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Global existence of strong solution for the Cucker–Smale–Navier–Stokes system

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Abstract

We present a global existence theory for strong solution to the Cucker–Smale–Navier–Stokes system in a periodic domain, when initial data is sufficiently small. To model interactions between flocking particles and an incompressible viscous fluid, we couple the kinetic Cucker–Smale model and the incompressible Navier–Stokes system using a drag force mechanism that is responsible for the global flocking between particles and fluids. We also revisit the emergence of time-asymptotic flocking via new functionals measuring local variances of particles and fluid around their local averages and the distance between local averages velocities. We show that the particle and fluid velocities are asymptotically aligned to the common velocity, when the viscosity of the incompressible fluid is sufficiently large compared to the sup-norm of the particles' local mass density.

Keywords: Cucker–Smale flocking particle; Compressible Navier–Stokes equations; Kinetic Cucker–Smale model; Strong solution; Asymptotic flocking behavior

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1. Introduction

We are concerned with the time-asymptotic interactions between Cucker–Smale (in short C–S) particles and incompressible viscous fluid, which can be effectively modeled by the coupled system of kinetic C–S model and incompressible Navier–Stokes equations. Consider a situation where many C–S particles are scattered inside a highly viscous incompressible fluid in a periodic spatial domain $\mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3 = [0,1]^3$. In this case, the dynamics of the particles and fluid can be described by the coupled system of a Vlasov type equation and the incompressible Navier–Stokes equations. More precisely, let $f = f(x,\xi,t)$ be the one-particle distribution function of the C–S particles with velocity $\xi \in \mathbb{R}^3$ at position $x \in \mathbb{T}^3$ at time $t > 0$, and $u = u(x,t)$ be the bulk velocity of the incompressible fluid. The coupled dynamics of (f,u) is then governed by the Cucker–Smale–Navier–Stokes (in short CS–NS) system [2]:

$$\begin{aligned} &\partial_t f + \nabla_x \cdot (\xi f) + \nabla_\xi \cdot \big(\big(F_a[f] + F_d\big) f\big) = 0, \quad (x,\xi) \in \mathbb{T}^3 \times \mathbb{R}^3,\ t > 0, \\ &\partial_t u + (u \cdot \nabla_x) u + \nabla_x p - \mu \Delta_x u = -\int_{\mathbb{R}^3} F_d f \, d\xi, \quad \nabla_x \cdot u = 0, \end{aligned} \tag{1.1}$$

subject to initial data:

$$(f,u)|_{t=0} = (f_0, u_0), \tag{1.2}$$

which is requested to satisfy

$$\begin{aligned} &(i) \quad \int_{\mathbb{T}^3 \times \mathbb{R}^3} f_0 \, d\xi \, dx = 1, \\ &(ii) \quad \operatorname{supp}_\xi f_0 \text{ is bounded in } \mathbb{R}^3 \text{ for each } x \in \mathbb{T}^3. \end{aligned} \tag{1.3}$$

Here F_a and F_d represent the alignment (flocking) force and the drag force per unit mass, respectively:

$$\begin{aligned} F_a[f](x,\xi,t) &:= \int_{\mathbb{T}^3 \times \mathbb{R}^3} \psi(x,y)(\xi_* - \xi) f(y,\xi_*,t) \, d\xi_* \, dy, \\ F_d(x,\xi,t) &:= u(x,t) - \xi. \end{aligned} \tag{1.4}$$

The kernel function $\psi : \mathbb{T}^3 \times \mathbb{T}^3 \to \mathbb{R}_+$ is a $\mathcal{C}^1$-function that satisfies the following conditions of symmetry and boundedness:

$$\begin{aligned} &(i) \quad \psi(x,y) = \psi(y,x). \\ &(ii) \quad m_\psi \le \psi(x,y) \le M_\psi, \quad M_\psi - m_\psi < 1, \end{aligned} \tag{1.5}$$

where e_i is the standard unit vector, m_ψ and M_ψ are positive constants.

Note that the first equation in (1.1) is the kinetic Cucker–Smale equation derived from the particle C–S model using the BBGKY hierarchy in [11]. This kinetic equation is coupled via a drag force which is the main mechanism responsible for the flocking between particles and fluids. For a detailed description of the modeling and related literature, we refer readers to the authors' paper [2] and the references therein. The global existence of a weak solution to the system (1.1) was studied in [2] using Boudin et al.'s framework for dilute spray dynamics, and a priori flocking estimate was also investigated in the class of smooth solutions. In contrast, the existence of more regular solutions, i.e., strong and classical solutions to the system (1.1) has not yet been investigated, which motivated this work. For other related particle-fluid interactions, we refer to [1,3–10,12–14].

The main results of this paper are two-fold. First, we present the global existence and uniqueness of small strong solution to the system (1.1) for a suitable class of small and regular initial data depending on the existence time interval. By increasing the regularity of the initial data, we can obtain classical solutions. Second, we revisit the asymptotic flocking problem via new Lyapunov functionals that measure the local velocity variances and the distance between local velocity averages of the particles and fluid. From the time-evolution estimates of these functionals, we show that the local fluctuations from the local averaged velocities of the particles and fluid tend to zero at least exponentially.

The rest of the paper is organized as follows. In Section 2, we present several simplified notations and state our main results. In Section 3, we present the global existence of strong solutions to the CS–NS system in a periodic domain. We study an a priori estimate for asymptotic flocking in Section 4, and finally give a summary of the main results in Section 5.

2. Statements of the main results

In this section, we present several simplified notations and discuss the main results without proofs. Detailed proofs will be presented in Sections 3 and 4.

Throughout the paper, we set $T \in (0,\infty)$ to be a positive constant, and let C be a generic constant that may differ from line to line, and is not dependent on T. For a given $t \in [0,T)$, the set $P(t)$ and $\eta(t)$ denote the particle velocity support and its size at time t:

$$\begin{aligned} &P(t) := \big\{\xi \in \mathbb{R}^3 \ : \ \exists (x,\xi) \in \mathbb{T}^3 \times \mathbb{R}^3 \text{ such that } f(x,\xi,t) \neq 0\big\}, \\ &\eta(t) := \max\big\{|\xi| \ : \ \xi \in P(t)\big\}. \end{aligned} \tag{2.1}$$

We also introduce some simplified notations. Throughout the paper, we denote $M_p(f(t))$ to be the p-th velocity moments of the kinetic density f and set several norms:

$$M_p\big(f(t)\big) := \int_{\mathbb{T}^3\times\mathbb{R}^3} |\xi|^p f \, d\xi\, dx, \quad p \in \mathbb{Z}_+ \cup \{0\},$$

$$\big\|u(t)\big\|_{L^p} := \big\|u(t)\big\|_{L^p(\mathbb{T}^3)}, \qquad \|u\|_{L^r(0,T;L^p)} := \|u\|_{L^r(0,T;L^p(\mathbb{T}^3))}.$$

We are now ready to state our main results. Our first result concerns the global existence and uniqueness of strong solution in any finite-time interval $[0,T)$.

Theorem 2.1. *Suppose that $T \in (0,\infty)$ and (f_0,u_0) satisfies (1.3) and an extra assumption:*

$$\exists \varepsilon > 0 \quad \textit{such that} \quad \|f_0\|_{W^{1,\infty}(\mathbb{T}^3\times\mathbb{R}^3)} + \|u_0\|_{H^2(\mathbb{T}^3)} < \varepsilon.$$

Then the system (1.1)–(1.5) has a unique strong solution (f, u) in the time interval $[0, T)$ satisfying the following estimates:

$$
\begin{aligned}
&(i) \quad f \in W^{1,\infty}\big(\mathbb{T}^3 \times \mathbb{R}^3 \times [0, T)\big),\\
&(ii) \quad u \in C^0\big(0, T; H^2\big(\mathbb{T}^3\big)\big) \cap L^2\big(0, T; H^3\big(\mathbb{T}^3\big)\big) \cap H^1\big(0, T; H^1\big(\mathbb{T}^3\big)\big),\\
&(iii) \quad p \in L^\infty\big(0, T; H^1\big(\mathbb{T}^3\big)\big),\\
&(iv) \quad (f, u) \text{ satisfies the system (1.1)–(1.5) in the distributional sense.}
\end{aligned}
$$

Remark 2.1. 1. In the proof of Theorem 2.1, the size of ε is the order of

$$
\varepsilon e^{\mathcal{X} T} = \mathcal{O}(1), \quad \varepsilon \ll \mu^{\frac{9}{2}},
$$

where $\mathcal{X}$ is a big positive constant depending on $P(0)$ and ψ, but is independent of ε and T.

2. For given $T \in (0, \infty)$, we assume that the initial data (f_0, u_0) satisfies

$$
\|f_0\|_{W^{2,\infty}(\mathbb{T}^3 \times \mathbb{R}^3)} + \|u_0\|_{H^3(\mathbb{T}^3)} \ll 1,
$$

Then the same arguments as in Theorem 2.1 yield a strong solution with the regularity:

$$
\begin{aligned}
&f \in W^{2,\infty}\big(\mathbb{T}^3 \times \mathbb{R}^3 \times [0, T)\big),\\
&u \in C\big(0, T; H^3\big(\mathbb{T}^3\big)\big) \cap L^2\big(0, T; H^4\big(\mathbb{T}^3\big)\big) \cap H^1\big(0, T; H^2\big(\mathbb{T}^3\big)\big).
\end{aligned}
$$

In fact, (f, u) is a classical solution according to the Sobolev embedding theorem:

$$
f \in C^1\big(\mathbb{T}^3 \times \mathbb{R}^3 \times [0, T)\big), \qquad u \in L^2\big(0, T; C^2\big(\mathbb{T}^3\big)\big) \cap C^1\big(0, T; L^2\big(\mathbb{T}^3\big)\big).
$$

The second result is concerned with the asymptotic flocking estimate for the system (1.1). For this, we introduce a Lyapunov functional $\mathcal{E}$ measuring the local velocity fluctuations and the distance between local velocity averages:

$$
\mathcal{E}(t) := 2 \int_{\mathbb{T}^3 \times \mathbb{R}^3} \big|\xi - \xi_c(t)\big|^2 f(t) d\xi dx + 2 \int_{\mathbb{T}^3} \big|u(t) - u_c(t)\big|^2 dx + \big|u_c(t) - \xi_c(t)\big|^2,
$$

where u_c and ξ_c are local velocity averages of the fluid and particles:

$$
u_c(t) := \int_{\mathbb{T}^3} u dx \quad \text{and} \quad \xi_c(t) := \frac{\int_{\mathbb{T}^3 \times \mathbb{R}^3} \xi f d\xi dx}{\int_{\mathbb{T}^3 \times \mathbb{R}^3} f d\xi dx}. \tag{2.2}
$$

Theorem 2.2. *Suppose that T, (f_0, u_0) and μ satisfy*

$$
T \in (0, \infty], \qquad \mathcal{E}(0) < \infty, \qquad \mu > \frac{\sup_{0 \le t \le \infty} \|\rho_p(t)\|_{L^\infty}}{\pi_3}. \tag{2.3}
$$

Then, for any classical solution (f, u) in the time-interval $[0, T)$ to the system (1.1)–(1.2), the following estimate of exponential convergence holds:

$$\mathcal{E}(t) \le \mathcal{E}(0) \exp\bigl(-\min\{2m_\psi + 1, K, 2\}t\bigr), \quad t \in [0, T),$$

where K is a positive constant given by $K := 2\mu\pi_3 - 2\sup_{0\le t\le\infty} \|\rho_p(t)\|_{L^\infty} > 0$, π_3 is a positive constant appearing in Poincaré's inequality for the torus $\mathbb{T}^3$ and ρ_p is the local particle density:

$$\rho_p(x, t) := \int_{\mathbb{R}^3} f(x, \xi, t)d\xi. \tag{2.4}$$

Remark 2.2. Note that our condition (2.3) depends on the sup-norm of the local particle density ρ_p which should be determined by the solution of the system (1.1). Thus the theorem has a natural a priori setting. However, this a priori condition is not that severe. For example, the classical solutions in Remark 2.1 have a uniform bound for the local particle density (see Lemma 3.1) for $t \in [0, T]$:

$$\sup_{0\le t<T} \bigl\| f(t) \bigr\|_{W^{2,\infty}} < \varepsilon^{\frac{2}{3}}, \qquad \sup_{0\le t<T} \eta(t) < \eta^\infty.$$

Thus the local mass density for the particles is uniformly bounded in the existence time-interval:

$$\sup_{0\le t<T} \bigl\|\rho_p(t)\bigr\| \le \varepsilon^{\frac{2}{3}} \sup_{0\le t<T} \eta(t) < \varepsilon^{\frac{2}{3}}\eta^\infty.$$

3. Global existence of strong solutions to the CS–NS system

In this section, we discuss the global existence of strong solutions to the CS–NS system (1.1)–(1.5). As this section is rather lengthy, we first briefly outline its layout, and then present all necessary estimates. The proof of Theorem 2.1 is divided into three steps.

- In Step A, we present approximate solutions (f^n, u^n) which are solutions of the linearized system by freezing the nonlinear coefficients by substituting the previous iterates. This iteration scheme is different from that for weak solutions in [2], where we used a regularized system by mollifying the bulk velocity u in the nonlinear convection term, i.e., $(\theta_\varepsilon * u) \cdot \nabla u$ instead of $(u \cdot \nabla)u$.
- In Step B, we show that (f^n) and (u^n) are Cauchy sequences in $L^\infty(\mathbb{T}^3 \times \mathbb{R}^3 \times [0, T])$ and $L^\infty(0, T; H^1(\mathbb{T}^3))$, respectively.
- In Step C, we show that the limit functions (f, u) of the Cauchy sequences (f^n) and (u^n) have the regularity desired for Theorem 2.1.

3.1. Construction of approximate solutions

We describe our iteration scheme as follows. For the first iterate (f^0, u^0), we set

$$\bigl(f^0, u^0\bigr) := (f_0, 0).$$

Note that if we take $u^0 = u_0$, which belongs to $H^2(\mathbb{T}^3)$, then u^0 does not satisfy the required regularity H^3 of iterations in Step A of Lemma 3.1. This is why we simply take u^0 to be zero. Of course, in this case, the zeroth-iterate (f^0, u^0) does not satisfy the given initial data, but this is O.K, because iterates (f^n, u^n), $n \geq 1$ described below will take the same initial data (f_0, u_0) at time $t = 0$, and our strong solution will be the limit of (f^n, u^n).

Suppose that the n-th iterate (f^n, u^n) is constructed. Then the $(n+1)$-th iterate (f^{n+1}, u^{n+1}) is defined as the unique solution of the following linear system:

$$\begin{aligned}
&\partial_t f^{n+1} + \nabla_x \cdot \left(\xi f^{n+1}\right) + \nabla_\xi \cdot \left[F\left(f^n, u^n\right) f^{n+1}\right] = 0, \quad (x, \xi) \in \mathbb{T}^3 \times \mathbb{R}^3,\ t > 0,\\
&\partial_t u^{n+1} + \left(u^n \cdot \nabla_x\right) u^{n+1} + \nabla_x p^{n+1} - \mu \Delta_x u^{n+1} = -\int_{\mathbb{R}^3} \left(u^n - \xi\right) f^{n+1} d\xi,\\
&\nabla_x \cdot u^{n+1} = 0,
\end{aligned} \tag{3.1}$$

subject to the initial data:

$$\left(f^{n+1}, u^{n+1}\right)\big|_{t=0} = (f_0, u_0), \quad (x, \xi) \in \mathbb{T}^3 \times \mathbb{R}^3,$$

where the forcing term $F(f^n, u^n)$ is defined as follows:

$$\begin{aligned}
F\left(f^n, u^n\right) &= F_a\left(f^n\right) + u^n - \xi\\
&= \int_{\mathbb{T}^3 \times \mathbb{R}^3} \psi(x, y)(\xi_* - \xi) f^n(y, \xi_*, t) d\xi_* dy + u^n - \xi.
\end{aligned}$$

Note that the equation for f^{n+1} is in a divergent form, so under a suitable decay condition for ξ, the total mass is conserved. Hence, without loss of generality, we set

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} f^n(x, \xi, t) d\xi dx = 1, \quad n \geq 0.$$

Next, we present several a priori estimates for the uniform boundedness of the velocity support in f^n and an H^2-estimate of u^n using the energy estimates.

For a given $(x, \xi) \in \mathbb{T}^3 \times \mathbb{R}^3$ at time t, we define a particle trajectory $(x^{n+1}(s), \xi^{n+1}(s)) := (x^{n+1}(s; t, x, \xi), \xi^{n+1}(s; t, x, \xi))$ to be the solution of the following ODEs:

$$\begin{aligned}
&\frac{d}{ds} x^{n+1}(s) = \xi^{n+1}(s),\\
&\frac{d}{ds} \xi^{n+1}(s) = F_a\left(f^n\right)\left(x^{n+1}(s), \xi^{n+1}(s), s\right) + u^n\left(x^{n+1}(s), s\right) - \xi^{n+1}(s),
\end{aligned} \tag{3.2}$$

subject to the initial data:

$$x^{n+1}(t) = x, \qquad \xi^{n+1}(t) = \xi.$$

In the following lemma, we study the uniform boundedness of the velocity support and energy of particles (second velocity moment of kinetic density) under the a priori assumption of the fluid velocity.

Lemma 3.1. *Let (f^k, u^k) be the solution to the system (3.1) with initial data (f_0, u_0) satisfying*

$$M_2(f_0) < \infty, \qquad \xi^k(0) \in P(0) \quad \text{for all } k \geq 1. \tag{3.3}$$

If there exists a positive constant $U^\infty > 0$ such that

$$\sup_{0\leq k\leq n} \left\| u^k \right\|_{L^\infty(\mathbb{T}^3\times[0,T))} \leq U^\infty.$$

Then, we have

$$\sup_{0\leq s\leq T} \max_{1\leq k\leq n+1} \left|\xi^k(s)\right| \leq \eta^\infty, \qquad \sup_{0\leq s\leq T} \max_{0\leq k\leq n+1} M_2\big(f^k(s)\big) \leq M_2^\infty,$$

where $P(0)$ is the initial velocity support defined by (2.1), η_0 and η^∞ are positive constants defined as follows.

$$M_2^\infty := \left[\max\left\{\sqrt{M_2(f_0)}, \frac{U^\infty}{m_\psi+1} + \frac{M_\psi\sqrt{M_2(f_0)}}{m_\psi+1}\right\}\right]^2 \left(\frac{m_\psi+1}{m_\psi+1-M_\psi}\right)^2,$$

$$\eta^\infty := 3\left(\eta_0 + \frac{M_\psi\sqrt{M_2^\infty}+U^\infty}{m_\psi+1}\right), \qquad \eta_0 := \max\big\{|\xi| \;:\; \xi \in P(0)\big\}.$$

Proof. We prove the desired estimate by the method of induction.

• Step A (initial step $n = 0$): In this case, we need to show that

$$\sup_{0\leq s\leq T} \left|\xi^1(s)\right| \leq \eta^\infty, \qquad \sup_{0\leq s\leq T} \max\big\{M_2\big(f^0(s)\big), M_2\big(f^1(s)\big)\big\} \leq M_2^\infty.$$

◇ Step A.1: We first note that for $i = 1, 2, 3$,

$$\begin{aligned}
&(F_a)_i\big(f^0\big)\big(x^1(s), \xi^1(s), s\big)\,\mathrm{sgn}\big(\xi_i^1(s)\big) \\
&\quad = \left(\int_{\mathbb{T}^3\times\mathbb{R}^3} \psi\big(x^1, y\big)\big(\xi_{*i} - \xi_i^1\big) f^0(y, \xi_*, s)\,d\xi_* dy\right)\mathrm{sgn}\big(\xi_i^1(s)\big) \\
&\quad \leq M_\psi\sqrt{M_2\big(f^0\big)} - m_\psi\left|\xi_i^1\right| \leq M_\psi\sqrt{M_2^\infty} - m_\psi\left|\xi_i^1\right|,
\end{aligned}$$

where we used the following relation:

$$M_2\big(f^0\big) = M_2(f_0) \leq M_2^\infty. \tag{3.4}$$

On the other hand, by our construction of $u^0 \equiv 0$,

$$
\begin{aligned}
\frac{d}{ds}\left|\xi_i^1(s)\right| &= \left[(F_a)_i\left(f^0\right)\left(x^1(s), \xi^1(s), s\right) + u_i^0\left(x^1(s), s\right) - \xi_i^1(s)\right]\operatorname{sgn}\left(\xi_i^1(s)\right) \\
&\le (F_a)_i\left(f^0\right)\left(x^1(s), \xi^1(s), s\right)\operatorname{sgn}\left(\xi_i^1(s)\right) + \left\|u^0\right\|_{L^\infty([0,T]\times\mathbb{T}^3)} - \left|\xi_i^1(s)\right| \\
&\le M_\psi\sqrt{M_2\left(f^0(s)\right)} + \left\|u^0\right\|_{L^\infty([0,T]\times\mathbb{T}^3)} - (m_\psi + 1)\left|\xi_i^1(s)\right| \\
&\le M_\psi\sqrt{M_2^\infty} + U^\infty - (m_\psi + 1)\left|\xi_i^1(s)\right|.
\end{aligned}
$$

Gronwall's lemma implies that:

$$
\begin{aligned}
\left|\xi_i^1(t)\right| &\le \left|\xi_i^1(0)\right|e^{-(m_\psi+1)t} + \frac{M_\psi\sqrt{M_2^\infty} + U^\infty}{m_\psi + 1}\left(1 - e^{-(m_\psi+1)t}\right) \\
&\le \left|\xi_i^1(0)\right| + \frac{M_\psi\sqrt{M_2^\infty} + U^\infty}{m_\psi + 1} \\
&\le \eta_0 + \frac{M_\psi\sqrt{M_2^\infty} + U^\infty}{m_\psi + 1} = \frac{\eta^\infty}{3}.
\end{aligned}
$$

Thus, we have

$$
\left|\xi^1(t)\right| \le \sum_{i=1}^{3}\left|\xi_i^1(t)\right| \le \eta^\infty. \tag{3.5}
$$

Therefore, it follows from (3.5) that

$$
\sup_{0\le s\le T}\left|\xi^1(s)\right| \le \eta^\infty.
$$

◇ Step A.2: We multiply the transport equation for f^1 by $|\xi|^2$ to obtain

$$
\partial_t\left(|\xi|^2 f^1\right) + \nabla_x\cdot\left(|\xi|^2\xi f^1\right) + \nabla_\xi\cdot\left[|\xi|^2 F\left(f^0, u^0\right) f^1\right] = 2\xi\cdot F\left(f^0, u^0\right) f^1.
$$

We can integrate the above relation using the spatial periodicity of f^1 and its compact support in ξ-variable (see (3.5)) to get:

$$
\begin{aligned}
\frac{d}{dt}M_2\left(f^1\right) &= 2\int_{\mathbb{T}^3\times\mathbb{R}^3} \xi\cdot F_a\left(f^0\right) f^1\,d\xi\,dx + 2\int_{\mathbb{T}^3\times\mathbb{R}^3} \xi\cdot\left(u^0 - \xi\right) f^1\,d\xi\,dx \\
&:= \mathcal{I}_{11} + \mathcal{I}_{12}.
\end{aligned} \tag{3.6}
$$

• (Estimate of $\mathcal{I}_{11}$): From the definition of F_a in (1.4) and $M_1(f^1) \le \sqrt{M_2(f^1)}$, we have

$$\begin{aligned}
\mathcal{I}_{11} &= 2 \int\limits_{\mathbb{T}^3\times\mathbb{R}^3} \Bigg(\int\limits_{\mathbb{T}^3\times\mathbb{R}^3} \psi(x,y)\xi\cdot(\xi_* - \xi) f^0(y,\xi_*) f^1(x,\xi)\, d\xi_* dy \Bigg) d\xi dx \\
&= 2 \int\limits_{\mathbb{T}^3\times\mathbb{R}^3} \Bigg(\int\limits_{\mathbb{T}^3\times\mathbb{R}^3} \psi(x,y)\xi\cdot\xi_* f^0(y,\xi_*) f^1(x,\xi)\, d\xi_* dy \Bigg) d\xi dx \\
&\quad - 2 \int\limits_{\mathbb{T}^3\times\mathbb{R}^3} \Bigg(\int\limits_{\mathbb{T}^3\times\mathbb{R}^3} \psi(x,y)|\xi|^2 f^0(y,\xi_*) f^1(x,\xi)\, d\xi_* dy \Bigg) d\xi dx \\
&\le 2M_\psi \sqrt{M_2(f_0)}\sqrt{M_2\big(f^1\big)} - 2m_\psi M_2\big(f^1\big). \qquad (3.7)
\end{aligned}$$

• (Estimate of $\mathcal{I}_{12}$): We use the assumption $\|u^0\|_{L^\infty(\mathbb{T}^3\times[0,T))} = 0 < U^\infty$ to get

$$\mathcal{I}_{12} \le 2U^\infty M_1\big(f^1\big) - 2M_2\big(f^1\big) \le 2U^\infty \sqrt{M_2\big(f^1\big)} - 2M_2\big(f^1\big). \qquad (3.8)$$

Combining (3.7) and (3.8) in (3.6), we find that

$$\frac{d}{dt} M_2\big(f^1\big) \le 2\big(M_\psi \sqrt{M_2(f_0)} + U^\infty\big)\sqrt{M_2\big(f^1\big)} - 2(m_\psi + 1) M_2\big(f^1\big).$$

We then set:

$$Y := \sqrt{M_2\big(f^1\big)}, \quad \text{i.e., } Y^2 = M_2\big(f^1\big).$$

Then we have

$$\frac{dY}{dt} + (m_\psi + 1)Y \le \big(M_\psi \sqrt{M_2(f_0)} + U^\infty\big),$$

which yields:

$$\begin{aligned}
\sqrt{M_2\big(f^1\big)} &\le \sqrt{M_2(f_0)} e^{-(m_\psi+1)t} + \int\limits_0^t e^{-(m_\psi+1)(t-s)} \big(M_\psi \sqrt{M_2(f_0)} + U^\infty\big) ds \\
&= \frac{U^\infty}{m_\psi + 1} + \left(\sqrt{M_2(f_0)} - \frac{U^\infty}{m_\psi+1} \right) e^{-(m_\psi+1)t} + M_\psi \int\limits_0^t e^{-(m_\psi+1)(t-s)} \sqrt{M_2(f_0)} ds \\
&= \frac{U^\infty}{m_\psi + 1} + \frac{M_\psi \sqrt{M_2(f_0)}}{m_\psi+1} + \left(\sqrt{M_2(f_0)} - \frac{U^\infty}{m_\psi+1} - \frac{M_\psi \sqrt{M_2(f_0)}}{m_\psi+1} \right) e^{-(m_\psi+1)t} \\
&\le \max\left\{ \sqrt{M_2(f_0)}, \frac{U^\infty}{m_\psi+1} + \frac{M_\psi \sqrt{M_2(f_0)}}{m_\psi+1} \right\} \\
&\le \sqrt{M_2^\infty}. \qquad (3.9)
\end{aligned}$$

We now combine (3.4) and (3.9) to find

$$\sup_{0\le s\le T} \max\{M_2(f^0), M_2(f^1)\} \le M_2^\infty.$$

• Step B (inductive step): Assume that the result is true for $n = N+1$. We now show that the result holds for $n = N+2$. Suppose that there exists a positive constant $U^\infty > 0$ such that

$$\sup_{0\le k\le N+1} \|u^k\|_{L^\infty(\mathbb{T}^3\times[0,T))} \le U^\infty.$$

Then because

$$\sup_{0\le k\le N} \|u^k\|_{L^\infty(\mathbb{T}^3\times[0,T))} \le \sup_{0\le k\le N+1} \|u^k\|_{L^\infty(\mathbb{T}^3\times[0,T))} \le U^\infty,$$

by our induction hypothesis, we have

$$\sup_{0\le s\le T} \max_{0\le k\le N+1} |\xi^k(s)| \le \eta^\infty, \qquad \sup_{0\le s\le T} \max_{0\le k\le N+1} M_2(f^k(s)) \le M_2^\infty.$$

To derive the desired result, we must show that

$$\sup_{0\le s\le T} |\xi^{N+2}(s)| \le \eta^\infty, \qquad \sup_{0\le s\le T} M_2(f^{N+2}(s)) \le M_2^\infty. \tag{3.10}$$

◇ Step B.1: By the same analysis as in Step A.1, we have

$$\sup_{0\le s\le T} |\xi^{N+2}(s)| \le \eta^\infty. \tag{3.11}$$

◇ Step B.2: By the same analysis as Step A.2, we have

$$\frac{d}{dt} M_2(f^{N+2}) \le 2\big(M_\psi \sqrt{M_2(f^{N+1})} + U^\infty\big)\sqrt{M_2(f^{N+2})} - 2(m_\psi+1) M_2(f^{N+2}).$$

This yields

$$\begin{aligned}
\sqrt{M_2(f^{N+2})} &\le \sqrt{M_2(f_0)} e^{-(m_\psi+1)t} + \int_0^t e^{-(m_\psi+1)(t-s)} \big(M_\psi \sqrt{M_2(f^{N+1})} + U^\infty\big) ds \\
&= \frac{U^\infty}{m_\psi+1} + \left(\sqrt{M_2(f_0)} - \frac{U^\infty}{m_\psi+1}\right) e^{-(m_\psi+1)t} \\
&\quad + M_\psi \int_0^t e^{-(m_\psi+1)(t-s)} \sqrt{M_2(f^{N+1})} ds.
\end{aligned} \tag{3.12}$$

We then set:

$$S_{n+1}(s) := \max_{0 \le k \le n+1} \sqrt{M_2\big(f^k(s)\big)}.$$

In (3.12), we have

$$\begin{aligned} S_{N+2} &\le \frac{U^\infty}{m_\psi + 1} + \left(\sqrt{M_2(f_0)} - \frac{U^\infty}{m_\psi + 1}\right) e^{-(m_\psi+1)t} + \frac{M_\psi S_{N+1}}{m_\psi + 1}\big(1 - e^{-(m_\psi+1)t}\big) \\ &\le \frac{U^\infty}{m_\psi + 1} + \left(\sqrt{M_2(f_0)} - \frac{U^\infty}{m_\psi + 1}\right) e^{-(m_\psi+1)t} + \frac{M_\psi S_{N+2}}{m_\psi + 1}, \end{aligned} \tag{3.13}$$

where we used $S_{N+1} \le S_{N+2}$.

Then the relation (3.13) yields

$$S_{N+2} \le \max\left\{\sqrt{M_2(f_0)}, \frac{U^\infty}{m_\psi + 1} + \frac{M_\psi \sqrt{M_2(f_0)}}{m_\psi + 1}\right\} \frac{m_\psi + 1}{m_\psi + 1 - M_\psi} = \sqrt{M_2^\infty}. \tag{3.14}$$

Finally, we combine (3.11) and (3.14) to show that (3.10) holds. □

Remark 3.1. Let $P^n(t)$ denote the n-th particle velocity support at time t, i.e.,

$$P^n(t) := \big\{\xi \in \mathbb{R}^3 \ : \ \exists (x, \xi) \in \mathbb{T}^3 \times \mathbb{R}^3 \text{ such that } f^n(x, \xi, t) \ne 0\big\}$$

Then, Lemma 3.1 implies that

$$\sup_{0 \le s \le T} \max_{1 \le k \le n+1} \big|\eta^k(s)\big| \le \eta^\infty,$$

where η^k is defined as follows:

$$\eta^k(s) := \max\big\{|\xi| \ : \ \xi \in P^k(s)\big\}.$$

In the following lemma, we study the H^2-energy estimate of the approximate solutions.

Lemma 3.2. *Let (f^n, u^n) be the sequence of approximate solutions with initial data (f_0, u_0) satisfying (3.3). If there exists a positive constant $\bar{U}^\infty > 0$ such that*

$$\max_{1 \le k \le n} \|u^k\|_{L^\infty(0,T;H^2)} \le \bar{U}^\infty,$$

then for $t \in [0, T)$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u^{n+1}\|_{H^2}^2 + \mu\|\nabla u^{n+1}\|_{H^2}^2 \\ &\quad \le C\big(\|u^n\|_{H^2}\|\nabla u^{n+1}\|_{H^2}^2 + \|f^{n+1}\|_{W^{1,\infty}}\big(\|u^n\|_{H^2} + 1\big)\|u^{n+1}\|_{H^3}\big), \end{aligned}$$

where C is a generic positive constant that depends on $\bar{U}^\infty$ and the initial data, but is independent of t and n.

Proof. Before we proceed with the H^2-estimates, note that Sobolev's inequality implies

$$\max_{1\le k\le n}\|u^k\|_{L^\infty(\mathbb{T}^3\times[0,T))}\le C\max_{1\le k\le n}\|u^k\|_{L^\infty(0,T;H^2)}\le C\bar{U}^\infty,\quad 0\le k\le n.$$

We then apply Lemma 3.1 and Remark 3.1 with $U^\infty=C\bar{U}^\infty$ to get

$$\sup_{0\le s\le T}\max_{0\le k\le n+1}\left|\eta^k(s)\right|\le\eta^\infty,\qquad \sup_{0\le s\le T}\max_{0\le k\le n+1}M_2\left(f^k(s)\right)\le M_2^\infty, \tag{3.15}$$

where M_2^∞ and η^∞ are positive constants appearing in Lemma 3.1.

• Case A (Zeroth-order estimate): It follows from $(3.1)_2$ that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u^{n+1}\|_{L^2}^2+\mu\|\nabla u^{n+1}\|_{L^2}^2\\&\quad=-\int_{\mathbb{T}^3}(u^n\cdot\nabla)u^{n+1}\cdot u^{n+1}dx-\int_{\mathbb{T}^3\times\mathbb{R}^3}u^{n+1}\cdot\left(u^n-\xi\right)f^{n+1}d\xi dx\\&\quad=:\mathcal{I}_{21}-3\mathcal{I}_{22}.r\end{aligned} \tag{3.16}$$

The terms on the right-hand-side can be estimated as follows.

◇ (Estimate of $\mathcal{I}_{21}$): Since $\nabla\cdot u^k=0$ for all $k\ge 0$, we have

$$\mathcal{I}_{21}=-\frac{1}{2}\int_{\mathbb{T}^3}u^n\cdot\nabla\left|u^{n+1}\right|^2dx=\frac{1}{2}\int_{\mathbb{T}^3}\left(\nabla\cdot u^n\right)\left|u^{n+1}\right|^2dx=0. \tag{3.17}$$

◇ (Estimate of $\mathcal{I}_{22}$): We will show that

$$\mathcal{I}_{22}\le\max\left\{\left(\eta^\infty\right)^3,\left(\eta^\infty\right)^{3/2}\sqrt{M_2^\infty}\right\}\|f^{n+1}\|_{L^\infty}\left(\|u^n\|_{L^2}+1\right)\|u^{n+1}\|_{L^2}. \tag{3.18}$$

First, note that

$$\begin{aligned}\mathcal{I}_{22}&=\int_{\mathbb{T}^3\times\mathbb{R}^3}u^{n+1}\cdot\left(u^n-\xi\right)f^{n+1}d\xi dx\\&=\int_{\mathbb{T}^3}u^{n+1}\cdot u^n\left(\int_{P^{n+1}(t)}f^{n+1}d\xi\right)dx-\int_{P^{n+1}(t)}\left(\int_{\mathbb{T}^3}u^{n+1}\cdot\xi f^{n+1}d\xi\right)dx.\end{aligned} \tag{3.19}$$

On the other hand, we have

$$\int_{P^{n+1}(t)} f^{n+1} d\xi \le \|f^{n+1}\|_{L^\infty} (\eta^\infty)^3,$$

$$\begin{aligned}\Bigg| \int_{P^{n+1}(t)} \Big(\int_{\mathbb{T}^3} u^{n+1} \cdot \xi f^{n+1} d\xi \Big) dx \Bigg| &\le \int_{P^{n+1}(t)} \Big(\int_{\mathbb{T}^3} |u^{n+1}| |\xi| f^{n+1} dx \Big) d\xi \\ &\le \|u^{n+1}\|_{L^2} \|f^{n+1}\|_{L^\infty} \int_{P^{n+1}(t)} \Big(\int_{\mathbb{T}^3} |\xi|^2 f^{n+1} dx \Big)^{1/2} d\xi \\ &\le \|u^{n+1}\|_{L^2} \|f^{n+1}\|_{L^\infty} (\eta^{n+1}(t))^{1/2} \sqrt{M_2(f^{n+1}(s))} \\ &\le (\eta^\infty)^{3/2} \sqrt{M_2^\infty} \|u^{n+1}\|_{L^2} \|f^{n+1}\|_{L^\infty},\end{aligned}$$

where we used particle energy bounds given by (3.15). We can combine the above estimates with (3.19) to get (3.18). Finally, in (3.16), we combine (3.17) and (3.18) to derive a zeroth-order energy estimate:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u^{n+1}\|_{L^2}^2 + \mu \|\nabla u^{n+1}\|_{L^2}^2 \\ &\quad \le \max\Big\{ (\eta^\infty)^3, (\eta^\infty)^{3/2}\sqrt{M_2^\infty} \Big\} \|f^{n+1}\|_{L^\infty} (\|u^n\|_{L^2} + 1) \|u^{n+1}\|_{L^2} \end{aligned} \tag{3.20}$$

• Case B (High-order estimates): For each multi-index α with $|\alpha| = 1, 2$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha u^{n+1}\|_{L^2}^2 + \mu \|\nabla \partial_x^\alpha u^{n+1}\|_{L^2}^2 \\ &= -\int_{\mathbb{T}^3} \partial_x^\alpha u^{n+1} \cdot \partial_x^\alpha (u^n \cdot \nabla_x u^{n+1}) dx - \int_{\mathbb{T}^3 \times \mathbb{R}^3} \partial_x^\alpha u^{n+1} \cdot \partial_x^\alpha \big((u^n - \xi) f^{n+1} \big) d\xi dx \\ &=: \mathcal{I}_{31} - 3\mathcal{I}_{32}. \end{aligned} \tag{3.21}$$

Next, we estimate $\mathcal{I}_{31}$ and $\mathcal{I}_{32}$ separately.

⋄ (Estimate of $\mathcal{I}_{31}$): We use Leibniz's rule and Sobolev embedding as in lower-order estimate to derive the desired estimate:

$$\begin{aligned}\mathcal{I}_{31} &= -\sum_{\beta < \alpha} \binom{\alpha}{\beta} \int_{\mathbb{T}^3} \partial_x^\alpha u^{n+1} \partial_x^\beta u^n \nabla_x \partial_x^{\alpha-\beta} u^{n+1} dx \\ &\quad - \int_{\mathbb{T}^3} \partial_x^\alpha u^{n+1} (\partial_x^\alpha u^n)(\nabla_x u^{n+1}) dx \\ &\le \sum_{\beta < \alpha} \binom{\alpha}{\beta} \|\partial_x^\alpha u^{n+1}\|_{L^6} \|\partial_x^\beta u^n\|_{L^3} \|\nabla \partial_x^{\alpha-\beta} u^{n+1}\|_{L^2}\end{aligned}$$

$$
\begin{aligned}
&+ \|\partial_x^\alpha u^{n+1}\|_{L^6} \|\partial_x^\alpha u^n\|_{L^2} \|\nabla u^{n+1}\|_{L^3} \\
&\le C\|u^n\|_{H^2} \|\nabla u^{n+1}\|_{H^2}^2,
\end{aligned}
$$

where we used the Sobolev embedding:

$$
\|u\|_{L^3} \le C\|u\|_{H^1}, \qquad \|u\|_{L^6} \le C\|u\|_{H^1}.
$$

$\diamond$ (Estimate of $\mathcal{I}_{32}$): Since $f \in W^{1,\infty}$, we need to move one-derivative of $\partial_x^\alpha((u^n - \xi) f^{n+1})$ to $\partial_x^\alpha u^{n+1}$ using integration by parts, i.e., for $|\gamma| = 1$,

$$
\begin{aligned}
\mathcal{I}_{32} &= \int_{\mathbb{T}^3 \times \mathbb{R}^3} \big(\partial_x^{\alpha+\gamma} u^{n+1}\big) \partial_x^{\alpha-\gamma} \big(\big(u^n - \xi\big) f^{n+1}\big) d\xi\, dx \\
&= \int_{\mathbb{T}^3 \times \mathbb{R}^3} \big(\partial_x^{\alpha+\gamma} u^{n+1}\big) \sum_{0<\beta\le\alpha-\gamma} \binom{\alpha-\gamma}{\beta} \big(\partial_x^\beta u^n\big)\big(\partial^{\alpha-\gamma-\beta} f^{n+1}\big) d\xi\, dx \\
&\quad + \int_{\mathbb{T}^3 \times \mathbb{R}^3} \big(\partial_x^{\alpha+\gamma} u^{n+1}\big)\big(u^n - \xi\big)\big(\partial^{\alpha-\gamma} f^{n+1}\big) d\xi\, dx \\
&=: \mathcal{I}_{32}^1 + \mathcal{I}_{32}^2.
\end{aligned}
$$

We use the same arguments as for Case A to find:

$$
\begin{aligned}
\mathcal{I}_{32}^1 &\le C\big(\eta^\infty\big)^3 \|f^{n+1}\|_{W^{1,\infty}} \|u^n\|_{H^2} \|\nabla u^{n+1}\|_{H^2}, \\
\mathcal{I}_{32}^2 &\le \max\Big\{\big(\eta^\infty\big)^3, \big(\eta^\infty\big)^{3/2}\sqrt{M_2^\infty}\Big\} \|f^{n+1}\|_{W^{1,\infty}} \big(\|u^n\|_{L^2} + 1\big) \|\nabla u^{n+1}\|_{H^2}.
\end{aligned}
$$

We combine the above estimates in (3.21) to find

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha u^{n+1}\|_{L^2}^2 + \mu\|\nabla \partial_x^\alpha u^{n+1}\|_{L^2}^2 \\
&\quad \le C\|u^n\|_{H^2}\|\nabla u^{n+1}\|_{H^2}^2 + C\big(\eta^\infty\big)^3 \|f^{n+1}\|_{W^{1,\infty}} \|u^n\|_{H^2} \|\nabla u^{n+1}\|_{H^2} \\
&\qquad + \max\Big\{\big(\eta^\infty\big)^3, \big(\eta^\infty\big)^{3/2}\sqrt{M_2^\infty}\Big\} \|f^{n+1}\|_{W^{1,\infty}} \big(\|u^n\|_{L^2} + 1\big) \|\nabla u^{n+1}\|_{H^2}. \qquad (3.22)
\end{aligned}
$$

Finally, we combine (3.20) and (3.22) to obtain the desired estimate. □

3.2. Convergence estimate

In this part, we study the uniform boundedness of f^n and u^n, when the initial data is sufficiently small. We also show that the approximate solutions are Cauchy sequences in suitable Banach spaces.

We first show that the approximate solutions are uniformly bounded for sufficiently regular and small initial data.

Lemma 3.3. *Suppose that the initial data* (f_0, u_0) *satisfy* (3.3) *and the following conditions:*

$$\|f_0\|_{W^{1,\infty}} + \|u_0\|_{H^2} < \varepsilon.$$

Then, for any approximate solutions (f^k, u^k) *in the time-interval* $[0, T)$*, we have*

$$\|f^k\|_{W^{1,\infty}(\mathbb{T}^3\times\mathbb{R}^3\times[0,T))} < \varepsilon^{\frac{2}{3}}, \qquad \|u^k\|_{\mathcal{C}^0([0,T];H^2(\mathbb{T}^3))} < \varepsilon^{\frac{1}{3}}, \qquad \|u^k\|_{L^2(0,T;H^3(\mathbb{T}^3))} < \varepsilon^{\frac{1}{3}}.$$

Proof. Let (f^n, u^n) be the approximate solutions constructed by the iteration scheme (3.1). To derive the desired estimate, we use the method of induction.

• Step A (Initial step): Since $(f^0, u^0) = (f_0, 0)$, we have:

$$\|f^0\|_{W^{1,\infty}} < \varepsilon \ll \varepsilon^{\frac{2}{3}}, \qquad \|u^0\|_{\mathcal{C}^0([0,T];H^2)} < \varepsilon^{\frac{1}{3}}, \qquad \|u^0\|_{L^2(0,T;H^3)} < \varepsilon^{\frac{1}{3}}.$$

• Step B (Inductive step): Suppose that the approximate solutions (f^k, u^k) satisfy

$$\|f^k\|_{W^{1,\infty}} < \varepsilon^{\frac{2}{3}}, \qquad \|u^k\|_{\mathcal{C}([0,T];H^2)} < \varepsilon^{\frac{1}{3}}, \qquad \|u^k\|_{L^2(0,T;H^3)} < \varepsilon^{\frac{1}{3}}, \quad 1 \le k \le n.$$

We next show that (f^{n+1}, u^{n+1}) satisfies the desired estimate.

◇ Step B-1 (Estimate of f^{n+1}): From now on, we drop the x-dependence in ∇_x, i.e., we use ∇ instead of ∇_x, however to avoid confusion, we retain ∇_ξ to denote the gradient with respect to ξ-variable. We introduce a nonlinear transport operator $\mathcal{N}$ on $\mathbb{T}^3 \times \mathbb{R}^3$ associated with (3.1):

$$\mathcal{N} := \partial_t + \xi \cdot \nabla + F\big(f^n, u^n\big) \cdot \nabla_\xi.$$

It follows from $(3.1)_1$ that

$$\begin{aligned}
\mathcal{N}\big(f^{n+1}\big) &= -\big(\nabla_\xi \cdot F\big(f^n, u^n\big)\big) f^{n+1} \le C f^{n+1}, \\
\mathcal{N}\big(\partial_{x_i} f^{n+1}\big) &= -\partial_{x_i} F\big(f^n, u^n\big) \cdot \nabla_\xi f^{n+1} - \big(\nabla_\xi \cdot \partial_{x_i} F\big(f^n, u^n\big)\big) f^{n+1} \\
&\quad - \big(\nabla_\xi \cdot F\big(f^n, u^n\big)\big) \partial_{x_i} f^{n+1} \\
&\le C\big(\big(1 + \|\partial_{x_i} u^n\|_{L^\infty}\big)\big|\nabla_\xi f^{n+1}\big| + \big|f^{n+1}\big| + \big|\partial_{x_i} f^{n+1}\big|\big), \\
\mathcal{N}\big(\partial_{\xi_i} f^{n+1}\big) &= -\partial_{x_i} f^{n+1} - \partial_{\xi_i} F\big(f^n, u^n\big) \cdot \nabla_\xi f^{n+1} - \big(\nabla_\xi \cdot F\big(f^n, u^n\big)\big) \partial_{\xi_i} f^{n+1} \\
&\le C\big(\big|\partial_{x_i} f^{n+1}\big| + \big|\nabla_\xi f^{n+1}\big|\big),
\end{aligned} \tag{3.23}$$

where the following estimates are used:

$$\begin{aligned}
&\nabla_\xi \cdot F\big(f^n, u^n\big) \le 3M_\psi + 3, \qquad \partial_{\xi_i} F\big(f^n, u^n\big) \le M_\psi + 1, \\
&\partial_{x_i} F\big(f^n, u^n\big) \le C\|\nabla\psi\|_{L^\infty} + \|\nabla u^n\|_{L^\infty}, \\
&\nabla_\xi \cdot \partial_{x_i} F\big(f^n, u^n\big) \le 3\|\nabla\psi\|_{L^\infty}.
\end{aligned} \tag{3.24}$$

We combine estimates (3.23) and (3.24) to get

$$\mathcal{N}\Big(\sum_{0\le|\alpha|+|\beta|\le 1} \nabla_x^\alpha \nabla_\xi^\beta f^{n+1}(t)\Big) \le C\big(1+\|\nabla u^n\|_{L^\infty}\big) \sum_{0\le|\alpha|+|\beta|\le 1} \nabla_x^\alpha \nabla_\xi^\beta f^{n+1}(t). \quad (3.25)$$

We set $\mathcal{F}^{n+1}(t)$ to measure the $W^{1,\infty}$-norm of f^{n+1}:

$$\mathcal{F}^{n+1}(t) := \sum_{0\le|\alpha|+|\beta|\le 1} \big\|\nabla_x^\alpha \nabla_\xi^\beta f^{n+1}(t)\big\|_{L^\infty}.$$

We integrate the relation (3.25) along the particle trajectory and take a supremum with respect to (x,ξ) to get

$$\mathcal{F}^{n+1}(t) \le \mathcal{F}^{n+1}(0) \exp C\big(T + \|\nabla u^n\|_{L^1(0,T;L^\infty)}\big), \quad t \in (0,T).$$

On the other hand, since $e^{\mathcal{X}T}\varepsilon = \mathcal{O}(1)$, we have

$$\begin{aligned}
&\mathcal{F}^{n+1}(0) < \varepsilon, \\
&\|\nabla u^n\|_{L^1(0,T;L^\infty(\mathbb{T}^3))} \le C\sqrt{T}\|u^n\|_{L^2(0,T;H^3(\mathbb{T}^3))} \le Ce^{\frac{T}{2}}\varepsilon^{\frac{1}{3}} = \mathcal{O}(1).
\end{aligned}$$

Thus, we have

$$\mathcal{F}^{n+1}(t) \le C\varepsilon e^{CT} < \varepsilon^{\frac{2}{3}}.$$

◇ Step B-2 (Estimate of u^{n+1}): Note that the estimate in Step B-1 and induction hypothesis imply

$$\|f^{n+1}\|_{W^{1,\infty}(\mathbb{T}^3\times\mathbb{R}^3\times[0,T))} < \varepsilon^{\frac{2}{3}}, \qquad \|u^n\|_{\mathcal{C}([0,T];H^2)} < \varepsilon^{\frac{1}{3}}, \qquad \|u^n\|_{L^2(0,T;H^3)} < \varepsilon^{\frac{1}{3}}.$$

Recall the energy estimate in Lemma 3.2 and we use the result of Step B-1 and the induction hypothesis on u^n to find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u^{n+1}\|_{H^2}^2 + \mu\|\nabla u^{n+1}\|_{H^2}^2 \\
&\quad \le C\big(\|u^n\|_{H^2}\|\nabla u^{n+1}\|_{H^2}^2 + \|f^{n+1}\|_{W^{1,\infty}}\big(\|u^n\|_{H^2}+1\big)\|u^{n+1}\|_{H^3}\big) \\
&\quad \le C\big(\varepsilon^{\frac{1}{3}}\|\nabla u^{n+1}\|_{H^2}^2 + \varepsilon^{\frac{2}{3}}\big(\varepsilon^{\frac{1}{3}}+1\big)\|u^{n+1}\|_{H^3}\big) \\
&\quad \le C\big[\varepsilon^{\frac{10}{9}} + \varepsilon^{\frac{2}{9}}\big(\varepsilon^{\frac{1}{3}}+1\big)^2\|u^{n+1}\|_{L^2}^2 + \varepsilon^{\frac{2}{9}}\big(\varepsilon^{\frac{1}{9}} + \big(\varepsilon^{\frac{1}{3}}+1\big)^2\big)\|\nabla u^{n+1}\|_{H^2}^2\big],
\end{aligned}$$

where we used Young's inequality:

$$\varepsilon^{\frac{2}{3}}\big(\varepsilon^{\frac{1}{3}}+1\big)\|u^{n+1}\|_{H^3} = \varepsilon^{\frac{5}{9}}\big(\varepsilon^{\frac{1}{9}}\big(\varepsilon^{\frac{1}{3}}+1\big)\|u^{n+1}\|_{H^3}\big) \le \frac{\varepsilon^{\frac{10}{9}}}{2} + \frac{\varepsilon^{\frac{2}{9}}(\varepsilon^{\frac{1}{3}}+1)^2}{2}\|u^{n+1}\|_{H^3}^2.$$

We use the smallness of ε to find

$$\begin{aligned}\frac{d}{dt}\|u^{n+1}\|_{H^2}^2+\|\nabla u^{n+1}\|_{H^2}^2 &\le C\varepsilon^{\frac{10}{9}}+C\varepsilon^{\frac{2}{9}}\big(\varepsilon^{\frac{1}{3}}+1\big)^2\|u^{n+1}\|_{L^2}^2\\ &\le C\big(\varepsilon^{\frac{10}{9}}+\varepsilon^{\frac{2}{9}}\|u^{n+1}\|_{H^2}^2\big).\end{aligned}\tag{3.26}$$

We now apply the Gronwall inequality to (3.26) to have

$$\begin{aligned}\|u^{n+1}(t)\|_{H^2}^2+\int_0^t\|\nabla u^{n+1}(s)\|_{H^2}^2ds &\le C\|u_0\|_{H^2}^2e^{C\varepsilon^{\frac{2}{9}}T}+C\varepsilon^{\frac{8}{9}}\big(e^{C\varepsilon^{\frac{2}{9}}T}-1\big)\\ &\le\frac{\varepsilon^{\frac{2}{3}}}{2}\big(C\varepsilon^{\frac{4}{3}}e^{C\varepsilon^{\frac{2}{9}}T}+C\varepsilon^{\frac{2}{9}}e^{C\varepsilon^{\frac{2}{9}}T}\big)\\ &\le\frac{\varepsilon^{\frac{2}{3}}}{2}.\end{aligned}$$

This implies

$$\|u^{n+1}\|_{L^\infty(0,T;H^2)}+\|u^{n+1}\|_{L^2(0,T;H^3)}\le\varepsilon^{\frac{1}{3}}.$$

We next show that

$$\|u^{n+1}\|_{C([0,T];H^2(\mathbb{T}^3))}\le\varepsilon^{\frac{1}{3}}.$$

For this, we use (3.26):

$$\frac{d}{dt}\|u^{n+1}\|_{H^2}^2\le C\big(\varepsilon^{\frac{10}{9}}+\varepsilon^{\frac{2}{9}}\times\varepsilon^{\frac{2}{3}}\big)\le C\sqrt{\varepsilon}$$

to find

$$\big|\|u^{n+1}(t)\|_{H^2}^2-\|u^{n+1}(s)\|_{H^2}^2\big|=\left|\int_s^t\frac{d}{d\tau}\|u^{n+1}(\tau)\|_{H^2}^2d\tau\right|\le C\sqrt{\varepsilon}(t-s),\quad 0\le s\le t\le T.$$

This implies the continuity of $\|u^{n+1}(t)\|_{H^2}^2$ in t. □

Next, we show that the approximate solution (f^n,u^n) is Cauchy in $L^\infty(\mathbb{T}^3\times\mathbb{R}^3\times[0,T))\times L^\infty(0,T;H^1(\mathbb{T}^3))\cap L^2(0,T;H^2(\mathbb{T}^3))$.

Proposition 3.1. *Let (f^n,u^n) be a sequence of approximate solutions with initial data (f_0,u_0) satisfying (3.3) and the smallness:*

$$\|f_0\|_{W^{1,\infty}}+\|u_0\|_{H^2}<\varepsilon.$$

Then f^n and u^n are Cauchy sequences in $L^\infty(\mathbb{T}^3\times\mathbb{R}^3\times[0,T])$ and $L^\infty(0,T;H^1(\mathbb{T}^3))\cap L^2(0,T;H^2(\mathbb{T}^3))$, respectively.

Proof. It follows from Lemma 3.3 that

$$\|f^k\|_{W^{1,\infty}(\mathbb{T}^3\times\mathbb{R}^3\times[0,T))} < \varepsilon^{\frac{2}{3}}, \qquad \|u^k\|_{\mathcal{C}^0([0,T];H^2(\mathbb{T}^3))} < \varepsilon^{\frac{1}{3}}, \qquad \|u^k\|_{L^2(0,T;H^3(\mathbb{T}^3))} < \varepsilon^{\frac{1}{3}}.$$

Thus, we have

$$\|u^k\|_{L^\infty(\mathbb{T}^3\times\mathbb{R}^3\times[0,T))} \le C\|u^k\|_{\mathcal{C}^0([0,T];H^2(\mathbb{T}^3))} \le C\varepsilon^{\frac{1}{3}}.$$

By applying Lemma 3.1 with $U^\infty = C\varepsilon^{\frac{1}{3}}$, we obtain the boundedness of the velocity support:

$$\max_{k\ge 1}\sup_{0\le t\le T}\eta^k(t) \le \eta^\infty.$$

We then set

$$\Delta_{n+1}(t) := \|(f^{n+1}-f^n)(t)\|_{L^\infty}^2 + \|(u^{n+1}-u^n)(t)\|_{H^1}^2 + \int_0^t \|\nabla(u^{n+1}-u^n)(s)\|_{H^1}^2 ds,$$

and claim:

$$\Delta_{n+1}(t) \le A(\varepsilon)\int_0^t \Delta_n(s)ds + B(\varepsilon)\int_0^t \Delta_{n+1}(s)ds, \tag{3.27}$$

where $A(\varepsilon)$ and $B(\varepsilon)$ are positive constants dependent on ε.

Proof of claim (3.27).

• Case A (Estimate of the time-evolution of $\|f^{n+1}-f^n\|_{L^\infty}^2$): It follows from (3.1) that

$$\begin{aligned}
&\partial_t(f^{n+1}-f^n) + \xi\cdot\nabla_x(f^{n+1}-f^n) + F(f^n,u^n)\cdot\nabla_\xi(f^{n+1}-f^n)\\
&\quad = -\big(F(f^n,u^n) - F(f^{n-1},u^{n-1})\big)\cdot\nabla_\xi f^n - (f^{n+1}-f^n)\nabla_\xi\cdot\big(F(f^{n-1},u^{n-1})\big)\\
&\qquad - f^{n+1}\nabla_\xi\cdot\big(F(f^n,u^n) - F(f^{n-1},u^{n-1})\big)\\
&\quad := \sum_{i=1}^3 \mathcal{I}_{4i}.
\end{aligned}\tag{3.28}$$

Below, we estimate the terms $\mathcal{I}_{4i}$ as follows. We first recall that

$$F(f^n,u^n) = \int_{\mathbb{T}^3\times\mathbb{R}^3} \psi(x,y)(\xi_*-\xi)f^n(y,\xi_*,t)d\xi_* dy + u^n - \xi. \tag{3.29}$$

◇ (Estimate of $\mathcal{I}_{41}$): Note that

$$\begin{aligned}
&\big|F\big(f^n,u^n\big)-F\big(f^{n-1},u^{n-1}\big)\big| \\
&\quad=\big|F\big(f^n,u^n\big)-F\big(f^{n-1},u^n\big)+F\big(f^{n-1},u^n\big)-F\big(f^{n-1},u^{n-1}\big)\big| \\
&\quad\le\bigg|\int\limits_{\mathbb{T}^3\times\mathbb{R}^3}\psi(x,y)(\xi_*-\xi)\big(f^n-f^{n-1}\big)d\xi_*dy\bigg|+\big|u^n-u^{n-1}\big| \\
&\quad\le C\big\|f^n-f^{n-1}\big\|_{L^\infty}+\big\|u^n-u^{n-1}\big\|_{L^\infty}.
\end{aligned}$$

This yields

$$\begin{aligned}
|\mathcal{I}_{41}|&\le\big|F\big(f^n,u^n\big)-F\big(f^{n-1},u^{n-1}\big)\big|\big|\nabla_\xi f^n\big| \\
&\le C\varepsilon^{\frac{2}{3}}\big(\big\|f^n-f^{n-1}\big\|_{L^\infty}+\big\|u^n-u^{n-1}\big\|_{L^\infty}\big) \\
&\le C\varepsilon^{\frac{2}{3}}\big(\big\|f^n-f^{n-1}\big\|_{L^\infty}+\big\|u^n-u^{n-1}\big\|_{H^2}\big). \qquad (3.30)
\end{aligned}$$

◇ (Estimate of $\mathcal{I}_{42}$): It follows from (3.29) that

$$\big|\nabla_\xi\cdot F\big(f^{n-1},u^{n-1}\big)\big|=\bigg|-3\int\limits_{\mathbb{T}^3\times\mathbb{R}^3}\psi(x,y)f^{n-1}(y,\xi_*,t)d\xi_*dy-3\bigg|\le 3(M_\psi+1).$$

Thus, we have

$$|\mathcal{I}_{42}|\le C\big\|f^{n+1}-f^n\big\|_{L^\infty}.$$

◇ (Estimate of $\mathcal{I}_{43}$): Note that

$$\begin{aligned}
&\big|\nabla_\xi\cdot\big(F\big(f^n,u^n\big)-F\big(f^{n-1},u^{n-1}\big)\big)\big| \\
&\quad\le 3\bigg|\int\limits_{\mathbb{T}^3\times\mathbb{R}^3}\psi(x,y)\big(f^n-f^{n-1}\big)(y,\xi_*,t)d\xi_*dy\bigg| \\
&\quad\le 3M_\psi\big\|f^n-f^{n-1}\big\|_{L^\infty}.
\end{aligned}$$

Again this yields

$$|\mathcal{I}_{43}|\le C\varepsilon^{\frac{2}{3}}\big\|f^n-f^{n-1}\big\|_{L^\infty}. \qquad (3.31)$$

We now combine (3.28) with (3.30)–(3.31) to get:

$$\begin{aligned}
&\partial_t\big|f^{n+1}-f^n\big|+\xi\cdot\nabla_x\big|f^{n+1}-f^n\big|+F\big(f^n,u^n\big)\cdot\nabla_\xi\big|f^{n+1}-f^n\big| \\
&\quad\le C\big(\varepsilon^{\frac{2}{3}}+1\big)\big\|f^n-f^{n-1}\big\|_{L^\infty}+C\varepsilon^{\frac{2}{3}}\big\|u^n-u^{n-1}\big\|_{H^2}. \qquad (3.32)
\end{aligned}$$

Multiplying (3.32) by $|f^{n+1}-f^n|$ and using Young's inequality, we can obtain

$$\begin{aligned}&\partial_t\big|f^{n+1}-f^n\big|^2+\xi\cdot\nabla_x\big|f^{n+1}-f^n\big|^2+F\big(f^n,u^n\big)\cdot\nabla_\xi\big|f^{n+1}-f^n\big|^2\\&\quad\le C\big(\varepsilon^{\frac{2}{3}}+1\big)\big(\big\|f^{n+1}-f^n\big\|_{L^\infty}^2+\big\|f^n-f^{n-1}\big\|_{L^\infty}^2\big)+C\varepsilon^{\frac{2}{3}}\big\|u^n-u^{n-1}\big\|_{H^2}^2.\end{aligned}\tag{3.33}$$

We integrate (3.33) along the particle trajectory to find

$$\begin{aligned}&\big\|\big(f^{n+1}-f^n\big)(t)\big\|_{L^\infty}^2\\&\quad\le C\big(\varepsilon^{\frac{2}{3}}+1\big)\int_0^t\big\|\big(f^{n+1}-f^n\big)(s)\big\|_{L^\infty}^2+\big\|\big(f^n-f^{n-1}\big)(s)\big\|_{L^\infty}^2ds\\&\qquad+C\varepsilon^{\frac{2}{3}}\int_0^t\big\|\big(u^n-u^{n-1}\big)(s)\big\|_{H^2}^2ds.\end{aligned}\tag{3.34}$$

Then we apply Gronwall's inequality to have

$$\begin{aligned}&\big\|\big(f^{n+1}-f^n\big)(t)\big\|_{L^\infty}^2\\&\quad\le C\big(\varepsilon^{\frac{2}{3}}+1\big)\int_0^t\big\|\big(f^n-f^{n-1}\big)(s)\big\|_{L^\infty}^2ds+C\varepsilon^{\frac{2}{3}}\int_0^t\int_0^s\big\|\big(u^n-u^{n-1}\big)(\tau)\big\|_{H^2}^2d\tau ds,\end{aligned}\tag{3.35}$$

where we also used

$$\int_0^t\int_0^s\big\|\big(f^n-f^{n-1}\big)(\tau)\big\|_{L^\infty}^2d\tau ds\le T\int_0^t\big\|\big(f^n-f^{n-1}\big)(s)\big\|_{L^\infty}^2ds.$$

• Case B (Estimate of the time-evolution of $\|u^{n+1}-u^n\|_{H^1}^2$): In order to estimate the sequence $(u^{n+1}-u^n)_{n\ge1}$, we take the difference of $(3.1)_2$ between the $(n+1)$-th and the n-th iterates to have

$$\begin{aligned}&\partial_t\big(u^{n+1}-u^n\big)-\mu\Delta\big(u^{n+1}-u^n\big)+\nabla_x\big(p^{n+1}-p^n\big)\\&\quad=-u^n\cdot\nabla_x\big(u^{n+1}-u^n\big)-\big(u^n-u^{n-1}\big)\cdot\nabla u^n-\int_{\mathbb{R}^3}\big(u^n-u^{n-1}\big)f^{n+1}d\xi\\&\qquad-\int_{\mathbb{R}^3}u^{n-1}\big(f^{n+1}-f^n\big)d\xi+\int_{\mathbb{R}^3}\xi\big(f^{n+1}-f^n\big)d\xi,\\&\nabla_x\cdot\big(u^{n+1}-u^n\big)=0,\quad t>0,\ x\in\mathbb{T}^3.\end{aligned}$$

• (Zeroth-order estimate): We take the inner product with $u^{n+1}-u^n$ and then integrate it over $\mathbb{T}^3\times\mathbb{R}^3$ using the same estimates as in Case A of Lemma 3.2 to find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u^{n+1}-u^n\|_{L^2}^2+\mu\|\nabla(u^{n+1}-u^n)\|_{L^2}^2 \\
&\le C\big(\|\nabla u^n\|_{L^2}\|u^n-u^{n-1}\|_{L^3}\|u^{n+1}-u^n\|_{L^6}+\|f^{n+1}\|_{L^\infty}\|u^n-u^{n-1}\|_{L^2}\|u^{n+1}-u^n\|_{L^2} \\
&\quad+\|f^{n+1}-f^n\|_{L^\infty}\|u^{n-1}\|_{L^2}\|u^{n+1}-u^n\|_{L^2}+\|f^{n+1}-f^n\|_{L^\infty}\|u^{n+1}-u^n\|_{L^2}\big) \\
&\le C\varepsilon^{\frac{1}{3}}\big(\|\nabla(u^{n+1}-u^n)\|_{L^2}^2+\|u^{n+1}-u^n\|_{L^2}^2\big)+C\varepsilon^{\frac{1}{3}}\big(\|u^n-u^{n-1}\|_{H^1}^2+\|u^{n+1}-u^n\|_{H^1}^2\big) \\
&\quad+C\varepsilon^{\frac{2}{3}}\big(\|u^n-u^{n-1}\|_{L^2}^2+\|u^{n+1}-u^n\|_{L^2}^2\big)+C\varepsilon^{\frac{1}{3}}\big(\|f^{n+1}-f^n\|_{L^\infty}^2+\|u^{n+1}-u^n\|_{L^2}^2\big) \\
&\quad+C\big(\|f^{n+1}-f^n\|_{L^\infty}^2+\|u^{n+1}-u^n\|_{L^2}^2\big) \\
&\le C\big(\varepsilon^{\frac{1}{3}}+1\big)\big(\|u^{n+1}-u^n\|_{H^1}^2+\|f^{n+1}-f^n\|_{L^\infty}^2\big)+C\varepsilon^{\frac{1}{3}}\|u^n-u^{n-1}\|_{H^1}^2. \qquad (3.36)
\end{aligned}$$

• (First-order estimate): Similar to the lower-order estimate, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla(u^{n+1}-u^n)\|_{L^2}^2+\mu\|\nabla^2(u^{n+1}-u^n)\|_{L^2}^2 \\
&=-\int_{\mathbb{T}^3}\nabla(u^{n+1}-u^n)\nabla\big(u^n\cdot\nabla_x(u^{n+1}-u^n)\big)dx \\
&\quad-\int_{\mathbb{T}^3}\nabla(u^{n+1}-u^n)\nabla\big((u^n-u^{n-1})\cdot\nabla u^n\big)dx \\
&\quad-\int_{\mathbb{T}^3\times\mathbb{R}^3}\nabla(u^{n+1}-u^n)\nabla\big((u^n-u^{n-1})f^{n+1}\big)d\xi\,dx \\
&\quad-\int_{\mathbb{T}^3\times\mathbb{R}^3}\nabla(u^{n+1}-u^n)\nabla\big(u^{n-1}(f^{n+1}-f^n)\big)d\xi\,dx \\
&\quad+\int_{\mathbb{T}^3\times\mathbb{R}^3}\nabla(u^{n+1}-u^n)\xi\nabla(f^{n+1}-f^n)d\xi\,dx \\
&=:\sum_{i=1}^{5}\mathcal{I}_{5i}.
\end{aligned}$$

Below, we estimate the terms $\mathcal{I}_{5i}$ separately. We use the same arguments as in Case B of Lemma 3.2:

$$\begin{aligned}
\diamond\ \mathcal{I}_{51}&\le\|\nabla(u^{n+1}-u^n)\|_{L^3}\big(\|\nabla u^n\|_{L^2}\|\nabla(u^{n+1}-u^n)\|_{L^6}+\|u^n\|_{L^6}\|\nabla^2(u^{n+1}-u^n)\|_{L^2}\big) \\
&\le C\|u^n\|_{H^1}\|\nabla(u^{n+1}-u^n)\|_{H^1}^2\le C\varepsilon^{\frac{1}{3}}\|\nabla(u^{n+1}-u^n)\|_{H^1}^2, \\
\diamond\ \mathcal{I}_{52}&\le\|\nabla(u^{n+1}-u^n)\|_{L^3}\big(\|\nabla u^n\|_{L^6}\|\nabla(u^n-u^{n-1})\|_{L^2}+\|\nabla^2u^n\|_{L^2}\|u^n-u^{n-1}\|_{L^6}\big) \\
&\le C\varepsilon^{\frac{1}{3}}\|u^n-u^{n-1}\|_{H^1}\|\nabla(u^{n+1}-u^n)\|_{H^1}, \\
\diamond\ \mathcal{I}_{53}&\le C\|\nabla(u^{n+1}-u^n)\|_{L^2}\big(\|f^{n+1}\|_{L^\infty}\|\nabla(u^n-u^{n-1})\|_{L^2}+\|\nabla f^{n+1}\|_{L^\infty}\|u^n-u^{n-1}\|_{L^2}\big)
\end{aligned}$$

$$
\begin{aligned}
&\le C\varepsilon^{\frac{2}{3}}\|u^n-u^{n-1}\|_{H^1}\|\nabla(u^{n+1}-u^n)\|_{L^2},\\
\diamond\, \mathcal{I}_{54} &= \int_{\mathbb{T}^3\times\mathbb{R}^3} \nabla^2(u^{n+1}-u^n)u^{n-1}(f^{n+1}-f^n)\,d\xi\,dx\\
&\le C\|f^{n+1}-f^n\|_{L^\infty}\|\nabla^2(u^{n+1}-u^n)\|_{L^2}\|u^{n-1}\|_{L^2}\\
&\le C\varepsilon^{\frac{1}{3}}\|f^{n+1}-f^n\|_{L^\infty}\|\nabla(u^{n+1}-u^n)\|_{H^1},\\
\diamond\, \mathcal{I}_{55} &= -\int_{\mathbb{T}^3\times\mathbb{R}^3} \nabla^2(u^{n+1}-u^n)\xi(f^{n+1}-f^n)\,d\xi\,dx\\
&\le C\|f^{n+1}-f^n\|_{L^\infty}\|\nabla^2(u^{n+1}-u^n)\|_{L^2}\le C\|f^{n+1}-f^n\|_{L^\infty}\|\nabla(u^{n+1}-u^n)\|_{H^1}.
\end{aligned}
\tag{3.37}
$$

Combining all estimates in (3.37), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla(u^{n+1}-u^n)\|_{L^2}^2+\mu\|\nabla^2(u^{n+1}-u^n)\|_{L^2}^2\\
&\quad\le C\varepsilon^{\frac{1}{3}}\|\nabla(u^{n+1}-u^n)\|_{H^1}^2+C\varepsilon^{\frac{1}{3}}\big(\|u^n-u^{n-1}\|_{H^1}^2+\|\nabla(u^{n+1}-u^n)\|_{H^1}^2\big)\\
&\qquad+C\varepsilon^{\frac{2}{3}}\big(\|u^n-u^{n-1}\|_{H^1}^2+\|\nabla(u^{n+1}-u^n)\|_{L^2}^2\big)\\
&\qquad+C\varepsilon^{\frac{1}{3}}\big(\|f^{n+1}-f^n\|_{L^\infty}^2+\|\nabla(u^{n+1}-u^n)\|_{H^1}^2\big)\\
&\qquad+C\big(\varepsilon^{-\frac{1}{3}}\|f^{n+1}-f^n\|_{L^\infty}^2+\varepsilon^{\frac{1}{3}}\|\nabla(u^{n+1}-u^n)\|_{H^1}^2\big)\\
&\quad\le C\varepsilon^{\frac{1}{3}}\|\nabla(u^{n+1}-u^n)\|_{H^1}^2+C\varepsilon^{\frac{1}{3}}\|u^n-u^{n-1}\|_{H^1}^2\\
&\qquad+C\big(\varepsilon^{\frac{1}{3}}+\varepsilon^{-\frac{1}{3}}\big)\|f^{n+1}-f^n\|_{L^\infty}^2.
\end{aligned}
\tag{3.38}
$$

We now combine (3.36) and (3.38) to get

$$
\begin{aligned}
&\frac{d}{dt}\|u^{n+1}-u^n\|_{H^1}^2+\mu\|\nabla(u^{n+1}-u^n)\|_{H^1}^2\\
&\quad\le C\big(\varepsilon^{\frac{1}{3}}+1\big)\|u^{n+1}-u^n\|_{H^1}^2+C\big(1+\varepsilon^{\frac{1}{3}}+\varepsilon^{-\frac{1}{3}}\big)\|f^{n+1}-f^n\|_{L^\infty}^2\\
&\qquad+C\varepsilon^{\frac{1}{3}}\|u^n-u^{n-1}\|_{H^1}^2+C\varepsilon^{\frac{1}{3}}\|\nabla^2(u^{n+1}-u^n)\|_{L^2}^2.
\end{aligned}
\tag{3.39}
$$

We integrate (3.39) and combine with (3.35) to obtain

$$
\begin{aligned}
&\|(f^{n+1}-f^n)(t)\|_{L^\infty}^2+\|(u^{n+1}-u^n)(t)\|_{H^1}^2+\mu\int_0^t\|\nabla(u^{n+1}-u^n)(s)\|_{H^1}^2ds\\
&\le C\big(1+\varepsilon^{\frac{1}{3}}+\varepsilon^{-\frac{1}{3}}\big)\int_0^t\big(\|(f^{n+1}-f^n)(s)\|_{L^\infty}^2+\|(f^n-f^{n-1})(s)\|_{L^\infty}^2\big)ds
\end{aligned}
$$

$$+ C\left(1+\varepsilon^{\frac{1}{3}}+\varepsilon^{-\frac{1}{3}}\right)\int_0^t \left(\left\|\left(u^{n+1}-u^n\right)(s)\right\|_{H^1}^2 + \left\|\left(u^n-u^{n-1}\right)(s)\right\|_{H^1}^2\right)ds$$

$$+ C\varepsilon^{\frac{2}{3}}\int_0^t\int_0^s \left\|\nabla\left(u^n-u^{n-1}\right)(\tau)\right\|_{H^1}^2 d\tau ds + C\varepsilon^{\frac{1}{3}}\int_0^t \left\|\nabla^2\left(u^{n+1}-u^n\right)(s)\right\|_{L^2}^2 ds.$$

By the smallness of ε, we can have

$$\left\|\left(f^{n+1}-f^n\right)(t)\right\|_{L^\infty}^2 + \left\|\left(u^{n+1}-u^n\right)(t)\right\|_{H^1}^2 + \int_0^t \left\|\nabla\left(u^{n+1}-u^n\right)(s)\right\|_{H^1}^2 ds$$

$$\le C\left(1+\varepsilon^{\frac{1}{3}}+\varepsilon^{-\frac{1}{3}}\right)\int_0^t \left(\left\|\left(f^{n+1}-f^n\right)(s)\right\|_{L^\infty}^2 + \left\|\left(f^n-f^{n-1}\right)(s)\right\|_{L^\infty}^2\right)ds$$

$$+ C\left(1+\varepsilon^{\frac{1}{3}}+\varepsilon^{-\frac{1}{3}}\right)\int_0^t \left(\left\|\left(u^{n+1}-u^n\right)(s)\right\|_{H^1}^2 + \left\|\left(u^n-u^{n-1}\right)(s)\right\|_{H^1}^2\right)ds$$

$$+ C\left(1+\varepsilon^{\frac{1}{3}}+\varepsilon^{-\frac{1}{3}}\right)\int_0^t \left(\int_0^s \left\|\nabla\left(u^{n+1}-u^n\right)(\tau)\right\|_{H^1}^2 d\tau + \int_0^s \left\|\nabla\left(u^n-u^{n-1}\right)(\tau)\right\|_{H^1}^2 d\tau\right)ds,$$

which proves the claim with $A(\varepsilon)=B(\varepsilon)=C(1+\varepsilon^{\frac{1}{3}}+\varepsilon^{-\frac{1}{3}})$.

Finally, we apply Gronwall's inequality [3] to derive

$$\Delta_{n+1}(t) = \left\|\left(f^{n+1}-f^n\right)(t)\right\|_{L^\infty}^2 + \left\|\left(u^{n+1}-u^n\right)(t)\right\|_{H^1}^2 + \int_0^t \left\|\nabla\left(u^{n+1}-u^n\right)(s)\right\|_{H^1}^2 ds$$

$$\le \frac{(C(1+\varepsilon^{\frac{1}{3}}+\varepsilon^{-\frac{1}{3}})T)^{n+1}}{(n+1)!}, \quad t \le T.$$

This implies that (f^n) and (u^n) are Cauchy sequences in $L^\infty(\mathbb{T}^3\times\mathbb{R}^3\times[0,T])$ and $L^\infty(0,T;H^1(\mathbb{T}^3))\cap L^2(0,T;H^2(\mathbb{T}^3))$, respectively. □

3.3. *Proof of Theorem 2.1*

In this part, we present the proof of Theorem 2.1. We divide its proof into two parts.

• Part A (Existence): Let (f^n) and (u^n) be sequences of approximate solutions, as constructed in Section 3.1. Then by Proposition 3.1, (f_n) and (u_n) are Cauchy sequences in $L^\infty(\mathbb{T}^3\times\mathbb{R}^3\times[0,T))$ and $L^\infty(0,T;H^1(\mathbb{T}^3))\cap L^2(0,T;H^2(\mathbb{T}^2))$, respectively. Thus they converge strongly to some pair of limit functions $(\bar f,\bar u)\in L^\infty(\mathbb{T}^3\times\mathbb{R}^3\times[0,T))\times L^\infty(0,T;H^1(\mathbb{T}^3))\cap L^2(0,T;H^2(\mathbb{T}^2))$ as $n\to\infty$:

$$\begin{aligned}
&f^n \to \bar{f} \quad \text{in } L^\infty\big(\mathbb{T}^3 \times \mathbb{R}^3 \times [0,T)\big),\\
&u^n \to \bar{u} \quad \text{in } L^\infty\big(0,T; H^1\big(\mathbb{T}^3\big)\big) \cap L^2\big(0,T; H^2\big(\mathbb{T}^2\big)\big).
\end{aligned}$$

It is easy to see that the pair of limit functions $(\bar{f}, \bar{u})$ is a weak solution to the CS–NS system (1.1)–(1.5) in a distributional sense (see [2]). We must verify that the limit functions $(\bar{f}, \bar{u})$ do have the required regularity of strong solution (see the required regularity in Theorem 2.1).

Note that f^n and u^n are bounded in $W^{1,\infty}(\mathbb{T}^3 \times \mathbb{R}^3 \times [0,T))$ and $C^0(0,T; H^2(\mathbb{T}^3)) \cap L^2(0,T; H^3(\mathbb{T}^3)) \cap H^1(0,T; H^1(\mathbb{T}^3))$, respectively (see Lemma 3.3). Hence by the weak compactness theorem for reflexive Banach spaces and the Banach–Alaoglu theorem, there exist subsequences (f^{n_k}, u^{n_k}) and the pair (f,u) of weak limits:

$$\begin{aligned}
&f \in W^{1,\infty}\big(\mathbb{T}^3 \times \mathbb{R}^3 \times [0,T)\big), \qquad f^{n_k} \rightharpoonup^* f \quad \text{in } W^{1,\infty}\big(\mathbb{T}^3 \times \mathbb{R}^3 \times [0,T)\big),\\
&u \in C^0\big(0,T; H^2\big(\mathbb{T}^3\big)\big) \cap L^2\big(0,T; H^3\big(\mathbb{T}^3\big)\big), \quad \text{and}\\
&u^{n_k} \rightharpoonup u \quad \text{in } C^0\big(0,T; H^2\big(\mathbb{T}^3\big)\big) \cap L^2\big(0,T; H^3\big(\mathbb{T}^3\big)\big).
\end{aligned}$$

By the uniqueness of weak limits, we have

$$\begin{aligned}
&f = \bar{f} \quad \text{in } L^\infty\big(\mathbb{T}^3 \times \mathbb{R}^3 \times [0,T)\big),\\
&u = \bar{u} \quad \text{in } L^\infty\big(0,T; H^1\big(\mathbb{T}^3\big)\big) \cap L^2\big(0,T; H^2\big(\mathbb{T}^2\big)\big).
\end{aligned}$$

Therefore, the pair of weak limits (f,u) is also the weak solution to the system (1.1)–(1.5) and (f,u) has the required regularity of a strong solution. The required regularity of $u_t \in L^2(0,T; H^1(\mathbb{T}^3))$ and $p \in L^\infty(0,T; H^1(\mathbb{T}^3))$ follow from the Navier–Stokes equations in (1.1) and the regularity of f and u. Hence, (f,u) is our strong solution to the system (1.1)–(1.5).

• Part B (Uniqueness): Let (f,u) and $(\bar{f},\bar{u})$ be the two strong solutions in Part A corresponding to the same initial data (f_0,u_0). We set:

$$\Delta_d(t) := \big\|f(t) - \bar{f}(t)\big\|_{L^\infty}^2 + \big\|u(t) - \bar{u}(t)\big\|_{H^1}^2.$$

Then, by the same arguments as in Proposition 3.1, $\Delta_d(t)$ satisfies Gronwall's inequality:

$$\Delta_d(t) + \int_0^t \big\|\nabla(u-\bar{u})(s)\big\|_{H^1}^2 ds \le \bar{A}(\varepsilon) \int_0^t \Delta_d(s)ds, \qquad \Delta_d(0) = 0,$$

and the standard Gronwall's lemma implies that

$$\begin{aligned}
&\Delta_d(t) = 0, \quad \text{i.e., } f \equiv \bar{f} \quad \text{in } L^\infty\big([0,T] \times \mathbb{T}^3 \times \mathbb{R}^3\big),\\
&u \equiv \bar{u} \quad \text{in } L^\infty\big(0,T; H^1\big(\mathbb{T}^3\big)\big) \cap L^2\big(0,T; H^2\big(\mathbb{T}^3\big)\big).
\end{aligned}$$

We now need to verify that

$$\begin{aligned} f &\equiv \bar{f} \quad \text{in } W^{1,\infty}\big([0,T]\times\mathbb{T}^3\times\mathbb{R}^3\big), \\ u &\equiv \bar{u} \quad \text{in } L^\infty\big(0,T;H^2\big(\mathbb{T}^3\big)\big)\cap L^2\big(0,T;H^3\big(\mathbb{T}^3\big)\big). \end{aligned}$$

For the uniqueness of f, we set that Ω denotes one of sets $[0,T)$, $\mathbb{T}^3$, $\mathbb{R}^3$ and ζ denotes one of variables t, x, ξ. Then, for any $\phi\in\mathcal{C}^1(\Omega)$, we have

$$\begin{aligned} \left|\int_\Omega \partial_\zeta(f-\bar{f})\phi\, d\zeta\right| &= \left|\int_\Omega (f-\bar{f})\partial_\zeta\phi\, d\zeta\right| \\ &\le C\|f-\bar{f}\|_{L^\infty([0,T]\times\mathbb{T}^3\times\mathbb{R}^3)}\|\partial_\zeta\phi\|_{L^\infty}=0. \end{aligned}$$

Thus we have $\partial_\zeta f=\partial_\zeta\bar{f}$ a.e. on $[0,T]\times\mathbb{T}^3\times\mathbb{R}^3$, which implies

$$f\equiv\bar{f} \quad \text{in } W^{1,\infty}\big([0,T]\times\mathbb{T}^3\times\mathbb{R}^3\big).$$

On the other hand, for any $\phi\in\mathcal{C}^1(\mathbb{T}^3)$, we have

$$\left|\int_{\mathbb{T}^3}\nabla_x^2(u-\bar{u})\phi\, dx\right| = \left|\int_{\mathbb{T}^3}\nabla_x(u-\bar{u})\nabla\phi\, dx\right| \le C\|u-\bar{u}\|_{L^\infty(0,T;H^1(\mathbb{T}^3))}\|\nabla_x\phi\|_{L^\infty}=0.$$

Thus we have $\nabla_x^2 u=\nabla_x^2\bar{u}$ a.e. on $[0,T]\times\mathbb{T}^3\times\mathbb{R}^3$, which implies

$$u\equiv\bar{u} \quad \text{in } L^\infty\big(0,T;H^2\big(\mathbb{T}^3\big)\big).$$

We can similarly show that

$$u\equiv\bar{u} \quad \text{in } L^2\big(0,T;H^3\big(\mathbb{T}^3\big)\big).$$

This completes the proof of Theorem 2.1.

4. An exponential flocking estimate

In this section, we revisit the asymptotic flocking problem for the system (1.1)–(1.5) on the domain $\mathbb{T}^3\times\mathbb{R}^3$, which has been studied in the authors' earlier work [2]. In fact, the analysis in this section can be generalized to any dimension, but for consistentency with the existence theory in Section 3, we restrict our analysis to the three dimensional spatial domain $\mathbb{T}^3$.

4.1. A Lyapunov functional

We showed in the previous section that if the initial data is sufficiently regular, a unique classical solution to (1.1)–(1.5) can be obtained. Thus it is natural to wonder whether these classical solutions exhibit asymptotic flocking behavior, a justification of the modeling of the interaction between particles and fluids. In [2], we used the following Lyapunov functionals to measure the deviations from the time-dependent averaged particle velocity ξ_c:

$$\mathcal{E}_p^o(t) := \int_{\mathbb{T}^3\times\mathbb{R}^3} \left|\xi - \xi_c(t)\right|^2 f\,d\xi\,dx, \qquad \mathcal{E}_f^o(t) := \frac{1}{3}\int_{\mathbb{T}^3} \left|u - \xi_c(t)\right|^2 dx,$$

where ξ_c is defined by (2.2).

To close the time-evolution estimate of $\mathcal{E}_f^o$, we need to control the term $\int_{\mathbb{T}^3} |u - \xi_c|^2 dx$ in terms of $\int_{\mathbb{T}^3} |\nabla(u - \xi_c)|^2 dx$. The proof of Lemma 3.2 in [2] is technically vague since $u - \xi_c$ does not have a zero mean, and as a result we cannot use Poincaré's inequality. To remedy this mistake in [2], we introduce the modified sub-functionals $\mathcal{E}_p$, $\mathcal{E}_f$ and $\mathcal{E}_d$:

$$\mathcal{E}(t) := 2\mathcal{E}_p(t) + 2\mathcal{E}_f(t) + \mathcal{E}_d(t), \qquad \mathcal{E}_p(t) := \int_{\mathbb{T}^3\times\mathbb{R}^3} \left|\xi - \xi_c(t)\right|^2 f\,d\xi\,dx,$$

$$\mathcal{E}_f(t) := \int_{\mathbb{T}^3} \left|u - u_c(t)\right|^2 dx, \qquad \mathcal{E}_d(t) := \left|u_c(t) - \xi_c(t)\right|^2, \quad t \geq 0,$$

where u_c and ξ_c are the mean bulk velocity of the fluid and the averaged C–S particle velocity defined in (2.2). More precisely, we will use Poincaré inequality to get dissipation corresponding to sub-functional $\mathcal{E}_f$ from the diffusion Δu. Next we investigate the evolution of distance between the averaged velocities $\xi_c(t)$ and $u_c(t)$ using the functional $\mathcal{E}_d$. Together with these observations, we finally show that the Lyapunov functional $\mathcal{E}(t)$ will be exponentially dissipated as time goes on. This is our new strategy for the estimate of asymptotic flocking behavior compared to the one in [2]. On the other hand, by the triangle inequality, $\mathcal{E}_f^o$ ($\mathcal{E}_f$ in Section 3, [2]) is bounded by $\mathcal{E}_f + \mathcal{E}_d$, which are defined above, therefore, the asymptotic result in [2] still works.

4.2. Time-decay estimates

In this part, we show that the functional $\mathcal{E}$ converges to zero exponentially fast, when the viscosity μ is sufficiently large.

Lemma 4.1. *Let (f, u) be any global classical solutions to the system (1.1)–(1.5). Then we have*

$$(i) \quad \frac{d\mathcal{E}_p}{dt} \leq -2m_\psi \mathcal{E}_p + 2\int_{\mathbb{T}^3\times\mathbb{R}^3} (\xi - \xi_c)\cdot(u - \xi) f\,d\xi\,dx.$$

$$(ii) \quad \frac{d\mathcal{E}_f}{dt} \leq -2\mu\pi_3 \mathcal{E}_f + 2\int_{\mathbb{T}^3\times\mathbb{R}^3} (u_c - u)\cdot(u - \xi) f\,d\xi\,dx.$$

$$(iii) \quad \frac{d\mathcal{E}_d}{dt} = -4\int_{\mathbb{T}^3\times\mathbb{R}^3} (u_c - \xi_c)\cdot(u - \xi) f\,d\xi\,dx,$$

where π_3 is an optimal constant appearing in Poincaré's inequality for the domain $\mathbb{T}^3$.

Proof. (i) Since the estimate for $\mathcal{E}_p$ can be found in Lemma 3.1 [2], we simply estimate the time-variation of the functionals $\mathcal{E}_f$ and $\mathcal{E}_d$.

(ii) We use (2.2) and the definition of $\mathcal{E}_f$ to get

$$\begin{aligned}\frac{d\mathcal{E}_f}{dt} &= 2\int_{\mathbb{T}^3}(u-u_c)\cdot\partial_t u dx - 2\dot{u}_c\cdot\int_{\mathbb{T}^3}(u-u_c)dx \\ &= 2\int_{\mathbb{T}^3}(u-u_c)\cdot\partial_t u dx \equiv 2\mathcal{T}.\end{aligned}$$

Below, we estimate the term $\mathcal{T}$. We use the second equation in (1.1) to find

$$\begin{aligned}\mathcal{T} &= -\int_{\mathbb{T}^3}(u\cdot\nabla)u\cdot(u-u_c)dx - \int_{\mathbb{T}^3}(u-u_c)\cdot\nabla p dx \\ &\quad + \mu\int_{\mathbb{T}^3}(u-u_c)\cdot\Delta u dx - \int_{\mathbb{T}^3\times\mathbb{R}^3}(u-u_c)\cdot(u-\xi)f dx d\xi \\ &=: \mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3+\mathcal{T}_4.\end{aligned}$$

• (Estimate of $\mathcal{T}_1$): We use integration by parts and $\nabla\cdot u=0$ to get

$$\begin{aligned}&\int_{\mathbb{T}^3}(u\cdot\nabla)u\cdot u dx = \frac{1}{2}\int_{\mathbb{T}^3}u\cdot\nabla|u|^2dx = -\frac{1}{2}\int_{\mathbb{T}^3}(\nabla\cdot u)|u|^2dx = 0, \\ &\int_{\mathbb{T}^3}(u\cdot\nabla)u dx = -\int_{\mathbb{T}^3}(\nabla\cdot u)u dx = 0.\end{aligned}$$

This implies

$$\mathcal{T}_1 = 0.$$

• (Estimate of $\mathcal{T}_2$): We use $\nabla\cdot u=0$ and the periodicity of p to find

$$\mathcal{T}_2 = \int_{\mathbb{T}^3}(u-u_c)\cdot\nabla p dx = -\int_{\mathbb{T}^3}(\nabla\cdot u)p dx = 0.$$

• (Estimate of $\mathcal{T}_3$): We use integration by parts and Poincaré's inequality to obtain

$$\mathcal{T}_3 = -\mu\int_{\mathbb{T}^3}|\nabla u|^2dx \le -\mu\pi_3\int_{\mathbb{T}^3}|u-u_c|^2dx = -\mu\pi_3\mathcal{E}_f,$$

where π_3 is the Poincaré constant for $\mathbb{T}^3$.

(iii) Because

$$\dot{\xi}_c = \int_{\mathbb{T}^3\times\mathbb{R}^3} (u-\xi) f d\xi dx \quad \text{and} \quad \dot{u}_c = -\int_{\mathbb{T}^3\times\mathbb{R}^3} (u-\xi) f d\xi dx,$$

we have

$$\begin{aligned}\frac{d\mathcal{E}_d}{dt} &= 2(u_c-\xi_c)\cdot(\dot{u}_c-\dot{\xi}_c)\\ &= -4(u_c-\xi_c)\cdot\int_{\mathbb{T}^3\times\mathbb{R}^3}(u-\xi) f d\xi dx. \qquad \square\end{aligned}$$

4.3. Proof of Theorem 2.2

We are now ready for the proof of Theorem 2.2. To get rid of the annoying term $\int_{\mathbb{T}^3\times\mathbb{R}^3}(u_c-\xi_c)\cdot(u-\xi) f d\xi dx$ in the time-decay estimates of $\mathcal{E}_p$, $\mathcal{E}_f$ and $\mathcal{E}_d$ in Lemma 4.1, we consider the following linear combination of sub-functionals:

$$\mathcal{E} := 2\mathcal{E}_p + 2\mathcal{E}_f + \mathcal{E}_d.$$

Then it follows from Lemma 4.1 that

$$\frac{d\mathcal{E}}{dt} \le -4m_\psi\mathcal{E}_p - 4\mu\pi_3\mathcal{E}_f - 4\int_{\mathbb{T}^3\times\mathbb{R}^3}|u-\xi|^2 f d\xi dx. \tag{4.1}$$

Note that our purpose is to derive Gronwall's inequality for $\mathcal{E}$, however, the estimates in (4.1) do not contain $\mathcal{E}_d$ on the right-hand-side, so in order to extract a good term $-|\mathcal{O}(1)|\mathcal{E}_d$ out of

$$-4\int_{\mathbb{T}^3\times\mathbb{R}^3}|u-\xi|^2 f d\xi dx,$$

we use the following standard interpolation technique:

$$\begin{aligned}&\int_{\mathbb{T}^3\times\mathbb{R}^3}|u-\xi|^2 f d\xi dx\\ &\quad= \int_{\mathbb{T}^3\times\mathbb{R}^3}|u-u_c+u_c-\xi_c+\xi_c-\xi|^2 f d\xi dx\\ &\quad= \int_{\mathbb{T}^3}\rho_p|u-u_c|^2 dx + |u_c-\xi_c|^2 + \int_{\mathbb{T}^3\times\mathbb{R}^3}|\xi_c-\xi|^2 f d\xi dx\\ &\qquad + 2\int_{\mathbb{T}^3\times\mathbb{R}^3}(u-u_c)\cdot(u_c-\xi_c) f d\xi dx + 2\int_{\mathbb{T}^3\times\mathbb{R}^3}(u_c-\xi_c)\cdot(\xi_c-\xi) f d\xi dx\end{aligned}$$

$$+2\int_{\mathbb{T}^3\times\mathbb{R}^3}(u-u_c)\cdot(\xi_c-\xi)f\,d\xi\,dx, \tag{4.2}$$

where $\rho_p(x,t)$ is the particle density defined by (2.4).

• (Second cross term in (4.2)):

$$\int_{\mathbb{T}^3\times\mathbb{R}^3}(u_c-\xi_c)\cdot(\xi_c-\xi)f\,d\xi\,dx=(u_c-\xi_c)\cdot\int_{\mathbb{T}^3\times\mathbb{R}^3}(\xi_c-\xi)f\,d\xi\,dx=0.$$

• (First and third cross terms in (4.2)): For the remaining two terms in (4.2), we estimate

$$\begin{aligned}
&\left|2\int_{\mathbb{T}^3\times\mathbb{R}^3}(u-u_c)\cdot(u_c-\xi)f\,d\xi\,dx\right|\\
&\le 2\delta\int_{\mathbb{T}^3}|u-u_c|^2\left(\int_{\mathbb{R}^3}f\,d\xi\right)dx+\frac{1}{2\delta}\int_{\mathbb{T}^3\times\mathbb{R}^3}|u_c-\xi|^2f\,d\xi\,dx\\
&=2\delta\int_{\mathbb{T}^3}\rho_p|u-u_c|^2dx+\frac{1}{2\delta}\left(|u_c-\xi_c|^2+\int_{\mathbb{T}^3\times\mathbb{R}^3}|\xi_c-\xi|^2f\,d\xi\,dx\right)\\
&=2\delta\int_{\mathbb{T}^3}\rho_p|u-u_c|^2dx+\frac{1}{2\delta}\big(\mathcal{E}_d(t)+\mathcal{E}_p(t)\big).
\end{aligned}\tag{4.3}$$

In (4.2), it follows from (4.3) that

$$\int_{\mathbb{T}^3\times\mathbb{R}^3}|u-\xi|^2f\,d\xi\,dx\ge(1-2\delta)\int_{\mathbb{T}^3}\rho_p|u-u_c|^2dx+\left(1-\frac{1}{2\delta}\right)(\mathcal{E}_d+\mathcal{E}_p). \tag{4.4}$$

Finally we combine (4.1) and (4.4) to obtain

$$\begin{aligned}
\frac{d\mathcal{E}}{dt}\le{}&-4\left(m_\psi+1-\frac{1}{2\delta}\right)\mathcal{E}_p-4\mu\pi_3\mathcal{E}_f-4\left(1-\frac{1}{2\delta}\right)\mathcal{E}_d\\
&-4(1-2\delta)\int_{\mathbb{T}^3}\rho_p|u-u_c|^2dx.
\end{aligned}$$

We now choose $\delta=1$ to satisfy

$$1-\frac{1}{2\delta}>0.$$

For such δ, we have

$$\begin{aligned}\frac{d\mathcal{E}}{dt} &\le -2(2m_\psi+1)\mathcal{E}_p - 2\big(2\mu\pi_3 - 2\big\|\rho_p(t)\big\|_{L^\infty}\big)\mathcal{E}_f - 2\mathcal{E}_d \\ &\le -2(2m_\psi+1)\mathcal{E}_p - 2\Big(2\mu\pi_3 - 2\sup_{0\le t\le\infty}\big\|\rho_p(t)\big\|_{L^\infty}\Big)\mathcal{E}_f - 2\mathcal{E}_d, \end{aligned} \tag{4.5}$$

where we used

$$\int_{\mathbb{T}^3} \rho_p|u-u_c|^2 dx \le \|\rho_p\|_{L^\infty}\mathcal{E}_f.$$

If the viscosity μ is sufficiently large so that

$$2\mu\pi_3 - 2\sup_{0\le t\le\infty}\big\|\rho_p(t)\big\|_{L^\infty} > 0, \quad \text{i.e., } \mu > \frac{\sup_{0\le t\le\infty}\|\rho_p(t)\|_{L^\infty}}{\pi_3},$$

then the relation (4.5) implies

$$\frac{d\mathcal{E}}{dt} \le -(2m_\psi+1)(2\mathcal{E}_p) - K(2\mathcal{E}_f) - 2\mathcal{E}_d \le -\min\{2m_\psi+1, K, 2\}\mathcal{E},$$

where $K = 2\mu\pi_3 - 2\sup_{0\le t\le\infty}\|\rho_p(t)\|_{L^\infty} > 0$. Then, Gronwall's lemma yields an exponential decay:

$$\mathcal{E}(t) \le \mathcal{E}(0)e^{-\gamma t}, \quad t \ge 0,$$

where $\gamma := \min\{2m_\psi+1, K, 2\}$. This completes the proof.

5. Conclusion

In this paper, we showed the global existence of the unique strong solution to the Cucker–Smale–Navier–Stokes system for a sufficiently regular and small data depending on the length of the existence time-interval $T \in [0,\infty)$. Theorem 2.1 implies that if the initial data is sufficiently regular and small, then the strong solution obtained are smooth enough to be classical solution. We also revisited the asymptotic flocking estimate for a family of classical solutions, in which the kinematic viscosity μ is sufficiently large compared to the sup-norm of the local particle density. Hence we corrected some mistakes in previous literature [2]. Of course our asymptotic flocking estimates are conducted in an a priori setting, but this is good enough to cover the classical solutions obtained in Theorem 2.1 (see Remark 2.1) as a corollary of Theorem 2.2.

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References

[1] C. Baranger, L. Desvillettes, Coupling Euler and Vlasov equation in the context of sprays: the local-in-time, classical solutions, J. Hyperbolic Differ. Equ. 3 (2006) 1–26.
[2] H.-O. Bae, Y.-P. Choi, S.-Y. Ha, M.-J. Kang, Time-asymptotic interaction of flocking particles and an incompressible viscous fluid, Nonlinearity 24 (2012) 1155.
[3] L. Boudin, L. Desvillettes, C. Grandmont, A. Moussa, Global existence of solution for the coupled Vlasov and Navier–Stokes equations, Differential Integral Equations 22 (2009) 1247–1271.
[4] M. Chae, K. Kang, J. Lee, Global existence of weak and classical solutions for the Navier–Stokes–Vlasov–Fokker–Planck equations, J. Differential Equations 251 (2011) 2431–2465.
[5] P. Constantin, C. Fefferman, E. Titi, A. Zarnescu, Regularity for coupled two dimensional nonlinear Fokker–Planck and Navier–Stokes systems, Comm. Math. Phys. 270 (2007) 789–811.
[6] P. Constantin, G. Seregin, Global regularity of solutions of coupled Navier–Stokes equations and nonlinear Fokker–Planck equations, Discrete Contin. Dyn. Syst. 26 (2010) 1185–1196.
[7] R. Duan, A. Lorz, P. Markowich, Global solutions to the coupled chemotaxis-fluid equations, Comm. PDE 35 (2010) 1635–1673.
[8] T. Goudon, L. He, A. Moussa, P. Zhang, The Navier–Stokes–Vlasov–Fokker–Planck system near equilibrium, SIAM J. Math. Anal. 42 (2010) 2177–2202.
[9] T. Goudon, P.-E. Jabin, A. Vasseur, Hydrodynamic limit for the Vlasov–Navier–Stokes equations I. Light particles regime, Indiana Univ. Math. J. 53 (2004) 1495–1515.
[10] T. Goudon, P.-E. Jabin, A. Vasseur, Hydrodynamic limit for the Vlasov–Navier–Stokes equations II. Fine particles regime, Indiana Univ. Math. J. 53 (2004) 1517–1536.
[11] S.-Y. Ha, E. Tadmor, From particle to kinetic and hydrodynamic description of flocking, Kinet. Relat. Models 1 (2008) 415–435.
[12] K. Hamdache, Global existence and large time behavior of solutions for the Vlasov–Stokes equations, Japan J. Indust. Appl. Math. 15 (1998) 51–74.
[13] A. Mellet, A. Vasseur, Global weak solutions for a Vlasov–Fokker–Planck/Navier–Stokes system of equations, Math. Models Methods Appl. Sci. 17 (2007) 1039–1063.
[14] F. Otto, A. Tzavaras, Continuity of velocity gradients in suspension of rod-like molecules, Comm. Math. Phys. 277 (2008) 729–758.