \tag{3.21}$$

$$\left|\arg\left(r^2 + a^2\right)\right| \leqslant 8\varepsilon, \qquad \left|\arg\left(\lambda + r^2 + a^2\right)\right| \leqslant \theta, \tag{3.22}$$

$$\left|e^{i|\tilde{x}|a}\right| \leqslant e^{-\varepsilon|\tilde{x}|(r+|s|)}, \tag{3.23}$$

$$\left|e^{-y\sqrt{r^2+a^2}}\right| \leqslant e^{-cy(r+|s|)} \quad \text{for all } y > 0, \tag{3.24}$$

$$\left|e^{-y'\sqrt{\lambda+r^2+a^2}}\right| \leqslant e^{-cy'(1+r+|s|)} \quad \text{for all } y' > 0, \tag{3.25}$$

$$c\big(1+r+|s|\big) \leqslant \left|\sqrt{\lambda+r^2+a^2}\right| \leqslant \frac{1}{c}\big(1+r+|s|\big), \tag{3.26}$$

$$\left|\sqrt{\lambda+r^2+a^2}-\sqrt{r^2+a^2}\right| = \frac{1}{|\sqrt{\lambda+r^2+a^2}+\sqrt{r^2+a^2}|} \geqslant \frac{c}{1+r+|s|} \tag{3.27}$$

for all $\lambda \in \Sigma_\theta$ with $|\lambda| = 1$. Moreover, we have

$$\left|e^{-y\sqrt{r^2+a^2}}\phi\big(y\big(\sqrt{\lambda+r^2+a^2}-\sqrt{r^2+a^2}\big)\big)\right| \leqslant \frac{1}{c}\cdot\frac{e^{-cy(r+|s|)}}{1+y|\sqrt{\lambda+r^2+a^2}-\sqrt{r^2+a^2}|} \tag{3.28}$$

for all $\lambda \in \Sigma_\theta$ with $|\lambda| = 1$. The proof of (3.28) is given in Appendix A. Hence, by shifting the path of integration for a to the contour $a \to s + i\varepsilon(r+|s|)$, $s \in \mathbb{R}$ for $\varepsilon \in (0, \varepsilon_0)$, we obtain

$$\begin{aligned}&\left|\partial_{y'}^k\partial_{\tilde{x}}^\alpha r_\nu(\tilde{x}, y, y', \lambda)\right| \\ &\quad\leqslant C\int_0^\infty dr\, r^{n-2}\int_{-\infty}^{\infty} e^{-c(|\tilde{x}|+y+y')(r+|s|)-cy'}\ \frac{y(1+r+|s|)^k}{1+r+|s|+y}\big(r+|s|\big)^{|\alpha|+1}\,ds.\end{aligned} \tag{3.29}$$

Hence, Proposition 2.1 yields

$$\int_0^\infty\int_{\mathbb{R}^n}\left|\partial_{y'}^k\partial_{\tilde{x}}^\alpha r_\nu(\tilde{x}, y, y', \lambda)\right| d\tilde{x}\,dy' \leqslant Cy\int_0^\infty e^{-cys}\frac{(1+s)^{k-1}s^{|\alpha|}}{1+y+s}\,ds.$$

Therefore, Proposition 2.2 yields

$$\int_0^\infty\int_{\mathbb{R}^n}\left|\partial_{y'}^k\partial_{\tilde{x}}^\alpha r_\nu(\tilde{x}, y, y', \lambda)\right| d\tilde{x}\,dy' \leqslant \begin{cases} Cy/(1+y)^2 & \text{if } (k,|\alpha|) = (0,0),\\ Cy\log(e+1/y)/(1+y)^3 & \text{if } (k,|\alpha|) = (0,1),\\ C/(1+y)^3 & \text{if } (k,|\alpha|) = (0,2),\\ C/[y(1+y)^3] & \text{if } (k,|\alpha|) = (0,3),\\ Cy\log(e+1/y)/(1+y)^2 & \text{if } (k,|\alpha|) = (1,0),\\ C/[y^{|\alpha|-1}(1+y)^2] & \text{if } (k,|\alpha|) = (1,1),(1,2),(1,3)\end{cases} \tag{3.30}$$

for all $\lambda \in \Sigma_\theta$ with $|\lambda| = 1$.

Since $\partial_{y'}^k\partial_{\tilde{x}}^\alpha r_\nu(\tilde{x}, y, y', \lambda) = |\lambda|^{\frac{n-1+|\alpha|+k}{2}}(\partial_{y'}^k\partial_{\tilde{x}}^\alpha r_\nu)(|\lambda|^{\frac12}\tilde{x}, |\lambda|^{\frac12}y, |\lambda|^{\frac12}y', \frac{\lambda}{|\lambda|})$, we have

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^n}\left|\partial_{y'}^k\partial_{\tilde{x}}^\alpha r_\nu(\tilde{x}, y, y', \lambda)\right| d\tilde{x}\,dy' \\ &\quad\leqslant |\lambda|^{-1+\frac{|\alpha|+k}{2}}\int_0^\infty\int_{\mathbb{R}^n}\left|\big(\partial_{y'}^k\partial_{\tilde{x}}^\alpha r_\nu\big)\Big(\tilde{x}, |\lambda|^{\frac12}y, y', \frac{\lambda}{|\lambda|}\Big)\right| d\tilde{x}\,dy'\end{aligned}$$

$$\leqslant |\lambda|^{-1+\frac{|\alpha|+k}{2}} \times \begin{cases} C|\lambda|^{\frac{1}{2}} y/(1+|\lambda|^{\frac{1}{2}} y)^2 & \text{if } (k,|\alpha|)=(0,0), \\ C|\lambda|^{\frac{1}{2}} y \log(e+1/(|\lambda|^{\frac{1}{2}} y))/(1+|\lambda|^{\frac{1}{2}} y)^3 & \text{if } (k,|\alpha|)=(0,1), \\ C/(1+|\lambda|^{\frac{1}{2}} y)^3 & \text{if } (k,|\alpha|)=(0,2), \\ C/[|\lambda|^{\frac{1}{2}} y(1+|\lambda|^{\frac{1}{2}} y)^3] & \text{if } (k,|\alpha|)=(0,3), \\ C|\lambda|^{\frac{1}{2}} y \log(e+1/(|\lambda|^{\frac{1}{2}} y))/(1+|\lambda|^{\frac{1}{2}} y)^2 & \text{if } (k,|\alpha|)=(1,0), \\ C/[|\lambda|^{\frac{|\alpha|-1}{2}} y^{|\alpha|-1}(1+|\lambda|^{\frac{1}{2}} y)^2] & \text{if } (k,|\alpha|)=(1,1),(1,2),(1,3) \end{cases}$$

$$\leqslant \begin{cases} \frac{Cy}{|\lambda|^{\frac{1}{2}}(1+|\lambda|^{\frac{1}{2}} y)^2} & \text{if } (k,|\alpha|)=(0,0), \\ \frac{Cy\log(e+1/(|\lambda|^{\frac{1}{2}} y))}{(1+|\lambda|^{\frac{1}{2}} y)^3} & \text{if } (k,|\alpha|)=(0,1), \\ \frac{C}{(1+|\lambda|^{\frac{1}{2}} y)^3} & \text{if } (k,|\alpha|)=(0,2), \\ \frac{C}{y(1+|\lambda|^{\frac{1}{2}} y)^3} & \text{if } (k,|\alpha|)=(0,3), \\ \frac{Cy\log(e+1/(|\lambda|^{\frac{1}{2}} y))}{(1+|\lambda|^{\frac{1}{2}} y)^2} & \text{if } (k,|\alpha|)=(1,0), \\ \frac{C}{y^{|\alpha|-1}(1+|\lambda|^{\frac{1}{2}} y)^2} & \text{if } (k,|\alpha|)=(1,1),(1,2),(1,3) \end{cases} \tag{3.31}$$

for all $\lambda \in \Sigma_\theta$. Similarly, the estimate (3.31) can be proven with r_v replaced by r_w. This proves (3.18).

Next, (3.19) and (3.20) are shown. Only r_v is estimated. By (3.16), we have

$$\partial_{y'}^k \partial_{\tilde{x}}^\alpha \partial_y^2 r_v(\tilde{x}, y, y', \lambda) = c_n \int_0^\infty dr\, r^{n-2} \int_{S^{n-1}} db \int_{-\infty}^\infty e^{i|\tilde{x}|a} \frac{(-\sqrt{\lambda+r^2+a^2})^k e^{-y'\sqrt{\lambda+r^2+a^2}}}{\sqrt{\lambda+r^2+a^2}\sqrt{r^2+a^2}}$$
$$\times h(y,r,a,\lambda)\big(i(a,rb)Q_{\tilde{x}}^T\big)^\alpha Q_{\tilde{x}} \begin{pmatrix} a \\ rb^T \end{pmatrix} (a,rb) Q_{\tilde{x}}^T\, da, \tag{3.32}$$

where

$$h(y,r,a,\lambda) = \partial_y^2\big\{ y e^{-y\sqrt{r^2+a^2}} \phi\big(y\big(\sqrt{\lambda+r^2+a^2} - \sqrt{r^2+a^2}\big)\big)\big\}.$$

It is assumed that $|\lambda| = 1$, and, again, the path of integration for a is shifted to the contour $a \to s + i\varepsilon(r+|s|)$, $s \in \mathbb{R}$ for $\varepsilon \in (0, \varepsilon_0)$. Then, we obtain

$$\big|\partial_{y'}^k \partial_{\tilde{x}}^\alpha \partial_y^2 r_v(\tilde{x}, y, y', \lambda)\big|$$
$$\leqslant C \int_0^\infty dr\, r^{n-2} \int_{-\infty}^\infty e^{-c(|\tilde{x}|+y')(r+|s|)-cy'} \big(r+|s|\big)^{|\alpha|+1} \big(1+r+|s|\big)^{k-1} \big|h(y,r,a,\lambda)\big|\, ds. \tag{3.33}$$

In addition,

$$\big|h(y,r,a,\lambda)\big| \leqslant \frac{1}{c} e^{-cy(r+|s|)} \min\big\{\big(1+r+|s|\big)^3, \big(1+r+|s|\big) + y\big(1+r+|s|\big)^2\big\}, \tag{3.34}$$

which will be shown in Appendix A. Then, we have

$$\begin{aligned}&\bigl|\partial_{y'}^k\partial_{\tilde{x}}^\alpha\partial_y^2 r_\nu(\tilde{x},y,y',\lambda)\bigr|\\&\quad\leqslant C\int_0^\infty dr\, r^{n-2}\int_{-\infty}^\infty e^{-c(|\tilde{x}|+y'+y)(r+|s|)-cy'}\bigl(r+|s|\bigr)^{|\alpha|+1}\bigl(1+r+|s|\bigr)^{k+2}\,ds\end{aligned}\tag{3.35}$$

and

$$\begin{aligned}&\bigl|\partial_{y'}^k\partial_{\tilde{x}}^\alpha\partial_y^2 r_\nu(\tilde{x},y,y',\lambda)\bigr|\\&\quad\leqslant C\int_0^\infty dr\, r^{n-2}\int_{-\infty}^\infty e^{-c(|\tilde{x}|+y'+y)(r+|s|)-cy'}\bigl(r+|s|\bigr)^{|\alpha|+1}\bigl\{\bigl(1+r+|s|\bigr)^k+y\bigl(1+r+|s|\bigr)^{k+1}\bigr\}\,ds.\end{aligned}\tag{3.36}$$

From (3.35) and Proposition 2.1, we obtain

$$\begin{aligned}\int_0^\infty dy'\int_{\mathbb{R}^n}\bigl|\partial_{y'}^k\partial_{\tilde{x}}^\alpha\partial_y^2 r_\nu(\tilde{x},y,y',\lambda)\bigr|\,d\tilde{x}&\leqslant C\int_0^\infty e^{-csy}s^{|\alpha|}(1+s)^{k+1}\,ds\\&\leqslant C\begin{cases}\frac{1}{y}+\frac{1}{y^2}+\frac{1}{y^3} & \text{if } (k,|\alpha|)=(0,0),(1,0),\\ \frac{1}{y^2}+\frac{1}{y^3} & \text{if } (k,|\alpha|)=(0,1).\end{cases}\end{aligned}\tag{3.37}$$

From (3.36), Propositions 2.1 and 2.2(3), we have

$$\begin{aligned}&\int_0^\infty dy'\int_{\mathbb{R}^n}\bigl|\partial_{y'}^k\partial_{\tilde{x}}^\alpha\partial_y^2 r_\nu(\tilde{x},y,y',\lambda)\bigr|\,d\tilde{x}\\&\quad\leqslant C\int_0^\infty e^{-csy}s^{|\alpha|}(1+s)^{k-1}\,ds+Cy\int_0^\infty e^{-csy}s^{|\alpha|}(1+s)^k\,ds\\&\quad\leqslant C\begin{cases}\frac{\log(e+1/y)}{1+y}+1 & \text{if } (k,|\alpha|)=(0,0),\\ \frac{1}{y}+1 & \text{if } (k,|\alpha|)=(1,0),\\ \frac{1}{y} & \text{if } (k,|\alpha|)=(0,1).\end{cases}\end{aligned}\tag{3.38}$$

Hence, using (3.37) for the case in which $y>1$ and using (3.38) for the case in which $y\leqslant 1$, we have

$$\int_0^\infty dy'\int_{\mathbb{R}^n}\bigl|\partial_{\tilde{x}}^\alpha\partial_y^2 r_\nu(\tilde{x},y,y',\lambda)\bigr|\,d\tilde{x}\leqslant C\begin{cases}\frac{\log(e+1/y)}{1+y} & \text{if } |\alpha|=0,\\ \frac{1}{y(1+y)} & \text{if } |\alpha|=1\end{cases}\tag{3.39}$$

and

$$\int_0^\infty dy'\int_{\mathbb{R}^n}\bigl|\partial_{y'}\partial_y^2 r_\nu(\tilde{x},y,y',\lambda)\bigr|\,d\tilde{x}\leqslant\frac{C}{y}\tag{3.40}$$

when $\lambda\in\Sigma_\theta$ with $|\lambda|=1$. The scaling argument is again used for general $\lambda\in\Sigma_\theta$. Since $\partial_{y'}^k\partial_{\tilde{x}}^\alpha\partial_y^2 r_\nu(\tilde{x},y,y',\lambda)=|\lambda|^{\frac{n+1+k+|\alpha|}{2}}(\partial_{y'}^k\partial_{\tilde{x}}^\alpha\partial_y^2 r_\nu)(|\lambda|^{\frac12}\tilde{x},|\lambda|^{\frac12}y,|\lambda|^{\frac12}y',\frac{\lambda}{|\lambda|})$, by (3.39) and (3.40) we have

$$\int_0^\infty dy' \int_{\mathbb{R}^n} \left|\partial_{\tilde{x}}^\alpha \partial_y^2 r_v(\tilde{x}, y, y', \lambda)\right| d\tilde{x} = |\lambda|^{\frac{n+1+|\alpha|}{2}} \int_0^\infty dy' \int_{\mathbb{R}^n} \left|\left(\partial_{\tilde{x}}^\alpha \partial_y^2 r_v\right)\left(|\lambda|^{\frac{1}{2}}\tilde{x}, |\lambda|^{\frac{1}{2}} y, |\lambda|^{\frac{1}{2}} y', \frac{\lambda}{|\lambda|}\right)\right| d\tilde{x}$$

$$= |\lambda|^{\frac{|\alpha|}{2}} \int_0^\infty dy' \int_{\mathbb{R}^n} \left|\left(\partial_{\tilde{x}}^\alpha \partial_y^2 r_v\right)\left(\tilde{x}, |\lambda|^{\frac{1}{2}} y, y', \frac{\lambda}{|\lambda|}\right)\right| d\tilde{x}$$

$$\leqslant C \begin{cases} \dfrac{\log(e+1/(|\lambda|^{\frac{1}{2}} y))}{1+|\lambda|^{\frac{1}{2}} y} & \text{if } |\alpha| = 0, \\ \dfrac{1}{y(1+|\lambda|^{\frac{1}{2}} y)} & \text{if } |\alpha| = 1 \end{cases} \tag{3.41}$$

and

$$\int_0^\infty dy' \int_{\mathbb{R}^n} \left|\partial_{y'} \partial_y^2 r_v(\tilde{x}, y, y', \lambda)\right| d\tilde{x} = |\lambda|^{\frac{n+2}{2}} \int_0^\infty dy' \int_{\mathbb{R}^n} \left|\left(\partial_{y'} \partial_y^2 r_v\right)\left(|\lambda|^{\frac{1}{2}}\tilde{x}, |\lambda|^{\frac{1}{2}} y, |\lambda|^{\frac{1}{2}} y', \frac{\lambda}{|\lambda|}\right)\right| d\tilde{x}$$

$$= |\lambda|^{\frac{1}{2}} \int_0^\infty dy' \int_{\mathbb{R}^n} \left|\left(\partial_{y'} \partial_y^2 r_v\right)\left(\tilde{x}, |\lambda|^{\frac{1}{2}} y, y', \frac{\lambda}{|\lambda|}\right)\right| d\tilde{x}$$

$$\leqslant \frac{C}{y} \tag{3.42}$$

for $\lambda \in \Sigma_\theta$. Similarly, the above estimates, namely, (3.41) and (3.42), with r_v replaced by r_w can also be proven. These estimates prove (3.19) and (3.20). □

4. Proof of Theorem 1

In this section, Theorem 1 is proven.

Proof of Theorem 1(1). First, it is shown that for $0 < \theta < \pi$, $|\arg\lambda| < \theta$, and $\lambda \neq 0$, it holds that

$$\left\|R(\lambda) f\right\|_{B^{-s}_{\infty,q,+}} \leqslant C_\theta \frac{1}{|\lambda|} \|f\|_{B^{-s}_{\infty,q,+,\sigma}}. \tag{4.1}$$

Let $\bar{\mathbf{e}}^+ r_v(\tilde{x}, y, y', \lambda)$ and $\bar{\mathbf{e}}^+ r_w(\tilde{x}, y, y', \lambda)$ be the even extensions with respect to y' of $r_v(\tilde{x}, y, y', \lambda)$ and $r_w(\tilde{x}, y, y', \lambda)$, i.e.,

$$\bar{\mathbf{e}}^+ r(\tilde{x}, y, y', \lambda) := \begin{cases} r(\tilde{x}, y, y', \lambda) & (y' \geqslant 0), \\ r(\tilde{x}, y, -y', \lambda) & (y' < 0). \end{cases}$$

Then, Lemma 3.1 yields

$$\left\|\bar{\mathbf{e}}^+ r_v(\tilde{x}, y, y', \lambda)\right\|_{L^1(\mathbb{R}^n_{\tilde{x}} \times \mathbb{R}_{y'})} + \left\|\bar{\mathbf{e}}^+ r_w(\tilde{x}, y, y', \lambda)\right\|_{L^1(\mathbb{R}^n_{\tilde{x}} \times \mathbb{R}_{y'})} \leqslant \frac{Cy}{|\lambda|^{\frac{1}{2}}(1+|\lambda|^{\frac{1}{2}} y)^2}, \tag{4.2}$$

$$\left\|\bar{\mathbf{e}}^+ r_v(\tilde{x}, y, y', \lambda)\right\|_{W^{1,1}(\mathbb{R}^n_{\tilde{x}} \times \mathbb{R}_{y'})} + \left\|\bar{\mathbf{e}}^+ r_w(\tilde{x}, y, y', \lambda)\right\|_{W^{1,1}(\mathbb{R}^n_{\tilde{x}} \times \mathbb{R}_{y'})}$$

$$\leqslant \frac{Cy}{|\lambda|^{\frac{1}{2}}(1+|\lambda|^{\frac{1}{2}} y)^2} \left(|\lambda|^{\frac{1}{2}} \log\left(e + \frac{1}{|\lambda|^{\frac{1}{2}} y}\right) + 1\right). \tag{4.3}$$

Setting $G(\tilde{x}, y, y', \lambda) = (g_{ij})_{i,j=1,2,\dots,n+1}$ to be an $(n+1)\times(n+1)$ matrix function on $\mathbb{R}^n_{\tilde{x}} \times \{y > 0\} \times \mathbb{R}_{y'} \times \Sigma_\theta$ defined by

$$G(\tilde{x}, y, y', \lambda) = \begin{pmatrix} & & 0 \\ \bar{\mathbf{e}}^+ r_v(\tilde{x}, y, y', \lambda) & & \vdots \\ & & 0 \\ \bar{\mathbf{e}}^+ r_w(\tilde{x}, y, y', \lambda) & & 0 \end{pmatrix},$$

from (3.10)–(3.14), we have

$$R(\lambda) f = (\lambda - \Delta)^{-1} \mathbf{e}^- f + T_\lambda f, \tag{4.4}$$

where

$$\begin{aligned} T_\lambda f(\tilde{x}, y) &= \int_0^\infty \int_{\mathbb{R}^n} G(\tilde{x} - \tilde{x}', y, y', \lambda) f(\tilde{x}', y')\, d\tilde{x}'\, dy' \\ &= \int_{-\infty}^\infty \int_{\mathbb{R}^n} G(\tilde{x} - \tilde{x}', y, y', \lambda) \chi(y') f(\tilde{x}', y')\, d\tilde{x}'\, dy'. \end{aligned}$$

The resolvent of Δ in the whole space $\mathbb{R}^{n+1}$ satisfies

$$\left\| (\lambda - \Delta)^{-1} g \right\|_{L^\infty(\mathbb{R}^{n+1})} \leqslant C \frac{1}{|\lambda|} \|g\|_{L^\infty(\mathbb{R}^{n+1})}.$$

Since $\varphi_j * (\lambda - \Delta)^{-1} \mathbf{e}^- f = (\lambda - \Delta)^{-1} (\varphi_j * \mathbf{e}^- f)$ and $\psi * (\lambda - \Delta)^{-1} \mathbf{e}^- f = (\lambda - \Delta)^{-1} (\psi * \mathbf{e}^- f)$, by (2.11) and the above estimate, we have

$$\left\| (\lambda - \Delta)^{-1} \mathbf{e}^- f \right\|_{B^{-s}_{\infty,q}(\mathbb{R}^{n+1})} \leqslant C \frac{1}{|\lambda|} \|\mathbf{e}^- f\|_{B^{-s}_{\infty,q}(\mathbb{R}^{n+1})} \leqslant C \frac{1}{|\lambda|} \|f\|_{B^{-s}_{\infty,q,+}}. \tag{4.5}$$

Since $L^1 \subset B^0_{1,\infty}$, $W^{1,1} \subset B^1_{1,\infty}$, and $B^s_{1,l}(\mathbb{R}^{n+1}) = (B^0_{1,\infty}(\mathbb{R}^{n+1}), B^1_{1,\infty}(\mathbb{R}^{n+1}))_{s,l}$ for $0 < s < 1$, $1 \leqslant l \leqslant \infty$, by (4.2) and (4.3), we have

$$\begin{aligned} \left\| G(\cdot, y, \cdot, \lambda) \right\|_{B^s_{1,l}(\mathbb{R}^n_{\tilde{x}} \times \mathbb{R}_{y'})} &\leqslant C(s,l) \frac{y}{|\lambda|^{\frac{1}{2}} (1 + |\lambda|^{\frac{1}{2}} y)^2} \left(|\lambda|^{\frac{1}{2}} \log\left(e + \frac{1}{|\lambda|^{\frac{1}{2}} y} \right) + 1 \right)^s \\ &\leqslant C(s,l) \frac{|\lambda|^{\frac{1}{2}} y}{|\lambda| (1 + |\lambda|^{\frac{1}{2}} y)^2} \left(|\lambda|^{\frac{s}{2}} \log^s\left(e + \frac{1}{|\lambda|^{\frac{1}{2}} y} \right) + 1 \right) \\ &\leqslant C(s,l) \frac{|\lambda|^{\frac{s}{2}} + 1}{|\lambda| (1 + |\lambda|^{\frac{1}{2}} y)} \cdot \frac{|\lambda|^{\frac{1}{2}} y \log^s (e + \frac{1}{|\lambda|^{1/2} y}) + |\lambda|^{\frac{1}{2}} y}{1 + |\lambda|^{\frac{1}{2}} y} \\ &= C(s,l) \frac{|\lambda|^{\frac{s}{2}} + 1}{|\lambda| (1 + |\lambda|^{\frac{1}{2}} y)} Q\left(|\lambda|^{\frac{1}{2}} y \right), \end{aligned} \tag{4.6}$$

where $Q(t) = \frac{t \log^s (e + \frac{1}{t}) + t}{1 + t}$ and $\log^s t = (\log t)^s$. Letting $q' = \frac{q}{q-1}$, by duality, we have

$$\begin{aligned}\bigl|T_\lambda f(\tilde{x},y)\bigr| &\leqslant \bigl\|G(\cdot,y,\cdot,\lambda)\bigr\|_{B^s_{1,q'}(\mathbb{R}^n_{\tilde{x}'}\times\mathbb{R}_{y'})}\bigl\|\chi(\cdot)f\bigr\|_{B^{-s}_{\infty,q}}\\ &\leqslant C\frac{1+|\lambda|^{s/2}}{|\lambda|}\cdot\frac{Q(|\lambda|^{\frac{1}{2}}y)}{1+|\lambda|^{1/2}y}\|f\|_{B^{-s}_{\infty,q,+,\sigma}}.\end{aligned}$$

Hence,

$$\bigl\|T_\lambda f(\cdot,y)\bigr\|_{L^\infty(\mathbb{R}^n_{\tilde{x}})}\leqslant C\frac{1+|\lambda|^{s/2}}{|\lambda|}\cdot\frac{Q(|\lambda|^{\frac{1}{2}}y)}{1+|\lambda|^{1/2}y}\|f\|_{B^{-s}_{\infty,q,+,\sigma}}. \tag{4.7}$$

From $\lim_{t\to 0}Q(t)=0$, we obtain

$$T_\lambda f(\tilde{x},0)=0. \tag{4.8}$$

Since $s>0$ and $M:=\sup_{t>0}Q(t)<\infty$, by (4.7) for $0<|\lambda|\leqslant 1$, we have

$$\|T_\lambda f\|_{B^{-s}_{\infty,q,+}}\leqslant C\|T_\lambda f\|_{L^\infty(\mathbb{R}^{n+1}_+)}=\sup_{y>0}\bigl\|T_\lambda f(\cdot,y)\bigr\|_{L^\infty(\mathbb{R}^n_{\tilde{x}})}\leqslant C\frac{1}{|\lambda|}\|f\|_{B^{-s}_{\infty,q,+,\sigma}}. \tag{4.9}$$

For all $\lambda\in\Sigma_\theta$, from a scaling argument, by Lemma 2.1 and (4.7), we have

$$\begin{aligned}\bigl\|\mathbf{e}^0T_\lambda f\bigr\|_{\dot{B}^{-s}_{\infty,q}} &= |\lambda|^{-s/2}\left\|\mathbf{e}^0T_\lambda f\left(\frac{\tilde{x}}{|\lambda|^{1/2}},\frac{y}{|\lambda|^{1/2}}\right)\right\|_{\dot{B}^{-s}_{\infty,q}}\\ &\leqslant C|\lambda|^{-s/2}\left\|T_\lambda f\left(\frac{\tilde{x}}{|\lambda|^{1/2}},\frac{y}{|\lambda|^{1/2}}\right)\right\|_{L^1_w(0,\infty;L^\infty(\mathbb{R}^n_{\tilde{x}}))\cap L^\infty(0,\infty;L^\infty(\mathbb{R}^n_{\tilde{x}}))}\\ &\leqslant C|\lambda|^{-s/2}\left\|\frac{1+|\lambda|^{s/2}}{|\lambda|}\cdot\frac{M}{1+y}\right\|_{L^1_w\cap L^\infty}\|f\|_{B^{-s}_{\infty,q,+,\sigma}}\\ &\leqslant C\frac{|\lambda|^{-s/2}+1}{|\lambda|}\|f\|_{B^{-s}_{\infty,q,+,\sigma}}.\end{aligned} \tag{4.10}$$

Then, for $|\lambda|\geqslant 1$, we have

$$\|T_\lambda f\|_{B^{-s}_{\infty,q,+}}\leqslant C\frac{1}{|\lambda|}\|f\|_{B^{-s}_{\infty,q,+,\sigma}}, \tag{4.11}$$

because $\|T_\lambda f\|_{B^{-s}_{\infty,q,+}}\leqslant C\|\mathbf{e}^0T_\lambda f\|_{B^{-s}_{\infty,q}}\leqslant C\|\mathbf{e}^0T_\lambda f\|_{\dot{B}^{-s}_{\infty,q}}$. From (4.9) and (4.11), we obtain

$$\|T_\lambda f\|_{B^{-s}_{\infty,q,+}}\leqslant C\frac{1}{|\lambda|}\|f\|_{B^{-s}_{\infty,q,+,\sigma}} \tag{4.12}$$

for all $\lambda\in\Sigma_\theta$. Combining (4.5) and (4.12) with (4.4), we obtain the desired estimate (4.1) and $R(\lambda)f\in B^{-s}_{\infty,q,+}$.

Next, it is shown that $R(\lambda)f$ satisfies the divergence-free condition and

$$R(\lambda)f\in B^{-s}_{\infty,q,+,\sigma}.$$

It is temporarily assumed that for $0<s<s'<1$,

$$\bigl\|\mathbf{e}^-R(\lambda)f\bigr\|_{B^{2-s'}_{\infty,\infty}}\leqslant C(\lambda,q,s,s')\|f\|_{B^{-s}_{\infty,q,+,\sigma}}, \tag{4.13}$$

which implies that $R(\lambda)f \in W^{1,\infty}(\mathbb{R}^{n+1}_+)$. Since (4.7) and $\lim_{t\to 0} Q(t) = 0$ imply $T_\lambda f(\tilde{x}, 0) = 0$ and since $((\lambda - \Delta)^{-1}\mathbf{e}^- f)(\tilde{x}, 0) = 0$, we have

$$R(\lambda)f|_{\partial\mathbb{R}^{n+1}_+} = 0.$$

We can also show that $\operatorname{div} R(\lambda)f = 0$ in $\mathbb{R}^{n+1}_+$ as follows. Desch, Hieber and Prüss [8] proved that $R(\lambda)$ is a bounded operator in L^∞_σ. Since $f_N := \psi_N * Ef \in W^{1,\infty}(\mathbb{R}^{n+1}_+)$, $f_N^{n+1}|_{\partial\mathbb{R}^{n+1}_+} = \psi_N * \mathbf{e}^- f^{n+1}|_{\partial\mathbb{R}^{n+1}_+} = 0$ and $\operatorname{div} f_N = 0$, f_N belongs to L^∞_σ. Hence, $R(\lambda)f_N \in L^\infty_\sigma \cap W^{1,\infty}(\mathbb{R}^{n+1}_+)$ with $R(\lambda)f_N|_{\partial\mathbb{R}^{n+1}_+} = 0$ and consequently $\operatorname{div} R(\lambda)f_N = 0$ in $\mathbb{R}^{n+1}_+$. Since $f_N \to f$ in $B^{-s}_{\infty,q,+}$, by (4.13), we have $\operatorname{div} R(\lambda)f = 0$ in $\mathbb{R}^{n+1}_+$. Then, $\operatorname{div} ER(\lambda)f = 0$ in $\mathbb{R}^{n+1}$ and $R(\lambda)f \in B^{-s}_{\infty,q,+,\sigma}$.

Then, (4.13) remains to be shown. It is known that

$$\big\|\mathbf{e}^-(\lambda - \Delta)^{-1}\mathbf{e}^- f\big\|_{B^{2-s}_{\infty,q}} = \big\|(\lambda - \Delta)^{-1}\mathbf{e}^- f\big\|_{B^{2-s}_{\infty,q}} \leqslant C(\lambda, s, q)\|\mathbf{e}^- f\|_{B^{-s}_{\infty,q}}. \tag{4.14}$$

Then, it suffices to show that

$$\|\mathbf{e}^- T_\lambda f\|_{B^{2-s'}_{\infty,\infty}} \leqslant C(\lambda, s', s, q)\|f\|_{B^{-s}_{\infty,q}(\mathbb{R}^{n+1}_+)}. \tag{4.15}$$

Since $C^2_\sigma(\mathbb{R}^{n+1}_+)$ is dense in $B^{-s}_{\infty,q,+,\sigma}$, it may be assumed that $f \in C^2_\sigma(\mathbb{R}^{n+1}_+)$. Since, for $y > 0$,

$$\Delta T_\lambda f(\tilde{x}, y) = \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} \big(\tilde{\Delta} + \partial_y^2\big)G(\tilde{x} - \tilde{x}', y, y', \lambda)\chi(y')f(\tilde{x}', y')\,d\tilde{x}'\,dy', \tag{4.16}$$

where $\tilde{\Delta} = \sum_{i=1}^n \partial_{x_j}^2$ and $\Delta = \tilde{\Delta} + \partial_y^2$. Let $b(y) := \frac{\log(e+1/y)}{1+y}$. Since Lemma 3.1 yields

$$\begin{aligned}
&\big\|\big(\tilde{\Delta} + \partial_y^2\big)G(\cdot, y, \cdot, \lambda)\big\|_{B^s_{1,q'}(\mathbb{R}^n_{\tilde{x}}\times\mathbb{R}_{y'})} \\
&\quad\leqslant C(s, q)\big\|\big(\tilde{\Delta} + \partial_y^2\big)G(\cdot, y, \cdot, \lambda)\big\|^{1-s}_{L^1(\mathbb{R}^n_{\tilde{x}}\times\mathbb{R}_{y'})}\big\|\big(\tilde{\Delta} + \partial_y^2\big)G(\cdot, y, \cdot, \lambda)\big\|^{s}_{W^{1,1}(\mathbb{R}^n_{\tilde{x}}\times\mathbb{R}_{y'})} \\
&\quad\leqslant \left(\frac{C\log(e + 1/(|\lambda|^{1/2}y))}{1 + |\lambda|^{\frac{1}{2}}y}\right)^{1-s}\left(\frac{C\log(e + 1/(|\lambda|^{1/2}y))}{1 + |\lambda|^{\frac{1}{2}}y} + \frac{C}{y}\right)^{s} \\
&\quad\leqslant C\big\{b\big(|\lambda|^{1/2}y\big) + \big(b\big(|\lambda|^{1/2}y\big)\big)^{1-s}y^{-s}\big\},
\end{aligned} \tag{4.17}$$

by duality, for $y > 0$, we have

$$\begin{aligned}
\big\|\Delta T_\lambda f(\cdot, y)\big\|_{L^\infty(\mathbb{R}^n_{\tilde{x}})} &\leqslant \big\|\big(\tilde{\Delta} + \partial_y^2\big)G(\cdot, y, \cdot, \lambda)\big\|_{B^s_{1,q'}(\mathbb{R}^n_{\tilde{x}}\times\mathbb{R}_{y'})}\|f\|_{B^{-s}_{\infty,q,+}} \\
&\leqslant C\big\{b\big(|\lambda|^{1/2}y\big) + \big(b\big(|\lambda|^{1/2}y\big)\big)^{1-s}y^{-s}\big\}\|f\|_{B^{-s}_{\infty,q,+}}.
\end{aligned} \tag{4.18}$$

Since $\{b(y) + (b(y))^{1-s}y^{-s}\} \in L^{1/s'}(0,\infty)$ for $s < s' < 1$, from Lemma 2.1(a), we have

$$\begin{aligned}
\|\mathbf{e}^\pm \Delta T_\lambda f\|_{\dot{B}^{-s'}_{\infty,\infty}(\mathbb{R}^{n+1})} &= C|\lambda|^{-s'/2}\left\|(\mathbf{e}^\pm \Delta T_\lambda f)\left(\frac{\tilde{x}}{|\lambda|^{1/2}}, \frac{y}{|\lambda|^{1/2}}\right)\right\|_{\dot{B}^{-s'}_{\infty,\infty}(\mathbb{R}^{n+1})} \\
&\leqslant C|\lambda|^{-s'/2}\big\|b(y) + \big(b(y)\big)^{1-s}y^{-s}|\lambda|^{s/2}\big\|_{L^{1/s'}}\|f\|_{B^{-s}_{\infty,q,+,\sigma}} \\
&\leqslant C|\lambda|^{-s'/2}\big(1 + |\lambda|^{s/2}\big)\|f\|_{B^{s}_{\infty,q,+}}.
\end{aligned} \tag{4.19}$$

Hence

$$\|\mathbf{e}^- \Delta T_\lambda f\|_{B^{-s'}_{\infty,\infty}(\mathbb{R}^{n+1})} \leqslant C\|\mathbf{e}^- \Delta T_\lambda f\|_{\dot{B}^{-s'}_{\infty,\infty}(\mathbb{R}^{n+1})} \leqslant C|\lambda|^{-s'/2}\bigl(1+|\lambda|^{s/2}\bigr)\|f\|_{B^s_{\infty,q,+}}. \tag{4.20}$$

Since $\|\partial_y^2 T_\lambda f(\cdot, y)\|_\infty \in L^{1/s'}(-\infty,\infty)$, we have $\partial_y T_\lambda f(\tilde{x}, y) \in C[0,\infty)$ for all $\tilde{x}$, and hence $\mathbf{e}^- \partial_y \partial_y T_\lambda f = \partial_y \mathbf{e}^+ \partial_y T_\lambda f$ in $\mathcal{S}'(\mathbb{R}^{n+1})$. Moreover, $T_\lambda f(\tilde{x}, 0) = 0$ implies that $\mathbf{e}^+ \partial_y T_\lambda f = \partial_y \mathbf{e}^- T_\lambda f$ in $\mathcal{S}'(\mathbb{R}^{n+1})$. Hence,

$$\mathbf{e}^- \partial_y^2 T_\lambda f = \partial_y^2 \mathbf{e}^- T_\lambda f \quad \text{in } \mathcal{S}'\bigl(\mathbb{R}^{n+1}\bigr),$$

which yields

$$\mathbf{e}^- \Delta T_\lambda f = \Delta \mathbf{e}^- T_\lambda f \quad \text{in } \mathcal{S}'\bigl(\mathbb{R}^{n+1}\bigr). \tag{4.21}$$

Then, from (4.20) and (4.21), we obtain

$$\|\Delta \mathbf{e}^- T_\lambda f\|_{B^{-s'}_{\infty,\infty}(\mathbb{R}^{n+1})} \leqslant C\|f\|_{B^{-s}_{\infty,q,+}}. \tag{4.22}$$

Since $\|g\|_{B^{2-s'}_{\infty,\infty}} \leqslant \|g\|_{B^{-s'}_{\infty,q}} + C\|\Delta g\|_{B^{-s'}_{\infty,\infty}}$ and $\|\mathbf{e}^- T_\lambda f\|_{B^{-s}_{\infty,q}} \leqslant C\|T_\lambda f\|_{B^{-s}_{\infty,q,+,\sigma}}$, we obtain (4.15) from (4.12) and (4.22), which proves (4.13).

Next, we shall show that $R(1)B^{-s}_{\infty,q,+,\sigma}$ is dense in $B^{-s}_{\infty,q,+,\sigma}$. In Appendix A, it is shown that $R(\lambda)$ satisfies the resolvent equation $R(\lambda_1) - R(\lambda_2) = (\lambda_1 - \lambda_2)R(\lambda_1)R(\lambda_2)$ on $B^{-s}_{\infty,q,+,\sigma}$. This implies that $R(1)B^{-s}_{\infty,q,+,\sigma} = R(\lambda)B^{-s}_{\infty,q,+,\sigma}$. Hence, it suffices to show that

$$\lambda R(\lambda)u \to u \quad \text{in } B^{-s}_{\infty,q,+,\sigma} \text{ as } \lambda \to \infty. \tag{4.23}$$

Let $\lambda > 0$. Desch, Hieber and Prüss [8, p. 127] proved that

$$\lambda T_\lambda f = \lambda \begin{pmatrix} v_2^T(\lambda) \\ w_2(\lambda) \end{pmatrix} \to 0 \quad \text{in } BUC \text{ as } \lambda \to \infty \tag{4.24}$$

for all $f \in BUC_\sigma := \{u \in BUC(\mathbb{R}^{n+1}_+);\ \operatorname{div} u = 0,\ u|_{\partial\mathbb{R}^{n+1}_+} = 0\}$. Their proof of (4.24) is also valid for the case in which $f \in B_\sigma$. Hence, for $f \in B^{-s}_{\infty,q,+,\sigma}$

$$\lambda T_\lambda f_N \to 0 \quad \text{in } BUC \text{ as } \lambda \to \infty, \tag{4.25}$$

where $f_N = \psi_N * Ef$, because $f_N \in B_\sigma$. On the other hand, it can be proven that

$$\lim_{\lambda\to\infty} \bigl\|\lambda(\lambda - \Delta)^{-1}\mathbf{e}^- f - \mathbf{e}^- f\bigr\|_{B^{-s}_{\infty,q}} = 0. \tag{4.26}$$

Since Δ generates an analytic C_0-semi-group on $BUC(\mathbb{R}^{n+1})$, we have

$$\lim_{\lambda\to\infty} \bigl\|\lambda(\lambda - \Delta)^{-1}\psi * \mathbf{e}^- f - \psi * \mathbf{e}^- f\bigr\|_\infty = 0, \qquad \lim_{\lambda\to\infty} \bigl\|\lambda(\lambda - \Delta)^{-1}\phi_j * \mathbf{e}^- f - \phi_j * \mathbf{e}^- f\bigr\|_\infty = 0$$

for all $j = 1, 2, \ldots$. Since

$$\begin{aligned}\sum_{j=1}^\infty 2^{-sq}\bigl\|\lambda(\lambda - \Delta)^{-1}\phi_j * \mathbf{e}^- f - \phi_j * \mathbf{e}^- f\bigr\|_\infty^q &\leqslant C\sum_{j=1}^\infty 2^{-sq}\bigl\|\phi_j * \mathbf{e}^- f\bigr\|_\infty^q \\ &\leqslant \|\mathbf{e}^- f\|^q_{B^{-s}_{\infty,q}} < \infty,\end{aligned} \tag{4.27}$$

we obtain (4.26). Finally, since (4.4) and (4.12) yield

$$\begin{aligned}
&\big\|\lambda R(\lambda)f - f\big\|_{B^{-s}_{\infty,q,+,\sigma}} \\
&= \big\|\lambda(\lambda-\Delta)^{-1}\mathbf{e}^- f - \mathbf{e}^- f + \lambda T_\lambda(f - f_N) + \lambda T_\lambda f_N\big\|_{B^{-s}_{\infty,q,+}} \\
&\leqslant \big\|\lambda(\lambda-\Delta)^{-1}\mathbf{e}^- f - \mathbf{e}^- f\big\|_{B^{-s}_{\infty,q}} + C\|f - f_N\|_{B^{-s}_{\infty,q,+}} + \|\lambda T_\lambda f_N\|_{B^{-s}_{\infty,q,+}},
\end{aligned} \tag{4.28}$$

from (4.25), (4.26), and $\lim_{N\to\infty}\|f - f_N\|_{B^{-s}_{\infty,q,+}} = 0$, we obtain (4.23). Hence, $R(1)B^{-s}_{\infty,q,+,\sigma}$ is dense in $B^{-s}_{\infty,q,+,\sigma}$.

Since $R(\lambda)$ satisfies the resolvent equation: $R(\lambda_1) - R(\lambda_2) = (\lambda_1 - \lambda_2)R(\lambda_1)R(\lambda_2)$, $\ker R(\lambda)$ is independent of λ. Then, (4.23) implies $\ker R(\lambda) = 0$ for all $\lambda \in \Sigma_\pi$, and hence, there exists a closed, densely defined operator $A = A_{s,q,+}$ in $B^{-s}_{\infty,q,+,\sigma}$ such that

$$R(\lambda) = (\lambda + A_{s,q,+})^{-1} \tag{4.29}$$

for all $\lambda \in \Sigma_\pi$, where the domain $D(A_{s,q,+}) = R(1)B^{-s}_{\infty,q,+,\sigma}$. Therefore, the estimate (4.1) proves Theorem 1(1). □

Proof of Theorem 1(2). (I) The estimate (a) follows from (4.1).

(II) Next, (b) is proven. First, the case in which $t = 1$ is considered, and the following is shown

$$\big\|\mathbf{e}^- e^{-A} f\big\|_{B^m_{\infty,1}} \leqslant C\|f\|_{B^{-s}_{\infty,q,+,\sigma}} \tag{4.30}$$

for $m < 2 - s$, $0 < s < 1$. Since e^{-tA} is an analytic semi-group on $B^{-s}_{\infty,q,+,\sigma}$, we have

$$\big\|Ae^{-tA} f\big\|_{B^{-s}_{\infty,q,+,\sigma}} \leqslant Ct^{-1}\|f\|_{B^{-s}_{\infty,q,+,\sigma}},$$

which yields

$$\big\|(1+A)e^{-A} f\big\|_{B^{-s}_{\infty,q,+,\sigma}} \leqslant C\|f\|_{B^{-s}_{\infty,q,+,\sigma}}.$$

On the other hand, let s' be a number satisfying $s < s' < 1$ and $m < 2 - s'$, then (4.13) implies that

$$\|\mathbf{e}^- g\|_{B^m_{\infty,1}} \leqslant \|\mathbf{e}^- g\|_{B^{2-s'}_{\infty,\infty}} \leqslant C\big\|(1+A)g\big\|_{B^{-s}_{\infty,q,+,\sigma}} \quad \text{for} \quad g \in D(A).$$

Then, we obtain (4.30).

When $0 < m < 2 - s$, since $\|\mathbf{e}^- e^{-A} f\|_{\dot{B}^m_{\infty,1}} \leqslant C\|\mathbf{e}^- e^{-A} f\|_{B^m_{\infty,1}}$, by (4.30), we have

$$\big\|\mathbf{e}^- e^{-A} f\big\|_{\dot{B}^m_{\infty,1}} \leqslant C\|f\|_{B^{-s}_{\infty,q,+,\sigma}}. \tag{4.31}$$

Next, recall that for $f_t(x) := f(t^{1/2}x)$,

$$\begin{aligned}
C_1 t^{k/2}\|f\|_{\dot{B}^k_{\infty,1}} \leqslant \|f_t\|_{\dot{B}^k_{\infty,1}} \leqslant C_2 t^{k/2}\|f\|_{\dot{B}^k_{\infty,1}} \quad &\text{for } t > 0,\ k \in \mathbb{R}, \\
\|f_t\|_{B^k_{\infty,q}} \leqslant C_3\big(1 + t^{k/2}\big)\|f\|_{B^k_{\infty,q}} \quad &\text{for } t > 0,\ k \in \mathbb{R},
\end{aligned} \tag{4.32}$$

where the constants C_1, C_2, and C_3 depend only on k, q, and n. Then, since $(e^{-tA} f)(x) = (e^{-A}(f_t))_{1/t}(x)$, by Theorem 1(a), (4.31), and (4.32), for $0 < m < 2 - s$, we have

$$\begin{aligned}\left\|\mathbf{e}^- e^{-tA} f\right\|_{B^m_{\infty,1}} &\leqslant \left\|\mathbf{e}^- e^{-tA} f\right\|_{B^{-s}_{\infty,q}} + \left\|\mathbf{e}^- e^{-tA} f\right\|_{\dot{B}^m_{\infty,1}} \\ &\leqslant C\big(\left\|e^{-tA} f\right\|_{B^{-s}_{\infty,q,+}} + t^{-m/2}\left\|\mathbf{e}^- e^{-A} f_t\right\|_{\dot{B}^m_{\infty,1}}\big) \\ &\leqslant C\big(\|f\|_{B^{-s}_{\infty,q,+}} + t^{-m/2}\|f_t\|_{B^{-s}_{\infty,q,+}}\big) \\ &\leqslant C\big(1+t^{-(m+s)/2}\big)\|f\|_{B^{-s}_{\infty,q,+,\sigma}}. \end{aligned} \tag{4.33}$$

For the case in which $-s < m \leqslant 0$, we use the interpolation $B^m_{\infty,1} = (B^{-s}_{\infty,q}, B^1_{\infty,1})_{\theta,1}$, where $m = -s(1-\theta)+\theta$. Based on the above estimate (4.33) with $m=1$ and estimate (a), we have

$$\begin{aligned}\left\|\mathbf{e}^- e^{-tA} f\right\|_{B^m_{\infty,1}} &\leqslant \left\|\mathbf{e}^- e^{-tA} f\right\|^{1-\theta}_{B^{-s}_{\infty,q}} \left\|\mathbf{e}^- e^{-tA} f\right\|^{\theta}_{B^1_{\infty,1}} \\ &\leqslant C\left\|e^{-tA} f\right\|^{1-\theta}_{B^{-s}_{\infty,q,+}} \big(1+t^{-(1+s)\theta/2}\big)\|f\|^{\theta}_{B^{-s}_{\infty,q,+}} \\ &= C\big(1+t^{-(m+s)/2}\big)\|f\|_{B^{-s}_{\infty,q,+,\sigma}}. \end{aligned} \tag{4.34}$$

This proves estimate (b).

(III) Estimate (c) is a consequence of estimate (b), which yields

$$\left\|\nabla\mathbf{e}^- e^{-tA} f\right\|_{B^{-s}_{\infty,1}} \leqslant \left\|\mathbf{e}^- e^{-tA} f\right\|_{B^{1-s}_{\infty,1}} \leqslant C\left(1+\frac{1}{t}\right)^{1/2}\|f\|_{B^{-s}_{\infty,q,+,\sigma}}. \tag{4.35}$$

For $v \in D(A)$, $\partial_y \mathbf{e}^- v(\tilde{x}, y) = \mathbf{e}^+ \partial_y v(\tilde{x}, y)$ and $\partial_j \mathbf{e}^- v(\tilde{x}, y) = \mathbf{e}^- \partial_j v(\tilde{x}, y)$ hold for $j = 1, 2, \ldots, n$, because $v \in W^{1,\infty}$ and $v|_{\partial\mathbb{R}^{n+1}_+} = 0$. Then, from the inequality $\|u\|_{B^{-s}_{\infty,1,+}} \leqslant \|\mathbf{e}^{\pm} u\|_{B^{-s}_{\infty,1}}$ and (4.35), we obtain estimate (c).

(IV) From the property for analytic semi-groups, we easily obtain estimate (d). Since, for $T > 0$, there exists a constant $C = C(T)$ such that

$$\left\|Ae^{-tA} f\right\|_{B^{-s}_{\infty,q,+,\sigma}} \leqslant Ct^{-1}\|f\|_{B^{-s}_{\infty,q,+,\sigma}} \quad \text{and} \quad \left\|\big(e^{-tA}-I\big)f\right\|_{B^{-s}_{\infty,q,+,\sigma}} \leqslant Ct\|Af\|_{B^{-s}_{\infty,q,+,\sigma}}$$

for $0 < t \leqslant T$, we have

$$\begin{aligned}\left\|\big(e^{-tA}-e^{-\tau A}\big)f\right\|_{B^{-s}_{\infty,q,+,\sigma}} &= \left\|\big(e^{-tA}-e^{-\tau A}\big)f\right\|^{1-\alpha}_{B^{-s}_{\infty,q,+,\sigma}} \left\|\big(e^{-(t-\tau)A}-I\big)e^{-\tau A} f\right\|^{\alpha}_{B^{-s}_{\infty,q,+,\sigma}} \\ &\leqslant C\|f\|^{1-\alpha}_{B^{-s}_{\infty,q,+,\sigma}} (t-\tau)^{\alpha}\left\|Ae^{-\tau A} f\right\|^{\alpha}_{B^{-s}_{\infty,q,+,\sigma}} \leqslant C(t-\tau)^{\alpha}\tau^{-\alpha}\|f\|_{B^{-s}_{\infty,q,+,\sigma}}\end{aligned}$$

for $0 < \tau < t \leqslant T$.

(V) By estimates (c) and (d), we easily obtain

$$\begin{aligned}\left\|\nabla\big(e^{-tA}-e^{-\tau A}\big)f\right\|_{B^{-s}_{\infty,1,+}} &= \left\|\nabla e^{-(\tau/2)A}\big(e^{-(t-\tau/2)A}-e^{-(\tau/2)A}\big)f\right\|_{B^{-s}_{\infty,1,+}} \\ &\leqslant C\big(1+\tau^{-1/2}\big)\left\|\big(e^{-(t-\tau/2)A}-e^{-(\tau/2)A}\big)f\right\|_{B^{-s}_{\infty,q,+,\sigma}} \\ &\leqslant C(t-\tau)^{\alpha}\tau^{-\alpha}\big(1+\tau^{-1/2}\big)\|f\|_{B^{-s}_{\infty,q,+,\sigma}},\end{aligned}$$

which yields the estimate (e). □

5. Proof of Theorem 2

Here, x_{n+1} is denoted by y, as in Section 3. Note that if $u \in W^{1,\infty}(\mathbb{R}^{n+1}_+)$ with $\operatorname{div} u = 0$ and $u|_{\partial\mathbb{R}^{n+1}_+} = 0$, then $E(u\cdot\nabla u) = \sum_{j=1}^{n+1} \partial_j((Eu)^j Eu) = \nabla\cdot(Eu\otimes Eu)$, and hence $P_+(u\cdot\nabla u) = P\nabla\cdot(Eu\otimes Eu)$. Furthermore, if $w \in W^{1,\infty}(\mathbb{R}^{n+1})$ is an odd function with respect to $y = x_{n+1}$, then $\mathbf{e}^-(u\cdot\nabla w) = \nabla\cdot((Eu)w)$.

Step 1. First, by the standard argument, a solution to the following system of integral equations is constructed:

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-\tau)A}P\nabla\cdot(Eu\otimes Eu)(\tau)\,d\tau + \int_0^t e^{-(t-\tau)A}P_+(g\bar\theta + g\bar S)(\tau)\,d\tau, \tag{5.1}$$

$$\bar\theta(t) = e^{t\Delta}\mathbf{e}^-\bar\theta_0 - \int_0^t e^{(t-\tau)\Delta}\big\{\nabla\cdot\big((Eu)\bar\theta\big) + \mathbf{e}^-(u\cdot\nabla\bar S) + \mathbf{e}^-(\partial_t\bar S - \Delta\bar S)\big\}(\tau)\,d\tau, \tag{5.2}$$

where $e^{t\Delta}$ is the heat semi-group in the whole space $\mathbb{R}^{n+1}$: $e^{t\Delta}f(x) = \int_{\mathbb{R}^{n+1}} \frac{1}{(4\pi t)^{(n+1)/2}} e^{\frac{-|x-z|^2}{4t}} f(z)\,dz$. Here, $\bar\theta$ is a function on the *whole* space $\mathbb{R}^{n+1}$. Let

$$F(u,v) = \int_0^t e^{-(t-\tau)A}P\nabla\cdot(Eu\otimes Ev)(\tau)\,d\tau,$$

$$I(w) = \int_0^t e^{-(t-\tau)A}P_+(gw)(\tau)\,d\tau,$$

$$J(u,w) = \int_0^t e^{(t-\tau)\Delta}\nabla\cdot\big((Eu)w\big)(\tau)\,d\tau,$$

$$K(w) = \int_0^t e^{(t-\tau)\Delta}\mathbf{e}^- w(\tau)\,d\tau,$$

where $\nabla\cdot(Eu\otimes Ev) = \sum_{j=1}^{n+1}\partial_j((Eu)^j Ev)$, and let

$$\Psi(u,w) = e^{-tA}u_0 - F(u,u) + I(w) + I(\bar S), \tag{5.3}$$

$$\Phi(u,w) = e^{t\Delta}\mathbf{e}^-\bar\theta_0 - J(u,w) - K(u\cdot\nabla\bar S) - K(\partial_t\bar S - \Delta\bar S), \tag{5.4}$$

$$\Pi(u,w) = \big(\Psi(u,w), \Phi(u,w)\big). \tag{5.5}$$

We shall construct a local-in-time solution to

$$\big(u(t),\bar\theta(t)\big) = \Pi\big(u(t),\bar\theta(t)\big).$$

To this end, it suffices to show that, for small $T > 0$, $\Pi(u,\bar\theta)$ is a contraction mapping on a subset of

$$Z_T = X_T \times Y_T,$$

where

$$X_T = \{v \in C([0,T]; B^{-s}_{\infty,q,+,\sigma}) \cap C((0,T); B^{1-s}_{\infty,1,+});\ \|v\|_{X_T} < \infty,\ v(t)|_{\partial\mathbb{R}^{n+1}_+} = 0 \text{ for all } 0 < t < T\},$$

$$Y_T = \{w \in C([0,T]; \dot{B}^0_{\infty,1}) \cap C((0,T); \dot{B}^1_{\infty,1});\ \|w\|_{Y_T} < \infty,\ w = \mathbf{e}^- w \text{ in } \mathbb{R}^{n+1}\times[0,T]\},$$

$$\|v\|_{X_T} = \sup_{0<\tau<T} \|v(\tau)\|_{B^{-s}_{\infty,q,+,\sigma}} + \sup_{0<\tau<T} \tau^{1/2}\|\nabla v(\tau)\|_{B^{-s}_{\infty,1,+}},$$

$$\|w\|_{Y_T} = \sup_{0<\tau<T} \|w(\tau)\|_{\dot{B}^0_{\infty,1}} + \sup_{0<\tau<T} \tau^{1/2}\|\nabla w(\tau)\|_{\dot{B}^0_{\infty,1}},$$

$$\|(u,w)\|_{Z_T} = \|u\|_{X_T} + \|w\|_{Y_T}.$$

Note that $B^{1-s}_{\infty,1,+} \subset C(\bar{\mathbb{R}}^{n+1}_+)$, and hence $v(t)|_{\partial\mathbb{R}^{n+1}_+}$ is well defined for $0 < t < T$. Since $v|_{\partial\mathbb{R}^{n+1}_+} = 0$, we have $\partial_y \mathbf{e}^{\pm} v = \mathbf{e}^{\mp}\partial_y v$ in $\mathcal{S}'$ and $\partial_j \mathbf{e}^{\pm} v = \mathbf{e}^{\pm}\partial_j v$ in $\mathcal{S}'$ for $1 \leqslant j \leqslant n$, which implies the following:

$$\|\nabla \mathbf{e}^{\pm} v\|_{B^{-s}_{\infty,1}} \cong \|\nabla v\|_{B^{-s}_{\infty,1,+}} \quad \text{and hence} \tag{5.6}$$

$$\|\mathbf{e}^{\pm} v\|_{B^{1-s}_{\infty,1}} \leqslant C\big(\|v\|_{B^{-s}_{\infty,q,+}} + \|\nabla v\|_{B^{-s}_{\infty,1,+}}\big), \tag{5.7}$$

because $\|\psi * \mathbf{e}^{\pm} v\|_{\infty} \leqslant \|\mathbf{e}^{\pm} v\|_{B^{-s}_{\infty,q}} \leqslant C\|v\|_{B^{-s}_{\infty,q,+}}$. Then, by (5.7) and the interpolation $(B^{-s}_{\infty,q}, B^{1-s}_{\infty,1})_{s,1} = B^0_{\infty,1}(\subset L^{\infty})$, we have

$$\begin{aligned}\|\mathbf{e}^{\pm} u\|_{L^{\infty}} &\leqslant \|\mathbf{e}^{\pm} u(t)\|_{B^0_{\infty,1}} \leqslant C\|\mathbf{e}^{\pm} u(t)\|^{1-s}_{B^{-s}_{\infty,q}} \|\mathbf{e}^{\pm} u(t)\|^{s}_{B^{1-s}_{\infty,1}} \\ &\leqslant C\big(\|u(t)\|_{B^{-s}_{\infty,q,+}} + \|u(t)\|^{1-s}_{B^{-s}_{\infty,q,+}} \|\nabla u(t)\|^{s}_{B^{-s}_{\infty,1,+}}\big) \leqslant C\big(1 + t^{-s/2}\big)\|u\|_{X_T}\end{aligned} \tag{5.8}$$

and

$$\|Eu\|_{B^{1-s}_{\infty,1}} \leqslant C\big(1 + t^{-1/2}\big)\|u\|_{X_T} \tag{5.9}$$

for $0 < t < T$.

Before estimating $\Pi(u, \bar{\theta})$, it is necessary to verify that if $u, v \in X_T$ and $w \in Y_T$, then $P\nabla\cdot(Eu \otimes Ev)$, $P_+(gw)$ and $P_+(g\bar{S})$ are well defined and belong to $B^{-s}_{\infty,q,+,\sigma}$. By the inequality:

$$\|fg\|_{B^{\gamma}_{p,q}} \leqslant C(\gamma, p, q, n)\big(\|f\|_{\infty}\|g\|_{B^{\gamma}_{p,q}} + \|f\|_{B^{\gamma}_{p,q}}\|g\|_{\infty}\big), \tag{5.10}$$

$$\|fg\|_{\dot{B}^{\gamma}_{p,q}} \leqslant C(\gamma, p, q, n)\big(\|f\|_{\infty}\|g\|_{\dot{B}^{\gamma}_{p,q}} + \|f\|_{\dot{B}^{\gamma}_{p,q}}\|g\|_{\infty}\big) \tag{5.11}$$

for $\gamma > 0$ (see, e.g., [25]) and (5.7) we have

$$\begin{aligned}\|\nabla\cdot(Eu \otimes Ev)\|_{\dot{B}^{-s}_{\infty,q}} &\leqslant \|Eu \otimes Ev\|_{\dot{B}^{1-s}_{\infty,q}} \leqslant \|Eu \otimes Ev\|_{B^{1-s}_{\infty,q}} \\ &\leqslant C\big(\|Eu\|_{\infty}\|Ev\|_{B^{1-s}_{\infty,q}} + \|Ev\|_{\infty}\|Eu\|_{B^{1-s}_{\infty,q}}\big) \\ &\leqslant C\big\{\|Eu\|_{\infty}\big(\|v\|_{B^{-s}_{\infty,q,+,\sigma}} + \|\nabla v\|_{B^{-s}_{\infty,1,+}}\big) \\ &\quad + \|Ev\|_{\infty}\big(\|u\|_{B^{-s}_{\infty,q,+,\sigma}} + \|\nabla u\|_{B^{-s}_{\infty,1,+}}\big)\big\}.\end{aligned} \tag{5.12}$$

Hence, by (5.8) and (5.12), $\nabla\cdot(Eu\otimes Ev)(t)$ belongs to the homogeneous Besov space $\dot B^{-s}_{\infty,q}$ for $0<t<T$. In addition, Lemma 2.1(b) shows that $\|E(g\bar S)\|_{\dot B^{-s}_{\infty,q}}\leqslant C\|(g\bar S)\|_{L^1_w(0,\infty;L^\infty(\mathbb{R}^n))\cap L^\infty}$. Since $w=\mathbf{e}^-w$ and $g=(0,\dots,0,|g|)$ is a constant vector, $E(gw)=g\mathbf{e}^-w=gw\in\dot B^0_{\infty,1}$. Then, $P\nabla\cdot(Eu\otimes Ev)\in\dot B^{-s}_{\infty,q}$, $P_+(g\bar S)=PE(g\bar S)\in\dot B^{-s}_{\infty,q}$ and $P_+(gw)=PE(gw)=P(gw)\in\dot B^0_{\infty,1}\subset L^\infty\subset B^{-s}_{\infty,q}$ for $0<t<T$, because the operator P is bounded in the homogeneous Besov spaces. Therefore, we have $P\nabla\cdot(Eu\otimes Ev)$, $P_+(g\bar S)$ and $P_+(gw)$ belong to $B^{-s}_{\infty,q,+,\sigma}$, since $\dot B^{-s}_{\infty,q}\subset B^{-s}_{\infty,q}$, $P\nabla\cdot(Eu\otimes Ev)=EP\nabla\cdot(Eu\otimes Ev)$, $P_+=PE=EPE$ and these three terms are divergence-free in the sense of $\mathcal{S}'(\mathbb{R}^{n+1})$.

Next, $\Psi(u,\bar\theta)$ is estimated. Let $0<T<1$, and let $0<\tau<t<T$. By (5.8) and (5.12), we have

$$\begin{aligned}
\big\|P\nabla\cdot(Eu\otimes Ev)(\tau)\big\|_{B^{-s}_{\infty,q,+,\sigma}}&\leqslant C\big\|P\nabla\cdot(Eu\otimes Ev)\big\|_{\dot B^{-s}_{\infty,q}}\\
&\leqslant C\big\|\nabla\cdot(Eu\otimes Ev)\big\|_{\dot B^{-s}_{\infty,q}}\\
&\leqslant C\big\{\big\|u(\tau)\big\|_\infty\big(\big\|v(\tau)\big\|_{B^{-s}_{\infty,q,+}}+\big\|\nabla v(\tau)\big\|_{B^{-s}_{\infty,1,+}}\big)\\
&\quad+\big\|v(\tau)\big\|_\infty\big(\big\|u(\tau)\big\|_{B^{-s}_{\infty,q,+}}+\big\|\nabla u(\tau)\big\|_{B^{-s}_{\infty,1,+}}\big)\big\}\\
&\leqslant C\tau^{-(s+1)/2}\|u\|_{X_T}\|v\|_{X_T}.
\end{aligned}\tag{5.13}$$

Hence, from Theorem 1(a), we obtain

$$\big\|F(u,v)\big\|_{B^{-s}_{\infty,q,+,\sigma}}\leqslant C\int_0^t\tau^{-(s+1)/2}\|u\|_{X_T}\|v\|_{X_T}\,d\tau\leqslant Ct^{(1-s)/2}\|u\|_{X_T}\|v\|_{X_T},\tag{5.14}$$

because $0<s<1$. From Theorem 1(c) and (5.13), we obtain

$$\begin{aligned}
t^{1/2}\big\|\nabla F(u,v)\big\|_{B^{-s}_{\infty,1,+}}&\leqslant t^{1/2}\int_0^t\big\|\nabla e^{-(t-\tau)A}P\nabla\cdot(Eu\otimes Ev)\big\|_{B^{-s}_{\infty,1,+}}d\tau\\
&\leqslant Ct^{1/2}\int_0^t(t-\tau)^{-1/2}\big\|P\nabla\cdot(Eu\otimes Ev)\big\|_{B^{-s}_{\infty,q,+}}d\tau\\
&\leqslant Ct^{1/2}\int_0^t(t-\tau)^{-1/2}\tau^{-(s+1)/2}d\tau\|u\|_{X_T}\|v\|_{X_T}\\
&\leqslant Ct^{(1-s)/2}\|u\|_{X_T}\|v\|_{X_T}.
\end{aligned}\tag{5.15}$$

Hence, (5.14) and (5.15) yield

$$\big\|F(u,v)\big\|_{X_T}\leqslant CT^{(1-s)/2}\|u\|_{X_T}\|v\|_{X_T}.\tag{5.16}$$

Since

$$\big\|P_+\big(gw(\tau)\big)\big\|_{B^{-s}_{\infty,q,+,\sigma}}\leqslant\big\|P\big(g\mathbf{e}^-w(\tau)\big)\big\|_\infty\leqslant C|g|\|\mathbf{e}^-w\|_{\dot B^0_{\infty,1}}=C|g|\|w\|_{\dot B^0_{\infty,1}}\leqslant C|g|\|w\|_{Y_T},\tag{5.17}$$

and

$$\begin{aligned}\big\|\nabla e^{-(t-\tau)A}P_+\big(gw(\tau)\big)\big\|_{B^{-s}_{\infty,1,+}} &\leqslant C(t-\tau)^{-1/2}\big\|P_+\big(gw(\tau)\big)\big\|_{B^{-s}_{\infty,q,+,\sigma}}\\ &\leqslant C(t-\tau)^{-1/2}|g|\|w\|_{Y_T},\end{aligned}$$

we have

$$\big\|I(w)\big\|_{X_T} \leqslant CT|g|\|w\|_{Y_T}. \tag{5.18}$$

Since

$$\begin{aligned}\big\|P_+(g\bar{S})\big\|_{B^{-s}_{\infty,q,+}} &\leqslant C\big\|PE(g\bar{S})\big\|_{\dot{B}^{-s}_{\infty,q}} \leqslant C\big\|E(g\bar{S})\big\|_{\dot{B}^{-s}_{\infty,q}}\\ &\leqslant C\big\|(g\bar{S})\big\|_{L^1_w(0,\infty;L^\infty(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^{n+1}_+)}\end{aligned} \tag{5.19}$$

and

$$\big\|\nabla e^{-(t-\tau)A}P_+(g\bar{S})\big\|_{B^{-s}_{\infty,1,+}} \leqslant C(t-\tau)^{-1/2}\big\|(g\bar{S})\big\|_{L^1_w(0,\infty;L^\infty(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^{n+1}_+)},$$

we have

$$\big\|I(\bar{S})\big\|_{X_T} \leqslant CT|g|\|\bar{S}\|, \tag{5.20}$$

where

$$\begin{aligned}\|\bar{S}\| := \sup_{0\leqslant t\leqslant\infty}\Big\{&\big\|\bar{S}(t)\big\|_{L^1_w(0,\infty;L^\infty(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^{n+1}_+)} + \big\|\nabla\bar{S}(t)\big\|_{L^1_w(0,\infty;L^\infty(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^{n+1}_+)}\\ &+ \big\|\partial_t\bar{S}(t)-\Delta\bar{S}(t)\big\|_{L^1_w(0,\infty;L^\infty(\mathbb{R}^n))\cap L^\infty(\mathbb{R}^{n+1}_+)}\Big\}.\end{aligned} \tag{5.21}$$

Note that

$$\|\bar{S}\| \leqslant C(h)\sup_{0\leqslant t\leqslant\infty}\Big\{\big\|S(t)\big\|_{W^{2,\infty}(\mathbb{R}^n_{\tilde{x}})} + \big\|\partial_t S(t)\big\|_{L^\infty(\mathbb{R}^n_{\tilde{x}})}\Big\},$$

where h is the function introduced in Section 1. By Theorem 1(2),

$$\big\|e^{-tA}u_0\big\|_{X_T} \leqslant C\|u_0\|_{B^{-s}_{\infty,q,+,\sigma}}. \tag{5.22}$$

Therefore, from (5.16), (5.18), (5.20) and (5.22), we have

$$\big\|\Psi(u,w)\big\|_{X_T} \leqslant C_0\|u_0\|_{B^{-s}_{\infty,q,+,\sigma}} + C_1T^{(1-s)/2}\|u\|^2_{X_T} + C_2T|g|\big(\|w\|_{Y_T} + \|\bar{S}\|\big) \tag{5.23}$$

and

$$\begin{aligned}\big\|\Psi(u_1,w_1)-\Psi(u_2,w_2)\big\|_{X_T} \leqslant{}& C_1T^{(1-s)/2}\|u_1-u_2\|_{X_T}\big(\|u_1\|_{X_T}+\|u_2\|_{X_T}\big)\\ &+ C_2T|g|\|w_1-w_2\|_{Y_T},\end{aligned} \tag{5.24}$$

because

$$\Psi(u_1,w_1)-\Psi(u_2,w_2) = -F(u_1-u_2,u_1) - F(u_2,u_1-u_2) + I(w_1-w_2).$$

Next, $\Phi(u,\bar{\theta})$ is estimated. By Proposition 2.3, (5.8), (5.9) and (5.11), we have

$$\begin{aligned}\big\|e^{(t-\tau)\Delta}\nabla\cdot\big((Eu)w\big)(\tau)\big\|_{\dot{B}^0_{\infty,1}} &\leqslant C(t-\tau)^{-1/2}\big\|(Eu)w\big\|_{\dot{B}^0_{\infty,\infty}}\\ &\leqslant C(t-\tau)^{-1/2}\|Eu\|_\infty\|w\|_\infty\\ &\leqslant C(t-\tau)^{-1/2}\tau^{-s/2}\|u\|_{X_T}\|w\|_{Y_T}\end{aligned}\tag{5.25}$$

and

$$\begin{aligned}\big\|\nabla e^{(t-\tau)\Delta}\nabla\cdot\big((Eu)w\big)(\tau)\big\|_{\dot{B}^0_{\infty,1}} &\leqslant C(t-\tau)^{-(s+1)/2}\big\|(Eu)w\big\|_{\dot{B}^{1-s}_{\infty,\infty}}\\ &\leqslant C(t-\tau)^{-(s+1)/2}\big(\big\|(Eu)\big\|_\infty\|w\|_{\dot{B}^{1-s}_{\infty,\infty}}+\big\|(Eu)\big\|_{\dot{B}^{1-s}_{\infty,\infty}}\|w\|_\infty\big)\\ &\leqslant C(t-\tau)^{-(s+1)/2}\tau^{-1/2}\|u\|_{X_T}\|w\|_{Y_T},\end{aligned}\tag{5.26}$$

where $\|w(\tau)\|_{\dot{B}^{1-s}_{\infty,\infty}}\leqslant C\|w\|^s_{\dot{B}^0_{\infty,1}}\|\nabla w\|^{1-s}_{\dot{B}^0_{\infty,1}}\leqslant C\tau^{-(1-s)/2}\|w\|_{Y_T}$ and $\|Eu\|_{\dot{B}^{1-s}_{\infty,\infty}}\leqslant\|Eu\|_{B^{1-s}_{\infty,\infty}}\leqslant C\|Eu\|_{B^{1-s}_{\infty,1}}$ were used. Hence, we obtain

$$\big\|J(u,w)\big\|_{Y_T}\leqslant CT^{(1-s)/2}\|u\|_{X_T}\|w\|_{Y_T}.\tag{5.27}$$

By Proposition 2.3 and Lemma 2.1, we have

$$\begin{aligned}\big\|e^{(t-\tau)\Delta}\mathbf{e}^-(u\cdot\nabla\bar{S})(\tau)\big\|_{\dot{B}^0_{\infty,1}} &\leqslant C(t-\tau)^{-s/2}\big\|\mathbf{e}^-(u\cdot\nabla\bar{S})\big\|_{\dot{B}^{-s}_{\infty,\infty}}\\ &\leqslant C(t-\tau)^{-s/2}\big\|(u\cdot\nabla\bar{S})\big\|_{L^1_w(0,\infty;L^\infty(\mathbb{R}^n_x))\cap L^\infty(\mathbb{R}^{n+1}_+)}\\ &\leqslant C(t-\tau)^{-s/2}\|u\|_\infty\|\nabla\bar{S}\|_{L^1_w(0,\infty;L^\infty(\mathbb{R}^n_x))\cap L^\infty(\mathbb{R}^{n+1}_+)}\\ &\leqslant C(t-\tau)^{-s/2}\tau^{-s/2}\|u\|_{X_T}|\bar{S}|\end{aligned}$$

and

$$\big\|\nabla e^{(t-\tau)\Delta}\mathbf{e}^-(u\cdot\nabla\bar{S})(\tau)\big\|_{\dot{B}^0_{\infty,1}}\leqslant C(t-\tau)^{-(s+1)/2}\tau^{-s/2}\|u\|_{X_T}|\bar{S}|,$$

which yield

$$\big\|K(u\cdot\nabla\bar{S})\big\|_{Y_T}\leqslant CT^{1-s}\|u\|_{X_T}|\bar{S}|.\tag{5.28}$$

Similarly, we obtain

$$\big\|K(\partial_t\bar{S}-\Delta\bar{S})\big\|_{Y_T}\leqslant CT^{1-s/2}|\bar{S}|.\tag{5.29}$$

It is straightforward to see that

$$\big\|e^{t\Delta}\mathbf{e}^-\bar{\theta}_0\big\|_{Y_T}\leqslant C\|\mathbf{e}^-\bar{\theta}_0\|_{\dot{B}^0_{\infty,1}}.\tag{5.30}$$

Hence, by (5.27)–(5.30) we have

$$\begin{aligned}\big\|\Phi(u,w)\big\|_{Y_T} &\leqslant C_0\|\bar\theta_0\|_{\dot B^0_{\infty,1}} + C_1T^{(1-s)/2}\|u\|_{X_T}\|w\|_{Y_T}\\ &\quad + C_2T^{1-s}\|u\|_{X_T}|\bar S| + C_3T^{1-s/2}|\bar S| \end{aligned}\tag{5.31}$$

and

$$\begin{aligned}\big\|\Phi(u_1,w_1)-\Phi(u_2,w_2)\big\|_{Y_T} &\leqslant C_1T^{(1-s)/2}\big(\|u_1-u_2\|_{X_T}\|w_1\|_{Y_T} + \|u_2\|_{X_T}\|w_1-w_2\|_{Y_T}\big)\\ &\quad + C_2T^{1-s}\|u_1-u_2\|_{X_T}|\bar S|,\end{aligned}\tag{5.32}$$

because

$$\Phi(u_1,w_1)-\Phi(u_2,w_2) = -J(u_1-u_2,w_1) - J(u_2,w_1-w_2) - K\big((u_1-u_2)\cdot\nabla\bar S\big).$$

Let

$$A_0 = C_0\big(\|u_0\|_{B^{-s}_{\infty,q,+,\sigma}} + \|\mathbf{e}^-\bar\theta_0\|_{\dot B^0_{\infty,1}}\big) + \big(C_2|g| + C_3\big)|\bar S|.$$

Then, from (5.23), (5.24), (5.31), and (5.32), for $U=(u,w)\in Z_T$

$$\big\|\Pi(U)\big\|_{Z_T} \leqslant A_0 + C_1T^{(1-s)/2}\|U\|^2_{Z_T} + C_2T^{1-s}\big(|g|+|\bar S|\big)\|U\|_{Z_T},\tag{5.33}$$

and for $U_1=(u_1,w_1), U_2=(u_2,w_2)\in Z_T$

$$\begin{aligned}\big\|\Pi(U_1)-\Pi(U_2)\big\|_{Z_T} &\leqslant C_1T^{(1-s)/2}\|U_1-U_2\|_{Z_T}\big(\|U_1\|_{Z_T}+\|U_2\|_{Z_T}\big)\\ &\quad + C_2T^{1-s}\big(|g|+|\bar S|\big)\|U_1-U_2\|_{Z_T}.\end{aligned}\tag{5.34}$$

An elementary consideration shows that if

$$0 < 1 - C_2\big(|g|+|\bar S|\big)T^{1-s} =: l_T,\tag{5.35}$$

$$0 < l_T^2 - 4A_0C_1T^{(1-s)/2} =: m_T\tag{5.36}$$

and if

$$\|U\|_{Z_T}, \|U_1\|_{Z_T}, \|U_2\|_{Z_T} \leqslant \frac{l_T-\sqrt{m_T}}{2C_1T^{(1-s)/2}} =: k_T,\tag{5.37}$$

then

$$\big\|\Pi(U)\big\|_{Z_T} \leqslant k_T,\tag{5.38}$$

$$\big\|\Pi(U_1)-\Pi(U_2)\big\|_{Z_T} \leqslant (1-\sqrt{m_T})\|U_1-U_2\|_{Z_T}.\tag{5.39}$$

Hence, Π is a contraction map on

$$\tilde Z_T = \big\{U\in Z_T;\ \|U\|_{Z_T}\leqslant k_T\big\},$$

which proves the existence of a solution $(u,\bar\theta)$ to the system of integral equations (5.1) and (5.2).

Note that (5.35) and (5.36) hold if we set

$$T = \min\biggl(1, \Bigl(\frac{1}{2C_2(|g| + |\bar{S}|)}\Bigr)^{\frac{1}{1-s}}, \Bigl(\frac{1}{32C_0C_1(\|u_0\|_{B^{-s}_{\infty,q,+,\sigma}} + \|\mathbf{e}^-\bar{\theta}_0\|_{\dot{B}^0_{\infty,1}} + (|g|+1)|\bar{S}|)}\Bigr)^{\frac{2}{1-s}}\biggr).$$

Step 2. Next, the mild solution $(u, \bar{\theta})$ given in Step 1 is shown to satisfy (AB). Since, for $0 < t < t' < T$,

$$\begin{aligned} u(t') - u(t) &= \bigl(e^{-t'A} - e^{-tA}\bigr)u_0 \\ &\quad - \int_0^t \bigl(e^{-(t'-\tau)A} - e^{-(t-\tau)A}\bigr)\bigl[P\bigl(\nabla\cdot Eu\otimes Eu - E(g\bar{\theta}) - E(g\bar{S})\bigr)\bigr](\tau)\,d\tau \\ &\quad - \int_t^{t'} e^{-(t'-\tau)A}\bigl[P\bigl(\nabla\cdot Eu\otimes Eu - E(g\bar{\theta}) - E(g\bar{S})\bigr)\bigr](\tau)\,d\tau, \end{aligned}$$

from Theorem 1(d), (5.13), (5.17) and (5.19), for $0 < \alpha < 1/2$, we have

$$\begin{aligned} &\bigl\|u(t') - u(t)\bigr\|_{B^{-s}_{\infty,q,+,\sigma}} \\ &\quad\leqslant C(T)(t'-t)^{\alpha}\bigl\{t^{-\alpha}\|u_0\|_{B^{-s}_{\infty,q,+,\sigma}} + t^{-\alpha-s/2+1/2}\|u\|^2_{X_T} + t^{-\alpha+1}|g|\bigl(\|\bar{\theta}\|_{Y_T} + |\bar{S}|\bigr)\bigr\} \\ &\qquad + C(T)(t'-t)\bigl\{t^{-s/2-1/2}\|u\|^2_{X_T} + |g|\bigl(\|\bar{\theta}\|_{Y_T} + |\bar{S}|\bigr)\bigr\}. \end{aligned} \tag{5.40}$$

Hence, for $0 < \alpha < 1/2$,

$$u \in C^{\alpha}\bigl([\epsilon, T]; B^{-s}_{\infty,q,+,\sigma}\bigr) \quad \text{for all } \epsilon \in (0,T). \tag{5.41}$$

Similarly, using Theorem 1(e), for $0 < \alpha < 1/2$,

$$\nabla u \in C^{\alpha}\bigl([\epsilon, T]; B^{-s}_{\infty,1,+}\bigr) \quad \text{for all } \epsilon \in (0,T). \tag{5.42}$$

Since, for $0 < \epsilon < t < t' < T$,

$$\begin{aligned} \bar{\theta}(t') - \bar{\theta}(t) &= \bigl(e^{(t'-\epsilon)\Delta} - e^{(t-\epsilon)\Delta}\bigr)\bar{\theta}(\epsilon) \\ &\quad - \int_{\epsilon}^{t} \bigl(e^{(t'-\tau)\Delta} - e^{(t-\tau)\Delta}\bigr)\bigl\{\nabla\cdot\bigl((Eu)\bar{\theta}\bigr) + \mathbf{e}^-(u\cdot\nabla\bar{S}) + \mathbf{e}^-(\partial_t\bar{S} - \Delta\bar{S})\bigr\}(\tau)\,d\tau \\ &\quad - \int_t^{t'} e^{(t'-\tau)\Delta}\bigl\{\nabla\cdot\bigl((Eu)\bar{\theta}\bigr) + \mathbf{e}^-(u\cdot\nabla\bar{S}) + \mathbf{e}^-(\partial_t\bar{S} - \Delta\bar{S})\bigr\}(\tau)\,d\tau, \end{aligned} \tag{5.43}$$

and since

$$\bigl\|\bigl(e^{t'\Delta} - e^{t\Delta}\bigr)f\bigr\|_{\dot{B}^{\beta}_{\infty,1}} \leqslant C(t'-t)^{\alpha}\bigl\|(-\Delta)^{\alpha}e^{t\Delta}f\bigr\|_{\dot{B}^{\beta}_{\infty,1}} \leqslant C(t'-t)^{\alpha}t^{-\alpha-\gamma/2}\|f\|_{\dot{B}^{\beta-\gamma}_{\infty,\infty}}$$

for $\alpha > 0$, $\gamma > -2\alpha$ and $\beta \in \mathbb{R}$, calculations similar to (5.25)–(5.32) yield

$$\begin{aligned}\big\|\bar\theta(t')-\bar\theta(t)\big\|_{\dot B^0_{\infty,1}} &\leqslant C(T)(t'-t)^\alpha\big\{(t-\epsilon)^{-\alpha}\big\|\bar\theta(\epsilon)\big\|_{\dot B^0_{\infty,1}}+t^{-\alpha+1/2-s/2}\|u\|_{X_T}\|\bar\theta\|_{Y_T}\\ &\quad+t^{1-\alpha-s}\|u\|_{X_T}|\bar S|+t^{1-\alpha-s/2}|\bar S|\big\}\\ &\quad+C(T)(t'-t)^{1/2}t^{-s/2}\|u\|_{X_T}\|\bar\theta\|_{Y_T}\\ &\quad+C(T)(t'-t)^{1-s/2}t^{-s/2}\big(\|u\|_{X_T}+1\big)|\bar S|\end{aligned}\tag{5.44}$$

for $0<\alpha<1/2$. Hence, for $0<\alpha<1/2$, we have

$$\bar\theta\in C^\alpha\big([\epsilon,T];\dot B^0_{\infty,1}\big)\quad\text{for all }\epsilon\in(0,T).\tag{5.45}$$

The assumption $S\in C^\alpha([0,\infty);L^\infty(\mathbb{R}^n_x))$ and (5.19) imply $P_+(g\bar S)\in C^\alpha([0,\infty);B^{-s}_{\infty,q,+,\sigma})$. Then, from (5.13), (5.17), (5.41), (5.42), and (5.45), for $0<\alpha<1/2$, we have

$$-P\nabla\cdot(Eu\otimes Eu)+P_+(g\theta+g\bar S)\in C^\alpha\big([\epsilon,T];B^{-s}_{\infty,q,+,\sigma}\big)\quad\text{for all }\epsilon\in(0,T).$$

This fact and the integral equation (5.1) imply that u satisfies (2.14) and

$$u_t+Au+P\nabla\cdot(Eu\otimes Eu)=P_+\big(g(\bar\theta+\bar S)\big).$$

(See, e.g., [38, Theorem 3.3.4].) The above equation is equivalent to the first equation of (AB), because $u\in C((0,T];D(A))\subset C((0,T];W^{1,\infty})$ implies $\nabla\cdot(Eu\otimes Eu)=E(u\cdot\nabla u)$. On the other hand, since

$$u,\bar\theta\in C\big([\epsilon,T];W^{1,\infty}\big)\quad\text{for all }\epsilon\in(0,T),$$

we have

$$\nabla\cdot\big((Eu)\bar\theta\big)=\mathbf{e}^-(u\cdot\nabla\bar\theta)\in C\big([\epsilon,T];L^\infty\big)\quad\text{for all }\epsilon\in(0,T).\tag{5.46}$$

Then, from (5.43), (5.46), and a calculation similar to (5.44), for $0<\alpha<1/2$, we have

$$\nabla\bar\theta\in C^\alpha\big([\epsilon,T];\dot B^0_{\infty,1}\big)\quad\text{for all }\epsilon\in(0,T).\tag{5.47}$$

By (5.47) and the assumption of Theorem 2, we have

$$\nabla\cdot\big((Eu)\bar\theta\big)+\mathbf{e}^-(u\cdot\nabla\bar S)+\mathbf{e}^-(\partial_t\bar S-\Delta\bar S)\in C^\alpha\big([\epsilon,T];L^\infty\big)\quad\text{for all }\epsilon\in(0,T),$$

which implies that $\bar\theta$ satisfies (2.15) and the second equation of (AB) in $\mathbb{R}^{n+1}_+$.

Step 3. The uniqueness of solutions satisfying (2.14)–(2.15) remains to be proven. First, the uniqueness of solutions that satisfy not only (2.14)–(2.15) but also (2.16)–(2.17) is shown. Let $U_1=(u_1,\bar\theta_1)$ and $U_2=(u_2,\bar\theta_2)$ be two solutions of the integral equations (5.1) and (5.2) with the same data $u_0,\bar\theta_0$ and $\bar S$, and let

$$M=\|U_1\|_{Z_T}+\|U_2\|_{Z_T}.$$

From (5.34), for $0<t\leqslant\min(T,1)$, we have

$$\|U_1-U_2\|_{Z_t}\leqslant Ct^{(1-s)/2}\big(M+|g|+|\bar S|\big)\|U_1-U_2\|_{Z_t}.$$

Hence, if we take

$$t_1=\min\bigg(\frac{1}{\{2C(M+|g|+|\bar S|)\}^{\frac{2}{1-s}}},1,T\bigg),$$

then

$$U_1(t) \equiv U_2(t) \quad \text{for all } t \in [0, t_1]$$

holds. For the case in which $t_1 < T$, since $U_1(t+t_1)$ and $U_2(t+t_1)$ are two solutions of (5.1) and (5.2) on $(0, T-t_1)$ with the same data $u_1(t_1), \bar{\theta}_1(t_1)$ and $\bar{S}(\cdot + t_1)$, we have

$$\big\|U_1(\cdot+t_1) - U_2(\cdot+t_1)\big\|_{Z_t} \leqslant Ct^{(1-s)/2}\big(M + |g| + |\bar{S}|\big)\big\|U_1(\cdot+t_1) - U_2(\cdot+t_1)\big\|_{Z_t}.$$

Then,

$$U_1(t) \equiv U_2(t) \quad \text{for all } t \in [t_1, t_2],$$

where $t_2 = t_1 + \min(t_1, T-t_1) = \min(2t_1, T)$. Repeating the above argument, we have

$$U_1(t) \equiv U_2(t) \quad \text{for all } t \in [0, T),$$

which proves the uniqueness of solutions satisfying (2.14)–(2.17).

Next, the uniqueness of solutions satisfying (2.14)–(2.15) is proven, without assuming (2.16)–(2.17). To this end, it suffices to show that solutions in the class (2.14)–(2.15) necessarily satisfy (2.16)–(2.17). This is shown using a method similar to that used by Brezis [3] and Miura [28]. (See also Kozono [24].) Let

$$M := \sup_{0<\tau<T} \big\|u(\tau)\big\|_{B^{-s}_{\infty,q,+,\sigma}} + \sup_{0<\tau<T} \big\|\bar{\theta}(\tau)\big\|_{\dot{B}^0_{\infty,1}} + \big(|g|+1\big)|\bar{S}|,$$

$$T_0 := \min\left(1, \left(\frac{1}{2C_2(|g|+|S|)}\right)^{1/(1-s)}, \left(\frac{1}{32C_0C_1M}\right)^{2/(1-s)}\right),$$

and let $0 < \tau_0 < T/2$. Then, from Step 1, we obtain a local solution $(v(t), \varphi(t))$ to (AB) on $[0, T_0]$ with the data $(u(\tau_0), \bar{\theta}(\tau_0), \bar{S}(\cdot+\tau_0))$ such that

$$\sup_{0<t<T_0} t^{1/2}\big\|\nabla v(t)\big\|_{B^{-s}_{\infty,1,+}} + \sup_{0<t<T_0} t^{1/2}\big\|\nabla \mathbf{e}^{-}\varphi(t)\big\|_{\dot{B}^0_{\infty,1}} \leqslant k_{T_0} \leqslant \frac{1}{2C_1T_0^{(1-s)/2}}.$$

On the other hand, $(u(\cdot+\tau_0), \bar{\theta}(\cdot+\tau_0))$ is also a solution with the same data $(u(\tau_0), \bar{\theta}(\tau_0), \bar{S}(\cdot+\tau_0))$. Since (2.14) and (2.15) imply

$$\sup_{0<t<T/2} t^{1/2}\big\|\nabla u(t+\tau_0)\big\|_{B^{-s}_{\infty,1,+}} + \sup_{0<t<T/2} t^{1/2}\big\|\nabla \mathbf{e}^{-}\bar{\theta}(t+\tau_0)\big\|_{\dot{B}^0_{\infty,1}} < \infty,$$

by the uniqueness of solutions with (2.14)–(2.17), we have $(v(t), \varphi(t)) = (u(t+\tau_0), \bar{\theta}(t+\tau_0))$ for $0 \leqslant t \leqslant \min(T_0, T/2)$, and hence

$$t^{1/2}\big\|\nabla u(t+\tau_0)\big\|_{B^{-s}_{\infty,1,+}} + t^{1/2}\big\|\nabla \mathbf{e}^{-}\bar{\theta}(t+\tau_0)\big\|_{\dot{B}^0_{\infty,1}} \leqslant \frac{1}{2C_1T_0^{(1-s)/2}} \quad \text{for } 0 < t \leqslant \min(T_0, T/2).$$

Since (2.14) and (2.15) imply that $\nabla u \in C((0,T]; B^{-s}_{\infty,1,+})$ and $\nabla \mathbf{e}^{-}\theta \in C((0,T]; \dot{B}^0_{\infty,1})$, letting $\tau_0 \to +0$, we obtain

$$t^{1/2}\big\|\nabla u(t)\big\|_{B^{-s}_{\infty,1,+}} + t^{1/2}\big\|\nabla \mathbf{e}^{-}\bar{\theta}(t)\big\|_{\dot{B}^0_{\infty,1}} \leqslant \frac{1}{2C_1T_0^{(1-s)/2}} \quad \text{for } 0 < t \leqslant \min(T_0, T/2),$$

which shows (2.16) and (2.17). This proves the uniqueness of solutions with (2.14) and (2.15) and completes the proof of Theorem 2.

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Appendix A

A.1. Proofs of (3.28) and (3.34)

Proof. First, (3.28) is proven. Let $\lambda \in \Sigma_\theta$ with $|\lambda| = 1$, $l := \sqrt{\lambda + r^2 + a^2}$ and $m := \sqrt{r^2 + a^2}$. By (3.22), we have $\Re m \geqslant c(r + |s|)\cos(4\varepsilon_0)$ and $\Re l \geqslant c(1 + r + |s|)\cos(\theta/2)$. Since $e^{-ym}\phi(y(l-m)) = \frac{e^{-ym} - e^{-yl}}{y(l-m)}$, we have

$$\bigl|e^{-ym}\phi\bigl(y(l-m)\bigr)\bigr| \leqslant \frac{e^{-y\Re m} + e^{-y\Re l}}{y|l-m|} \leqslant \frac{e^{-cy(r+|s|)} + e^{-cy(1+r+|s|)}}{y|l-m|} \leqslant \frac{2e^{-cy(r+|s|)}}{y|l-m|}. \tag{A.1}$$

On the other hand, when $y|l-m| \leqslant 1$, we have

$$\bigl|e^{-ym}\phi\bigl(y(l-m)\bigr)\bigr| = \bigl|e^{-ym}\bigr| \sup_{|z|\leqslant 1}\bigl|\phi(z)\bigr| \leqslant Ce^{-y\Re m} \leqslant Ce^{-cy(r+|s|)}. \tag{A.2}$$

Hence, (A.1) and (A.2) yield

$$\bigl|e^{-ym}\phi\bigl(y(l-m)\bigr)\bigr| \leqslant \frac{e^{-cy(r+|s|)}}{c(1 + y|l-m|)},$$

which proves (3.28).

Next, (3.34) is proven. Since

$$\begin{aligned} h(y,r,a,\lambda) &= \partial_y^2\bigl\{ye^{-ym}\phi\bigl(y(l-m)\bigr)\bigr\} = \partial_y^2\left\{\frac{e^{-ym} - e^{-yl}}{l-m}\right\} \\ &= \frac{m^2e^{-ym} - l^2e^{-yl}}{l-m}, \end{aligned} \tag{A.3}$$

by (3.24)–(3.27), we have

$$\bigl|h(y,r,a,\lambda)\bigr| \leqslant \frac{|m|^2e^{-y\Re m} + |l|^2e^{-y\Re l}}{|l-m|} \leqslant C\bigl(1 + r + |s|\bigr)^3e^{-cy(r+|s|)}. \tag{A.4}$$

On the other hand, letting $I = \{z \in \mathbb{C};\ z = tyl + (1-t)ym,\ 0 \leqslant t \leqslant 1\}$, we have

$$\begin{aligned} \bigl|h(y,r,a,\lambda)\bigr| &= \frac{1}{y}\left|\frac{(ym)^2e^{-ym} - (yl)^2e^{-yl}}{yl - ym}\right| \leqslant \frac{1}{y}\sup_{z\in I}\left|\frac{d}{dz}\bigl(z^2e^{-z}\bigr)\right| \\ &\leqslant \frac{1}{y}\sup_{z\in I}\bigl|2ze^{-z} - z^2e^{-z}\bigr| \leqslant \frac{2}{y}\sup_{z\in I}\bigl(|z| + |z|^2\bigr)e^{-\Re z} \\ &\leqslant \frac{2}{y}\bigl\{y\bigl(1 + r + |s|\bigr) + y^2\bigl(1 + r + |s|\bigr)^2\bigr\}e^{-y\min(\Re m, \Re l)} \\ &\leqslant C\bigl\{\bigl(1 + r + |s|\bigr) + y\bigl(1 + r + |s|\bigr)^2\bigr\}e^{-cy(r+|s|)}. \end{aligned} \tag{A.5}$$

Then, (A.4) and (A.5) yield the desired estimate (3.34). □

A.2. Space $B^{-s}_{\infty,q,+,\sigma}$

Lemma A.1. *Let $0 < s < 1$ and $1 < q < \infty$. If $u \in B^{-s}_{\infty,q,+,\sigma}$, then*

$$(Eu)^{n+1}(x_{n+1}) \in C\big((-\infty,\infty); B^{-s}_{\infty,q}\big(\mathbb{R}^n_{\tilde{x}}\big)\big).$$

Proof. Let $q' = \frac{q}{q-1}$ and

$$B^{-s}_{\infty,q,\sigma}\big(\mathbb{R}^{n+1}\big) := \big\{ f \in \big(B^{-s}_{\infty,q}\big(\mathbb{R}^{n+1}\big)\big)^{n+1};\ \operatorname{div} u = 0 \text{ in } \mathcal{S}'\big(\mathbb{R}^{n+1}\big)\big\},$$
$$C^1_\sigma\big(\mathbb{R}^{n+1}\big) := \left\{ f \in \big(C^1\big(\mathbb{R}^{n+1}\big)\big)^{n+1};\ \sum_{|\alpha|\leqslant 1} \sup_{x\in\mathbb{R}^{n+1}} \big|\partial^\alpha_x u(x)\big| < \infty,\ \operatorname{div} u = 0 \text{ in } \mathbb{R}^{n+1} \right\}.$$

In order to prove the above lemma, it suffices to show that if $f \in B^{-s}_{\infty,q,\sigma}(\mathbb{R}^{n+1})$, then

$$f^{n+1}(x_{n+1}) \in C\big((-\infty,\infty); B^{-s}_{\infty,q}\big(\mathbb{R}^n\big)\big). \tag{A.6}$$

If $h \in B^{s+1}_{1,q'}(\mathbb{R}^{n+1})$, then the trace operator

$$\operatorname{Tr} : h(\tilde{x}, x_{n+1}) \to h(\tilde{x}, 0)$$

makes sense, and it holds that

$$\|\operatorname{Tr} h\|_{B^s_{1,q'}(\mathbb{R}^n_{\tilde{x}})} \leqslant C\|h\|_{B^{s+1}_{1,q'}(\mathbb{R}^{n+1})},$$

see [41, p. 212]. Moreover, for $\phi(\tilde{x}) \in C^\infty_0(\partial\mathbb{R}^{n+1}_+) = C^\infty_0(\mathbb{R}^n)$, there exists $\bar{\phi}(\tilde{x}, x_{n+1}) \in B^{s+1}_{1,q'}(\mathbb{R}^{n+1})$, such that

$$\|\nabla\bar{\phi}\|_{B^s_{1,q'}(\mathbb{R}^{n+1})} \leqslant C\|\phi\|_{B^s_{1,q'}(\mathbb{R}^n)} \quad \text{and} \quad \bar{\phi}(\tilde{x}, 0) = \phi(\tilde{x}). \tag{A.7}$$

(See [41, p. 212, Remark 2].) Here, the constant C is independent of ϕ. Then, for $g = (g^1, g^2, \dots, g^{n+1}) \in C^1_\sigma(\mathbb{R}^{n+1})$,

$$-\int_{\mathbb{R}^n} g^{n+1}(\tilde{x}, 0)\phi(\tilde{x})\,d\tilde{x} = \int_{\mathbb{R}^{n+1}_+} \operatorname{div}(g\bar{\phi})\,d\tilde{x}\,dx_{n+1} = \int_{\mathbb{R}^{n+1}_+} g\cdot\nabla\bar{\phi}\,d\tilde{x}\,dx_{n+1}.$$

Hence, by (A.7), we have

$$\big\|g^{n+1}(\tilde{x}, 0)\big\|_{B^{-s}_{\infty,q}(\mathbb{R}^n_{\tilde{x}})} \leqslant C\|g\|_{B^{-s}_{\infty,q}(\mathbb{R}^{n+1})}.$$

Since $C^1_\sigma(\mathbb{R}^{n+1})$ is dense in $B^{-s}_{\infty,q,\sigma}(\mathbb{R}^{n+1})$, if $f \in B^{-s}_{\infty,q,\sigma}(\mathbb{R}^{n+1})$,

$$(\operatorname{Tr}\cdot)^{n+1} : f(\tilde{x}, x_{n+1}) \to f^{n+1}(\tilde{x}, 0)$$

makes sense and the following holds:

$$\left\|(\operatorname{Tr} f)^{n+1}\right\|_{B^{-s}_{\infty,q}(\mathbb{R}^n)} \leqslant C\|f\|_{B^{-s}_{\infty,q,\sigma}(\mathbb{R}^{n+1})}. \tag{A.8}$$

Moreover, it holds that

$$-\int_{\mathbb{R}^n} (\operatorname{Tr} f)^{n+1}\psi(\tilde{x},0)\,d\tilde{x} = \int_{\mathbb{R}^{n+1}_+} f\cdot\nabla\psi\,d\tilde{x}\,dx_{n+1} \quad \text{for all } \psi\in C_0^\infty\bigl(\mathbb{R}^{n+1}\bigr),$$

because $-\int_{\mathbb{R}^n} g^{n+1}(\tilde{x},0)\psi(\tilde{x},0)\,d\tilde{x} = \int_{\mathbb{R}^{n+1}_+} g\cdot\nabla\psi\,d\tilde{x}\,dx_{n+1}$. Let

$$\tau_h f(\tilde{x},x_n) = f(\tilde{x},x_n+h) \quad \text{in } \mathcal{S}'.$$

Then, (A.8) yields

$$\left\|f^{n+1}(\tilde{x},h') - f^{n+1}(\tilde{x},h)\right\|_{B^{-s}_{\infty,q}(\mathbb{R}^n_{\tilde{x}})} = \left\|\bigl(\operatorname{Tr}(\tau_{h'}f-\tau_h f)\bigr)^{n+1}\right\|_{B^{-s}_{\infty,q}(\mathbb{R}^n)} \leqslant C\|\tau_{h'}f-\tau_h f\|_{B^{-s}_{\infty,q,\sigma}(\mathbb{R}^{n+1})}.$$

Since $\psi * f\in BUC$ and $\varphi_j * f\in BUC$, we have

$$\begin{aligned}
&\left\|\psi*(\tau_{h'}f)-\psi*(\tau_h f)\right\|_{L^\infty(\mathbb{R}^{n+1})} = \left\|\tau_{h'}(\psi*f)-\tau_h(\psi*f)\right\|_{L^\infty(\mathbb{R}^{n+1})}\to 0 \quad \text{as } h'\to h,\\
&\left\|\varphi_j*(\tau_{h'}f)-\varphi_j*(\tau_h f)\right\|_{L^\infty(\mathbb{R}^{n+1})} = \left\|\tau_{h'}(\varphi_j*f)-\tau_h(\varphi_j*f)\right\|_{L^\infty(\mathbb{R}^{n+1})}\to 0 \quad \text{as } h'\to h
\end{aligned} \tag{A.9}$$

for all $j=0,1,\ldots$. Hence, it holds that

$$\begin{aligned}
\|\tau_{h'}f-\tau_h f\|_{B^{-s}_{\infty,q}(\mathbb{R}^{n+1})} &= \left\|\psi*(\tau_{h'}f-\tau_h f)\right\|_\infty\\
&\quad+\left(\sum_{j\geqslant 0}2^{-sjq}\left\|\varphi_j*(\tau_{h'}f-\tau_h f)\right\|_\infty^q\right)^{1/q}\to 0 \quad \text{as } h'\to h,
\end{aligned}$$

because $\sum_{j\geqslant0}2^{-sjq}\|\varphi_j*(\tau_{h'}f-\tau_h f)\|_\infty^q \leqslant \sum_{j\geqslant0}2\cdot 2^{-sjq}\|\varphi_j*f\|_\infty^q<\infty$. Therefore,

$$\left\|f^{n+1}(\tilde{x},h')-f^{n+1}(\tilde{x},h)\right\|_{B^{-s}_{\infty,q}(\mathbb{R}^n_{\tilde{x}})}\to 0 \quad \text{as } h'\to h,$$

which implies (A.6). □

A.3. Resolvent equation on $B^{-s}_{\infty,q,+,\sigma}$

Here, the resolvent equation

$$R(\lambda_1)-R(\lambda_2) = (\lambda_1-\lambda_2)R(\lambda_1)R(\lambda_2) \tag{A.10}$$

on $B^{-s}_{\infty,q,+,\sigma}$ is shown. Before showing the resolvent equation, the following lemma is proven.

Lemma A.2. (i) *Let $a\in C^1_\sigma(\mathbb{R}^{n+1}_+)$. Then, there exist $\{a_m\}_{m=1}^\infty$ defined on the whole space $\mathbb{R}^{n+1}$ and a constant vector $\vec{b}$ such that*

$$a_m \in BUC\big(\mathbb{R}^{n+1}\big) \cap L^r_\sigma\big(\mathbb{R}^{n+1}_+\big) \cap B_\sigma \quad \text{for all } r > 1, \tag{A.11}$$

$$a_m \to Ea + \vec{b} \quad \text{weak-}* \text{ in } L^\infty\big(\mathbb{R}^{n+1}\big), \tag{A.12}$$

$$a_m \to Ea + \vec{b} \quad \text{strongly in } L^\infty(\Omega) \text{ for all compact sets } \Omega \Subset \mathbb{R}^{n+1}, \quad \text{and} \tag{A.13}$$

$$\|a_m\|_{L^\infty(\mathbb{R}^{n+1})} \leqslant C\|a\|_{L^\infty(\mathbb{R}^{n+1}_+)}, \tag{A.14}$$

where C depends only on n.

(ii) *Let $f \in C([0,T]; C^1_\sigma(\mathbb{R}^{n+1}_+))$. Then, there exist $\{f_m\}_{m=1}^\infty$ defined on $[0,T]\times\mathbb{R}^{n+1}$ and a vector function $\vec{b}(t) \in C[0,T]$ independent of the x-variable such that*

$$f_m \in C\big([0,T]; BUC\big(\mathbb{R}^{n+1}\big) \cap L^r_\sigma\big(\mathbb{R}^{n+1}_+\big) \cap B_\sigma\big) \quad \text{for all } r > 1, \tag{A.15}$$

$$f_m \to Ef + \vec{b} \quad \text{weak-}* \text{ in } L^\infty\big((0,T)\times\mathbb{R}^{n+1}\big), \tag{A.16}$$

$$f_m \to Ef + \vec{b} \quad \text{strongly in } L^\infty\big([0,T]\times\Omega\big) \text{ for all compact sets } \Omega \Subset \mathbb{R}^{n+1}, \tag{A.17}$$

$$\|f_m\|_{L^\infty(0,T;L^\infty(\mathbb{R}^{n+1}))} \leqslant C\|f\|_{L^\infty(0,T;L^\infty(\mathbb{R}^{n+1}_+))}, \tag{A.18}$$

where C depends only on n.

Remark A.1. If $f \in C^\alpha([0,T]; L^\infty(\mathbb{R}^{n+1}_+)) \cap C([0,T]; C^1_\sigma(\mathbb{R}^{n+1}_+))$ for some $\alpha > 0$, then there exist $\{f_m\}_{m=1}^\infty$ and $\vec{b}(t) \in C^\alpha[0,T]$ such that (A.15)–(A.18) and

$$f_m \in C^\alpha\big([0,T]; L^\infty \cap L^r_\sigma\big(\mathbb{R}^{n+1}_+\big)\big) \quad \text{for all } r > 1. \tag{A.19}$$

Proof of Lemma A.2. Only (ii) of Lemma A.2 is proven, because the proof of (i) is quite similar that of (ii). Let Φ and Φ_m be smooth radial functions defined on $\mathbb{R}^{n+1}$ such that

$$\Phi(x) = \Phi\big(|x|\big) = \begin{cases} 1 & (|x| \leqslant 1), \\ 0 & (|x| \geqslant 2), \end{cases} \qquad \Phi_m(x) = \Phi\left(\frac{|x|}{m}\right)$$

and let

$$Kg = \nabla(-\Delta)^{-1} g.$$

Note that

$$\|Kg\|_{BMO(\mathbb{R}^{n+1})} \leqslant C\|g\|_{L^{n+1}(\mathbb{R}^{n+1})}, \tag{A.20}$$

$$\|Kg\|_{L^\infty(\mathbb{R}^{n+1})} \leqslant C\|g\|_{L^{n+1,1}(\mathbb{R}^{n+1})}, \tag{A.21}$$

$$\|K\nabla g\|_{L^r(\mathbb{R}^{n+1})} \leqslant C(r)\|g\|_{L^r(\mathbb{R}^{n+1})} \quad \text{for } 1 < r < \infty, \tag{A.22}$$

$$\|\nabla Kg\|_{BMO(\mathbb{R}^{n+1})} \leqslant C\|g\|_{L^\infty(\mathbb{R}^{n+1})}, \tag{A.23}$$

where $L^{p,q}$ is the Lorentz space. (See, e.g., [2].) Let

$$b_m = K(Ef \cdot \nabla\Phi_m), \qquad f_m = \Phi_m Ef + b_m.$$

Since $(Ef)\Phi_m \in C([0,T]; L^1 \cap L^\infty(\mathbb{R}^{n+1}))$ and $Ef \cdot \nabla\Phi_m = \nabla\cdot((Ef)\Phi)$, by (A.22), we have

$$b_m,\, f_m \in C\big([0,T]; L^r\big(\mathbb{R}^{n+1}\big)\big) \quad \text{for all } 1 < r < \infty,$$

and $\operatorname{div} f_m(t) = 0$ in $\mathcal{S}'(\mathbb{R}^{n+1})$ for all $t \in [0,T]$. Moreover, since $Ef \cdot \nabla \Phi_m \in C([0,T]; L^{n+1,1} \cap L^\infty)$, we have $b_m, f_m \in C([0,T]; BUC(\mathbb{R}^{n+1}))$. In addition, $b_m^{n+1}|_{\partial\mathbb{R}^{n+1}_+} = f_m^{n+1}|_{\partial\mathbb{R}^{n+1}_+} = 0$, because $\partial_{n+1}(-\Delta)^{-1}(Ef \cdot \nabla\Phi_m)$ is an odd function with respect to x_{n+1}. Hence, we have $b_m, f_m \in C([0,T]; B_\sigma \cap L^r_\sigma(\mathbb{R}^{n+1}_+))$ for all $1 < r < \infty$. By (A.21), we have

$$\big\|b_m(t)\big\|_{L^\infty(\mathbb{R}^{n+1})} \leqslant C\big\|f(t)\big\|_{L^\infty(\mathbb{R}^{n+1})}\|\nabla\Phi_m\|_{L^{n+1,1}} \leqslant C\|f\|_{L^\infty([0,T]\times\mathbb{R}^{n+1})}\|\nabla\Phi\|_{L^{n+1,1}}, \tag{A.24}$$

where the constant C is independent of m. Then, (A.18) is obtained, and a subsequence $\{b_{m'}\}$ of $\{b_m\}$ and a function $b(x,t) \in L^\infty([0,T]\times\mathbb{R}^{n+1})$ exist such that

$$b_{m'} \to b \quad \text{weak-}* \text{ in } L^\infty\big([0,T]\times\mathbb{R}^{n+1}\big) \text{ as } m' \to \infty. \tag{A.25}$$

Similarly, by (A.21) and (A.23), we have

$$\big\|b_m(t) - b_m(s)\big\|_{L^\infty(\mathbb{R}^{n+1})} \leqslant C\big\|f(t) - f(s)\big\|_{L^\infty(\mathbb{R}^{n+1})}\|\nabla\Phi\|_{L^{n+1,1}}, \tag{A.26}$$

$$\big\|\nabla b_m(t)\big\|_{BMO(\mathbb{R}^{n+1})} \leqslant C\|f\|_{L^\infty([0,T]\times\mathbb{R}^{n+1})}\|\nabla\Phi_m\|_\infty \leqslant C\frac{1}{m}\|f\|_{L^\infty([0,T]\times\mathbb{R}^{n+1})}, \tag{A.27}$$

where the constant C is independent of m. Since $\|b_m\|_{C^\alpha} \cong \|b_m\|_{B^\alpha_{\infty,\infty}} \leqslant C(\|b_m\|_\infty + \|\nabla b_m\|_{BMO})$ for $0 < \alpha < 1$, and since $f \in C([0,T]; L^\infty)$, the inequalities (A.24), (A.26), and (A.27) imply that

$$\big\{b_m(t)\big\}_{m=1}^\infty \text{ is bounded on } [0,T] \text{ in } C^\alpha\big(\mathbb{R}^{n+1}\big) \quad (0 < \alpha < 1), \tag{A.28}$$

$$\big\{b_m(t)\big\}_{m=1}^\infty \text{ is equicontinuous on } [0,T] \text{ in } L^\infty\big(\mathbb{R}^{n+1}\big). \tag{A.29}$$

Hence, by the Ascoli theorem, we have

$$b \in C\big([0,T]; L^\infty(\Omega)\big) \quad \text{and} \quad b_{m'} \to b \quad \text{strongly in } C\big([0,T]; L^\infty(\Omega)\big) \tag{A.30}$$

for all compact sets $\Omega \Subset \mathbb{R}^{n+1}$. On the other hand, for all $\Psi \in (\mathcal{S}_0)^{n+1} := (\{v \in \mathcal{S}(\mathbb{R}^{n+1});\ \mathcal{F}v(\eta) \equiv 0 \text{ near } \eta = 0\})^{n+1}$,

$$\begin{aligned}\left|\int_{\mathbb{R}^{n+1}} b_m(x,t)\cdot\Psi(x)\,dx\right| &= \left|\int_{\mathbb{R}^{n+1}} \big(Ef(t)\cdot\nabla\Phi_m\big)\big((-\Delta)^{-1}\operatorname{div}\Psi\big)\,dx\right| \\ &\leqslant \|f\|_\infty\|\nabla\Phi_m\|_\infty\big\|(-\Delta)^{-1}\operatorname{div}\Psi\big\|_{L^1} \\ &\to 0 \quad \text{as } m \to \infty.\end{aligned} \tag{A.31}$$

Since $\mathcal{S}_0$ is dense in the Hardy space $\mathcal{H}^1$, and since $\{b_m(t)\}_{m=1}^\infty$ is bounded in $BMO(\mathbb{R}^{n+1})$, we have

$$b_m(t) \to 0 \quad \text{weak-}* \text{ in } BMO\big(\mathbb{R}^{n+1}\big). \tag{A.32}$$

Hence, (A.30) and (A.32) yield $\int_{\mathbb{R}^{n+1}} b(x,t)\partial_j\psi(x)\,dx = 0$ for all $\psi \in C_0^\infty(\mathbb{R}^{n+1})$, $j = 1,2,\dots,n+1$, and all $t \in [0,T]$, which implies

$$\nabla b(x,t) = 0. \tag{A.33}$$

Then, b is independent of x, and (A.30) implies $b \in C([0,T])$.

Here, $\Phi_m Ef \to Ef$ weak-$*$ in $L^\infty(0,T;L^\infty(\mathbb{R}^{n+1}))$ and strongly in $C([0,T];L^\infty(\Omega))$ for all compact sets Ω. Therefore, (A.25) and (A.30) prove Lemma A.2(ii). □

Next, (A.10) is proven on $B^{-s}_{\infty,q,+,\sigma}$ for all $\lambda_1,\lambda_2\in\Sigma_\pi$. Let $\lambda_1,\lambda_2\in\Sigma_\pi$. Since $C^1_\sigma(\mathbb{R}^{n+1}_+)$ is dense in $B^{-s}_{\infty,q,+,\sigma}$, it suffices to show that

$$R(\lambda_1)f - R(\lambda_2)f = (\lambda_1-\lambda_2)R(\lambda_1)R(\lambda_2)f \tag{A.34}$$

for all $f\in C^1_\sigma(\mathbb{R}^{n+1}_+)$. Lemma A.2(i) indicates that there exist $\{f_m\}_{m=1}^\infty$ and a constant vector $\vec{b}$ such that

$$f_m \to Ef+\vec{b} \quad \text{weak-}* \text{ in } L^\infty\big(\mathbb{R}^{n+1}\big) \quad \text{and} \quad f_m\in L^r_\sigma\big(\mathbb{R}^{n+1}_+\big) \quad (1<r<\infty). \tag{A.35}$$

Since the resolvent equation (A.10) holds on $L^r_\sigma(\mathbb{R}^{n+1}_+)$ $(1<r<\infty)$, the following holds:

$$R(\lambda_1)f_m - R(\lambda_2)f_m = (\lambda_1-\lambda_2)R(\lambda_1)R(\lambda_2)f_m. \tag{A.36}$$

Let $\lambda\in\Sigma_\pi$ and y denote x_{n+1} as in Section 3. Based on (4.4), for $x=(\tilde{x},y)\in\mathbb{R}^{n+1}_+$

$$\begin{aligned} R(\lambda)f(x) &= \big((\lambda-\Delta)^{-1}\mathbf{e}^-f\big)(x) + (T_\lambda f)(x) \\ &= \big((\lambda-\Delta)^{-1}\mathbf{e}^-f\big)(x) + \int_0^\infty\int_{\mathbb{R}^n} G(\tilde{x}-\tilde{x}',y,y',\lambda)f(\tilde{x}',y')\,d\tilde{x}'\,dy', \end{aligned} \tag{A.37}$$

where $G(\cdot,y,\cdot,\lambda)\in L^1(\mathbb{R}^n_{\tilde{x}}\times\mathbb{R}_{y'})$ for all $y>0$. Note that $R(\lambda)f$ is well defined for all $f\in L^\infty(\mathbb{R}^{n+1}_+)$ without any boundary conditions, and $R(\lambda)$ is a bounded operator in $L^\infty(\mathbb{R}^{n+1}_+)$. Since $(\lambda-\Delta)^{-1}$ is a bounded operator in $L^1(\mathbb{R}^{n+1})$,

$$\begin{aligned} &\text{if } g_m\to g \quad \text{weak-}* \text{ in } L^\infty\big(\mathbb{R}^{n+1}_+\big), \\ &\quad \text{then } (\lambda-\Delta)^{-1}\mathbf{e}^-g_m \to (\lambda-\Delta)^{-1}\mathbf{e}^-g \quad \text{weak-}* \text{ in } L^\infty\big(\mathbb{R}^{n+1}\big). \end{aligned} \tag{A.38}$$

Since (3.18) implies $\sup_{y\in\mathbb{R}}\|G(\cdot,y,\cdot,\lambda)\|_{L^1(\mathbb{R}^n_{\tilde{x}}\times\mathbb{R}_{y'})}\leqslant C(\lambda)<\infty$, we have

$$\begin{aligned} &\text{if } g_m\to g \quad \text{weak-}* \text{ in } L^\infty\big(\mathbb{R}^{n+1}_+\big), \\ &\quad \text{then } (T_\lambda g_m)(x)\to(T_\lambda g)(x) \end{aligned} \tag{A.39}$$

for all $x\in\mathbb{R}^{n+1}_+$, and, consequently,

$$\begin{aligned} &\text{if } g_m\to g \quad \text{weak-}* \text{ in } L^\infty\big(\mathbb{R}^{n+1}_+\big), \\ &\quad \text{then } (T_\lambda g_m)\to(T_\lambda g) \quad \text{weak-}* \text{ in } L^\infty\big(\mathbb{R}^{n+1}_+\big). \end{aligned} \tag{A.40}$$

Then, we have

$$\begin{aligned} &\text{if } g_m\to g \quad \text{weak-}* \text{ in } L^\infty\big(\mathbb{R}^{n+1}_+\big), \\ &\quad \text{then } R(\lambda)g_m\to R(\lambda)g \quad \text{weak-}* \text{ in } L^\infty\big(\mathbb{R}^{n+1}_+\big). \end{aligned} \tag{A.41}$$

Hence, letting $m\to\infty$, from (A.35), (A.36), and (A.41), we obtain

$$R(\lambda_1)(f+\vec{b}) - R(\lambda_2)(f+\vec{b}) = (\lambda_1-\lambda_2)R(\lambda_1)R(\lambda_2)(f+\vec{b}). \tag{A.42}$$

Since $\int_{\mathbb{R}^n} G(\tilde{x}, y, y', \lambda)d\tilde{x} = 0$ for all $y > 0$ and all $y' \in \mathbb{R}$, and since $((\lambda_2 - \Delta)^{-1}\mathbf{e}^-\vec{b})$ is independent of $\tilde{x}$, we have

$$T_{\lambda_1}\vec{b} = 0, \qquad T_{\lambda_2}\vec{b} = 0, \qquad T_{\lambda_1}(\lambda_2 - \Delta)^{-1}\mathbf{e}^-\vec{b} = 0.$$

Hence, (A.42) yields

$$\begin{aligned} &R(\lambda_1)f - R(\lambda_2)f + (\lambda_1 - \Delta)^{-1}\mathbf{e}^-\vec{b} - (\lambda_2 - \Delta)^{-1}\mathbf{e}^-\vec{b} \\ &\quad = (\lambda_1 - \lambda_2)R(\lambda_1)R(\lambda_2)f + (\lambda_1 - \lambda_2)(\lambda_1 - \Delta)^{-1}(\lambda_2 - \Delta)^{-1}\mathbf{e}^-\vec{b}. \end{aligned} \tag{A.43}$$

Since $(\lambda - \Delta)^{-1}$ satisfies the resolvent equation on $L^\infty(\mathbb{R}^{n+1})$, from (A.43) the desired equation (A.34) is obtained.

A.4. Stokes equation on $B^{-s}_{\infty,q,+,\sigma}$

Let $L^r_{uloc}(\mathbb{R}^{n+1}_+) := \{g \in L^r_{loc}(\mathbb{R}^{n+1}_+);\ \|g\|_{L^r_{uloc}(\mathbb{R}^{n+1}_+)} < \infty\}$, where

$$\|g\|_{L^r_{uloc}(\mathbb{R}^{n+1}_+)} := \sup_x \left(\int_{|x-z|<1,\, z\in\mathbb{R}^{n+1}_+} \left|g(z)\right|^r dz \right)^{1/r}.$$

Lemma A.3. (i) *Let $0 < s < 1$, $1 \leqslant q < \infty$ and $a \in B^{-s}_{\infty,q,+,\sigma}$. Then, $u(t) = e^{-tA}a$ is a solution of the Stokes equation*:

$$(\mathrm{S})_1 \quad \begin{cases} \partial_t u - \Delta u + \nabla p = 0 & \text{in } \mathcal{D}'\big((0,\infty)\times\mathbb{R}^{n+1}_+\big), \\ \operatorname{div} u = 0 & \text{in } \mathcal{D}'\big((0,\infty)\times\mathbb{R}^{n+1}_+\big), \\ u|_{t=0} = a \end{cases}$$

with associate pressure p, which satisfies $\nabla p \in L^\infty_{loc}((0,\infty); L^{1/s'}_{uloc}(\mathbb{R}^{n+1}_+))$ for $1 > s' > s$.

(ii) *Let $0 < s < 1$, $1 \leqslant q < \infty$, and $f \in C^\alpha([0,T]; B^{-s}_{\infty,q,+,\sigma})$ for some $\alpha > 0$. Then, $u(t) = \int_0^t e^{-(t-\tau)A} f(\tau)\, d\tau$ is a solution of the Stokes equation:*

$$(\mathrm{S})_2 \quad \begin{cases} \partial_t u - \Delta u + \nabla p = f & \text{in } \mathcal{D}'\big((0,\infty)\times\mathbb{R}^{n+1}_+\big), \\ \operatorname{div} u = 0 & \text{in } \mathcal{D}'\big((0,\infty)\times\mathbb{R}^{n+1}_+\big), \\ u|_{t=0} = 0 \end{cases}$$

with associate pressure p, which satisfies $\nabla p \in L^\infty(0,T; L^{1/s'}_{uloc}(\mathbb{R}^{n+1}_+))$ for $1 > s' > s$.

Proof. Only (ii) is proven, because the proof of (i) is similar to that of (ii).

Step 1. First, the case in which $f \in C^\alpha([0,T]; C^1_\sigma(\mathbb{R}^{n+1}_+))$ is considered. Recall the representation of e^{-tA} by means of the resolvent $R(\lambda) = (\lambda + A)^{-1}$. Let $\pi/2 < \theta < \pi$, and let δ be an arbitrary positive number. In addition, let $\Gamma_\delta := \{re^{i\theta}; 1 \leqslant r < \infty\} \cup \{e^{i\psi}; -\theta \leqslant \psi \leqslant \theta\} \cup \{re^{-i\theta}; 1 \leqslant r < \infty\}$. By (4.4), for all $g \in B^{-s}_{\infty,q,+,\sigma}$, we have

$$\big(e^{-tA}g\big) = \frac{1}{2\pi i}\int_{\Gamma_\delta} e^{\lambda t}(\lambda + A)^{-1} g\, d\lambda$$

$$= \frac{1}{2\pi i}\int_{\Gamma_\delta} e^{\lambda t}(\lambda-\Delta)^{-1}\mathbf{e}^- g\,d\lambda + \frac{1}{2\pi i}\int_{\Gamma_\delta} e^{\lambda t}T_\lambda g\,d\lambda$$

$$= e^{t\Delta}\mathbf{e}^- g + H_t(g), \tag{A.44}$$

where

$$H_t(g) := \frac{1}{2\pi i}\int_{\Gamma_\delta} e^{\lambda t}T_\lambda g\,d\lambda.$$

Note that $H_t(g)$ is independent of δ, because $e^{-tA}g$ and $e^{t\Delta}\mathbf{e}^- g$ are independent of δ. Note also that the equalities (A.44) and $e^{0A}g = g = e^{0\Delta}\mathbf{e}^- g$ imply $H_0(g) = 0$. By (A.44), for all $x = (\tilde{x}, y) \in \mathbb{R}^{n+1}_+$, we have

$$\begin{aligned} u(x,t) &= \int_0^t \big(e^{-(t-\tau)A}f(\tau)\big)(x)\,d\tau = \int_0^t \big(e^{(t-\tau)\Delta}\mathbf{e}^- f(\tau)\big)(x)\,d\tau + \int_0^t H_{t-\tau}\big(f(\tau)\big)(x)\,d\tau \\ &= \int_0^t \frac{1}{(4\pi(t-\tau))^{(n+1)/2}}\int_{\mathbb{R}^{n+1}} e^{\frac{-|x-z|^2}{4(t-\tau)}}\mathbf{e}^- f(z,\tau)\,dz\,d\tau \\ &\quad + \frac{1}{2\pi i}\int_0^t\int_{\Gamma_\delta} e^{\lambda(t-\tau)}\int_{\mathbb{R}^{n+1}} G(\tilde{x}-\tilde{x}', y, y', \lambda)\chi(y')f(\tilde{x}', y', \tau)\,d\tilde{x}'\,dy'\,d\lambda\,d\tau. \end{aligned} \tag{A.45}$$

From (4.2),

$$\int_0^t\int_{\mathbb{R}^{n+1}}\int_{\Gamma_\delta} \big|e^{\lambda(t-\tau)}\big|\big|G(\tilde{x}-\tilde{x}', y, y', \lambda)\big||d\lambda|\,d\tilde{x}'\,dy'\,d\tau \leqslant C(t, y, \delta) < \infty \tag{A.46}$$

for all $\tau > 0$, $y > 0$, and $\tilde{x} \in \mathbb{R}^n$. By Lemma A.2 and Remark A.1, there exist $\{f_m\}$ and a vector function $\vec{b}(t) \in C^\alpha([0,T])$ with (A.15)–(A.19). Since $T_\lambda\vec{b} = 0$ implies $\int_0^t H_{t-\tau}(\vec{b}(\tau))\,d\tau = 0$, from (A.45), (A.46), and (A.16), we have

$$\begin{aligned} &\int_0^t \big(e^{-(t-\tau)A}f_m(\tau)\big)(x)\,d\tau \\ &\quad\to \int_0^t \big(e^{(t-\tau)\Delta}\mathbf{e}^- f(\tau)\big)(x)\,d\tau + \int_0^t \big(e^{(t-\tau)\Delta}\mathbf{e}^-\vec{b}(\tau)\big)(x)\,d\tau + \int_0^t H_{t-\tau}\big(f(\tau)\big)(x)\,d\tau \\ &\quad= u(x,t) + \int_0^t \big(e^{(t-\tau)\Delta}\mathbf{e}^-\vec{b}(\tau)\big)(x)\,d\tau \end{aligned} \tag{A.47}$$

as $m \to \infty$ for all $t > 0$ and $x = (\tilde{x}, y) \in \mathbb{R}^{n+1}_+$.

Let $u_m := \int_0^t e^{-(t-\tau)A} f_m(\tau)\,d\tau$. Since $f_m \in C^\alpha([0,T]; L^r_\sigma(\mathbb{R}^{n+1}_+))$ for all $1 < r < \infty$, we see that $u_m = \int_0^t e^{-(t-s)A_r} f_m(\tau)\,d\tau$ for all $1 < r < \infty$, where A_r is the usual Stokes operator on $L^r_\sigma(\mathbb{R}^{n+1}_+)$. (See [8].) Then, u_m satisfies the Stokes equation (S) with an associated pressure p_m, i.e.,

$$\partial_t u_m - \Delta u_m + \nabla p_m = f_m \quad \text{in } C\big(0,\infty; L^r(\mathbb{R}^{n+1}_+)\big) \tag{A.48}$$

for all $1 < r < \infty$. Hereinafter, for simplicity, let $\|g\|_{L^q_y(L^r_{\tilde{x}})}$ denote

$$\left(\int_0^\infty \big\|g(\tilde{x}, y)\big\|^q_{L^r(\mathbb{R}^n_{\tilde{x}})}\,dy\right)^{1/q}.$$

Then, from (4.7), for $s < s' < 1$, we have

$$\|T_\lambda f_m\|_{L^{1/s'}_y(L^\infty_{\tilde{x}})} \leqslant C\frac{1+|\lambda|^{2/s}}{|\lambda|^{1+s'/2}}\|f_m\|_{B^{-s}_{\infty,q,+,\sigma}}.$$

Then, based on the above and a simple calculation, we have

$$\begin{aligned}\big\|H_t(f_m)\big\|_{L^{1/s'}_y(L^\infty_{\tilde{x}})} &\leqslant C\int_{\Gamma_\delta} e^{t\Re\lambda}\frac{1+|\lambda|^{2/s}}{|\lambda|^{1+s'/2}}|d\lambda|\|f_m\|_{B^{-s}_{\infty,q,+,\sigma}}\\ &\leqslant C\|f_m\|_{B^{-s}_{\infty,q,+,\sigma}}\left(\frac{1+\delta^{s/2}}{\delta^{1+s'/2}t}+\frac{1+\delta^{s/2}}{\delta^{s'/2}}e^{t\delta}\right)\end{aligned}$$

for all $0 < t < T$ and $\delta > 0$, where C is independent of m, t and δ. Hence, letting $t = 1/\delta$, we obtain

$$\big\|H_t(f_m)\big\|_{L^{1/s'}_y(L^\infty_{\tilde{x}})} \leqslant Ct^{s'/2}\big(1+t^{-s/2}\big)\|f_m\|_{B^{-s}_{\infty,q,+,\sigma}},$$

and

$$\left\|\frac{1}{2\pi i}\int_0^t H_{t-\tau}\big(f_m(\tau)\big)\,d\tau\right\|_{L^{1/s'}_y(L^\infty_{\tilde{x}})} \leqslant CT^{s'/2}\big(T+T^{1-s/2}\big)\sup_{0\leqslant\tau<T}\|f_m\|_{B^{-s}_{\infty,q,+,\sigma}}$$

for all $0 < t < T$. Since

$$u_m(t) = \int_0^t e^{(t-\tau)\Delta}\mathbf{e}^- f_m(\tau)\,d\tau + \int_0^t H_{t-\tau}\big(f_m(\tau)\big)\,d\tau,$$

$\|g\|_{L^r_{uloc}(\mathbb{R}^{n+1}_+)} \leqslant C\|g\|_{L^r_y(L^\infty_{\tilde{x}})}$ and $\|e^{t\Delta}\mathbf{e}^- f_m\|_{L^r_{uloc}} \leqslant C\|e^{t\Delta}\mathbf{e}^- f_m\|_\infty \leqslant C(t^{-s/2}+1)\|\mathbf{e}^- f_m\|_{B^{-s}_{\infty,q}} \leqslant C(t^{-s/2}+1)\|f_m\|_{B^{-s}_{\infty,q,+}}$, by (A.18) we have

$$\sup_{0<t<T}\|u_m\|_{L^{1/s'}_{uloc}(\mathbb{R}^{n+1}_+)} \leqslant C(T)\sup_{0<t<T}\|f_m\|_{B^{-s}_{\infty,q,+,\sigma}} \leqslant C(T)\sup_{0<t<T}\|f\|_\infty. \tag{A.49}$$

Therefore, from (A.47) and (A.49), there exists a subsequence $\{u_{m'}\}$ of $\{u_m\}$ such that, for all $v \in C_0^\infty((0,T)\times\mathbb{R}^{n+1}_+)$,

$$\int_0^T\int_{\mathbb{R}^{n+1}_+} (-\partial_t v - \Delta v)\cdot u_{m'}\,dx\,d\tau \to \int_0^T\int_{\mathbb{R}^{n+1}_+} \big((-\partial_t v - \Delta v)\cdot u + v\cdot\vec{b}\big)\,dx\,d\tau \quad \text{as } m'\to\infty, \tag{A.50}$$

since $(\partial_t - \Delta)\int_0^t e^{(t-\tau)\Delta}\mathbf{e}^-\vec{b}(\tau)\,d\tau = \mathbf{e}^-\vec{b}$.

Next, the pressure term p_m is estimated. Since $(\partial_t - \Delta)\int_0^t e^{(t-\tau)\Delta}\mathbf{e}^- f_m(\tau)\,d\tau = \mathbf{e}^- f_m$ and $H_0(f_m(t)) = 0$, we have

$$\begin{aligned}\nabla p_m &= -(\partial_t - \Delta)u_m + f_m\\ &= -(\partial_t - \Delta)\int_0^t H_{t-\tau}\big(f_m(\tau)\big)\,d\tau\\ &= -\int_0^t (\partial_t - \Delta)H_{t-\tau}\big(f_m(\tau)\big)\,d\tau \quad \text{in } \mathbb{R}^{n+1}_+.\end{aligned} \tag{A.51}$$

By (4.7) and (4.18), for $s < s' < 1$, we have

$$\begin{aligned}\big\|(\partial_t - \Delta)H_{t-\tau}(g)\big\|_{L_y^{1/s'}(L^\infty_{\tilde{x}})} &\leqslant \left\|\int_{\Gamma_\delta}\lambda e^{(t-\tau)\lambda}T_\lambda g\,d\lambda\right\|_{L_y^{1/s'}(L^\infty_{\tilde{x}})} + \left\|\int_{\Gamma_\delta} e^{(t-\tau)\lambda}\Delta T_\lambda g\,d\lambda\right\|_{L_y^{1/s'}(L^\infty_{\tilde{x}})}\\ &\leqslant C(n,s,s')\|g\|_{B^{-s}_{\infty,q,+,\sigma}}\left(\frac{1+\delta^{s/2}}{\delta^{s'/2}(t-\tau)} + \frac{1+\delta^{s/2}}{\delta^{s'/2}}\delta e^{\delta(t-\tau)}\right)\end{aligned}$$

for all $\delta > 0$ and $t > \tau > 0$. Then, letting $\delta = 1/(t-\tau)$, we obtain

$$\big\|(\partial_t - \Delta)H_{t-\tau}\big(f_m(\tau)\big)\big\|_{L_y^{1/s'}(L^\infty_{\tilde{x}})} \leqslant C\big\|f_m(\tau)\big\|_{B^{-s}_{\infty,q,+,\sigma}}\left(\frac{1}{(t-\tau)^{1-s'/2}} + \frac{1}{(t-\tau)^{1-s'/2+s/2}}\right)$$

and consequently

$$\big\|\nabla p_m(t)\big\|_{L_y^{1/s'}(L^\infty_{\tilde{x}})} \leqslant C\big(t^{s'/2} + t^{s'/2-s/2}\big)\sup_{0<\tau<T}\big\|f_m(\tau)\big\|_{B^{-s}_{\infty,q,+,\sigma}} \leqslant C(T)\sup_{0<\tau<T}\big\|f(\tau)\big\|_\infty$$

for all $0 < t < T$, where the constant $C(T)$ is independent of m. Hence, there exist subsequence $\{p_{m'}\}$ of $\{p_m\}$ and a function p such that

$$\begin{gathered}\nabla p \in L^\infty\big(0,T;L^{1/s'}_{uloc}(\mathbb{R}^{n+1}_+)\big),\\ \int_0^T\int_{\mathbb{R}^{n+1}_+} v\cdot\nabla p_{m'}\,dx\,d\tau \to \int_0^T\int_{\mathbb{R}^{n+1}_+} v\cdot\nabla p\,dx\,d\tau \quad \text{as } m'\to\infty\end{gathered} \tag{A.52}$$

for all $v \in (C_0^\infty((0,T)\times\mathbb{R}^{n+1}_+))^{n+1}$. Therefore, from (A.16), (A.48), (A.50), and (A.52), letting $m'\to\infty$, we have $\partial_t u - \Delta u + \vec{b} + \nabla p = f + \vec{b}$ in $\mathcal{D}'$, and, consequently,

$$\partial_t u - \Delta u + \nabla p = f \quad \text{in } \mathcal{D}'\big(\mathbb{R}^{n+1}_+\times(0,\infty)\big). \tag{A.53}$$

Step 2. Next, the general case $f \in C^\alpha([0,T]; B^{-s}_{\infty,q,+,\sigma})$ is considered. Since $C^1_\sigma(\mathbb{R}^{n+1}_+)$ is dense in $B^{-s}_{\infty,q,+,\sigma}$, there exists a sequence $f_m \in C^\alpha([0,T]; C^1_\sigma(\mathbb{R}^{n+1}_+))$, such that $f_m \to f$ in $C([0,T]; B^{-s}_{\infty,q,+,\sigma})$. Let $u_m = \int_0^t e^{-(t-\tau)A} f_m(\tau)\,d\tau$. Then, u_m satisfies (A.53), as shown in Step 1. Letting $m \to \infty$, by an argument similar to Step 1, $u(t) = \int_0^t e^{-(t-\tau)A} f(\tau)\,d\tau$ also satisfies (A.53). This proves Lemma A.3. □

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