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Existence and uniqueness of solution for a generalized nonlinear derivative Schrödinger equation

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Abstract

In this work we study the well-posedness for the initial value problem associated to a generalized derivative Schrödinger equation for small size initial data in weighted Sobolev space. The techniques used include parabolic regularization method combined with sharp linear estimates. An important point in our work is that the contraction principle is likely to fail but gives us inspiration to obtain certain uniform estimates that are crucial to obtain the main result. To prove such uniform estimates we assume smallness on the initial data in weighted Sobolev spaces.

Keywords: Derivative Schrödinger equation; Parabolic regularization; Weighted Sobolev spaces; Contraction principle

1. Introduction

We shall study the following initial value problem (IVP)

$$\begin{cases} i\partial_t u + \partial_x^2 u + i|u|^\alpha \partial_x u = 0 \\ u(\cdot, 0) = u_0 \end{cases} \tag{1.1}$$

where u is a complex valued function of $(x, t) \in \mathbb{R} \times \mathbb{R}$ and $\alpha > 0$.

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The equation in (1.1) is a generalization of the derivative nonlinear Schrödinger equation, (DNLS)

$$i\partial_t u + \partial_x^2 u + i\partial_x(|u|^2 u) = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}. \tag{1.2}$$

The DNLS equation appears in physics as a model that describes the propagation of Alfvén waves in plasma (see [20,21,23]). In mathematics, this equation has also been extensively studied in regard to well-posedness for the associated IVP [5,7,9–12,27,28]. Tsutsumi and Fukuda [29], using parabolic regularization, proved local well-posedness in Sobolev spaces $H^s(\mathbb{R})$, $s > 3/2$. Hayashi [10] proved well-posedness for initial data $u_0 \in H^1(\mathbb{R})$ satisfying the smallness condition

$$\|u_0\|_{L^2} < \sqrt{2\pi}. \tag{1.3}$$

His idea was to use a gauge transformation to turn the DNLS equation into a system of nonlinear Schrödinger equations without derivative in the nonlinearity. This system, in turn, can be treated using Strichartz estimates. It is known that DNLS equation can be written as a Hamiltonian system

$$\frac{du(t)}{dt} = -iE'(u(t)) \tag{1.4}$$

where $E(u)$ is the energy of u defined by

$$E(u)(t) = \frac{1}{2}\int |\partial_x u|^2 dx + \frac{1}{4}\mathrm{Im}\int |u|^2 \bar{u}\partial_x u dx. \tag{1.5}$$

As a consequence of (1.4) it follows that E is a conserved quantity. In particular, the result of Hayashi is global in time. Later on Hayashi and Ozawa [11] based on the same gauge transformation proved global well-posedness for initial condition in $H^m(\mathbb{R})$, $m \in \mathbb{N}$, also satisfying the smallness condition (1.3). The best result regarding local well-posedness was obtained by Takaoka in [27]. He proved that the IVP associated to DNLS equation is well-posed in $H^s(\mathbb{R})$, for $s \geq 1/2$. He considered the gauge transformation

$$v(x,t) = u(x,t)\exp\left(\frac{i}{2}\int_{-\infty}^{x} |u(y,t)|^2 dy\right).$$

to transform DNLS equation into

$$i\partial_t v + \partial_x^2 v = -iv^2\partial_x\bar{v} - \frac{1}{2}|v|^4 v \tag{1.6}$$

and used the Fourier restriction norm method introduced by Bourgain [3]. Biagioni and Linares [2] proved that the IVP associated to DNLS equation is not well-posed in $H^s(\mathbb{R})$ for $s < 1/2$, which implies that Takaoka's result is sharp. Using the I-method Colliander, Keel, Staffilani, Takaoka and Tao [4,5] showed that the IVP associated to DNLS equation is globally well-posed for $s > 1/2$.

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The main difficulty to deal with DNLS equation is the presence of the derivative in the nonlinearity, which causes the so called loss of derivatives. This means that the standard way of proving existence of solution of

$$u(t) = U(t)u_0 - \int_0^t U(t-t')\partial_x(|u|^2 u)dt' \tag{1.7}$$

cannot be accomplished only by using the property of unitary group and Strichartz estimates of the Schrödinger propagator $U(t) = e^{it\partial_x^2}$. In fact the right hand side of (1.7) has less derivative than the left hand side and Strichartz estimates do not provide us gain of derivatives. This is one point that makes the study of DNLS equation more difficult than the corresponding cubic nonlinear Schrödinger equation (NLS), namely

$$i\partial_t u + \partial_x^2 u + \lambda|u|^2 u = 0.$$

The equation in (1.1) admits a family of solitary waves solutions given explicitly by

$$\psi(x,t) = \varphi_{\omega,c}(x-ct)\exp i\left\{\omega t + \frac{c}{2}(x-ct) - \frac{1}{\alpha}\int_{-\infty}^{x-ct}\varphi_{\omega,c}^{\alpha}(y)dy\right\},$$

where $\omega > c^2/4$ and

$$\varphi_{\omega,c}(y)^{\alpha} = \frac{(2+\alpha)(4\omega - c^2)}{4\sqrt{\omega}\left(\cosh(\frac{\alpha}{2}\sqrt{4\omega - c^2}y) - \frac{c}{2\sqrt{\omega}}\right)}.$$

Liu, Simpson and Sulem in [19] studied the orbital stability of these solitary waves for the equation in (1.1). More precisely,

Definition 1. The solitary wave $\psi_{\omega,c}$ is said to be *orbitally stable* for (1.1) if for each initial data $u_0 \in H^1(\mathbb{R})$ sufficiently close to $\psi_{\omega,c}(0)$ corresponds a unique u global solution of (1.1) and

$$\sup_{t\geq 0}\inf_{(\theta,y)\in\mathbb{R}\times\mathbb{R}}\|u(t) - e^{i\theta}\psi_{\omega,c}(t,\cdot - y)\|_{H^1}$$

is sufficiently small. Otherwise $\psi_{\omega,c}$ is said to be *orbitally unstable*.

Results of orbital stability or instability were obtained according to the value of α. It was assumed the existence of solution of the IVP (1.1) for arbitrary $\alpha > 0$, and initial data $u_0 \in H^1(\mathbb{R})$. However, besides the case $\alpha = 2$ there is only a few well-posedness results for (1.1). For integer powers $\alpha \geq 5$, Hao [8] proved local well-posedness in $H^{1/2}(\mathbb{R})$. There is also the work of Ambrose and Simpson [1], who established existence and uniqueness of solution $u \in L^\infty([0,T]; H^1(\mathbb{T}))$ for all values $\alpha \geq 1$. Nevertheless, to our knowledge there is no result dealing with the case $\alpha < 2$. This case possesses a lot of difficulties. In fact, the gauge transformation used in the work of Hayashi [9] cannot be applied, because it is required that u actually satisfies the differential equation, that is $u \in C^2$. Turns out we cannot expect such smoothness from u

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when $\alpha < 2$, since the nonlinear term $|u|^\alpha$ is only C^1. In the present work we carry out the case $\alpha > 1$. The main result regards the most interesting case $1 < \alpha < 2$.

Theorem 1. *Let $1 < \alpha < 2$. Consider $X = H^{3/2}(\mathbb{R}) \cap \{f \in S'(\mathbb{R}); xf \in H^{1/2}(\mathbb{R})\}$. If $u_0 \in X$ and*

$$\|u_0\|_X = \|u_0\|_{H^{3/2}} + \|xu_0\|_{H^{1/2}}$$

is sufficiently small then there exist $T = T(\|u_0\|_X) > 0$ and a unique

$$u \in C([0,T]; H^{3/2-}) \cap L^\infty([0,T]; X)$$

solution of the integral equation

$$u(t) = U(t)u_0 - \int_0^t U(t-t')(|u|^\alpha \partial_x u)(t')dt'. \tag{1.8}$$

Our strategy to prove Theorem 1 will be based on the technique known as parabolic regularization (or viscosity argument) introduced by T. Kato [14]. It consists in the following: First we prove that for each positive parameter ϵ (viscosity) the initial value problem

$$\begin{cases} i\partial_t u_\epsilon + \partial_x^2 u_\epsilon + i|u_\epsilon|^\alpha \partial_x u_\epsilon = i\epsilon \partial_x^2 u_\epsilon \\ u_\epsilon(\cdot, 0) = u_0 \end{cases} \tag{1.9}$$

has solution in $[0, T_\epsilon]$. The second step is to prove that all the solutions u_ϵ can be defined at the same interval of time and they converge as ϵ goes to zero and the limit solves (1.1). The first step is quite simple due to the presence of the parabolic term $i\epsilon\partial_x^2 u$. As usual we can obtain solution $u_\epsilon \in C([0, T_\epsilon]; H^s(\mathbb{R}))$, with $T_\epsilon = \epsilon/\|u_0\|_{H^s}$, in some Sobolev space $H^s(\mathbb{R})$. The difficulty lies in the second step. To prove that the maximal existing time of definition of each solution u_ϵ is independent of epsilon we need

$$\sup_{\epsilon>0} \|u_\epsilon\|_{L^\infty_{T_\epsilon} H^s_x} < \infty. \tag{1.10}$$

The estimate (1.10) permits to extend all the solutions to an interval of time $[0, T]$ independent of ϵ and the extension still satisfies $\sup_{\epsilon>0} \|u_\epsilon\|_{L^\infty_T H^s_x} < \infty$. Using compactness, for each $t \in [0, T]$ we have weak convergence $u_\epsilon(t)$ of some subsequence to an element $u(t)$ in $H^s(\mathbb{R})$. So, u is the candidate to be a solution. Thus we have a great chance to be successful with the parabolic regularization if we can prove the uniform estimate (1.10). In [13] it is already presented how (1.10) can be obtained in $H^s(\mathbb{R})$, $s > 3/2$, for nonlinearities like $u^m \partial_x u$, $m \in \mathbb{N}$. There, the argument can also be adapted to the nonlinearity $|u|^\alpha \partial_x u$ whenever we have some smoothness of the nonlinearity, for example $\alpha \geq 2$, that allows us to prove

$$\||u|^\alpha\|_{H^s} \leq c\|u\|^\alpha_{H^s}. \tag{1.11}$$

However, (1.11) is no longer true for low powers α since $|z|^\alpha$ does not have enough regularity. In this work we obtain a uniform estimate like (1.10) in $H^{3/2}(\mathbb{R})$ when $1 < \alpha < 2$

(see Theorem 4). Our argument was inspired by trying to use the contraction mapping principle. Using sharp smoothing properties associated to the linear equation we define a subspace $E \subset C([0,T]; H^s(\mathbb{R}))$ and prove that the associated integral operator

$$\Psi(u) = U(t)u_0 - \int_0^t U(t-t')(|u|^\alpha \partial_x u)dt' \tag{1.12}$$

is well defined, i.e., $\Psi : E \to E$ for small data. However, the argument we used to prove Ψ is well defined in E does not provide contraction when considering $\alpha < 2$ (see Remark 2).

We will also study the case $\alpha > 2$. The difficulties we have to perform contraction argument in the case $1 < \alpha < 2$, disappear when considering $\alpha > 2$. This is because the contraction principle argument yields us to consider maximal function

$$\Big\| \sup_{t\in[0,T]} |u(\cdot,t)| \Big\|_{L^\alpha} \tag{1.13}$$

and there is no estimate for the norm (1.13) when $\alpha < 2$.

Theorem 2. *Let $\alpha > 2$ and $T > 0$ be given. There exists $\delta > 0$ such that for all initial data $u_0 \in H^{1/2}(\mathbb{R})$ with $\|u_0\|_{H^{1/2}} \le \delta$ there exists one, and only one $u \in C([0,T]; H^{1/2}(\mathbb{R}))$ solution of*

$$u(t) = U(t)u_0 - \int_0^t U(t-t')(|u|^\alpha \partial_x u)(t')dt' \tag{1.14}$$

satisfying

$$\|\partial_x u\|_{L^\infty_x L^2_T}, \quad \|u\|_{L^\alpha_x L^\infty_T} < \infty, \quad \|u\|_{L^{2\alpha}_x L^\infty_T} < \infty.$$

The main point in the proof of Theorem 2 is the fact that we do have maximal estimates when $\alpha \ge 2$ (see Lemma 3).

The remainder of our work is organized as follows: In Section 2 we present a list of results that will be used along the work; In Section 3 we prove existence of solutions u_ϵ to the problem (3.1), and establish some uniform estimates for these solutions; In Section 4 we prove Theorem 1 by proving the convergence of the sequence u_ϵ as the parameter goes to zero and the limit satisfies the integral equation (1.8); In Section 5 we study the case $\alpha > 2$. In this case we establish well-posedness for the IVP (1.1) in $H^{1/2}(\mathbb{R})$, for small data via contraction argument.

1.1. Notation

Given $A, B > 0$ the notation $A \lesssim B$ means $A \le cB$ for some constant $c > 0$. $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. The notation $r+$ stands $r + \epsilon$, with $\epsilon > 0$ sufficiently small. Also, we write T^+, to denote T raised to some positive power. We denote $\langle x \rangle = (1 + x^2)^{1/2}$. The Fourier transform of a function f, and its inverse Fourier transform are denoted by $\hat{f}$ and $\check{f}$ respectively. For an $s \in \mathbb{R}$, $J^s = (1 - \partial_x^2)^{s/2}$, and $D^s = (-\partial_x^2)^{s/2}$ stand respectively for the Riesz and Bessel potential of order $-s$. The space $H^s(\mathbb{R}) = (1-\partial_x^2)^{-s/2}(L^2(\mathbb{R}))$ denotes the Sobolev space of order s endowed with the norm $\|f\|_{H^s} = \|J^s f\|_{L^2}$. For two variable functions $f = f(x,t)$, with $(x,t) \in \mathbb{R} \times [0,T]$, we consider the mixed spaces $L^q([0,T]; L^p(\mathbb{R}))$ and $L^p(\mathbb{R}; L^q([0,T]))$, corresponding to the norms

$$\|f\|_{L_T^q L_x^p} = \left(\int_0^T \Big(\int_{\mathbb{R}} |f(x,t)|^p dx \Big)^{q/p} dt \right)^{1/q}$$

and

$$\|f\|_{L_x^p L_T^q} = \left(\int_{\mathbb{R}} \Big(\int_0^T |f(x,t)|^q dt \Big)^{p/q} dx \right)^{1/p},$$

respectively.

2. Preliminary estimates

We consider the IVP associated to the free Schrödinger equation

$$\begin{cases} i\partial_t u + \partial_x^2 u = 0, & x,t \in \mathbb{R} \\ u(\cdot,0) = f \end{cases} \tag{2.1}$$

whose solution is given by

$$u(x,t) = \{e^{-it\xi^2} \hat{f}\}^\vee(x)$$

and is denoted by $U(t)f$. The family $\{U(t)\}_{t\in\mathbb{R}}$ forms a unitary group in $H^s(\mathbb{R})$, for all $s \in \mathbb{R}$. In the following we will list some estimates satisfied for solutions of IVP (2.1). We begin by the so called $L^q L^p$ estimates or Strichartz estimates.

Lemma 1 *(Strichartz estimates). For all pairs (p,q) satisfying*

$$2 \le p \le \infty \ \text{ and } \ 2/q = 1/2 - 1/p$$

we have

$$\|U(t)f\|_{L_T^q L_x^p} \le c\|f\|_{L^2} \tag{2.2}$$

and

$$\left\| \int_0^t U(t-t')F(x,t')dt' \right\|_{L_T^q L_x^p} \le c\|F\|_{L_T^{q'} L_x^{p'}} \tag{2.3}$$

where $1/p' + 1/p = 1/q' + 1/q = 1$, for some constant $c > 0$.

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Proof. See Ginibre and Velo [6]. □

Next we have the smoothing effects estimates.

Lemma 2 *(Smoothing effects). There exists a constant* $c > 0$*, such that:*
i)

$$\|D^{1/2}U(t)f\|_{L_x^\infty L_T^2} \leq c\|f\|_{L^2}; \tag{2.4}$$

for all $f \in L^2(\mathbb{R})$*, and*
ii)

$$\left\| D^{1/2} \int_{\mathbb{R}} U(t')F(\cdot,t')dt' \right\|_{L_T^\infty L_x^2} \leq c\|F\|_{L_x^1 L_T^2}; \tag{2.5}$$

iii)

$$\left\| \partial_x \int_0^t U(t-t')F(\cdot,t')dt' \right\|_{L_x^\infty L_T^2} \leq c\|F\|_{L_x^1 L_T^2}; \tag{2.6}$$

for all $F \in L_x^1 L_T^2$.

The above estimates were proved by Kenig, Ponce and Vega [15]. For a detailed proof of Lemma 1 to (2.6) see [18, Chapter 4].

Finally, we present the following maximal function estimates for the Schrödinger propagator.

Lemma 3 *(Maximal function estimate). Let* $p \in (2, \infty)$ *and* $s \geq \max\{1/2 - 1/p, 1/p\}$*. Then there exists a constant* $c > 0$ *such that*

$$\|U(t)f\|_{L_x^p L_T^\infty} \leq c\|f\|_{H^s}. \tag{2.7}$$

Proof. See Keith M. Rogers and Paco Villarroya [24]. □

Remark 1. Estimate (2.7) was established by Kenig and Ruiz [17] in the case $p = 4$ and later on by Kenig, Ponce and Vega [16] for $p = 2$ and $s > 1/2$.

Lemma 4. *Given* s *and* p *as in Lemma 3 we have*

$$\left\| \int_0^t U(t-t')F(\cdot,t')dt' \right\|_{L_x^p L_T^\infty} \leq c(\|F\|_{L_x^1 L_T^2} + \|F\|_{L_T^1 L_x^2}) \tag{2.8}$$

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Proof. Using Lemma 3 and then applying estimate (2.5) in Lemma 2 we get

$$\left\| \int_0^1 U(t-t')F(\cdot,t')dt' \right\|_{L_x^p L_T^\infty} \le c(\|F\|_{L_x^1 L_T^2} + \|F\|_{L_T^1 L_x^2}). \tag{2.9}$$

According to the proof of Lemma 3 in [22], the nonretarded estimate (2.9) implies the retarded one in (2.8). □

3. The regularized problem

In this section we shall consider for each $\epsilon > 0$ the problem

$$\begin{cases} i\partial_t u + \partial_x^2 u + i|u|^\alpha \partial_x u = i\epsilon\partial_x^2 u \\ u(\cdot,0) = u_0 \end{cases} \tag{3.1}$$

where $1 < \alpha < 2$.

3.1. Linear estimates

We shall denote by $U_\epsilon(t)f = e^{(i+\epsilon)t\partial_x^2} f$ the solution of the linear problem associated to (3.1), which is, via Fourier transform, given by $U_\epsilon(t)f = \{e^{-(i+\epsilon)t\xi^2}\widehat{f}\}^\vee$.

Lemma 5. *Given $s > 0$, there exists a constant $c = c_s > 0$ such that*

$$\|U_\epsilon(t)f\|_{H_x^s} \le c_s\left(1 + \frac{1}{(\epsilon t)^{s/2}}\right)\|f\|_{L^2}.$$

Proof. Indeed, using Plancherel Theorem we get

$$\begin{aligned} \|D^s U_\epsilon(t)f\|_{L_x^2} &= \||\xi|^{s/2} e^{-(i+\epsilon)t\xi^2}\widehat{f}\|_{L_x^2} \\ &\le \frac{1}{|\epsilon t|^{s/2}} \sup_{y>0}\{y^{s/2}e^{-y}\}\|f\|_{L^2}. \qquad \Box \end{aligned}$$

Next we establish properties that are true for $\{U(t)\}$ and still hold for $U_\epsilon(t)$ uniformly in the parameter ϵ. We notice that $U_\epsilon(t) = E_\epsilon(t)U(t)$ where $E_\epsilon(t)g = \{e^{-\epsilon t\xi^2}\widehat{g}\}^\vee$, is the heat flow. Let $\varphi(x) = e^{-\pi x^2}$ and denote by φ_ρ the function $\varphi_\rho(x) = \rho\varphi(\rho x)$. So the heat flow can be written as

$$E_\epsilon(t)g = \varphi_{\rho_{\epsilon,t}} * g, \tag{3.2}$$

where $\rho_{\epsilon,t} = \sqrt{\frac{\pi}{\epsilon t}}$. The formula (3.2) together with the Young inequality and Strichartz estimates for the Schrödinger group $U(t)$ (Lemma 1) give us

Lemma 6. *There exists a constant $c > 0$ independent of ϵ such that*

$$\|U_\epsilon(t) f\|_{L_T^q L_x^p} \leq c\|f\|_{L^2}$$

for all pairs (p, q) satisfying $2 \leq p \leq \infty$, $1/2 = 2/q + 1/p$.

In order to prove a smoothing effect for $U_\epsilon(t)$, with constants independent of ϵ we consider the following norm

$$\|\cdot\|_{l_j^\infty(L^2(Q_j))} = \sup_{j\in\mathbb{Z}} \|\cdot\|_{L^2(Q_j)}$$

where Q_j is the rectangle $Q_j = [0, T] \times [j, j+1]$.

Lemma 7. *Let $T > 0$ and $\epsilon > 0$ satisfying $0 < \epsilon < \pi/T$. Then there exists $c > 0$ independent of ϵ such that*

$$\|D^{1/2} U_\epsilon(t) f\|_{l_j^\infty(L^2(Q_j))} \leq c\|f\|_{L^2}$$

for all $f \in L^2(\mathbb{R})$.

Proof. It is enough to prove that

$$\|E_\epsilon(t) g\|_{l_j^\infty(L^2(Q_j))} \leq c\|g\|_{l_j^\infty(L^2(Q_j))}$$

with c independent of ϵ. Writing $E_\epsilon(t)$ as the convolution in (3.2) we have

$$\begin{aligned} E_\epsilon(t) g(\cdot, t)(x) &= \int_{|y|<1} \varphi_{\rho_{\epsilon,t}}(y) g(x-y, t) dy + \int_{|y|>1} \varphi_{\rho_{\epsilon,t}}(y) g(x-y, t) dy \\ &= I_0(x, t) + I_\infty(x, t). \end{aligned}$$

By using the Minkowsky inequality, a change of variables and the fact $[j-y, j+1-y] \subset [j-1, j+2]$ for all $|y| < 1$, we get

$$\|I_0(\cdot, t)\|_{L^2([j, j+1])} \leq \|g(\cdot, t)\|_{L^2([j-1, j+2])} \int_{\mathbb{R}} \varphi(y) dy,$$

for all $j \in \mathbb{Z}$. Thus

$$\|I_0\|_{l_j^\infty(L^2(Q_j))} \lesssim \|g\|_{l_j^\infty(L^2(Q_j))}.$$

To bound $\|I_\infty\|_{l_j^\infty(L^2(Q_j))}$, notice that $\rho_{\epsilon,t} \leq \rho_{\epsilon,t}^2$ for $\epsilon \leq \pi/T$. Then for ϵ sufficiently small

$$
\begin{aligned}
|I_\infty(x,t)| &\leq \int\limits_{|y|>1} \rho_{\epsilon,t}^2 e^{-|\rho_{\epsilon,t} y|^2} |g(x-y,t)| dy \\
&\leq \sup\{\delta e^{-\delta} : \delta > 0\} \int\limits_{|y|>1} \frac{1}{y^2} |g(x-y,t)| dy.
\end{aligned}
$$

Using change of variables we have

$$
\|I_\infty\|_{L^2(Q_j)} \lesssim \int\limits_{|y|>1} \frac{1}{y^2} \|g\|_{L^2([0,T]\times[m_{j,y}, m_{j,y}+2])} dy
$$

where $m_{j,y}$ denotes the integer number such that $m_{j,y} \leq j - y < m_{j,y} + 1$. Then we conclude

$$
\|I_\infty\|_{l_j^\infty(L^2(Q_j))} \lesssim \|g\|_{l_j^\infty(L^2(Q_j))}. \qquad \square
$$

As a consequence of Lemma 7 we have the dual version:

Lemma 8. *There exists $c > 0$ independent of ϵ such that*

$$
\|D^{1/2} \int\limits_0^t U_\epsilon(t-t') F(\cdot, t') dt'\|_{L_T^\infty L_x^2} \leq c \|F\|_{l_j^1(L^2(Q_j))}. \tag{3.3}
$$

Proof. Consider $G \in L^2(\mathbb{R})$, such that $\|G\|_{L^2} = 1$. Using Fubini's theorem, Parserval's identity and the definition of $U_\epsilon(t)$ we have

$$
\begin{aligned}
&\int\limits_{\mathbb{R}} \left(D^{1/2} \int\limits_0^t U_\epsilon(t-t') F(\cdot, t')(x) dt' \right) \bar{G}(x) dx \\
&= \int\limits_0^t \int\limits_{\mathbb{R}} F(x,t') \overline{D^{1/2} E_\epsilon(t-t') U(t'-t) G(x)} dx dt'.
\end{aligned} \tag{3.4}
$$

Splitting the integral with respect to the variable x in (3.4) into a sum of the integral over the intervals $[j, j+1]$, using the Hölder inequality and then Lemma 7 it follows

$$
\begin{aligned}
&\left| \int\limits_{\mathbb{R}} \left(D^{1/2} \int\limits_0^t U_\epsilon(t-t') F(\cdot, t')(x) dt' \right) G(x) dx \right| \\
&\sum_{j\in\mathbb{Z}} \|F\|_{L^2([0,t]\times[j,j+1])} \|D^{1/2} E_\epsilon(t-\cdot) U(\cdot - t) \bar{G}\|_{L^2([0,t]\times[j,j+1])} \\
&\lesssim \sum_{j\in\mathbb{Z}} \|F\|_{L^2([0,t]\times[j,j+1])}.
\end{aligned} \tag{3.5}
$$

By duality (3.3) holds. $\square$

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We are also able to prove the following inhomogeneous smoothing effect.

Lemma 9. *There exists $c > 0$ independent of ϵ such that*

$$\|\partial_x \int_0^t U_\epsilon(t-t')F(\cdot,t')dt'\|_{l_j^\infty(L^2(Q_j))} \leq c\|F\|_{l_j^1(L(Q_j))}.$$

The proof of this lemma follows basically the ideas presented in [18]. The additional ingredient is the following:

Lemma 10. *For each $w \in \mathbb{C}$ define the function $f_w(x) = \dfrac{x}{x^2+w^2}$. Then there exists a constant $c > 0$ such that*

$$\|\widehat{f_w}\|_{L^\infty} \leq c$$

for all $w \in \mathbb{C}$.

Proof. Indeed, for $\mathrm{Re}(w) > 0$, we consider for each $\xi \in \mathbb{R}$ the complex function

$$F(z) = \frac{e^{i|\xi|z}}{z^2+w^2},$$

defined in $D_R^+ = \{z \in \mathbb{C}; \mathrm{Im}(z) > 0, |z| \leq R\}$. We use residue calculus to compute the integral of F along the boundary of D_R^+. On the other hand, the integral along the curve $\{z \in \mathbb{C}; |z| = R\}$, goes to zero as R goes to zero. Following these ideas we end up with

$$\left\{\frac{1}{x^2+w^2}\right\}^\vee = \frac{\pi}{w}e^{-w|\xi|}.$$

Using the relation between Fourier transform and differentiation we obtain $\widehat{f_w}(\xi) = -i\pi\,\mathrm{sgn}(\xi)e^{-w|\xi|}$. The case $\mathrm{Re}(w) < 0$ follows from the fact $f_{-w} = f_w$. Finally, when $\mathrm{Re}(w) = 0$ we can write f_w as

$$f_w = \frac{1}{2}\left[\frac{1}{x-\mathrm{Im}(w)} + \frac{1}{x+\mathrm{Im}(w)}\right],$$

and then we compute $\widehat{f_w}$ as a sum of Fourier transform of translations of the principal value distribution $p.v.\left(\dfrac{1}{x}\right)$. Thus we get $\widehat{f_w} = -i\pi\cos(\mathrm{Im}(w)\xi)\mathrm{sgn}(\xi)$. In any case it follows $\|\widehat{f_w}\|_{L^\infty} \leq \pi$. □

Proof of Lemma 9. Denote by v the following function

$$v(x,t) = \int_0^t U_\epsilon(t-t')F(\cdot,t')(x)dt'.$$

The function v can be written as

$$v(x,t) = -\int\int \frac{e^{it\tau} - e^{-it(1-i\epsilon)\xi^2}}{\tau + (1-i\epsilon)\xi^2} e^{ix\xi} \tilde{F}(\xi,\tau) d\xi d\tau, \tag{3.6}$$

where $\tilde{F}$ denotes the Fourier transform of F with respect to both variables x and t. Then differentiating v with respect to the variable x we obtain from (3.6)

$$\begin{aligned}\partial_x v(x,t) &= -\int\int \frac{i\xi e^{it\tau}}{\tau + (1-i\epsilon)\xi^2} e^{ix\xi} \tilde{F}(\xi,\tau) d\xi d\tau \\ &\quad + \int\int \frac{i\xi e^{-it(1-i\epsilon)\xi^2}}{\tau + (1-i\epsilon)\xi^2} e^{ix\xi} \tilde{F}(\xi,\tau) d\xi d\tau \\ &= \partial_x v_1(x,t) + \partial_x v_2(x,t).\end{aligned}$$

Write

$$\partial_x v_1(x,t) = -\frac{1}{1-i\epsilon} \int\int \frac{i\xi}{w^2 + \xi^2} \tilde{F}(\xi,\tau) e^{ix\xi} e^{it\tau} d\xi d\tau \tag{3.7}$$

where $w = w_{\epsilon,\tau}$ is a complex number such that $w^2 = \dfrac{\tau}{1-i\epsilon}$. Denoting

$$K_w(x) = \left\{ \frac{i\xi}{w^2 + \xi^2} \right\}^{\vee}(x),$$

and using the properties of convolution in the integral (3.7) and then applying Plancherel Theorem with respect to the variable t it follows

$$\|\partial_x v_1(x,\cdot)\|_{L^2_t} = \frac{1}{\sqrt{1+\epsilon^2}} \|\{K_{w_{\tau,\epsilon}} * \widehat{F}^{(t)}(\tau)\}(x)\|_{L^2_\tau}.$$

Then applying the Minkowsky inequality for integrals and finally Lemma 10 we obtain

$$\begin{aligned}\|\partial_x v_1(x,\cdot)\|_{L^2_t} &\le \sup_{\tau,\epsilon} \|K_{w_{\tau,\epsilon}}\|_{L^\infty} \int \|\widehat{F(y,\cdot)}^{(t)}\|_{L^2_\tau} dy \\ &\lesssim \|F\|_{L^1_x L^2_t}.\end{aligned}$$

Since $\|\cdot\|_{l^\infty_j(L^2(Q_j))} \le \|\cdot\|_{L^\infty_x L^2_T}$ and $\|\cdot\|_{L^1_x L^2_T} \le \|\cdot\|_{l^1_j(L^2(Q_j))}$ we conclude

$$\|\partial_x v_1\|_{l^\infty_j(L^2(Q_j))} \lesssim \|F\|_{l^1_j(L^2(Q_j))}.$$

Now we turn our attention to $\partial_x v_2$. First of all note that one can be written as

$$\partial_x v_2(x,t) = D^{1/2} U_\epsilon(t) G(x) \tag{3.8}$$

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where G is defined via Fourier Transform by

$$\widehat{G}(\xi) = \int \frac{i\,\mathrm{sgn}(\xi)|\xi|^{1/2}\tilde{F}(\xi,\tau)}{\tau + (1-i\epsilon)\xi^2} d\tau.$$

Notice that

$$\left\{\mathrm{p.v.}\frac{1}{\tau + (1-i\epsilon)\xi^2}\right\}^{\vee}(t) = c\ \ \mathrm{sgn}(t)e^{-it(1-i\epsilon)\xi^2}. \tag{3.9}$$

So using Parseval Identity and (3.9) we have

$$\widehat{G}(\xi) = \left\{\mathcal{H}D^{1/2}\int U_\epsilon(t)F(\cdot,t)\mathrm{sgn}(t)dt\right\}^{\wedge}(\xi) \tag{3.10}$$

where $\mathcal{H}$ is the Hilbert transform. Using (3.8), Lemma 7, (3.10) and Lemma 8 we conclude

$$\|\partial_x v_2\|_{l_j^\infty(L^2(Q_j))} \lesssim \|F\|_{l_j^1(L^2(Q_j))}. \qquad \square$$

Lemma 11. *If f is differentiable then for all x, $t \in \mathbb{R}$ we have*

$$xU_\epsilon(t)f(x) = U_\epsilon(t)(xf)(x) - 2(i+\epsilon)tU_\epsilon(t)(\partial_x f)(x)$$

Proof. By using properties of the Fourier transform and the definition of $U_\epsilon(t)$ we have

$$\begin{aligned} xU_\epsilon(t)f(x) &= i\{\partial_\xi(e^{-(\epsilon+i)t\xi^2}\hat{f})\}^\vee(x) \\ &= -2(i+\epsilon)t\{e^{-(\epsilon+i)t\xi^2}i\xi\hat{f}\}^\vee(x) + \{e^{-(\epsilon+i)t\xi^2}\partial_\xi\hat{f}\}^\vee(x) \\ &= -2(i+\epsilon)tU_\epsilon(t)(\partial_x f)(x) + U_\epsilon(t)(xf)(x). \qquad \square \end{aligned}$$

3.2. Local existence theory

Now we deal with the problem of existence of solution to (3.1). We shall prove the existence of solution to the corresponding integral equation

$$u(t) = U_\epsilon(t)u_0 - \int_0^t U_\epsilon(t-t')(|u|^\alpha\partial_x u)(t')dt'. \tag{3.11}$$

Theorem 3. *Let $\epsilon > 0$ and $u_0 \in H^{3/2}(\mathbb{R})$ be given. Then there exists one, and only one, $u_\epsilon \in C([0,T]; H^{3/2}(\mathbb{R}))$ solution of the integral equation (3.11), where*

$$T_\epsilon = \frac{\epsilon^3}{c\|u_0\|_{H^{3/2}}^{4\alpha}}$$

for some constant c, independent of ϵ.

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Proof. We define the integral operator Ψ_ϵ given on the right hand side of (3.11). Using Lemma 5 it follows

$$\begin{aligned}\|\Psi_\epsilon(u)\|_{H^{3/2}_x} &\le \|u_0\|_{H^{3/2}} + \int_0^t \left(1 + \frac{c}{(\epsilon|t-t'|)^{3/4}}\right) \|(|u|^\alpha \partial_x u)(t')\|_{L^2_x} dt' \\ &\le \|u_0\|_{H^{3/2}} + \left(T + \frac{4cT^{1/4}}{\epsilon^{3/4}}\right) \||u|^\alpha \partial_x u\|_{L^\infty_T L^2_x}.\end{aligned}$$

Using the Sobolev embedding $H^{1/2+}(\mathbb{R}) \subset L^\infty(\mathbb{R})$ we conclude

$$\|\Psi_\epsilon(u)\|_{L^\infty_T H^{3/2}_x} \le \|u_0\|_{H^{3/2}} + \left(T + \frac{4cT^{1/4}}{\epsilon^{3/4}}\right) \|u\|^{\alpha+1}_{L^\infty_T H^{3/2}_x}. \tag{3.12}$$

Therefore the operator Ψ_ϵ maps

$$E_T = \{u \in C([0,T]; H^{3/2}(\mathbb{R})); \|u\|_{L^\infty_T H^{3/2}_x} \le 2\|u_0\|_{H^{3/2}}\}$$

into itself. Next, we prove

$$\Psi_\epsilon : E_T \longrightarrow E_T$$

is a contraction with respect to the norm $\|\cdot\|_{L^\infty_{T_\epsilon} H^{3/2}_x}$. Indeed, consider $u, v \in E_T$ and denote $G(u,v) = |u|^\alpha \partial_x u - |v|^\alpha \partial_x v$. We have

$$\Psi_\epsilon(u) - \Psi_\epsilon(v) = -\int_0^t U_\epsilon(t-t') G(u,v)(t') dt'.$$

Then using Lemma 5 we obtain similarly

$$\|\Psi_\epsilon(u) - \Psi_\epsilon(v)\|_{H^{3/2}_x} \lesssim \left(T + \frac{4cT^{1/4}}{\epsilon^{3/4}}\right) \|G(u,v)\|_{L^\infty_T L^2_x}. \tag{3.13}$$

Using the property

$$\left||u|^\alpha - |v|^\alpha\right| \lesssim \left(|u|^{\alpha-1} + |v|^{\alpha-1}\right) |u - v| \tag{3.14}$$

combined with the Sobolev embedding we can obtain

$$\|G(u,v)\|_{L^\infty_T L^2_x} \lesssim (\|u\|^{\alpha-1}_{H^{3/2}} + \|u\|^{\alpha-1}_{H^{3/2}}) \|u - v\|_{L^\infty_T H^{3/2}_x}. \tag{3.15}$$

Plugging estimate (3.15) into (3.13) we conclude

$$\|\Psi_\epsilon(u) - \Psi_\epsilon(v)\|_{L^\infty_{T_\epsilon} H^{3/2}_x} \le \frac{1}{2} \|u - v\|_{L^\infty_{T_\epsilon} H^{3/2}_x}.$$

From the Banach fixed point theorem for contractions we conclude our proof. □

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3.3. *Uniform estimates for the solutions of the regularized problem*

We consider the norm

$$\begin{aligned}\Omega(u) &\equiv \|u\|_{L^\infty_T H^{3/2}_x} + \|xu\|_{L^\infty_T H^{1/2}_x} + \|\partial_x^2 u\|_{l^\infty_j(L^2(Q_j))} + \|\partial_x(xu)\|_{l^\infty_j(L^2(Q_j))}\\ &\equiv \Omega_1(u) + \Omega_2(u) + \Omega_3(u) + \Omega_4(u)\end{aligned}$$

We intend to prove that $\sup_{\epsilon>0}\Omega(u_\epsilon)$ is finite whenever the initial data belongs to the weighted space

$$X = H^{3/2}(\mathbb{R}) \cap \{f \in \mathcal{S}'(\mathbb{R});\, xf \in H^{1/2}(\mathbb{R})\}$$

and

$$\|u_0\|_X = \|u_0\|_{H^{3/2}} + \|xu_0\|_{H^{1/2}}$$

is sufficiently small.
I-*The norms* Ω_1 *and* Ω_3:
Using Lemma 7, Lemma 8 and Lemma 9 we obtain

$$\begin{aligned}\Omega_1(u_\epsilon) + \Omega_3(u_\epsilon) \lesssim \|u_0\|_{H^{3/2}} &+ \||u_\epsilon|^\alpha \partial_x^2 u_\epsilon\|_{l^1_j(L^2(Q_j))}\\ &+ \||u_\epsilon|^{\alpha-1}(\partial_x u_\epsilon)^2\|_{l^1_j(L^2(Q_j))}.\end{aligned} \tag{3.16}$$

Let us take care of each term in (3.16) separately. First use the Hölder inequality to obtain

$$\||u_\epsilon|^\alpha \partial_x^2 u_\epsilon\|_{l^1_j(L^2(Q_j))} \le \|u_\epsilon\|^\alpha_{l^\alpha_j(L^\infty(Q_j))}\Omega_3(u).$$

To control $\|\cdot\|_{l^\alpha_j(L^\infty(Q_j))}$ we introduce a weight. Indeed, by the Hölder inequality we have

$$\|u_\epsilon\|_{l^{1+}_j(L^\infty(Q_j))} \lesssim \|\langle x\rangle^{1-} u_\epsilon\|_{L^\infty_T L^\infty_x}.$$

Then, the Sobolev embedding gives us

$$\|u_\epsilon\|_{l^{1+}_j(L^\infty(Q_j))} \lesssim \|J^{1/2+}\big(\langle x\rangle^{1-} u_\epsilon\big)\|_{L^\infty_T L^2_x}.$$

Combining Lemma 14, and Lemma 16 we deduce

$$\|u_\epsilon\|_{l^{1+}_j(L^2(Q_j))} \lesssim \Omega(u_\epsilon). \tag{3.17}$$

Consequently

$$\||u_\epsilon|^\alpha \partial_x^2 u_\epsilon\|_{l^1_j(L^\infty(Q_j))} \lesssim \Omega(u_\epsilon)^{\alpha+1}. \tag{3.18}$$

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Now we turn our attention to $\||u|^{\alpha-1}(\partial_x u)^2\|_{l^1_j(L^2(Q_j))}$. First we use the Hölder inequality and (3.17) to obtain

$$\||u_\epsilon|^{\alpha-1}(\partial_x u_\epsilon)^2\|_{l^1_j(L^2(Q_j))} \le \Omega(u_\epsilon)^{\alpha-1}\|\partial_x u_\epsilon\|^2_{l_j^{\frac{2}{2-\alpha}-}(L^4(Q_j))}. \tag{3.19}$$

To examine (3.19) we consider two cases:
Case 1: $3/2 < \alpha < 2$.
In this case $\|\cdot\|_{L^4(Q_j)} \le T^+\|\cdot\|_{L^{\frac{2}{2-\alpha}-}(Q_j)}$, so and then

$$\|\partial_x u_\epsilon\|_{l_j^{\frac{2}{2-\alpha}-}(L^4(Q_j))} \le T^+\|\partial_x u_\epsilon\|_{L_T^{\frac{2}{2-\alpha}-}L_x^{\frac{2}{2-\alpha}-}}. \tag{3.20}$$

Applying the Sobolev embedding to (3.20) we conclude

$$\|\partial_x u_\epsilon\|_{l_j^{\frac{2}{2-\alpha}-}(L^4(Q_j))} \lesssim T^+\Omega_1(u_\epsilon).$$

Case 2: $1 < \alpha \le 3/2$.
Notice that

$$\begin{aligned}\|\partial_x u_\epsilon\|_{l_j^{\frac{2}{2-\alpha}-}(L^4(Q_j))} &\lesssim \|\langle x\rangle^\rho \partial_x u_\epsilon\|_{L^4_T L^4_x}\\ &\lesssim T^{1/4}\|J^{1/4}(\langle x\rangle^\rho \partial_x u_\epsilon)\|_{L^\infty_T L^2_x},\end{aligned}$$

for $\rho > \frac{3-2\alpha}{4}$. To finish the analysis of this case, we claim that

$$\|D^{1/4}(\langle x\rangle^\rho \partial_x u_\epsilon)\|_{L^2} \lesssim \|J^{1/4}(xu_\epsilon)\|_{L^2} + \|J^{3/2}u_\epsilon\|_{L^2}. \tag{3.21}$$

In fact, note that

$$\|D^{1/4}(\langle x\rangle^\rho \partial_x u_\epsilon)\|_{L^2} \le \|D^{5/4}(\langle x\rangle^\rho u_\epsilon)\|_{L^2} + \|D^{1/4}\left(u_\epsilon \partial_x(\langle x\rangle^\rho)\right)\|_{L^2}. \tag{3.22}$$

Since $\partial_x(\langle x\rangle^\rho)$ and $\partial_x^2(\langle x\rangle^\rho)$ are bounded we have

$$\begin{aligned}\|D^{1/4}\left(u_\epsilon \partial_x(\langle x\rangle^\rho)\right)\|_{L^2} &\lesssim \|u_\epsilon \partial_x(\langle x\rangle^\rho)\|_{L^2} + \|\partial_x\left(u_\epsilon \partial_x(\langle x\rangle^\rho)\right)\|_{L^2}\\ &\lesssim \|u_\epsilon\|_{H^1}.\end{aligned} \tag{3.23}$$

Applying Lemma 14 with $\theta = 1-\rho$ we have

$$\begin{aligned}\|D^{5/4}(\langle x\rangle^\rho u_\epsilon)\|_{L^2} &\le \|J^{1/2+\theta}(\langle x\rangle^{1-\theta}u_\epsilon)\|_{L^2}\\ &\lesssim \|J^{1/2}(\langle x\rangle u_\epsilon)\|^{1-\theta}_{L^2}\|J^{3/2}u_\epsilon\|^\theta_{L^2}.\end{aligned} \tag{3.24}$$

Using Lemma 16 we conclude the proof of the claim. Therefore

$$\||u_\epsilon|^{\alpha-1}(\partial_x u_\epsilon)^2\|_{l^1_j(L^2(Q_j))} \le \Omega(u_\epsilon)^{\alpha+1}, \tag{3.25}$$

and so

$$\Omega_1(u_\epsilon) + \Omega_3(u_\epsilon) \lesssim \|u_0\|_{H^{3/2}} + (1+T^+)\Omega(u_\epsilon)^{\alpha+1}. \tag{3.26}$$

II-*Norms* Ω_2 *and* Ω_4:
From Lemma 11 we have

$$\begin{aligned} xu(x,t) &= U_\epsilon(t)(xu_0) - 2(i+\epsilon)tU_\epsilon(t)(\partial_x u_0) \\ &\quad - \int_0^t U_\epsilon(t-t')(x|u_\epsilon|^\alpha \partial_x u_\epsilon)(t')dt' \\ &\quad - 2(1-i\epsilon)\int_0^t (t-t')U_\epsilon(t-t')\big(\partial_x(|u_\epsilon|^\alpha \partial_x u_\epsilon)\big)(t')dt' \\ &= L + NL_1 + NL_2 \end{aligned}$$

Thus

$$\begin{aligned} \Omega_2(u_\epsilon) + \Omega_4(u_\epsilon) \le{}& \|L\|_{L^\infty_T H^{1/2}_x} + \|\partial_x L\|_{l^\infty_j(L^2(Q_j))} \\ &+ \|NL_1\|_{L^\infty_T H^{1/2}_x} + \|\partial_x NL_1\|_{l^\infty_j(L^2(Q_j))} \\ &+ \|NL_2\|_{L^\infty_T H^{1/2}_x} + \|\partial_x NL_2\|_{l^\infty_j(L^2(Q_j))}. \end{aligned}$$

Applying Lemma 8 to the linear term it follows

$$\|L\|_{L^\infty_T H^{1/2}_x} + \|\partial_x L\|_{l^\infty_j(L^2(Q_j))} \lesssim \|u_0\|_X.$$

Regarding the nonlinear terms, Lemma 8 implies for each $t \in [0,T]$

$$\begin{aligned} \|NL_1(t)\|_{H^{1/2}_x} \le{}& \Big\|\int_0^t U_\epsilon(t-t')(x|u_\epsilon|^\alpha \partial_x u_\epsilon)(t')dt'\Big\|_{L^2_x} \\ &+ \Big\|D^{1/2}\int_0^t U_\epsilon(t-t')(x|u_\epsilon|^\alpha \partial_x u_\epsilon)(t')dt'\Big\|_{L^2_x} \\ \lesssim{}& T^{1/2}\|x|u_\epsilon|^\alpha \partial_x u_\epsilon\|_{L^2_T L^2_x} + \|x|u_\epsilon|^\alpha \partial_x u_\epsilon\|_{l^1_j(L^2(Q_j))}. \end{aligned}$$

Note that

$$\|x|u_\epsilon|^\alpha \partial_x u_\epsilon\|_{L^2_T L^2_x} = \|x|u_\epsilon|^\alpha \partial_x u_\epsilon\|_{l^2_j(L^2(Q_j))}.$$

Thus

$$\|NL_1\|_{L^\infty_T H^{1/2}_x} \lesssim T^{1/2}\|x|u_\epsilon|^\alpha \partial_x u_\epsilon\|_{l^2_j(L^2(Q_j))} + \|x|u_\epsilon|^\alpha \partial_x u_\epsilon\|_{l^1_j(L^2(Q_j))}. \tag{3.27}$$

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Using Hölder's inequality and the estimate in (3.17) we obtain

$$\begin{aligned}\|x|u_\epsilon|^\alpha \partial_x u_\epsilon\|_{l^1_j(L^2(Q_j))} &\le \|u_\epsilon\|^\alpha_{l^\alpha_j(L^\infty(Q_j))}\|x\partial_x u_\epsilon\|_{l^\infty_j(L^2(Q_j))} \\ &\le \Omega(u_\epsilon)^\alpha \|x\partial_x u_\epsilon\|_{l^\infty_j(L^2(Q_j))}.\end{aligned}$$

We conclude

$$\|x|u_\epsilon|^\alpha \partial_x u_\epsilon\|_{l^1_j(L^2(Q_j))} \lesssim (1+T^+)\Omega(u_\epsilon)^{\alpha+1} \tag{3.28}$$

The term $\|x|u_\epsilon|^\alpha \partial_x u_\epsilon\|_{l^2_j(L^2(Q_j))}$ can be estimated similarly. We bound the term $\|\partial_x NL_1\|_{l^\infty_j(L^2(Q_j))}$ applying Lemma 9 followed by estimate (3.28). Finally,

$$\begin{aligned}\|NL_2\|_{H^{1/2}_x} &\le 2(1+\epsilon)\|\int_0^t (t-t')U_\epsilon(t-t')\big(|u_\epsilon|^\alpha \partial_x u_\epsilon\big)(t')dt'\|_{H^{3/2}_x} \\ &\lesssim 2(1+\epsilon)T^+\Big[\||u_\epsilon|^\alpha\partial_x u_\epsilon\|_{L^\infty_T L^2_x} + \|\partial_x(|u_\epsilon|^\alpha \partial_x u_\epsilon)\|_{l^1_j(L^2(Q_j))}\Big].\end{aligned}$$

Estimates (3.18) and (3.25) provide us

$$\|\partial_x(|u_\epsilon|^\alpha \partial_x u_\epsilon)\|_{l^1_j(L^2(Q_j))} \lesssim (1+T^+)\Omega(u_\epsilon)^{\alpha+1} \tag{3.29}$$

and the Sobolev embedding

$$\begin{aligned}\||u_\epsilon|^\alpha \partial_x u_\epsilon\|_{L^\infty_T L^2_x} &\le \|u_\epsilon\|^\alpha_{L^\infty_T L^\infty_x}\|\partial_x u_\epsilon\|_{L^\infty_T L^2_x} \\ &\lesssim \Omega(u_\epsilon)^{\alpha+1}.\end{aligned}$$

Then we conclude

$$\|NL_2\|_{L^\infty_T H^{1/2}_x} \lesssim (1+T^+)\Omega(u_\epsilon)^{2+a}.$$

By Lemma 9 and estimate (3.29) we get

$$\|\partial_x NL_2\|_{l^\infty_j(L^2(Q_j))} \lesssim (1+T^+)\Omega(u_\epsilon)^{\alpha+1}.$$

Therefore

$$\Omega_2(u_\epsilon) + \Omega_4(u_\epsilon) \lesssim \|u_0\|_X + (1+T^+)\Omega(u_\epsilon)^{\alpha+1}. \tag{3.30}$$

Gathering the estimates (3.26) and (3.30) we conclude that there exist positive constants β and c, independent of ϵ such that

$$\Omega_T(u_\epsilon) \le c\|u_0\|_X + c(1+T^\beta)\Omega_T(u_\epsilon)^{\alpha+1} \tag{3.31}$$

for all $\epsilon > 0$ whenever we have u_ϵ solution of (3.11) defined in $[0, T]$. As a consequence we have the following:

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Theorem 4. *Let $u_0 \in X$, with $\|u_0\|_X < 1/4c$. Considering the constants c and β in (3.31) we define*

$$T_* = \left(\frac{1 - 4c\|u_0\|_X}{4c\|u_0\|_X}\right)^{1/\beta}.$$

If u_ϵ is a solution of (3.11) defined in $[0, T]$ with $T < T_*$ we have

$$\Omega_T(u_\epsilon) \leq \frac{1 - \sqrt{1 - 4c(1 + T^\beta)\|u_0\|_X}}{2c}. \tag{3.32}$$

Proof. For $0 \leq \tilde{T} \leq T$ consider the polynomial $p(x) = c_{\tilde{T}} x^2 - x + c\|u_0\|_X$, $c_{\tilde{T}} = c(1 + \tilde{T}^\beta)$, and note that $p(x)$ has two roots

$$r_0 = \frac{1 - \sqrt{1 - 4c_{\tilde{T}}\|u_0\|_X}}{2c_{\tilde{T}}} \quad \text{and} \quad r_1 = \frac{1 + \sqrt{1 - 4c_{\tilde{T}}\|u_0\|_X}}{2c_{\tilde{T}}}$$

that, in its turn, belong to $(0, 1)$. Since $c_{\tilde{T}} x^{\alpha+1} - x + c\|u_0\|_X \leq p(x)$ in $(0, 1)$ and $p(x)$ is negative in (r_0, r_1) we have $c_{\tilde{T}} x^{\alpha+1} - x + c\|u_0\|_X < 0$ for all $x \in (r_0, r_1)$. Thus, from (3.31) we conclude $\Omega_{\tilde{T}}(u_\epsilon) \in (0, r_0] \cup [r_1, +\infty)$. Using the fact that $\Omega_{\tilde{T}}(u_\epsilon)$ depends continuously on $\tilde{T}$, it follows that either

$$\Omega_{\tilde{T}}(u_\epsilon) \leq r_0, \quad \text{for all} \quad 0 \leq \tilde{T} \leq T \tag{3.33}$$

or

$$\Omega_{\tilde{T}}(u_\epsilon) \geq r_1, \quad \text{for all} \quad 0 \leq \tilde{T} \leq T \tag{3.34}$$

However, (3.34) cannot happen because

$$\lim_{\tilde{T} \to 0} \Omega_{\tilde{T}}(u_\epsilon) = \|u_0\|_X$$

and $\|u_0\|_X \leq r_0$. □

Corollary 1. *Given $u_0 \in X$ and T_* as in Theorem 4, then, all the solutions u_ϵ can be extended to $[0, T_*]$ and satisfy (3.32) for all $0 < T < T_*$.*

Remark 2. Applying the argument presented in this section to the operator $\Psi(u)$ (defined in (1.12)) we may have

$$\Omega_T(\Psi(u)) \leq c\|u_0\|_X + c(1 + T^\beta)\Omega_T(u)^{\alpha+1}. \tag{3.35}$$

However, applying the same ideas to $\Psi(u) - \Psi(v)$ instead of $\Psi(u)$ would only lead us to conclude that

$$\Omega_T(\Psi(u) - \Psi(v)) \lesssim (\Omega_T(u) + \Omega_T(v))\Omega_T(u - v)^{\alpha-1}. \tag{3.36}$$

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Indeed, because of (3.16) we would have

$$\| |u|^{\alpha-1}(\partial_x u)^2 - |v|^{\alpha-1}(\partial_x v)^2 \|_{l^1_j(L^2(Q_j))} \tag{3.37}$$

and it is not obvious how to bound (3.37) by a factor $\Omega(u-v)$ when $\alpha < 2$.

4. Proof of Theorem 1

Consider the solutions u_ϵ all defined in the same interval of time. Since for each $t \in [0, T]$ the family of solutions $\{u_\epsilon(t)\}_{\epsilon>0}$ is bounded in $H^{3/2}(\mathbb{R})$ there must exist, for each t, a sequence $\{\epsilon_j\}_{j=1}^{\infty}$ and an element $u(t) \in H^{3/2}(\mathbb{R})$ such that $u_{\epsilon_j}(t)$ converges weakly in $H^{3/2}(\mathbb{R})$ to $u(t)$ as $j \longrightarrow \infty$. Throughout this section we shall prove that the function u in fact satisfies (1.8).

4.1. Convergence in L^2

To prove the convergence in L^2 it will be necessary the following lemma:

Lemma 12. *There exists a constant $c > 0$ such that*

$$\|\partial_x u_\epsilon\|_{L^4_T L^\infty_x} < c$$

for all $\epsilon > 0$.

Proof. First we differentiate the integral equation (3.11) and we apply Lemma 6 to the pair $(4, \infty)$ to obtain

$$\begin{aligned}\|\partial_x u_\epsilon\|_{L^4_T L^\infty_x} &\lesssim \|\partial_x u_0\|_{L^2} + \alpha \int_0^T \| |u_\epsilon|^{\alpha-1}(\partial_x u_\epsilon)^2\|_{L^2_x} dt' \\ &\quad + \int_0^T \| |u_\epsilon|^{\alpha} \partial_x^2 u_\epsilon\|_{L^2_x} dt' \\ &\lesssim \|\partial_x u_0\|_{L^2} + T\|u_\epsilon\|^{\alpha+1}_{L^\infty_T H^{3/2}_x} + T^{1/2}\| |u_\epsilon|^{\alpha}\partial_x^2 u_\epsilon\|_{L^2_T L^2_x}.\end{aligned} \tag{4.1}$$

Using the computation (3.17) we have

$$\begin{aligned}\| |u_\epsilon|^{\alpha}\partial_x^2 u_\epsilon\|_{L^2_T L^2_x} &= \| |u_\epsilon|^{\alpha}\partial_x^2 u_\epsilon\|_{l^2_j(L^2(Q_j))} \\ &\le \|u_\epsilon\|^{\alpha}_{l^{2\alpha}_j(L^\infty(Q_j))} \|\partial_x^2 u_\epsilon\|_{l^\infty_j(L(Q_j))} \\ &\le \Omega(u_\epsilon)^{\alpha+1}.\end{aligned} \tag{4.2}$$

Then substituting (4.2) into (4.1) we conclude

$$\|\partial_x u_\epsilon\|_{L^4_T L^\infty_x} \lesssim \|u_0\|_{H^1} + T^+\Omega(u)^{\alpha+1} \qquad \square$$

Fix $u_\epsilon, u_{\epsilon'}$ and consider $w = w_{\epsilon,\epsilon'} = u_\epsilon - u_{\epsilon'}$. A straightforward calculation gives us

$$\begin{aligned} i\frac{d}{dt}\|w(\cdot,t)\|_{L^2_x} + 2i\epsilon\|\partial_x w(\cdot,t)\|^2_{L^2_x} &= 2i(\epsilon-\epsilon')\mathrm{Re}\int \bar{w}\partial_x^2 u_{\epsilon'}dx \\ &\quad + 2i\mathrm{Re}\int (|u_{\epsilon'}|^\alpha - |u_\epsilon|^\alpha)\bar{w}\partial_x u_\epsilon dx \\ &\quad + i\int \partial_x(|u_{\epsilon'}|^\alpha)|w|^2 dx \\ &= I + II + III. \end{aligned}$$

So

$$\frac{d}{dt}\|w(\cdot,t)\|^2_{L^2_x} \le |I| + |II| + |III|. \tag{4.3}$$

Next we are going to estimate each of the terms I, II and III. First we take care of I. Indeed, using integration by parts and the Hölder inequality

$$|I| \le 2|\epsilon-\epsilon'|\|\partial_x\bar{w}\|_{L^2_x}\|\partial_x u_{\epsilon'}\|_{L^2_x} \tag{4.4}$$

In the terms II and III we apply the Hölder inequality as well. So

$$\begin{aligned} |II| &\lesssim \int (|u_{\epsilon'}|^{\alpha-1} + |u_\epsilon|^{\alpha-1})|w|^2|\partial_x u_\epsilon| dx \\ &\lesssim (\Omega(u_\epsilon) + \Omega(u_{\epsilon'}))^{\alpha-1}\|\partial_x u_\epsilon\|_{L^\infty_x}\|w\|^2_{L^2_x}, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} |III| &\le \|\partial_x(|u_{\epsilon'}|^\alpha)\|_{L^\infty_x}\|w\|^2_{L^2_x} \\ &\lesssim \Omega(u_{\epsilon'})^{\alpha-1}\|\partial_x u_{\epsilon'}\|_{L^\infty_x}\|w\|^2_{L^2_x}. \end{aligned} \tag{4.6}$$

Gathering the estimates (4.4), (4.5) and (4.6) and recalling $\sup_{\mu>0}\Omega(u_\mu) < \infty$ we obtain

$$\frac{d}{dt}\|w(\cdot,t)\|^2_{L^2_x} \lesssim |\epsilon-\epsilon'| + (\|\partial_x u_\epsilon(\cdot,t)\|_{L^\infty_x} + \|\partial_x u_{\epsilon'}(\cdot,t)\|_{L^\infty_x})\|w(\cdot,t)\|^2_{L^2_x}. \tag{4.7}$$

Finally, applying the Gronwall inequality we obtain from (4.7)

$$\|w(\cdot,t)\|^2_{L^2_x} \lesssim |\epsilon-\epsilon'|T\exp\left[\tilde{c}T^{3/4}(\|\partial_x u_\epsilon\|_{L^4_T L^\infty_x} + \|\partial_x u_{\epsilon'}\|_{L^4_T L^\infty_x})\right], \tag{4.8}$$

for some constant $\tilde{c} > 0$. Therefore, there exists $\tilde{u} \in C([0,T]; L^2(\mathbb{R}))$ such that

$$\lim_{\epsilon\to 0}\|u_\epsilon - \tilde{u}\|_{L^\infty_T L^2_x} = 0.$$

Also, the limit $\tilde{u}$ coincides with the weak limit u. Consequently $u_\epsilon \longrightarrow u$ in $L^\infty_T L^2_x$ as $\epsilon \to 0$.

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4.2. Existence

We have proved that there exists $u \in C([0,T]; L^2(\mathbb{R}))$ to which u_ϵ converges strongly in $L^\infty_T L^2_x$ and weakly in $H^{3/2}(\mathbb{R})$ for each $t \in [0,T]$. Using that

$$\|\cdot\|_{H^s} \le \|\cdot\|_{L^2}^{1-\theta} \|\cdot\|_{H^{3/2}}^{\theta}, \quad \theta = \frac{3/2 - s}{3/2}$$

we obtain that u_ϵ converges strongly to u in $L^\infty_T H^s_x$ as ϵ goes to zero, for all $0 \le s < 3/2$. In particular $u \in C([0,T]; H^s(\mathbb{R}))$, for all $0 \le s < 3/2$.

We shall prove that u is solution of the integral equation (1.8). Indeed, using Plancherel's Theorem and Dominated Convergence Theorem we have the convergence of the linear part. To investigate the nonlinear part denote

$$v(x,t) = \int_0^t U(t-t')F(t')dt' \quad \text{and} \quad v_\epsilon(x,t) = \int_0^t U_\epsilon(t-t')F_\epsilon(t')dt'$$

where $F(x,t) = |u|^\alpha \partial_x u$ and $F_\epsilon(x,t) = |u_\epsilon|^\alpha \partial_x u_\epsilon$. We have

$$\|v - v_\epsilon\|_{L^2_x} \le 2T \|F - F_\epsilon\|_{L^\infty_T L^2_x} + \int_0^t \|(1 - e^{-\epsilon(t-t')\xi^2})\widehat{F(t')}\|_{L^2_\xi} dt'. \tag{4.9}$$

Since $t' \in [0,t] \longmapsto |(1 - e^{-\epsilon(t-t')\xi^2})\widehat{F(t')}|$ is bounded by $2|\widehat{F}|$ which in turn belongs to $L^1_T L^2_\xi$, it follows from the Dominated Convergence Theorem that

$$\lim_{\epsilon \to 0} \int_0^t \|(1 - e^{-\epsilon(t-t')\xi^2})\widehat{F(t')}\|_{L^2_\xi} dt' = 0.$$

Finally, we take care of $\|F - F_\epsilon\|_{L^\infty_T L^2_x}$. Adding and subtracting $|u_\epsilon|^{1+a}\partial_x u$ we have

$$\|F - F_\epsilon\|_{L^\infty_T L^2_x} \le \||u|^\alpha - |u_\epsilon|^\alpha\|_{L^\infty_T L^\infty_x} \|\partial_x u\|_{L^\infty_T L^2_x} + \|u_\epsilon\|^\alpha_{L^\infty_T L^\infty_x} \|\partial_x(u - u_\epsilon)\|_{L^\infty_T L^2_x}.$$

Using (3.14), the Sobolev embedding and that $\sup_{\epsilon > 0} \|u_\epsilon\|_{L^\infty_T H^{3/2}_x} < \infty$ we can obtain

$$\|F - F_\epsilon\|_{L^\infty_T L^2_x} \lesssim \|u - u_\epsilon\|_{L^\infty_T H^1_x}.$$

Then we conclude

$$\lim_{\epsilon \to 0} \|v_\epsilon(t) - v(t)\|_{L^\infty_T L^2_x} = 0 \tag{4.10}$$

for each $t \in [0,T]$. It follows

$$u(t) = \lim_{\epsilon \to 0} u_\epsilon(t) = U(t)u_0 - \int_0^t U(t-t')(|u|^\alpha \partial_x u)(t')dt'$$

in $L^2(\mathbb{R})$, for each $t \in [0,T]$.

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Theorem 5. *Given $u_0 \in X$, and T_* satisfying the hypothesis of Theorem 4, then there exists u belonging to $L^\infty([0,T_*];X) \cap C([0,T_*];H^{3/2-}(\mathbb{R}))$ solution of (1.8).*

4.3. Uniqueness

Here we will prove that the function u is the unique solution of the integral equation (1.8) in the class $L^\infty([0,T];X) \cap C([0,T];H^{3/2-}(\mathbb{R}))$.

Lemma 13. *If $v \in L^\infty([0,T];X) \cap C([0,T];H^{3/2-}(\mathbb{R}))$ is a solution of the integral equation (1.8) then $\partial_x v \in L^\infty_x L^2_T$.*

Proof. In fact, using Lemma 2.4 and Lemma 2.5 we have

$$\begin{aligned}\|\partial_x v\|_{L^\infty_x L^2_T} &\lesssim \|D^{1/2}u_0\|_{L^2} + \||v|^\alpha \partial_x v\|_{L^1_x L^2_T} \\ &\lesssim \|u_0\|_{H^{3/2}} + T^{1/2}\||v|^\alpha\|_{L^2_x L^\infty_T}\|\partial_x v\|_{L^\infty_T L^2_x}. \end{aligned} \tag{4.11}$$

Therefore

$$\|\partial_x v\|_{L^\infty_x L^2_T} \lesssim \|u_0\|_{H^{3/2}} + T^{1/2}\|v\|^{\alpha+1}_{L^\infty([0,T];X)}. \qquad \square$$

To help us to prove uniqueness we consider the norm

$$|||v||| = \|v\|_{L^\infty_T H^{1/2}_x} + \|\partial_x v\|_{L^\infty_x L^2_T}$$

Let $\tilde{u}$ be another solution of (1.8) in $L^\infty([0,T];X) \cap C([0,T];H^{3/2-}(\mathbb{R}))$. Using Lemma 2.4 it follows

$$|||u - \tilde{u}||| \lesssim \||u|^\alpha \partial_x u - |\tilde{u}|^\alpha \partial_x \tilde{u}\|_{L^1_T L^2_x} + \||u|^\alpha \partial_x u - |\tilde{u}|^\alpha \partial_x \tilde{u}\|_{L^1_x L^2_T}.$$

Using (3.14), the Hölder inequality with $\frac{1}{2} = \frac{1}{p} + \frac{1}{q}$ and $q \geq \frac{2}{\alpha - 1}$ and then the Sobolev embedding we get

$$\begin{aligned}\||u|^\alpha \partial_x u - |\tilde{u}|^\alpha \partial_x \tilde{u}\|_{L^1_T L^2_x} &\lesssim T\||\tilde{u}|^{\alpha-1} + |u|^{\alpha-1}\|_{L^\infty_T L^q_x}\|\tilde{u} - u\|_{L^\infty_T L^p_x}\|\partial_x \tilde{u}\|_{L^\infty_T L^2_x} \\ &\quad + T^{1/2}\||u|^\alpha \partial_x(\tilde{u} - u)\|_{L^2_T L^2_x} \\ &\lesssim T(\|\tilde{u}\|_{L^\infty_T H^{3/2}_x} + \|u\|_{L^\infty_T H^{3/2}_x})^\alpha \|\tilde{u} - u\|_{L^\infty_T H^{1/2}_x} \\ &\quad + \||u|^\alpha\|_{L^2_x L^\infty_T}\|\partial_x(\tilde{u} - u)\|_{L^\infty_T L^2_T}. \end{aligned}$$

The argument we used to obtain (3.17) also gives us

$$\|u\|_{L^{1+}_x L^\infty_T} \lesssim \|u\|_{L^\infty([0,T];X)}. \tag{4.12}$$

Thus

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$$\||u|^\alpha \partial_x u - |\tilde{u}|^\alpha \partial_x \tilde{u}\|_{L^1_T L^2_x} \lesssim T(\|\tilde{u}\|_{L^\infty_T H^{3/2}_x} + \|u\|_{L^\infty_T H^{3/2}_x})^\alpha \|\tilde{u} - u\|_{L^\infty_T H^{1/2}_x} + \|u\|_{L^\infty([0,T];X)} \||\tilde{u} - u\||. \tag{4.13}$$

Regarding the norm $L^1_x L^2_T$ we have

$$\begin{aligned}
\||u|^\alpha \partial_x u - |\tilde{u}|^\alpha \partial_x \tilde{u}\|_{L^1_x L^2_T} &\lesssim \||\tilde{u}|^{\alpha-1} + |u|^{\alpha-1}\|_{L_x^{\frac{1}{\alpha-1}+} L^\infty_T} \||\tilde{u} - u|\partial_x \tilde{u}\|_{L_x^{\frac{1}{2-\alpha}-} L^2_T} \\
&\quad + \|u\|^\alpha_{L^\alpha_x L^\infty_T} \|\partial_x(\tilde{u} - u)\|_{L^\infty_x L^2_T} \\
&\lesssim (\|u\|^{\alpha-1}_{L^{1+}_x L^\infty_T} + \|\tilde{u}\|^{\alpha-1}_{L^{1+}_x L^\infty_T}) \||\tilde{u} - u|\partial_x \tilde{u}\|_{L_x^{\frac{1}{2-\alpha}-} L^2_T} \\
&\quad + \|u\|^\alpha_{L^\alpha_x L^\infty_T} \||\tilde{u} - u\||.
\end{aligned}$$

Now choose n sufficiently large and use the Hölder inequality with

$$\frac{1}{\frac{1}{2-\alpha}-} = \frac{1}{\frac{1}{2-\alpha}} + \frac{1}{n}, \quad \text{and} \quad \frac{1}{2} = \frac{1}{2+} + \frac{1}{n}$$

and then we use the Sobolev embedding to obtain

$$\begin{aligned}
\||\tilde{u} - u|\partial_x \tilde{u}\|_{L_x^{\frac{1}{2-\alpha}-} L^2_T} &\le \|\partial_x \tilde{u}\|_{L_x^{\frac{1}{2-\alpha}} L^{2+}_T} \|\tilde{u} - u\|_{L^n_x L^n_T} \\
&\lesssim \|\partial_x \tilde{u}\|_{L_x^{\frac{1}{2-\alpha}} L^{2+}_T} \|\tilde{u} - u\|_{L^\infty_T H^{1/2}_x}
\end{aligned}$$

Using one more time $\|\cdot\|_{L^{1+}_x L^\infty_T} \lesssim \|\cdot\|_{L^\infty([0,T];X)}$ we obtain

$$\begin{aligned}
&\||u|^\alpha \partial_x u - |\tilde{u}|^\alpha \partial_x \tilde{u}\|_{L^1_x L^2_T} \\
&\quad \lesssim \left(\|\tilde{u}\|^{\alpha-1}_{L^\infty([0,T];X)} + \|u\|^{\alpha-1}_{L^\infty([0,T];X)}\right) \|\partial_x \tilde{u}\|_{L_x^{\frac{1}{2-\alpha}} L^{2+}_T} \||\tilde{u} - u\|| \\
&\quad + \|u\|^\alpha_{L^\infty([0,T];X)} \||\tilde{u} - u\||.
\end{aligned}$$

It remains to estimate $\|\partial_x \tilde{u}\|_{L_x^{\frac{1}{1-a}} L^{2+}_T}$. We consider two cases.

Case 1: $3/2\alpha < 2$.

In this case we can take $2+ = \frac{1}{2-\alpha}$ and then

$$\|\partial_x \tilde{u}\|_{L_x^{\frac{1}{2-\alpha}} L^{2+}_T} \lesssim T^+ \|\tilde{u}\|_{L^\infty_T H^{3/2}_x}.$$

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Case 2: $1 < \alpha \leq 3/2$.

In this case we consider $\rho > \dfrac{3-2\alpha}{2}$. Then using the Hölder inequality with

$$\frac{1}{\frac{1}{2-\alpha}} = \frac{1}{2+} + \frac{2}{3-2\alpha}$$

and that the function $\langle \cdot \rangle^{-\rho} \in L^{\frac{2}{3-2\alpha}-}(\mathbb{R})$, since $\rho > \dfrac{3-2\alpha}{2}$, we obtain

$$\begin{aligned}
\|\partial_x \tilde{u}\|_{L_x^{\frac{1}{2-\alpha}} L_T^{2+}} &\leq \|\langle \cdot \rangle^{\rho} \partial_x \tilde{u}\|_{L_x^{2+} L_T^{2+}} \|\langle \cdot \rangle^{-\rho}\|_{L^{\frac{2}{3-2\alpha}-}} \\
&\lesssim T^{+} \|\langle \cdot \rangle^{\rho} \partial_x \tilde{u}\|_{L_T^{\infty} L_x^{2+}} \\
&\lesssim T^{+} \|D^{\epsilon}(\langle \cdot \rangle^{\rho} \partial_x \tilde{u})\|_{L_T^{\infty} L_x^{2}},
\end{aligned} \tag{4.14}$$

where $\epsilon = \frac{1}{2} - \frac{1}{2+}$ is sufficiently small. Applying the same argument we used to justify (3.21) it follows

$$\|D^{\epsilon}(\langle x \rangle^{\rho} \partial_x \tilde{u})\|_{L^2} \lesssim \|J^{1/2}(x\tilde{u})\|_{L^2} + \|J^{3/2}\tilde{u}\|_{L^2}. \tag{4.15}$$

Replacing (4.15) in (4.14) we deduce that

$$\|\partial_x \tilde{u}\|_{L_x^{\frac{1}{2-\alpha}} L_T^{2+}} \lesssim T^{+} \|\tilde{u}\|_{L^{\infty}([0,T];X)}.$$

Hence

$$\begin{aligned}
&\| |u|^{\alpha} \partial_x u - |\tilde{u}|^{\alpha} \partial_x \tilde{u} \|_{L_x^1 L_T^2} \\
&\quad \lesssim T^{+} \left(\|\tilde{u}\|_{L^{\infty}([0,T];X)} + \|u\|_{L^{\infty}([0,T];X)} \right)^{\alpha} |||u - \tilde{u}||| \\
&\quad\quad + \|u\|^{\alpha}_{L^{\infty}([0,T];X)} |||\tilde{u} - u|||.
\end{aligned} \tag{4.16}$$

Therefore

$$\begin{aligned}
|||u - \tilde{u}||| &\lesssim T^{+} \left(\|\tilde{u}\|_{L^{\infty}([0,T];X)} + \|u\|_{L^{\infty}([0,T];X)} \right)^{\alpha} |||u - \tilde{u}||| \\
&\quad + \|u\|^{\alpha}_{L^{\infty}([0,T];X)} |||u - \tilde{u}|||.
\end{aligned} \tag{4.17}$$

Consider

$$J = \{T \in [0, T_*]; u(t) = \tilde{u}(t), t \in [0, T]\}.$$

Estimate (4.17) implies $T \in J$ for all T sufficiently small. We claim $T'_* := \sup J = T_*$. In fact, if $T'_* < T_*$ we can consider $T \in (T'_*, T_*)$. Repeating the argument we presented to obtain (4.17) and noticing $u - \tilde{u} = 0$ in $[0, T'_*]$ we get

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$$|||u - \tilde{u}||| \lesssim (T - T'_*)^+ \left(\|\tilde{u}\|_{L^\infty([0,T];X)} + \|u\|_{L^\infty([0,T];X)}\right)^\alpha |||u - \tilde{u}||| \\ + \|u\|^\alpha_{L^\infty([0,T];X)} |||u - \tilde{u}|||.$$

It follows $T \in J$ if T is sufficiently close to T'_*. That is a contradiction. So $T'_* = T_*$.

5. Proof of Theorem 2

In this section we carry out the proof of Theorem 2. A great difficulty we had in the low power case $1 < \alpha < 2$ was to control the term $\|u\|_{L^\alpha_x L^\infty_T}$. Since we do not have maximal estimates for $\|U(t)f\|_{L^p_x L^\infty_T}$ when $1 \le p < 2$, we were obligated to introduce weights and many other difficulties arose from that. But now, for $\alpha \ge 2$ we are in a more comfortable situation because of Lemma 2.7. This maximal estimate will be our main ingredient in our approach here.

Given $u_0 \in H^{1/2}$ we consider the integral operator $\Psi = \Psi_{u_0}$ defined as in (1.12). Also we consider the norm

$$\Omega(u) = \|u\|_{L^\infty_T H^{1/2}_x} + \|\partial_x u\|_{L^\infty_x L^2_T} + \|u\|_{L^\alpha_x L^\infty_T} + \|u\|_{L^{2\alpha}_x L^\infty_T}$$

and the space $E_{A,T} = \{u \in C([0,T]; H^{1/2}(\mathbb{R})); \Omega(u) \le A\}$. Applying Lemma 2.5 and Lemma 2.6 in the nonlinear part of $\Psi(u)$ we have

$$\|\Psi(u)\|_{L^\infty_T H^{1/2}_x} + \|\partial_x \Psi(u)\|_{L^\infty_x L^2_T} \lesssim \|u_0\|_{H^{1/2}} + \||u|^\alpha \partial_x u\|_{L^1_x L^2_T} + \||u|^\alpha \partial_x u\|_{L^1_T L^2_x}.$$

We obtain the same bound for $\|\Psi(u)\|_{L^\alpha_x L^\infty_T} + \|\Psi(u)\|_{L^{2\alpha}_x L^\infty_T}$ by using Lemma 3 and Lemma 4. Therefore

$$\Omega(\Psi(u)) \lesssim \|u_0\|_{H^{1/2}} + \||u|^\alpha \partial_x u\|_{L^1_x L^2_T} + \||u|^\alpha \partial_x u\|_{L^1_T L^2_x}. \tag{5.1}$$

On the other hand we have

$$\||u|^\alpha \partial_x u\|_{L^1_x L^2_T} \le \|u\|^\alpha_{L^\alpha_x L^\infty_T} \|\partial_x u\|_{L^\infty_x L^2_T} \tag{5.2}$$

and, using that $\|\cdot\|_{L^1_T L^2_x} \le T^{1/2} \|\cdot\|_{L^2_x L^2_T}$,

$$\||u|^\alpha \partial_x u\|_{L^1_T L^2_x} \le T^{1/2} \|u\|^\alpha_{L^{2\alpha}_x L^\infty_T} \|\partial_x u\|_{L^\infty_x L^2_T}. \tag{5.3}$$

Therefore

$$\Omega(u) \le c\|u_0\|_{H^{1/2}} + c(1 + T^{1/2})\Omega(u)^{\alpha+1}. \tag{5.4}$$

Taking $A = 2c\|u_0\|_{H^{1/2}}$ sufficiently small (5.4) implies $\Psi(E_{A,T}) \subset E_{A,T}$ for some $T = T(\|u_0\|_{H^{1/2}})$. Finally, using the property

$$||u|^\alpha - |v|^\alpha| \lesssim (|u|^{\alpha-1} + |v|^{\alpha-1})|u - v| \tag{5.5}$$

it follows from the same argument that

$$\Omega(\Psi(u) - \Psi(v)) \lesssim (\Omega(u)^{\alpha-1} + \Omega(v)^{\alpha-1})\Omega(u - v). \tag{5.6}$$

That completes the proof of Theorem 2.

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Appendix A

Lemma 14. *For $\theta \in [0, 1]$ we have*

$$\|J^{1/2+\theta}(\langle x\rangle^{1-\theta} f)\|_{L^2} \lesssim \|J^{1/2}(\langle x\rangle f)\|_{L^2}^{1-\theta} \|J^{3/2} f\|_{L^2}^{\theta}.$$

To prove this lemma we will use the following characterization of Sobolev spaces.

Theorem 6 *(Characterization I). Let $0 < \sigma < 2$ and $1 < p < \infty$. Consider*

$$\mathcal{D}_\sigma f(x) = \lim_{\epsilon\to 0} \int\limits_{|y|>\epsilon} \frac{f(x+y) - f(x)}{|y|^{1+\sigma}} dy. \tag{A.1}$$

Then $f \in L^p_\sigma(\mathbb{R})$ if, only if, $f \in L^p(\mathbb{R})$ and the limit defined in (A.1) converges in L^p norm. In this case

$$\|J^\sigma f\|_{L^p} \approx \|f\|_{L^p} + \|\mathcal{D}_\sigma f\|_{L^p}.$$

The proof of this theorem can be found in [25] or [26].

Before proving Lemma 14 we also need the following.

Lemma 15. *Let $0 < \sigma < 1$ and $1 < p < \infty$. Then there exists a constant $c > 0$ such that*

$$(c(1 + |t|))^{-1}\|J^\sigma(\langle\cdot\rangle^{it} f)\|_{L^p} \le \|J^\sigma f\|_{L^p} \le c(1 + |t|)\|J^\sigma(\langle\cdot\rangle^{it} f)\|_{L^p}.$$

Proof. Denote $\varphi(x) = \log\langle x\rangle$.

$$\mathcal{D}_\sigma(\langle\cdot\rangle^{it} f)(x) = e^{it\varphi(x)}\phi_{\sigma,t}(f)(x) + e^{it\varphi(x)}\mathcal{D}_\sigma f(x).$$

So

$$\|\mathcal{D}_\sigma(\langle\cdot\rangle^{it} f)(x)\|_{L^p} \le \|\phi_{\sigma,t}(f)\|_{L^p} + \|\mathcal{D}_\sigma f\|_{L^p}.$$

Let us estimate $\|\phi_{\sigma,t}(f)\|_{L^p}$.

$$\begin{aligned}\phi_{\sigma,t}(f)(x) &= \lim_{\epsilon\to 0}\int\limits_{\epsilon<|y|<1}\frac{e^{it\varphi(x+y)}f(x+y)-e^{it\varphi(x)}f(x)}{|y|^{1+\sigma}}dy\\ &\quad+\int\limits_{|y|>1}\frac{e^{it\varphi(x+y)}f(x+y)-e^{it\varphi(x)}f(x)}{|y|^{1+\sigma}}dy\\ &= I+II.\end{aligned}$$

We have from the Minkowsky inequality

$$\begin{aligned}\|II\|_{L^p} &\le 2\int\limits_{|y|>1}\frac{1}{|y|^{1+\sigma}}\|f(\cdot+y)\|_{L^p}dy\\ &= c_\sigma\|f\|_{L^p}.\end{aligned}$$

Since φ is Lipschitz we have

$$|e^{it(\varphi(x+y)-\varphi(x))}-1|\le |t||y|. \tag{A.2}$$

Applying the Minkowsky inequality again we get

$$\begin{aligned}\|I\|_{L^p} &\le |t|\lim_{\epsilon\to 0}\int\limits_{\epsilon<|y|<1}\frac{1}{|y|^{\sigma}}\|f(\cdot+y)\|_{L^p}dy\\ &= c_\sigma|t|\|f\|_{L^p}.\end{aligned}$$

Therefore applying Theorem 6

$$\begin{aligned}\|J^\sigma(\langle\cdot\rangle^{it}f)\|_{L^p} &\lesssim \|\mathcal{D}_\sigma(\langle\cdot\rangle^{it}f)\|_{L^p}+\|f\|_{L^p}\\ &\lesssim (1+|t|)\|J^\sigma f\|_{L^p}.\end{aligned} \tag{A.3}$$

The opposite inequality follows immediately by applying (A.3) to the function $\langle\cdot\rangle^{it}f$ instead of f. $\square$

Proof of Lemma 14. Given $g\in L^2(\mathbb{R})$ such that $\|g\|_{L^2}=1$ define $F_g: S\longrightarrow\mathbb{C}$ by

$$F_g(z)=e^{z^2-1}\int\limits_{\mathbb{R}}J^{1/2+z}(\langle x\rangle^{1-z}f)(x)\bar{g}(x)dx$$

where S is the strip $S=\{z\in\mathbb{C}; 0\le \mathrm{Re}(z)\le 1\}$. Using the Cauchy–Schwartz inequality and Lemma 15

$$\begin{aligned}|F_g(iy)| &\lesssim (1+|y|)e^{-(y^2+1)}\|J^{1/2}(\langle x\rangle f)\|_{L^2}\\ &\lesssim \|J^{1/2}(\langle x\rangle f)\|_{L^2}\end{aligned} \tag{A.4}$$

Similarly we have

$$|F_g(1+iy)| \lesssim \|J^{3/2}f\|_{L^2}.$$

Then using the three lines theorem we obtain

$$|F_g(\theta)| \lesssim \|J^{1/2}(\langle x\rangle f)\|_{L^2}^{1-\theta}\|J^{3/2}f\|_{L^2}^{\theta} \tag{A.5}$$

for all $\theta \in [0,1]$. Taking the supremum over all $g \in L^2(\mathbb{R})$ such that $\|g\|_{L^2}=1$ in (A.5) we conclude

$$\|J^{1/2+\theta}(\langle x\rangle^{1-\theta}f))\|_{L^2} \le c\|J^{1/2}(\langle x\rangle f)\|_{L^2}^{1-\theta}\|J^{3/2}f\|_{L^2}^{\theta}. \qquad \square$$

Lemma 16. *Given $0<s<1$ we have*

$$\|J^s(\langle x\rangle f)\|_{L^2} \lesssim \|J^s(xf)\|_{L^2} + \|J^s f\|_{L^2}.$$

To prove this lemma we will use another characterization of the Sobolev spaces whose proof can also be found in [25] or [26].

Theorem 7 *(Characterization II). Let $s \in (0,1)$ and $2/(1+2s) < p < \infty$. Consider*

$$\mathcal{D}^s f(x) = \left(\int_{\mathbb{R}} \frac{|f(x+y)-f(x)|^2}{|y|^{1+2s}}dy\right)^{1/2}. \tag{A.6}$$

Then $f \in L_s^p(\mathbb{R})$ if, and only if f and $\mathcal{D}^s f$ belong to $L^p(\mathbb{R})$. Moreover

$$\|J^s f\|_{L^p} \approx \|f\|_{L^p} + \|\mathcal{D}^s f\|_{L^p}$$

for all $f \in L_s^p(\mathbb{R})$.

Proof of Lemma 16. It is enough to prove

$$\|D^s(|x|f)\|_{L^2} \lesssim \|J^s(xf)\|_{L^2} + \|J^s f\|_{L^2}.$$

Consider the cut-off function $\chi \in C_c^\infty(\mathbb{R})$, supported in $[-2,2]$ and identically 1 in $[-1,1]$. We have

$$\begin{aligned}\|D^s(|x|f)\|_{L^2} &\le \|D^s(|x|\chi(x)f)\|_{L^2} + \|D^s(|x|(1-\chi(x))f)\|_{L^2}\\ &\le I + II.\end{aligned}$$

From Theorem 7 we have

$$\begin{aligned}\|D^s(|x|\chi(x)f)\|_{L^2} &\lesssim \||x|\chi(x)f\|_{L^2} + \|\mathcal{D}^s(|x|\chi(x)f)\|_{L^2}\\ &\lesssim \|f\|_{L^2} + \|\mathcal{D}^s(|x|\chi(x))\|_{L^\infty}\|f\|_{L^2} + \||x|\chi(x)\|_{L^\infty}\|\mathcal{D}^s f\|_{L^2}\\ &\lesssim \|f\|_{L^2} + \|\mathcal{D}^s f\|_{L^2}.\end{aligned}$$

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Then we conclude $I \lesssim \|J^s f\|_{L^2}$. Similarly, using Theorem 7,

$$\|D^s(|x|(1-\chi(x))f)\|_{L^2} \lesssim \|J^s(xf)\|_{L^2} + \|xf\|_{L^2}$$

Then $II \lesssim \|J^s(xf)\|_{L^2}$. This finishes the proof of Lemma 16. □

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