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# Stability for the multifrequency inverse medium problem

Gang Bao [a], Faouzi Triki [b,*]

[a] *Department of Mathematics, Zhejiang University, Hangzhou, Zhejiang, 310027, China*
[b] *Laboratoire Jean Kuntzmann, UMR CNRS 5224, Université Grenoble-Alpes, 700 Avenue Centrale, 38401 Saint-Martin-d'Hères, France*

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**Abstract**

The solution of a multi-frequency 1d inverse medium problem consists of recovering the refractive index of a medium from measurements of the scattered waves for multiple frequencies. In this paper, rigorous stability estimates are derived when the frequency takes value in a bounded interval. It is showed that the ill-posedness of the inverse medium problem decreases as the width of the frequency interval becomes larger. More precisely, under certain regularity assumptions on the refractive index, the estimates indicate that the power in Hölder stability is an increasing function of the largest value in the frequency band. Finally, a Lipschitz stability estimate is obtained for the observable part of the medium function defined through a truncated trace formula.


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## 1. Introduction

This paper is concerned with the stability for determining the refractive index of an one-dimensional (1d) medium from boundary measurements. For a fixed frequency, it is known that

* Corresponding author.
*E-mail addresses:* baog@zju.edu.cn (G. Bao), faouzi.triki@univ-grenoble-alpes.fr (F. Triki).

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this inverse problem is severely ill-posed and suffers from the lack of uniqueness. Several numerical results show that in the case of multiple frequencies, in contrast with the single frequency case, the ill-posedness decreases dramatically when the frequency band increases and covers the resonance region of the medium ([14], [11], [6], [5] and references therein). However, little is known about the stability for the inverse problem or the convergence issues for the numerical methods. Our goal of the present paper is to prove stability results for the multifrequency inverse medium scattering problem. Such results would be essential for a rigorous justification of the numerical observations.

Consider the 1d scalar Helmholtz equation

$$\phi''(x,k) + k^2(1+q(x))\phi(x,k) = 0, \tag{1}$$

where the real-valued $(1+q(x))$ is the refractive index of the medium. For any real number $k$, we look for a solution of the form

$$\phi_\pm(x,k) = \psi_\pm(x,k) + e^{\pm ikx},$$

where the scattered wave $\psi_+, \psi_-$ corresponding to the left excitation $e^{ikx}$, and the right excitation $e^{-ikx}$, respectively, satisfy the outgoing radiation conditions

$$\begin{aligned} \psi'(x,k) - ik\psi(x,k) &= 0 \qquad \text{for} \quad x \geq 1, \\ \psi'(x,k) + ik\psi(x,k) &= 0 \qquad \text{for} \quad x \leq 0. \end{aligned}$$

The sum of the incident wave and its corresponding scattered wave, $\phi(x,k)$, is called the total wave. Throughout, it is assumed that the medium function $q(x)$ has the regularity $C_0^{m+1}([0,1])$ with $m \geq 4$, and satisfies

$$1 + q(x) \geq n_0, \qquad \text{for } x \in \mathbb{R}, \tag{2}$$

with $n_0 \in (0,1)$ a fixed constant. The scattered wave $\psi(x,k)$ satisfies the Helmholtz equation

$$\psi_\pm''(x,k) + k^2(1+q(x))\psi_\pm(x,k) = -k^2 q(x) e^{\pm ikx}, \tag{3}$$

for all $x \in (0,1)$.

Since $q(x)$ vanishes outside $(0,1)$, it is easy to see that for $k \in \mathbb{R}$, there exist complex numbers $\mu_\pm$ known as the reflection coefficients, such that

$$\begin{aligned} \psi_+(x,k) &= \mu_+(k)e^{-ikx} \qquad \text{for} \quad x \leq 0, \\ \psi_-(x,k) &= \mu_-(k)e^{ikx} \qquad \text{for} \quad x \geq 1. \end{aligned} \tag{4}$$

The existence and uniqueness of the solutions $\psi_\pm \in C([0,1])$ are well known for any real $k$ [8].

Therefore, the function $k \to \mu_\pm(k)$ are well defined on $\mathbb{R}$. The outgoing radiation conditions imply

$$\begin{aligned} \phi_+(x,k) &= \phi_+(1,k)e^{ikx} \qquad \text{for} \quad x \geq 1, \\ \phi_-(x,k) &= \phi_-(0,k)e^{-ikx} \qquad \text{for} \quad x \leq 0. \end{aligned}$$

Furthermore, the constants $\phi_+(1,k)$ and $\phi_-(0,k)$ are nonzero. If they are zero then Cauchy theorem implies that $\phi_\pm(x,k)=0$ for all $x\in\mathbb{R}$, which means that $\psi_\pm=-e^{\mp ikx}$ on the whole space. This is in contradiction with the outgoing radiation conditions. In fact, $\phi_\pm(x,k)\neq 0$ holds for all $x\in\mathbb{R}$ and for all $k\in\mathbb{C}$, satisfying $\mathrm{Im}(k)\geq 0$ (Corollary 4.1 [14]).

The multifrequency inverse medium problem may be stated as follows:

*Given one of the reflection coefficients $\mu_+(k)$ and $\mu_-(k)$ for $k\in(0,k_0)$, to reconstruct the refractive index $1+q(x)$ for $x\in[0,1]$.*

Define the impedance functions $p_\pm(x,k)$ associated with $\psi_\pm(x,k)$, respectively, by

$$p_\pm(x,k)=\pm\frac{\phi_\pm'(x,k)}{ik\phi_\pm(x,k)}. \tag{5}$$

It is shown in [14] that these functions are well defined and verify in addition the nonlinear Ricatti equation

$$p_\pm'(x,k)\pm ikp_\pm^2(x,k)=\mp ik(1+q(x)), \tag{6}$$

subject to the boundary conditions

$$\begin{aligned} p_-(0,k) &= 1; & p_-(1,k) &= d_-(k),\\ p_+(0,k) &= d_+(k); & p_+(1,k) &= \quad 1, \end{aligned} \tag{7}$$

for all $x\in(0,1)$, $k\in\mathbb{R}$, where

$$d_\pm(k)=\frac{1-\mu_\pm(k)}{1+\mu_\pm(k)}. \tag{8}$$

The inverse problem may be restated as:

*Given the data $d_-(k)$, $k\in(0,k_0)$ or $d_+(k)$, $k\in(0,k_0)$, to reconstruct the medium function $q(x)$ for $x\in[0,1]$.*

It is well known that in the case where the data is given for all frequencies, this inverse problem has a unique solution, and a number of algorithms have been proposed for its numerical treatment [28]. However, in applications, the reflection coefficients $\mu_\pm(k)$ are usually measured with finite-accuracy at a finite number of the frequencies $k$. Hence, the well-posedness of the inverse problem when the measurements are taken over a finite interval is of critical importance. It is well known that the ill-posedness of the inverse scattering problem decreases as the frequency increases [3]. However, at high frequencies, the nonlinear equation becomes extremely oscillatory and possesses many more local minima. A challenge for solving this problem is to develop a solution method that takes advantages of the regularity of the problem for high frequencies without being undermined by local minima. To overcome the difficulties, a recursive linearization method was proposed in [14,15,13] for solving the inverse problem of the two-dimensional Helmholtz equation. Based on the Riccati equations for the scattering matrices, the method requires full aperture data and needs to solve a sensitivity matrix equation at each iteration. The numerical results were very successful to address the ill-posedness computationally. However, there are two serious issues remain to be resolved. Due to the high computational cost, it is numerically difficult to extend the method to the three-dimensional problems. Recently, new and more efficient recursive linearization methods have been developed for solving the two-dimensional Helmholtz equation and the three dimensional Maxwell equations for both full and limited aperture data by directly using the differential equation formulation [11], [6], [5], [7]. Theoretically, little is known about the stability for the inverse problem with multiple frequency data. Our main objective of this work is to establish stability estimates for the inverse problem with multiple frequency data.

We state here our first main result associated to the inversion with boundary measurements on a band of frequencies. For $m \geq 4$, $M > 0$, and $q_0 \in C_0^{m+1}([0,1])$ satisfying (2), we further denote the set $\mathcal{Q} = \mathcal{Q}(n_0, m, M)$, by

$$\mathcal{Q} := \{q \in C_0^{m+1}([0,1]) :\ \|q - q_0\|_{C^{m+1}([0,1])} \leq M,\ n_0 \leq 1 + q\}. \tag{9}$$

We next give our first main stability estimate for the multifrequency inverse medium problem. In what follows $c_{\mathcal{Q}}$ and $k_{\mathcal{Q}}$ denote generic strictly positives constants depending only on $\mathcal{Q}$.

**Theorem 1.1.** *Assume that $q, \widetilde{q}$ be two medium functions in $\mathcal{Q}$. Let $d = d_\pm$ and $\widetilde{d} = \widetilde{d}_\pm$ be the boundary measurements associated respectively to $q$ and $\widetilde{q}$ as defined in (7), satisfying $\|d - \widetilde{d}\|_{L^\infty(0,+\infty)} < 1$. Let $k^\star \in \mathbb{R}_+$ be the smallest value satisfying*

$$|d(k^\star) - \widetilde{d}(k^\star)| = \|d - \widetilde{d}\|_{L^\infty(0,+\infty)}.$$

*Then, there exist constants $c_{\mathcal{Q}} > 0$, and $n_{\mathcal{Q}} \in \mathbb{N}^*$, such that the following estimate holds*

$$\|q - \widetilde{q}\|_{L^\infty(\mathbb{R})} \leq c_{\mathcal{Q}} \|d - \widetilde{d}\|_{L^\infty(0,k_0)}^{\frac{m}{m+1} w_0(k^\star, k_0)}, \tag{10}$$

*for all $k_0 > 0$, where the function $w_0(k^\star, k_0)$ is continuous on $(\mathbb{R}_+^*)^2$, and verifies*

$$\frac{2}{\pi}\arctan\Big(\frac{(e^{k_0}-1)^{n_{\mathcal{Q}}}}{\sqrt{(e^{k^\star}-1)^{2n_{\mathcal{Q}}} - (e^{k_0}-1)^{2n_{\mathcal{Q}}}}}\Big)$$
$$\leq w_0(k^\star, k_0) \leq \frac{2}{\pi}\arctan\left(\inf\{\frac{k_0}{\sqrt{(k^\star)^2 - k_0^2}}, \frac{e^{k_0 n_{\mathcal{Q}}}}{\sqrt{e^{2k^\star n_{\mathcal{Q}}} - e^{2k_0 n_{\mathcal{Q}}}}}\}\right),$$

*for all $k_0 \in (0, k^\star]$.*

**Remark 1.1.** The Hölder exponent $\frac{m}{m+1} w_0(k^\star, k_0)$ in the estimate (1.1) is an increasing function of $k_0$. It tends to zero when $k_0$ goes to zero which shows as expected that the ill-posedness of the inversion increases when the band of frequency shrinks. On the other hand, the function $w_0(k^\star, k_0)$ approaches to its upper bound $\frac{m}{m+1}$ when $k_0$ tends to $k^\star$, which is the global Hölder stability estimate obtained in Corollary 3.1.

The value $k^\star$ represents the frequency at which the noise is the most important. We observe that the Hölder exponent $\frac{m}{m+1} w_0(k^\star, k_0)$ is a decreasing function of $k^\star$, and tends to zero when $k^\star$ approaches $+\infty$.

By considering the stability estimate (1.1), we conclude that the reconstruction of the medium function is accurate when the frequency band is large enough and contains the noise frequency ($k^\star \in (0, k_0]$), while it deteriorates when the frequency band shrinks toward zero. These theoretical results confirm the numerical observations and the physical expectations for the increasing stability phenomena by taking multifrequency data.

**Theorem 1.2.** *Assume that $q, \widetilde{q}$ be two medium functions in $\mathcal{Q}$. Let $d = d_\pm$ and $\widetilde{d} = \widetilde{d}_\pm$ be the boundary measurements associated respectively to $q$ and $\widetilde{q}$ as defined in (7), satisfying $\varepsilon := \|d - \widetilde{d}\|_{L^\infty(0,+\infty)} < 1$.*

*Then, there exist constants $c_{\mathcal{Q}} > 0$, $k_{\mathcal{Q}} > 0$ and $n_{\mathcal{Q}} \in \mathbb{N}^*$, such that the following estimates hold*

$$\|q - \widetilde{q}\|_{L^\infty(\mathbb{R})} \leq c_{\mathcal{Q}} \varepsilon^{\frac{m}{m+1}}, \qquad \text{if } k_0 \geq \frac{k_{\mathcal{Q}}}{\varepsilon^{\frac{1}{m}}}, \tag{11}$$

$$\|q - \widetilde{q}\|_{L^\infty(\mathbb{R})} \leq \frac{c_{\mathcal{Q}}}{|\ln(\eta(k_0)|\ln(\varepsilon)|)|^{\frac{m^2}{m+1}}} \qquad \text{if } k_0 < \frac{k_{\mathcal{Q}}}{\varepsilon^{\frac{1}{m}}}, \tag{12}$$

*where the function $\eta$ is given by*

$$\eta(k_0) = \frac{(e^{k_0} - 1)^{n_{\mathcal{Q}}}}{1 + 2\sqrt{1 + (e^{k_0} - 1)^{2n_{\mathcal{Q}}}}}.$$

**Remark 1.2.** The estimates (1.2) and (1.2) show that the stability is Hölder when the largest value in the frequency band $k_0$ is larger than a critical limit, and is of logarithmic type when $k_0$ becomes small. Hence for a limited band of frequencies one can improve the stability of the inverse problem by increasing the largest frequency. The critical limit only depends on the noise in the measurement and the set of medium functions $\mathcal{Q}$. When $k_0$ tends to zero the function $\eta(k_0)$ approaches zero, and right-hand side term blows up. This behavior demonstrate again that the inverse problem is severely ill-posed when $k_0$ is close to zero, and confirms the observations made in Remark 1.1.

Based on the high frequency asymptotic expansions of the fields $\phi_\pm$, Chen and Rokhlin [14] introduced the observable part of the medium $q(x)$ on the band of frequency $(0, k_0)$, as the function $q_{k_0}(x)$ unique solution to the truncated version of the trace formula (3.1), that is

$$p'_{k_0,\pm}(x,k) \pm ikp^2_{k_0,\pm}(x,k) \pm ik(1 + q(x)) = 0, \tag{13}$$

$$q'(x) - \frac{2}{\pi}(1 + q(x)) \int_{-k_0}^{k_0} (p_{k_0,+}(x,k) - p_{k_0,-}(x,k))dk = 0, \tag{14}$$

for all $x \in (0, 1)$, subject to the boundary conditions

$$p_{k_0,+}(0,k) = d_+(k); \quad p_{k_0,-}(0,k) = 1; \quad q_{k_0}(0) = 0, \tag{15}$$

for all $k \in \mathbb{C}_+$. They also have derived error estimates of the approximation of the medium function $q(x)$ by its observable part $q_{k_0}(x)$ on the frequency band $(0, k_0)$ [14].

Our third main result is to characterize $q_{k_0}(x)$ in terms of the frequency band $(0, k_0)$, and to show that the recovery of $q_{k_0}(x)$ is not sensitive to errors in the measurements if $k_0$ is large enough.

**Theorem 1.3.** *Assume that $q, \widetilde{q}$ be two medium functions in $\mathcal{Q}$. Let $d = d_\pm$ and $\widetilde{d} = \widetilde{d}_\pm$ be the boundary measurements associated respectively to $q$ and $\widetilde{q}$ as defined in (7). Let $q_{k_0}$ and $\widetilde{q}_{k_0}$ be the observable parts of respectively $q$ and $\widetilde{q}$ on $(0, k_0)$ solutions to the system (13)-(14)-(15). Then there exist constants $\rho_{\mathcal{Q}} > 0$ and $k_{\mathcal{Q}} > 0$ such that*

$$\left\| q_{k_0} - \widetilde{q}_{k_0} \right\|_{L^\infty(\mathbb{R})} \leq \rho_{\mathcal{Q}} \| d(k) - \widetilde{d}(k) \|_{L^1(0,k_0)},$$

*is satisfied for all $k_0 \geq k_{\mathcal{Q}}$.*

For higher dimension, to the best of our knowledge, this inverse problem is still open. This is due to the difficulties in the analysis of the scattering data as a function of the frequency, which are related to the strong nonlinearity for high frequencies and the existence of trapped rays. From a physical point of view, the situation is better understood. According to Uncertainty Principle there exists a resolution limit to the sharpness of details of the medium that can be observed from measurements in the far field region. This limit known as the diffraction limit is about one half of the wavelength. Consequently the reconstruction of the medium can be then reduced by increasing the magnitude of the frequency [12]. Mathematically, the inverse medium problem with full measurements at a fixed frequency is notoriously ill-posed [27,35]. In fact, Alessandrini proved that the stability estimates in 3d is of logarithmic type [2], and Mandache showed later the optimality of such estimates [29]. Recent studies have been conducted on the behavior of the constant in the logarithmic stability in terms of the fixed frequency [3,23,31]. Several other results in inverse scattering problems that are related to the increasing stability phenomena by increasing the frequency were obtained in different settings [1,4,23,33]. All of these results demonstrate the increasing stability phenomena when the frequency becomes larger. For the case of the inverse source problem for Helmholtz equation and an homogeneous background it was shown in [8–10,16,25,26] that the ill-posedness of the inverse problem decreases as the frequency increases. Convergence results for iterative algorithms solving the multi-frequency inverse medium problem are obtained in [12,22]. Finally, we refer the reader to the topical review on inverse scattering problems [7] with multifrequencies on other related topics.

The rest of the paper is structured as follows. Auxiliary results related to the behavior of the impedance functions as functions of the frequency are provided in Section 2. The stability estimate for the observable part of the medium is given in Section 3. Finally, the proof of the main stability estimates for the multifrequency inverse medium problem is provided in Sections 4 and 5.

## 2. Properties of the impedance functions

A major difficulty in studying the multifrequency inverse medium problem is the fact that the partial differential equation describing the scattering phenomena involves a product of the frequency and the refractive index. In the 1d case, Gel'fand-Levitan techniques can be employed when the medium function is smooth to convert the Helmholtz equation into a Schrödinger equation. In the obtained Schrödinger equation, the frequency and the refractive index are separated, which allows a better understanding of the behavior of the solutions as functions of the frequency.

This approach was used to study the 1d inverse spectral problem [36]. It also led the authors in [14] to derive high-frequency asymptotic expansions of the impedance functions. Here, we first present some of these useful asymptotic results and our further analysis. In addition, we also study the meromorphic extensions of the impedance functions to the lower half complex plane.

For convenience, we complexify $k$. Denote $\mathbb{C}_\pm$ the upper half and lower half of the complex plane, that is

$$\mathbb{C}_+ = \{k \in \mathbb{C} : \mathrm{Im}(k) \geq 0\}; \quad \mathbb{C}_- = \{k \in \mathbb{C} : \mathrm{Im}(k) < 0\}.$$

It is easy to check from the uniqueness of the equations (6) with the boundary conditions (7), that

$$\overline{p_\pm(x,k)} = p_\pm(x, -\overline{k}), \tag{16}$$

for all $x \in \mathbb{R}$ and $k \in \mathbb{C}_+$.

*Low frequency behavior*

We next present the behavior of the impedances functions when the frequency $k$ is close to 0. In Lemma 4.1 and 4.2 of [13], the author derived the first term in the asymptotic expansion $p_\pm$ when $k$ approaches 0. Here we provide explicit bounds in a given frequency neighborhood of 0.

**Proposition 2.1.** *The following estimate*

$$|d_\pm(k)| \leq 2,$$

*holds for all $k \in \mathbb{C}$ satisfying $|k| \leq 1/M_1$, with*

$$M_1 = 2(\|q_0\|_{L^\infty(0,1)} + M). \tag{17}$$

**Proof.** Since the proofs of the estimates for $d_+$ and $d_-$ are identical, we only provide the proof for $d_+$.

Let

$$g_0(x,y) = \frac{e^{-ik|x-y|}}{-2ik},$$

be the Green function of the one dimension Helmholtz equation with the same radiation conditions as $\psi_+$. Multiplying the equation (3) by $g_0(x,y)$ and integrating by parts yield the following Lippmann-Schwinger integral equation

$$(I_d - K_q)[\psi_+] = K_q[e^{-ik\cdot}], \tag{18}$$

where $I_d$ is the identity operator from $L^\infty(0,1)$ to itself, and $K_q$ is a linear integral operator on $L^\infty(0,1)$, defined by

$$K_q[\psi](x) = -k^2 \int_0^1 g_0(x,y)q(y)\psi(y),$$

for all $\psi \in L^\infty(0,1)$. Therefore for $2|k|(\|q_0\|_{L^\infty(0,1)} + M) \le 1$, the operator $K_q$ becomes a contraction, and we deduce from the convergence of the Neumann series

$$|\mu_+(k)| \le \|\psi_+\|_{L^\infty(0,1)} \le 1/3.$$

Hence $|d_+(k)| \le 2$ for $|k| \le 1/M_1$, which finishes the proof. □

**Remark 2.1.** (Born approximation) Using the Neumann series and after a forward calculation, we obtain

$$\mu_+(k) = -\frac{k}{2i}\mathcal{F}(q)(-2k) + \sum_{p=2}^{\infty}\left(\frac{ik}{2}\right)^p \int_{(0,1)^p} e^{ik\kappa_p(\xi)} Q_p(\xi)d\xi,$$

for all $k \in (0,k_0)$, where $k_0 < 1/M_1$, $\kappa_p(\xi) = \xi_1 + \sum_{l=1}^{p-1}|\xi_{l+1} - \xi_l| + \xi_p$ for all $\xi \in \mathbb{R}^p$, and $Q_p(\xi) = \prod_{j=1}^p q(\xi_j)$. Since the first term in the low frequency expansion is the Fourier transform $\mathcal{F}(q)(2k)$, $k \in (-k_0,k_0)$, it seems natural to try to reconstruct the medium function from this term by considering the rest as a small perturbation ($O(k_0^2)$), and by using the same techniques as in [8]. It turns out that this approach fails to give any approximation of the medium function. The Born approximation error $O(k_0^2)$ is a higher order differential operator that is exponentially amplified in the inversion of the first term, and the final term does not vanish when $k_0$ tends to zero.

*High frequency behavior*

The following result was obtained in [14].

**Proposition 2.2.** *Assume that $q \in \mathcal{Q}$. The impedances $p_\pm(x,k)$ are continuous functions of $(x,k) \in [0,1]\times\mathbb{C}_+$, and analytic functions of $k \in \mathbb{C}_+$. Moreover there exists a constant $c_\mathcal{Q} > 0$ such that the following estimates*

$$\left\| p_\pm(x,k) - \sqrt{1+q(x)} \pm \frac{q'(x)}{4i(1+q(x))}\frac{1}{k}\right\|_{L^\infty} \le \frac{c_\mathcal{Q}}{|k|^2}, \tag{19}$$

$$\left\| \overline{p_+(x,k)} - p_-(x,k)\right\|_{L^\infty} \le \frac{c_\mathcal{Q}}{|k|^m}, \tag{20}$$

*hold for all $k \in \mathbb{C}_+^*$.*

We remark that the estimate (2.2) provides the two first terms in WKB expansions of the functions $p_\pm$. For large real $k$, the difference between $\overline{p}_+$ and $p_-$ is extremely small, which decays as $1/k^m$ where $m$ is the smoothness of the medium $q(x)$.

*Meromorphic extension*

It is known that the impedance functions $p_{\pm}(x,k)$ and in particular the reflexion coefficients $\mu_{\pm}(k)$ are holomorphic in $\mathbb{C}_+$, and have meromorphic extensions in $\mathbb{C}_-$. The poles of $\mu_{\pm}$ are called the scattering resonances of the medium. Here, we establish the existence of a scattering resonances-free strip in the complex plane. The proof is based on a similar result for the 1d Schrödinger equation derived in [21].

From (1) it follows that the poles can be characterized in the following way: $k \in \mathbb{C}_-$ is a scattering pole if and only if there exists a nontrivial function $\phi$, such that

$$\phi''(x,k) + k^2(1+q(x))\phi(x,k) = 0, \quad x \in (0,1), \tag{21}$$

with

$$\phi'(0,k) = -ik\phi(0,k),\;\; \phi'(1) = ik\phi(1,k), \tag{22}$$

We now present a connection between the solution of the Helmholtz equation (21) and the one of an equivalent Schrödinger equation. This will allow us to relate our scattering resonances to the well studied poles of the resolvent of the Schrödinger operator. This approach has been also used to derive the high frequency asymptotic expansions in Theorem 2.2.

Define further the functions $n, x, N, r, \xi : \mathbb{R} \to \mathbb{R}$ by the following expressions:

$$n(x) = \sqrt{1+q(x)},\; t(x) = \int_0^x n(s)ds,\; N(t) = n(x(t))^{-1/4}, \tag{23}$$

$$r(t) = \frac{N''(t)}{N(t)} - \frac{n'(x)}{2(n(x))^2} = \frac{1}{4}n^{-2}(x)\left(q''(x) - nq'(x) - \frac{5}{4}n^{-1}(q'(x))^2\right). \tag{24}$$

Then $\xi(t,k)$ defined by the Liouville transformation

$$\xi(t,k) := N^{-1}(t)\phi(x(t),k),$$

satisfies the Schrödinger equation:

$$\xi''(t,k) + (r(t)+k^2)\xi(t,k) = 0, \quad t \in (0,T), \tag{25}$$

with

$$\xi(0,k) = 1 + \mu_+(k), \tag{26}$$

where $T = t(1) = \int_0^1 n(s)ds$ is the travel time needed for the wave with speed $\frac{1}{n}$ to propagate from one end to another. We remark that $r(t)$ has a compact support in $(0,T)$. Consequently $k$ is a scattering resonance of (21)-(22) iff it is a resonance of the system (25)-(26).

The pole distribution of the resolvent for the Schrödinger operator has been the subject of extensive investigations due to the continuous advance of quantum mechanics. Many studies have focused on the problem of locating poles in the complex plane for different classes of potentials [18,20,37,17]. For the one dimensional Schrödinger operator with super-exponentially decaying potentials, more precise results are possible. Particularly, using the representation of the scattering matrix given by Melin [30], Hitrik [21] derived an explicit pole-free strip for the Schrödinger operator in the case of compactly supported potentials. The following result is a direct consequence of Hitrik's result and the observation that scattering resonances of the system (21)-(22) are also the poles of the Schrödinger operator (25)-(26).

**Proposition 2.3.** *Let $r : \mathbb{R} \to \mathbb{R}$ be defined by (24), and $h(r) := \frac{1}{4T} e^{-2T\|r\|_{L^1(0,T)}}$. Then the strip*

$$S_q = \{k \in \mathbb{C};\ -h(r) \le Im(k) \le 0, Re(k) \neq 0\}, \tag{27}$$

*is free from scattering resonances of the system (21)-(22).*

**Corollary 2.1.** *Let $c_{\mathcal{Q},1} = \max_{q\in\mathcal{Q}} \|n(x)\|_{L^\infty}$, and $c_{\mathcal{Q},2} = \max_{q\in\mathcal{Q}} \|r(t)\|_{L^\infty}$. Then it follows from Proposition 2.3 that the strip of width $h_{\mathcal{Q},1} = \frac{1}{4c_{\mathcal{Q},1}} e^{-2c_{\mathcal{Q},1}^2 c_{\mathcal{Q},2}}$, defined by*

$$S^*_{\mathcal{Q}} = \left\{k \in \mathbb{C};\ -h_{\mathcal{Q},1} \le Im(k) \le 0, Re(k) \neq 0\right\}, \tag{28}$$

*is free from scattering resonances of (21)-(22) for all $q \in \mathcal{Q}$.*

We also deduce from Proposition 2.3 and Proposition 2.1 that the coefficients $d_\pm(k)$ have holomorphic extensions in the strip $S_{\mathcal{Q}}$ defined by

$$S_{\mathcal{Q}} := \left\{k \in \mathbb{C};\ |\mathrm{Im}(k)| < h_{\mathcal{Q}}\right\}, \tag{29}$$

where

$$h_{\mathcal{Q}} = \min\{h_{\mathcal{Q},1}, \frac{1}{M_1}\}.$$

We next obtain global bounds of these functions in the strip.

**Proposition 2.4.** *There exist constants $k_{\mathcal{Q}} > 0$, $c_{\mathcal{Q}} > 0$, $d_{\mathcal{Q}} > 0$ that only depend on $\mathcal{Q}$, such that the following inequality hold*

$$|d_\pm(k) - 1| \le \frac{c_{\mathcal{Q}}}{|Re(k)|^2}, \quad \forall k \in S_{\mathcal{Q}},\ Re(k) \ge k_{\mathcal{Q}}, \tag{30}$$

$$|d_\pm(k)| \le d_{\mathcal{Q}}, \quad \forall k \in S_{\mathcal{Q}}. \tag{31}$$

**Proof.** Since the proofs of the bounds for $\mu_+(k)$ and $\mu_-(k)$ are identical we only provide the proof for the second scattering coefficient. The proof may be given by combining the general idea in the proof of Lemma 4.12 in [15] and the meromorphic extension result above.

Applying the Liouville transformation to (1), we find that $\xi_-(t,k) := N^{-1}(t)\phi_-(x(t),k)$ satisfies the Schrödinger equation:

$$\xi''(t,k) + (r(t) + k^2)\xi(t,k) = 0, \quad t \in (0,T), \tag{32}$$

with

$$\xi(t,k) = e^{ikt}, \quad t \le 0. \tag{33}$$

The impedance function $p_-(x,k)$ is then given by

$$p_-(x,k) = -n(x)\frac{\xi_-'(t,k)}{ik\xi_-(t,k)} + \frac{n'(x)}{2ikn(x)}.$$

Introducing now the auxiliary functions $\mathfrak{m}(t,k) = e^{ikt}\xi_-(t,k)$ and $\mathfrak{n}(t,k) = -\frac{1}{ik}e^{ikt}\xi_-'(t,k)$. A forward calculation yields

$$p_-(x,k) = \frac{\mathfrak{m}(t,k)}{\mathfrak{n}(t,k)}.$$

We deduce from the system (32)-(33), that $\mathfrak{m}(t,k)$ satisfies

$$\mathfrak{m}''(t,k) - 2ik\mathfrak{m}'(t,k) = -r(t)\mathfrak{m}(t,k), \quad t \in (0,T), \tag{34}$$

with the initial conditions

$$\mathfrak{m}(0,k) = 1 \ \mathfrak{m}'(0,k) = 0. \tag{35}$$

Multiplying (34) by $e^{-2ikt}$ and integrating, we get

$$\mathfrak{m}'(t,k) = -\int_0^t r(s)e^{2ik(t-s)}\mathfrak{m}(s,k)ds \tag{36}$$

Integrating the equation (36), we obtain

$$\mathfrak{m} = \frac{1}{2ik}\int_0^t r(s)(1-e^{2ik(t-s)})\mathfrak{m}(s,k)ds + 1, \tag{37}$$

$$= \mathcal{M}_k[\mathfrak{m}] + 1, \tag{38}$$

where $\mathcal{M}_k : C(0,T) \to C(0,T)$ is a compact operator defined by

$$\mathcal{M}_k[f](t) = \frac{1}{2ik}\int_0^t r(s)(1-e^{2ik(t-s)})f(s)ds. \tag{39}$$

Since $q$ belongs to $\mathcal{Q}$ there exist constants $k_{\mathcal{Q}} > 0$, and $c_{\mathcal{Q}} > 0$ such that

$$\|\mathcal{M}_k\| \le \frac{c_{\mathcal{Q}}}{|\mathrm{Re}(k)|}, \quad \forall k \in S_{\mathcal{Q}},\ |\mathrm{Re}(k)| \ge k_{\mathcal{Q}}.$$

Then, the Fredholm equation (37) has a unique solution satisfying

$$|\mathfrak{m}(t,k)-1| \leq \frac{2\, c_{\mathcal{Q}}}{|\mathrm{Re}(k)|}, \quad \forall t \in (0,T),\ \forall k \in S_{\mathcal{Q}},\ |\mathrm{Re}(k)| \geq k_{\mathcal{Q}}.$$

It can be approximated by the Neumann's series truncated at the second term

$$\mathfrak{m}(t,k) = 1 + \frac{1}{2ik}\int_0^t r(s)ds + O(\frac{1}{|\mathrm{Re}(k)|^2}),\ \forall k \in S_{\mathcal{Q}},\ |\mathrm{Re}(k)| \geq k_{\mathcal{Q}},$$

uniformly in $t \in (0,T)$.

Similarly, following the same approach, we have

$$\mathfrak{n}(t,k) = 1 + \frac{1}{2ik}\int_0^t r(s)ds + O(\frac{1}{|\mathrm{Re}(k)|^2}),\ \forall k \in S_{\mathcal{Q}},\ |\mathrm{Re}(k)| \geq k_{\mathcal{Q}},$$

uniformly in $t \in (0,T)$.

Consequently

$$\left|\frac{\mathfrak{m}(T,k)}{\mathfrak{n}(T,k)} - 1\right| = |p_-(1,k)-1| = O(\frac{1}{|\mathrm{Re}(k)|^2}),\ \forall k \in S_{\mathcal{Q}},\ |\mathrm{Re}(k)| \geq k_{\mathcal{Q}}, \tag{40}$$

Combining (40) with Proposition 2.1, and the fact that $d_-(k) = p_-(1,k)$ is holomorphic in $S_{\mathcal{Q}}$, we deduce the bound (2.4) for $d_-$. □

## 3. Observable part of the medium

Recall from (13)-(15) that the observable part of the medium $q_{k_0}(x)$ for $k \in (0,k_0)$. In this section using the truncated trace formula introduced in [14], we characterize $q_{k_0}(x)$ in terms of the frequency band $(0,k_0)$, and study how its determination is sensitive to errors in the measurements.

The following trace formula is on the asymptotic behavior in Proposition 2.2.

**Lemma 3.1.** *(Trace formula, [14])Let $q \in \mathcal{Q}$. Then the following trace formula holds*

$$q'(x) = \frac{2}{\pi}(1+q(x))\int_{-\infty}^{\infty}(p_+(x,k)-p_-(x,k))dk. \tag{41}$$

*More precisely, there exists a constant $c_{\mathcal{Q}} > 0$ such that the estimate*

$$\left\| q'(x) - \frac{2}{\pi}(1+q(x))\int_{-k_0}^{k_0}(p_+(x,k)-p_-(x,k))dk \right\|_{L^\infty(\mathbb{R})} \leq \frac{c_{\mathcal{Q}}}{k_0^m}, \tag{42}$$

*holds for all $k_0 \in \mathbb{C}^*$.*

The truncated version of the trace formula (3.1) means that the function

$$\frac{2}{\pi}\int_{-k_0}^{k_0}(p_+(x,k)-p_-(x,k))dk,$$

provides a good approximation of $\log(1+q(x))'$ as long as $k_0$ is large and the medium $q(x)$ is smooth.

**Lemma 3.2.** *Let $q\in\mathcal{Q}$. Then, there exist constants $c_{\mathcal{Q}}>0$ and $k_{\mathcal{Q}}>0$ such that truncated trace formula system (13)-(14)-(15) has a unique solution $q_{k_0}$. In addition the following estimates hold*

$$\left\|p_\pm-p_{k_0,\pm}\right\|_{C([0,1]\times[-k_0,k_0])},\ \left\|q-q_{k_0}\right\|_{L^\infty(\mathbb{R})}\le\frac{c_{\mathcal{Q}}}{k_0^m},$$

*for all $k_0\ge k_{\mathcal{Q}}$.*

Our second main result of this paper is to characterize $q_{k_0}(x)$ in terms of the frequency band $(0,k_0)$, and to show that the recovery of $q_{k_0}(x)$ is not sensitive to errors in the measurements.

We are now ready to give the proof of Theorem 1.3.

**Proof.** Let $p_{k_0,\pm}(x,k)$ and $\widetilde{p}_{k_0,\pm}(x,k)$ be the impedance functions solutions to the system (13)-(14)-(15) related respectively to the observable mediums $q_{k_0}$ and $\widetilde{q}_{k_0}$. To simplify the notation we introduce the impedance perturbations $u_\pm(x,k)=p_{k_0,\pm}(x,k)-\widetilde{p}_{k_0,\pm}(x,k)$ due to the measurements difference on the boundary $\epsilon(k)=d_+(k)-\widetilde{d}_+(k)$.

Then $u_\pm(x,k)$, $q_{k_0}$ and $\widetilde{q}_{k_0}$ verify

$$u_+'+ik(p_{k_0,+}+\widetilde{p}_{k_0,+})u_+-ik(q_{k_0}-\widetilde{q}_{k_0})=0, \tag{43}$$

$$u_-'-ik(p_{k_0,-}+\widetilde{p}_{k_0,-})u_-+ik(q_{k_0}-\widetilde{q}_{k_0})=0, \tag{44}$$

$$\left(\log\left|\frac{1+q_{k_0}}{1+\widetilde{q}_{k_0}}\right|\right)'-\frac{2}{\pi}\int_{-k_0}^{k_0}(u_+(x,k)-u_-(x,k))dk=0, \tag{45}$$

subject to the boundary conditions

$$u_+(0,k)\ =\ \epsilon(k);\quad u_-(0,k)\ =\ 0;\quad q_{k_0}(0)=\widetilde{q}_{k_0}=0, \tag{46}$$

for all $x\in(0,1)$, $k\in\mathbb{C}_+$.

Integrating the equation (42) over $(0,x)$, we obtain

$$\log\left|\frac{1+q_{k_0}}{1+\widetilde{q}_{k_0}}\right|=\frac{2}{\pi}\int_0^x\int_{-k_0}^{k_0}(u_+(t,k)-u_-(t,k))dkdt. \tag{47}$$

Solving the equations (42) and (42) gives

$$u_-(x,k) = -ik\int_0^x \widehat{q}(t)e^{ik\int_t^x (p_{k_0,-}(\tau,k)+\widetilde{p}_{k_0,-}(\tau,k))d\tau}dt$$

$$u_+(x,k) = \epsilon(k)e^{-ik\int_0^x (p_{k_0,+}(t,k)+\widetilde{p}_{k_0,+}(t,k))dt} + ik\int_0^x \widehat{q}(t)e^{-ik\int_t^x (p_{k_0,+}(\tau,k)+\widetilde{p}_{k_0,+}(\tau,k))d\tau}dt,$$

where $\widehat{q}(t) = q_{k_0}(t) - \widetilde{q}_{k_0}(t)$.

Substituting the new expressions of $u_\pm(x,k)$ into the equality (42), we find

$$\log\left|\frac{1+q_{k_0}}{1+\widetilde{q}_{k_0}}\right| = \tag{48}$$

$$\frac{2}{\pi}\int_{-k_0}^{k_0} \epsilon(k)e^{-ik\int_0^x (p_{k_0,+}(t,k)+\widetilde{p}_{k_0,+}(t,k))dt}dk + \frac{2i}{\pi}\int_0^x\int_0^r \widehat{q}(t)K(r,t,k_0)dtdr,$$

where

$$K(r,t,k_0) = \int_{-k_0}^{k_0} k\left(e^{-ik\int_t^r (p_{k_0,+}(\tau,k)+\widetilde{p}_{k_0,+}(\tau,k))d\tau} + e^{ik\int_t^r (p_{k_0,-}(\tau,k)+\widetilde{p}_{k_0,-}(\tau,k))d\tau}\right)dk,$$

for $r, t \in (0,1)$. □

**Lemma 3.3.** *Under the same conditions as in Theorem 1.3, there exist constants $c_Q > 0$ and $k_Q > 0$ such that*

$$|K(r,t,k_0)| \le c_Q,$$

*for all $r, t \in (0,1)$ and $k_0 \ge k_Q$.*

**Proof.** (Lemma 3.3) First we remark from the uniqueness of solution to the system (13)-(14)-(15) that $p_{k_0,+}$ like the impedance function $p_+(x,k)$, satisfies

$$\overline{p_{k_0,+}(x,k)} = p_{k_0,+}(x,-k),$$

for all $x \in (0,1)$. Then, by a change of variables ($k \to -k$), we obtain

$$\int_{-k_0}^{k_0} ke^{-ik\int_t^r (p_{k_0,+}(\tau,k)+\widetilde{p}_{k_0,+}(\tau,k))d\tau}dk = -\int_{-k_0}^{k_0} ke^{ik\int_t^r (\overline{p_{k_0,+}(\tau,k)}+\overline{\widetilde{p}_{k_0,+}(\tau,k)})d\tau}dk$$

Hence, $K$ can be rewritten as

$$K(r,t,k_0) = -\int_{-k_0}^{k_0} k\left(e^{ik\int_t^r(\overline{p_{k_0,+}(\tau,k)}+\overline{\widetilde{p}_{k_0,+}(\tau,k)})d\tau} - e^{ik\int_t^r(p_{k_0,-}(\tau,k)+\widetilde{p}_{k_0,-}(\tau,k))d\tau}\right)dk,$$

Now, let $\widetilde{K}$ be defined as follows

$$K(r,t,k_0) = -\int_{-k_0}^{k_0} k\left(e^{ik\int_t^r(\overline{p_+(\tau,k)}+\overline{\widetilde{p}_+(\tau,k)})d\tau} - e^{ik\int_t^r(p_-(\tau,k)+\widetilde{p}_-(\tau,k))d\tau}\right)dk,$$

According to Lemma 3.2, there the integrand of $K(r,t,k_0) - \widetilde{K}(r,t,k_0)$ decays like $\frac{1}{k_0^{m-1}}$ uniformly with respect to $r, t \in [0,1]$. Therefore there exist constants $c_Q > 0$ and $k_Q > 0$ such that

$$\left|K(r,t,k_0) - \widetilde{K}(r,t,k_0)\right| \leq c_Q,$$

for all $k \geq k_Q$.

The asymptotic expansions (2.2) and (2.2) in Theorem 2.2 imply that

$$\left|e^{ik\int_t^r \overline{p_\pm(\tau,k)}d\tau}\right|, \left|e^{ik\int_t^r \overline{p_\pm(\tau,k)}d\tau}\right| \leq c_Q$$

for all $t, r \in [0,1]$ and $k \in \mathbb{C}^+$. Furthermore

$$\left|e^{ik\int_t^r \overline{p_\pm(\tau,k)}d\tau} - e^{ik\int_t^r \overline{p_\pm(\tau,k)}d\tau}\right| \leq \frac{c_Q}{k_0^m}$$

all $t, r \in [0,1]$ and $k_0 \geq k_Q$. Combining the previous inequalities we finally obtain that $\widetilde{K}(r,t,k_0)$ is uniformly bounded over $[0,1]^2$ for all $k_0 \geq k_Q$, which finishes the proof of the lemma. □

Back to the equation (42), by combining the integral equation with the estimates of Lemma 3.3 and the bounds over the functions $p_{k_0,+}$ and $\overline{p}_{k_0,+}$, we obtain

$$\begin{aligned}\left|\log\left|\frac{1+q_{k_0}}{1+\widetilde{q}_{k_0}}\right|\right| &\leq c_Q\left(\|\epsilon(k)\|_{L^1(-k_0,k_0)} + \int_0^x\int_0^r |\widehat{q}(t)|dtdr\right),\\ &\leq c_Q\left(\|\epsilon(k)\|_{L^1(-k_0,k_0)} + \int_0^x |\widehat{q}(t)|dt\right),\end{aligned} \tag{49}$$

for all $x \in (0,1)$.

Observing that the fact that $q_{k_0} \to q$ and $\widetilde{q}_{k_0} \to \widetilde{q}$ in $L^\infty(0,1)$ combined with inequalities (2) imply that the functions $1+q_{k_0}$ and $1+\widetilde{q}_{k_0}$ are lower ad upper bounded for large $k_0$, that is, there exist a constant $k_Q > 0$ such that

$$\frac{n_0}{2} \leq 1 + q_{k_0}(x),\ 1+\widetilde{q}_{k_0} \leq 2n_0$$

for all $x \in [0, 1]$ and $k_0 \geq k_Q$. Therefore

$$\widehat{q}(x) \leq \frac{1}{2n_0} \left| \log \left| \frac{1+q_{k_0}}{1+\widetilde{q}_{k_0}} \right| \right|,$$

for all $x \in [0, 1]$. Combining the last inequality with (49) gives

$$|\widehat{q}(x)| \leq c_Q \left( \|\epsilon(k)\|_{L^1(0,k_0)} + \int_0^x |\widehat{q}(t)| dt \right), \tag{50}$$

for all $x \in (0, 1)$ and $k_0 \geq k_Q$.

Applying Gronwall's inequality (Lemma 6.1) on (50), with the choice of $\rho_Q = c_Q + c_Q^2 e^{c_Q}$, we find

$$|\widehat{q}(x)| \leq \rho_Q \|\epsilon(k)\|_{L^1(0,k_0)}$$

for all $x \in \mathbb{R}$ and $k_0 \geq k_Q$, which finishes the proof of the Theorem 1.3. $\square$

**Remark 3.1.** The estimate of Theorem 1.3 provides a basis for excellent numerical results to reconstruct the observable part of the medium. In addition, it is an integral part of the proof of Theorem 1.1.

Now, we go back to the proof of the main theorems. Lemma 3.2 implies that if $k_0$ is large enough we have the existence of $q_{k_0}$ and $\widetilde{q}_{k_0}$. By splitting the difference $q - \widetilde{q}$ into three parts we have

$$\|q - \widetilde{q}\|_{L^\infty(0,1)} \leq \|q - q_{k_0}\|_{L^\infty(0,1)} + \|q_{k_0} - \widetilde{q}_{k_0}\|_{L^\infty(0,1)} + \|\widetilde{q} - \widetilde{q}_{k_0}\|_{L^\infty(0,1)}.$$

Using now the results of Lemma 3.2 and Theorem 1.3 to estimate each part of the right hand side we finish the proof of Theorem 1.1. $\square$

**Theorem 3.1.** *Assume that $q, \widetilde{q}$ be two medium functions in $\mathcal{Q}$. Let $d_+(k)$ and $\widetilde{d}_+(k)$ be the boundary measurements associated respectively to $q$ and $\widetilde{q}$ as defined in (7). Then, there exist constants $c_Q > 0$ and $k_Q$ such that*

$$\|q - \widetilde{q}\|_{L^\infty(\mathbb{R})} \leq c_Q \left( \|d_\pm - \widetilde{d}_\pm\|_{L^1(0,k_0)} + \frac{1}{k_0^m} \right), \tag{51}$$

*for all $k_0 \geq k_Q$.*

Obviously this result implies the uniqueness of the multi-frequency inverse medium, and a conditional Lipschitz stability estimate when the band of frequency is large enough.

**Corollary 3.1.** *Assume that $q, \widetilde{q}$ be two medium functions in $\mathcal{Q}$. Let $d_+(k)$ and $\widetilde{d}_+(k)$ be the boundary measurements associated respectively to $q$ and $\widetilde{q}$ as defined in (7), satisfying $\|d_\pm - \widetilde{d}_\pm\|_{L^\infty(0,+\infty)} < 1$. Then, there exists a constant $c_Q > 0$ such that the following Lipschitz stability*

$$\|q-\widetilde{q}\|_{L^\infty(\mathbb{R})} \le c_{\mathcal{Q}} \|d_+(k)-\widetilde{d}_+(k)\|_{L^\infty(0,+\infty)}^{\frac{m}{m+1}},$$

*holds.*

**Proof.** Under the same assumptions of Theorem 3.1, we have

$$\|q-\widetilde{q}\|_{L^\infty(\mathbb{R})} \le c_{\mathcal{Q}} \left( k_0 \|d_+-\widetilde{d}_+\|_{L^\infty(0,k_0)} + \frac{1}{k_0^m} \right), \tag{52}$$

for all $k_0 = sk_{\mathcal{Q}}$ with $s > 1$. By taking $s = \|d_+-\widetilde{d}_+\|_{L^\infty(0,k_0)}^{-\frac{1}{m+1}}$, we get the wanted estimate. □

**Remark 3.2.** The estimate (3.1) has two parts: the first is Lipschitz in terms of the errors in measurements, and the second decays as the size of the frequency interval takes larger values. Clearly, this shows that as the frequency increases a conditional Hölder stability in $L^\infty$ norm can be reached as illustrated in Corollary 3.1.

## 4. Proof of Theorem 1.1

In this section we prove the stability estimate (1.1). We first provide the following conditional stability estimate for the unique continuation of $d_\pm$ on a line.

**Theorem 4.1.** *Let $k_0 > 0$, $d_\pm$ and $\widetilde{d}_\pm$ be the impedance coefficients given in (7) for respectively $q$ and $\widetilde{q}$ in $\mathcal{Q}$.. Then the following estimate hold*

$$|d_\pm - \widetilde{d}_\pm|(k) \le 2d_{\mathcal{Q}} \|d_\pm-\widetilde{d}_\pm\|_{L^\infty(0,k_0)}^{w_0(k,k_0)}, \tag{53}$$

*for all $k \ge k_0$, where $d_{\rm II}$ is the constant appearing in Proposition 2.4.*

**Proof.** We deduce from Proposition 2.4 that

$$|d_-(k) - \widetilde{d}_-(k)| \le 2d_{\mathcal{Q}}, \tag{54}$$

for all $k \in S_{\mathcal{Q}}$.

Without loss of generality we can assume that $h_{\mathcal{Q}} = \frac{\pi}{2n_{\mathcal{Q}}}$, where $n_{\mathcal{Q}} \in \mathbb{N}^*$. Let $S_{h_{\mathcal{Q}}} = \{k \in \mathbb{C}; \mathrm{Re}(k) > 0,\ |\mathrm{Im}(k)| < h_{\mathcal{Q}}\}$, be half a strip, and et $w_0(k;k_0)$ be the harmonic measure of the complex open domain $S_{h_{\mathcal{Q}}} \setminus [0,k_0]\times\{0\}$. It is the unique solution to the system:

$$\begin{aligned}
\Delta w(k;k_0) &= 0 \quad k \in S_{h_{\mathcal{Q}}} \setminus [0,k_0]\times\{0\},\\
w(k;k_0) &= 0 \quad k \in \partial S_{h_{\mathcal{Q}}},\\
w(k;k_0) &= 1 \quad k \in (0,k_0]\times\{0\}.
\end{aligned}$$

The holomorphic unique continuation of the functions $d_\pm - \widetilde{d}_\pm$ using the Two constants Theorem [24,32], gives

$$\|d_\pm-\widetilde{d}_\pm\|_{L^\infty(0,k)} \le (2d_{\mathcal{Q}})^{1-w_0(k,k_0)} \|d_\pm-\widetilde{d}_\pm\|_{L^\infty(0,k_0)}^{w_0(k,k_0)}, \qquad \forall k \ge k_0.$$

Finally, the bounds satisfied by $w(k;k_0)$ are obtained from Lemma 6.2. □

We deduce again from Proposition 2.4 the existence of $k^\star \in \mathbb{R}_+$ satisfying

$$\|d_\pm - \widetilde{d}_\pm\|_{L^\infty(0,+\infty)} = |d_-(k^\star) - \widetilde{d}_-(k^\star)|.$$

We then deduce from Theorem 4.1 the following estimate

$$\|d_\pm - \widetilde{d}_\pm\|_{L^\infty(0,+\infty)} = |d_-(k^\star) - \widetilde{d}_-(k^\star)| \leq 2d_Q \|d_\pm - \widetilde{d}_\pm\|^{w_0(k^\star,k_0)}_{L^\infty(0,k_0)}.$$

Considering the global stability estimate in Corollary 3.1, we obtain

$$\|q - \widetilde{q}\|_{L^\infty(\mathbb{R})} \leq c_Q \|d_+ - \widetilde{d}_+\|^{\frac{m}{m+1}}_{L^\infty(0,+\infty)} \leq 2\, c_Q d_Q \|d_\pm - \widetilde{d}_\pm\|^{\frac{m}{m+1} w_0(k^\star,k_0)}_{L^\infty(0,k_0)},$$

which finishes the proof of the theorem. □

## 5. Proof of Theorem 1.2

In this section we prove the stability estimates (1.2)-(1.2). We start by deriving a lower bound to the harmonic measure $w_0$ on $\mathbb{R}_+$.

**Proposition 5.1.** *The harmonic measure $w_0(k, k_0)$ satisfies*

$$w_0(k, k_0) \geq \frac{6}{\pi}\eta(k_0)e^{-n_Q k},$$

**Proof.** It is known in the literature that the following inequality [34]

$$\arctan(x) \geq 3\widehat{\eta}(x),$$

holds for all $x > 0$, where

$$\widehat{\eta}(x) = \frac{x}{1 + 2\sqrt{1 + x^2}}.$$

Hence

$$\frac{2}{\pi}\arctan\Big(\frac{(e^{k_0} - 1)^{n_Q}}{\sqrt{(e^k - 1)^{2n_Q} - (e^{k_0} - 1)^{2n_Q}}}\Big) \geq \frac{2}{\pi}\arctan\left((e^{k_0} - 1)^{n_Q} e^{-n_Q k}\right) \geq \frac{6}{\pi}\eta(k_0)e^{-n_Q k},$$

where $\eta(k_0) = \widehat{\eta}((e^{k_0} - 1)^{n_Q})$. □

We deduce from Proposition 2.2 that

$$|d_+(k) - \widetilde{d}_+(k)| \leq |d_+(k) - 1| + |\widetilde{d}_+(k) - 1| \leq \frac{c_Q}{k^m}, \tag{55}$$

for all $k \in \mathbb{R}_+^*$, with $c_Q \geq 2d_Q$.

Theorem 4.1 and the last inequality lead to

$$|d_+(k) - \widetilde{d}_+(k)| \leq \min\{2d_Q \varepsilon^{w_0(k,k_0)}; \frac{c_Q}{k^m}\},$$

for all $k \in \mathbb{R}_+^*$.

Now we consider the two following cases.

Case 1: assume that $\frac{c_{Q,1}}{k_0^m} \leq \varepsilon$ holds.

Hence $\|d_\pm - \widetilde{d}_\pm\|_{L^\infty(0,+\infty)} \leq \varepsilon$ is satisfied, and we immediately get the first stability estimate (1.2).

Case 2: assume that $\frac{c_Q}{k_0^m} > \varepsilon$ holds. Due to the monotonicity of the functions $w_0(k_1, k_0)$ and $\frac{1}{k^m}$, there exists a unique $k_1 \in (k_0, +\infty)$ satisfying

$$\frac{c_Q}{k_1^m} = 2d_Q \varepsilon^{w_0(k_1,k_0)}, \tag{56}$$

and

$$\|d_+ - \widetilde{d}_+\|_{L^\infty(0,+\infty)} \leq \frac{c_Q}{k_1^m}. \tag{57}$$

Since $0 < \varepsilon < 1$, and $c_Q \geq 2d_Q$, we have $k_1 > 1$.

On the other hand combining (56), and Proposition 5.1, gives

$$\frac{c_Q}{k_1^m} \leq \varepsilon^{\frac{6}{\pi}\eta(k_0)e^{-n_Q k}},$$

which in turn leads to

$$e^{n_Q k_1}\left(\ln(2d_Q) - \ln(c_Q) + m\ln(k_1)\right) \geq \frac{6}{\pi}\eta(k_0)|\ln(\varepsilon)|.$$

Since $c_Q \geq 2d_Q$, and $k_1 > 1$, we deduce from the last inequality the existence of $c_Q > 0$ such that

$$e^{c_Q k_1} \geq \eta(k_0)|\ln(\varepsilon)|,$$

holds. Hence

$$k_1 c_Q \geq \ln\left(\eta(k_0)|\ln(\varepsilon)|\right).$$

Combining now the last inequality and estimate (57), we find

$$\|d_+ - \widetilde{d}_+\|_{L^\infty(0,+\infty)} \leq \frac{c_Q}{\left(\ln\left(\eta(k_0)|\ln(\varepsilon)|\right)\right)^m}.$$

By Corollary 3.1, and the last inequality, we obtain the desired stability estimate (1.2), with $k_Q = c_Q^{\frac{m}{m+1}}$. □

## 6. Appendix

We first recall the Gornwall's inequality.

**Lemma 6.1.** *Assume that $u, v$ and $w : [0,1] \to \mathbb{R}_+$ are continuous functions satisfying the inequality*

$$u(x) \leq v(x) + \int_0^x u(t)w(t)dt,$$

*for all $x \in [0,1]$. Then*

$$u(x) \leq v(x) + \int_0^x v(t)w(t)e^{\int_t^x w(\tau)d\tau}dt.$$

We next give upper and lower estimates of a harmonic measure in a complex strip containing a slit.

**Lemma 6.2.** *Fix $n^\star \in \mathbb{N}^*$, and let $h^\star = \frac{\pi}{2n^\star}$, $k_0 > 0$ be two fixed real constants, $S_{h^\star} = \{k \in \mathbb{C}; Re(k) > 0,\ |Im(k)| < h^\star\}$ be half a strip. Denote $w_0(k, k_0)$ the harmonic measure of $S_{h^\star} \setminus (0, k_0] \times \{0\}$, Then*

$$\frac{2}{\pi}\arctan\left(\frac{(e^{k_0}-1)^{n^\star}}{\sqrt{(e^k-1)^{2n^\star} - (e^{k_0}-1)^{2n^\star}}}\right)$$
$$\leq w_0(k, k_0) \leq \frac{2}{\pi}\arctan\left(\inf\{\frac{k_0}{\sqrt{k^2-k_0^2}}, \frac{e^{k_0 n^\star}}{\sqrt{e^{2kn^\star} - e^{2k_0 n^\star}}}\}\right),$$

*for all $k \geq k_0$.*

**Proof.** For $n \in \mathbb{N}^*$, denote by $w_n(k, k_0)$ the harmonic measure of $[0, k_0] \times \{0\}$ in the sector $\mathbb{S}_{\frac{\pi}{2n}} = \{k \in \mathbb{C}; |\arg(k)| < \frac{\pi}{2n})\}$.

Let $\Xi_n(k, k_0) = \sqrt{k^{2n} - k_0^{2n}}$ be the conformal mapping of the domain $\mathbb{S}_{\frac{\pi}{2n}} \setminus [0, k_0] \times \{0\}$ onto the right half-plane $\mathbb{S}_{\frac{\pi}{2}}$. Here $\sqrt{k}$ is the principal branch of square root function on $\mathbb{C} \setminus (-\infty, 0)$ satisfying $\sqrt{1} = 1$. The parts of the boundary $[0, k_0] \times \{0\}|_\pm$ are then mapped onto $[-ik_0^n, ik_0^n]$.

Now define $w^\star(z, k_0^n)$ to be the harmonic measure of the right half-plane $\mathbb{S}_{\frac{\pi}{2}} \setminus [-ik_0^n, ik_0^n]$. The explicit expression of $w^*$ is well known [19]

$$w^*(z, k_0^n) = \frac{2}{\pi}\arctan\left(\frac{k_0^n}{z}\right), \text{ for } z \in (0, +\infty).$$

Since $w_n(k, k_0) = w^*(\Xi_n(k, k_0), k_0^n)$ for $k \in \mathbb{S}_{\frac{\pi}{2n}} \setminus [0, k_0] \times \{0\}$, we also obtain

$$w_n(k,k_0) = \frac{2}{\pi}\arctan(\frac{k_0^n}{\sqrt{k^{2n}-k_0^{2n}}}),\ \text{for } k \in (k_0,+\infty). \tag{58}$$

Let $\Xi_{-1}(k) = e^k$, be the conformal mapping of the domain $S_{\frac{\pi}{2n^\star}} \setminus [0,k_0]\times\{0\}$ onto the domain $\mathbb{S}_{\frac{\pi}{2n^\star}} \setminus \overline{B_1(0)} \cup \big([1,e^{k_0}]\times\{0\}\big)$.

Since $w_0(k,k_0) \le w_{n^\star}(\Xi_{-1}(k),\Xi_{-1}(k_0))$ on $\partial\left(\mathbb{S}_{\frac{\pi}{2n^\star}} \setminus \overline{B_1(0)} \cup \big([1,e^{k_0}]\times\{0\}\big)\right)$, we deduce from the maximum principle

$$w_0(k,k_0) \le \frac{2}{\pi}\arctan(\frac{e^{k_0 n^\star}}{\sqrt{e^{2kn^\star}-e^{2k_0n^\star}}}), \tag{59}$$

for all $k \ge k_0$. By construction we have $S_{\frac{\pi}{2n^\star}} \subset \mathbb{S}_{\frac{\pi}{2}}$, and consequently $0 = w_0(k,k_0) \le w_2(k,k_0)$ on $\{|\mathrm{Im}(k)| = \frac{\pi}{2n^\star}\}$. Then again by the maximum principle we obtain

$$w_0(k,k_0) \le w_1(k,k_0) = \frac{2}{\pi}\arctan(\frac{k_0}{\sqrt{k^2-k_0^2}}), \tag{60}$$

for all $k \ge k_0$.

Combining inequalities (59) and (60), we finally find

$$w_0(k,k_0) \le \frac{2}{\pi}\arctan\left(\inf\{\frac{k_0}{\sqrt{k^2-k_0^2}}, \frac{e^{k_0n^\star}}{\sqrt{e^{2kn^\star}-e^{2k_0n^\star}}}\}\right), \tag{61}$$

which gives the right-hand side inequality.

Let $\Xi_{-2}(k) = e^k - 1$, be the conformal mapping of the domain $\mathbb{S}_{\frac{\pi}{n}} \setminus [0,k_0]\times\{0\}$ onto the domain $D_{n^\star} \setminus [0, e^{k_0}-1]\times\{0\}$, where $D_{n^\star} = \{z \in \mathbb{C};\, z+1 \in \mathbb{S}_{\frac{\pi}{2n^\star}}, \mathrm{Re}(z)+1 > 0\}$. Then $w_0(\Xi_{-2}^{-1}(k),k_0)$ is the harmonic measure of $[0,e^{k_0}-1]\times\{0\}$ in the domain $D_{n^\star}$. Now since $[0,e^{k_0}-1]\times\{0\} \subset \mathbb{S}_{\frac{\pi}{2n^\star}} \subset D_{n^\star}$, we have $0 = w_{n^\star}(k,k_0) \le w_0(\Xi_{-2}^{-1}(k),k_0)$ on $\partial\mathbb{S}_{\frac{\pi}{2n^\star}}$.

The maximum principle implies that $w_{n^\star}(k, e^{k_0}-1) \le w_0(\Xi_{-2}^{-1}(k),k_0)$ holds on $\mathbb{S}_{\frac{\pi}{2n^\star}}$, and particularly, we have

$$\frac{2}{\pi}\arctan(\frac{(e^{k_0}-1)^{n^\star}}{\sqrt{k^{2n^\star}-(e^{k_0}-1)^{2n^\star}}}) \le w_0(\Xi_{-2}^{-1}(k),k_0),\ \text{for all } k \in (e^{k_0}-1,+\infty),$$

or equivalently

$$\frac{2}{\pi}\arctan(\frac{(e^{k_0}-1)^{n^\star}}{\sqrt{(e^k-1)^{2n^\star}-(e^{k_0}-1)^{2n^\star}}}) \le w_0(k,k_0),\ \text{for all } k \in (k_0,+\infty), \tag{62}$$

which provides the desired left-hand inequality. $\square$

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