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Extremal solutions of multi-valued variational inequalities in plane exterior domains

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Abstract

Let $\Omega = \mathbb{R}^2 \setminus \overline{B(0,1)}$ be the exterior of the closed unit disc in the plane. In this paper we prove existence and enclosure results of multi-valued variational inequalities in Ω of the form: Find $u \in K$ and $\eta \in F(u)$ such that

$$\langle -\Delta u, v-u\rangle \geq \langle a\eta, v-u\rangle, \quad \forall\, v \in K,$$

where K is a closed convex subset of the Hilbert space $X = D_0^{1,2}(\Omega)$ which is the completion of $C_c^\infty(\Omega)$ with respect to the $\|\nabla\cdot\|_{2,\Omega}$-norm. The lower order multi-valued operator F is generated by an upper semicontinuous multi-valued function $f : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$, and the (single-valued) coefficient $a : \Omega \to \mathbb{R}_+$ is supposed to decay like $|x|^{-2-\alpha}$ with $\alpha > 0$. Unlike in the situation of higher-dimensional exterior domain, that is $\mathbb{R}^N \setminus \overline{B(0,1)}$ with $N \geq 3$, the borderline case $N = 2$ considered here requires new tools for its treatment and results in a qualitatively different behaviour of its solutions. We establish a sub-supersolution principle for the above multi-valued variational inequality and prove the existence of extremal solutions. Moreover, we are going to show that classes of generalized variational-hemivariational inequalities turn out to be merely special cases of the above multi-valued variational inequality.

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Keywords: Variational inequality; Exterior plane domain; Extremal solution; Multivalued mapping; Variational-hemivariational inequalities

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1. Introduction

Let $\Omega = \mathbb{R}^2 \setminus \overline{B(0,1)} = \{x \in \mathbb{R}^2 : |x| > 1\}$ be the exterior of the closed unit disc $B = B(0,1)$ in the plane, and let $X = D_0^{1,2}(\Omega)$ be the Beppo-Levi space, which is the completion of $\mathcal{D} = C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_X^2 = \int_\Omega |\nabla u|^2 \, dx.$$

Our main goal is to prove existence and enclosure results of the following multi-valued variational inequality (MVI for short) in the unbounded domain Ω: Find $u \in K \subset X$ and $\eta \in F(u)$ such that

$$\langle -\Delta u, v - u\rangle \geq \langle a\eta, v - u\rangle, \quad \forall\, v \in K, \tag{1.1}$$

where K is a closed convex subset of the Hilbert space X, and $\langle \cdot, \cdot \rangle$ stands for the duality pairing between X and its dual space X^*. The lower order multi-valued operator F, which will be specified later, is generated by an upper semicontinuous multi-valued function $f : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$, and the (single-valued) coefficient $a : \Omega \to \mathbb{R}_+$ appearing in (1.1) is supposed to decay like $|x|^{-2-\alpha}$ with $\alpha > 0$. A precise description of the inequality (1.1) and its solutions will be discussed in the sequel.

Our present work is motivated by the recent paper [5] where extremal solutions of logistic-type equations in exterior plane domains were studied. However, due to the general multi-valued lower order terms, unlike in [5] the problem investigated here does not have a variational structure, and therefore variational approaches are not directly applicable. To the best of our knowledge, variational inequalities with multi-valued terms have not been studied in a systematic way on (unbounded) exterior domains.

Our primary goal here is not on the most general settings and conditions on the exterior domain Ω or on the principal and lower order terms in the variational inequality, but instead on a nonlinear functional analytic framework for inequalities and nonsmooth problems with multi-valued mappings on exterior domains. We should also mention that exterior domain problems in dimension $N = 2$ considered here represent a borderline case in comparison with exterior problems in higher dimension $N \geq 3$, that requires a different treatment and which results in a qualitatively different behaviour of its solution. The main reason for this is that the underlying solution space $X = D_0^{1,2}(\Omega)$ for $N = 2$ is qualitatively different from the corresponding space in dimension $N \geq 3$, which is readily seen by the following characterization of $X = D_0^{1,2}(\Omega)$ for $N = 2$ and $N \geq 3$. As for the borderline case $N = 2$, in [11, Theorems I.2.7, I.2.16] it is shown that X coincides with $\hat{D}_0^{1,2}(\Omega)$ which is given by

$$\hat{D}_0^{1,2}(\Omega) = \left\{ \begin{array}{l} u \in L^{1,2}(\Omega) : u \in L^2(\Omega \cap B_R), \ \forall\, R > 1, \\ \text{and } \eta\, u \in H_0^1(\Omega) \ \text{ for any } \eta \in C_c^\infty(\mathbb{R}^2) \end{array} \right\}, \tag{1.2}$$

where $B_R = B(0, R)$ is the open ball of radius R centered at the origin, and

$$L^{1,2}(\Omega) = \{u \in L^1_{\text{loc}}(\Omega) : \nabla u \in [L^2(\Omega)]^2\},$$

and $H_0^1(\Omega)$ denotes the usual Sobolev space of square integrable functions on Ω with zero traces on $\partial\Omega$. Note that $\eta u \in H_0^1(\Omega)$ for any $\eta \in C_c^\infty(\mathbb{R}^2)$ implies $u|_{|x|=1} = 0$ in the sense of traces. In case $N \geq 3$, due to the Sobolev embedding, we have $X = D_0^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$ with $2^* = \frac{2N}{N-2}$, and therefore, $X = D_0^{1,2}(\Omega)$ can be characterized as

$$D_0^{1,2}(\Omega) = \{u \in L^{1,2}(\Omega) : u \in L^{2^*}(\Omega)\}. \tag{1.3}$$

In view of (1.3), $u \in D_0^{1,2}(\Omega)$ is 2^*-integrable for $N \geq 3$, while due to (1.2), u need not be p-integrable for any $1 \leq p < \infty$ in case $N = 2$. To give an idea of the qualitatively different behaviour of solutions of (1.1) for $N = 2$ and $N \geq 3$, respectively, let us consider the following simple example, which is a special case of (1.1).

$$-\Delta u(x) = \frac{2}{|x|^4} \quad \text{in } \Omega = \mathbb{R}^N \setminus \overline{B(0,1)}, \qquad u(x) = 0 \ \text{ for } |x| = 1. \tag{1.4}$$

Note, $u \in D_0^{1,2}(\Omega)$ is a (weak) solution of (1.4) if

$$\int_\Omega \nabla u \nabla \varphi \, dx = \int_\Omega \frac{2}{|x|^4} \varphi \, dx, \quad \forall \varphi \in D_0^{1,2}(\Omega).$$

By elementary calculation the unique solution of (1.4) is

$$u(x) = \frac{1}{2}\Big(1 - \frac{1}{|x|^2}\Big) \ \text{ if } N = 2, \qquad u(x) = \frac{1}{|x|}\Big(1 - \frac{1}{|x|}\Big) \ \text{ if } N = 3,$$

which shows that

$$\lim_{|x|\to\infty} u(x) = \frac{1}{2} \ (N = 2), \qquad \lim_{|x|\to\infty} u(x) = 0 \ (N = 3),$$

and in the case $N = 2$, the solution u is not even integrable on Ω for any $1 \leq p < \infty$. Moreover, taking advantage of results from [4] one can show that for any $N \geq 3$, problem (1.4) has a unique solution in $D_0^{1,2}(\Omega)$ that decays like $\frac{1}{|x|^{N-2}}$ as $|x| \to \infty$, and is 2^*-integrable on Ω.

The sub-supersolution method to be established in this paper for the MVI (1.1) will enable us to treat general variational-hemivariational inequalities in (unbounded) exterior plane domains of the form: Find $u \in K$ such that

$$\langle -\Delta u, v - u\rangle + \int_\Omega a(-j)^o(u, u; v - u) \, dx \geq 0, \quad \forall v \in K, \tag{1.5}$$

where $j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(r, s) \mapsto j(r, s)$, is supposed to be locally Lipschitz continuous with respect to s, and where $j^o(r, s; \varrho)$ denotes Clarke's generalized directional derivatives at s in the direction ϱ for fixed r, defined by

$$j^o(r, s; \varrho) = \limsup_{y \to s, \ \varepsilon \downarrow 0} \frac{j(r, y + \varepsilon\varrho) - j(r, y)}{\varepsilon}. \tag{1.6}$$

We are going to show that under some regularity assumptions on the function $s \mapsto j^o(s, s; \varrho)$ to be specified later, the variational-hemivariational inequality (1.5) turns out to be only a special case of the MVI (1.1). In particular, for a locally Lipschitz function $s \mapsto j(s)$, that is, j is independent of r, (1.5) reduces to the following variational-hemivariational inequality: Find $u \in K$ such that

$$\langle -\Delta u, v-u\rangle + \int_\Omega a(-j)^o(u; v-u)\,dx \ge 0, \quad \forall\, v \in K, \tag{1.7}$$

which is easily seen to be equivalent to the MVI (1.1) with $f(s) = \partial j(s)$, where $\partial j : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ denotes Clarke's generalized gradient that is known to be upper semicontinuous, see [6, Chap. 2].

The paper is organized as follows: in Section 2 we introduce basic notations and function spaces. Our main existence and enclosure result is proved in Section 3 along with a characterization of the solution set. In Section 4, as an application of our main result, we deal with a multi-valued obstacle problem. Finally, in Section 5, we clarify the connection between general variational-hemivariational inequalities of the form (1.5) and the multi-valued variational inequality (1.1).

2. Assumptions, notations and preliminaries

Let $X = D_0^{1,2}(\Omega)$ with $\Omega = \mathbb{R}^2 \setminus \overline{B(0,1)}$, and denote by X^* the dual space of X and $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{X^*,X}$ the dual pairing between X^* and X. Note that although X is a Hilbert space with the inner product $\int_\Omega \nabla u \nabla v\,dx$ ($u, v \in X$), as will be seen later, it is more convenient for our treatment to identify another Hilbert space Y containing X with its dual Y^*, instead of identifying X with X^*. Throughout this paper we assume the following hypothesis on the nonnegative coefficient a:

(Ha) Let $a : \Omega \to \mathbb{R}$ be a nonnegative measurable function with positive measure support such that for some $c_0,\ \alpha > 0$:

$$0 \le a(x) \le \frac{c_0}{|x|^{2+\alpha}} \quad \text{for a.e. } x \in \Omega.$$

Concerning f, we denote by $L^0(\Omega)$ the set of all real valued measurable functions defined on Ω and use the notation

$$\mathcal{K}(Z) = \{A \subset Z : A \neq \emptyset,\ A \text{ is closed and convex}\},$$

where Z is a normed vector space. We impose the following hypothesis on the multi-valued function f.

(HF) Let $f : \mathbb{R} \to \mathcal{K}(\mathbb{R})$ be an upper semicontinuous function, that is, for each $u \in \mathbb{R}$ and each open $U \subset \mathbb{R}$ such that $f(u) \subset U$, there exists $\delta > 0$ such that if $|v-u| < \delta$ then $f(v) \subset U$.

Remark 2.1. We remark that for single-valued function $f : \mathbb{R} \to \mathbb{R}$, upper semicontinuous as defined in (HF) is identical with continuous. Further, from the definition of upper semicontinuous multi-valued functions of the form $f : \mathbb{R} \to \mathcal{K}(\mathbb{R})$ we readily observe that also $-f : \mathbb{R} \to \mathcal{K}(\mathbb{R})$ is a upper semicontinuous multi-valued function.

Note that since $f(u)$ is a closed and bounded interval in $\mathbb{R}$, (HF) is equivalent to the Hausdorff upper semicontinuity (h-u.s.c.) of f (cf. [7, Theorem 2.68, Chap. 1]). There are two properties of f that are consequences of assumption (HF), which are important for our later discussions.

Corollary 2.2. *Let f satisfy (HF) then f has the following property:*

(F1) *f is measurable and thus graph-measurable (in the sense of multi-valued functions).*

Proof. The measurability property (F1) readily follows from (HF). □

Consequently, f is superpositionally measurable, that is, if $u \in L^0(\Omega)$ then $f(u) = f(u(\cdot)) : \Omega \to \mathcal{K}(\mathbb{R})$ is measurable. Let $F(u)$ be the set of all measurable selections of $f(u)$, that is,

$$F(u) = \left\{ \eta \in L^0(\Omega) : \eta(x) \in f(u(x)) \text{ for a.e. } x \in \Omega \right\}. \tag{2.1}$$

Due to the measurability of $x \mapsto f(u(x))$ on Ω, we have $F(u) \neq \emptyset$ whenever $u \in L^0(\Omega)$. The second useful property of f that follows from condition (HF) is given by the following corollary.

Corollary 2.3. *Let f satisfy (HF) then f has the following property:*

(F2) *f is bounded on $\mathbb{R}$, that is, if $S \subset \mathbb{R}$ is bounded then $f(S) = \bigcup_{s\in S} f(s)$ is also a bounded subset of $\mathbb{R}$.*

Proof. In fact, since $f(s) \in \mathcal{K}(\mathbb{R})$, it follows from (HF) and [9, Proposition 4.2] that $f(s) = [\alpha(s), \beta(s)]$, $\forall s \in \mathbb{R}$, where $\alpha, \beta : \mathbb{R} \to \mathbb{R}$, $\alpha \le \beta$ on $\mathbb{R}$, and α is lower semicontinuous, β is upper semicontinuous on $\mathbb{R}$, in the usual semicontinuity sense of single-valued real functions. Let S be a bounded subset of $\mathbb{R}$. Since $I = \overline{S}$ is compact and α is lower semicontinuous on I, $\alpha_0 = \inf \alpha(I) = \min \alpha(I)$ is a real number. Similarly, from the upper semicontinuity of β on I, $\beta_0 = \sup \beta(I) = \max \beta(I) \in \mathbb{R}$. For all $s \in I$, we have $f(s) = [\alpha(s), \beta(s)] \subset [\alpha_0, \beta_0]$. Hence, $f(S) \subset f(I) \subset [\alpha_0, \beta_0]$. □

From (HF) it follows in view of (F1) and (F2) that $\emptyset \neq F(u) \subset L^\infty(\Omega)$ whenever $u \in L^\infty(\Omega)$. For a precise formulation of solutions of (1.1), we need some auxiliary definitions.

Definition 2.4.

(a) A function $g \in L^0(\Omega)$ is said to define a bounded linear functional on X if $gv \in L^1(\Omega)$ for all $v \in X$ and the functional $v \mapsto \int_\Omega gv\,dx$ belongs to X^*, that is, there exists a constant $c_g \in (0, \infty)$ such that $\left|\int_\Omega gv\,dx\right| \le c_g \|v\|_X$, $\forall v \in X$.

(b) We denote by X_0^* the set of all functions g that defines a bounded linear functional on X, $X_0^* = \{g \in L^0(\Omega) : gv \in L^1(\Omega), \forall v \in X \text{ and } \exists\, c_g > 0 : \left|\int_\Omega gv\,dx\right| \le c_g \|v\|_X,\ \forall v \in X\}$.

If $g \in X_0^*$, then $\hat{g}$ is the bounded linear functional generated by g given by

$$\hat{g}(v) = \langle \hat{g}, v \rangle_{X^*,X} = \int_\Omega g v \, dx, \ \forall v \in X.$$

In order to simplify notations we agree on the following identification.

Identification. In what follows, in order to simplify the notations, we are going to use for $\hat{g}$ again g, that is, the function $g \in X_0^*$ stands likewise also for the associated linear functional generated by g.

We are now ready for a precise formulation of (1.1). (Note for $u, v \in X : \langle -\Delta u, v \rangle = \int_\Omega \nabla u \nabla v \, dx$.)

Definition 2.5. A function $u \in K$ is a solution of (1.1) if there exist $\eta \in F(u)$, that is, $\eta \in L^0(\Omega)$,

$$\eta(x) \in f(u(x)), \quad \text{for a.e. } x \in \Omega, \tag{2.2}$$

such that $a\eta \in X_0^*$, and

$$\int_\Omega \nabla u (\nabla v - \nabla u) dx \geq \int_\Omega a\eta (v - u) \, dx, \ \forall v \in K. \tag{2.3}$$

Inequality (1.1) can also be stated as an inclusion in the dual space X^* as follows. Let I_K be the indicator functional of K, $I_K : X \to [0, \infty]$,

$$I_K(u) = \begin{cases} 0 & \text{if} \quad u \in K \\ \infty & \text{if} \quad u \notin K, \end{cases}$$

which is a proper, convex, and lower semicontinuous functional on X with effective domain $D(I_K) = K$. Let ∂I_K be the subdifferential of I_K (in the sense of Convex Analysis), and define the multi-valued functions f_a and F_a by

$$\begin{aligned} & f_a(x, s) = a(x) f(s), \quad \text{for a.e. } x \in \Omega \text{ and for all } s \in \mathbb{R}, \\ & F_a(u) = a F(u), \end{aligned} \tag{2.4}$$

then (1.1) can be reformulated as the following inclusion: Find $u \in K$ such that

$$-\Delta u - F_a(u) + \partial I_K(u) \ni 0. \tag{2.5}$$

Finally, let us introduce the concepts of sub- and supersolution of (2.2)-(2.3). To this end let us first introduce the following notation: For functions w, z and sets W and Z of functions defined on Ω we use the notations: $w \wedge z = \min\{w, z\}$, $w \vee z = \max\{w, z\}$, $W \wedge Z = \{w \wedge z : w \in W, \ z \in Z\}$, $W \vee Z = \{w \vee z : w \in W, \ z \in Z\}$, and $w \wedge Z = \{w\} \wedge Z$, $w \vee Z = \{w\} \vee Z$. In particular, we denote $w^+ = w \vee 0$, and $w^- = (-w) \vee 0$. For functions $\underline{u} \leq \overline{u}$, we denote by

$$[\underline{u}, \overline{u}] = \{u : \underline{u}(x) \le u(x) \le \overline{u}(x) \text{ for a.e. } x \in \Omega\}$$

the ordered interval formed by $\underline{u}$ and $\overline{u}$.

Definition 2.6. A function $\underline{u} \in X$ is called a subsolution of (1.1), or more precisely of (2.2)-(2.3), if

$$\underline{u} \vee K \subset K,$$

and if there exists $\underline{\eta} \in F(\underline{u})$, i.e.,

$$\underline{\eta}(x) \in f(\underline{u}(x)) \text{ for a.e. } x \in \Omega, \tag{2.6}$$

such that $a\underline{\eta} \in X_0^*$ and

$$\int_\Omega \nabla\underline{u}(\nabla v - \nabla\underline{u})\, dx \ge \int_\Omega a\underline{\eta}(v - \underline{u})\, dx, \quad \forall\, v \in \underline{u} \wedge K. \tag{2.7}$$

Similarly, $\overline{u} \in X$ is called a supersolution of (2.2)-(2.3) if

$$\overline{u} \wedge K \subset K,$$

and if there exists $\overline{\eta} \in F(\overline{u})$, i.e.,

$$\overline{\eta}(x) \in f(\overline{u}(x)) \text{ for a.e. } x \in \Omega, \tag{2.8}$$

such that $a\overline{\eta} \in X_0^*$ and

$$\int_\Omega \nabla\overline{u}(\nabla v - \nabla\overline{u})\, dx \ge \int_\Omega a\overline{\eta}(v - \overline{u})\, dx, \quad \forall\, v \in \overline{u} \vee K. \tag{2.9}$$

Finally let us introduce the weighted Lebesgue space $Y = L^2(\Omega; w)$, which will be used in the proof of our main result (see Section 3). Let $w_0(r) = \frac{1}{r^{2+\alpha}} (r \in (1, \infty))$, we shall use the weight $w(x) = w_0(|x|) = \frac{1}{|x|^{2+\alpha}}$ $(x \in \Omega)$, and the corresponding weighted Lebesgue space $Y = L^2(\Omega; w)$, which is a Hilbert space with the usual norm

$$\|u\|_Y^2 = \|u\|_{L^2(\Omega; w)}^2 = \int_\Omega |u(x)|^2 w(x) dx$$

and inner product

$$\langle u, v\rangle_Y = \langle u, v\rangle_{L^2(\Omega; w)} = \int_\Omega u(x) v(x) w(x)\, dx, \quad u, v \in Y.$$

Since $w_0 \in L^1((1,\infty); r\ln r)$, according to [1, Lemma 2.2], the embedding $i_w : X \hookrightarrow Y, u \mapsto i_w(u) = u$, is compact. Moreover, we identify Y with its dual Y^*, which yields the embeddings: $X \hookrightarrow Y = Y^* \hookrightarrow X^*$. The adjoint $i_w^* : Y \hookrightarrow X^*$ is also compact and for all $u \in Y, v \in X$, the following relations are valid:

$$\langle i_w^*(u), v\rangle_{X^*,X} = \langle u, i_w(v)\rangle_Y = \langle u, v\rangle_Y = \int_\Omega uvw\,dx.$$

We have the following simple connection between Y, and X_0^* defined in Definition 2.4, which will be useful in our proofs later. (Note: we identify $g \in X_0^*$ with the corresponding linear functional.)

Lemma 2.7. *If $u \in Y$ then $uw \in X_0^*$ and $uw = i_w^*(u)$.*

Proof. Let $v \in X$, since $i_w(v) = v \in Y$, we have $uv \in L^1(\Omega; w)$, i.e., $(uw)v \in L^1(\Omega)$. Furthermore, $\left|\int_\Omega (uw)v dx\right| = |\langle u, v\rangle_Y| \le \|u\|_Y\|v\|_Y \le C\|u\|_Y\|v\|_X$. Since this is true for all $v \in X$, we see that $uw \in X_0^*$. Moreover, $\langle uw, v\rangle_{X^*,X} = \int_\Omega (uw)v\,dx = \langle u, v\rangle_Y = \langle i_w^*(u), v\rangle_{X^*,X}$, $\forall v \in X$, which means that $uw = i_w^*(u)$. $\square$

3. Main result: existence and enclosure of solutions

In this section we are going to prove our main existence and enclosure result, and qualitatively characterize the solution set. We have the following general existence and enclosure/comparison theorem for (1.1) when bounded and ordered sub- and supersolutions exist.

Theorem 3.1. *Let a and f satisfy (Ha) and (HF), respectively. Suppose there are subsolutions $\underline{u}_i \in X \cap L^\infty(\Omega)$, $i = 1, \dots, k$, and supersolutions $\overline{u}_j \in X \cap L^\infty(\Omega)$, $j = 1, \dots, m$, of (2.2)-(2.3) such that*

$$\underline{u} := \max\{\underline{u}_i : 1 \le i \le k\} \le \overline{u} := \min\{\overline{u}_j : 1 \le j \le m\}. \tag{3.1}$$

Then, there exists a solution u of (2.2)-(2.3) such that

$$\underline{u} \le u \le \overline{u} \text{ a.e. on } \Omega. \tag{3.2}$$

Proof. The proof of this theorem is divided into 5 steps.

Step 1. *Auxiliary MVI*

We shall define in this step some functions to truncate and regularize the original MVI and define next an auxiliary MVI.

Note that since $\underline{u}_i$ and $\overline{u}_j$ are bounded functions on Ω, according to property (F2), the sets $F(\underline{u}_i)$ and $F(\overline{u}_j)$ are bounded subsets of $L^\infty(\Omega)$ (with respect to the usual $L^\infty(\Omega)$-norm). For each $i \in \{1, \dots, k\}$ (resp. $j \in \{1, \dots, m\}$), $\underline{\eta}_i$ (resp. $\overline{\eta}_j$) is a function in $L^\infty(\Omega)$ satisfying (2.6) and (2.7) with $\underline{\eta}_i$ instead of $\underline{\eta}$ (resp. (2.8) and (2.9) with $\overline{\eta}_j$ instead of $\overline{\eta}$). We construct families $\{\Omega_i : 1 \le i \le k\}$ and $\{\Omega^j : 1 \le j \le m\}$ of subsets of Ω inductively as follows. Let $\Omega_1 = \{x \in \Omega : \underline{u}(x) = \underline{u}_1(x)\}$, and

$$\Omega_i = \left\{ x \in \Omega \setminus \bigcup_{l=1}^{i-1} \Omega_l : \underline{u}(x) = \underline{u}_i(x) \right\} \quad \text{for } i = 2, \ldots, k.$$

Similarly, let $\Omega^1 = \{x \in \Omega : \overline{u}(x) = \overline{u}_1(x)\}$, and

$$\Omega^j = \left\{ x \in \Omega \setminus \bigcup_{l=1}^{j-1} \Omega^l : \overline{u}(x) = \overline{u}^j(x) \right\} \quad \text{for } j = 2, \ldots, m.$$

It is clear that $\Omega_i (1 \le i \le k)$ (resp. $\Omega^j (1 \le j \le m)$) are disjoint measurable subsets of Ω and $\Omega = \bigcup_{i=1}^k \Omega_i = \bigcup_{j=1}^m \Omega^j$. Let us define

$$\underline{\eta} = \sum_{i=1}^{k} \underline{\eta}_i \chi_{\Omega_i} \text{ and } \overline{\eta} = \sum_{j=1}^{m} \overline{\eta}_j \chi_{\Omega^j},$$

where χ_A $(A \subset \Omega)$ is the characteristic function of A. From their definitions, we see that $\underline{\eta}, \overline{\eta} \in L^\infty(\Omega)$. Moreover, since $\underline{\eta}(x) = \underline{\eta}_i(x)$ and $\underline{u}(x) = \underline{u}_i(x)$ for a.e. $x \in \Omega_i$ $(1 \le i \le k)$, we have

$$\underline{\eta}(x) \in f(\underline{u}(x)) \text{ for a.e. } x \in \Omega. \tag{3.3}$$

Similarly, $\overline{\eta}(x) \in f(\overline{u}(x))$ for a.e. $x \in \Omega$. Next, let us define a truncated function for $f(u)$. Let $f_0 : \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ be given by

$$f_0(x,u) = \begin{cases} \{\underline{\eta}(x)\} & \text{if} \quad u < \underline{u}(x) \\ f(u) & \text{if} \quad \underline{u}(x) \le u \le \overline{u}(x) \\ \{\overline{\eta}(x)\} & \text{if} \quad u > \overline{u}(x). \end{cases} \tag{3.4}$$

We see directly from this definition that $f_0(x,s) \in \mathcal{K}(\mathbb{R})$ for a.e. $x \in \Omega$, all $s \in \mathbb{R}$. Also, (3.3) and property (F1) imply that for a.e. $x \in \Omega$, the function $f_0(x,\cdot)$ is also upper semicontinuous from $\mathbb{R}$ to $\mathcal{K}(\mathbb{R})$. Moreover, we see from the graph-measurability of f that f_0 is also graph-measurable. In particular, f_0 is superpositionally measurable. Therefore, for any $u \in L^0(\Omega)$, the set $F_0(u)$ of all measurable selections of $f_0(\cdot, u(\cdot))$,

$$F_0(u) = \{\eta \in L^0(\Omega) : \eta(x) \in f_0(x, u(x)) \text{ for a.e. } x \in \Omega\},$$

is nonempty. Furthermore, since $\underline{u}, \overline{u}, \underline{\eta}, \overline{\eta} \in L^\infty(\Omega)$, we see from property (F2) that for any $u \in L^0(\Omega)$, the set $F_0(u)$ is a subset of $L^\infty(\Omega)$. Moreover, the range $F_0(L^0(\Omega))$ of the mapping F_0 is a bounded subset of $L^\infty(\Omega)$, that is

$$c_1 := \sup\{\|\eta\|_{L^\infty(\Omega)} : \eta \in F_0(u), u \in L^0(\Omega)\} < \infty. \tag{3.5}$$

Next, we need the following single-valued regularizing function $b : \Omega \times \mathbb{R} \to \mathbb{R}$,

$$b(x,u) = \begin{cases} [u - \overline{u}(x)]w(x) & \text{if} \quad u > \overline{u}(x) \\ 0 & \text{if} \quad \underline{u}(x) \le u \le \overline{u}(x) \\ [u - \underline{u}(x)]w(x) & \text{if} \quad u < \underline{u}(x), \quad \text{for } x \in \Omega, u \in \mathbb{R}, \end{cases} \tag{3.6}$$

and denote by B the associated Nemytskii operator, that is, $B(u)(x) = b(x, u(x))$. It is clear that b is a Carathéodory function and since $\underline{u}, \overline{u} \in L^\infty(\Omega)$, there exists $b_1 > 0$ such that

$$|b(x,u)| \leq b_1(|u|+1)w(x), \ (x \in \Omega, u \in \mathbb{R}). \tag{3.7}$$

Moreover, for some constants $b_2, b_3 > 0$,

$$\int_\Omega b(x,u(x))u(x)dx \geq b_2\|u\|_Y^2 - b_3, \ \forall u \in Y. \tag{3.8}$$

Next, let us define certain truncation functions needed for the regularization of the involved multivalued mappings. For $i \in \{1, \ldots, k\}$, let $T_i(x, u)$ be a Carathéodory function such that for $x \in \Omega$, $u \in \mathbb{R}$,

$$T_i(x,u) = \begin{cases} |\underline{\eta}_i(x) - \underline{\eta}(x)| & \text{if} \quad u \leq \underline{u}_i(x) \\ 0 & \text{if} \quad u \geq \underline{u}(x), \end{cases} \tag{3.9}$$

and

$$0 \leq T_i(x,u) \leq |\underline{\eta}_i(x) - \underline{\eta}(x)|, \text{ for a.e. } x \in \Omega, \text{ all } u \in \mathbb{R}. \tag{3.10}$$

A simple choice of such function is

$$T_i(x,u) = |\underline{\eta}_i(x) - \underline{\eta}(x)|\sigma\left(\frac{u - \underline{u}_i(x)}{\underline{u}(x) - \underline{u}_i(x)}\right), \tag{3.11}$$

for $x \in \Omega$, $u \in \mathbb{R}$, where $\sigma \in C(\mathbb{R}, \mathbb{R})$, $0 \leq \sigma(s) \leq 1$, $\forall s \in \mathbb{R}$, $\sigma(s) = 1$ if $s \leq 0$, and $\sigma(s) = 0$ if $s \geq 1$, such as for example

$$\sigma(s) = \begin{cases} 1, & s \leq 0 \\ 1-s, & 0 \leq s \leq 1 \\ 0, & s \geq 1. \end{cases} \tag{3.12}$$

It is clear that T_i given by (3.11)-(3.12) is a Carathéodory function satisfying (3.9) and (3.10). Similarly, for $j = 1, \ldots, m$, we define $T^j : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$T^j(x,u) = |\overline{\eta}_j(x) - \overline{\eta}(x)|\left[1 - \sigma\left(\frac{u - \overline{u}(x)}{\overline{u}_j(x) - \overline{u}(x)}\right)\right]. \tag{3.13}$$

T^j is a Carathéodory function with

$$T^j(x,u) = \begin{cases} |\overline{\eta}_j(x) - \overline{\eta}(x)| & \text{if} \quad u \geq \overline{u}_j(x) \\ 0 & \text{if} \quad u \leq \overline{u}(x), \end{cases} \tag{3.14}$$

and similarly to (3.10),

$$0 \leq T^j(x,u) \leq |\overline{\eta}_j(x) - \overline{\eta}(x)|, \text{ for a.e. } x \in \Omega, \text{ all } u \in \mathbb{R}. \tag{3.15}$$

Consequently,

$$T_i(\cdot,u),\, T^j(\cdot,u) \in L^\infty(\Omega),\ \forall\, u \in L^0(\Omega), \tag{3.16}$$

and the mappings $u \mapsto T_i(\cdot,u)$ and $u \mapsto T^j(\cdot,u)$ are bounded mappings from $L^0(\Omega)$ to $L^\infty(\Omega)$. Moreover, there exists $c_2 > 0$ such that

$$0 \le T_i(x,u),\, T^j(x,u) \le c_2, \tag{3.17}$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}$, $i \in \{1,\dots,k\}$, $j \in \{1,\dots,m\}$.

Let us consider the following auxiliary variational inequality related to (2.3): Find $u \in K$, $\eta \in L^0(\Omega)$ such that $\eta \in F_0(u)$, i.e.,

$$\eta(x) \in f_0(x,u(x)) \text{ a.e. } x \in \Omega \tag{3.18}$$

and

$$\begin{aligned}&\int_\Omega \nabla u(\nabla v - \nabla u)dx - \int_\Omega a(x)\eta(x)(v-u)dx + \int_\Omega b(x,u)(v-u)dx\\ &- \textstyle\sum_{i=1}^k \int_\Omega a(x)T_i(x,u)(v-u)dx + \sum_{j=1}^m \int_\Omega a(x)T^j(x,u)(v-u)dx \ge 0,\\ &\forall v \in K.\end{aligned} \tag{3.19}$$

Note that for any $u \in X$, any $\eta \in F_0(u)$, the functions $\frac{a}{w}\eta$, $\frac{1}{w}B(u)$, $\frac{a}{w}T_i(\cdot,u)$, and $\frac{a}{w}T^j(\cdot,u)$ all belong to $L^\infty(\Omega)(\subset Y)$. Thus by Lemma 2.7, $a\eta$, $B(u)$, $aT_i(\cdot,u)$, $aT^j(\cdot,u)$ belong to X_0^*. Since for any $\eta \in F_0(u)$ we have $a\eta \in X_0^*$, which generates the linear functional $v \mapsto \int_\Omega a\eta v\,dx$ that is again denoted by $a\eta \in X^*$, we see that $aF_0(u) \subset X^*$ for any $u \in X$. Therefore, the multi-valued mapping $F_{0,a} : X \to 2^{X^*} \setminus \{\emptyset\}$ given by

$$F_{0,a}(u) = aF_0(u) \tag{3.20}$$

is well defined. Using the multi-valued operator $F_{0,a}$ the auxiliary MVI (3.18), (3.19) is equivalent to the following: Find $u \in K$ and $\eta \in F_0(u)$ such that

$$\langle -\Delta u - a\,\eta + B(u) - T(u), v-u\rangle \ge 0, \quad \forall\, v \in K, \tag{3.21}$$

where

$$T(u) = \sum_{i=1}^k aT_i(\cdot,u) - \sum_{j=1}^m aT^j(\cdot,u).$$

The auxiliary MVI (3.19) can be rewritten as the following inclusion: Find $u \in X$ such that

$$-\Delta u - F_{0,a}(u) + B(u) - T(u) + \partial I_K(u) \ni 0. \tag{3.22}$$

Step 2. $F_{0,a} : X \to \mathcal{K}(X^*)$ *is pseudomonotone*

We show in this step that the multi-valued mapping $F_{0,a}$ is a pseudomonotone mapping from X to $\mathcal{K}(X^*)$. We shall use here and also in the next steps the definitions and properties of pseudomonotone and generalized pseudomonotone mappings in the original works [2], [3], and standard references such as [10] or [12].

Let us consider the multi-valued function $f_1(x,s) = \frac{a(x)}{w(x)} f_0(x,s)$ for $x \in \Omega$, $s \in \mathbb{R}$. It follows from the corresponding properties of f_0 that $f_1(x,s) \in \mathcal{K}(\mathbb{R})$ for a.e. $x \in \Omega$, all $s \in \mathbb{R}$, and for a.e. $x \in \Omega$, the function $f_1(x,\cdot)$ is upper semicontinuous from $\mathbb{R}$ to $\mathcal{K}(\mathbb{R})$. Moreover, since f_0 is graph-measurable, so is f_1. For $u \in L^0(\Omega)$, let $F_1(u)$ denote the set of all measurable selections of $f_1(\cdot, u)$,

$$\begin{aligned} F_1(u) &= \{\zeta \in L^0(\Omega) : \zeta(x) \in f_1(x,u(x)) \text{ for a.e. } x \in \Omega\} \\ &= \{\tfrac{a}{w}\eta \in L^0(\Omega) : \eta(x) \in f_0(x,u(x)) \text{ for a.e. } x \in \Omega\} \\ &= \{\tfrac{a}{w}\eta : \eta \in F_0(u)\} \\ &= \tfrac{a}{w} F_0(u). \end{aligned}$$

Moreover, according to (3.5) and condition (Ha), for any $u \in Y$ and $\eta \in F_0(u)$,

$$\left| \frac{a(x)}{w(x)} \eta(x) \right| \le c_0 c_1 \text{ for a.e. } x \in \Omega. \tag{3.23}$$

Since the constant function c_0c_1 belongs to Y, we see that $\frac{a}{w}\eta$ also belongs to Y. Hence, for any $u \in Y$, $F_1(u)$ is a (nonempty) bounded subset of Y. Since $f_0(x,\cdot)$ and thus $f_1(x,\cdot)$ are intervals, we see immediately that $F_1(u)$ is convex. Let us prove that $F_1(u)$ is closed in Y. In fact, let $\{\eta_n\}$ be a sequence in $F_0(u)$ such that $\frac{a}{w}\eta_n \to p$ in Y for some $p \in Y$. By passing to a subsequence if necessary, we have

$$\frac{a(x)}{w(x)} \eta_n(x) \to p(x) \text{ for a.e. } x \in \Omega. \tag{3.24}$$

Let $\Omega_0 = \{x \in \Omega : a(x) = 0\}$ and let s_0 be any element of $F_0(u)$, i.e. $s_0(x) \in f_0(x,u(x))$ for a.e. $x \in \Omega$, and thus, for a.e. $x \in \Omega_0$. Define the following function:

$$\eta_0 = \begin{cases} s_0(x), & x \in \Omega_0 \\ \frac{w(x)}{a(x)} p(x), & x \in \Omega \setminus \Omega_0. \end{cases}$$

For a.e. $x \in \Omega_0$, it follows from (3.24) that $p(x) = 0 = \frac{a(x)}{w(x)}\eta_0(x)$. Thus, $p(x) = \frac{a(x)}{w(x)}\eta_0(x)$ for a.e. $x \in \Omega$. Let us verify that $\eta_0 \in F_0(u)$. In fact, for $x \in \Omega_0$, $\eta_0(x) = s_0(x) \in f_0(x,u(x))$ by the choice of s_0. For a.e. $x \in \Omega \setminus \Omega_0$, since $a(x) \neq 0$, (3.24) implies that $\eta_n(x) \to \frac{w(x)}{a(x)} p(x) = \eta_0(x)$ as $n \to \infty$. Since $\eta_n(x) \in f_0(x,u(x))$ for all n, and $f_0(x,u(x))$ is a closed interval, we obtain that $\eta_0(x) \in f_0(x,u(x))$. This shows that $\eta_0 \in F_0(u)$ and, consequently, that $p \in F_1(u)$. Thus, $F_1(u)$ is a closed subset of Y. We have shown that $F_1(u) \in \mathcal{K}(Y)$ for every $u \in Y$ and thus $F_1 : u \mapsto F_1(u)$ defines a mapping from Y to $\mathcal{K}(Y)$.

Also, since $F_1(u)$ is convex, we have that $F_1(u)$ is weakly closed, which, together with its boundedness, proves that $F_1(u)$ is weakly compact in Y. From the continuity of i_w^* from Y to X^*, both spaces equipped with weak topologies, we see from Lemma 2.7 that for any $u \in X$, $F_{0,a}(u) = aF_0(u) = i_w^* F_1(u)$ is a weakly compact subset of X^*. Together with the convexity of $F_{0,a}(u)$, which is a direct consequence of the convexity of $f_0(x,\cdot)$, we see that $F_{0,a}(u) \in \mathcal{K}(X^*)$ for all $u \in X$. In particular, $F_{0,a}$ has effective domain $D(F_{0,a}) = X$. Moreover, (3.5) shows that $F_{0,a}$ is a bounded mapping.

The properties of f_1 established above also show that all assumptions of [7, Theorem 7.26, Chap. 2] are satisfied by the function f_1 on Ω with the weighted measure $d\mu = w dx$ and $Y = L^2(\Omega; w)$. According to this theorem, the mapping F_1 is Hausdorff upper semicontinuous (h-upper semicontinuous, cf. [7, Definition 2.60, Chap. 1]) on Y.

Next, let $\{u_n\} \subset X$ and $\{u_n^*\} \subset X^*$ be sequences such that

$$u_n \rightharpoonup u_0 \text{ in } X, \tag{3.25}$$

$$u_n^* \rightharpoonup u_0^* \text{ in } X^*, \tag{3.26}$$

and

$$u_n^* \in F_{0,a}(u_n) = aF_0(u_n), \quad \forall\, n \in \mathbb{N}. \tag{3.27}$$

First, let us prove that

$$u_0^* \in F_{0,a}(u_0). \tag{3.28}$$

In fact, for each $n \in \mathbb{N}$, there is $\eta_n \in F_0(u_n)$ such that $u_n^* = a\eta_n$. From (3.25) we have $u_n = i_w(u_n) \to i_w(u_0) = u_0$ in Y. Because of the h-uppersemicontinuity of F_1 on Y noted above, this implies that

$$\lim_{n\to\infty} h_Y^*(F_1(u_n), F_1(u_0)) = 0, \tag{3.29}$$

where h_Y^* is the Hausdorff semi-distance between subsets of Y defined by

$$h_Y^*(A,B) := \sup_{u\in A}(\mathrm{dist}_Y(u,B)) = \sup_{u\in A}\left(\inf_{v\in B}\|u-v\|_Y\right) \text{ for } A, B \subset Y.$$

Since $\eta_n \in F_0(u_n)$, we have $\frac{a}{w}\eta_n \in F_1(u_n)$, and it follows from (3.29) that

$$\lim_{n\to\infty} \mathrm{dist}_Y\left(\frac{a}{w}\eta_n, F_1(u_0)\right) = 0.$$

Consequently, there exists a sequence $\{\zeta_n\}$ in $F_1(u_0)$ such that

$$\lim_{n\to\infty}\left\|\frac{a}{w}\eta_n - \zeta_n\right\|_Y = 0. \tag{3.30}$$

On the other hand, since $F_1(u_0)$ is closed, bounded, and convex in Y, it is bounded and weakly closed, and thus weakly compact in Y. Thus, by passing to a subsequence if necessary, we can assume that $\zeta_n \rightharpoonup \zeta_0$ in Y for some $\zeta_0 = \frac{a}{w}\eta_0 \in F_1(u_0)$, with $\eta_0 \in F_0(u_0)$. This limit and (3.30) imply that

$$\frac{a}{w}\eta_n \rightharpoonup \zeta_0 \left(= \frac{a}{w}\eta_0\right) \text{ in } Y. \tag{3.31}$$

Since i_w^* is continuous from Y to X^*, both equipped with weak topologies, Lemma 2.7 gives

$$u_n^* = a\eta_n = i_w^*\left(\frac{a}{w}\eta_n\right) \rightharpoonup i_w^*\left(\frac{a}{w}\eta_0\right) = a\eta_0 \in X^*$$

in X^*. Together with (3.26), this yields $u_0^* = a\eta_0$. Since $\eta_0 \in F_0(u_0)$, we have $u_0^* \in aF_0(u_0) = F_{0,a}(u_0)$, proving (3.28).

Next, let us check that

$$\langle u_n^*, u_n\rangle_{X^*,X} \to \langle u_0^*, u_0\rangle_{X^*,X}. \tag{3.32}$$

In fact, we have

$$\langle u_n^*, u_n\rangle_{X^*,X} - \langle u_0^*, u_0\rangle_{X^*,X} = \langle u_n^*, u_n - u_0\rangle_{X^*,X} + \langle u_n^* - u_0^*, u_0\rangle_{X^*,X}.$$

From (3.26), it follows

$$\langle u_n^* - u_0^*, u_0\rangle_{X^*,X} \to 0. \tag{3.33}$$

On the other hand, from condition (Ha), (3.5), and Lemma 2.7,

$$\begin{aligned}
|\langle u_n^*, u_n - u_0\rangle_{X^*,X}| &= \langle a\eta_n, u_n - u_0\rangle_{X^*,X} \\
&= |\langle i_w^*(\tfrac{a}{w}\eta_n), u_n - u_0\rangle_{X^*,X}| \\
&= |\langle (\tfrac{a}{w}\eta_n), i_w(u_n - u_0)\rangle_Y| \\
&\le \|\tfrac{a}{w}\eta_n\|_Y \|u_n - u_0\|_Y \\
&\le c_0 c_1 \|u_n - u_0\|_Y.
\end{aligned}$$

Since $\|u_n - u_0\|_Y \to 0$ in Y, as a consequence of (3.25), we see that $\langle u_n^*, u_n - u_0\rangle_{X^*,X} \to 0$, which together with (3.33), implies (3.32). The conclusions in (3.28) and (3.32) from the assumptions (3.25)–(3.27) show that $F_{0,a}$ is generalized pseudomonotone. Since $F_{0,a}(u) \in \mathcal{K}(X^*)$, $\forall u \in X$, and $F_{0,a}$ is bounded, we see that $F_{0,a}$ is also pseudomonotone.

Step 3. $-\Delta : X \to X^*$, *$B : X \to X^*$ and $T : X \to X^*$ are pseudomonotone*

We show in this step the pseudomonotonicity and some other needed properties of the other single-valued mappings in (3.21).

First, the mapping $-\Delta$ defined by $\langle -\Delta u, v\rangle_{X^*,X} = \int_\Omega \nabla u \nabla v\, dx$ $(u, v \in X)$, is a linear bounded and monotone mapping from X to X^* with domain $D(-\Delta) = X$. Hence, $-\Delta$ is pseudomonotone.

Next, let us consider the mapping $u \mapsto B(u)$. It follows from the estimate (3.7) that $\frac{1}{w}B(u) \in Y$ whenever $u \in Y$. Moreover standard convergence arguments in Lebesgue spaces show that if $u_n \to u$ in Y then $\frac{1}{w}B(u_n) \to \frac{1}{w}B(u)$ in Y, in other words, the Nemytskii operator $u \mapsto \frac{1}{w}B(u)$ is bounded and continuous from Y into itself. Let $u \in X$. Since $u = i_w(u) \in Y$, it follows from Lemma 2.7 that $B(u) \in X_0^*$, which according to the identification with the corresponding element of X^*, is given by $B(u) = i_w^*(\frac{1}{w}B(u))$. Moreover, if $u_n \rightharpoonup u$ in X then $u_n = i_w(u_n) \to i_w(u) = u$ in Y. Hence, as noted above, $\frac{1}{w}B(u_n) \to \frac{1}{w}B(u)$ in Y, which implies that

$$X^* \ni B(u_n) = i_w^*\left(\frac{1}{w}B(u_n)\right) \to i_w^*\left(\frac{1}{w}B(u)\right) = B(u) \text{ in } X^*.$$

This limit shows that the mapping $u \mapsto B(u)$ is completely continuous and thus (single-valued) pseudomonotone on X.

By analogous proofs, we see that the mappings $u \mapsto \frac{a}{w} T_i(\cdot, u)$, $i = 1, \dots, k$, and $u \mapsto \frac{a}{w} T^j(\cdot, u)$, $j = 1, \dots, m$, are also continuous from Y into itself. Hence, from the compactness of the embedding i_w we see that the mappings

$$u \mapsto aT_i(\cdot, u) = i_w^* \circ \left(\frac{a}{w} T_i\right) \circ i_w, \quad u \mapsto aT^j(\cdot, u) = i_w^* \circ \left(\frac{a}{w} T^j\right) \circ i_w$$

are completely continuous and are thus (single-valued) pseudomonotone mappings from X to X^*, which implies that $u \mapsto T(u)$ is a pseudomonotone mapping from X to X^*.

The results above, together with that in Step 2, imply that the mapping

$$u \mapsto -\Delta u - F_{0,a}(u) + B(u) - T(u)$$

is pseudomonotone from X to $\mathcal{K}(X^*)$.

Step 4. *Coercivity of* $-\Delta - F_{0,a} + B - T : X \to \mathcal{K}(X^*)$

Next, we prove in this step that the multi-valued operator $-\Delta - F_{0,a} + B - T : X \to \mathcal{K}(X^*)$ of (3.21) satisfies the following coercivity condition for some $u_0 \in K$

$$\lim_{\|u\|_X \to \infty} \left\{ \inf_{\eta \in F_0(u)} \left\langle -\Delta u - a\eta + B(u) - T(u), u - u_0 \right\rangle \right\} = \infty, \tag{3.34}$$

which, together with the pseudomonotone property established in Step 3, implies the existence of solutions of (3.21), and equivalently, of (3.19).

Letting u_0 be any (fixed) element of K, we have

$$\left| \int_\Omega \nabla u \nabla u_0 \, dx \right| \le \|u_0\|_X \|u\|_X \tag{3.35}$$

In the next estimates, c stands for a generic positive constant that does not depend on u and $\eta \in F_0(u)$, and may change from line to line. For any $\eta \in F_0(u)$, we have from (3.5) and property (F2) that $\frac{a}{w}\eta \in Y$. Hence, from Lemma 2.7,

$$\begin{aligned} |\langle a\eta, u - u_0\rangle| &= \left| \left\langle i_w^* \left(\tfrac{a}{w}\eta\right), u - u_0 \right\rangle_{X^*, X} \right| \\ &= \left| \left\langle \tfrac{a}{w}\eta, i_w(u - u_0) \right\rangle_Y \right| \\ &\le \left\| \tfrac{a}{w}\eta \right\|_Y (\|u\|_Y + \|u_0\|_Y) \\ &\le c(\|u\|_X + 1). \end{aligned} \tag{3.36}$$

Similarly, from (Ha) and (3.17), we see for $i \in \{1, \dots, k\}$ that

$$\begin{aligned} |\langle aT_i(\cdot, u), u - u_0\rangle| &= \left| \left\langle i_w^* \left(\tfrac{a}{w} T_i(\cdot, u)\right), u - u_0 \right\rangle_{X^*, X} \right| \\ &= \left| \left\langle \tfrac{a}{w} T_i(\cdot, u), i_w(u - u_0) \right\rangle_Y \right| \\ &\le \left\| \tfrac{a}{w} T_i(\cdot, u) \right\|_Y (\|u\|_Y + \|u_0\|_Y) \\ &\le c(\|u\|_X + 1). \end{aligned}$$

Hence,

$$\sum_{i=1}^{k} |\langle aT_i(\cdot,u), u-u_0\rangle| \le c(\|u\|_X+1) \tag{3.37}$$

and analogously,

$$\sum_{j=1}^{m} |\langle aT^j(\cdot,u), u-u_0\rangle| \le c(\|u\|_X+1). \tag{3.38}$$

Lastly, for any $u \in X(\subset Y)$, it follows from (3.7) that

$$\begin{aligned} \left|\int_\Omega b(x,u)u_0\,dx\right| &\le b_1\int_\Omega(|u|+1)|u_0|w\,dx \\ &\le c(\|u_0\|_Y\|u\|_Y+1) \\ &\le c(\|u\|_X+1). \end{aligned}$$

Together with (3.8), this yields

$$\int_\Omega b(x,u)(u-u_0)dx \ge b_2\|u\|_Y^2 - c(\|u\|_X+1). \tag{3.39}$$

Combining the estimates from (3.35) to (3.39) shows that for any $u \in X$, $\eta \in F_0(u)$ we get

$$\left\langle -\Delta u - a\eta + B(u) - T(u), u-u_0\right\rangle \ge \|u\|_X^2 - c(\|u\|_X+1),$$

which immediately implies the coercivity condition (3.34).

Step 5. *Existence and enclosure*

It follows from the conclusions in Steps 1–4 and [8, Corollary 2.6] that the auxiliary MVI (3.21), or equivalently (3.18)-(3.19), has solutions. Let $u \in K$ be any solution of (3.18)-(3.19). In this step, we prove that

$$\underline{u} \le u \le \overline{u} \ \text{ a.e. on } \Omega, \tag{3.40}$$

and next that u is indeed a solution of the original MVI (1.1). To verify the first inequality, we let s be any number in $\{1,\dots,k\}$ and prove that

$$\underline{u}_s \le u \ \text{ a.e. on } \Omega. \tag{3.41}$$

By definition of subsolutions we have $\underline{u}_s \vee u \in K$. Letting $v = \underline{u}_s \vee u = u + (\underline{u}_s - u)^+$ in (3.19) yields

$$\begin{aligned} &\int_\Omega \nabla u\nabla[(\underline{u}_s-u)^+]\,dx - \int_\Omega a\eta(\underline{u}_s-u)^+\,dx + \int_\Omega b(x,u)(\underline{u}_s-u)^+\,dx \\ &- \sum_{i=1}^k\int_\Omega aT_i(x,u)(\underline{u}_s-u)^+\,dx + \sum_{j=1}^m\int_\Omega aT^j(x,u)(\underline{u}_s-u)^+\,dx \ge 0. \end{aligned} \tag{3.42}$$

From (2.7) with $\underline{u}_s$ and $\underline{\eta}_s$ instead of $\underline{u}$ and $\underline{\eta}$, and $v = \underline{u}_s - (\underline{u}_s - u)^+ = \underline{u}_s \wedge u \in \underline{u}_s \wedge K$, we obtain

$$-\int_\Omega \nabla \underline{u}_s \nabla[(\underline{u}_s - u)^+]\,dx + \int_\Omega a\underline{\eta}_s(\underline{u}_s - u)^+\,dx \geq 0. \tag{3.43}$$

Adding inequalities (3.42) and (3.43) yields

$$\begin{aligned}&\textstyle\int_\Omega (\nabla u - \nabla \underline{u}_s)\nabla[(\underline{u}_s - u)^+]\,dx - \int_\Omega a(\eta - \underline{\eta}_s)(\underline{u}_s - u)^+\,dx \\ &\textstyle+ \int_\Omega b(x,u)(\underline{u}_s - u)^+\,dx - \sum_{i=1}^k \int_\Omega aT_i(x,u)(\underline{u}_s - u)^+\,dx \\ &\textstyle+ \sum_{j=1}^m \int_\Omega aT^j(x,u)(\underline{u}_s - u)^+\,dx \geq 0.\end{aligned} \tag{3.44}$$

Stampacchia's theorem gives

$$\int_\Omega (\nabla u - \nabla \underline{u}_s)\nabla[(\underline{u}_s - u)^+]\,dx = -\int_{\{x\in\Omega:\underline{u}_s(x)>u(x)\}} |\nabla(\underline{u}_s - u)|^2\,dx \leq 0. \tag{3.45}$$

At $x \in \Omega$ such that $\underline{u}_s(x) > u(x)$, since $\underline{u}_s(x) \leq \underline{u}(x) \leq \overline{u}(x)$, we have from (3.14) that $T^j(x,u(x)) = 0$ and thus

$$\int_\Omega aT^j(x,u)(\underline{u}_s - u)^+\,dx = \int_{\{x\in\Omega:\underline{u}_s(x)>u(x)\}} aT^j(x,u)(\underline{u}_s - u)\,dx = 0, \tag{3.46}$$

for all $j \in \{1,\dots,m\}$. Furthermore, for $x \in \Omega$ such that $\underline{u}_s(x) > u(x)$, we have $u(x) < \underline{u}(x)$ which, together with (3.18) and (3.4), implies that $\eta(x) \in \{\underline{\eta}(x)\}$, i.e.,

$$\eta(x) = \underline{\eta}(x). \tag{3.47}$$

Also, for such x, (3.9) gives

$$T_s(x,u(x)) = |\underline{\eta}_s(x) - \underline{\eta}(x)|. \tag{3.48}$$

As a direct consequence of (3.10),

$$\int_\Omega aT_i(x,u)(\underline{u}_s - u)^+\,dx \geq 0,\ \forall i \in \{1,\dots,k\}.$$

Thanks to (3.47) and (3.48), we get

$$\begin{aligned}&\textstyle -\int_\Omega a(\eta - \underline{\eta}_s)(\underline{u}_s - u)^+\,dx - \sum_{i=1}^k \int_\Omega aT_i(x,u)(\underline{u}_s - u)^+\,dx \\ \leq&\textstyle -\int_\Omega a(\eta - \underline{\eta}_s)(\underline{u}_s - u)^+\,dx - \int_\Omega aT_s(x,u)(\underline{u}_s - u)^+\,dx \\ =&\textstyle \int_{\{x\in\Omega:\underline{u}_s(x)>u(x)\}} a\{(\underline{\eta}_s(x) - \underline{\eta}(x)) - |\underline{\eta}(x) - \underline{\eta}_s(x)|\}(\underline{u}_s - u)\,dx \\ \leq&\ 0.\end{aligned} \tag{3.49}$$

Combining (3.44) with (3.45), (3.46), and (3.49), we obtain

$$0 \le \int_{\Omega} b(x,u)(\underline{u}_s - u)^+ \, dx = \int_{\{x\in\Omega:\underline{u}_s(x)>u(x)\}} b(x,u)(\underline{u}_s - u)\, dx.$$

From (3.6), if $\underline{u}_s(x) > u(x)$ then $\underline{u}(x) > u(x)$ and $b(x,u(x)) = [u(x) - \underline{u}(x)]w(x)$. Hence,

$$0 \le - \int_{\{x\in\Omega:\underline{u}_s(x)>u(x)\}} [\underline{u}(x) - u(x)][\underline{u}_s(x) - u(x)]w(x)\, dx.$$

Since $\underline{u}(x) - u(x) > 0$ and $\underline{u}_s(x) - u(x) > 0$ on the set $\{x \in \Omega : \underline{u}_s(x) > u(x)\}$, this inequality implies that this set must have measure 0, which means that $u(x) \ge \underline{u}_s(x)$ for a.e. $x \in \Omega$, and hence it follows (3.41). Since (3.41) holds for all $s \in \{1, \ldots, k\}$, we get the first inequality of (3.40). The second inequality of (3.40) is verified analogously.

From (3.40) and (3.6)-(3.9)-(3.14), we have

$$b(\cdot, u) = T_i(\cdot, u) = T^j(\cdot, u) = 0 \text{ a.e. on } \Omega,$$

for all $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, m\}$. Also, from (3.40) and (3.4), together with (3.18), we see that $\eta(x) \in f_0(x, u(x)) = f(u(x))$ for a.e. $x \in \Omega$. In view of these observations, (3.18)-(3.19) reduce to (2.2)-(2.3). Our proof of Theorem 3.1 is complete. □

Let $\underline{u}_i$, $1 \le i \le k$ and $\overline{u}_j$, $1 \le j \le m$, be sub- and supersolutions that satisfy the conditions in Theorem 3.1. We have proved that the set $\mathcal{S}$ of solutions of (1.1) (i.e. of (2.2)-(2.3)) between $\underline{u}$ and $\overline{u}$,

$$\mathcal{S} = \{u \in K : u \text{ satisfies (2.2)-(2.3) and } \underline{u} \le u \le \overline{u} \text{ a.e. on } \Omega\},$$

is nonempty. As consequences of Theorem 3.1, some further qualitative properties of $\mathcal{S}$ are given in the following result.

Corollary 3.2. *The solution set $\mathcal{S}$ possesses the following properties:*

(a) *$\mathcal{S}$ is a compact subset of X.*
(b) *If*

$$K \wedge K \subset K \text{ and } K \vee K \subset K, \tag{3.50}$$

then
  (i) *any $u \in \mathcal{S}$ is a subsolution and a supersolution of (1.1).*
  (ii) *$\mathcal{S}$ is directed downward and upward, that is, for all $u_1, u_2 \in \mathcal{S}$, there exist $u \in \mathcal{S}$ and $\tilde{u} \in \mathcal{S}$ such that*

$$u \le \min\{u_1, u_2\} \text{ and } \tilde{u} \ge \max\{u_1, u_2\}, \text{ respectively.}$$

(iii) *$\mathcal{S}$ has smallest and greatest elements, that is, there are $u_*, u^* \in \mathcal{S}$ such that $u_* \le u \le u^*$ for all $u \in \mathcal{S}$.*

Proof. Ad (a). Since $\underline{u}, \overline{u} \in L^\infty(\Omega)$, it follows from (3.2) that the set $\{\|u\|_{L^\infty(\Omega)} : u \in \mathcal{S}\}$ is bounded. Let $\{u_n\}$ be a sequence in $\mathcal{S}$. For each $n \in \mathbb{N}$, let η_n be the corresponding function in $F(u_n)$ that satisfies (2.2) and (2.3) (for each $u = u_n$ and $\eta = \eta_n$). Let f_0 be defined by (3.4). Since $\mathcal{S} \subset [\underline{u}, \overline{u}]$, we see from (3.4) that f can be replaced by f_0 in (2.2) and (2.3). In particular, $\eta_n \in F_0(u_n)$, $\forall n \in \mathbb{N}$. From (3.4), $\{\eta_n\}$ is a bounded sequence in $L^\infty(\Omega)$. Using (2.3) with u_n, η_n, and $v = v_0$, a fixed element of K, we see that $\{\int_\Omega |\nabla u_n|^2 dx\}$ is a bounded sequence, i.e., $\{u_n\}$ is a bounded sequence in X. Therefore, there exists a subsequence $\{u_{n_l}\}$ of $\{u_n\}$ such that

$$u_{n_l} \rightharpoonup u_0 \text{ in } X, \tag{3.51}$$

for some $u_0 \in K$ (as K is weakly closed in X). Thus, $u_{n_l} = i_w(u_{n_l}) \to i_w(u_0) = u_0$ in Y. In particular, $u_0 \in [\underline{u}, \overline{u}]$.

Arguing as in Step 2 in the proof of Theorem 3.1 (cf. (3.31)), we see that

$$\frac{a}{w}\eta_{n_l} \rightharpoonup \frac{a}{w}\eta_0 \text{ in } Y, \tag{3.52}$$

for some $\eta_0 \in F_0(u_0)$. For any $v \in K$, we have from Lemma 2.7 and (3.52) that

$$\begin{aligned} \textstyle\int_\Omega a\eta_{n_l}(v - u_{n_l})\,dx &= \langle i_w^*(\tfrac{a}{w}\eta_{n_l}), v - u_{n_l}\rangle_{X^*,X} \\ &= \langle \tfrac{a}{w}\eta_{n_l}, i_w(v - u_{n_l})\rangle_Y \\ &\to \langle \tfrac{a}{w}\eta_0, i_w(v - u_0)\rangle_Y \\ &= \langle i_w^*(\tfrac{a}{w}\eta_0), v - u_0\rangle_Y \\ &= \textstyle\int_\Omega a\eta_0(v - u_0)\,dx. \end{aligned}$$

Letting $l \to \infty$ in (2.3) with $u = u_{n_l}$ and $\eta = \eta_{n_l}$, and noting the limits

$$\lim \int_\Omega \nabla u_{n_l} \nabla v\,dx = \int_\Omega \nabla u_0 \nabla v\,dx$$

and

$$\liminf \int_\Omega |\nabla u_{n_l}|^2\,dx = \liminf \|u_{n_l}\|_X^2 \ge \|u_0\|_X^2 = \int_\Omega |\nabla u_0|^2\,dx,$$

we obtain

$$\begin{aligned} \textstyle\int_\Omega \nabla u_0(\nabla v - \nabla u_0)dx &\ge \limsup\left[\textstyle\int_\Omega \nabla u_{n_l}(\nabla v - \nabla u_{n_l})dx\right] \\ &\ge \liminf\left[\textstyle\int_\Omega \nabla u_{n_l}(\nabla v - \nabla u_{n_l})dx\right] \\ &\ge \lim \textstyle\int_\Omega a\eta_{n_l}(v - u_{n_l})dx \\ &= \textstyle\int_\Omega a\eta_0(v - u_0)dx. \end{aligned} \tag{3.53}$$

Hence, u_0 is a solution of (1.1), i.e., $u_0 \in \mathcal{S}$. Using $v = u_0$ in (3.53) also yields $\limsup \int_\Omega |\nabla u_{n_l}|^2 dx \le \int_\Omega |\nabla u_0|^2 dx$. Hence, $\lim \int_\Omega |\nabla u_{n_l}|^2 dx = \int_\Omega |\nabla u_0|^2 dx$, that is, $\|u_{n_l}\|_X \to \|u_0\|_X$, which, together with (3.51), implies that $u_{n_l} \to u_0$ in X. This completes the proof of the compactness of $\mathcal{S}$ in X.

Ad (b). The proof of (i) is straightforward from the definitions of solutions and sub-supersolutions of (1.1) given in Definitions 2.5 and 2.6. To prove (ii), assume $u_1, u_2 \in \mathcal{S}$. Since they are also subsolutions of (1.1), Theorem 3.1 thus implies the existence of a solution $\tilde{u}$ of (1.1) such that $\max\{u_1, u_2\} \le \tilde{u} \le \min\{\overline{u}_j : 1 \le j \le m\} = \overline{u}$, which shows that $\mathcal{S}$ is upward directed. But $u_1, u_2 \in \mathcal{S}$ are also supersolutions of (1.1), Theorem 3.1 thus implies the existence of a solution u of (1.1) such that $\max\{\underline{u}_i : 1 \le i \le k\} = \underline{u} \le u \le \min\{u_1, u_2\}$, which proves that $\mathcal{S}$ is downward directed.

As for (iii) let us show the existence of a greatest element only, since the existence of the smallest element is proved analogously. Since X is separable with the metric generated by $\|\cdot\|_X$, so is $\mathcal{S}$. Let $\{w_n\}$ be a dense sequence in $\mathcal{S}$. Using the directedness of $\mathcal{S}$, we can construct inductively a sequence $\{u_n\}$ in $\mathcal{S}$ such that $w_n \le u_n \le u_{n+1}, \forall n \in \mathbb{N}$. Let $u^*(x) = \sup\{u_n(x) : n \in \mathbb{N}\} = \lim_{n\to\infty} u_n(x),\ x \in \Omega$. As a consequence of the compactness of $\mathcal{S}$, $u_n \to u^*$ in X and $u^* \in \mathcal{S}$. Since $u^* \ge w_n$ a.e. in Ω for all $n \in \mathbb{N}$, from the density of $\{w_n\}$ in $\mathcal{S}$, we see that $u^* \ge u$ a.e. in Ω for all $u \in \mathcal{S}$. $\square$

Remark 3.3. Examples of closed convex sets of X that satisfy (3.50) are given by

$$K = \{u \in X : u \ge \phi \text{ a.e. on } \Omega\},$$

which represents an obstacle problem, or

$$K = \{u \in X : |\nabla u| \le C \text{ a.e. on } \Omega\},$$

with the constraint on the gradient, like in an elasto-plastic torsion problem. In order for $K \ne \emptyset$ in the obstacle problem, we need to impose conditions on ϕ such as for instance $\phi \in L^0(\Omega)$ with $\phi(x) \le 0$ for a.e. $x \in \Omega$, or $\phi \in H^{1,2}(\Omega)$ with $trace(\phi) \le 0$ on $\partial\Omega = \partial B(0, 1)$.

4. Multi-valued obstacle problem

As an application of the theory developed in Section 3, here we consider the obstacle problem: Find $u \in K$ and $\eta \in F(u)$ such that

$$\langle -\Delta u, v - u\rangle \ge \langle a\eta, v - u\rangle, \quad \forall\, v \in K, \tag{4.1}$$

where

$$K = \{u \in X : u \ge \phi \text{ a.e. on } \Omega\},$$

and $\phi \in L^0(\Omega)$ with $\phi(x) \le 0$ for a.e. $x \in \Omega$, and $a : \Omega \to \mathbb{R}_+$ satisfies (Ha). The multi-valued operator F is the Nemytskii operator generated by the multi-valued function $f : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ that is supposed to fulfill the following hypotheses:

(H1) $f : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ is upper semicontinuous and uniformly bounded, that is,

$$\exists\, C > 0 : \ |\eta| \le C, \quad \forall\, \eta \in f(s), \quad \text{and } \forall\, s \in \mathbb{R}.$$

As seen from the proof of Corollary 2.3, the multi-valued function f satisfying (H1) has the representation

$$f(s) = [\alpha(s), \beta(s)], \quad \forall\, s \in \mathbb{R}, \tag{4.2}$$

where the single-valued function $\alpha : \mathbb{R} \to \mathbb{R}$ is bounded and lower semicontinuous, and $\beta : \mathbb{R} \to \mathbb{R}$ is bounded and upper semicontinuous.

(H2) Let $\alpha : \mathbb{R} \to \mathbb{R}$ of (4.2) satisfy

$$\lim_{s \to 0} \frac{\alpha(s)}{s} = \mu > \lambda_1, \tag{4.3}$$

where $\lambda_1 > 0$ is the first eigenvalue of the eigenvalue problem:

$$-\Delta u = \lambda\, a(x) u \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega. \tag{4.4}$$

Note, (H1) implies hypothesis (HF), which along with (H2) will allow us to construct a pair of sub-supersolutions such that our main existence and enclosure result is applicable. Regarding the eigenvalue problem (4.4) let us recall [5, Corollary 3.3].

Corollary 4.1. *Under hypothesis (Ha) the eigenvalue problem (4.4) has a sequence of eigenvalues* $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \to \infty$*, with corresponding eigenfunctions* $\psi_j \in X \cap L^\infty(\Omega)$ *for all* $j \in \mathbb{N}$.

Next let us consider the linear problem

$$-\Delta u = a(x) \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega. \tag{4.5}$$

From [5, Theorem 4.1] we deduce the following result.

Corollary 4.2. *Assume hypothesis (Ha). Then (4.5) has a unique solution* $\Upsilon \in X$ *which enjoys the following properties:*

(i) $\Upsilon(x) > 0$ *for all* $x \in \Omega$.
(ii) $\Upsilon \in C^{1,\beta}_{0,\mathrm{loc}}(\overline{\Omega})$ *and* $\frac{\partial \Upsilon(x)}{\partial n} < 0$ *for* $x \in \partial\Omega$*, with* $\partial/\partial n$ *denoting the outward normal derivative.*
(iii) $0 \le \Upsilon(x) \le c$ *for all* $x \in \overline{\Omega}$.

Finally, from [5, Theorem 2.5] and [5, Corollary 6.1] we obtain

Corollary 4.3. *Assume (Ha) and let* ψ_1 *be the eigenfunction belonging to the first eigenvalue* λ_1 *of (4.4). Then* $\psi_1(x) \approx \Upsilon(x)$ *for all* $x \in \overline{\Omega}$*, that is, there are positive constants* $c_1,\ c_2 > 0$ *such that*

$$c_1 \psi_1(x) \le \Upsilon(x) \le c_2 \psi_1(x), \quad x \in \overline{\Omega}. \tag{4.6}$$

By means of the above Corollary 4.1–Corollary 4.3 as well as Theorem 3.1 and Corollary 3.2 of Section 3 we are now able to prove the following existence and enclosure result for the multi-valued obstacle problem (4.1).

Theorem 4.4. *Assume hypotheses (Ha), (H1), and (H2). Then for $\varepsilon > 0$ sufficiently small and $M > 0$ sufficiently large, $\underline{u} = \varepsilon\,\psi_1$ and $\overline{u} = M\,\Upsilon$ are sub- and supersolutions, respectively, of the multi-valued obstacle problem (4.1) satisfying $\underline{u} \le \overline{u}$. Moreover, there exist extremal solutions of (4.1) within the interval $[\underline{u}, \overline{u}]$.*

Proof. We only need to show that $\underline{u} = \varepsilon\,\psi_1$ is a subsolution for $\varepsilon > 0$ sufficiently small, and $\overline{u} = M\,\Upsilon$ is a supersolution for $M > 0$ sufficiently large of (4.1). The existence of extremal solutions of (4.1) then follows immediately from Theorem 3.1 and Corollary 3.2 of Section 3.

Clearly, $\underline{u} \vee K = \varepsilon\,\psi_1 \vee K \subset K$. Let $\underline{\eta} = \alpha(\varepsilon\,\psi_1)$, then $\underline{\eta}(x) \in f(\underline{u}(x))$, and since $\underline{\eta}$ is bounded we have $a\underline{\eta} \in X_0^*$. To finish the proof for $\underline{u} = \varepsilon\,\psi_1$ being a subsolution we need to verify the inequality (2.7), that is,

$$\int_\Omega \nabla\underline{u}(\nabla v - \nabla\underline{u})\,dx \ge \int_\Omega a\underline{\eta}(v - \underline{u})\,dx, \quad \forall\, v \in \underline{u} \wedge K. \tag{4.7}$$

Note, $v \in \underline{u} \wedge K$ is represented by $v = \underline{u} \wedge \varphi = \underline{u} - (\underline{u} - \varphi)^+$ for any $\varphi \in K$, and thus (4.7) is equivalent to

$$\int_\Omega \nabla\underline{u}\nabla(\underline{u} - \varphi)^+\,dx \le \int_\Omega a\underline{\eta}(\underline{u} - \varphi)^+\,dx, \quad \forall\, \varphi \in K. \tag{4.8}$$

Since $(\underline{u} - \varphi)^+ \in X_+ = \{u \in X : u(x) \ge 0 \text{ for a.e. } x \in \Omega\}$, the proof of (4.8) is done provided $\underline{u} = \varepsilon\,\psi_1 \in X$ is a subsolution of the equation

$$-\Delta u = a\,\alpha(u). \tag{4.9}$$

Using the properties of the first eigenfunction ψ_1 along with hypotheses (Ha), (H1), and (H2), we have for $\varepsilon > 0$ small $\frac{\alpha(\varepsilon\,\psi_1)}{\varepsilon\,\psi_1} \ge \lambda_1 + \delta$ for some $0 < \delta < \mu - \lambda_1$ and thus

$$\begin{aligned}
-\Delta\underline{u} - a\,\alpha(\underline{u}) &= \lambda_1 a\,\varepsilon\,\psi_1 - a\,\alpha(\varepsilon\,\psi_1) \\
&= a\,\varepsilon\,\psi_1\Big(\lambda_1 - \frac{\alpha(\varepsilon\,\psi_1)}{\varepsilon\,\psi_1}\Big) \\
&\le a\,\varepsilon\,\psi_1\Big(\lambda_1 - (\lambda_1 + \delta)\Big) \le 0,
\end{aligned}$$

which shows that $\underline{u} = \varepsilon\,\psi_1$ is a subsolution of (4.1).

To prove that $\overline{u} = M\,\Upsilon$ is a supersolution of (4.1) for $M > 0$ sufficiently large, note that $\overline{u} \wedge K = M\,\Upsilon \wedge K \subset K$ is trivially satisfied. Let $\overline{\eta} = \beta(M\,\Upsilon)$, then $\overline{\eta}(x) \in f(\overline{u}(x))$, and since $\overline{\eta}$ is bounded we have $a\,\overline{\eta} \in X_0^*$. So it remains to verify inequality (2.9), that is,

$$\int_\Omega \nabla\overline{u}(\nabla v - \nabla\overline{u})\,dx \ge \int_\Omega a\overline{\eta}(v - \overline{u})\,dx, \quad \forall\, v \in \overline{u} \vee K. \tag{4.10}$$

Clearly, $v \in \overline{u} \vee K$ is represented by $v = \overline{u} + (\varphi - \overline{u})^+$ for any $\varphi \in K$, and thus (4.10) is equivalent to

$$\int_\Omega \nabla \overline{u}\, \nabla(\varphi - \overline{u})^+\, dx \geq \int_\Omega a\overline{\eta}(\varphi - \overline{u})^+\, dx, \quad \forall\, \varphi \in K. \tag{4.11}$$

Since $(\varphi - \overline{u})^+ \in X_+$ for all $\varphi \in K$, the proof of (4.11) is finished provided $\overline{u} = M\,\Upsilon$ is a supersolution of the semilinear equation

$$-\Delta u = a\,\beta(u), \tag{4.12}$$

that is $-\Delta\overline{u} - a\,\beta(\overline{u}) \geq 0$ in the weak sense. Taking into account that $\Upsilon \in X$ is a weak solution of (4.5) we see

$$\begin{aligned} -\Delta\overline{u} - a\,\beta(\overline{u}) &= -\Delta(M\,\Upsilon) - a\,\beta(M\,\Upsilon) = a\,M - a\,\beta(M\,\Upsilon) \\ &= a\,(M - \beta(M\,\Upsilon)) \geq 0, \end{aligned}$$

for $M > 0$ sufficiently large, since by (H1), $\beta : \mathbb{R} \to \mathbb{R}$ is uniformly bounded. Thus $\overline{u} = M\,\Upsilon$ is a supersolution of (4.1), which according to Corollary 4.3 satisfies $\overline{u} = M\,\Upsilon \geq \varepsilon\,\psi_1$ for a possibly even larger M. This completes the proof of the theorem. □

Remark 4.5. In Theorem 4.4 the obstacle function $\phi \in L^0(\Omega)$ of $K = \{u \in X : u \geq \phi \text{ a.e. on } \Omega\}$, is supposed to satisfy $\phi(x) \leq 0$ for a.e. $x \in \Omega$. In view of (4.6), more general obstacle functions ϕ are admissible such that Theorem 4.4 remains true. For instance, any function $\phi \in L^0(\Omega)$ satisfying

$$\operatorname*{ess\,sup}_{x\in\Omega} \frac{\phi(x)}{\Upsilon(x)} < \infty \;\text{ or }\; \operatorname*{ess\,sup}_{x\in\Omega} \frac{\phi(x)}{\psi_1(x)} < \infty \tag{4.13}$$

In case of an obstacle problem with an obstacle being from above, that is,

$$K = \{u \in X : u \leq \phi \text{ a.e. on } \Omega\},$$

and $\phi \in L^0(\Omega)$, Theorem 4.4 still remains true for obstacle functions satisfying one of the following conditions:

$$\operatorname*{ess\,inf}_{x\in\Omega} \frac{\phi(x)}{\Upsilon(x)} > 0 \;\text{ or }\; \operatorname*{ess\,inf}_{x\in\Omega} \frac{\phi(x)}{\psi_1(x)} > 0 \;\text{ or simply }\operatorname*{ess\,inf}_{x\in\Omega} \phi(x) > 0. \tag{4.14}$$

5. Generalized variational-hemivariational inequalities

The sub-supersolution method for the MVI (1.1) established in Section 3 will enable us to treat general variational-hemivariational inequalities in (unbounded) exterior plane domains of the form: Find $u \in K$ such that

$$\langle -\Delta u, v - u\rangle + \int_\Omega a(-j)^o(u, u; v - u)\, dx \geq 0, \quad \forall\, v \in K, \tag{5.1}$$

where $j^o(r, s; \varrho)$ denotes Clarke's generalized directional derivatives at s in the direction ϱ for fixed r of some function $j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(r, s) \mapsto j(r, s)$, that is supposed to satisfy the following hypothesis:

(HJ) For each $r \in \mathbb{R}$, the function $s \mapsto j(r, s)$ is locally Lipschitz, and for $\varrho = \pm 1$, the function $s \mapsto j^o(s, s; \varrho)$ is locally bounded and upper semicontinuous.

We recall here for convenience the definition of Clarke's generalized directional derivatives at s in the direction ϱ for fixed r, $j^o(r, s; \varrho)$,

$$j^o(r, s; \varrho) = \limsup_{y \to s,\ \varepsilon \downarrow 0} \frac{j(r, y + \varepsilon \varrho) - j(r, y)}{\varepsilon}, \tag{5.2}$$

see [6, Chap. 2]. The main goal of this section is to pave the way for dealing with variational-hemivariational type inequalities in unbounded domains. We are going to show that under hypothesis (HJ) the general variational-hemivariational inequality (5.1) turns out to be only a special case of the MVI (1.1). To this end let us recall the notion of Clarke's generalized gradient of the locally Lipschitz function $s \mapsto j(r, s)$ at s for fixed r, which is denoted by $\partial j(r, s)$ and defined by

$$\partial j(r, s) = \{\xi \in \mathbb{R} : j^o(r, s; \varrho) \ge \xi\, \varrho,\ \forall\, \varrho \in \mathbb{R}\}, \tag{5.3}$$

see [6, Chap. 2]. Let us introduce the multi-valued function $f : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ given by

$$f(s) = \partial j(s, s). \tag{5.4}$$

Lemma 5.1. *If $(r, s) \mapsto j(r, s)$ satisfies hypothesis (HJ), then the multi-valued function $f : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ given by (5.4) satisfies hypothesis (HF).*

Proof. From the definition of $\partial j(r, s)$ and the positive homogeneity of the mapping $\varrho \mapsto j^o(r, s; \varrho)$ for r fixed we see that for all $r, s \in \mathbb{R}$,

$$\partial j(r, s) = [-j^o(r, s; -1),\, j^o(r, s; 1)], \tag{5.5}$$

which shows that $f : \mathbb{R} \to \mathcal{K}(\mathbb{R})$. To prove that $f : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ is a multi-valued upper semicontinuous function, we make use of the upper semicontinuity of $s \mapsto j^o(s, s; \pm 1)$. Let $s_0 \in \mathbb{R}$ and U be an open neighborhood of $f(s_0) = \partial j(s_0, s_0)$. In view of (5.5), there exists $\varepsilon > 0$ such that

$$(-j^o(s_0, s_0; -1) - \varepsilon,\, j^o(s_0, s_0; 1) + \varepsilon) \subset U.$$

From the upper semicontinuity of the (single-valued) functions $s \mapsto j^o(s, s; \pm 1)$ at s_0, there exists an open neighborhood O of s_0 such that for all $s \in O$ we have

$$j^o(s, s; 1) < j^o(s_0, s_0; 1) + \varepsilon, \text{ and } j^o(s, s; -1) < j^o(s_0, s_0; -1) + \varepsilon,$$

and hence it follows for all $s \in O$,

$$f(s) = \partial j(s,s) = [-j^o(s,s;-1), j^o(s,s;1)]$$
$$\subset (-j^o(s_0,s_0;-1) - \varepsilon, j^o(s_0,s_0;1) + \varepsilon) \subset U,$$

which proves the upper semicontinuity of f at s_0. $\square$

From Lemma 5.1 we see that f given by (5.4) is a special multi-valued function that satisfies (HF). Let us now consider the MVI associated with f, that is: Find $u \in K \subset X$ and $\eta \in F(u)$, i.e., $\eta(x) \in f(u(x)) = \partial j(u(x), u(x))$ such that

$$\langle -\Delta u, v - u\rangle \geq \langle a\,\eta, v - u\rangle, \quad \forall\, v \in K. \tag{5.6}$$

The following equivalence of problems (5.1) and (5.6) holds.

Theorem 5.2. *Assume hypotheses (Ha), (HJ) and let the lattice condition $K \wedge K \subset K$ and $K \vee K \subset K$ be satisfied. Then $u \in X \cap L^\infty(\Omega)$ is a solution of (5.1) if and only if u is a solution of (5.6).*

Proof. Let u be a solution of (5.6), that is,

$$\langle -\Delta u, v - u\rangle + \int_\Omega a(-\eta)\,(v-u) \geq 0, \quad \forall\, v \in K, \tag{5.7}$$

with $-\eta(x) \in -f(u(x)) = -\partial j(u(x), u(x))$, and due to the calculus for Clarke's gradient we have

$$-\eta(x) \in \partial(-j)(u(x), u(x)),$$

which by the definition of Clarke's gradient of $(-j)$ yields

$$-\eta(x)(v(x) - u(x)) \leq (-j)^o(u(x), u(x); v(x) - u(x)) \ \text{ for a.e. } x \in \Omega. \tag{5.8}$$

Since $a \geq 0$, we get from (5.7) and the last inequality (5.8) that u satisfies (5.1).

Let us prove the reverse, and assume that u is a solution of (5.1). In order to show that u is a solution of the multi-valued variational inequality (5.6), we are going to show that u is both a subsolution and a supersolution for the multi-valued problem (5.6), which then by applying Theorem 3.1 completes the proof.

Since K has the lattice condition, we can use in (5.1), in particular, $v \in u \wedge K$, i.e., $v = u \wedge \varphi = u - (u - \varphi)^+$ with $\varphi \in K$, which yields

$$\langle -\Delta u, -(u-\varphi)^+\rangle + \int_\Omega a(-j)^o(u, u; -(u-\varphi)^+)\,dx \geq 0, \quad \forall\, \varphi \in K. \tag{5.9}$$

Since $(-j)^o(s,s;\cdot)$ is positive homogeneous, (5.9) becomes

$$\langle -\Delta u, -(u-\varphi)^+\rangle + \int_\Omega a(-j)^o(u, u; -1)(u-\varphi)^+\,dx \geq 0, \quad \forall\, \varphi \in K. \tag{5.10}$$

Further, by applying Clarke' gradient calculus from [6, Chap. 2] we see

$$\begin{aligned}(-j)^o(u(x),u(x);-1) &= \max\{-\zeta(x) : \zeta(x) \in \partial(-j)(u(x),u(x))\} \\ &= -\min\{\zeta(x) : \zeta(x) \in \partial(-j)(u(x),u(x))\} \\ &= -\underline{\eta}(x),\end{aligned}$$

where

$$\underline{\eta}(x) \in \partial(-j)(u(x),u(x)) = -\partial j(u(x),u(x). \tag{5.11}$$

From (5.10) and (5.11) we get by using $v \in u \wedge K$, that is $v = u - (u-\varphi)^+$ for $\varphi \in K$

$$\langle -\Delta u, v-u\rangle \geq \int_\Omega a(-\underline{\eta})(v-u)\,dx, \quad \forall\, v \in u \wedge K, \tag{5.12}$$

where $-\underline{\eta}(x) = \hat{\underline{\eta}}(x) \in \partial j(u(x),u(x) = f(u(x))$, which proves that u is a subsolution of (5.6). In a similar way one can show that u is also a supersolution of (5.6). Applying Theorem 3.1, there exists a solution $\tilde{u}$ of (5.6) such that $u \leq \tilde{u} \leq u$, and hence $u = \tilde{u}$, that is u is a solution of the MVI (5.6). This completes the equivalence proof. □

Remark 5.3. We are not seeking here to establish existence and enclosure results for the inequalities (1.1) and (5.1) in the most general settings on the exterior domain Ω or on the operators involved in the inequalities, but concentrate instead on a new working framework for such inequalities on exterior domains. More general nonsmooth problems with quasilinear elliptic operators of monotone type can therefore be studied by the theory and tools developed here. Moreover, the connection between hemi-variational inequalities and variational inequalities with multi-valued terms elaborated in Section 5 allows extending their application scope further to hemi-variational inequalities on exterior domains.

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