Global existence, uniqueness and stability for nonlinear dissipative bulk-interface interaction systems

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Abstract

We show global well-posedness and exponential stability of equilibria for a general class of nonlinear dissipative bulk-interface systems. They correspond to thermodynamically consistent gradient structure models of bulk-interface interaction. The setting includes nonlinear slow and fast diffusion in the bulk and nonlinear coupled diffusion on the interface. Additional driving mechanisms can be included and non-smooth geometries and coefficients are admissible, to some extent. An important application are volume-surface reaction-diffusion systems with nonlinear coupled diffusion.
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1. Introduction

In this paper, we consider nonlinear bulk-interface systems that arise from thermodynamically consistent gradient structure modelling of the interaction of dissipative dynamics on bulk domains and on and across interfaces or surfaces. The modelling concept was introduced in [22], [31] and it was shown that it applies to many different processes (examples and references below in Subsection 1.2). Here, the aim is to show global well-posedness and stability, including in particular slow and fast diffusion in the bulk, coupled nonlinear diffusion on the interface

and general reaction-type nonlinearities on the interface and as boundary conditions. The results are based on a specific functional analytic framework that uses maximal parabolic regularity in $W^{-1,q}$ and global L^∞-bounds that derive from the dissipative structure.

1.1. Model equations

The bulk $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded set with boundary $\partial\Omega$ and it can be separated into two open disjoint domains Ω_+ and Ω_- by an interface Γ of dimension $d - 1$. The domains Ω_+, Ω_- and manifold Γ have unit outer normal vector fields ν_+, ν_-, ν_Γ at their boundaries. The scalar quantities $u_+ : (0, T) \times \Omega_+ \to \mathbb{R}$, $u_- : (0, T) \times \Omega_- \to \mathbb{R}$ and $u_\Gamma : (0, T) \times \Gamma \to \mathbb{R}$ interact on and across Γ, and they satisfy the three sets of equations

$$
\begin{cases}
\dot{u}_+ - \mathrm{div}(k_+(u_+)\nabla u_+) = f_+(u_+), & \text{in } (0, T) \times \Omega_+, \\
(k_+(u_+)\nabla u_+)\nu_+ + m_+(u)(u_+ - u_\Gamma) + m_\Gamma(u)(u_+ - u_-) = g_+(u), & \text{on } (0, T) \times \Gamma, \\
(k_+(u_+)\nabla u_+)\nu_+ = h_+(u_+), & \text{on } (0, T) \times \{\partial\Omega_+ \backslash \Gamma\},
\end{cases}
\tag{1.1}
$$

and,

$$
\begin{cases}
\dot{u}_- - \mathrm{div}(k_-(u_-)\nabla u_-) = f_-(u_-), & \text{in } (0, T) \times \Omega_-, \\
(k_-(u_-)\nabla u_-)\nu_- + m_-(u)(u_- - u_\Gamma) + m_\Gamma(u)(u_- - u_+) = g_-(u), & \text{on } (0, T) \times \Gamma, \\
(k_-(u_-)\nabla u_-)\nu_- = h_-(u_-), & \text{on } (0, T) \times \{\partial\Omega_- \backslash \Gamma\},
\end{cases}
\tag{1.2}
$$

and,

$$
\begin{cases}
\dot{u}_\Gamma - \mathrm{div}_\Gamma(k_\Gamma(u)\nabla_\Gamma u_\Gamma) - m_+(u)(u_+ - u_\Gamma) - m_-(u)(u_- - u_\Gamma) = f_\Gamma(u), & \text{in } (0, T) \times \Gamma, \\
(k_\Gamma(u)\nabla_\Gamma u_\Gamma)\nu_\Gamma = h_\Gamma(u_\Gamma), & \text{on } (0, T) \times \partial\Gamma.
\end{cases}
\tag{1.3}
$$

To improve the presentation, we write

$$
u = (u_+, u_-, u_\Gamma), \quad f = (f_+, f_-, f_\Gamma), \dots
$$

The coefficient matrices k, the scalar transmission coefficients m and the external forces and inhomogeneous boundary conditions f, g, h may have (locally Lipschitz) dependence of the solution u and continuous dependence of the space variables with

$$
k_\pm(x, u_\pm) \in \mathbb{R}_{\geq 0}^{d \times d} \qquad \text{and} \qquad k_\Gamma(y, u) \in \mathbb{R}_{\geq 0}^{(d-1) \times (d-1)},
\tag{1.4}
$$

and

$$
m_\Gamma(y, u), m_+(y, u), m_-(y, u) \in \mathbb{R}_{\geq 0}.
\tag{1.5}
$$

More precise assumptions on k, m, f, g, h are given in Section 2. Examples are slow or fast diffusion $k_+(u_+) = \kappa_0 u_+^{\rho-1}$ with constants $\rho \in \mathbb{R}$ and $\kappa_0 > 0$, where the case $k_+(u_+) = \frac{1}{u_+^2}$ is motivated by the entropic structure for the system in [31] (see Subsection 5.1). The diffusion coefficient k_Γ can depend on all three unknowns u in a nonlinear way and the same is true for the transmission coefficients m. An example is $m_+(u) = \frac{1}{u_+^2 u_\Gamma}$ (motivated in Subsection 5.1).

For the global equilibration result on all three components, it is needed that $m_\pm$, m_Γ are not only non-negative, but that at least two of them are positive, so every pair of components interacts at least indirectly across the interface.

1.2. Applications

The system (1.1) – (1.3) can be related to several types of applications:

- An important example is that f, g are given by chemical reaction rates for the densities u_+, u_- and u_Γ of species that interact and react at the interface. A concrete example is

$$f_+(u_+) = 0, g_+(u) = -k\alpha(u_+^\alpha - \kappa u_\Gamma^\beta), \quad f_\Gamma(u) = k\beta(u_+^\alpha - \kappa u_\Gamma^\beta) \tag{1.6}$$

  with $k, \kappa > 0$ and $\alpha, \beta \geq 1$. This situation is typical for many (cell) biological, ecological and technological processes [2,22,28,31] and these systems have gained increasing attention in analysis and numerical analysis [4,17–19,26,30,36]. It is shown in [9] how the main result here, Theorem 3.1, applies in this case and allows to extend some of the previous results in [4,17,19,26,36] to the case of nonlinear and nonlinearly coupled diffusion and to the case that only part of the boundary of Ω is interactive. Nonlinear and coupled diffusion often appear in the modelling of reaction-diffusion systems, and they can be naturally associated to bulk-interface systems [2,3,22,29,31]. It is the aim of future work to use the analysis started here and in [9] to study more complex systems of multicomponent reaction and diffusion.
- A second example is the modelling of heat conduction within a bulk material that is separated into two parts by a thin heat-conducting plate. Particularly at high and low temperatures, thermal conductivity of bulk and plate materials become nonlinear in their dependence of temperature and non-equilibrium modelling of heat conduction across the plate leads to nonlinear transmission coefficients m of the type above, see for example [38] for material parameters and Subsection 5.1 on the associated Onsager structure. In technological applications, the geometry often includes sharp edges and singularities where interface and boundary meet. These non-smooth settings are essentially included in the analysis here. To the author's knowledge, it is the first result on this particular quasilinear transmission-type problem in a non-smooth setting.
- For models of diffusion and transport of electrical charges in semiconductor devices, like solar cells, active interfaces often play a crucial role, cf. [21]. In particular for the case of three spatial dimensions, well-posedness for these systems in non-smooth geometric settings is still hard to achieve but highly relevant in modelling, simulation and optimization [12]. In future work, the method developed here can be adapted to provide well-posedness for systems that include coupled diffusion and transport along active interfaces.
- The quasi-linear structure of (1.1) – (1.3) may appear after a change of coordinates that transforms a free boundary problem to a fixed domain, [35].
- Here, global existence and uniqueness is shown including semilinear f, g, h as long as boundedness is preserved. This includes, for example, driving mechanisms of Allen-Cahn-type associated to phase separation, cf. Corollary 3.6 and the example in (3.7). The result here applies if phase-separation occurs both in a bulk and along a (lower-dimensional) surface part of the material, coupled by (non-linear) transmission terms. Well-posedness for Allen-Cahn equations with dynamic interface conditions was shown in [6].

1.3. Functional analytic setting

We use that system (1.1) – (1.3) can be recast as a quasilinear abstract Cauchy problem of the form

$$\dot{u}(t) + \mathcal{A}_{u(t)}u(t) = \mathcal{F}(u(t)),$$
$$u(0) = u^0,$$

in the space W_0^{-1,q,q_Γ} of functionals on

$$W^{1,q,q_\Gamma} = W^{1,q'}(\Omega_+) \times W^{1,q'}(\Omega_-) \times W^{1,q'_\Gamma}(\Gamma)$$

with $q > d$, $q_\Gamma > 2$ in spatial dimensions $d = 2, 3$, where $\mathcal{A}_{u(t)}$ is a second-order divergence-form elliptic operator (details in Subsection 2.2). The main result is that this problem is globally well-posed. This may seem like a special choice of spaces at first, but actually, there is a good reason for this choice in that the quasilinear map

$$W^{1,q,q_\Gamma} \ni w \mapsto L^{\infty,\infty} \ni w \mapsto k(w) \in L^{\infty,\infty} \mapsto \mathcal{A}_w \in \mathcal{B}(W^{1,q,q_\Gamma}, W_0^{-1,q,q_\Gamma}) \qquad (1.7)$$

is well-defined, locally Lipschitz, and compact (estimate (3.10) and Lemma 3.2). In the usual weak setting with $w \in W^{1,2,2}$, without the first embedding, the second mapping is not Lipschitz, so local well-posedness is less clear. On the other hand, if $W_0^{-1,q,q}$ is replaced by a smaller space like $L^{2,2}$ (the strong setting), the third mapping is not bounded, so more than global L^∞-estimates are needed for global existence.

The W_0^{-1,q,q_Γ}-setting is also well adapted to non-smooth situations like non-smooth coefficients, non-smooth boundaries and mixed boundary conditions. In [11], there is a survey on results on the corresponding elliptic isomorphy that is used for the quasilinear theory. Here, this means that it is not necessary that all of the surface of the bulk domain is active and that local well-posedness in this setting extends to mixed Dirichlet and Neumann boundary conditions [10] and that Ω_+ and Ω_- and the coefficients k, m are allowed to be (spatially) non-smooth to some extent. This is natural from a modelling point of view, as the separation of a smooth domain by an interface, even a plane, will usually create a kink. It can also be highly relevant in semiconductor device modelling [12] and for catalysis modelling [4]. A general advantage of the quasilinear approach to (1.1) – (1.3) is that it has a very good perturbation theory. It is a well-studied problem how to include lower-order terms, time-dependence of coefficients and external forces, cf. Subsection 5.2. For example, in [8], local well-posedness of similar systems is shown in an L^p-setting with smooth coefficients and geometries.

Here, in the proof of global well-posedness, a Schaefer fixed point argument is used to prove existence of solutions, and the local Lipschitzianity of the map (1.7) is used afterwards, to show uniqueness. This method needs an explanation as a maximal regularity approach with Lipschitzian nonlinearity is already tailored to a contraction mapping argument that provides both existence and uniqueness. The point here is that we are interested in global existence. The usual quasilinear maximal parabolic regularity approach provides well-posedness up to possible blow-up in the time-trace norm (the space of local semiflow). In the present weak setting, the trace space X_{q,q_Γ}^r of solutions $u \in C^0(J_T, X_{q,q_\Gamma}^r)$ embeds into the space of Hölder-continuous functions, $X_{q,q_\Gamma}^r \hookrightarrow C^{\beta,\beta_\Gamma}$ (Lemma 2.4), so for global existence, with the usual arguments, (at least)

global Hölder bounds would be needed. The method presented here circumvents the need for uniform higher-order a priori bounds by splitting existence and uniqueness proofs. To apply Schaefer's fixed point result, it is sufficient to know *a posteriori* L^∞-bounds, rather than *a priori* bounds. This appears to be a technical simplification only, but it is a standard method used for quasilinear elliptic problems [20], so it may in principle be rewarding for parabolic systems as well.

Regarding the nonlinearities in f, g, h in the system (1.1) – (1.3), the need for global L^∞-estimates can be a severe restriction, in particular, on reaction-diffusion problems. On the other hand, it is not a stronger condition than what is used in the semilinear theory [17,19,36], so the main result here can be applied in these situations. The boundedness-by-entropy method developed by Jüngel [27] shows that there is a connection between the entropic gradient structures of a system and L^∞-estimates. A direct argument for this in a simple nonlinear diffusion-reaction case was also made in [9]. The proof of L^∞-bounds, equilibration and exponential rates for system (1.1) – (1.3), Lemma 3.5 and Theorem 4.1, make exact use of the gradient structure of the bulk-interface interaction in that for convex energies, the linearization of the dual dissipation potentials acts like a discrete gradient on the components.

Outline The paper is organized as follows. The next section contains basic assumptions on the geometry and coefficient functions in (1.1) – (1.3) and collects preliminary results on the bilinear form and linearized operator associated to the system. In Section 3, the main result on existence and uniqueness of global solutions is proved. In Section 4, the bulk-interface Poincaré inequality and exponential stability for the global equilibrium under mass conservation are shown. In the last Section 5, the relation of the model to the entropic Onsager system of heat diffusion and transfer derived in [31] are discussed and extensions of the main results like the case of $\Omega_- = \emptyset$, higher regularity, dependence of coefficients on time, and the inclusion of lower-order perturbations are given.

2. Basic assumptions and functional analytic framework

2.1. Assumptions on geometry and coefficients

Assumption 2.1. The bulk domains Ω_+ and Ω_- are bounded Lipschitz domains [23, Def. 1.2.12]. The interface Γ is a $d-1$-dimensional C^1-part of the boundary of both Ω_+ and Ω_- with Lipschitz boundary $\partial\Gamma$ if $d=3$.

For $q \in [1, \infty]$, $L^q(\omega)$ denotes the usual real Lebesgue space of q-integrable functions on a domain or manifold ω, $W^{m,q}(\omega)$ denote the usual L^q-Sobolev spaces of order $m \in \mathbb{N}$ and $C^\alpha(\omega)$ are the uniform Hölder spaces of exponent $\alpha \geq 0$ with $C^0(\omega) = C(\overline{\omega})$ if ω is bounded. Function spaces related to (1.1) – (1.3) are: for $q, q_\Gamma \in [1, \infty]$, $\alpha, \alpha_\Gamma \geq 0$ and $\mathcal{H}_{d-1}$ the $d-1$-dimensional Hausdorff measure on Γ,

$$\mathbf{L}^{q,q_\Gamma} := L^q(\Omega_+) \times L^q(\Omega_-) \times L^{q_\Gamma}(\Gamma),$$

$$\mathbf{W}^{1,q,q_\Gamma} := W^{1,q}(\Omega_+) \times W^{1,q}(\Omega_-) \times W^{1,q_\Gamma}(\Gamma), \quad \text{and}$$

$$\mathbf{C}^{\alpha,\alpha_\Gamma} := C^\alpha(\Omega_+) \times C^\alpha(\Omega_-) \times C^{\alpha_\Gamma}(\Gamma),$$

where $W^{1,q}(\Gamma)$ is defined through the standard notion of weak differentiability on the manifold Γ that preserves embedding and trace theorems, integration by parts and Poincaré inequalities.

The trace operator

$$\text{tr}_\Gamma : W^{1,q}(\Omega_+) \to L^q(\Gamma) \tag{2.1}$$

is well-defined and continuous (likewise for Ω_-). Set

$$\text{tr}_\Gamma u = (\text{tr}_\Gamma u_+, \text{tr}_\Gamma u_-, u_\Gamma) \tag{2.2}$$

for the trace components of u on the interface Γ. Often we omit the operator tr_Γ (like in the statement of the model equations (1.3) on Γ). Dual Sobolev spaces are denoted by

$$W_0^{-1,q}(\omega) := (W^{1,q'}(\omega))' \text{ and } \mathbf{W}_0^{-1,q,q_\Gamma} := (\mathbf{W}^{1,q',q_\Gamma'})'$$

with $\frac{1}{q} + \frac{1}{q'} = \frac{1}{q_\Gamma} + \frac{1}{q_\Gamma'} = 1$. For $-\infty < l \leq L < +\infty$ and $n \in \mathbb{N}$, let

$$(\mathbb{R}_l^L)^n := \{v \in \mathbb{R}^n : l \leq v_i \leq L \text{ for } i = 1, \ldots, n\}, \quad \text{and}$$
$$\mathbf{C}_l^L = \{u \in \mathbf{C}^{0,0} : l \leq u_\pm(x), u_\Gamma(y) \leq L \text{ for all } x \in \Omega_\pm, y \in \Gamma\}.$$

Assumption 2.2. Let k and m be given as in (1.4) and (1.5) and let $-\infty < l < L < +\infty$ be given constants.

(1) Uniformly in $u \in (\mathbb{R}_l^L)^3$, the coefficient matrices $k(\cdot, u)$ are measurable, bounded and elliptic, i.e. there are constants $\underline{k}, \overline{k} > 0$ such that

$$\|k(\cdot, u)\|_{L^\infty} \leq \overline{k}, \tag{2.3}$$

and such that for all $x \in \mathbb{R}^d$, $y \in \mathbb{R}^{d-1}$,

$$x \cdot k_\pm(\cdot, u)x \geq \underline{k}|x|^2 \quad \text{and} \quad y \cdot k_\Gamma(\cdot, u)y \geq \underline{k}|y|^2, \tag{2.4}$$

almost everywhere in $\Omega_\pm, \Gamma$. In particular, $\underline{k}$ and $\overline{k}$ may depend on l, L, but not on $u \in (\mathbb{R}_l^L)^3$.

(2) Uniformly in $u \in (\mathbb{R}_l^L)^3$, the transmission coefficients $m_\pm, m_\Gamma$ are measurable and there are constants $\underline{m}, \overline{m} > 0$ such that

$$\|m(\cdot, u)\|_{L^\infty(\Gamma)} \leq \overline{m} \tag{2.5}$$

and such that at least two of the three transmission functions, e.g. m_+, m_Γ are positively bounded from below,

$$\underline{m} \leq m_+(\cdot, u), m_\Gamma(\cdot, u), \tag{2.6}$$

and the third transmission function is non-negative,

$$0 \leq m_-(\cdot, u),$$

almost everywhere in Γ. Note that again, $\underline{m}, \overline{m}$ may depend on l, L, but not on $u \in (\mathbb{R}_l^L)^3$.

(3) The functions $\mathbb{R} \ni u_\pm \mapsto k_\pm(x, u_\pm)$, $\mathbb{R}^3 \ni u \mapsto k_\Gamma(y, u)$ and $\mathbb{R}^3 \ni u \mapsto m(y, u)$ are locally Lipschitz uniformly in $y \in \Gamma$, $x \in \Omega_\pm$.

(4) If $d = 3$, then $k_\pm$ are of the form $k_\pm(x, u_\pm) = \kappa_\pm(x, u_\pm)\varkappa_\pm(x)$. The functions $\kappa_\pm \colon \Omega_\pm \times \mathbb{R} \to \mathbb{R}$ are scalar, satisfy 2.2(3) and for all $u_\pm \in \mathbb{R}$, we have $\kappa_\pm(\cdot, u_\pm) \in C^0(\Omega_\pm)$ with $\underline{k} \leq \kappa_\pm(\cdot, u_\pm)$. The functions $\varkappa_\pm \colon \Omega_\pm \to \mathbb{R}_{\geq 0}^{3 \times 3}$ satisfy 2.2(1) and are uniformly continuous on $\Omega_\pm$.

Some examples of coefficients k, m that satisfy Assumption 2.2 are in Subsections 5.1 and 1.1. Assumption 2.2(4) is used to guarantee that if $d = 3$, the non-autonomous operator $\mathcal{A}_{u(t)}$ that defines the system, does not change its domain of definition, see Lemma 3.2. Note that the condition $\varkappa_\pm \in C(\overline{\Omega}_\pm)^{3 \times 3}$ may be relaxed considerably, e.g. to hold only piecewise on layers. For a detailed discussion of necessary and sufficient conditions for this property, we refer to [11].

2.2. Bilinear form and elliptic operator

The dissipation in (1.1) – (1.3) across Γ is governed by the transmission coefficient matrix $\mathbf{m}$ given by

$$\mathbf{m} = \begin{pmatrix} m_+ + m_\Gamma & -m_\Gamma & -m_+ \\ -m_\Gamma & m_- + m_\Gamma & -m_- \\ -m_+ & -m_- & m_+ + m_- \end{pmatrix}.$$

With Assumption 2.2(2), $\mathbf{m}$ is positive semi-definite and for $r = (r_+, r_-, r_\Gamma) \in \mathbb{R}^3$,

$$r \cdot \mathbf{m}r = 0 \text{ a.e., if and only if } r_+ = r_- = r_\Gamma. \tag{2.7}$$

Let $-\infty < l \leq L < +\infty$. For fixed $u \in C_l^L$, define the bilinear form

$$\mathfrak{a}_u : \mathrm{W}^{1,2,2} \times \mathrm{W}^{1,2,2} \to \mathbb{R}$$

by

$$\mathfrak{a}_u(\psi, \varphi) := \mathfrak{l}_u(\psi, \varphi) + \mathfrak{m}_u(\psi, \varphi),$$

where

$$\mathfrak{l}_u(\psi, \varphi) := \int_{\Omega_+} \nabla \psi_+ \cdot k_+(u_+) \nabla \varphi_+ \, \mathrm{d}x + \int_{\Omega_-} \nabla \psi_- \cdot k_-(u_-) \nabla \varphi_- \, \mathrm{d}x$$

$$+ \int_\Gamma \nabla_\Gamma \psi_\Gamma \cdot k_\Gamma(u) \nabla_\Gamma \varphi_\Gamma \, \mathrm{d}\mathcal{H}_{d-1}$$

$$=: \mathfrak{l}_{u,+}(\psi_+, \varphi_+) + \mathfrak{l}_{u,-}(\psi_-, \varphi_-) + \mathfrak{l}_{u,\Gamma}(\psi_\Gamma, \varphi_\Gamma),$$

and

$$\mathfrak{m}_u(\psi, \varphi) = \int_\Gamma \mathrm{tr}_\Gamma \, \psi \cdot \mathbf{m} \mathrm{tr}_\Gamma \, \varphi \, \mathrm{d}\mathcal{H}_{d-1}.$$

By (2.1), the form $\mathfrak{a}_u$ is well-defined and continuous. Due to (2.7) and Assumption 2.2,

$$\mathfrak{a}_u(\varphi, \varphi) \geq 0 \quad \text{and} \quad \mathfrak{a}_u(\varphi, \varphi) = 0 \text{ if and only if } \varphi_+ = \varphi_- = \varphi_\Gamma \equiv \text{const.} \tag{2.8}$$

The form $\mathfrak{a}_u$ induces an operator $\mathcal{A}_u : \mathrm{W}^{1,2,2} \to \mathrm{W}_0^{-1,2,2}$ by

$$\mathcal{A}_u(\psi)(\varphi) := \mathfrak{a}_u(\psi, \varphi), \quad \text{for all } \psi, \varphi \in W^{1,2,2}.$$

For $q, q_\Gamma \in [2, \infty)$, let $\mathcal{A}_u^{q,q_\Gamma}$ be the maximal restriction of $\mathcal{A}_u$ to $\mathrm{W}_0^{-1,q,q_\Gamma}$ with

$$\mathrm{dom}(\mathcal{A}_u^{q,q_\Gamma}) = \{ \psi \in \mathrm{W}_0^{-1,q,q_\Gamma} \cap \mathrm{W}^{1,2,2} : \mathcal{A}_u \psi \in \mathrm{W}_0^{-1,q,q_\Gamma} \},$$

and $\mathcal{A}_u^{q,q_\Gamma} = \mathcal{A}_u|_{\mathrm{dom}(\mathcal{A}_u^{q,q_\Gamma})}$. It will be shown in Lemma 3.2 below that $\mathrm{dom}(\mathcal{A}_u^{q,q_\Gamma}) = \mathrm{W}^{1,q,q_\Gamma}$. Let $\mathcal{L}_u^{q,q_\Gamma}$ be the divergence operator in $\mathrm{W}_0^{-1,q,q_\Gamma}$ analogously induced by $\mathfrak{l}_u$ and let $\mathcal{L}_{u,+}^{q,q_\Gamma}$, $\mathcal{L}_{u,-}^{q,q_\Gamma}$ and $\mathcal{L}_{u,\Gamma}^{q,q_\Gamma}$ be the Neumann operators induced by $\mathfrak{l}_{u,+}$, $\mathfrak{l}_{u,-}$ and $\mathfrak{l}_{u,\Gamma}$ on the domains Ω_+, Ω_- and Γ, respectively. We write $\mathcal{M}_u^{q,q_\Gamma}$ for the bounded transmission operator given by

$$\mathcal{M}_u^{q,q_\Gamma}(\psi)(\varphi) := \mathfrak{m}_u(\psi, \varphi), \qquad \psi \in \mathrm{dom}(\mathcal{L}_u^{q,q_\Gamma}), \varphi \in \mathrm{W}^{1,q',q_\Gamma'}, \tag{2.9}$$

so that

$$\mathcal{A}_u^{q,q_\Gamma} = \mathcal{L}_u^{q,q_\Gamma} + \mathcal{M}_u^{q,q_\Gamma}.$$

The external forces and inhomogeneous Neumann boundary conditions f, g, h in (1.1)–(1.3) are realized as a $\mathrm{W}_0^{-1,q,q_\Gamma}$-functional $\mathcal{F}(u)$ with components $\mathcal{F}_+(u) \in \mathrm{W}_0^{-1,q}(\Omega_+)$, $\mathcal{F}_-(u) \in \mathrm{W}_0^{-1,q}(\Omega_-)$ and $\mathcal{F}_\Gamma(u) \in \mathrm{W}_0^{-1,q_\Gamma}$ given by

$$\mathcal{F}_+(u)(\varphi_+) = \int_{\Omega_+} f_+(u_+)\varphi_+ \, \mathrm{d}x + \int_\Gamma g_+(u)\mathrm{tr}_\Gamma \, \varphi_+ \, \mathrm{d}\mathcal{H}_{d-1} + \int_{\partial\Omega_+ \setminus \Gamma} h_+(u_+)\mathrm{tr}_{\partial\Omega_+ \setminus \Gamma} \, \varphi_+ \, \mathrm{d}\mathcal{H}_{d-1}$$

$$\mathcal{F}_-(u)(\varphi_-) = \int_{\Omega_-} f_-(u_-)\varphi_- \, \mathrm{d}x + \int_\Gamma g_-(u)\mathrm{tr}_\Gamma \, \varphi_- \, \mathrm{d}\mathcal{H}_{d-1} + \int_{\partial\Omega_- \setminus \Gamma} h_-(u_-)\mathrm{tr}_{\partial\Omega_- \setminus \Gamma} \, \varphi_- \, \mathrm{d}\mathcal{H}_{d-1}$$

$$\mathcal{F}_\Gamma(u)(\varphi_\Gamma) = \int_\Gamma f_\Gamma(u)\varphi_\Gamma \, \mathrm{d}\mathcal{H}_{d-1} + \int_{\partial\Gamma} h_\Gamma(u_\Gamma)\mathrm{tr}_{\partial\Gamma} \, \varphi_\Gamma \, \mathrm{d}\mathcal{H}_{d-2},$$

for all $\varphi \in \mathrm{W}^{1,q',q_\Gamma'}$. Using trace and embedding results, it follows that $\mathcal{F}(u)$ is well-defined, if

$$f(u) \in \mathrm{L}^{p,p_\Gamma}, \text{ and } g_\pm(u), h_\pm(u) \in L^p(\Gamma),$$

where

$$p > \frac{d}{d+1-\frac{d}{q'}}, \ p_\Gamma > \frac{d-1}{d-\frac{d-1}{q'_\Gamma}} \text{ and } \rho > \frac{d-1}{d-\frac{d}{q'}}.$$

If $d = 2$, then $p_\Gamma > 1$ is sufficient. If $d = 3$, then $h_\Gamma(u_\Gamma) \in L^{\rho_\Gamma}(\partial\Gamma)$ with $\rho_\Gamma > 1$ is a sufficient condition. With the assumptions in this section, the system (1.1), (1.2) and (1.3) can be considered as the quasilinear abstract Cauchy problem

$$\dot{u}(t) + \mathcal{A}_{u(t)} u(t) = \mathcal{F}(u(t)) \in W_0^{-1,q,q_\Gamma}, \quad u(0) = u_0, \tag{2.10}$$

for $q, q_\Gamma \geq 2$.

2.3. Maximal parabolic regularity and embedding theorems

In the proof of the main result, we use non-autonomous maximal parabolic regularity of $\mathcal{A}_{u(\cdot)}^{q,q_\Gamma}$ in W_0^{-1,q,q_Γ}. To make this more precise, this subsection contains some definitions and preliminary results. For $T > 0$, let in the following $J_T = (0, T)$. For two Banach spaces X, Y that form an interpolation couple, $(X, Y)_{\theta,p}$ denotes the real interpolation spaces with parameters $\theta \in (0, 1)$, $p \in [1, \infty]$.

Definition 2.3. Let $1 < r < \infty$, let X be a Banach space and assume that B is a closed operator in X with dense domain $\mathrm{dom}(B) \subset X$, equipped with the graph norm. We say that B satisfies maximal $L^r(J_T; X)$-regularity if for all $u^0 \in (\mathrm{dom}(B), X)_{1-\frac{1}{r},r}$ and $f \in L^r(0, T; X)$ there is a unique solution

$$u \in L^r(J_T; \mathrm{dom}(B)) \cap W^{1,r}(J_T; X)$$

of the abstract Cauchy problem

$$\begin{cases} \dot{u} + Bu &= f, \\ u(0) &= u^0, \end{cases}$$

posed in X, satisfying

$$\|\dot{u}\|_{L^r(J_T;X)} + \|Bu\|_{L^r(J_T;X)} \leq C(\|u^0\|_{(\mathrm{dom}(B),X)_{1-\frac{1}{r},r}} + \|f\|_{L^r(J_T;X)})$$

with a constant $C > 0$ independent of u^0 and f (see e.g. [1, Ch. III.1]).

Note that the notion of maximal $L^r(J_T; X)$-regularity is independent of $1 < r < \infty$ and $T > 0$, cf. [14]. In the following, for $q, q_\Gamma \geq 2$, $1 < r < \infty$ and given $u \in C^{0,0}$, we consider maximal regularity of $\mathcal{A}_u^{q,q_\Gamma}$. Thus,

$$\mathrm{MR}_{q,q_\Gamma}^r := L^r(J_T; \mathrm{dom}(\mathcal{A}_u^{q,q_\Gamma})) \cap W^{1,r}(J_T; W_0^{-1,q,q_\Gamma})$$

is the corresponding solution space and

$$\mathrm{X}_{q,q_\Gamma}^r := (\mathrm{dom}(\mathcal{A}_u^{q,q_\Gamma}), W_0^{-1,q,q_\Gamma})_{1-\frac{1}{r},r}$$

is the corresponding time trace space.

In Lemma 3.2 below, it is shown that there are $q > d$, $q_\Gamma > d - 1$, such that $\mathrm{dom}(\mathcal{A}_u^{q,q_\Gamma}) = W^{1,q,q_\Gamma}$. The following lemma summarizes useful embeddings for the corresponding function spaces.

Lemma 2.4. *If* $\mathrm{dom}(\mathcal{A}_u^{q,q_\Gamma}) = W^{1,q,q_\Gamma}$, *then*

(1) for $\alpha \leq 1 - \frac{d}{q}$ *and* $\alpha_\Gamma \leq 1 - \frac{d-1}{q_\Gamma}$,

$$\mathrm{dom}(\mathcal{A}_u^{q,q_\Gamma}) \hookrightarrow C^{\alpha,\alpha_\Gamma}, \tag{2.11}$$

(2) for any $1 < r < \infty$,

$$\mathrm{MR}_{q,q_\Gamma}^r \hookrightarrow C^0(J_T; X_{q,q_\Gamma}^r). \tag{2.12}$$

If $q > d$, $q_\Gamma > d - 1$, *and* $r > \max(\frac{2q}{q-d}, \frac{2q_\Gamma}{q_\Gamma-d+1})$, *then*

$$X_{q,q_\Gamma}^r \hookrightarrow C^{\beta,\beta_\Gamma}, \tag{2.13}$$

where $0 < \beta \leq 1 - \frac{d}{q} - \frac{2}{r}$ *and* $0 < \beta_\Gamma \leq 1 - \frac{d-1}{q_\Gamma} - \frac{2}{r}$.

(3) for $q > d$, $q_\Gamma > d - 1$, *let* $0 < \delta < \min(\frac{q-d}{2q}, \frac{q_\Gamma-d+1}{2q_\Gamma})$ *and* $r > \max(\frac{2q}{q-2\delta q-d}, \frac{2q_\Gamma}{q_\Gamma-2\delta q_\Gamma-d+1})$, *then*

$$\mathrm{MR}_{q,q_\Gamma}^r \hookrightarrow C^\delta(J_T; C^{\gamma,\gamma_\Gamma}) \tag{2.14}$$

with $0 < \gamma \leq 1 - \frac{d}{q} - \frac{2}{r} - 2\delta$ *and* $0 < \gamma_\Gamma \leq 1 - \frac{d-1}{q_\Gamma} - \frac{2}{r} - 2\delta$. *In particular, the embedding*

$$\mathrm{MR}_{q,q_\Gamma}^r \hookrightarrow C^0(J_T; C^{0,0}) \tag{2.15}$$

is compact.

Proof. Note that Ω_+, Ω_- and Γ are sufficiently regular for embedding and interpolation results to work "as usual". The first embedding (2.11) is standard, cf. e.g. [39, 2.8.1(c)]. For embedding (2.12), cf. [1, Section III.4.10]. Embedding (2.13) follows by definition of X_{q,q_Γ}^r, combining e.g. the interpolation result [39, p. 186, (14)] and the embedding [39, 2.8.1]. From [13, Lemma 3.4(b)], it follows that

$$\mathrm{MR}_{q,q_\Gamma}^r \hookrightarrow C^\delta(J_T; (W_0^{-1,q,q_\Gamma}, W^{1,q,q_\Gamma})_{\theta,1})$$

with $0 < \theta \leq 1 - \frac{1}{r} - \delta$. Embedding (2.14) then follows again by combining [39, p. 186, (14)] and [39, 2.8.1]. □

For uniqueness, an assumption on the dependence of $\mathcal{F}$ of u is needed. By embedding, the assumption is satisfied, e.g. if the dependence of $\mathcal{F}$ of u in an L^p-norm is locally Lipschitz (for example, it is straightforward to see that semilinear terms of reaction-diffusion type with arbitrary powers are included [9]).

Assumption 2.5. Given $1 < r < \infty$ and $q, q_\Gamma > 2$, the function $\mathcal{F}: X^r_{q,q_\Gamma} \to W_0^{-1,q,q_\Gamma}$ is boundedly Lipschitz in the sense that for all $\tilde{L} > 0$, there exists a constant $C_{\tilde{L}} > 0$ such that for all $u_1, u_2 \in X^r_{q,q_\Gamma}$ with $\|u_1\|_{X^r_{q,q_\Gamma}}, \|u_2\|_{X^r_{q,q_\Gamma}} \le \tilde{L}$,

$$\|\mathcal{F}(u_1) - \mathcal{F}(u_2)\|_{W_0^{-1,q,q_\Gamma}} \le C_{\tilde{L}} \|u_1 - u_2\|_{X^r_{q,q_\Gamma}}.$$

3. Global existence and uniqueness

The main result of the paper is global existence and uniqueness of solutions of (2.10). For local well-posedness, it is sufficient that $\mathcal{F}$ satisfies Assumption 2.5. For global existence, it is required that $\mathcal{F}$ preserves the L^∞-bounds for (2.10). This requirement on $\mathcal{F}$ is defined more precisely in Step (3) in the proof of Theorem 3.1. Examples of $\mathcal{F}$ that satisfy these assumptions are given in Corollary 3.6 and it is shown that this includes coupled volume-surface reaction-diffusion in [9] (the case $\mathcal{F} = 0$ corresponds to the original gradient structure).

Theorem 3.1. *There exist $q > d$, $q_\Gamma > d - 1$ such that for all $r > \max(\frac{2q}{q-d}, \frac{2q_\Gamma}{q_\Gamma - d + 1})$, $u^0 \in X^r_{q,q_\Gamma}$, $T > 0$ and $\mathcal{F}$ satisfying Assumption 2.5 and preserving L^∞-bounds, there is a unique global solution*

$$u \in W^{1,r}(J_T; W_0^{-1,q,q_\Gamma}) \cap L^r(J_T; W^{1,q,q_\Gamma})$$

of (2.10). In particular, the solution is Hölder continuous in time and space,

$$u \in C^\delta(J_T; C^{\gamma,\gamma_\Gamma}),$$

with $\delta, \gamma, \gamma_\Gamma$ as in Lemma 2.4 and (2.10) is well-posed.

The proof of Theorem 3.1 is divided into four steps:

(1) provisional reduction to bounded coefficients,
(2) preliminary results on the linearized non-autonomous problem,
(3) a priori L^∞-bounds,
(4) Schaefer argument and proof of the theorem.

(1) *Provisional reduction to bounded coefficients* By Lemma 2.4, $u^0 \in C^{\beta,\beta_\Gamma} \subset C^{0,0}$. Let $-\infty < l_0 \le L_0 < +\infty$ be such that $u^0 \in C_{l_0}^{L_0}$. Define

$$[f]_l^L(x) := \begin{cases} L, & f(x) \ge L, \\ l, & f(x) \le l, \\ f(x), & \text{otherwise,} \end{cases}$$

and let $L := L_0 + 1$, $l = l_0/2$. Instead of the coefficient functions k and m, consider

$$k_l^L(\cdot, u(\cdot)) = k(\cdot, [u]_l^L(\cdot)) \quad \text{and} \quad m_l^L(\cdot, u(\cdot)) = m(\cdot, [u]_l^L(\cdot)) \tag{3.1}$$

in the following. In Step (4) below, it is shown that $k_l^L = k$ and $m_l^L = m$ along the orbits of u^0. Since the solution is unique and regular and the dependence of k, m of the solution is Lipschitz, this is sufficient to prove the theorem. Clearly, if k, m satisfy Assumption 2.2, then also k_l^L, m_l^L satisfy Assumption 2.2 and the bounds in 2.2(1) and 2.2(2) hold uniformly in $u \in C^{0,0}$ for k_l^L, m_l^L.

(2) *Preliminary results on the linearized non-autonomous problem*   In this step of the proof and in Step (3), using Step (1), assume additionally that all coefficient functions are such that the bounds in 2.2(1) and 2.2(2) hold uniformly in $u \in C^{0,0}$.

Lemma 3.2. *There exist $q > d$ and $q_\Gamma > d - 1$ such that for any $u \in C^0(J_T; C^{0,0})$, for all $t \in \overline{J_T}$, for any $\lambda > 0$, the operator $\mathcal{A}_{u(t)}^{q,q_\Gamma} + \lambda$ is an isomorphism*

$$\mathcal{A}_{u(t)}^{q,q_\Gamma} + \lambda \colon W^{1,q,q_\Gamma} \to W_0^{-1,q,q_\Gamma}. \tag{3.2}$$

Proof. First note that $\mathcal{A}_{u(t)}^{q,q_\Gamma} \colon \mathrm{dom}(\mathcal{A}_{u(t)}^{q,q_\Gamma}) \to W_0^{-1,q,q_\Gamma}$ is well-defined for all $t \in \overline{J_T}$. In the case $d = 2$, by the Lax-Milgram theorem, the claim holds for $q = q_\Gamma = 2$. By Sneiberg's theorem [37], the isomorphism property extrapolates to a neighbourhood of $W^{1,2,2}$ in the complex interpolation scale $[W^{1,p,p_\Gamma}, W^{1,p',p_\Gamma'}]_{1/p} = W^{1,2,2_\Gamma}$, $1 < p, p_\Gamma < \infty$, see [24].

If $d = 3$, then Assumption 2.2(4) holds. If $\kappa_\pm \equiv 1$, then $k_\pm = \varkappa_\pm$ is independent of u and then by [15, Theorem 1.1], [25, Lemma 6.5], there is a $q > 3$ such that the isomorphism property $\mathcal{L}_{u(t),\pm} + \lambda \colon W^{1,q}(\Omega_\pm) \to W_0^{-1,q}(\Omega_\pm)$ holds true. Using the same extrapolation argument as in the case $d = 2$, there exists a $q_\Gamma > 2$ such that $\mathcal{L}_{u(t),\Gamma} + \lambda \colon W^{1,q_\Gamma}(\Gamma) \to W_0^{-1,q_\Gamma}(\Gamma)$ is an isomorphism. In [11, Theorem 6.3] it was shown that the domains of $\mathcal{L}_{u(t),\pm}, \mathcal{L}_{u(t),\Gamma}$ remain unchanged by a scalar multiplicative perturbation $\kappa_\pm \in C^0(\Omega_\pm)$ that is positively bounded from below. This proves the result for the operators $\mathcal{L}_{u(t)}, t \in \overline{J_T}$. By relative boundedness of $\mathcal{M}_{u(t)}$, [10, Lemma 3.4], the domains of $\mathcal{L}_{u(t)} + \lambda$ and $\mathcal{A}_{u(t)} + \lambda$ coincide. This proves the claim. □

Lemma 3.3. *Let $2 \leq q, q_\Gamma < \infty$, $1 < r < \infty$ and let $u \in C^0(J_T; C^{0,0})$. Then for all $t \in \overline{J_T}$, $\mathcal{A}_{u(t)}^{q,q_\Gamma}$ has maximal $L^r(J_T; W_0^{-1,q,q_\Gamma})$-regularity.*

Proof. The result was shown in [10] if Γ is flat. It remains to check the maximal regularity of the Neumann operator $\mathcal{L}_{u(t),\Gamma}$ on C^1-boundaries. This follows from maximal regularity for flat domains [24], using the usual localization methods, i.e. exploiting that the property of maximal regularity is preserved under perturbations that occur when locally flattening the domain and straightening the boundary with respect to a sufficiently fine covering and a corresponding partition of unity, see [7] for the general strategy and [8] for this argument in a similar context. □

Lemma 3.4. *Let $w \in C^0(J_T; C^{0,0})$, q, q_Γ be as in Lemma 3.2. Then for every r and $u^0 \in X_{q,q_\Gamma}^r$ as in Theorem 3.1, for all $f \in L^r(J_T; W_0^{-1,q,q_\Gamma})$, there exists a unique global solution $v \in MR_{q,q_\Gamma}^r$ of*

$$\dot{v}(t) + \mathcal{A}_{w(t)}^{q,q_\Gamma} v(t) = f(t), \quad in \ W_0^{-1,q,q_\Gamma}, \tag{3.3}$$
$$v(0) = u^0.$$

The solution operator

$$(\partial_t + \mathcal{A}_{w(\cdot)}^{q,q_\Gamma})^{-1} \colon f \in L^r(J_T; W_0^{-1,q,q_\Gamma}) \mapsto v \in MR_{q,q_\Gamma}^r \qquad (3.4)$$

is bounded.

Proof. For two Banach spaces X, Y, let $\mathcal{B}(X, Y)$ denote the space of bounded linear operators $B \colon X \to Y$. By continuity of w, k_l^L and m_l^L and by Lemma 3.2, the map $J_T \ni t \mapsto \mathcal{A}_w^{q,q_\Gamma}(t) \in \mathcal{B}(W^{1,q,q_\Gamma}, W_0^{-1,q,q_\Gamma})$ is uniformly continuous. By Lemma 3.4, for all $t \in \overline{J_T}$, $\mathcal{A}_w^{q,q_\Gamma}(t)$ has maximal $L^r(J_T; W_0^{-1,q,q_\Gamma})$-regularity, so existence and boundedness of the solution operator follow from [34, Theorem 2.5]. $\quad\square$

(3) L^∞-*bounds on* u If $\mathcal{F} = 0$, uniform $L^{\infty,\infty}$-bounds on u can be proved by a bulk-interface comparison principle. With respect to results for the bulk problem, the point is to show that the nonlinear bulk-interface interaction terms derived from a generalized gradient structure preserve this property.

Lemma 3.5. *Let* r, q, q_Γ *as in Theorem 3.1,* $\mathcal{F} = 0$ *and* $u^0 \in X_{q,q_\Gamma}^r$ *with* $u^0 \in C_l^L$ *for some* $-\infty < l \le L < +\infty$. *Assume that* $u \in MR_{q,q_\Gamma}^r$ *is a solution of* (2.10). *Then for all* $t \in \overline{J_T}$, $u(t) \in C_l^L$.

Proof. Define $\zeta_l(t) = [(u(t) - l)^-]$ and $\zeta^L(t) = [(L - u(t))^-]$, where

$$[f^-](x) := \begin{cases} 0, & f(x) \ge 0, \\ -f(x), & f(x) < 0. \end{cases}$$

Since $[\cdot^-]$ is Lipschitz and $r, q, q_\Gamma \ge 2$, we have $\zeta_l, \zeta^L \in L^r(J_T; W^{1,q,q_\Gamma}) \hookrightarrow L^{r'}(J_T; W^{1,q',q_\Gamma'})$ with

$$\nabla \zeta^L(t, x) = \begin{cases} 0, & u(t, x) \le L, \\ \nabla u(t, x), & u(t, x) > L, \end{cases}$$

and $\zeta_l(0) = \zeta^L(0) \equiv 0$. Thus, for all $s \in \overline{J_T}$, testing (2.10) with ζ^L in space and time gives $\zeta^L \equiv 0$ as

$$\int_0^s \dot{u}(t)(\zeta^L(t)) \, dt = \frac{1}{2} \|\zeta^L(s)\|_{L^{2,2}}^2 \ge 0$$

and

$$\int_0^s \mathcal{A}_{u(t)}^{q,q_\Gamma} u(t)(\zeta^L(t)) \, dt = \int_0^s \mathfrak{l}_{u(t)}(u(t), \zeta^L(t)) + \mathfrak{m}_{u(t)}(u(t), \zeta^L(t)) \, dt \ge 0. \qquad (3.5)$$

To show the estimate from below in (3.5), note that

$$\int_0^s \mathfrak{l}_{u(t)}(u(t), \zeta^L(t)) \, dt = \int_0^s \mathfrak{l}_{u(t)}(\zeta^L(t), \zeta^L(t)) \, dt \geq 0$$

as k is bounded below by $\underline{k}$ and that

$$\int_0^s \mathfrak{m}_{u(t)}(u(t), \zeta^L(t)) \, dt = \int_0^s \int_\Gamma m_+(u)(u_+ - u_\Gamma)(\zeta_+^L - \zeta_\Gamma^L)(t) \tag{3.6}$$

$$+ m_-(u)(u_- - u_\Gamma)(\zeta_-^L - \zeta_\Gamma^L)(t)$$

$$+ m_\Gamma(u)(u_+ - u_-)(\zeta_+^L - \zeta_-^L)(t) \, d\mathcal{H}_{d-1} \, dt,$$

where m is bounded below by $\underline{m}$ and where

$$\int_\Gamma (u_+ - u_\Gamma)(\zeta_+^L - \zeta_\Gamma^L)(t) \, d\mathcal{H}_{d-1} = \int_{\{x \in \Gamma : \, u_+(x) > L > u_\Gamma(x)\}} (u_+ - u_\Gamma)(u_+ - L)(t) \, d\mathcal{H}_{d-1}$$

$$+ \int_{\{x \in \Gamma : \, u_+(x) < L < u_\Gamma(x)\}} (u_+ - u_\Gamma)(L - u_\Gamma)(t) \, d\mathcal{H}_{d-1}$$

$$+ \int_{\{x \in \Gamma : \, u_+(x), u_\Gamma(x) > L\}} (u_+ - u_\Gamma)(u_+ - u_\Gamma)(t) \, d\mathcal{H}_{d-1} \geq 0,$$

and non-negativity of the remaining terms on the right-hand-side of (3.6) follows analogously. The proof of the lower bound, i.e. $\zeta^l \equiv 0$ follows analogously by testing (2.10) with ζ^l. $\quad\square$

If $\mathcal{F} \neq 0$, the proof of Theorem 3.1 still requires that $\mathcal{F}$ preserves L^∞-bounds in a suitable sense. More precisely, it is straightforward to see that the proof of Lemma 3.5 still works if $\mathcal{F}$ is such that for given $u_0 \in C_l^L$, there are constants $l_F < L_F \in \mathbb{R}$ such that for all $u \in \mathrm{MR}_{q,q_\Gamma}^r$ such that $u(0) = u_0$, for all $s \in J_T$,

$$\int_0^s \mathcal{F}(u(t))(\zeta^{L_f}(t)) \, dt \leq 0 \quad \text{and} \quad \int_0^s \mathcal{F}(u(t))(\zeta_{l_f}(t)) \, dt \geq 0.$$

It is shown in [9] how chemical reaction rates of type (1.6) fit into the framework of Theorem 3.1. Another general example is given by terms of Allen-Cahn-type, treated in the following corollary. A concrete example is

$$f_+(u_+) = -u_+^3, \, g_+(u_+) = (1 - u_+)^3, \quad f_\Gamma(u_\Gamma) = -u_\Gamma^5. \tag{3.7}$$

Corollary 3.6. *Let $\mathcal{F}$ satisfy Assumption 2.5 and let all the components φ of $\mathcal{F}$, e.g. $\varphi = f_+, g_-, \dots$ in (1.1)–(1.3) be independent of $x \in \Omega_+, \Omega_-$, $y \in \Gamma$, respectively. Assume that φ are continuously differentiable in u and that $g_\pm$ depend only on $u_\pm$, respectively, whereas f_Γ depends only on u_Γ. Assume that all φ satisfy the dissipativity condition*

$$\liminf_{|v| \to \infty} -\varphi'(v) > 0. \tag{3.8}$$

Then, under the assumptions of Lemma 3.5, given a solution $u \in \mathrm{MR}^r_{q,q_\Gamma}$ of (2.10), there are constants $-\infty < l_f \leq L_f < +\infty$, such that for all $t \in \overline{J_T}$, $u(t) \in \mathrm{C}^{L_f}_{l_f}$.

Proof. Condition (3.8) guarantees that for every component φ, there exist constants $-\infty < l_\varphi \leq L_\varphi < +\infty$ such that $\varphi(v) > 0$ for all $v < l_\varphi$ and $\varphi(v) < 0$ for all $v > L_\varphi$. Let $l_f := \min_\varphi(l_\varphi)$ and $L^f := \max_\varphi(L_\varphi)$. In the choice of test functions ζ_l, ζ^L in the proof of Lemma 3.5, replace l, L by l_f, L_f. It is then straightforward to check that for all $s \in \overline{J_T}$, $\int_0^s \mathcal{F}(u(t))(\zeta^{L_f}(t)) \, \mathrm{d}t \leq 0$ and that $\int_0^s \mathcal{F}(u(t))(\zeta_{l_f}(t)) \, \mathrm{d}t \geq 0$. Combined with the calculations in the proof of Lemma 3.5, this proves the claim. $\square$

(4) *Schaefer argument and proof of Theorem 3.1* Let q, q_Γ be given by Lemma 3.2 and let r and $u^0 \in \mathrm{X}^r_{q,q_\Gamma}$ be given as in Theorem 3.1. By embedding (2.13), $u^0 \in \mathrm{C}^L_l$ for some $-\infty < l \leq L < +\infty$. In the following, let

$$C^0_{u^0}(J_T; \mathrm{C}^{0,0}) := \{u \in C^0(J_T; \mathrm{C}^{0,0}) : u(0) = u^0\}. \tag{3.9}$$

Define

$$\mathcal{T} : C^0_{u^0}(J_T; \mathrm{C}^{0,0}) \to C^0_{u^0}(J_T; \mathrm{C}^{0,0})$$

by $\mathcal{T}w = v \in \mathrm{MR}^r_{q,q_\Gamma}$ the solution of (3.3) with $v(0) = u^0$ given by Lemma 3.4. By embedding (2.15), Lemma 3.4 and Assumption 2.5,

$$\mathcal{T}w = \mathrm{Id}_{\mathrm{MR}^r_{q,q_\Gamma} \to C^0_{u^0}(J_T; \mathrm{C}^{0,0})}(\partial_t + \mathcal{A}^{q,q_\Gamma}_{w(\cdot)})^{-1}(\mathcal{F}(w))$$

is well-defined and compact. A fixed point of $\mathcal{T}$ would solve (2.10). To obtain existence of a fixed point by Schaefer's Theorem [16, Theorem 9.2.4], it suffices to show that

(1) $\mathcal{T}$ is continuous, and that
(2) the Schaefer set

$$S := \{u \in C^0_{u^0}(J_T; \mathrm{C}^{0,0}) : u = \lambda \mathcal{T}(u) \text{ for some } 0 \leq \lambda \leq 1\}$$

is bounded.

To show continuity of $\mathcal{T}$, we show continuity of the map

$$C^0_{u^0}(J_T; \mathrm{C}^{0,0}) \ni w \mapsto (\partial_t + \mathcal{A}^{q,q_\Gamma}_{w(\cdot)})^{-1} \in \mathcal{B}(L^r(J_T; \mathrm{W}^{-1,q,q_\Gamma}_0); \mathrm{MR}^r_{q,q_\Gamma}) =: \mathcal{B}^r_{q,q_\Gamma}.$$

This follows from bounded Lipschitzianity of the map $w \mapsto \mathcal{A}^{q,q_\Gamma}_{w(\cdot)}$ and continuity of the inversion. A detailed proof is the following: for given $w \in C^0_{u^0}(J_T; \mathrm{C}^{0,0})$, there are constants $-\infty < l_0 < L_0 < +\infty$ (possibly dependent on T), such that $w \in C^0_{u^0}(J_T; \mathrm{C}^{L_0}_{l_0})$. Define

$$C_w := \|(\partial_t + \mathcal{A}_{w(\cdot)}^{q,q_\Gamma})^{-1}\|_{\mathcal{B}_{q,q_\Gamma}^r}$$

(Lemma 3.4) and let C_δ be a Lipschitz constant for k and m on $(\mathbb{R}_{l_0-\delta}^{L_0+\delta})^3$ with $\delta > 0$ (δ possibly small such that Assumption 2.2(3) applies). Let $w_n \to w$ in $C_{u^0}^0(J_T; C^{0,0})$ and let n be so large that $w_n \in C_{u^0}^0(J_T; C_{l_0-\delta}^{L_0+\delta})$ and $\|w_n - w\|_{C_{u^0}^0(J_T;C^{0,0})} \leq \frac{1}{2C_w C_\delta T^{1/r}}$. Then by Assumption 2.2(3) and the definition of $\mathcal{A}_w^{q,q_\Gamma}$,

$$\|\mathcal{A}_{w(\cdot)}^{q,q_\Gamma} - \mathcal{A}_{w_n(\cdot)}^{q,q_\Gamma}\|_{L^r(J_T;\mathcal{B}(W^{1,q,q_\Gamma},W_0^{-1,q,q_\Gamma}))} \leq C_\delta T^{1/r}\|w - w_n\|_{C_{u^0}^0(J_T;C^{0,0})}, \qquad (3.10)$$

and thus

$$\|(\partial_t + \mathcal{A}_{w(\cdot)}^{q,q_\Gamma})^{-1} - (\partial_t + \mathcal{A}_{w_n(\cdot)}^{q,q_\Gamma})^{-1}\|_{\mathcal{B}_{q,q_\Gamma}^r}$$

$$= \|(\partial_t + \mathcal{A}_{w(\cdot)}^{q,q_\Gamma})^{-1}(\partial_t + \mathcal{A}_{w_n(\cdot)}^{q,q_\Gamma} - \partial_t + \mathcal{A}_{w(\cdot)}^{q,q_\Gamma})(\partial_t + \mathcal{A}_{w_n(\cdot)}^{q,q_\Gamma})^{-1}\|_{\mathcal{B}_{q,q_\Gamma}^r}$$

$$\leq \|(\partial_t + \mathcal{A}_{w(\cdot)}^{q,q_\Gamma})^{-1}(\mathcal{A}_{w_n(\cdot)}^{q,q_\Gamma} - \mathcal{A}_{w(\cdot)}^{q,q_\Gamma})\|_{\mathcal{B}(\mathrm{MR}_{q,q_\Gamma}^r;\mathrm{MR}_{q,q_\Gamma}^r)} \cdot$$

$$\cdot \|(\partial_t + \mathcal{A}_{w(\cdot)}^{q,q_\Gamma})^{-1} + (\partial_t + \mathcal{A}_{w_n(\cdot)}^{q,q_\Gamma})^{-1} - (\partial_t + \mathcal{A}_{w(\cdot)}^{q,q_\Gamma})^{-1}\|_{\mathcal{B}_{q,q_\Gamma}^r}$$

$$\leq C_w^2 C_\delta T^{1/r}\|w_n - w\|_{C_{u^0}^0(J_T;C^{0,0})} + \frac{1}{2}\|(\partial_t + \mathcal{A}_{w(\cdot)}^{q,q_\Gamma})^{-1} - (\partial_t + \mathcal{A}_{w_n(\cdot)}^{q,q_\Gamma})^{-1}\|_{\mathcal{B}_{q,q_\Gamma}^r},$$

so $\mathcal{T}$ is continuous.

To show (2), note that if $u_\lambda = \lambda\mathcal{T}(u_\lambda)$ for some $0 < \lambda \leq 1$, then by definition of $\mathcal{T}$, $u_\lambda \in \mathrm{MR}_{q,q_\Gamma}^r$ and u_λ satisfies (2.10) with initial value $u_\lambda(0) = \lambda u^0$ and right-hand-side $\lambda\mathcal{F}(u_\lambda)$. Thus, if $\mathcal{F} \equiv 0$ or if $\mathcal{F}$ is as in Corollary 3.6, then S is bounded.

In addition, the L^∞-bounds are such that $k_l^L = k$ and $m_l^L = m$ along orbits of u^0, justifying step (1) a posteriori with possible adjustments to the choice of l and L by Corollary 3.6. This concludes the proof of existence in Theorem 3.1.

Local well-posedness and hence also uniqueness follow from the Lipschitz dependence (3.10) that provides a contraction for small $T > 0$, see [33, Theorem 3.1] for the abstract result in the theory of maximal parabolic regularity for quasilinear abstract Cauchy problems and see [10] for a proof in a very similar setting. Global stability in $L^{2,2}$ is shown in the next section.

4. Exponential decay to equilibrium and stability

We prove that under Assumption 2.2(2), the interaction on and across the interface Γ is sufficiently strong to force the system into the uniform equilibrium given by

$$u^\infty = \frac{1}{V}\left(\int_{\Omega_+} u_+^0(x)\,dx + \int_{\Omega_-} u_-^0(x)\,dx + \int_\Gamma u_\Gamma^0(y)\,d\mathcal{H}_{d-1}\right)$$

associated to u^0, where

$$V = |\Omega_+| + |\Omega_-| + |\Gamma|_{\mathcal{H}_{d-1}}.$$

The quasilinear gradient structure combined with the L^∞-bounds provide an exponential rate in $L^{2,2}$. Here, by a slight abuse of notation, u^∞ also denotes the constant vector function $u^\infty = u^\infty(1, 1, 1)^T \in C^{0,0}$.

Theorem 4.1. *Under the assumptions of Theorem 3.1 with $\mathcal{F} \equiv 0$, given $u^0 \in X^r_{q,q_\Gamma}$, the solution u converges to u^∞ at an exponential rate, in the sense that there is a $\delta > 0$ depending only on u^0, k, m, Ω and Γ, such that for all $s \geq 0$,*

$$\|u(s) - u^\infty\|_{L^{2,2}} \leq e^{-\delta s}\|u^0 - u^\infty\|_{L^{2,2}}. \tag{4.1}$$

Proof. Since for every solution $u \in MR^r_{q,q_\Gamma}$ and $T > 0$, $\mathfrak{a}_{u(s)}(u(s), u^\infty) = 0$, applying (2.10) to $u - u^\infty$ shows the energy balance

$$\|u(s) - u^\infty\|^2_{L^{2,2}} + \int_0^s \mathfrak{a}_{u(t)}(u(t), u(t))\, dt = \|u^0 - u^\infty\|^2_{L^{2,2}}, \tag{4.2}$$

for all $s > 0$. By Lemma 3.5 and Assumption 2.2,

$$\mathfrak{l}_{u(t)}(u(t), u(t)) \geq C\|\nabla u(t)\|^2_{L^{2,2}}, \text{ and}$$

$$\mathfrak{m}_{u(t)}(u(t), u(t)) \geq \underline{m}\left(\int_\Gamma (u_+ - u_\Gamma)^2(t) + (u_- - u_\Gamma)^2(t) + (u_+ - u_-)^2(t)\, d\mathcal{H}_{d-1}\right).$$

Hence, with the following Poincaré-type inequality, the claim follows directly from Gronwall's inequality. $\square$

Lemma 4.2. *(Bulk-Interface Poincaré Inequality) Let $u \in W^{1,2,2}$ and u^∞ the equilibrium associated to u. Then there is a constant $C > 0$, independent of u, such that*

$$\|u - u^\infty\|^2_{L^{2,2}} \leq C(\|\nabla u\|^2_{L^{2,2}} + \|u_+ - u_\Gamma\|^2_{L^2(\Gamma)} + \|u_- - u_\Gamma\|^2_{L^2(\Gamma)} + \|u_+ - u_-\|^2_{L^2(\Gamma)}). \tag{4.3}$$

Proof. For any $u \in L^{1,1}$, let in the following $\bar{u}_+ := \frac{1}{|\Omega_+|}\int_{\Omega_+} u_+$, $\bar{u}_- := \frac{1}{|\Omega_-|}\int_{\Omega_-} u_-$ and $\bar{u}_\Gamma := \frac{1}{|\Gamma|}\int_\Gamma u_\Gamma$ and let $\bar{u} = (\bar{u}_+, \bar{u}_-, \bar{u}_\Gamma) \in \mathbb{R}^3$. To prove (4.3), we use the two (standard) versions of Poincaré's inequality, see e.g. [5, Theorem 1 and Corollary 3]. For all $u_+ \in W^{1,p}(\Omega_+)$,

(1) there is a constant $\bar{C}_+ > 0$, such that

$$\|u_+ - \bar{u}_+\|^2_{L^2(\Omega_+)} \leq \bar{C}_+\|\nabla u_+\|^2_{L^2(\Omega_+)}, \quad \text{and,} \tag{4.4}$$

(2) there is a constant $C^\Gamma_+ > 0$, such that

$$\|u_+\|^2_{L^2(\Omega_+)} \leq C^\Gamma_+\left(\|\nabla u_+\|^2_{L^2(\Omega_+)} + \frac{1}{|\Gamma|}|\int_\Gamma u_+|^2\right). \tag{4.5}$$

Clearly, analogous statements hold for Ω_- with constants $\bar{C}_- > 0$ and $C_-^\Gamma > 0$ and (4.4) holds for u_Γ on the manifold Γ with constant $\bar{C}_\Gamma > 0$. An elementary calculation shows that

$$\|u - u^\infty\|_{L^{2,2}}^2 = \|u - \bar{u}\|_{L^{2,2}}^2 - V(u^\infty)^2 + |\Omega_+|\bar{u}_+^2 + |\Omega_-|\bar{u}_-^2 + |\Gamma|\bar{u}_\Gamma^2.$$

Inserting $Vu^\infty = |\Omega_+|\bar{u}_+ + |\Omega_-|\bar{u}_- + |\Gamma|\bar{u}_\Gamma$ gives

$$\|u - u^\infty\|_{L^{2,2}}^2 = \|u - \bar{u}\|_{L^{2,2}}^2 + \frac{|\Omega_+||\Omega_-|}{V}(\bar{u}_+ - \bar{u}_-)^2 + \frac{|\Omega_+||\Gamma|}{V}(\bar{u}_+ - \bar{u}_\Gamma)^2 + \frac{|\Omega_-||\Gamma|}{V}(\bar{u}_- - \bar{u}_\Gamma)^2.$$
$$(4.6)$$

By (4.4), $\|u - \bar{u}\|_{L^{2,2}}^2 \leq (\bar{C}_+ + \bar{C}_- + \bar{C}_\Gamma)\|\nabla u\|_{L^{2,2}}^2$, so it remains to estimate the last three terms in (4.6) by the right-hand-side in (4.3). By Hölder's inequality and by (4.5),

$$(\bar{u}_+ - \bar{u}_\Gamma)^2 = \frac{1}{|\Omega_+|^2}\left(\int_{\Omega_+} u_+ - \bar{u}_\Gamma\right)^2 \leq \frac{1}{|\Omega_+|}\|u_+ - \bar{u}_\Gamma\|_{L^2(\Omega_+)}^2$$

$$\leq \frac{C_+^\Gamma}{|\Omega_+|}\left(\|\nabla u_+\|_{L^2(\Omega_+)}^2 + \frac{1}{|\Gamma|}\left|\int_\Gamma u_+ - \bar{u}_\Gamma\right|^2\right)$$

$$\leq \frac{C_+^\Gamma}{|\Omega_+|}\left(\|\nabla u_+\|_{L^2(\Omega_+)}^2 + \|u_+ - u_\Gamma\|_{L^2(\Gamma)}^2\right).$$

The term $(\bar{u}_- - \bar{u}_\Gamma)^2$ can be estimated analogously. In order to estimate the last term $(\bar{u}_+ - \bar{u}_-)^2$, insert $-\bar{u}_\Gamma + \bar{u}_\Gamma$ and use the previous estimates. With this strategy, it is clear that for (4.3) to hold, it is sufficient that two of the three coefficient functions m_+, m_-, m_Γ are positive, so not every pair of unknowns needs to interact across Γ. It is also sufficient for two of the coefficients to be positive to guarantee the structure of the kernel of $\mathfrak{a}_u$ in (2.8). This concludes the proof of Lemma 4.2 and thus of Theorem 4.1. $\quad\square$

In addition to exponential stability of u^∞ within the sets of initial data with equal mass, Theorem 4.1 immediately implies stability of u^∞ in X_{q,q_Γ}^r:

Corollary 4.3. *For every $v^\infty \in \mathbb{R}_+$, $\varepsilon > 0$, if $u^0 \in X_{q,q_\Gamma}^r$ with $\|u^0 - v^\infty\|_{L^{1,1}} < \varepsilon V$, then $|u^\infty - v^\infty| < \varepsilon$.*

Proof. A direct calculation shows that

$$|u^\infty - v^\infty| = \frac{1}{V}\left|\int_{\Omega_+} u_+^0(x) - v^\infty \, dx + \int_{\Omega_-} u_-^0(x) - v^\infty \, dx + \int_\Gamma u_\Gamma^0(y) - v^\infty \, dy\right|$$

$$\leq \frac{1}{V}\|u^0 - v^\infty\|_{L^{1,1}}. \quad\square$$

5. Onsager modelling, extensions and concluding remarks

5.1. Entropic gradient structure for heat transfer (Onsager model)

The system in (1.1)–(1.3)was motivated by non-equilibrium thermodynamical modelling of heat transfer and diffusion processes across interfaces, [29], [32], and based on the results in [22] and [31]. For example, in [31], it is shown that for *flat* interfaces Γ, the heat transfer *Onsager* or *gradient system* associated to

$$\dot{\theta} = \mathcal{K}(\theta)\mathrm{D}\mathcal{S}(\theta) \tag{5.1}$$

is represented by the set of equations

$$
\begin{cases}
\dot{\theta}_\pm + \frac{1}{c_\pm}\mathrm{div}(K_\pm(\theta_\pm)\nabla\frac{1}{\theta_\pm}) &= 0, \quad \text{in } (0,T)\times\Omega_\pm, \\
(\frac{K_\pm(\theta_\pm)}{c_\pm}\nabla\frac{1}{\theta_\pm})\nu_\pm + M_\pm(\theta)(\frac{1}{\theta_\pm} - \frac{1}{\theta_\Gamma}) + M_\Gamma(\theta)(\frac{1}{\theta_\pm} - \frac{1}{\theta_{m_+}}) &= 0, \quad \text{on } (0,T)\times\Gamma, \\
(K_\pm(\theta_\pm)\nabla\frac{1}{\theta_\pm})\nu_\pm &= 0, \\
\text{on } (0,T)\times\{\partial\Omega_\pm\backslash\Gamma\},
\end{cases}
\tag{5.2}
$$

on the bulk parts, and

$$
\begin{cases}
\dot{\theta}_\Gamma + \frac{1}{c_\Gamma}\mathrm{div}(K_\Gamma(\theta)\nabla\frac{1}{\theta_\Gamma}) - M_+(\theta)(\frac{1}{\theta_+} - \frac{1}{\theta_\Gamma}) - M_-(\theta)(\frac{1}{\theta_-} - \frac{1}{\theta_\Gamma}) &= 0, \quad \text{in } (0,T)\times\Gamma, \\
(K_\Gamma(\theta)\nabla\frac{1}{\theta_\Gamma})\nu_\Gamma &= 0, \quad \text{on } (0,T)\times\partial\Gamma,
\end{cases}
\tag{5.3}
$$

on the flat interface Γ, where $c_\pm, c_\Gamma > 0$ are the specific heats of bulk and interface materials, respectively, and the coefficients K, M specify thermal conductivity within materials and across Γ in an entropic modelling. In (5.1), $\mathcal{S}$ is the total entropy functional

$$\mathcal{S}(\theta) = \int_{\Omega_+} c_+ \log\theta_+ \,\mathrm{d}x + \int_{\Omega_-} c_- \log\theta_- \,\mathrm{d}x + \int_\Gamma c_\Gamma \log\theta_\Gamma \,\mathrm{d}y,$$

and $\mathcal{K}$ is the Onsager operator corresponding to the dual dissipation potential

$$
\begin{aligned}
2\Psi^*(\theta, \phi) &= 2\Psi_+^*(\theta_+, \phi_+) + 2\Psi_-^*(\theta_-, \phi_-) + 2\Psi_\Gamma^*(\mathrm{tr}_\Gamma\,\theta, \mathrm{tr}_\Gamma\,\phi) \\
&= \int_{\Omega_+} \nabla\frac{\phi_+}{c_+} \cdot K_+(\theta_+)\nabla\frac{\phi_+}{c_+} \,\mathrm{d}x + \int_{\Omega_-} \nabla\frac{\phi_-}{c_-} \cdot K_-(\theta_-)\nabla\frac{\phi_-}{c_-} \,\mathrm{d}x \\
&+ \int_\Gamma \nabla_\Gamma\frac{\phi_\Gamma}{c_\Gamma} \cdot K_\Gamma(\mathrm{tr}_\Gamma\,\theta)\nabla\frac{\phi_\Gamma}{c_\Gamma} \,\mathrm{d}y + \int_\Gamma M_\Gamma(\mathrm{tr}_\Gamma\,\theta)(\frac{\mathrm{tr}_\Gamma\,\phi_+}{\mathrm{tr}_\Gamma\,c_+} - \frac{\mathrm{tr}_\Gamma\,\phi_-}{\mathrm{tr}_\Gamma\,c_-})^2 \,\mathrm{d}y \\
&+ \int_\Gamma M_+(\mathrm{tr}_\Gamma\,\theta)(\frac{\mathrm{tr}_\Gamma\,\phi_+}{\mathrm{tr}_\Gamma\,c_+} - \frac{\mathrm{tr}_\Gamma\,\phi_\Gamma}{\mathrm{tr}_\Gamma\,c_\Gamma})^2 + M_-(\mathrm{tr}_\Gamma\,\theta)(\frac{\mathrm{tr}_\Gamma\,\phi_-}{\mathrm{tr}_\Gamma\,c_-} - \frac{\mathrm{tr}_\Gamma\,\phi_\Gamma}{\mathrm{tr}_\Gamma\,c_\Gamma})^2 \,\mathrm{d}y. \quad (5.4)
\end{aligned}
$$

Equations (5.2) and (5.3) are equivalent to (1.1)–(1.3) by differentiating $\nabla\frac{1}{\theta}$ to $-\frac{1}{\theta^2}\nabla\theta$ and writing $\frac{1}{\theta_\Gamma\theta_+}(\theta_+ - \theta_\Gamma)$ instead of $(\frac{1}{\theta_\Gamma} - \frac{1}{\theta_+})$, for every term of this kind. The coefficients K

and k and M and m are then related via $m_\pm(\mathrm{tr}_\Gamma\,\theta) = \frac{M_\pm(\mathrm{tr}_\Gamma\,\theta)}{\theta_\Gamma\mathrm{tr}_\Gamma\,\theta_\pm}$, $m_\Gamma(\mathrm{tr}_\Gamma\,\theta) = \frac{M_\Gamma(\mathrm{tr}_\Gamma\,\theta)}{\mathrm{tr}_\Gamma\,\theta_+\mathrm{tr}_\Gamma\,\theta_-}$, $k_\pm(\theta_\pm) = \frac{K_\pm(\theta_\pm)}{\theta_\pm^2}$ and $k_\Gamma(\mathrm{tr}_\Gamma\,\theta) = \frac{K_\Gamma(\mathrm{tr}_\Gamma\,\theta)}{\theta_\Gamma^2}$.

We check the applicability and the implications of Theorem 3.1: It is straightforward to see that K, M satisfy Assumption 2.2 if and only if k, m satisfy Assumption 2.2. So if Assumption 2.2 on K, M is respected in an entropic modelling, well-posedness and exponential stability are obtained. In particular, the positivity of two components of M guarantees entropy production of the bulk-interface interaction and information on the Onsager system given by $\mathcal{S}$ and Ψ^* is retrieved: Starting from positive initial values, $l > 0$, the regularity in Theorem 3.1 and the maximum principle justify rigorously the equivalence of (5.2), (5.3) and (1.1)–(1.3) and the solution provides the gradient flow of $\mathcal{S}$ with respect to the dual dissipation metric Ψ^*. The entropy $\mathcal{S}(\theta(t))$ is well-defined along orbits and $-\mathcal{S}$ provides a strict Lyapunov functional by the energy balance $-\frac{d}{dt}\mathcal{S}(\theta(t)) + 2\Psi^*\big(\theta(t), \frac{c}{\theta(t)}\big) = 0$ and the fact that $2\Psi^*\big(\theta(t), \frac{c}{\theta(t)}\big) = 0$ implies $\mathfrak{a}_{\theta(t)}(\theta(t), \theta(t)) = 0$ along the positive orbits of θ. By Theorem 4.1, exponential stability holds in the sense that $\|c\theta(t) - c\theta^\infty\|_{L^{2,2}} \le e^{-\delta t}\|c\theta^0 - c\theta^\infty\|_{L^{2,2}}$ for some $\delta > 0$.

5.2. Small extensions and further remarks

The next remarks concern extensions of Theorem 3.1, mostly based on perturbation theory for maximal parabolic regularity.

Remark 5.1. Clearly, the analysis above includes the simpler case of bulk-surface interaction with $\Omega_- = \emptyset$, without the variable u_- and with $m_- = m_\Gamma = 0$.

Remark 5.2. If the Lipschitz dependence of k, m and $\mathcal{F}$ on u in Assumptions 2.2(3) and 2.5 is improved to C^n, $n \in \mathbb{N} \cup \{\infty, \omega\}$, then the solution u in Theorem 3.1 gains time regularity by [33, Theorem 5.1], i.e. it follows that

$$u \in C^n(J_T; X_{r,q,q_\Gamma}) \cap C^{n+1-1/r}(J_T; W_0^{-1,q,q_\Gamma}) \cap C^{n-1/r}(J_T; W^{1,q,q_\Gamma})$$

and that $u \in C^\infty(J_T; W^{1,q,q_\Gamma})$ if $n = \infty$ and u is real analytic on J_T if $n = \omega$.

Remark 5.3. The coefficient functions k, m and external forces and inhomogeneous boundary conditions f, g, h may additionally depend on time. For example, Theorems 3.1 and 4.1 continue to hold if Assumption 2.2 holds uniformly in $t \in (0, \infty)$ for k, m and $t \mapsto \mathcal{A}_{u(t)} \in \mathcal{L}(W^{1,q,q_\Gamma}, W_0^{-1,q,q_\Gamma})$ is continuous for all $u \in X_{q,q_\Gamma}^r$ and if Assumption 2.5 holds where $t \mapsto \mathcal{F}(t, u)$ is in $L^r(J_T; W_0^{-1,q,q_\Gamma})$, cf. [33, Section 3].

Remark 5.4. The results in Theorem 3.1 extend to perturbations of $\mathcal{A}_q$ by lower-order terms like transport terms $b \cdot \nabla u_\pm$, $b \in \mathbb{R}^d$. In particular, with suitable regularity assumptions, the coefficients $c_\pm : \Omega_\pm \to \mathbb{R}_+\backslash\{0\}$ and $c_\Gamma : \Gamma \to \mathbb{R}_+\backslash\{0\}$ in Subsection 5.1 can be chosen to depend on the spatial variables.

Remark 5.5. An exponential convergence rate as in (4.6) also holds in $L^{p,p}$ for $p > 2$ due to interpolation of the p-norms since

$$\|u(s) - u^\infty\|_{L^{\infty,\infty}} \le 3\max(|l - u^\infty|, |L - u^\infty|) =: C_\infty$$

is bounded uniformly in time. For all $s \geq 0$, with $\theta = 1 - \frac{2}{p}$, we get

$$
\begin{aligned}
\|u(s) - u^\infty\|_{L^{p,p}} &\leq C_\infty^\theta \|u(s) - u^\infty\|_{L^{2,2}}^{1-\theta} \\
&\leq C_\infty^\theta e^{-\frac{2\delta}{p}s} \|u^0 - u^\infty\|_{L^{2,2}}^{\frac{2}{p}} \\
&\leq e^{-\frac{2\delta}{p}s} \|u^0 - u^\infty\|_{L^{p,p}}.
\end{aligned}
$$

Additional rates that exploit the gradient structure of system (1.1) – (1.3) depend on the choice of k, m or of the energy and dissipation functionals $\mathcal{S}$ and Ψ, see e.g. [17] for exponential L^1-rates for volume-surface reaction-diffusion based on entropy production.

Acknowledgments

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