ARTICLE IN PRESS YJDEQ:9244

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J. Differential Equations ●●● (●●●●) ●●●–●●●

Journal of Differential Equations

www.elsevier.com/locate/jde

Ground state solution for a class of indefinite variational problems with critical growth

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Received 5 April 2017; revised 24 February 2018

Abstract

In this paper we study the existence of ground state solution for an indefinite variational problem of the type

$$\begin{cases} -\Delta u + (V(x) - W(x))u = f(x,u) \quad \text{in} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{P}$$

where $N \geq 2$, $V, W : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are continuous functions verifying some technical conditions and f possesses a critical growth. Here, we will consider the case where the problem is asymptotically periodic, that is, V is $\mathbb{Z}^N$-periodic, W goes to 0 at infinity and f is asymptotically periodic.

MSC: 35B33; 35A15; 35J15

Keywords: Critical growth; Variational methods; Elliptic equations; Indefinite strongly functional

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1 C.O. Alves was partially supported by CNPq/Brazil 301807/2013-2 and INCT-MAT.

2 G.F. Germano was partially supported by CAPES.

https://doi.org/10.1016/j.jde.2018.02.039

1. Introduction

In this paper we study the existence of ground state solution for an indefinite variational problem of the type

$$\begin{cases} -\Delta u + (V(x) - W(x))u = f(x,u), \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{P}$$

where $N \geq 2$, $V, W : \mathbb{R}^N \to \mathbb{R}$ are continuous functions verifying some technical conditions and f has a critical growth. Here, we will consider the case where the problem is asymptotically periodic, that is, V is $\mathbb{Z}^N$-periodic, W goes to 0 at infinity and f is asymptotically periodic.

In [13], Kryszewski and Szulkin have studied the existence of ground state solution for an indefinite variational problem of the type

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \quad \text{in} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{P_1}$$

where $V : \mathbb{R}^N \to \mathbb{R}$ is a $\mathbb{Z}^N$-periodic continuous function such that

$$0 \notin \sigma(-\Delta + V), \text{ the spectrum of } -\Delta + V. \tag{V_1}$$

Related to the function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, they assumed that f is continuous, $\mathbb{Z}^N$-periodic in x with

$$|f(x,t)| \leq c(|t|^{q-1} + |t|^{p-1}), \quad \forall t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^N \tag{h_1}$$

and

$$0 < \alpha F(x,t) \leq t f(x,t) \quad \forall t \in \mathbb{R}, \quad F(x,t) = \int_0^t f(x,s)\,ds \tag{h_2}$$

for some $c > 0$, $\alpha > 2$ and $2 < q < p < 2^*$ where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = +\infty$ if $N = 2$. The above hypotheses guarantee that the energy functional associated with (P_1) given by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2\,dx) - \int_{\mathbb{R}^N} F(x,u)\,dx, \ u \in H^1(\mathbb{R}^N),$$

is well defined and belongs to $C^1(H^1(\mathbb{R}^N), \mathbb{R})$. By ($V_1$), there is an equivalent inner product $\langle\ ,\ \rangle$ in $H^1(\mathbb{R}^N)$ such that

$$J(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}^N} F(x,u)\,dx,$$

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where $\|u\| = \sqrt{\langle u, u\rangle}$ and $H^1(\mathbb{R}^N) = E^+ \oplus E^-$ corresponds to the spectral decomposition of $-\Delta + V$ with respect to the positive and negative part of the spectrum with $u = u^+ + u^-$, where $u^+ \in E^+$ and $u^- \in E^-$. In order to show the existence of solution for (P_1), Kryszewski and Szulkin introduced a new and interesting generalized link theorem. In [16], Li and Szulkin have improved this generalized link theorem to prove the existence of solution for a class of indefinite problem with f being asymptotically linear at infinity.

The link theorems above mentioned have been used in a lot of papers, we would like to cite Chabrowski and Szulkin [5], do Ó and Ruf [8], Furtado and Marchi [9], Tang [30,31] and their references.

Pankov and Pflüger [21] also have considered the existence of solution for problem (P_1) with the same conditions considered in [13], however the approach is based on an approximation technique of periodic function together with the linking theorem due to Rabinowitz [22]. After, Pankov [20] has studied the existence of solution for problems of the type

$$\begin{cases} -\Delta u + V(x)u = \pm f(x,u), & \text{in} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{P_2}$$

by supposing (V_1), (h_1)–(h_2) and employing the same approach explored in [21]. In [20] and [21], the existence of ground state solution has been established by supposing that f is C^1 and there is $\theta \in (0,1)$ such that

$$0 < t^{-1} f(x,t) \le \theta f_t'(x,t), \quad \forall t \neq 0 \quad \text{and} \quad x \in \mathbb{R}^N. \tag{h_3}$$

However, in [20], Pankov has found a ground state solution by minimizing the energy functional J on the set

$$\mathcal{O} = \left\{ u \in H^1(\mathbb{R}^N) \setminus E^- \,;\; J'(u)u = 0 \text{ and } J'(u)v = 0, \forall\, v \in E^- \right\}.$$

The reader is invited to see that if J is definite strongly, that is, when $E^- = \{0\}$, the set $\mathcal{O}$ is exactly the Nehari manifold associated with J. Hereafter, we say that $u_0 \in H^1(\mathbb{R}^N)$ is called a *ground state solution* if

$$J'(u_0) = 0, \quad u_0 \in \mathcal{O} \quad \text{and} \quad J(u_0) = \inf_{w \in \mathcal{O}} J(w).$$

In [25], Szulkin and Weth have established the existence of ground state solution for problem (P_1) by completing the study made in [20], in the sense that, they also minimize the energy function on $\mathcal{O}$, however they have used more weaker conditions on f, for example f is continuous, $\mathbb{Z}^N$-periodic in x and satisfies

$$|f(x,t)| \le C(1 + |t|^{p-1}), \quad \forall t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^N \tag{f_1}$$

for some $C > 0$ and $p \in (2, 2^*)$.

$$f(x,t) = o(t) \ \text{ uniformly in } \ x \text{ as } |t| \to 0 \tag{f_2}$$

$$F(x,t)/|t|^2 \to +\infty \ \text{ uniformly in } \ x \text{ as } |t| \to +\infty \tag{f_3}$$

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and

$$t \mapsto f(x,t)/|t| \text{ is strictly increasing on } \mathbb{R}\setminus\{0\}. \tag{f_4}$$

The same approach has been used by Zhang, Xu and Zhang [36,37] to study a class of indefinite and asymptotically periodic problem.

After a review bibliography, we have observed that there are few papers involving indefinite problem whose the nonlinearity has a critical growth. For example, the critical case for $N \geq 4$ was considered in [5], [29] and [37] when f is given by

$$f(x,t) = g(x,t) + k(x)|t|^{2^*-2}t,$$

with $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ being a function with subcritical growth and $k : \mathbb{R}^N \to \mathbb{R}$ be a continuous function satisfying some conditions. For the case $N = 2$, we know only the paper [8] which considered the periodic case with f having an exponential critical growth, namely there is $\alpha_0 > 0$ such that

$$\lim_{|t|\to+\infty} \frac{|f(t)|}{e^{\alpha|t|^2}} = 0, \ \ \forall\alpha > \alpha_0, \ \lim_{|t|\to+\infty} \frac{|f(t)|}{e^{\alpha|t|^2}} = +\infty, \ \ \forall\alpha < \alpha_0.$$

Motivated by ideas found in Szulkin and Weth [25,26] together with the fact that there are few papers involving critical growth for $N = 2$ and $N \geq 3$ and indefinite problem, we intend in the present paper to study the existence of ground state solution for (P), with the nonlinearity f having critical growth and the problem being asymptotically periodic. Since we will work with the dimensions $N = 2$ and $N \geq 3$, we will state our conditions in two blocks, however the conditions on V and W are the same for any these dimensions.

The conditions on V and W.

On the functions V and W, we have assumed the following conditions:

(V_1) $V : \mathbb{R}^N \to \mathbb{R}$ is continuous and $\mathbb{Z}^N$-periodic.
(V_2) $\underline{\Lambda} := \sup(\sigma(-\nabla + V) \cap (-\infty, 0]) < 0 < \overline{\Lambda} := \inf(\sigma(-\nabla + V) \cap [0, +\infty))$.
(W_1) $W : \mathbb{R}^N \to \mathbb{R}$ is continuous and $\lim_{|x|\to+\infty} W(x) = 0$.
(W_2) $0 \leq W(x) \leq \Theta = \sup_{x\in\mathbb{R}^N} W(x) < \overline{\Lambda}, \quad \forall x \in \mathbb{R}^N$.

With relation to the function f, we have assumed the following conditions:

The dimension $N \geq 3$:

For this case, we suppose that f is the form

$$f(x,t) = h(x)|t|^{q-1}t + k(x)|t|^{2^*-2}t$$

with $1 < q < 2^* - 1$ and

(C_1) $h(x) = h_0(x) + h_*(x)$ and $k(x) = k_0(x) + k_*(x)$, where $h_0, h_*, k_0, k_* : \mathbb{R}^N \to \mathbb{R}$ are continuous function, h_0, k_0 are $\mathbb{Z}^N$-periodic, $\lim\limits_{|x|\to+\infty} h_*(x) = \lim\limits_{|x|\to+\infty} k_*(x) = 0$ and h_0, h_*, k_0, k_* are nonnegative.

(C_2) There is $x_0 \in \mathbb{R}^N$ such that

$$k(x_0) = \max_{x\in\mathbb{R}^N} k(x) \quad \text{and} \quad k(x) - k(x_0) = o(|x - x_0|^2) \quad \text{as} \quad x \to x_0.$$

(C_3) If $\inf_{x\in\mathbb{R}^N} h(x) = 0$, we assume that $V(x_0) < 0$.

The dimension $N = 2$:

(f_1) There exist functions f_0, $f^* : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ such that

$$f(x,t) = f_0(x,t) + f^*(x,t),$$

where f_0 and f^* are continuous functions, f_0 is $\mathbb{Z}^2$-periodic with respect to x and f^* is nonnegative.

(f_2) $\frac{f(x,t)}{t}, \frac{f_0(x,t)}{t} \to 0$ as $t \to 0$ uniformly with respect to $x \in \mathbb{R}^2$.

(f_3) For each fixed $x \in \mathbb{R}^2$, the functions $t \mapsto \frac{f(x,t)}{t}$ and $t \mapsto \frac{f_0(x,t)}{t}$ are increasing on $(0, +\infty)$ and decreasing on $(-\infty, 0)$.

(f_4) There exist $\theta, \mu > 2$ such that

$$0 < \theta F_0(x,t) \le t f_0(x,t) \qquad \text{and} \qquad 0 < \mu F(x,t) \le t f(x,t)$$

for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}^*$, where

$$F_0(x,t) := \int_0^t f_0(x,s)ds \quad \text{and} \quad F(x,t) := \int_0^t f(x,s)ds.$$

(f_5) There exist $\Gamma > 0$ and $\tau \in (1,2)$ such that $|f_0(x,t)| \le \Gamma e^{4\pi t^2}$ and $|f^*(x,t)| \le \Gamma H(x) e^{4\pi |t|^{\tau-2} t}$ for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$, where $H \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$.

(f_6) $F_0(x,t) \ge D(x)|t|^q$, $\forall\ (x,t) \in \mathbb{R}^2 \times \mathbb{R}$, for some positive continuous function D with $\inf_{x\in\mathbb{R}^2} D(x) > 0$ and $q > 2$.

An example of a function f verifying (f_1)–(f_6) is

$$f(x,t) = \lambda(3 - sen((x_1 + x_2)2\pi))|t|^{q-2} t e^{\alpha_0 t^2} + \frac{1}{x_1^2 + x_2^2 + 1}|t|^{p-2} t e^{4\pi |t|^{\tau-1} t}, \quad \forall t \in \mathbb{R}$$

with $x = (x_1, x_2)$, $\lambda > 0$, $\alpha_0 \in (0, 4\pi)$, $q, p \in (2, +\infty)$ and $\tau \in (1,2)$.

The above conditions imply that f has a critical growth if $N = 2$ or $N \ge 3$.

Our main theorem is the following:

Theorem 1.1. *Assume that* (V_1)–(V_2), (W_1)–(W_2), (C_1)–(C_3) *and* (f_1)–(f_6) *hold. Then, problem* (P) *has a ground state solution for* $N \geq 4$. *If* $N = 2, 3$, *there is* $\lambda^* > 0$ *such that if* $\inf_{x\in\mathbb{R}^2} D(x), \inf_{x\in\mathbb{R}^N} h(x) \geq \lambda^*$, *then problem* (P) *has a ground state solution.*

The Theorem 1.1 completes the study made in some of the papers above mentioned, in the sense that we are considering others conditions on V and f. For example, for the case $N \geq 3$, it completes the study made in [25], because the critical case was not considered for $N \geq 3$ or $N = 2$, and the case asymptotically periodic was not also analyzed. The Theorem 1.1 also completes [8], because in that paper was proved the existence of a solution only for the periodic case, while that we are finding ground state solution for the periodic and asymptotically periodic case by using a different method. Finally, the above theorem completes the main result of [29] and [36], because the authors considered only the case $W = 0$, and also the paper [5], because the dimension $N = 3$ was not considered as well as the asymptotically periodic case. Moreover, in [5] and [29] the authors considered only the case

$$V(x_0) < 0 \quad \text{and} \quad k(x) - k(x_0) = o(|x - x_0|^2) \quad \text{as} \quad x \to x_0.$$

In Theorem 1.1 this condition was not assumed if $\inf_{x\in\mathbb{R}^N} h(x) > 0$.

Before concluding this introduction, we would like point out that the reader can find others interesting results involving indefinite variational problem in Jeanjean [12], Schechter [27,28], Lin and Tang [17], Willem and Zou [34], Yang [35] and their references.

Notation. In this paper, we use the following notations:

- The usual norms in $H^1(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ will be denoted by $\| \ \|_{H^1(\mathbb{R}^N)}$ and $| \ |_p$ respectively.
- C denotes (possible different) any positive constant.
- $B_R(z)$ denotes the open ball with center z and radius R in $\mathbb{R}^N$.
- We say that $u_n \to u$ in $L^p_{loc}(\mathbb{R}^N)$ when

$$u_n \to u \quad \text{in} \quad L^p(B_R(0)), \quad \forall R > 0.$$

- If g is a mensurable function, the integral $\int_{\mathbb{R}^N} g(x)\,dx$ will be denoted by $\int g(x)\,dx$.

The plan of the paper is as follows: In Section 2 we will show some technical lemmas and prove the Theorem 1.1 for $N \geq 3$, while in Section 3 we will focus our attention to the dimension $N = 2$.

2. The case $N \geq 3$

In this section, our intention is to prove the Theorem 1.1 for the case $N \geq 3$. Some technical lemmas in this section are also true for dimension $N = 2$ and they will be used in Section 3.

In this section, our focus is the indefinite problem

$$\begin{cases} -\Delta u + (V(x) - W(x))u = h(x)|u|^{q-1}u + k(x)|u|^{2^*-2}u, & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{2.1}$$

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whose energy functional $\Phi_W : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\Phi_W(u) = \frac{1}{2}B(u,u) - \frac{1}{2}\int W(x)|u|^2 dx - \frac{1}{q+1}\int h(x)|u|^{q+1} dx - \frac{1}{2^*}\int k(x)|u|^{2^*} dx \quad (2.2)$$

is well defined, $\Phi_W \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and its critical points are precisely weak solutions of (2.1). Here, B is the bilinear form

$$B(u,v) = \int (\nabla u \nabla v + V(x)uv)\, dx. \quad (2.3)$$

Note that the bilinear form B is not positive definite, therefore it does not induce a norm. As in [25], there is an inner product $\langle\ ,\ \rangle$ in $H^1(\mathbb{R}^N)$ such that

$$\Phi_W(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \frac{1}{2}\int W(x)|u|^2\, dx - \int F(x,u)\, dx, \quad (2.4)$$

where $\|u\| = \sqrt{\langle u,u\rangle}$ and $H^1(\mathbb{R}^N) = E^+ \oplus E^-$ corresponds to the spectral decomposition of $-\Delta + V$ with respect to the positive and negative part of the spectrum with $u = u^+ + u^-$, where $u^+ \in E^+$ and $u^- \in E^-$. It is well known that B is positive definite on E^+, B is negative definite on E^- and the norm $\|\ \|$ is an equivalent norm to the usual norm in $H^1(\mathbb{R}^N)$, that is, there are $a, b > 0$ such that

$$b||u|| \le ||u||_{H^1(\mathbb{R}^N)} \le a||u||, \quad \forall\, u \in H^1(\mathbb{R}^N). \quad (2.5)$$

Hereafter, we denote by $\Phi : H^1(\mathbb{R}^N) \to \mathbb{R}$ the functional defined by

$$\Phi(u) = \frac{1}{2}B(u,u) - \frac{1}{q+1}\int h_0(x)|u|^{q+1} dx - \frac{1}{2^*}\int k_0(x)|u|^{2^*} dx,$$

or equivalently,

$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \frac{1}{q+1}\int h_0(x)|u|^{q+1} dx - \frac{1}{2^*}\int k_0(x)|u|^{2^*} dx. \quad (2.6)$$

Note that the critical points of Φ are weak solutions of the periodic problem

$$\begin{cases} -\Delta u + V(x)u = h_0(x)|u|^{q-1}u + k_0(x)|u|^{2^*-2}u, & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \quad (2.7)$$

In the sequel, $\mathcal{M}$, $E(u)$ and $\hat{E}(u)$ denote the following sets

$$\mathcal{M} := \{u \in H^1(\mathbb{R}^N) \setminus E^- \ ;\ \Phi'_W(u)u = 0 \text{ and } \Phi'_W(u)v = 0, \forall\, v \in E^-\}$$

and

$$E(u) := E^- \oplus \mathbb{R}u \quad\text{and}\quad \hat{E}(u) := E^- \oplus [0, +\infty)u.$$

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Therefore

$$E(u) = E^- \oplus \mathbb{R}u^+ \text{ and } \hat{E}(u) = E^- \oplus [0, +\infty)u^+.$$

Moreover, we denote by γ_W and γ the real numbers

$$\gamma_W := \inf_{\mathcal{M}} \Phi_W \quad \text{and} \quad \gamma := \inf_{\mathcal{M}} \Phi. \tag{2.8}$$

2.1. Technical lemmas

In this section we are going to show some lemmas which will be used in the proof of main Theorem 1.1.

Lemma 2.1. *If $u \in \mathcal{M}$ and $w = su + v$ where $s \geq 1$, $v \in E^-$ and $w \neq 0$, then*

$$\Phi_W(u + w) < \Phi_W(u).$$

Proof. In the sequel, we fix

$$G(x,t) := \frac{1}{2}W(x)t^2 + \frac{1}{q+1}h(x)|t|^{q+1} + \frac{1}{2^*}k(x)|t|^{2^*}$$

and

$$g(x,t) := W(x)t + h(x)|t|^{q-1}t + k(x)|t|^{2^*-2}t.$$

Then by a simple computation,

$$\Phi_W(u+w) - \Phi_W(u) =$$
$$-\frac{1}{2}||v||^2 + \int \left(g(x,u)\left[\left(\frac{s^2}{2} + s\right)u + (s+1)v\right] G(x,u) - G(x,u+w)\right) dx.$$

Now, the proof follows by adapting the ideas explored in [25, Proposition 2.3]. □

Lemma 2.2. *Let $\mathcal{K} \subset E^+ \setminus \{0\}$ be a compact subset, then there exists $R > 0$ such that $\Phi_W(w) \leq 0$, $\forall\, w \in E(u) \setminus B_R(0)$ and $u \in \mathcal{K}$.*

Proof. Setting the functional

$$\Psi_*(u) = \frac{1}{2}B(u,u) - \frac{1}{2^*}\int |u|^{2^*}\, dx$$

we have

$$\Phi_W(u) \leq \Psi_*(u), \quad \forall u \in H^1(\mathbb{R}^N).$$

Now, we apply the same idea from [25, Lemma 2.2] to the functional Ψ_* to get the desired result. □

Lemma 2.3. *For all $u \in H^1(\mathbb{R}^N)$, the functional $\Phi_W|_{E(u)}$ is weakly upper semicontinuous.*

Proof. First of all, note that $E(u)$ is weakly closed, because it is convex strongly closed. Now, we claim that the functional

$$\begin{aligned} \widetilde{\Phi} : E(u) &\to \mathbb{R} \\ w &\mapsto \frac{1}{2}\int W(x)|w|^2\,dx + \frac{1}{q+1}\int h(x)|w|^{q+1}\,dx + \frac{1}{2^*}\int k(x)|w|^{2^*}\,dx \end{aligned}$$

is weakly lower semicontinuous. Indeed, if $w_n \rightharpoonup w$ on $E(u)$, passing to a subsequence, we can assume that $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^N$. Then by Fatou's Lemma,

$$\begin{aligned} \widetilde{\Phi}(w) = \int W(x)w^2\,dx + \frac{1}{q+1}\int h(x)|w|^{q+1}\,dx + \frac{1}{2^*}\int k(x)|w|^{2^*}\,dx \le \\ \le \liminf_{n\to+\infty}\left[\int W(x)w_n^2\,dx + \frac{1}{q+1}\int h(x)|w_n|^{q+1}\,dx + \frac{1}{2^*}\int k(x)|w_n|^{2^*}\,dx\right], \end{aligned}$$

leading to

$$\widetilde{\Phi}(w) \le \liminf_{n\to+\infty}\widetilde{\Phi}(w_n).$$

Furthermore, the functional

$$\begin{aligned} \widetilde{\Psi} : E(u) &\to \mathbb{R} \\ w &\mapsto \frac{1}{2}B(w,w) \end{aligned}$$

is weakly upper semicontinuous. In fact, since

$$\widetilde{\Psi}(w) = \frac{1}{2}(||w^+||^2 - ||w^-||^2),$$

if $w_n = s_n u^+ + v_n \rightharpoonup w = su^+ + v$ with $v_n, v \in E^-$, then $s_n \to s$ in $\mathbb{R}$ and $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$. Thus,

$$\widetilde{\Psi}(w) = \frac{1}{2}(s^2||u^+||^2 - ||v||^2) \ge \limsup_{n\to+\infty}\frac{1}{2}(s_n^2||u^+||^2 - ||v_n||^2) = \limsup_{n\to+\infty}\widetilde{\Psi}(w_n).$$

As $\Phi_W|_{E(u)} = \widetilde{\Psi} - \widetilde{\Phi}$, the result is proved. □

Lemma 2.4. *For each $u \in H^1(\mathbb{R}^N) \setminus E^-$, $\mathcal{M} \cap \hat{E}(u)$ is a singleton set and the element of this set is the unique global maximum of $\Phi_W|_{\hat{E}(u)}$.*

Proof. The proof follows very closely the proof of [25, Lemma 2.6]. □

Lemma 2.5. *There exists $\rho > 0$ such that $\inf_{B_\rho(0)\cap E^+} \Phi_W > 0$.*

Proof. In what follows, let us fix $\overline{h} := \sup_{x\in\mathbb{R}^N} h(x)$ and $\overline{k} := \sup_{x\in\mathbb{R}^N} k(x)$. For $u \in E^+$,

$$\begin{aligned}\Phi_W(u) &= \frac{1}{2}||u||^2 - \frac{1}{2}\int W(x)|u|^2dx - \frac{1}{q+1}\int h(x)|u|^{q+1}dx - \frac{1}{2^*}\int k(x)|u|^{2^*}dx \\ &\geq \frac{1}{2}||u||^2 - \frac{\Theta}{2}\int |u|^2dx - \frac{\overline{h}}{q+1}\int |u|^{q+1}dx - \frac{\overline{k}}{2^*}\int |u|^{2^*}dx \\ &\geq \frac{1}{2}||u||^2 - \frac{\Theta}{2\overline{\Lambda}}||u||^2 - \frac{\overline{h}c_1}{q+1}||u||^{q+1} - \frac{\overline{k}c_2}{2^*}||u||^{2^*} \\ &= \frac{1}{2}\left(1 - \frac{\Theta}{\overline{\Lambda}}\right)||u||^2 - \frac{\overline{h}c_1}{q+1}||u||^{q+1} - \frac{\overline{k}c_2}{2^*}||u||^{2^*}.\end{aligned}$$

Thereby, the lemma follows by taking $\rho > 0$ satisfying

$$\frac{1}{2}\left(1 - \frac{\Theta}{\overline{\Lambda}}\right)\rho^2 - \frac{\overline{h}c_1}{q+1}\rho^{q+1} - \frac{\overline{k}c_2}{2^*}\rho^{2^*} > 0. \quad \square$$

Lemma 2.6. *The real number γ_W given in (2.8) is positive. In addition, if $u \in \mathcal{M}$ then $||u^+|| \geq \max\{||u^-||, \sqrt{2\gamma_W}\}$.*

Proof. By Lemma 2.5, there is $\rho > 0$ such that

$$l := \inf_{B_\rho(0)\cap E^+} \Phi_W > 0.$$

For all $u \in \mathcal{M}$, we know that $u^+ \neq 0$, then by Lemma 2.4,

$$\Phi_W(u) \geq \Phi_W\left(\frac{\rho}{||u^+||}u^+\right) \geq l,$$

from where it follows that

$$\gamma_W = \inf_{\mathcal{M}} \Phi_W \geq l > 0.$$

In addition, for all $u \in \mathcal{M}$,

$$\gamma_W \leq \Phi_W(u) \leq \frac{1}{2}B(u,u) = \frac{1}{2}(||u^+||^2 - ||u^-||^2),$$

implying that $||u^+|| \geq \max\{||u^-||, \sqrt{2\gamma_W}\}$. $\square$

Next we will show a boundedness from above for γ_W which will be crucial in our approach. However, before doing this we need to prove two technical lemmas. The first one is true for $N \geq 2$ and it has the following statement

Please cite this article in press as: C.O. Alves, G.F. Germano, Ground state solution for a class of indefinite variational problems with critical growth, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.02.039

Lemma 2.7. *Consider $N \geq 2$ and let $u \in E^+ \setminus \{0\}$, $p \in (2, 2^*)$ and $r, s_0 > 0$. Then there exists $\xi > 0$ such that*

$$\xi |su|_p \leq |su + v|_p, \tag{2.9}$$

for all $s \geq s_0$ and $v \in E^-$ with $||su + v|| \leq r$.

Proof. If the lemma does not hold, there are $s_n \geq s_0$ and $v_n \in E^-$ satisfying

$$||s_n u + v_n|| \leq r \quad \text{and} \quad |s_n u|_p \geq n|s_n u + v_n|_p, \ \forall n \in \mathbb{N}.$$

Setting $\alpha_n := |s_n u|_p$, we obtain

$$\left| \frac{u}{|u|_p} + \frac{v_n}{\alpha_n} \right|_p \leq \frac{1}{n}.$$

Thus, passing to a subsequence if necessary,

$$w_n := \frac{v_n}{\alpha_n} \to -\frac{u}{|u|_p} \qquad \text{a.e. in} \quad \mathbb{R}^N. \tag{2.10}$$

On the other hand,

$$||w_n||^2 = \frac{||v_n||^2}{s_n^2 |u|_p^2} \leq \frac{||s_n u + v_n||^2}{s_0^2 |u|_p^2} \leq \frac{r^2}{s_0^2 |u|_p^2} \qquad \forall n \in \mathbb{N},$$

showing that (w_n) is a bounded sequence in $H^1(\mathbb{R}^N)$. As $w_n \in E^-$, there is $w \in E^-$ such that for some subsequence (not renamed) $w_n \rightharpoonup w$ in E^-. Then by (2.10),

$$\frac{u}{|u|_p} = -w \in E^-,$$

which is absurd, since $u \in E^+ \setminus \{0\}$. $\square$

Lemma 2.8. *Let $u \in E^+ \setminus \{0\}$ be fixed. Then there are $r, s_0 > 0$ satisfying*

$$\sup_{w \in \widehat{E}(u)} \Phi_W(w) = \sup_{\substack{||su+v|| \leq r \\ s \geq s_0,\, v \in E^-}} \Phi_W(su + v). \tag{2.11}$$

Proof. From Lemma 2.2,

$$\sup_{\widehat{E}(u)} \Phi_W = \sup_{\widehat{E}(u) \cap B_r(0)} \Phi_W$$

for some $r > 0$. Hence, there are $(s_n) \subset [0, +\infty)$ and $(v_n) \subset E^-$ with $||s_n u + v_n|| \leq r$ and

$$\Phi_W(s_n u + v_n) \to \sup_{\widehat{E}(u) \cap B_r(0)} \Phi_W. \tag{2.12}$$

Next, we will prove that there exists $s_0 > 0$ such that

$$\sup_{\widehat{E}(u)\cap B_r(0)} \Phi_W = \sup_{\substack{||su+v|| \le r \\ s \ge s_0,\, v \in E^-}} \Phi_W(su+v).$$

Arguing by contradiction, suppose that for all $s_0 > 0$

$$\sup_{\widehat{E}(u)\cap B_r(0)} \Phi_W > \sup_{\substack{||su+v|| \le r \\ s \ge s_0,\, v \in E^-}} \Phi_W(su+v). \tag{2.13}$$

Such supposition permit us to conclude that $s_n \to 0$. On the other hand, recalling that

$$\Phi_W(s_n u + v_n) \le \frac{1}{2} s_n^2 ||u||^2,$$

we are leading to

$$0 < \gamma_W = \inf_{\mathcal{M}} \Phi_W \le \sup_{\widehat{E}(u)} \Phi_W = \Phi_W(s_n u + v_n) + o_n(1) \le \frac{1}{2} s_n^2 ||u||^2 + o_n(1),$$

which is a contradiction. This completes the proof. □

Now, we are ready to show the estimate from above involving the number γ_W given in (2.8)

Proposition 2.9. *Assume the conditions of Theorem 1.1. If $N \ge 4$, then*

$$\gamma_W < \frac{1}{N|k_0|_\infty^{\frac{N-2}{2}}} S^{N/2}. \tag{2.14}$$

If $N = 3$, there is $\lambda^* > 0$ such that the estimate (2.14) holds for $\inf_{x\in\mathbb{R}^N} h(x) > \lambda^*$.

Proof. Since $\gamma_W \le \gamma$, it is enough to prove that

$$\gamma < \frac{1}{N|k_0|_\infty^{\frac{N-2}{2}}} S^{N/2}.$$

If $N \ge 4$ and $\inf_{x\in\mathbb{R}^N} h(x) = 0$, the estimate is made in [5, Proposition 4.2]. Next we will do the proof for $N \ge 4$ and $\inf_{x\in\mathbb{R}^N} h(x) > 0$. To this end, we follow the same notation used in [5]. Let

$$\varphi_\epsilon(x) = \frac{c_N \psi(x) \epsilon^{\frac{N-2}{2}}}{(\epsilon^2 + |x|^2)^{\frac{N-2}{2}}}$$

where $c_N = (N(N-2))^{\frac{N-2}{4}}$, $\epsilon > 0$, and $\psi \in C_0^\infty(\mathbb{R}^N)$ is such that

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$$\psi(x)=1 \quad \text{for} \quad |x|\le \frac{1}{2} \quad \text{and} \quad \psi(x)=0 \quad \text{for} \quad |x|\ge 1.$$

From [33], we know that the estimates below hold

$$\begin{aligned} &|\nabla\varphi_\epsilon|_2^2 = S^{\frac{N}{2}} + O(\epsilon^{N-2}), \quad |\nabla\varphi_\epsilon|_1 = O(\epsilon^{\frac{N-2}{2}}), \quad |\varphi_\epsilon|_{2^*}^{2^*} = S^{\frac{N}{2}} + O(\epsilon^{N}), \\ &|\varphi_\epsilon|_{2^*-1}^{2^*-1} = O(\epsilon^{\frac{N-2}{2}}), \quad |\varphi_\epsilon|_q^q = O(\epsilon^{\frac{N-2}{2}}), \quad |\varphi_\epsilon|_1 = O(\epsilon^{\frac{N-2}{2}}) \end{aligned} \tag{2.15}$$

and

$$|\varphi_\epsilon|_2^2 = \begin{cases} b\epsilon^2|log\epsilon| + O(\epsilon^2), & \text{if} \quad N=4 \\ b\epsilon^2 + O(\epsilon^{N-2}), & \text{if} \quad N\ge 5. \end{cases} \tag{2.16}$$

Adapting the same idea explored in [5, Proposition 4.2], for each $u\in E^-$ we obtain

$$\Phi(s\varphi_\epsilon+u)\le \Phi(s\varphi_\epsilon)+O(\epsilon^{N-2}), \quad \forall s\ge 0,$$

where $O(\epsilon^{N-2})$ does not depend on u. Now, arguing as in [1], we get

$$\sup_{s\ge 0}\Phi(s\varphi_\epsilon)\le \frac{1}{N|k_0|_\infty^{\frac{N-2}{2}}}S^{N/2}+O(\epsilon^{N-2})+c_1\int_{B_1(0)}|\varphi_\epsilon|^2\,dx - c_2\int_{B_1(0)}|\varphi_\epsilon|^{q+1}\,dx,$$

implying that

$$\sup_{s\ge 0,\ u\in E^-}\Phi(s\varphi_\epsilon+u)\le \frac{1}{N|k_0|_\infty^{\frac{N-2}{2}}}S^{N/2}+c_1\int_{B_1(0)}|\varphi_\epsilon|^2\,dx - c_2\int_{B_1(0)}|\varphi_\epsilon|^{q+1}\,dx+O(\epsilon^{N-2}).$$

Moreover, in [1], we also find that

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^{N-2}}\left(c_1\int_{B_1(0)}|\varphi_\epsilon|^2\,dx - c_2\int_{B_1(0)}|\varphi_\epsilon|^{q+1}\,dx\right)=-\infty,$$

from where it follows that there exists $\epsilon>0$ small enough verifying

$$c_1\int_{B_1(0)}|\varphi_\epsilon|^2\,dx - c_2\int_{B_1(0)}|\varphi_\epsilon|^{q+1}\,dx+O(\epsilon^{N-2})<0,$$

and so,

$$\sup_{s\ge 0,\ u\in E^-}\Phi(s\varphi_\epsilon+u)<\frac{1}{N|k_0|_\infty^{\frac{N-2}{2}}}S^{N/2}$$

for some $\epsilon>0$ small enough.

Now, we will consider the case $N = 3$. For each $u \in E^+ \setminus \{0\}$, the Lemma 2.8 guarantees the existence of $r, s_0 > 0$ satisfying

$$\sup_{w \in \widehat{E}(u)} \Phi(w) = \sup_{\substack{||su+v|| \leq r \\ s \geq s_0,\, v \in E^-}} \Phi(su+v).$$

Therefore, applying Lemma 2.7,

$$\begin{aligned}
\sup_{\widehat{E}(u)} \Phi &= \sup_{\substack{||su+v|| \leq r \\ s \geq s_0,\, v \in E^-}} \Phi(su+v) \\
&\leq \sup_{\substack{||su+v|| \leq r \\ s \geq s_0,\, v \in E^-}} \left(\frac{s^2||u||^2}{2} - \frac{\lambda}{q+1} \int |su+v|^{q+1} dx \right) \\
&\leq \sup_{\substack{||su+v|| \leq r \\ s \geq s_0,\, v \in E^-}} \left(\frac{s^2||u||^2}{2} - \frac{\lambda \xi}{q+1} \int |su|^{q+1} dx \right) \\
&\leq \max_{s \geq 0} (As^2 - \lambda B s^{q+1}),
\end{aligned}$$

where

$$\lambda = \inf_{x \in \mathbb{R}^N} h(x), \quad A = \frac{||u||^2}{2} \quad \text{and} \quad B = \frac{\xi}{q+1} \int |u|^{q+1} dx.$$

As

$$\max_{s \geq 0} (As^2 - \lambda B s^{q+1}) \to 0 \quad \text{as} \quad \lambda \to +\infty,$$

there is $\lambda^* > 0$ such that

$$\sup_{w \in \widehat{E}(u)} \Phi(w) < \frac{1}{N|k_0|_\infty^{\frac{N-2}{2}}} S^{N/2} \quad \forall \lambda \geq \lambda^*,$$

showing the desired result. □

Lemma 2.10. *Let $(u_n) \subset H^1(\mathbb{R}^N)$ be a sequence verifying*

$$\Phi_W(u_n) \leq d, \quad \pm \Phi'_W(u_n) u_n^\pm \leq d||u_n|| \quad \textit{and} \quad -\Phi'_W(u_n) u_n \leq d||u_n||$$

for some $d > 0$. Then, (u_n) is bounded in $H^1(\mathbb{R}^N)$.

Proof. In the sequel, let $\theta := \chi_{[-1,1]} : \mathbb{R} \to \mathbb{R}$ be the characteristic function on interval $[-1, 1]$,

$$g(x,t) := \theta(t) f(x,t) \quad \text{and} \quad j(x,t) := (1 - \theta(t)) f(x,t),$$

where $f(x,t) = h(x)|t|^{q-1}t + k(x)|t|^{2^*-2}t$. Fixing

$$r := \frac{q+1}{q} \quad \text{and} \quad s = \frac{2^*}{2^*-1},$$

it follows that

$$(r-1)q = (s-1)(2^*-1) = 1.$$

Note that

$$\begin{aligned} |g(x,t)|^{r-1} &= \theta(t)^{r-1}|f(x,t)|^{r-1} \leq \theta(t)(|h|_\infty|t|^q + |k|_\infty|t|^{2^*-1})^{r-1} \\ &\leq \theta(t)2^{r-1}C(|t|^{(r-1)q} + |t|^{(r-1)(2^*-1)}) \leq K|t| \end{aligned}$$

for some $C > 0$ sufficiently large. So

$$|g(x,t)|^{r-1} \leq C|t|, \forall\ (x,t) \in \mathbb{R}^{N+1}. \tag{2.17}$$

Analogously,

$$|j(x,t)|^{s-1} \leq C|t|, \forall\ (x,t) \in \mathbb{R}^{N+1}. \tag{2.18}$$

Since $tf(x,t) \geq 0,\ (x,t) \in \mathbb{R}^{N+1}$, the inequalities (2.17) and (2.18) give

$$|g(x,t)|^r \leq Ctg(x,t) \text{ and } |j(x,t)|^s \leq Ctj(x,t), \quad \forall (x,t) \in \mathbb{R}^{N+1}. \tag{2.19}$$

The last two inequalities lead to

$$\begin{aligned} & d + d||u_n|| \geq \Phi_W(u_n) - \frac{1}{2}\Phi'_W(u_n)u_n = \\ & \left(\frac{1}{2} - \frac{1}{q+1}\right)\int h(x)|u|^{q+1}dx + \left(\frac{1}{2} - \frac{1}{2^*}\right)\int k(x)|u|^{2^*}dx \geq \\ & \left(\frac{1}{2} - \frac{1}{q+1}\right)\int h(x)|u|^{q+1}dx + \left(\frac{1}{2} - \frac{1}{q+1}\right)\int k(x)|u|^{2^*}dx = \\ & \left(\frac{1}{2} - \frac{1}{q+1}\right)\int (g(x,u_n)u_n + j(x,u_n)u_n)dx \geq \\ & \left(\frac{1}{2} - \frac{1}{q+1}\right)\frac{1}{C}\left(\int |g(x,u_n)|^r dx + \int |j(x,u_n)|^s dx\right), \end{aligned}$$

from where it follows

$$|g(x,u_n)|_r^r + |j(x,u_n)|_s^s \leq C(1 + ||u_n||) \tag{2.20}$$

for some $C > 0$. On the other hand,

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$$
\begin{aligned}
||u_n^-||^2 &= -\Phi'_W(u_n)u_n^- - \int W(x)u_n u_n^- dx - \int f(x,u_n)u_n^- dx \\
&\le d||u_n^-|| - \int W(x)u_n u_n^- dx + |g(x,u_n)|_r |u_n^-|_{q+1} + |j(x,u_n)|_s |u_n^-|_{2^*} \\
&\le -\int W(x)u_n u_n^- dx + C||u_n^-||\,(1 + |g(x,u_n)|_r + |j(x,u_n)|_s) \\
&\le -\int W(x)u_n u_n^- dx + C||u_n^-||\left(1 + (1+||u_n||)^{1/r} + (1+||u_n||)^{1/s}\right) \\
&\le -\int W(x)u_n u_n^- dx + C||u_n^-||\left(1 + ||u_n||^{1/r} + ||u_n||^{1/s}\right).
\end{aligned}
$$

Thus,

$$
||u_n^-||^2 \le -\int W(x)u_n u_n^- dx + C||u_n||\left(1 + ||u_n||^{1/r} + ||u_n||^{1/s}\right). \tag{2.21}
$$

The same argument works to prove that

$$
||u_n^+||^2 \le \int W(x)u_n u_n^+ dx + C||u_n||\left(1 + ||u_n||^{1/r} + ||u_n||^{1/s}\right). \tag{2.22}
$$

Recalling that $||u_n||^2 = ||u_n^+||^2 + ||u_n^-||^2$, the estimates (2.21) and (2.22) combined give

$$
||u_n||^2 \le \int W(x)u_n(u_n^+ - u_n^-)dx + C||u_n||\left(1 + ||u_n||^{1/r} + ||u_n||^{1/s}\right). \tag{2.23}
$$

On the other hand, we know that

$$
\begin{aligned}
\int W(x)u_n(u_n^+ - u_n^-)dx &= \int W(x)(u_n^+ + u_n^-)(u_n^+ - u_n^-)dx \\
&= \int W(x)(u_n^+)^2 dx - \int W(x)(u_n^-)^2 dx \\
&\le \int W(x)(u_n^+)^2 dx \le \Theta \int (u_n^+)^2 dx \le \frac{\Theta}{\overline{\Lambda}}||u_n^+||^2
\end{aligned}
$$

that is,

$$
\int W(x)u_n(u_n^+ - u_n^-)dx \le \frac{\Theta}{\overline{\Lambda}}||u_n||^2, \tag{2.24}
$$

where $\overline{\Lambda}$ was fixed in (W_2). Now, (2.23) combines with (2.24) to give

$$
\left(1 - \frac{\Theta}{\overline{\Lambda}}\right)||u_n||^2 \le C||u_n||\left(1 + ||u_n||^{1/r} + ||u_n||^{1/s}\right).
$$

This concludes the verification of Lemma 2.10. □

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As a byproduct of the last lemma, we have the corollaries below

Corollary 2.11. *If (u_n) is a (PS) sequence for Φ_W, then (u_n) is bounded. In addition, if $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, then u is a solution of (2.1).*

Corollary 2.12. *Φ_W is coercive on $\mathcal{M}$, that is, $\Phi_W(u) \to +\infty$ as $||u|| \to +\infty$ and $u \in \mathcal{M}$.*

The Lemma 2.4 permits to consider a function

$$m : E^+ \setminus \{0\} \to \mathcal{M} \ \text{ where } \ m(u) \in \hat{E}(u) \cap \mathcal{M}, \quad \forall u \in E^+ \setminus \{0\}. \tag{2.25}$$

The above function will be crucial in our approach. Next, we establish its continuity.

Lemma 2.13. *The function m is continuous.*

Proof. Suppose $u_n \to u$ in $E^+ \setminus \{0\}$. Since

$$\frac{u_n}{||u_n||} \to \frac{u}{||u||}, \quad m\left(\frac{u_n}{||u_n||}\right) = m(u_n) \quad \text{and} \quad m\left(\frac{u}{||u||}\right) = m(u),$$

without loss of generality, we may assume that $||u_n|| = ||u|| = 1$.

There are $t_n, t \in [0, +\infty)$ and $v_n, v \in E^-$ such that

$$m(u_n) = t_n u_n + v_n \quad \text{and} \quad m(u) = tu + v.$$

Note that $K := \{u_n\}_{n\in\mathbb{N}} \cup \{u\}$ is a compact set. Thereby, by Lemma 2.2, there exists $R > 0$ such that $\Phi_W(w) \le 0$ in $E(z) \setminus B_R(0)$ for all $z \in K$. Hence,

$$0 < \Phi_W(m(u_n)) = \sup_{\widehat{E}(u_n)} \Phi_W = \sup_{\widehat{E}(u_n)\cap B_R(0)} \Phi_W \le \sup_{w\in\widehat{E}(u_n)\cap B_R(0)} \frac{1}{2}||w^+||^2 \le \frac{1}{2}R^2,$$

showing that $(\Phi_W(m(u_n)))$ is a bounded sequence, and so, by Corollary 2.12, $(m(u_n))$ is a bounded sequence. The boundedness of $(m(u_n))$ implies that (t_n) and (v_n) are also bounded. Then, for some subsequence (not renamed),

$$t_n \to t_0 \ \text{ in } \ \mathbb{R}, \ v_n \rightharpoonup v_0 \ \text{ in } \ E^- \quad \text{and} \quad m(u_n) \rightharpoonup t_0 u + v_0 \ \text{ in } \ E^-. \tag{2.26}$$

Recalling that $\Phi_W(m(u_n)) \ge \Phi_W(tu_n + v)$, we obtain

$$\liminf_{n\to+\infty} \Phi_W(m(u_n)) \ge \Phi_W(m(u)).$$

Thus, the Fatou's Lemma combined with the weakly lower semicontinuous of the norm gives

$$
\begin{aligned}
\Phi_W(m(u)) &\le \liminf_{n\to+\infty} \Phi_W(m(u_n)) \le \limsup_{n\to+\infty} \Phi_W(m(u_n)) \\
&\limsup_{n\to+\infty} \left[\frac{1}{2} t_n^2 ||u_n||^2 - \frac{1}{2}||v_n||^2 - \frac{1}{2}\int W(x) m(u_n)^2 dx \right. \\
&\left. - \frac{1}{q+1}\int h(x)|m(u_n)|^{q+1} dx - \frac{1}{2^*}\int k(x)|m(u_n)|^{2^*} dx \right] \\
&\le \frac{1}{2} t_0^2 - \frac{1}{2}||v_0||^2 - \frac{1}{2}\int W(x)|t_0 u + v_0|^2 dx \\
&- \frac{1}{q+1}\int h(x)|t_0 u + v_0|^{q+1} dx - \frac{1}{2^*}\int k(x)|t_0 u + v_0|^{2^*} dx \\
&= \Phi_W(t_0 u + v_0) \le \Phi_W(m(u)),
\end{aligned}
$$

implying that

$$
\lim_{n\to+\infty} ||v_n|| = ||v_0|| \quad \text{and} \quad \Phi_W(t_0 u + v_0) = \Phi_W(m(u)). \tag{2.27}
$$

From (2.26) and (2.27), $v_n \to v_0$ in E^-. Now, the Lemma 2.1 together with (2.27) guarantees that $t_0 u + v_0 = m(u)$. Consequently,

$$
m(u_n) = t_n u_n + v_n \to t_0 u + v_0 = m(u),
$$

finishing the proof. $\square$

Hereafter, we consider the functional $\hat{\Psi} : E^+ \setminus \{0\} \to \mathbb{R}$ defined by $\hat{\Psi}(u) := \Phi_W(m(u))$. We know that $\hat{\Psi}$ is continuous by previous lemma. In the sequel, we denote by $\Psi : S^+ \to \mathbb{R}$ the restriction of $\hat{\Psi}$ to $S^+ = B_1(0) \cap E^+$.

The next three results establish some important properties involving the functionals Ψ and $\hat{\Psi}$ and their proofs follow as in [25].

Lemma 2.14. *$\hat{\Psi} \in C^1(E^+ \setminus \{0\}, \mathbb{R})$, and*

$$
\hat{\Psi}'(y)z = \frac{||m(y)^+||}{||y||} \Phi_W'(m(y))z, \ \forall y, z \in E^+, \ y \neq 0. \tag{2.28}
$$

Corollary 2.15. *The following assertions hold:*

(a) $\Psi \in C^1(S^+)$*, and*

$$
\Psi'(y)z = ||m(y)^+|| \Phi_W'(m(y))z, \ \textit{for } z \in T_y S^+.
$$

(b) (w_n) *is a $(PS)_c$ sequence for Ψ if and only if $(m(w_n))$ is a $(PS)_c$ sequence for Φ_W.*
(c) If $\gamma_W = \inf_{\mathcal{M}} \Phi_W$ is attained by $u \in \mathcal{M}$, then $\Phi_W'(u) = 0$.

Proposition 2.16. *There exists a $(PS)_{\gamma_W}$ sequence for Φ_W.*

Our next lemma will be used to prove the existence of ground state solution for the periodic case.

Lemma 2.17. *Let (u_n) be a $(PS)_c$ sequence for functional Φ given in (2.6) with $c \neq 0$. Then, there are $r, \epsilon > 0$ and (y_n) in $\mathbb{Z}^N$ satisfying*

$$\limsup_{n\in\mathbb{N}} \int_{B_r(y_n)} |u_n|^{2^*} dx \geq \epsilon. \tag{2.29}$$

In addition, if $c \in (-\infty, S^{N/2}|k_0|_\infty^{\frac{2-N}{2}}/N) \setminus \{0\}$, the sequence $v_n = u_n(\cdot - y_n)$ is also a $(PS)_c$ sequence for Φ, and for some subsequence, $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ with $v \neq 0$.

Proof. By Corollary 2.11, the sequence (u_n) is bounded in $H^1(\mathbb{R}^N)$. Arguing by contradiction, we suppose that

$$\limsup_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^{2^*} dx = 0,$$

for some $R > 0$. Applying [23, Lemma 2.1], it follows that $u_n \to 0$ in $L^{2^*}(\mathbb{R}^N)$, and so, by interpolation on the Lebesgue spaces, $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for all $p \in (2, 2^*]$. As

$$\Phi'(u_n)(u_n^-) = -||u_n^-||^2 - \int h_0(x)|u_n|^{q-1}u_n u_n^- dx - \int k_0(x)|u_n|^{2^*-2}u_n u_n^- dx,$$

we deduce that $u_n^- \to 0$ in $H^1(\mathbb{R}^N)$. By a similar argument $u_n^+ \to 0$ in $H^1(\mathbb{R}^N)$. Hence

$$u_n \to 0 \text{ in } \ H^1(\mathbb{R}^N).$$

Thereby, by continuity of Φ, $c = \lim \Phi(u_n) = \Phi(0) = 0$, which is absurd. Thus, there are $(z_n) \subset \mathbb{R}^N$ and $\eta > 0$ satisfying

$$\int_{B_R(z_n)} |u_n^+|^{2^*} dx \geq \eta > 0, \quad \forall n \in \mathbb{N}.$$

Recalling that for each $n \in \mathbb{N}$ there is $y_n \in \mathbb{Z}^N$ such that

$$B_R(z_n) \subset B_{R+\sqrt{N}}(y_n),$$

we have

$$\int_{B_{R+\sqrt{N}}(y_n)} |u_n^+|^{2^*} dx \geq \eta > 0, \quad \forall n \in \mathbb{N},$$

finishing the proof of (2.29).

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Now, assume $c \in (-\infty, S^{N/2}|k_0|_\infty^{\frac{2-N}{2}}/N) \setminus \{0\}$ and set $v_n := u_n(\cdot - y_n)$. By a simple computation, we see that (v_n) is also a $(PS)_c$ sequence for Φ with

$$\limsup_{n\to+\infty} \int_{B_r(0)} |v_n^+|^{2^*}\, dx \geq \epsilon. \tag{2.30}$$

By Corollary 2.12, (v_n) is bounded, and so, for some subsequence (still denoted by (v_n)), $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ for some $v \in H^1(\mathbb{R}^N)$. Suppose by contradiction $v = 0$ and assume that

$$|\nabla v_n|^2 \rightharpoonup \mu \qquad \text{and} \qquad |v_n|^{2^*} dx \rightharpoonup \nu \text{ in } \mathcal{M}^+(\mathbb{R}^N). \tag{2.31}$$

By Concentration-Compactness Principle II due to Lions [15], there exist a countable set $\mathcal{J}$, $(x_j)_{j\in\mathcal{J}} \subset \mathbb{R}^N$ and $(\mu_j)_{j\in\mathcal{J}}, (\nu_j)_{j\in\mathcal{J}} \subset [0, +\infty)$ such that

$$\nu = \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j} \quad \mu \geq \sum_{j\in\mathcal{J}} \mu_j \delta_{x_j} \text{ with } \mu_j \geq S\nu_j^{\frac{2}{2^*}}. \tag{2.32}$$

Now, our goal is to show that $\nu_j = 0$ for all $j \in \mathcal{J}$. First of all, note that

$$c = \lim_{n\to+\infty} \left[\Phi(v_n) - \frac{1}{2}\Phi'(v_n)v_n\right] \geq \frac{1}{N} \sum_{j\in\mathcal{J}} k_0(x_j)\nu_j. \tag{2.33}$$

On the other hand, setting $\psi_\epsilon(x) := \psi((x - x_j)/\epsilon), \forall\, x \in \mathbb{R}^N, \forall\, \epsilon > 0$, where $\psi \in C_c^\infty(\mathbb{R}^N)$ is such that $\psi \equiv 1$ in $B_1(0)$, $\psi \equiv 0$ in $\mathbb{R}^N \setminus B_2(0)$ and $|\nabla\psi| \leq 2$, with $0 \leq \psi \leq 1$, we have that $\psi_\epsilon v_n \in H^1(\mathbb{R}^N)$ and $(\psi_\epsilon v_n)$ is bounded in $H^1(\mathbb{R}^N)$. So

$$\Phi'(v_n)(\psi_\epsilon v_n) \to 0$$

or equivalently

$$\int \nabla v_n \nabla(\psi_\epsilon v_n)\, dx + \int V(x)\psi_\epsilon v_n^2\, dx - \int h_0(x)\psi_\epsilon |v_n|^{q+1} dx - \int k_0(x)|v_n|^{2^*}\psi_\epsilon dx \to 0.$$

By using the definition of ν and μ together with the last limit, we derive

$$\int \nabla v(\nabla\psi_\epsilon)v\, dx + \int V(x)\psi_\epsilon v^2\, dx - \int h_0(x)\psi_\epsilon |v|^{q+1} dx + \int \psi_\epsilon d\mu - \int k_0\psi_\epsilon d\nu = 0.$$

Now, taking the limit $\epsilon \to 0$, we find

$$\mu(x_j) = k_0(x_j)\nu_j.$$

By (2.32), $\mu_j \leq \mu(x_j)$. Then,

$$S\nu_j^{2/(2^*)} = \mu_j \leq \mu(x_j) = k_0(x_j)\nu_j.$$

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If $\nu_j \neq 0$, the last inequality gives

$$\nu_j \geq \frac{S^{N/2}}{|k_0|_\infty^{\frac{N-2}{2}}}. \tag{2.34}$$

Thereby, by (2.33) and (2.34), if there exists $j \in \mathcal{J}$ such that $\nu_j \neq 0$, we would have

$$c \geq \frac{S^{N/2}}{N|k_0|_\infty^{\frac{N-2}{2}}}$$

which is absurd. Hence $\nu_j = 0$ for all $j \in \mathcal{J}$, so $\nu \equiv 0$, and by (2.31), $|v_n|^{2^*} \rightharpoonup 0$ in $\mathcal{M}^+(\mathbb{R}^N)$. Consequently $v_n \to 0$ in $L^{2^*}_{loc}(\mathbb{R}^N)$ which contradicts (2.30), showing that $v \neq 0$. □

2.2. *Proof of Theorem 1.1: the case $N \geq 3$*

The proof will be divided into two cases, more precisely, the Periodic Case and the Asymptotically Periodic Case.

1- The periodic case:

Proof. From Proposition 2.16, there exists a $(PS)_\gamma$ sequence (u_n) for Φ, where γ was given in (2.8). By Lemma 2.17, passing to a subsequence if necessary, $u_n \rightharpoonup u \neq 0$ and $u \in H^1(\mathbb{R}^N)$ is a solution of problem (2.7), and so, $\Phi(u) \geq \gamma$. On the other hand

$$\begin{aligned}
\gamma &= \lim_{n\to+\infty}\left[\Phi(u_n) - \frac{1}{2}\Phi'(u_n)(u_n)\right] \\
&= \liminf_{n\to+\infty}\left[\left(\frac{1}{2} - \frac{1}{q+1}\right)\int h(x)|u_n|^{q+1}dx + \left(\frac{1}{2} - \frac{1}{2^*}\right)\int k(x)|u_n|^{2^*}\,dx\right] \\
&\geq \left[\left(\frac{1}{2} - \frac{1}{q+1}\right)\int h(x)|u|^{q+1}dx + \left(\frac{1}{2} - \frac{1}{2^*}\right)\int k(x)|u|^{2^*}\,dx\right] \\
&= \Phi(u) - \frac{1}{2}\Phi'(u)u = \Phi(u).
\end{aligned}$$

From this, $u \in H^1(\mathbb{R}^N)$ is a ground state solution for the problem (2.7). □

2- Asymptotically periodic case:

Proof. From definition of Φ_W and Φ, we have the inequality

$$\gamma_W \leq \gamma.$$

Next, our analysis will be divide into two cases, more precisely, $\gamma_W = \gamma$ and $\gamma_W < \gamma$.

Assume firstly $\gamma_W = \gamma$. Let $u \in H^1(\mathbb{R}^N)$ be a ground state solution of (2.7) for the periodic case and $v \in \widehat{E}(u)$ such that

$$\Phi_W(v) = \sup_{\widehat{E}(u)} \Phi_W.$$

Then,

$$\gamma_W \leq \Phi_W(v) \leq \Phi(v) \leq \Phi(u) = \gamma = \gamma_W,$$

implying that $\Phi_W(v) = \gamma_W$ with $v \in \mathcal{M}$. By Corollary 2.15, part (c), we deduce that v is a ground state solution of (2.1).

Now, assume $\gamma_W < \gamma$ and let (u_n) be a $(PS)_{\gamma_W}$ sequence for Φ_W given by Proposition 2.16. By Lemma 2.10, (u_n) is a bounded sequence, then for some subsequence (still denoted by (u_n)) $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. We claim that $u \neq 0$. Indeed, if $u = 0$ it is easy to see that

$$\int W(x)u_n^2 dx \to 0 \text{ and } \sup_{\|\psi\|\leq 1} \left| \int W(x)u_n \psi dx \right| \to 0.$$

In addiction, by (C_1), we also have

$$\int h^*(x)|u_n|^{q+1} dx \to 0 \quad \text{and} \quad \sup_{\|\psi\|\leq 1} \left| \int h^*(x)|u_n|^{q-1} u\psi dx \right| \to 0.$$

Arguing as in Lemma 2.17, we derive that $u_n \to 0$ in $L^{2^*}_{loc}(\mathbb{R}^N)$, and so,

$$\int k^*(x)|u_n|^{2^*} dx \to 0 \quad \text{and} \quad \sup_{\|\psi\|\leq 1} \left| \int k^*(x)|u_n|^{2^*-2} u_n \psi dx \right| \to 0.$$

Hence

$$\Phi_W(u_n) \to \gamma_W \text{ and } ||\Phi'_W(u_n)|| \to 0,$$

that is, (u_n) is a $(PS)_{\gamma_W}$ sequence for Φ_W. By Proposition 2.9,

$$\gamma_W < \frac{S^{N/2}}{N|k_0|_\infty^{\frac{N-2}{2}}}.$$

Then, Proposition 2.17 guarantees the existence of $(y_n) \subset \mathbb{Z}^N$ such that $v_n := u_n(\cdot - y_n) \rightharpoonup v \neq 0$ in $H^1(\mathbb{R}^N)$ and $\Phi'(v) = 0$. Consequently

$$\begin{aligned} \gamma_W &= \lim_{n\to+\infty} \Phi_W(u_n) = \lim_{n\to+\infty} \Phi(u_n) \\ &= \lim_{n\to+\infty} \Phi(v_n) = \lim_{n\to+\infty} \left[\Phi(v_n) - \frac{1}{2}\Phi'(v_n)v_n \right] \\ &\geq \Phi(v) - \frac{1}{2}\Phi'(v)v = \Phi(v) \geq \gamma \end{aligned}$$

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which is absurd, proving that $u \neq 0$. Now, we repeat the same argument explored in the periodic case to conclude that u is a ground state solution of (2.1). □

3. The case $N = 2$

In this section we are going to show the existence of ground state solution for the following indefinite problem

$$\begin{cases} -\Delta u + (V(x) - W(x))u = f(x,u), & \text{in } \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2), \end{cases} \tag{3.35}$$

by assuming (V_1), (V_2), (W_1), (W_2) and (f_1)–(f_6). Since we will work with exponential critical growth, in the next subsection we recall some facts involving this type of growth.

3.1. Results involving exponential critical growth

The exponential critical growth on f is motivated by the following estimates proved by Trudinger [32] and Moser [19].

Lemma 3.1 *(Trudinger–Moser inequality for bounded domains). Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Given any $u \in H_0^1(\Omega)$, we have*

$$\int_\Omega e^{\alpha |u|^2} dx < \infty, \quad \textit{for every } \alpha > 0.$$

Moreover, there exists a positive constant $C = C(|\Omega|)$ such that

$$\sup_{||u|| \leq 1} \int_\Omega e^{\alpha |u|^2} dx \leq C, \quad \textit{for all } \alpha \leq 4\pi.$$

The next result is a version of the Trudinger–Moser inequality for whole $\mathbb{R}^2$, and its proof can be found in Cao [4] (see also Ruf [24]).

Lemma 3.2 *(Trudinger–Moser inequality for unbounded domains). For all $u \in H^1(\mathbb{R}^2)$, we have*

$$\int \left(e^{\alpha |u|^2} - 1\right) dx < \infty, \quad \textit{for every } \alpha > 0.$$

Moreover, if $|\nabla u|_2^2 \leq 1$, $|u|_2 \leq M < \infty$ and $\alpha < 4\pi$, then there exists a positive constant $C = C(M, \alpha)$ such that

$$\int \left(e^{\alpha |u|^2} - 1\right) dx \leq C.$$

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The Trudinger–Moser inequalities will be strongly utilized throughout this section in order to deduce important estimates. The reader can find more recent results involving this inequality in [6], [10], [11], [18] and references therein.

In the sequel, we state some technical lemmas found in [3] and [7], which will be essential to carry out the proof of our results.

Lemma 3.3. *Let $\alpha > 0$ and $t \geq 1$. Then, for every each $\beta > t$, there exists a constant $C = C(\beta, t) > 0$ such that*

$$\left(e^{4\pi|s|^2} - 1\right)^t \leq C\left(e^{\beta 4\pi|s|^2} - 1\right), \quad \forall s \in \mathbb{R}.$$

Lemma 3.4. *Let (u_n) be a sequence such that $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^2$ and $(f(x,u_n)u_n)$ is bounded in $L^1(\mathbb{R}^2)$. Then, $f(x,u_n) \to f(x,u)$ in $L^1(B_R(0))$ for all $R > 0$, and so,*

$$\int f(x,u_n)\phi\,dx \to \int f(x,u)\phi\,dx, \quad \forall \phi \in C_0^\infty(\mathbb{R}^2).$$

3.2. Technical lemmas

In this subsection we have used the same notations of Section 2, however we will recall some of them for the convenience of the reader. In what follows, we denote by $\Phi_W : H^1(\mathbb{R}^2) \to \mathbb{R}$ the energy functional given by

$$\Phi_W(u) := \frac{1}{2}B(u,u) - \frac{1}{2}\int W(x)|u|^2 dx - \int F(x,u)dx,$$

where $B : H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \to \mathbb{R}$ is the bilinear form

$$B(u,v) := \int (\nabla u \nabla v + V(x)uv)dx, \;\; \forall\, u, v \in H^1(\mathbb{R}^2).$$

It is well known that $\Phi_W \in C^1(H^1(\mathbb{R}^2), \mathbb{R})$ with

$$\Phi'_W(u)v = B(u,v) - \int W(x)uv dx - \int f(x,u)v dx, \quad \forall u, v \in H^1(\mathbb{R}^2).$$

Therefore critical points of Φ_W are solutions of (3.35). Moreover, we can rewrite the functional Φ_W of the form

$$\Phi_W(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \frac{1}{2}\int W(x)|u|^2\,dx - \int F(x,u)\,dx.$$

In what follows, we also consider the C^1-functional $\Phi : H^1(\mathbb{R}^2) \to \mathbb{R}$

$$\Phi(u) := \frac{1}{2}B(u,u) - \int F_0(x,u)dx$$

or equivalently

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$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int F_0(x,u)\,dx,$$

whose the critical points are weak solutions of periodic problem

$$\begin{cases} -\Delta u + V(x) = F_0(x,u), & \text{in } \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2) \end{cases} \tag{3.36}$$

As in Section 2, we will consider the sets

$$\mathcal{M} := \{u \in H^1(\mathbb{R}^2) \setminus E^- \ ; \ \Phi'_W(u)u = 0 \text{ and } \Phi'_W(u)v = 0, \forall\, v \in E^-\},$$
$$E(u) := E^- \oplus \mathbb{R}u \ \text{ and } \ \hat{E}(u) := E^- \oplus [0,+\infty)u$$

Hence

$$E(u) = E^- \oplus \mathbb{R}u^+ \ \text{ and } \ \hat{E}(u) = E^- \oplus [0,+\infty)u^+.$$

Moreover, we fix the real numbers

$$\gamma_W := \inf_{\mathcal{M}} \Phi_W \quad \text{and} \quad \gamma := \inf_{\mathcal{M}} \Phi.$$

Lemma 3.5. *If $u \in \mathcal{M}$ and $w = su + v$ where $s \geq 1$ and $v \in E^-$ such that $w \neq 0$, then*

$$\Phi_W(u+w) < \Phi_W(u)$$

Proof. The proof follows as in Lemma 2.1. □

Lemma 3.6. *Let $\mathcal{K} \subset E^+ \setminus \{0\}$ be a compact subset, then there exists $R > 0$ such that $\Phi_W(w) \leq 0$, $\forall\, w \in E(u) \setminus B_R(0)$ and $u \in \mathcal{K}$.*

Proof. We repeat the argument used in the proof of [25, Lemma 2.2]. □

Lemma 3.7. *For all $u \in H^1(\mathbb{R}^2)$, the functional $\Phi_W|_{E(u)}$ is weakly upper semicontinuous.*

Proof. See proof of Lemma 2.3. □

Lemma 3.8. *For all $u \in H^1(\mathbb{R}^2) \setminus E^-$, $\mathcal{M} \cap \hat{E}(u)$ is a singleton set and the element of this set is the unique global maximum of $\Phi_W|_{\hat{E}(u)}$.*

Proof. See proof of Lemma 2.4. □

In the proof of the next lemma the fact that f has an exponential critical growth brings some difficulty and we will do its proof.

Lemma 3.9. *There exists $\rho > 0$ such that $\inf_{B_\rho(0)\cap E^+} \Phi_W > 0$.*

Proof. Given $p > 2$ and $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$|F(x,t)| \leq \epsilon|t|^2 + C_\epsilon|t|^p(e^{4\pi t^2} - 1), \quad \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}.$$

Then, for all $u \in E^+$, the Lemmas 3.2 and 3.3 lead to

$$\begin{aligned}
\Phi_W(u) &= \frac{1}{2}||u||^2 - \frac{1}{2}\int W(x)|u|^2 dx - \int F(x,u)dx \\
&\geq \frac{1}{2}||u||^2 - \frac{\Theta}{2}\int |u|^2 dx - \epsilon \int |u|^2 dx - C_\epsilon \int |u|^p (e^{4\pi u^2} - 1)dx \\
&= \frac{1}{2}||u||^2 - \frac{\Theta}{2\Lambda}|||u||^2 - \frac{\epsilon}{\Lambda}||u||^2 - C_\epsilon |u|_{2p}^p \left(\int (e^{8\pi u^2} - 1)dx\right)^{\frac{1}{2}} \\
&\geq \left[\frac{1}{2}\left(1 - \frac{\Theta}{\Lambda}\right) - \frac{\epsilon}{\Lambda}\right]||u||^2 - C||u||^p \left(\int (e^{8\pi u^2} - 1)dx\right)^{\frac{1}{2}}.
\end{aligned}$$

By Lemma 3.2, if $\rho < \frac{\sqrt{3}}{2\sqrt{2}}$,

$$\sup_{\|u\|=\rho} \int (e^{8\pi u^2} - 1)dx \leq \sup_{\|v\|\leq 1} \int (e^{3\pi u^2} - 1)dx = C < \infty.$$

So,

$$\Phi_W(u) \geq \left[\frac{1}{2}\left(1 - \frac{\Theta}{\Lambda}\right) - \frac{\epsilon}{\Lambda}\right]||u||^2 - C||u||^p.$$

Hence, decreasing ρ if necessary and fixing ϵ small enough, we get

$$\Phi_W(u) \geq \left[\frac{1}{2}\left(1 - \frac{\Theta}{\Lambda}\right) - \frac{\epsilon}{\Lambda}\right]\rho^2 - C\rho^p = \beta > 0. \quad \square$$

Lemma 3.10. *The real number γ_W is positive. In addition, if $u \in \mathcal{M}$ then $||u^+|| \geq \max\{||u^-||, \sqrt{2\gamma_W}\}$.*

Proof. See proof of Lemma 2.6. $\square$

The next lemma shows that (PS) sequences of Φ_W are bounded, as we are working with the exponential critical growth the arguments explored in Section 2 do not work in this case and a new proof must be done.

Lemma 3.11. *If (u_n) is a sequence such that*

$$\Phi_W(u_n) \leq d, \quad \pm\Phi'_W(u_n)u_n^{\pm} \leq d||u_n|| \quad \text{and} \quad -\Phi'_W(u_n)u_n \leq d$$

for some $d > 0$, then (u_n) is bounded in $H^1(\mathbb{R}^2)$ and $(f(u_n)u_n)$ is bounded in $L^1(\mathbb{R}^2)$.

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Proof. First of all, note that

$$\left(\frac{1}{2}-\frac{1}{\theta}\right)\int f(x,u_n)u_n dx \le \Phi_W(u_n)-\frac{1}{2}\Phi_W'(u_n)u_n \le 2d.$$

Hence, $\left(\int f(x,u_n)u_n dx\right)$ is bounded. Recalling that $f(x,t)t \ge 0$ for all $t\in\mathbb{R}$ and $x\in\mathbb{R}^2$, it follows that $(f(x,u_n)u_n)$ is bounded in $L^1(\mathbb{R}^2)$. On the other hand, we know that

$$||u_n^+||^2 \le d||u_n^+|| + \int f(x,u_n)u_n^+ dx + \int W(x)u_n u_n^+ dx$$

and so,

$$||u_n^+||^2 \le d||u_n^+|| + \left(\int f(x,u_n)v_n dx\right)||u_n^+||_{H^1(\mathbb{R}^2)} + \int W(x)u_n u_n^+ dx \tag{3.37}$$

where $v_n := \frac{u_n^+}{||u_n^+||_{H^1(\mathbb{R}^2)}}$.

Claim 3.12. *$\left(\int f(x,u_n)v_n dx\right)$ is a bounded sequence.*

Indeed, by a direct computation, there exists $K>0$ such that

$$|f(x,t)| \le \Gamma e^{1/4} \text{ implies } |f(x,t)|^2 \le K f(x,t)t, \quad \text{uniformly in} \quad x. \tag{3.38}$$

Moreover, by [8, Lemma 2.11],

$$rs \le (e^{r^2}-1) + s(log^+ s)^{1/2} + \frac{1}{4}s^2\chi_{[0,e^{1/4}]}(s) \quad \forall r,s\ge 0. \tag{3.39}$$

Now, the Lemma 3.2 combined with the above inequalities for $r=|v_n|$ and $s=\frac{1}{\Gamma}|f(u_n)|$ leads to

$$\begin{aligned}
&\left|\int f(x,u_n)v_n dx\right| \le \Gamma\int \frac{1}{\Gamma}|f(u_n)||v_n|dx \le \Gamma\int (e^{v_n^2}-1)dx+\\
&+\int |f(x,u_n)|\left(log^+\left(\frac{1}{\Gamma}|f(x,u_n)|\right)\right)^{1/2}dx+\\
&\frac{1}{4\Gamma}\int |f(x,u_n)|^2\chi_{[0,e^{1/4}]}\left(\frac{1}{\Gamma}|f(x,u_n)|\right)dx \le\\
&\Gamma T + \int |f(x,u_n)|\left(log^+\left(e^{4\pi u_n^2}\right)\right)^{1/2}dx + \frac{1}{4\Gamma}\int_{|f(x,u_n)|\le\Gamma e^{1/4}} |f(x,u_n)|^2 dx \le\\
&\Gamma T + \int |f(x,u_n)||u_n|\sqrt{4\pi}dx + \frac{1}{4\Gamma}\int_{|f(x,u_n)|\le\Gamma e^{1/4}} K f(x,u_n)u_n dx.
\end{aligned}$$

As $(f(x,u_n)u_n)$ is bounded in $L^1(\mathbb{R}^2)$, the last inequality yields $\left(\int f(x,u_n)v_n dx\right)$ is bounded. Consequently, there exists $A_0>0$ satisfying

$$\left|\int f(x,u_n)v_n dx\right| \le A_0, \quad \forall n \in \mathbb{N}.$$

Thereby, by (3.37),

$$||u_n^+||^2 \le d||u_n^+|| + A_0||u_n^+||_{H^1(\mathbb{R}^2)} + \int W(x)u_n u_n^+ dx. \tag{3.40}$$

Analogously, there is $B_0 > 0$ such that

$$||u_n^-||^2 \le d||u_n^-|| + B_0||u_n^-||_{H^1(\mathbb{R}^2)} - \int W(x)u_n u_n^- dx. \tag{3.41}$$

The inequalities (3.40) and (3.41) combine to give

$$\begin{aligned} &||u_n||^2 \le C||u_n|| + C||u_n|| + \int W(x)(u_n u_n^+ - u_n u_n^-)dx = 2C||u_n|| + \\ &+ \int W(x)((u_n^+)^2 - (u_n^-)^2)dx \le 2C||u_n|| + \int W(x)(u_n^+)^2 dx \le 2C||u_n|| + \frac{\Theta}{\Lambda}||u_n^+||^2 \end{aligned}$$

for some $C > 0$. Hence,

$$\left(1 - \frac{\Theta}{\Lambda}\right)||u_n||^2 \le 2\widetilde{C}||u_n||,$$

from where it follows that (u_n) is bounded. □

As a byproduct of the last lemma we have the corollary below

Corollary 3.13. *Φ_W is coercive on $\mathcal{M}$, that is, $\Phi_W(u) \to +\infty$ as $||u|| \to +\infty,\ u \in \mathcal{M}$.*

As in Section 2, the Lemma 3.8 permits to define a function

$$m : E^+ \setminus \{0\} \to \mathcal{M} \ \text{ where } \ m(u) \in \hat{E}(u) \cap \mathcal{M} \ \ \forall u \in E^+ \setminus \{0\}.$$

Now, we invite the reader to observe that the same approach used in Section 2 works to guarantee that the proposition below holds

Proposition 3.14. *There exists a (PS)$_{\gamma_W}$ sequence for Φ_W.*

Our next proposition is crucial when f has an exponential critical growth.

Proposition 3.15. *Fixed $\widetilde{A} \in (0, 1/a)$, there is $\lambda^* > 0$ such that $\gamma_W < \frac{\widetilde{A}^2}{2}$ for $\inf_{\mathbb{R}^2} D(x) > \lambda^*$, where a was given in (2.5).*

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Proof. Let $u \in E^+$ with $u \neq 0$ and set

$$h_D(s) := As^2 - \lambda B s^q,$$

where

$$\lambda = \inf_{x\in\mathbb{R}^2} D(x), \quad A = \frac{1}{2}||u||^2 \quad \text{and} \quad B = \xi \int |u|^q dx,$$

with ξ given in Lemma 2.7. Then, a straightforward computation leads to

$$\max_{s\geq 0} h_D(s) = \left(A - \frac{2A}{q}\right)\left(\sqrt[q-2]{\frac{2A}{qB\lambda}}\right)^2.$$

Thereby, by (f_6) and Lemma 2.7,

$$\begin{aligned}
c &\leq \sup_{\substack{s\in[0,+\infty)\\ v\in E^-}} \Phi_W(su+v) = \sup_{\substack{||su+v||\leq r\\ s\geq s_0,\, v\in E^-}} \Phi_W(su+v)\\
&\leq \sup_{\substack{||su+v||\leq r\\ s\geq s_0,\, v\in E^-}} \left[\frac{1}{2}s^2||u||^2 - \int F(x, su+v)dx\right]\\
&\leq \sup_{\substack{||su+v||\leq r\\ s\geq s_0,\, v\in E^-}} \left[\frac{1}{2}s^2||u||^2 - \lambda\int |su+v|^q dx\right]\\
&\leq \sup_{\substack{||su+v||\leq r\\ s\geq s_0,\, v\in E^-}} \left[\frac{1}{2}s^2||u||^2 - \lambda\xi s^q\int |u|^q dx\right]\\
&= \sup_{\substack{||su+v||\leq r\\ s\geq s_0,\, v\in E^-}} h_D(s)\\
&\leq \max_{s\geq 0} h_D(s) = \left(A - \frac{2A}{q}\right)\left(\sqrt[q-2]{\frac{2A}{qB\lambda}}\right)^2.
\end{aligned}$$

From the last inequality there is $\lambda^* > 0$ such that

$$\left(A - \frac{2A}{q}\right)\left(\sqrt[q-2]{\frac{2A}{qB\lambda}}\right)^2 < \frac{\widetilde{A}^2}{2}, \quad \forall\lambda \geq \lambda^*,$$

finishing the proof. □

Proposition 3.16. *Fix* $\inf_{x\in\mathbb{R}^2} D(x) \geq \lambda^*$ *and* $r > 0$*. Then, there exist a sequence* $(y_n) \subset \mathbb{R}^2$ *and* $\eta > 0$ *such that*

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$$\int_{B_r(y_n)} |u_n^+|^2 dx \geq \eta > 0, \quad \forall n \in \mathbb{N}.$$

Moreover, increasing r if necessary, the sequence (y_n) can be chosen in $\mathbb{Z}^2$.

Proof. Suppose by contradiction that the lemma does not hold for some $r > 0$. Then, by a lemma due to Lions [14],

$$u_n^+ \to 0 \text{ in } L^p(\mathbb{R}^2), \ \forall\, p \in (2, +\infty).$$

Define $w_n := \widetilde{A}\frac{u_n^+}{||u_n^+||}$. Since $u_n \in \mathcal{M}$ for all $n \in \mathbb{N}$, from Lemma 3.10 we have $\liminf\limits_{n\to+\infty} ||u_n^+|| > 0$, and so,

$$w_n \to 0 \text{ in } L^p(\mathbb{R}^2), \ \forall\, p \in (2, +\infty).$$

On the other hand, we also know that

$$||w_n||_{H^1(\mathbb{R}^2)} = \widetilde{A}\frac{||u_n^+||_{H^1(\mathbb{R}^2)}}{||u_n^+||} \leq \widetilde{A}a\frac{||u_n^+||}{||u_n^+||} = \widetilde{A}a < 1.$$

As $w_n \in \widehat{E}(u_n)$ and $u_n \in \mathcal{M}$, we derive that

$$\Phi(u_n) \geq \Phi(w_n) = \frac{1}{2}\widetilde{A}^2 - \int F(x, w_n) dx. \tag{3.42}$$

By [2, Proposition 2.3], we have $\int F(x, w_n)dx \to 0$. Therefore, passing to the limit in (3.42) as $n \to +\infty$, we obtain

$$\gamma w \geq \frac{\widetilde{A}^2}{2},$$

which contradicts the Proposition 3.15. Thus, there are $(z_n) \subset \mathbb{R}^2$ and $\eta > 0$ such that

$$\int_{B_r(z_n)} |u_n^+|^2 dx \geq \eta > 0, \quad \forall n \in \mathbb{N}.$$

Now, we repeat the same idea explored in Lemma 2.17 to conclude the proof. □

3.3. *Proof of Theorem 1.1: the case $N = 2$*

As in Section 2, the proof will be divided into two cases, the Periodic Case and the Asymptotically Periodic Case.

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3.4. Periodic case

Proof. First of all, we recall there is a $(PS)_{\gamma_W}$ sequence (u_n) for Φ which must be bounded. Thus, there is $u \in H^1(\mathbb{R}^2)$ such that for some subsequence of (u_n), still denoted by itself, we have

$$u_n \rightharpoonup u \quad \text{in} \quad H^1(\mathbb{R}^2)$$

and

$$u_n(x) \to u(x) \quad \text{a.e. in} \quad \mathbb{R}^2.$$

Moreover, by Lemma 3.11 the sequence $(f(x,u_n)u_n)$ is bounded in $L^1(\mathbb{R}^2)$. Therefore, by Lemma 3.4,

$$\Phi'(u)\phi = 0, \quad \forall \phi \in C_0^\infty(\mathbb{R}^2).$$

If we combine the Lemma 3.2 with the density of $C_0^\infty(\mathbb{R}^2)$ in $H^1(\mathbb{R}^2)$, we see that u is a critical point of Φ, that is,

$$\Phi'(u)v = 0, \quad \forall v \in H^1(\mathbb{R}^2).$$

Moreover, by Fatou's Lemma, we also have

$$\Phi(u) \leq \gamma.$$

If $u \neq 0$, we must have

$$\Phi(u) \geq \gamma,$$

showing that $\Phi(u) = \gamma$, and so, u is a ground state solution.

If $u = 0$, we can apply Lemma 3.16 to get a sequence $(y_n) \subset \mathbb{Z}^2$ and real numbers $r, \eta > 0$ verifying

$$\int_{B_r(y_n)} |u_n^+|^2 dx \geq \eta > 0, \quad \forall n \in \mathbb{N}.$$

Setting $v_n(x) = u_n(x + y_n)$, a direct computation gives that (v_n) is also a $(PS)_\gamma$ for Φ. Moreover, for some subsequence, there is $v \in H^1(\mathbb{R}^2)$ such that

$$v_n \rightharpoonup v \quad \text{in} \quad H^1(\mathbb{R}^2) \quad \text{and} \quad \int_{B_r(0)} |v^+|^2 dx \geq \eta > 0,$$

showing that $v \neq 0$. Therefore, arguing as above, v is a ground state solution for Φ. □

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3.5. The asymptotically periodic case

Proof. First of all, we recall that $\Phi_W \leq \Phi$, and so, $\gamma_W \leq \gamma$. As in Section 2, we will consider the cases $\gamma_W = \gamma$ and $\gamma_W < \gamma$. The first one follows as in Section 2, and we will omit its proof.

In what follows, we are considering $\gamma_W < \gamma$ and (u_n) be a $(PS)_{\gamma_W}$ sequence for Φ_W which was given in Lemma 3.14. The sequence (u_n) is bounded by Lemma 3.11. Thus, there is $u \in H^1(\mathbb{R}^2)$ and a subsequence of (u_n), still denoted by itself, such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$. Suppose by contradiction $u = 0$. Repeating the arguments explored in the case $N \geq 3$, we have

$$\int W(x)|u_n|^2 dx \to 0 \quad \text{and} \quad \sup_{\|\psi\| \leq 1} \left| \int W(x) u_n \psi \, dx \right| \to 0.$$

From (f_1), given $\epsilon > 0$ and $\beta > 0$ such that

$$\beta < \frac{2\pi}{\sup_{n \in \mathbb{N}} ||u_n||^2},$$

it must exist $\eta > 0$ satisfying

$$|f^*(x,s)| \leq \epsilon(e^{\beta s^2} - 1) \text{ for } \quad |t| \geq \eta \quad \text{and} \quad \forall x \in \mathbb{R}^2.$$

Therefore, by Lemma 3.2, for each $R > 0$ we have

$$\int\limits_{[|x|\geq R]\cap[|u_n|\geq\eta]} |f^*(x,u_n)||\psi| dx \leq \int\limits_{[|x|\geq R]\cap[|u_n|\geq\eta]} \epsilon|e^{\beta u_n^2} - 1||\psi| dx \leq$$

$$\leq \epsilon \left(\int\limits_{\mathbb{R}^2} |e^{\beta u_n^2} - 1|^2 dx \right)^{1/2} \left(\int\limits_{\mathbb{R}^2} |\psi|^2 dx \right)^{1/2} dx \leq \epsilon K ||\psi||_{H^1(\mathbb{R}^2)}.$$

On the other hand, fixing R large enough,

$$\begin{aligned} \int\limits_{[|x|\geq R]\cap[|u_n|\leq\eta]} |f^*(x,u_n)||\psi| dx &\leq C \int\limits_{|x|\geq R} H(x)|\psi| dx \\ &\leq \left(\int\limits_{|x|\geq R} |H(x)|^2 dx \right)^{1/2} \left(\int\limits_{\mathbb{R}^2} |\psi|^2 dx \right)^{1/2} \\ &\leq \epsilon C ||\psi||_{H^1(\mathbb{R}^2)}. \end{aligned}$$

Thus,

$$\sup_{\|\psi\| \leq 1} \left| \int\limits_{|x|\geq R} f^*(x,u_n)\psi \, dx \right| \leq \epsilon(C + K)||\psi||_{H^1(\mathbb{R}^2)}.$$

Now, as f^* has a subcritical growth and $u_n \to 0$ in $L^2(B_R(0))$, we also have

$$\sup_{\|\psi\|\leq 1}\left|\int_{|x|\leq R} f^*(x,u_n)\psi\,dx\right| \to 0.$$

Therefore,

$$\sup_{\|\psi\|\leq 1}\left|\int_{\mathbb{R}^2} f^*(x,u_n)\psi\,dx\right| \to 0.$$

A similar argument works to prove that

$$0 \leq \int F^*(x,u_n)dx \leq \int f^*(x,u_n)u_n dx \to 0.$$

The above limits yield

$$\Phi(u_n) \to \gamma_W \quad \text{and} \quad ||\Phi'(u_n)|| \to 0.$$

Arguing as in the periodic case, without loss of generality, we can assume that

$$u_n \rightharpoonup u \quad \text{in} \quad H^1(\mathbb{R}^2), u \neq 0 \text{ and } \Phi'(u) = 0.$$

Thus, $\Phi(u) \geq \gamma$. On the other hand, by Fatou's Lemma,

$$\Phi(u) \leq \liminf_{n\to+\infty} \Phi(u_n) = \gamma_W,$$

which is absurd, because we are supposing $\gamma_W < \gamma$. Thereby, $u \neq 0$ and since $(f(x,u_n)u_n)$ is bounded in $L^1(\mathbb{R}^2)$, we can conclude that u is a ground state solution of Φ_W. $\square$

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