

Asymptotic stability for a free boundary tumor model with angiogenesis

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Abstract

In this paper, we study a free boundary problem modeling solid tumor growth with vasculature which supplies nutrients to the tumor; this is characterized in the Robin boundary condition. It was recently established [Discrete Cont. Dyn. Syst. 39 (2019) 2473–2510] that for this model, there exists a threshold value μ^* such that the unique radially symmetric stationary solution is linearly stable under non-radial perturbations for $0 < \mu < \mu^*$ and linearly unstable for $\mu > \mu^*$. In this paper we further study the nonlinear stability of the radially symmetric stationary solution, which introduces a significant mathematical difficulty: the center of the limiting sphere is not known in advance owing to the perturbation of mode 1 terms. We prove a new fixed point theorem to solve this problem, and finally obtain that the radially symmetric stationary solution is nonlinearly stable for $0 < \mu < \mu^*$ when neglecting translations.

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1. Introduction

Over the past few decades, a variety of PDE models describing tumor growth in the form of free boundary problems have been proposed, developed and studied; see [2–4,6,22,23,29,30] and references therein. These models are based on mass conservation laws for cell densities and reaction-diffusion processes for nutrient concentrations within the tumor. Rigorous mathematical analysis and numerical simulation of these models have drawn considerable attention, and many interesting results have been established. Mathematical analysis of such free boundary problems not only provides important insight into the growing mechanism of tumors, but may also aid in the understanding of experimental and clinical observations.

In this paper, we consider a free boundary tumor model with angiogenesis, consisting of proliferating cells only. This model was proposed by Friedman and Lam [20] as an extension to the model of Byrne and Chaplain [3]. Let $\Omega(t)$ denote the tumor region at time t , σ and p be the concentration of nutrients and the pressure resulting from movement of cells, respectively. The equations are given by (see [20]):

$$\sigma_t = \Delta \sigma - \sigma, \quad x \in \Omega(t), \quad t > 0, \quad (1.1)$$

$$-\Delta p = \mu(\sigma - \tilde{\sigma}), \quad x \in \Omega(t), \quad t > 0, \quad (1.2)$$

where the positive parameter μ measures the aggressiveness of the tumor, and the positive constant $\tilde{\sigma}$ is a threshold concentration for proliferation. The equation (1.2) for the pressure is obtained by the conservation of mass (i.e., the cell proliferation rate $\mu(\sigma - \tilde{\sigma}) = \nabla \cdot \mathbf{V}$, here we assume a linear relationship between the cell proliferation rate and the nutrient concentration, \mathbf{V} denotes the velocity field of the tumor cell movement) and the Darcy's law assumption (i.e., $\mathbf{V} = -\nabla p$, where the extracellular matrix is regarded as porous medium).

These equations are supplemented with the boundary conditions:

$$\frac{\partial \sigma}{\partial \mathbf{n}} + \beta(\sigma - 1) = 0, \quad x \in \partial\Omega(t), \quad t > 0, \quad (\tilde{\sigma} < 1), \quad (1.3)$$

$$p = \gamma\kappa, \quad x \in \partial\Omega(t), \quad t > 0, \quad (1.4)$$

where in (1.3) the impact from angiogenesis is incorporated (see [20]), \mathbf{n} is the outward normal, (1.4) describes the cell-to-cell adhesiveness (see [2]) with γ being the adhesiveness coefficient and κ being the mean curvature.

If the velocity field is continuous up to the boundary, the velocity of the free boundary is given by (see [20]):

$$V_n = -\nabla p \cdot \mathbf{n} = -\frac{\partial p}{\partial \mathbf{n}}, \quad x \in \partial\Omega(t), \quad t > 0, \quad (1.5)$$

where V_n is the velocity of the free boundary in the direction \mathbf{n} .

We finally impose the following initial conditions:

$$\sigma|_{t=0} = \sigma_0 \quad \text{in } \Omega(0), \quad \text{where } \Omega(0) \text{ is given.} \quad (1.6)$$

Angiogenesis is a process that tumor cells secrete chemicals called tumor growth factors to stimulate the formation of new blood vessels penetrating into the tumor. In this model, nutrients

enter the tumor only through these new blood vessels. The positive constant β in (1.3) reflects strength of the blood vessel system of the tumor: $\beta = 0$ means that the tumor does not have its own blood vessel system, $\beta = \infty$ indicates that the tumor is all surrounded by the blood vessels which reduces to the Dirichlet boundary condition $\sigma = 1$.

The special case of this model $\beta = \infty$, corresponding to the Dirichlet boundary condition $\sigma = 1$, has drawn great attention of many researchers. Indeed, it was proved in [15] that a branch of symmetry-breaking stationary solutions bifurcates from the radially symmetric stationary solutions for each $\mu_n(R_S)$ ($n \geq 2$) with free boundary

$$r = R_S + \varepsilon Y_{n,0}(\theta) + O(\varepsilon^2), \quad (1.7)$$

where $Y_{n,0}$ is the spherical harmonic of order $(n, 0)$. In [1] it was proved that if μ is sufficiently small, then under any small perturbations the stationary solution is asymptotically stable. This work was improved by Friedman and Hu [16,17]. They determined a threshold value μ^* for which the radially symmetric stationary solution changes from stability to instability under non-radial perturbations. Furthermore, linear stability analysis of the stationary solution at the bifurcation point $\mu = \mu^*$ was studied in [19]. For the case that the nutrient consumption rate and the tumor cell proliferate rate are general functions, by using a center manifold analysis, Cui and Escher [8,10] found a positive critical value γ^* (for the adhesiveness constant γ in (1.4)) such that for $\gamma > \gamma^*$ the radially symmetric stationary solution is asymptotically stable with respect to small non-radial perturbations, while for $0 < \gamma < \gamma^*$ this stationary solution is unstable.

Recently, Zhou and Wu [40] considered the case that the nutrient concentration satisfies the Gibbs-Thomson relation on the boundary, and studied the existence and stability of the flat stationary solution. For the corresponding radially symmetric stationary solutions, it was proved [31] that the stationary solution with larger radius is asymptotically stable and the other smaller one is unstable with respect to radial perturbations. In the sequel [36], they further refined the above result, and proved that under non-radial perturbations there exists a positive threshold value γ^* such that the radially symmetric stationary solution with larger radius bifurcates from instability to stability. Moreover, considerable research works on asymptotic stability have been established for various tumor growth models, for instance, see [7,9,11–14,18,24,25,28,32–35,37–39,41].

The radially symmetric version of the system (1.1)–(1.5) was studied by Friedman and Lam [20], in which they established the asymptotic behavior of the global solution and the existence of a unique radially symmetric stationary solution given by (see also [26, Section 2])

$$\sigma_S(r) = \frac{\beta}{\beta + R_S P_0(R_S)} \frac{R_S^{\frac{1}{2}} I_{1/2}(r)}{r^{\frac{1}{2}} I_{1/2}(R_S)}, \quad 0 < r < R_S, \quad (P_n(R_S) \text{ is defined in (2.1)}), \quad (1.8)$$

$$p_S(r) = -\mu \sigma_S(r) + \frac{1}{6} \mu \tilde{\sigma} r^2 + C_1, \quad 0 < r < R_S, \quad (1.9)$$

where

$$C_1 = \frac{1}{R_S} + \frac{\mu \beta}{\beta + R_S P_0(R_S)} - \frac{1}{6} \mu \tilde{\sigma} R_S^2, \quad (1.10)$$

and R_S is the unique solution of

$$\frac{\beta P_0(R_S)}{\beta + R_S P_0(R_S)} = \frac{1}{3} \tilde{\sigma}. \quad (1.11)$$

Recently, it was established in [26] that $\mu_n(R_S)$ ($n \geq 2$ even) given by

$$\mu_n(R_S) = \frac{3n(n-1)(n+2)}{2\tilde{\sigma}R_S^4} \cdot \frac{\frac{n}{R_S} + \beta + R_S P_n(R_S)}{(n + \beta R_S)P_1(R_S) - (1 + \beta R_S)P_n(R_S)} \quad (1.12)$$

are bifurcation points of the symmetry-breaking stationary solution of the system (1.1)–(1.5) with free boundary

$$r = R_S + \varepsilon Y_{n,0}(\theta) + o(\varepsilon).$$

Moreover, $\mu_n(R_S)$ is monotonically increasing in n :

$$\mu_n(R_S) < \mu_{n+1}(R_S). \quad (1.13)$$

In a recent paper [27], we considered the linear stability of the stationary solution under non-radial perturbations and found a critical value $\mu^* = \mu^*(R_S)$ ($\leq \mu_2(R_S)$) such that the stationary solution is stable for the linearized problem if $0 < \mu < \mu^*(R_S)$, and it is unstable for the linearized problem if $\mu > \mu^*(R_S)$. With the linear stability of the radially symmetric stationary solution (σ_S, p_S, R_S) already established, the present paper addresses the following question: *Is this stationary solution also stable for the original fully nonlinear free boundary problem for $0 < \mu < \mu^*(R_S)$?*

Throughout the paper, by a rescaling if necessary we take $\gamma = 1$. We assume the initial conditions are perturbed as follows:

$$\partial\Omega(0) : r = R_S + \varepsilon\rho_0(\theta, \varphi), \quad \sigma|_{t=0} = \sigma_S(r) + \varepsilon w_0(r, \theta, \varphi), \quad (1.14)$$

where ρ_0 and w_0 are bounded functions.

For the linearized problem, the translation of the origin resulting from mode 1 is easily seen. However, for the fully nonlinear problem, the perturbation of mode 1 is “hidden” in the equation and it is not clear that mode 1 can be separated from other modes. Our grand challenge is to find a “correct” translation of the origin to take care of perturbations from the mode 1 terms; this is characterized in the following (Theorem 4.1) fixed point theorem.

The structure of this paper is as follows. In Section 2, we recall a few results from [16,27] which will be needed in the sequel. In Section 3, we transform the nonlinear problem into a new PDE system with a fixed boundary by Hanzawa transformation. In order to establish the stability result, it is crucial to find a new center of the sphere resulting from the perturbation of initial values; this is carried out in Section 4. This determination is crucial to our results. In Section 5, we derive the necessary estimates for all modes and complete the asymptotic stability in Section 6.

2. Preliminaries

In this section, we shall collect the properties of the function $P_n(\xi)$ that is introduced by Friedman and Hu [16] and some results derived in the argument of the linear stability [27], which are needed in our discussion.

The function $P_n(\xi)$ is defined by (see [16])

$$P_n(\xi) = \frac{I_{n+3/2}(\xi)}{\xi I_{n+1/2}(\xi)} = 2 \sum_{m=1}^{\infty} \frac{1}{\xi^2 + j_{n+1/2,m}^2}, \quad n = 0, 1, 2, 3, \dots \quad (2.1)$$

where $j_{n,m}$ are the m^{th} real positive zeros of Bessel functions $J_n(x)$, and $I_n(\xi)$ are the modified Bessel functions.

Recall [16] that, for any $n \geq 0$,

$$P_0(\xi) = \frac{1}{\xi} \coth \xi - \frac{1}{\xi^2}, \quad (2.2)$$

$$P_n(\xi) = \frac{1}{\xi^2 P_{n+1}(\xi) + (2n+3)}, \quad (2.3)$$

$$\frac{d}{dr} \left(\frac{I_{n+1/2}(\sqrt{s+1}r)}{r^{1/2}} \right) = \frac{\sqrt{s+1} I_{n+3/2}(\sqrt{s+1}r) + \frac{n}{r} I_{n+1/2}(\sqrt{s+1}r)}{r^{1/2}}. \quad (2.4)$$

As in [27], we have

$$\lambda \triangleq \left(\frac{\partial^2 \sigma_S}{\partial r^2} + \beta \frac{\partial \sigma_S}{\partial r} \right) \Big|_{r=R_S} = \frac{\beta P_0(R_S)}{\beta + R_S P_0(R_S)} [R_S^2 P_1(R_S) + 1 + \beta R_S] \quad (2.5)$$

and

$$\frac{\partial^2 p_S}{\partial r^2} \Big|_{r=R} = -\mu \left(\frac{\beta}{\beta + R_S P_0(R_S)} - \tilde{\sigma} \right). \quad (2.6)$$

It is shown in [27] that the linear stability depends on the zeros of the following function:

$$\begin{aligned} h_n(s) \triangleq h_n(s, \mu, R_S) &= \frac{1}{\mu} \frac{\beta + R_S P_0(R_S)}{\beta R_S P_0(R_S)} \left[s + \frac{n}{R_S^3} \left(\frac{n(n+1)}{2} - 1 \right) \right] - R_S P_1(R_S) \\ &+ \frac{R_S^2 P_1(R_S) + 1 + \beta R_S}{(s+1)R_S + (\frac{n}{R_S} + \beta)/P_n(\sqrt{s+1}R_S)}. \end{aligned} \quad (2.7)$$

For $0 < \mu < \mu^* = \mu^*(R_S)$, all zeros of $h_n(s)$ lie in $\text{Re } s < 0$ ($n \neq 1$); $s = 0$ is a zero of $h_1(s)$, all other zeros lie in $\text{Re } s < 0$ for $h_1(s)$. Furthermore,

$$\mu^*(R_S) < \mu_1(R_S), \quad (2.8)$$

where

$$\mu_1(R_S) = \frac{\beta + R_S P_0(R_S)}{\beta R_S^3 P_0(R_S)} \frac{R_S^2 P_1(R_S) + 1 + \beta R_S}{R_S P_1^2(R_S) - \frac{1+\beta R_S}{2} P_1'(R_S)}. \quad (2.9)$$

As in the proof of [5], we have the following local existence theorem:

Theorem 2.1. *If*

$$(\sigma_0, \Gamma_0) \in C^{1+\gamma}(\overline{\Omega}(0)) \times C^{4+\alpha}(\partial\Omega(0)) \quad \text{and} \quad \frac{\partial\sigma_0}{\partial\mathbf{n}} + \beta(\sigma_0 - 1) = 0 \quad \text{on} \quad \partial\Omega(0) \quad (2.10)$$

for some $\alpha, \gamma \in (0, 1)$, then there exists a unique solution (σ, p, Γ) of (1.1)–(1.5) for $t \in [0, T]$ with some $T > 0$, and

$$\begin{aligned} \sigma &\in C^{1+\gamma, (1+\gamma)/2} \left(\bigcup_{t \in [0, T]} \overline{\Omega}(t) \times \{t\} \right) \cap C^{2+2\alpha/3, 1+\alpha/3} \left(\bigcup_{t \in [t_0, T]} \overline{\Omega}(t) \times \{t\} \right), \quad \text{for any } t_0 > 0, \\ p &\in C^{2+\alpha, \alpha/3} \left(\bigcup_{t \in [0, T]} \overline{\Omega}(t) \times \{t\} \right), \quad \Gamma \in C^{4+\alpha, 1+\alpha/3}. \end{aligned}$$

3. The nonlinearly perturbed problem

The standard way to deal with free boundary problems is to transform it into a new PDE system with a fixed boundary. In this section, by Hanzawa transformation, we transform the free boundary problem (1.1)–(1.5) into a nonlinearly perturbed problem in a fixed domain. For simplicity, we denote R_S by R .

To begin with, let us assume the solution of the system (1.1)–(1.5) is of the form (which will be verified in Section 6)

$$\begin{aligned} \partial\Omega(t) : r &= R + \varepsilon\rho(\theta, \varphi, t), \\ \sigma(r, \theta, \varphi, t) &= \sigma_S(r) + \varepsilon w(r, \theta, \varphi, t), \\ p(r, \theta, \varphi, t) &= p_S(r) + \varepsilon q(r, \theta, \varphi, t), \end{aligned}$$

and then the system (1.1)–(1.5) can be written in terms of (w, q, ρ) as follows:

$$w_t - \Delta w + w = 0 \quad \text{in } \Omega(t), \quad t > 0, \quad (3.1)$$

$$-\Delta q = \mu w \quad \text{in } \Omega(t), \quad t > 0, \quad (3.2)$$

$$\frac{d\rho}{dt} = - \left(\frac{1}{\varepsilon} \frac{\partial p_S}{\partial \mathbf{n}} + \frac{\partial q}{\partial \mathbf{n}} \right) \sqrt{1 + \frac{|\varepsilon \nabla_\omega \rho|^2}{(R + \varepsilon\rho)^2}} \quad \text{on } \partial\Omega(t), \quad t > 0, \quad (3.3)$$

$$\frac{\partial w}{\partial \mathbf{n}} + \beta w = - \frac{1}{\varepsilon} \left[\frac{\partial \sigma_S}{\partial \mathbf{n}} + \beta \sigma_S - \beta \right] \quad \text{on } \partial\Omega(t), \quad t > 0, \quad (3.4)$$

$$q = - \frac{1}{\varepsilon} [p_S - \kappa] \quad \text{on } \partial\Omega(t), \quad t > 0, \quad (3.5)$$

where $\nabla_\omega = \vec{e}_\theta \partial_\theta + \vec{e}_\varphi \frac{1}{\sin\theta} \partial_\varphi$.

Recall [26], that

$$\mathbf{n} = \frac{1}{\sqrt{1 + |\varepsilon \nabla_{\omega} \rho|^2 / (R + \varepsilon \rho)^2}} \left(\vec{e}_r - \frac{\varepsilon \rho \theta}{R + \varepsilon \rho} \vec{e}_{\theta} - \frac{\varepsilon \rho \varphi}{(R + \varepsilon \rho) \sin \theta} \vec{e}_{\varphi} \right),$$

and

$$\nabla = \vec{e}_r \partial_r + \vec{e}_{\theta} \frac{1}{r} \partial_{\theta} + \vec{e}_{\varphi} \frac{1}{r \sin \theta} \partial_{\varphi} = \vec{e}_r \partial_r + \frac{1}{r} \nabla_{\omega}.$$

By the Taylor expansion, some of the right-hand side terms of (3.3) and (3.5) can be written in the following way, respectively,

$$\begin{aligned} \sqrt{1 + \frac{|\varepsilon \nabla_{\omega} \rho|^2}{(R + \varepsilon \rho)^2}} \frac{\partial p_S(R + \varepsilon \rho)}{\partial \mathbf{n}} &= \sqrt{1 + \frac{|\varepsilon \nabla_{\omega} \rho|^2}{(R + \varepsilon \rho)^2}} \cdot \nabla p_S|_{R + \varepsilon \rho} \mathbf{n} = \frac{\partial p_S(R + \varepsilon \rho)}{\partial r} \\ &= \frac{\partial p_S(R)}{\partial r} + \frac{\partial^2 p_S(R)}{\partial r^2} \varepsilon \rho + \varepsilon^2 P_{\varepsilon} \\ &= -\mu \left(\frac{\beta}{\beta + R P_0(R)} - \tilde{\sigma} \right) \varepsilon \rho + \varepsilon^2 P_{\varepsilon}, \quad (\text{by (2.6)}), \end{aligned} \quad (3.6)$$

and

$$p_S(R + \varepsilon \rho) - \kappa = \frac{\varepsilon}{R^2} (\rho + \frac{1}{2} \Delta_{\omega} \rho) + \varepsilon^2 K_{\varepsilon}, \quad (3.7)$$

(see [21, Theorem 8.1]) where

$$\Delta_{\omega} \rho = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \rho}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \rho}{\partial \varphi^2}.$$

We now proceed to deal with (3.4). In fact, (3.4) is equivalent to

$$\begin{aligned} \frac{\partial w}{\partial r} \Big|_{R + \varepsilon \rho} - \frac{\varepsilon}{(R + \varepsilon \rho)^2} \frac{\partial \rho}{\partial \theta} \frac{\partial w}{\partial \theta} - \frac{\varepsilon}{(R + \varepsilon \rho)^2 \sin^2 \theta} \frac{\partial \rho}{\partial \varphi} \frac{\partial w}{\partial \varphi} + \beta w|_{R + \varepsilon \rho} \cdot \sqrt{1 + \frac{|\varepsilon \nabla_{\omega} \rho|^2}{(R + \varepsilon \rho)^2}} \\ = -\frac{1}{\varepsilon} \left\{ \frac{\partial \sigma_S(R + \varepsilon \rho)}{\partial r} + \beta [\sigma_S(R + \varepsilon \rho) - 1] \cdot \sqrt{1 + \frac{|\varepsilon \nabla_{\omega} \rho|^2}{(R + \varepsilon \rho)^2}} \right\} \end{aligned}$$

By the Taylor expansion, we have

$$\begin{aligned} \frac{\partial \sigma_S(R + \varepsilon \rho)}{\partial r} + \beta [\sigma_S(R + \varepsilon \rho) - 1] \cdot \sqrt{1 + \frac{|\varepsilon \nabla_{\omega} \rho|^2}{(R + \varepsilon \rho)^2}} \\ = \frac{\partial \sigma_S(R + \varepsilon \rho)}{\partial r} + \beta [\sigma_S(R + \varepsilon \rho) - 1] + \varepsilon^2 \tilde{S}_{\varepsilon} \\ = \left(\frac{\partial^2 \sigma_S(R)}{\partial r^2} + \beta \frac{\partial \sigma_S(R)}{\partial r} \right) \varepsilon \rho + \varepsilon^2 S_{\varepsilon} \\ = \varepsilon \lambda \rho + \varepsilon^2 S_{\varepsilon}. \end{aligned} \quad (3.8)$$

As in [17], by Hanzawa transformation which is defined by

$$r = r' + \chi(R - r')\varepsilon\rho(\theta, \varphi, t), \quad t = t', \quad \theta = \theta', \quad \varphi = \varphi'$$

with

$$\chi(z) \in C^\infty, \quad \chi(z) = \begin{cases} 0, & \text{if } |z| \geq \frac{3}{4}\delta_0 \\ 1, & \text{if } |z| < \frac{1}{4}\delta_0 \end{cases}, \quad \left| \frac{d^k \chi}{dz^k} \right| \leq \frac{C}{\delta_0^k}, \quad (\delta_0 \text{ positive and small}),$$

and the expansions (3.6)-(3.8), the system (3.1)-(3.5) is transformed into the following system which is the nonlinearly perturbed problem of the system (1.1)-(1.5) in a fixed domain,

$$w'_t - \Delta w' + w' = \varepsilon[-A_\varepsilon^1 w' + A_\varepsilon w'] \quad \text{in } B_R, \quad t > 0, \quad (3.9)$$

$$\Delta' q' + \mu w' = -\varepsilon A_\varepsilon q' \quad \text{in } B_R, \quad t > 0, \quad (3.10)$$

$$\frac{d\rho'}{dt'} = \mu \left(\frac{\beta}{\beta + RP_0(R)} - \tilde{\sigma} \right) \rho' - \frac{\partial q'}{\partial r'} + \varepsilon B_\varepsilon^1 \quad \text{on } \partial B_R, \quad t > 0, \quad (3.11)$$

$$\frac{\partial w'}{\partial r'} + \beta w' = -\lambda \rho' + \varepsilon B_\varepsilon^2 \quad \text{on } \partial B_R, \quad t > 0, \quad (3.12)$$

$$q' = -\frac{1}{R^2} \left(\rho' + \frac{1}{2} \Delta_\omega \rho' \right) + \varepsilon B_\varepsilon^3 \quad \text{on } \partial B_R, \quad t > 0, \quad (3.13)$$

where B_R is the ball with radius R , A_ε and A_ε^1 were given in [17], and

$$\begin{aligned} B_\varepsilon^1 &= -\frac{1}{\varepsilon^2} \frac{\partial p_S(R + \varepsilon \rho')}{\partial r'} - \frac{1}{\varepsilon} \mu \left(\frac{\beta}{\beta + RP_0(R)} - \tilde{\sigma} \right) \rho' \\ &\quad + \frac{1}{(R + \varepsilon \rho')^2} \frac{\partial \rho'}{\partial \theta'} \frac{\partial q'}{\partial \theta'} + \frac{1}{(R + \varepsilon \rho')^2 \sin^2 \theta'} \frac{\partial \rho'}{\partial \varphi'} \frac{\partial q'}{\partial \varphi'}, \\ B_\varepsilon^2 &= -\frac{1}{\varepsilon^2} \left\{ \frac{\partial \sigma_S(R + \varepsilon \rho')}{\partial r'} + \beta [\sigma_S(R + \varepsilon \rho') - 1] \cdot \sqrt{1 + \frac{|\varepsilon \nabla_\omega \rho'|^2}{(R + \varepsilon \rho')^2}} \right\} \\ &\quad + \frac{1}{\varepsilon} \lambda \rho' + \frac{1}{\varepsilon} \left\{ -\beta w' \sqrt{1 + \frac{|\varepsilon \nabla_\omega \rho'|^2}{(R + \varepsilon \rho')^2}} + \beta w' \right\} \\ &\quad + \frac{1}{(R + \varepsilon \rho')^2} \frac{\partial \rho'}{\partial \theta'} \frac{\partial w'}{\partial \theta'} + \frac{1}{(R + \varepsilon \rho')^2 \sin^2 \theta'} \frac{\partial \rho'}{\partial \varphi'} \frac{\partial w'}{\partial \varphi'}, \\ B_\varepsilon^3 &= -\frac{1}{\varepsilon^2} [p_S(R + \varepsilon \rho') - \kappa] + \frac{1}{\varepsilon R^2} \left(\rho' + \frac{1}{2} \Delta_\omega \rho' \right). \end{aligned}$$

By the definition of A_ε and A_ε^1 in [17], we obtain that A_ε is a second order differential operator in (r', θ', φ') , and A_ε^1 involves $\frac{\partial \rho}{\partial t}$ and a first order differential operator in r' . Furthermore, all terms of A_ε^1 and A_ε do not involve any singularity, then it follows from Theorem 2.1 that, for $T > 1$,

$$\begin{aligned} -A_\varepsilon^1 w' + A_\varepsilon w' &\in C^{2\alpha/3, \alpha/3}(B_R \times [0, T]), \\ A_\varepsilon q' &\in C^{\alpha, \alpha/3}(B_R \times [0, T]). \end{aligned}$$

Notice that the term $\frac{1}{\varepsilon}$ is canceled out by the coefficient that accompanies it, so that $C^{2\alpha/3, \alpha/3}$ norm of $-A_\varepsilon^1 w' + A_\varepsilon w'$ and $C^{\alpha, \alpha/3}$ norm of $A_\varepsilon q'$ are uniformly bounded in ε .

On the other hand, although $\sin^2 \theta'$ appears in the denominator in the last term of B_ε^1 , one can simply choose a different coordinate system to deal with this problem. As functions on $\Sigma = \{|x| = 1\}$ (rather than as functions of (θ, φ)), there are no singularities, and $B_\varepsilon^1 \in C^{1+\alpha, \alpha/3}(\partial B_R \times [0, T])$. By (3.6), $C^{1+\alpha, \alpha/3}$ norm of B_ε^1 is uniformly bounded in ε . By (1.8) and Theorem 2.1, we obtain that $B_\varepsilon^2 \in C^{1+2\alpha/3, 1+\alpha/3}(\partial B_R \times [0, T])$. Moreover, we derive, by (3.8), that $C^{1+2\alpha/3, 1+\alpha/3}$ norm of B_ε^2 is uniformly bounded in ε . Similarly, by (3.7), we find that $B_\varepsilon^3 \in C^{2+\alpha, \alpha/3}(\partial B_R \times [0, T])$, and $C^{2+\alpha, \alpha/3}$ norm of B_ε^3 is uniformly bounded in ε .

For simplicity of notation, we shall denote functions $w'(r', \theta', \varphi', t')$, $q'(r', \theta', \varphi', t')$ and $\rho'(\theta', \varphi', t')$ again by $w(r, \theta, \varphi, t)$, $q(r, \theta, \varphi, t)$ and $\rho(\theta, \varphi, t)$, respectively, in the rest of this paper.

We need to estimate (w, q, ρ) of the system (3.9)–(3.13) to prove asymptotic stability. However this system is nonlinear, the method we shall use is to study the inhomogeneous linear system instead of this nonlinearly perturbed system itself, namely, we consider the system (3.9)–(3.13) where the ε terms of any order are replaced by given functions,

$$w_t - \Delta w + w = \varepsilon f^1(r, \theta, \varphi, t) \quad \text{in } B_R, \quad t > 0, \quad (3.14)$$

$$\Delta q + \mu w = -\varepsilon f^2(r, \theta, \varphi, t) \quad \text{in } B_R, \quad t > 0, \quad (3.15)$$

$$\frac{d\rho}{dt} = \mu \left(\frac{\beta}{\beta + R P_0(R)} - \tilde{\sigma} \right) \rho - \frac{\partial q}{\partial r} + \varepsilon b^1(\theta, \varphi, t) \quad \text{on } \partial B_R, \quad t > 0, \quad (3.16)$$

$$\frac{\partial w}{\partial r} + \beta w = -\lambda \rho + \varepsilon b^2(\theta, \varphi, t) \quad \text{on } \partial B_R, \quad t > 0, \quad (3.17)$$

$$q = -\frac{1}{R^2} \left(\rho + \frac{1}{2} \Delta_\omega \rho \right) + \varepsilon b^3(\theta, \varphi, t) \quad \text{on } \partial B_R, \quad t > 0, \quad (3.18)$$

with initial conditions

$$w|_{t=0} = w_0, \quad \rho|_{t=0} = \rho_0. \quad (3.19)$$

Then for given functions f^i, b^j satisfying additional assumptions, we solve the inhomogeneous linear system (3.14)–(3.18), and derive the estimate of (w, q, ρ) . After that we want to define the new functions \tilde{f}^i, \tilde{b}^j by

$$\tilde{f}^1 = -A_\varepsilon^1 w + A_\varepsilon w, \quad \tilde{f}^2 = -A_\varepsilon^1 w, \quad \tilde{b}^1 = B_\varepsilon^1, \quad \tilde{b}^2 = B_\varepsilon^2, \quad \tilde{b}^3 = B_\varepsilon^3.$$

We shall show that the mapping $S : (f^i, b^j) \rightarrow (\tilde{f}^i, \tilde{b}^j)$ admits a fixed point.

Assume that the functions f^i, b^j satisfy

$$\sqrt{|\varepsilon|} \left(\int_0^\infty e^{2\delta_1 t} \|f^1(\cdot, t)\|_{L^2(B_R)}^2 dt \right)^{1/2} \leq 1, \quad (3.20)$$

$$\sqrt{|\varepsilon|} \left(\int_0^\infty e^{2\delta_1 t} \|f^2(\cdot, t)\|_{L^2(B_R)}^2 dt \right)^{1/2} \leq 1, \quad (3.21)$$

$$\sqrt{|\varepsilon|} \left(\int_0^\infty e^{2\delta_1 t} \|b^1(\cdot, t)\|_{H^{1/2}(\partial B_R)}^2 dt \right)^{1/2} \leq 1, \quad (3.22)$$

$$\sqrt{|\varepsilon|} \left(\int_0^\infty e^{2\delta_1 t} \|b^2(\cdot, t)\|_{H^{1/2}(\partial B_R)}^2 dt \right)^{1/2} \leq 1, \quad (3.23)$$

$$\sqrt{|\varepsilon|} \left(\int_0^\infty e^{2\delta_1 t} \|b^3(\cdot, t)\|_{H^{3/2}(\partial B_R)}^2 dt \right)^{1/2} \leq 1, \quad (3.24)$$

where δ_1 is positive and sufficiently small, and that, for some $\alpha \in (0, 1)$,

$$\sqrt{|\varepsilon|} \|f^1\|_{C^{2\alpha/3, \alpha/3}(B_R \times [0, \infty))} \leq 1, \quad (3.25)$$

$$\sqrt{|\varepsilon|} \|f^2\|_{C^{\alpha, \alpha/3}(B_R \times [0, \infty))} \leq 1, \quad (3.26)$$

$$\sqrt{|\varepsilon|} \|b^1\|_{C^{1+\alpha, \alpha/3}(\partial B_R \times [0, \infty))} \leq 1, \quad (3.27)$$

$$\sqrt{|\varepsilon|} \|b^2\|_{C^{1+2\alpha/3, 1+\alpha/3}(\partial B_R \times [0, \infty))} \leq 1, \quad (3.28)$$

$$\sqrt{|\varepsilon|} \|b^3\|_{C^{2+\alpha, \alpha/3}(\partial B_R \times [0, \infty))} \leq 1. \quad (3.29)$$

For simplicity, we also assume that

$$\rho_0 \in C^{4+\alpha}(\overline{B_R}), \quad w_0 \in C^{2+2\alpha/3}(\overline{B_R}), \quad (3.30)$$

and the compatibility condition of order 2 for w is satisfied.

We formally expand all the functions f^i , b^j in terms of spherical harmonics

$$f^i(r, \theta, \varphi, t) = \sum_{n=0}^{\infty} \sum_{m=-n}^n f_{n,m}^i(r, t) Y_{n,m}(\theta, \varphi), \quad i = 1, 2,$$

$$b^j(\theta, \varphi, t) = \sum_{n=0}^{\infty} \sum_{m=-n}^n b_{n,m}^j(t) Y_{n,m}(\theta, \varphi), \quad j = 1, 2, 3.$$

Recall [17] that (3.20)–(3.24) imply

$$|\varepsilon| \int_0^\infty e^{2\delta_1 t} \|f_{n,m}^1(\cdot, t)\|_{L^2(B_R)}^2 dt = F_{n,m}^1, \quad \sum_{n,m} F_{n,m}^1 \leq 1, \quad (3.31)$$

$$|\varepsilon| \int_0^\infty e^{2\delta_1 t} \|f_{n,m}^2(\cdot, t)\|_{L^2(B_R)}^2 dt = F_{n,m}^2, \quad \sum_{n,m} F_{n,m}^2 \leq 1, \quad (3.32)$$

$$|\varepsilon|(n+1) \int_0^\infty e^{2\delta_1 t} |b_{n,m}^1(t)|^2 dt = B_{n,m}^1, \quad \sum_{n,m} B_{n,m}^1 \leq C, \quad (3.33)$$

$$|\varepsilon|(n+1) \int_0^\infty e^{2\delta_1 t} |b_{n,m}^2(t)|^2 dt = B_{n,m}^2, \quad \sum_{n,m} B_{n,m}^2 \leq C, \quad (3.34)$$

$$|\varepsilon|(n+1)^3 \int_0^\infty e^{2\delta_1 t} |b_{n,m}^3(t)|^2 dt = B_{n,m}^3, \quad \sum_{n,m} B_{n,m}^3 \leq C. \quad (3.35)$$

Now we proceed to solve the inhomogeneous linear system (3.14)–(3.18). We look for a solution of the form:

$$\begin{aligned} w(r, \theta, \varphi, t) &= \sum_{n=0}^\infty \sum_{m=-n}^n w_{n,m}(r, t) Y_{n,m}(\theta, \varphi), \\ q(r, \theta, \varphi, t) &= \sum_{n=0}^\infty \sum_{m=-n}^n q_{n,m}(r, t) Y_{n,m}(\theta, \varphi), \\ \rho(\theta, \varphi, t) &= \sum_{n=0}^\infty \sum_{m=-n}^n \rho_{n,m}(t) Y_{n,m}(\theta, \varphi). \end{aligned}$$

Then, by the relation

$$\Delta_\omega Y_{n,m}(\theta, \varphi) + n(n+1) Y_{n,m}(\theta, \varphi) = 0 \quad \text{and} \quad \Delta = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \Delta_\omega,$$

$w_{n,m}(r, t)$, $q_{n,m}(r, t)$ and $\rho_{n,m}(t)$ satisfy the following inhomogeneous linear system

$$\partial_t w_{n,m}(r, t) - \Delta w_{n,m}(r, t) + \left(\frac{n(n+1)}{r^2} + 1 \right) w_{n,m}(r, t) = \varepsilon f_{n,m}^1(r, t) \quad \text{in } B_R \times \{t > 0\}, \quad (3.36)$$

$$-\Delta q_{n,m}(r, t) + \frac{n(n+1)}{r^2} q_{n,m}(r, t) = \mu w_{n,m}(r, t) + \varepsilon f_{n,m}^2(r, t) \quad \text{in } B_R \times \{t > 0\}, \quad (3.37)$$

$$\frac{d\rho_{n,m}(t)}{dt} = \mu \left(\frac{\beta}{\beta + R P_0(R)} - \tilde{\sigma} \right) \rho_{n,m}(t) - \frac{\partial q_{n,m}(R, t)}{\partial r} + \varepsilon b_{n,m}^1(t), \quad t > 0, \quad (3.38)$$

$$\frac{\partial w_{n,m}(R, t)}{\partial r} + \beta w_{n,m}(R, t) = -\lambda \rho_{n,m}(t) + \varepsilon b_{n,m}^2(t), \quad t > 0, \quad (3.39)$$

$$q_{n,m}(R, t) = -\frac{1}{R^2} \left(1 - \frac{n(n+1)}{2} \right) \rho_{n,m}(t) + \varepsilon b_{n,m}^3(t), \quad t > 0, \quad (3.40)$$

$$\rho_{n,m}|_{t=0} = \rho_{0,n,m}, \quad w_{n,m}|_{t=0} = w_{0,n,m}(r) \quad \text{in } B_R. \quad (3.41)$$

Introduce the Laplace transform,

$$\widehat{w}_{n,m}(r, s) = \int_0^\infty e^{-st} w_{n,m}(r, t) dt, \quad \widehat{q}_{n,m}(r, s) = \int_0^\infty e^{-st} q_{n,m}(r, t) dt,$$

$$\widehat{\rho}_{n,m}(s) = \int_0^\infty e^{-st} \rho_{n,m}(t) dt.$$

Taking formally the Laplace transform of (3.36)-(3.40), we get

$$-\Delta \widehat{w}_{n,m}(r, s) + \left(\frac{n(n+1)}{r^2} + s + 1 \right) \widehat{w}_{n,m}(r, s) = w_{0,n,m}(r) + \varepsilon \widehat{f}_{n,m}^1(r, s) \quad \text{in } B_R, \quad (3.42)$$

$$-\Delta \widehat{q}_{n,m}(r, s) + \frac{n(n+1)}{r^2} \widehat{q}_{n,m}(r, s) = \mu \widehat{w}_{n,m}(r, s) + \varepsilon \widehat{f}_{n,m}^2(r, s) \quad \text{in } B_R, \quad (3.43)$$

$$s \widehat{\rho}_{n,m}(s) - \rho_{0,n,m} = \mu \left(\frac{\beta}{\beta + R P_0(R)} - \widetilde{\sigma} \right) \widehat{\rho}_{n,m}(s) - \frac{\partial \widehat{q}_{n,m}}{\partial r} \Big|_{r=R} + \varepsilon \widehat{b}_{n,m}^1(s). \quad (3.44)$$

$$\frac{\partial \widehat{w}_{n,m}(R, s)}{\partial r} + \beta \widehat{w}_{n,m}(R, s) = -\lambda \widehat{\rho}_{n,m}(s) + \varepsilon \widehat{b}_{n,m}^2(s), \quad (3.45)$$

$$\widehat{q}_{n,m}(R, s) = \frac{1}{R^2} \left(\frac{n(n+1)}{2} - 1 \right) \widehat{\rho}_{n,m}(s) + \varepsilon \widehat{b}_{n,m}^3(s). \quad (3.46)$$

As in [27], using (2.4), we can solve (3.42) and (3.45) in the form

$$\widehat{w}_{n,m}(r, s) = \frac{(-\lambda \widehat{\rho}_{n,m}(s) + \varepsilon \widehat{b}_{n,m}^2(s)) R^{\frac{1}{2}}}{\sqrt{s+1} I_{n+3/2}(\sqrt{s+1} R) + \left(\frac{n}{R} + \beta \right) I_{n+1/2}(\sqrt{s+1} R)} \frac{I_{n+1/2}(\sqrt{s+1} r)}{r^{\frac{1}{2}}} \\ + \xi_{1,n,m}(r, s) + \varepsilon \xi_{2,n,m}(r, s), \quad (3.47)$$

where $\xi_{1,n,m}(r, s)$ is the solution of

$$-\Delta \xi_{1,n,m}(r, s) + \left(\frac{n(n+1)}{r^2} + s + 1 \right) \xi_{1,n,m}(r, s) = w_{0,n,m}(r) \quad \text{in } B_R, \\ \left(\frac{\partial \xi_{1,n,m}(r, s)}{\partial r} + \beta \xi_{1,n,m}(r, s) \right) \Big|_{r=R} = 0, \quad (3.48)$$

and $\xi_{2,n,m}(r, s)$ is the solution of

$$-\Delta \xi_{2,n,m}(r, s) + \left(\frac{n(n+1)}{r^2} + s + 1 \right) \xi_{2,n,m}(r, s) = \widehat{f}_{n,m}^1(r, s) \quad \text{in } B_R, \\ \left(\frac{\partial \xi_{2,n,m}(r, s)}{\partial r} + \beta \xi_{2,n,m}(r, s) \right) \Big|_{r=R} = 0. \quad (3.49)$$

Let

$$\phi = \widehat{q}_{n,m} + \frac{\mu}{s+1} \widehat{w}_{n,m}, \quad (3.50)$$

then ϕ satisfies

$$-\Delta\phi + \frac{n(n+1)}{r^2}\phi = \frac{\mu}{s+1}(w_{0,n,m} + \varepsilon \widehat{f}_{n,m}^1) + \varepsilon \widehat{f}_{n,m}^2 \quad \text{in } B_R, \quad (3.51)$$

and by (3.44), (3.46) and (3.45), ϕ satisfies the following boundary condition

$$\begin{aligned} & \left(\frac{\partial\phi}{\partial r} + \beta\phi \right) \Big|_{r=R} \\ &= \mu \left(\frac{\beta}{\beta + RP_0(R)} - \widetilde{\sigma} \right) \widehat{\rho}_{n,m}(s) - (s\widehat{\rho}_{n,m}(s) - \rho_{0,n,m}) + \varepsilon \widehat{b}_{n,m}^1(s) \\ & \quad + \frac{\beta}{R^2} \left(\frac{n(n+1)}{2} - 1 \right) \widehat{\rho}_{n,m}(s) + \beta \varepsilon \widehat{b}_{n,m}^3(s) - \frac{\mu\lambda}{s+1} \widehat{\rho}_{n,m}(s) + \frac{\mu}{s+1} \varepsilon \widehat{b}_{n,m}^2(s). \end{aligned} \quad (3.52)$$

The solution of the problem (3.51)-(3.52) is given by

$$\begin{aligned} \phi(r, s) &= \frac{r^n}{nR^{n-1} + \beta R^n} \left[\mu \left(\frac{\beta}{\beta + RP_0(R)} - \widetilde{\sigma} \right) \widehat{\rho}_{n,m}(s) - (s\widehat{\rho}_{n,m}(s) - \rho_{0,n,m}) \right. \\ & \quad + \frac{\beta}{R^2} \left(\frac{n(n+1)}{2} - 1 \right) \widehat{\rho}_{n,m}(s) - \frac{\mu\lambda}{s+1} \widehat{\rho}_{n,m}(s) + \varepsilon \widehat{b}_{n,m}^1(s) \\ & \quad \left. + \frac{\mu}{s+1} \varepsilon \widehat{b}_{n,m}^2(s) + \beta \varepsilon \widehat{b}_{n,m}^3(s) \right] + \phi_{1,n,m}(r, s) + \varepsilon \phi_{2,n,m}(r, s), \end{aligned} \quad (3.53)$$

where $\phi_{1,n,m}(r, s)$ is the solution of

$$\begin{aligned} -\Delta\phi_{1,n,m} + \frac{n(n+1)}{r^2}\phi_{1,n,m} &= \frac{\mu}{s+1}w_{0,n,m} \quad \text{in } B_R, \\ \left(\frac{\partial\phi_{1,n,m}}{\partial r} + \beta\phi_{1,n,m} \right) \Big|_{r=R} &= 0, \end{aligned} \quad (3.54)$$

and $\phi_{2,n,m}(r, s)$ is the solution of

$$\begin{aligned} -\Delta\phi_{2,n,m} + \frac{n(n+1)}{r^2}\phi_{2,n,m} &= \frac{\mu}{s+1}\widehat{f}_{n,m}^1 + \widehat{f}_{n,m}^2 \quad \text{in } B_R, \\ \left(\frac{\partial\phi_{2,n,m}}{\partial r} + \beta\phi_{2,n,m} \right) \Big|_{r=R} &= 0. \end{aligned} \quad (3.55)$$

By (3.50), (3.53), (3.47) and (2.4), we have

$$\frac{\partial\widehat{q}_{n,m}}{\partial r} \Big|_{r=R} = \frac{\partial\phi}{\partial r} \Big|_{r=R} - \frac{\mu}{s+1} \frac{\partial\widehat{w}_{n,m}}{\partial r} \Big|_{r=R}$$

$$\begin{aligned}
&= \frac{n}{n + \beta R} \left\{ \mu \left(\frac{\beta}{\beta + R P_0(R)} - \tilde{\sigma} \right) \widehat{\rho}_{n,m}(s) - (s \widehat{\rho}_{n,m}(s) - \rho_{0,n,m}) \right. \\
&\quad + \frac{\beta}{R^2} \left(\frac{n(n+1)}{2} - 1 \right) \widehat{\rho}_{n,m}(s) - \frac{\mu \lambda}{s+1} \widehat{\rho}_{n,m}(s) \\
&\quad + \varepsilon \widehat{b}_{n,m}^1(s) + \frac{\mu}{s+1} \varepsilon \widehat{b}_{n,m}^2(s) + \beta \varepsilon \widehat{b}_{n,m}^3(s) \left. \right\} + \frac{\partial \phi_{1,n,m}}{\partial r} \Big|_{r=R} + \varepsilon \frac{\partial \phi_{2,n,m}}{\partial r} \Big|_{r=R} \\
&\quad - \frac{\mu}{s+1} \left\{ \frac{(-\lambda \widehat{\rho}_{n,m}(s) + \varepsilon \widehat{b}_{n,m}^2) [\sqrt{s+1} I_{n+3/2}(\sqrt{s+1}R) + \frac{n}{R} I_{n+1/2}(\sqrt{s+1}R)]}{\sqrt{s+1} I_{n+3/2}(\sqrt{s+1}R) + (\frac{n}{R} + \beta) I_{n+1/2}(\sqrt{s+1}R)} \right. \\
&\quad \left. + \frac{\partial \xi_{1,n,m}}{\partial r} \Big|_{r=R} + \varepsilon \frac{\partial \xi_{2,n,m}}{\partial r} \Big|_{r=R} \right\}.
\end{aligned}$$

Inserting the above expression into (3.44), we obtain

$$\begin{aligned}
&- \frac{\beta R}{n + \beta R} \mu \left(\frac{\beta}{\beta + R P_0(R)} - \tilde{\sigma} \right) \widehat{\rho}_{n,m}(s) + \frac{\beta R}{n + \beta R} (s \widehat{\rho}_{n,m}(s) - \rho_{0,n,m}) \\
&+ \frac{n}{n + \beta R} \frac{\beta}{R^2} \left(\frac{n(n+1)}{2} - 1 \right) \widehat{\rho}_{n,m}(s) - \frac{n}{n + \beta R} \frac{\mu \lambda}{s+1} \widehat{\rho}_{n,m}(s) \\
&+ \varepsilon \frac{n}{n + \beta R} [\widehat{b}_{n,m}^1 + \frac{\mu}{s+1} \widehat{b}_{n,m}^2 + \beta \widehat{b}_{n,m}^3] + \frac{\partial \phi_{1,n,m}}{\partial r} \Big|_{r=R} + \varepsilon \frac{\partial \phi_{2,n,m}}{\partial r} \Big|_{r=R} \\
&- \frac{\mu}{s+1} (-\lambda \widehat{\rho}_{n,m}(s) + \varepsilon \widehat{b}_{n,m}^2) \frac{\sqrt{s+1} I_{n+3/2}(\sqrt{s+1}R) + \frac{n}{R} I_{n+1/2}(\sqrt{s+1}R)}{\sqrt{s+1} I_{n+3/2}(\sqrt{s+1}R) + (\frac{n}{R} + \beta) I_{n+1/2}(\sqrt{s+1}R)} \\
&- \frac{\mu}{s+1} \left(\frac{\partial \xi_{1,n,m}}{\partial r} \Big|_{r=R} + \varepsilon \frac{\partial \xi_{2,n,m}}{\partial r} \Big|_{r=R} \right) - \varepsilon \widehat{b}_{n,m}^1 = 0.
\end{aligned}$$

After a direct computation, this equality is equivalent to

$$\begin{aligned}
&\left\{ \frac{\beta R}{n + \beta R} s + \frac{n}{n + \beta R} \frac{\beta}{R^2} \left(\frac{n(n+1)}{2} - 1 \right) - \frac{\beta R}{n + \beta R} \mu \left(\frac{\beta}{\beta + R P_0(R)} - \tilde{\sigma} \right) \right. \\
&\quad \left. - \frac{n}{n + \beta R} \frac{\mu \lambda}{s+1} + \frac{\mu \lambda}{s+1} \cdot \frac{\sqrt{s+1} I_{n+3/2}(\sqrt{s+1}R) + \frac{n}{R} I_{n+1/2}(\sqrt{s+1}R)}{\sqrt{s+1} I_{n+3/2}(\sqrt{s+1}R) + (\frac{n}{R} + \beta) I_{n+1/2}(\sqrt{s+1}R)} \right\} \widehat{\rho}_{n,m}(s) \\
&= \frac{\beta R}{n + \beta R} \rho_{0,n,m} - \varepsilon \frac{n}{n + \beta R} [\widehat{b}_{n,m}^1 + \frac{\mu}{s+1} \widehat{b}_{n,m}^2 + \beta \widehat{b}_{n,m}^3] \\
&\quad + \varepsilon \frac{\mu}{s+1} \widehat{b}_{n,m}^2 \frac{\sqrt{s+1} I_{n+3/2}(\sqrt{s+1}R) + \frac{n}{R} I_{n+1/2}(\sqrt{s+1}R)}{\sqrt{s+1} I_{n+3/2}(\sqrt{s+1}R) + (\frac{n}{R} + \beta) I_{n+1/2}(\sqrt{s+1}R)} \\
&\quad - \left(\frac{\partial \phi_{1,n,m}}{\partial r} \Big|_{r=R} + \varepsilon \frac{\partial \phi_{2,n,m}}{\partial r} \Big|_{r=R} \right) + \frac{\mu}{s+1} \left(\frac{\partial \xi_{1,n,m}}{\partial r} \Big|_{r=R} + \varepsilon \frac{\partial \xi_{2,n,m}}{\partial r} \Big|_{r=R} \right) + \varepsilon \widehat{b}_{n,m}^1.
\end{aligned}$$

By [27, page 2482] and (2.7), the brace in left-hand side is equal to

$$\frac{\beta R}{n + \beta R} \frac{\beta R P_0(R)}{\beta + R P_0(R)} \mu h_n(s, \mu, R),$$

and the right-hand side is equal to

$$\begin{aligned} & \frac{\beta R}{n + \beta R} \rho_{0,n,m} + \frac{\mu}{s+1} \frac{\partial \xi_{1,n,m}}{\partial r} \Big|_{r=R} - \frac{\partial \phi_{1,n,m}}{\partial r} \Big|_{r=R} + \varepsilon \left\{ \frac{\mu}{s+1} \frac{\partial \xi_{2,n,m}}{\partial r} \Big|_{r=R} - \frac{\partial \phi_{2,n,m}}{\partial r} \Big|_{r=R} \right. \\ & \left. + \frac{\beta R}{n + \beta R} \widehat{b}_{n,m}^1 + \frac{\beta R^2}{n + \beta R} \mu \widehat{b}_{n,m}^2 \frac{1}{(s+1)R + (\frac{n}{R} + \beta)/P_n(\sqrt{s+1}R)} - \frac{n\beta}{n + \beta R} \widehat{b}_{n,m}^3 \right\}, \end{aligned}$$

where we have used the fact

$$\begin{aligned} & \frac{\mu}{s+1} \widehat{b}_{n,m}^2 \frac{\sqrt{s+1}I_{n+3/2}(\sqrt{s+1}R) + \frac{n}{R}I_{n+1/2}(\sqrt{s+1}R)}{\sqrt{s+1}I_{n+3/2}(\sqrt{s+1}R) + (\frac{n}{R} + \beta)I_{n+1/2}(\sqrt{s+1}R)} - \frac{\mu}{s+1} \widehat{b}_{n,m}^2 \frac{n}{n + \beta R} \\ & = \frac{\mu}{s+1} \widehat{b}_{n,m}^2 \frac{\beta R}{n + \beta R} \frac{\sqrt{s+1}I_{n+3/2}(\sqrt{s+1}R)}{\sqrt{s+1}I_{n+3/2}(\sqrt{s+1}R) + (\frac{n}{R} + \beta)I_{n+1/2}(\sqrt{s+1}R)} \\ & = \frac{\beta R^2}{n + \beta R} \mu \widehat{b}_{n,m}^2 \frac{1}{(s+1)R + (\frac{n}{R} + \beta)/P_n(\sqrt{s+1}R)}, \quad (\text{by (2.1)}). \end{aligned}$$

It follows that

$$\begin{aligned} \widehat{\rho}_{n,m}(s) &= \frac{n + \beta R}{\beta R} \frac{\beta + RP_0(R)}{\beta RP_0(R)} \frac{1}{\mu h_n(s, \mu, R)} \\ & \cdot \left\{ \frac{\beta R}{n + \beta R} \rho_{0,n,m} + \frac{\mu}{s+1} \frac{\partial \xi_{1,n,m}}{\partial r} \Big|_{r=R} - \frac{\partial \phi_{1,n,m}}{\partial r} \Big|_{r=R} \right. \\ & + \varepsilon \left[\frac{\mu}{s+1} \frac{\partial \xi_{2,n,m}}{\partial r} \Big|_{r=R} - \frac{\partial \phi_{2,n,m}}{\partial r} \Big|_{r=R} + \frac{\beta R}{n + \beta R} \widehat{b}_{n,m}^1 \right. \\ & \left. \left. + \frac{\beta R^2}{n + \beta R} \mu \widehat{b}_{n,m}^2 \frac{1}{(s+1)R + (\frac{n}{R} + \beta)/P_n(\sqrt{s+1}R)} - \frac{n\beta}{n + \beta R} \widehat{b}_{n,m}^3 \right] \right\}, \end{aligned} \quad (3.56)$$

which is equivalent to

$$\begin{aligned} \widehat{\rho}_{n,m}(s) &= M_{n,m}(s) + \frac{n + \beta R}{\beta R} \frac{\beta + RP_0(R)}{\beta RP_0(R)} \frac{\varepsilon}{\mu h_n(s, \mu, R)} \\ & \cdot \left\{ \frac{\mu}{s+1} \frac{\partial \xi_{2,n,m}}{\partial r} \Big|_{r=R} - \frac{\partial \phi_{2,n,m}}{\partial r} \Big|_{r=R} + \frac{\beta R}{n + \beta R} \widehat{b}_{n,m}^1 \right. \\ & \left. + \frac{\beta R^2}{n + \beta R} \mu \widehat{b}_{n,m}^2 \frac{1}{(s+1)R + (\frac{n}{R} + \beta)/P_n(\sqrt{s+1}R)} - \frac{n\beta}{n + \beta R} \widehat{b}_{n,m}^3 \right\}, \end{aligned} \quad (3.57)$$

where

$$\begin{aligned} M_{n,m}(s) &= \frac{n + \beta R}{\beta R} \frac{\beta + RP_0(R)}{\beta RP_0(R)} \frac{1}{\mu h_n(s, \mu, R)} \\ & \cdot \left\{ \frac{\beta R}{n + \beta R} \rho_{0,n,m} + \frac{\mu}{s+1} \frac{\partial \xi_{1,n,m}}{\partial r} \Big|_{r=R} - \frac{\partial \phi_{1,n,m}}{\partial r} \Big|_{r=R} \right\}. \end{aligned} \quad (3.58)$$

4. A choice of a new center

After the initial values perturbed, say (1.14), the domain of the system (3.1)–(3.5) (or (1.1)–(1.5)) undergoes a translation owing to the perturbation of “hidden” mode 1 contributions. In order to prove asymptotic stability we need to determine the center of the limiting sphere. Unlike the linear stability, it is a grand challenge to find the new center. In this section, we shall accomplish this through the following contraction mapping of theorem:

Theorem 4.1. *Let $(X, \|\cdot\|)$ be a Banach space and let $\overline{B}_K(a_0)$ denote the closed ball in X with center a_0 and radius K . Let F be a mapping from $\overline{B}_K(a_0)$ into X and*

$$F(x) = F_1(x) + \varepsilon G(x),$$

such that

- (i) $F_1'(x)$ and $G'(x)$ are both continuous for $x \in \overline{B}_K(a_0)$,
- (ii) $F_1(a_0) = 0$ and the operator $F_1'(a_0)$ is invertible.

Then for small $|\varepsilon|$, the equation $F(x) = 0$ admits a unique solution x in $\overline{B}_K(a_0)$.

Proof. $F(x) = 0$ if and only if $F_1(x) = -\varepsilon G(x)$. By implicit function theorem, for ε small, this is equivalent to

$$x = F_1^{-1}(-\varepsilon G(x)).$$

Let $g(x) = F_1^{-1}(-\varepsilon G(x))$, then

$$\|g'(x)\| \leq \|(F_1^{-1})'(-\varepsilon G(x))\| \cdot |\varepsilon| \cdot \|G'(x)\|.$$

Under our assumptions, $\|G'(x)\| \leq C$ for $x \in \overline{B}_K(a_0)$ and $(F_1^{-1})'(y)$ is continuous for y in a small neighborhood of 0, therefore

$$\|(F_1^{-1})'(-\varepsilon G(x))\| \leq C, \quad x \in \overline{B}_K(a_0)$$

for $|\varepsilon|$ sufficiently small. It follows that $\|g'(x)\| \leq C|\varepsilon|$ for $x \in \overline{B}_K(a_0)$. Taking $|\varepsilon|$ to be small enough, we get a contraction, which provides a unique fixed point for $g(x)$ in $\overline{B}_K(a_0)$. \square

For $n \neq 1$, all zeros of $h_n(s)$ lie in $\operatorname{Re} s < -\delta(n+1)$ with some $\delta > 0$ (see [27]), which enables us to change the contour Γ of integration to $-\delta(n+1)$ in the inverse Laplace transform of the various functions (3.57). However, for $n = 1$, since $h_1(s)$ has a simple zero at $s = 0$ (see also [27]), we cannot move the contour Γ to $\operatorname{Re} s = -\delta$ with some $\delta > 0$ in the inverse Laplace transform and obtain the decay rate in t . In the sequel, we will make use of Theorem 4.1 to show that there is a translation of coordinates

$$0 \rightarrow \varepsilon a^*(\varepsilon),$$

where $a^*(\varepsilon)$ is uniformly bounded, such that for $n = 1$, the expression in the brace in (3.56) vanishes at $s = 0$ and thus the singularity of $1/h_1(s)$ at $s = 0$ will be canceled. Hence, in the new

coordinate system, we can take the inverse Laplace transform in (3.56) and move the contour Γ to $\operatorname{Re} s = -\delta$.

In the process of using the contraction mapping (Theorem 4.1), we adjust the center of the domain to an optimal location a at each iteration. For notational convenience, we use the same letter r, θ, φ to denote the new variables after the translation of coordinates $0 \rightarrow \varepsilon a$ with $a = (a_1, a_2, a_3)$. As in the proof of [17, (6.4) and (6.5) in Lemma 6.1], the initial data w_0, ρ_0 in the new coordinates are transformed into, respectively,

$$w_0(r, \theta, \varphi) + \frac{\partial \sigma_S(r)}{\partial r} (a_1 \cos \varphi \sin \theta + a_2 \sin \varphi \sin \theta + a_3 \cos \theta) + \varepsilon A(r, \theta, \varphi, \varepsilon; a), \quad (4.1)$$

$$\rho_0(\theta, \varphi) - (a_1 \cos \varphi \sin \theta + a_2 \sin \varphi \sin \theta + a_3 \cos \theta) + \varepsilon B(r, \theta, \varphi, \varepsilon; a), \quad (4.2)$$

where A and B are bounded functions. Recall that

$$Y_{1,-1} = \sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\varphi}, \quad Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos \theta, \quad Y_{1,1} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\varphi},$$

and $w_{0,1,m}, \rho_{0,1,m}$ ($m = -1, 0, 1$) are changed into (see [17, (6.6) and (6.7)])

$$\tilde{w}_{0,1,m} \triangleq w_{0,1,m} + b_{m+2} \frac{\partial \sigma_S(r)}{\partial r} + \varepsilon A_m, \quad (4.3)$$

$$\tilde{\rho}_{0,1,m} \triangleq \rho_{0,1,m} - b_{m+2} + \varepsilon B_m, \quad (4.4)$$

where A_m and B_m are bounded functions of (a, ε) , and b_{m+2} satisfies (see [17, (6.8)])

$$b_1 - b_3 = a_1 \sqrt{\frac{8\pi}{3}}, \quad i(b_1 + b_3) = -a_2 \sqrt{\frac{8\pi}{3}}, \quad b_2 = a_3 \sqrt{\frac{4\pi}{3}}.$$

Note that a translation of the origin does not change the equations (3.1)–(3.5), but change the initial data. Namely, moving the origin of the system (3.1)–(3.5) to εa is equivalent to keep the origin fixed at 0 but replace the initial values w_0 and ρ_0 by (4.1) and (4.2), respectively.

As stated in [17, page 632], we note that the functions (f^i, b^j) satisfying (3.20)–(3.29) depend on the new center a , i.e., $(f^i(r, \theta, \varphi, t; a), b^j(\theta, \varphi, t; a))$, to ensure the consistency condition of order 2.

In the new coordinate system, for $n = 1, s = 0$, the expression in the braces in (3.56) is given by

$$\begin{aligned} F_m(a) \equiv & \left\{ \frac{\beta R}{1 + \beta R} \tilde{\rho}_{0,1,m} + \frac{\mu}{s+1} \frac{\partial \xi_{1,1,m}}{\partial r} \Big|_{r=R} - \frac{\partial \phi_{1,1,m}}{\partial r} \Big|_{r=R} \right. \\ & + \varepsilon \left[\frac{\mu}{s+1} \frac{\partial \xi_{2,1,m}}{\partial r} \Big|_{r=R} - \frac{\partial \phi_{2,1,m}}{\partial r} \Big|_{r=R} + \frac{\beta R}{1 + \beta R} \widehat{b}_{1,m}^1 \right. \\ & \left. \left. + \frac{\beta R^2}{1 + \beta R} \mu \widehat{b}_{1,m}^2 \frac{1}{(s+1)R + (\frac{1}{R} + \beta)/P_1(\sqrt{s+1}R)} - \frac{\beta}{1 + \beta R} \widehat{b}_{1,m}^3 \right] \right\} \Big|_{s=0}, \end{aligned} \quad (4.5)$$

where $m = -1, 0, 1$. In fact, $F_m(a)$ depends implicitly on a through the dependence of each term of the right-hand side on a .

We first establish the following lemma.

Lemma 4.1. Let $\mu < \mu^*$, then there exists a translation of the origin $0 \rightarrow \varepsilon a$ with $a = (a_1, a_2, a_3)$ uniformly bounded, i.e. $|a| \leq Q_0$, such that in the new coordinate system,

$$\left\{ \frac{\beta R}{1 + \beta R} \tilde{\rho}_{0,1,m} + \frac{\mu}{s+1} \frac{\partial \xi_{1,1,m}}{\partial r} \Big|_{r=R} - \frac{\partial \phi_{1,1,m}}{\partial r} \Big|_{r=R} \right\} \Big|_{s=0} = \varepsilon C_m, \quad m = -1, 0, 1, \quad (4.6)$$

where C_m is a bounded function of (a, ε) .

Proof. By (3.48), (3.54) and (4.3), in the new coordinate system, the functions $\xi_{1,1,m}$ and $\phi_{1,1,m}$ at $s = 0$ satisfy

$$-\Delta \xi_{1,1,m} + \left(\frac{2}{r^2} + 1 \right) \xi_{1,1,m} = w_{0,1,m}(r) + b_{m+2} \frac{\partial \sigma_S(r)}{\partial r} + \varepsilon A_m \quad \text{in } B_R, \quad (4.7)$$

$$\left(\frac{\partial \xi_{1,1,m}}{\partial r} + \beta \xi_{1,1,m} \right) \Big|_{r=R} = 0,$$

$$-\Delta \phi_{1,1,m} + \frac{2}{r^2} \phi_{1,1,m} = \mu \left(w_{0,1,m}(r) + b_{m+2} \frac{\partial \sigma_S(r)}{\partial r} + \varepsilon A_m \right) \quad \text{in } B_R, \quad (4.8)$$

$$\left(\frac{\partial \phi_{1,1,m}}{\partial r} + \beta \phi_{1,1,m} \right) \Big|_{r=R} = 0.$$

The functions $\xi_{1,1,m}$ and $\phi_{1,1,m}$ can be rewritten as

$$\xi_{1,1,m} = b_{m+2} \xi_{1,1,m}^{(b)} + \tilde{\xi}_{1,1,m}, \quad \phi_{1,1,m} = b_{m+2} \phi_{1,1,m}^{(b)} + \tilde{\phi}_{1,1,m},$$

where $\tilde{\xi}_{1,1,m}$, $\tilde{\phi}_{1,1,m}$ are the solutions of (4.7) and (4.8) corresponding to the right-hand side terms $w_{0,1,m} + \varepsilon A_m$ and $\mu(w_{0,1,m} + \varepsilon A_m)$, respectively.

Substituting these results and (4.4) into (4.6), we obtain

$$\begin{aligned} & \left\{ \frac{\beta R}{1 + \beta R} \tilde{\rho}_{0,1,m} + \frac{\mu}{s+1} \frac{\partial \xi_{1,1,m}}{\partial r} \Big|_{r=R} - \frac{\partial \phi_{1,1,m}}{\partial r} \Big|_{r=R} \right\} \Big|_{s=0} \\ &= \left\{ -\frac{\beta R}{1 + \beta R} + \mu \frac{\partial \xi_{1,1,m}^{(b)}}{\partial r} \Big|_{r=R} - \frac{\partial \phi_{1,1,m}^{(b)}}{\partial r} \Big|_{r=R} \right\} b_{m+2} + G_m^1 + \varepsilon G_m^2 \\ &\equiv Q_m b_{m+2} + G_m^1 + \varepsilon G_m^2, \end{aligned} \quad (4.9)$$

where G_m^1 is independent of b_{m+2} and G_m^2 is a bounded function of a and ε . If we can show $Q_m \neq 0$, then we can choose b_{m+2} to cancel G_m^1 , leaving out the expression εG_m^2 , which is the right-hand side of (4.6). In fact, we have the following lemma:

Lemma 4.2. $Q_m = 0$ if and only if $\mu = \mu_1(R)$, where $\mu_1(R)$ is defined by (2.9).

Proof. In order to compute Q_m , we need to compute the functions $\xi_{1,1,m}^{(b)}$ and $\phi_{1,1,m}^{(b)}$ which satisfy

$$-\Delta \xi_{1,1,m}^{(b)} + \left(\frac{2}{r^2} + 1 \right) \xi_{1,1,m}^{(b)} = \frac{\partial \sigma_S(r)}{\partial r} \quad \text{in } B_R, \quad \left(\frac{\partial \xi_{1,1,m}^{(b)}}{\partial r} + \beta \xi_{1,1,m}^{(b)} \right) \Big|_{r=R} = 0, \quad (4.10)$$

and

$$-\Delta\phi_{1,1,m}^{(b)} + \frac{2}{r^2}\phi_{1,1,m}^{(b)} = \mu \frac{\partial\sigma_S(r)}{\partial r} \text{ in } B_R, \quad \left(\frac{\partial\phi_{1,1,m}^{(b)}}{\partial r} + \beta\phi_{1,1,m}^{(b)}\right)\Big|_{r=R} = 0. \quad (4.11)$$

Using the equation for $\sigma'_S(r)$, i.e. $\sigma'''_S + \frac{2}{r}\sigma''_S - \frac{2}{r^2}\sigma'_S = \sigma'_S$, one can easily verify that

$$\phi_{1,1,m}^{(b)} = -\mu\sigma'_S(r) + \frac{\mu\lambda}{1+\beta R}r, \quad (\lambda = \sigma''_S(R) + \beta\sigma'_S(R) \text{ by (2.5)}),$$

then by (1.8) and (2.5), we have

$$\begin{aligned} \frac{\partial\phi_{1,1,m}^{(b)}(R)}{\partial r} &= -\mu\sigma''_S(R) + \frac{\mu\lambda}{1+\beta R} \\ &= -\mu \frac{\beta[1-2P_0(R)]}{\beta+RP_0(R)} + \frac{\mu}{1+\beta R} \frac{\beta P_0(R)}{\beta+RP_0(R)} [R^2P_1(R) + 1 + \beta R]. \end{aligned} \quad (4.12)$$

We next proceed to compute $\xi_{1,1,m}^{(b)}$. As in [17, page 630], the equation for $\xi_{1,1,m}^{(b)}$ can be rewritten in the following form:

$$-r^2[\sigma'_S(r)]^2 \frac{\partial}{\partial r} \left(\frac{\xi_{1,1,m}^{(b)}}{\sigma'_S(r)} \right) = r^2\sigma_S(r)\sigma'_S(r) - \int_0^r \tau^2\sigma_S^2(\tau)d\tau,$$

then it follows that

$$\frac{\partial_r \xi_{1,1,m}^{(b)} \sigma'_S(r) - \xi_{1,1,m}^{(b)} \sigma''_S(r)}{[\sigma'_S(r)]^2} = -\frac{\sigma_S(r)}{\sigma'_S(r)} + \frac{1}{r^2[\sigma'_S(r)]^2} \int_0^r \tau^2\sigma_S^2(\tau)d\tau. \quad (4.13)$$

By (4.13), the boundary condition of $\xi_{1,1,m}^{(b)}$ in (4.10) and (2.5), we obtain

$$\frac{\partial\xi_{1,1,m}^{(b)}(R)}{\partial r} = \frac{\beta+RP_0(R)}{P_0(R)[R^2P_1(R)+1+\beta R]} \left[-\sigma_S(R)\sigma'_S(R) + \frac{1}{R^2} \int_0^R \tau^2\sigma_S^2(\tau)d\tau \right]. \quad (4.14)$$

It follows from (1.8) and (2.2) that the second term of the right-hand side of (4.14) is equal to

$$\begin{aligned} \int_0^R \tau^2\sigma_S^2(\tau)d\tau &= \left(\frac{\beta}{\beta+RP_0(R)} \frac{R^{1/2}}{I_{1/2}(R)} \right)^2 \int_0^R \tau I_{1/2}^2(\tau)d\tau \\ &= \left(\frac{\beta}{\beta+RP_0(R)} \frac{R^{1/2}}{I_{1/2}(R)} \right)^2 \int_0^R \frac{2}{\pi} \sinh^2 \tau d\tau \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{\beta}{\beta + R P_0(R)} \frac{R^{1/2}}{I_{1/2}(R)} \right)^2 \frac{2}{\pi} \frac{\sinh^2 R}{2} R [-P_0(R) - R^2 P_0^2(R) + 1] \\
&= \left(\frac{\beta}{\beta + R P_0(R)} \right)^2 \frac{R^3}{2} [-P_0(R) - R^2 P_0^2(R) + 1],
\end{aligned}$$

where $I_{1/2}^2(\tau) = \frac{2}{\pi\tau} \sinh^2 \tau$ is used, so that, by (1.8) and (2.4),

$$\begin{aligned}
\frac{\partial \xi_{1,1,m}^{(b)}(R)}{\partial r} &= \frac{\beta + R P_0(R)}{P_0(R)[R^2 P_1(R) + 1 + \beta R]} \left(\frac{\beta}{\beta + R P_0(R)} \right)^2 \\
&\quad \cdot \left[-\frac{3}{2} R P_0(R) - \frac{1}{2} R^3 P_0^2(R) + \frac{R}{2} \right].
\end{aligned} \tag{4.15}$$

Substituting the above results (4.12) and (4.15) into the expression Q_m , we derive

$$\begin{aligned}
Q_m &= -\frac{\beta R}{1 + \beta R} \\
&\quad + \mu \frac{\beta + R P_0(R)}{P_0(R)[R^2 P_1(R) + 1 + \beta R]} \left(\frac{\beta}{\beta + R P_0(R)} \right)^2 \left[-\frac{3}{2} R P_0(R) - \frac{1}{2} R^3 P_0^2(R) + \frac{R}{2} \right] \\
&\quad + \mu \frac{\beta}{\beta + R P_0(R)} [1 - 2 P_0(R)] - \frac{\mu}{1 + \beta R} \frac{\beta P_0(R)}{\beta + R P_0(R)} [R^2 P_1(R) + 1 + \beta R] \\
&= -\frac{\beta R}{1 + \beta R} + \mu \frac{\beta R}{1 + \beta R} \frac{\beta}{\beta + R P_0(R)} \\
&\quad \cdot \left\{ \frac{1 + \beta R}{R^2 P_1(R) + 1 + \beta R} \left[-\frac{3}{2} - \frac{1}{2} R^2 P_0(R) + \frac{1}{2 P_0(R)} \right] \right. \\
&\quad \left. + \frac{1 + \beta R}{\beta R} [1 - 2 P_0(R)] - \frac{R^2 P_0(R) P_1(R)}{\beta R} - \frac{1 + \beta R}{\beta R} P_0(R) \right\} \\
&= -\frac{\beta R}{1 + \beta R} + \mu \frac{\beta R}{1 + \beta R} \frac{\beta}{\beta + R P_0(R)} \\
&\quad \cdot \left\{ \frac{1 + \beta R}{R^2 P_1(R) + 1 + \beta R} \left[-\frac{3}{2} - \frac{1}{2} R^2 P_0(R) + \frac{1}{2 P_0(R)} \right] + R^2 P_0(R) P_1(R) \right\} \quad (\text{by (2.3)}) \\
&= \frac{\beta R}{1 + \beta R} \left\{ -1 + \mu \frac{\beta}{\beta + R P_0(R)} \frac{1}{R^2 P_1(R) + 1 + \beta R} \right. \\
&\quad \cdot \left[(1 + \beta R) \left(-\frac{3}{2} - \frac{1}{2} R^2 P_0(R) + \frac{1}{2 P_0(R)} \right) + R^2 P_0(R) P_1(R) (R^2 P_1(R) + 1 + \beta R) \right] \Big\}.
\end{aligned}$$

Furthermore, the bracket in the right-hand side can simplify to

$$\begin{aligned}
&(1 + \beta R) \left(-\frac{3}{2} - \frac{1}{2} R^2 P_0(R) + \frac{1}{2 P_0(R)} \right) + R^2 P_0(R) P_1(R) (R^2 P_1(R) + 1 + \beta R) \\
&= (1 + \beta R) \left(-\frac{3}{2} - \frac{1}{2} R^2 P_0(R) + \frac{1}{2 P_0(R)} \right) + R^4 P_0(R) P_1^2(R)
\end{aligned}$$

$$\begin{aligned}
& + (1 + \beta R) R^2 P_0(R) P_1(R) \\
& = (1 + \beta R) \left(-\frac{3}{2} - \frac{1}{2} R^2 P_0(R) + \frac{1}{2 P_0(R)} \right) + R^4 P_0(R) P_1^2(R) \\
& \quad + (1 + \beta R)(1 - 3 P_0(R)) \quad (\text{by (2.3)}) \\
& = (1 + \beta R) \left(-\frac{R^2 P_0(R)}{2} - \frac{1}{2} - 3 P_0(R) + \frac{1}{2 P_0(R)} \right) + R^4 P_0(R) P_1^2(R) \\
& = -\frac{1 + \beta R}{2} \left[R^2 P_0(R) + 1 + 6 P_0(R) - \frac{1}{P_0(R)} \right] + R^4 P_0(R) P_1^2(R) \\
& = -\frac{1 + \beta R}{2} R^3 P_0(R) P_1'(R) + R^4 P_0(R) P_1^2(R) \\
& = R^3 P_0(R) \left[R P_1^2(R) - \frac{1 + \beta R}{2} P_1'(R) \right],
\end{aligned}$$

where we have used the fact (see [17, page 631])

$$R^2 P_0(R) + 1 + 6 P_0(R) - \frac{1}{P_0(R)} = R^3 P_0(R) P_1'(R).$$

Hence, by (2.9), we obtain

$$\begin{aligned}
Q_m &= \frac{\beta R}{1 + \beta R} \left\{ -1 + \mu \frac{\beta R^3 P_0(R)}{\beta + R P_0(R)} \cdot \frac{R P_1^2(R) - \frac{1 + \beta R}{2} P_1'(R)}{R^2 P_1(R) + 1 + \beta R} \right\} \\
&= \frac{\beta R}{1 + \beta R} \left\{ -1 + \frac{\mu}{\mu_1(R)} \right\},
\end{aligned} \tag{4.16}$$

so that $Q_m = 0$ if and only if $\mu = \mu_1(R)$, which completes the proof of this lemma. \square

Hence, recalling (2.8), i.e., $\mu < \mu^* < \mu_1(R)$, we complete the proof of Lemma 4.1. \square

Let $F(a) = (F_{-1}(a), F_0(a), F_1(a))$. Now we proceed to give the main result of this section.

Theorem 4.2. *There exists a new center $\varepsilon a^*(\varepsilon)$ such that $F(a^*(\varepsilon)) = 0$.*

Proof. By (4.5) and (4.9),

$$F(a) = E(a) + \varepsilon G(a).$$

It follows from Lemma 4.1 that we can choose a_0 such that $E(a_0) = 0$. Clearly, $E'(a)$ and $G'(a)$ are continuous. Since $Q_m \neq 0$, $E'(a_0)$ is invertible. Therefore, by Theorem 4.1, the proof is complete. \square

5. The inhomogeneous linear system

In this section, we shall derive the estimates of $\rho(\theta, \varphi, t)$ for the inhomogeneous linear system (3.14)–(3.18) which is in the new coordinate system. In order to do so, we need to take the inverse Laplace transform of $\widehat{\rho}_{n,m}(s)$ which is of the form (3.56) (or (3.57)). Since for $n = 1$, $h_1(s)$ has a simple root at $s = 0$ that is different from the situation $n \neq 1$, the computation is divided into two cases: $n \neq 1$ and $n = 1$.

5.1. Case 1: $n \neq 1$

In this case, the choice of new center does not change the proof. Introduce the following inverse Laplace transforms of various terms in (3.57):

$$\rho_{M,n,m}(t) = \frac{1}{2\pi i} \int_{\Gamma} M_{n,m}(s) e^{st} ds, \quad (5.1)$$

$$\rho_{f,n,m}(t) = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\mu h_n(s, \mu, R)} \left[\frac{\mu}{s+1} \frac{\partial \xi_{2,n,m}}{\partial r} \Big|_{r=R} - \frac{\partial \phi_{2,n,m}}{\partial r} \Big|_{r=R} \right] e^{st} ds, \quad (5.2)$$

$$E_{1n}(t) = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\mu h_n(s, \mu, R)} e^{st} ds, \quad \left(\frac{1}{\mu h_n(s, \mu, R)} = \widehat{E}_{1n}(s) \right) \quad (5.3)$$

$$\begin{aligned} E_{2n}(t) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{P_n(\sqrt{s+1}R)}{h_n(s, \mu, R)} \cdot \frac{1}{(s+1)RP_n(\sqrt{s+1}R) + \frac{n}{R} + \beta} e^{st} ds \\ &\triangleq \frac{1}{2\pi i} \int_{\Gamma} \frac{P_n(\sqrt{s+1}R)}{h_n(s, \mu, R)} \cdot \frac{1}{\phi_n(s)} e^{st} ds, \end{aligned} \quad (5.4)$$

where $\Gamma: s = J + i\tau$, $-\infty < \tau < \infty$, and $J > \max\{\operatorname{Re}(\text{roots of } h_n)\}$. Note that if $f_{n,m}^1 \equiv f_{n,m}^2 \equiv 0$ and $b_{n,m}^1 \equiv b_{n,m}^2 \equiv b_{n,m}^3 \equiv 0$, then $\rho_{M,n,m}$ is the function $\rho_{n,m}$. The corresponding $w_{n,m}$ and $q_{n,m}$ will be denoted by $w_{M,n,m}$ and $q_{M,n,m}$. We further define $\rho_M = \sum_{n=0}^{\infty} \sum_{m=-n}^n \rho_{M,n,m} Y_{n,m}$, and similarly define $w_M = \sum_{n=0}^{\infty} \sum_{m=-n}^n w_{M,n,m} Y_{n,m}$, $q_M = \sum_{n=0}^{\infty} \sum_{m=-n}^n q_{M,n,m} Y_{n,m}$.

By (3.57), we have

$$\begin{aligned} \widehat{\rho}_{n,m}(s) &= \widehat{\rho}_{M,n,m}(s) + \frac{n + \beta R}{\beta R} \frac{\beta + RP_0(R)}{\beta RP_0(R)} \varepsilon \widehat{\rho}_{f,n,m}(s) \\ &\quad + \frac{\beta + RP_0(R)}{\beta RP_0(R)} \varepsilon (\widehat{b}_{n,m}^1 - \frac{n}{R} \widehat{b}_{n,m}^3) \widehat{E}_{1n} + \frac{\beta + RP_0(R)}{\beta P_0(R)} \varepsilon \widehat{b}_{n,m}^2 \widehat{E}_{2n} \\ &\triangleq \widehat{\rho}_{M,n,m}(s) + A_1 \varepsilon \widehat{\rho}_{f,n,m}(s) + A_2 \varepsilon (\widehat{b}_{n,m}^1 - \frac{n}{R} \widehat{b}_{n,m}^3) \widehat{E}_{1n} + A_3 \varepsilon \widehat{b}_{n,m}^2 \widehat{E}_{2n}. \end{aligned} \quad (5.5)$$

Then it follows that

$$\rho_{n,m} = \rho_{M,n,m} + A_1 \varepsilon \rho_{f,n,m} + A_2 \varepsilon (\widehat{b}_{n,m}^1 - \frac{n}{R} \widehat{b}_{n,m}^3) * E_{1n} + A_3 \varepsilon \widehat{b}_{n,m}^2 * E_{2n}. \quad (5.6)$$

Furthermore, we derive that

$$|\rho_{n,m} - \rho_{M,n,m}| \leq |\varepsilon| \left\{ A_1 |\rho_{f,n,m}| + A_2 \left| \left(b_{n,m}^1 - \frac{n}{R} b_{n,m}^3 \right) * E_{1n} \right| + A_3 |b_{n,m}^2 * E_{2n}| \right\}. \quad (5.7)$$

To gain the estimate of $\rho_{n,m}(t)$, we need to estimate the three terms on the right-hand side of (5.7), respectively.

We now proceed to estimate the last two terms on the right-hand side of (5.7). To begin with, we shall need an improvement of [27, Lemma 5.5].

Lemma 5.1. *There exists $\delta_0 > 0$, independent of n , such that the real parts of the roots of $h_n(s, \mu, R)$ are less than $-\delta_0 n^2$ for all sufficiently large n .*

Proof. $j_{n+1/2,m}$ satisfies

$$j_{n+1/2,m} > (m-1)\pi + \sqrt{\left(n + \frac{1}{2}\right)\left(n + \frac{5}{2}\right)},$$

which can be found in [27, page 2495].

Then by the definition of $P_n(\xi)$, we derive

$$\begin{aligned} \phi_n\left(-\frac{1}{3R^2}n^2\right) &= 2\left(-\frac{1}{3R^2}n^2 + 1\right)R \sum_{m=1}^{\infty} \frac{1}{\left(-\frac{1}{3R^2}n^2 + 1\right)R^2 + j_{n+1/2,m}^2} + \frac{n}{R} + \beta \\ &> 2\left(-\frac{1}{3R^2}n^2 + 1\right)R \\ &\quad \cdot \sum_{m=1}^{\infty} \frac{1}{\left(-\frac{1}{3R^2}n^2 + 1\right)R^2 + \left[(m-1)\pi + \sqrt{\left(n + \frac{1}{2}\right)\left(n + \frac{5}{2}\right)}\right]^2} + \frac{n}{R} + \beta \\ &> 2\left(-\frac{1}{3R^2}n^2 + 1\right)R \\ &\quad \cdot \left\{ \frac{1}{\left(n + \frac{1}{2}\right)\left(n + \frac{5}{2}\right) - \frac{1}{3}n^2 + R^2} + \sum_{m=2}^{\infty} \frac{1}{(m-1)^2\pi^2 + \frac{2}{3}n^2} \right\} + \frac{n}{R} + \beta \\ &> 2\left(-\frac{1}{3R^2}n^2 + 1\right)R \left\{ \frac{1}{\frac{2}{3}n^2} + \sum_{m=1}^{\infty} \frac{1}{m^2\pi^2 + \frac{2}{3}n^2} \right\} + \frac{n}{R} + \beta \\ &> 2\left(-\frac{1}{3R^2}n^2 + 1\right)R \left\{ \frac{1}{\frac{2}{3}n^2} + \frac{1}{\sqrt{\frac{2}{3}n}} \right\} + \frac{n}{R} + \beta \\ &> \frac{n - (1 + \frac{2}{3}\sqrt{\frac{3}{2}n})}{R} + \beta \\ &> \beta \end{aligned}$$

for n sufficiently large. Together with $\phi_n(\beta_1 + 0) = -\infty$, where $\beta_1 = -1 - (j_{n+1/2,1}/R)^2$ is the pole of the function $P_n(\sqrt{s+1}R)$, we deduce that the first zero γ_1 lies in the interval $(-1 - (j_{n+1/2,1}/R)^2, -n^2/(3R^2))$. Thus it is possible to choose δ_0 so small that $\gamma_1 < -\delta_0 n^2$.

The rest of the proof is similar with that of [27, Lemma 5.5], so we omit the details here. \square

Lemma 5.2. *Let $\mu < \mu^*(R)$. For $n \neq 1$, there exists a small positive number δ , depending only on μ , R such that*

$$|E_{1n}| = \left| \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{st} ds}{\mu h_n(s, \mu, R)} \right| \leq C e^{-\delta(n^3+1)t} + C(n+1)^{-3} e^{-\delta(n^2+1)t}, \quad (5.8)$$

$$|E_{2n}| = \left| \frac{1}{2\pi i} \int_{\Gamma} \frac{P_n(\sqrt{s+1}R)}{h_n(s, \mu, R)} \frac{e^{st}}{\phi_n(s)} ds \right| \leq C(n+1)^{-1} e^{-\delta(n^2+1)t}. \quad (5.9)$$

Proof. The function $h_n(s, \mu, R)$ can be rewritten as $h_n(s, \mu, R) = c_1(s + c(n) + k_n(s))$, where $c_1 k_n(s) = \frac{R^2 P_1(R) + 1 + \beta R}{(s+1)R + (\frac{n}{R} + \beta)/P_n(\sqrt{s+1}R)} - R P_1(R)$ and $c(n) \approx c_2 n^3$ as $n \rightarrow \infty$ with c_1, c_2 positive constants. By Lemma 5.1, $|k_n(s)| \leq \text{const} = c_3$ if $\text{Re } s > -\delta(n^2+1)$, provided δ is small enough. Then we can use the same argument as in the proof of [17, Lemma 4.1], and obtain the estimate (5.8).

We now proceed to prove (5.9). By (2.1), we have $|P_n(\sqrt{s+1}R)| \leq \frac{C}{\sqrt{|s|+1}}$ for $\text{Re } s \geq -\delta(n^2+1)$. Since $|\frac{1}{\phi_n(s)}| \leq C$ if $\text{Re } s \geq -\delta(n^2+1)$, it follows that

$$\begin{aligned} |E_{2n}| &= \left| \frac{1}{2\pi i} \int_{\Gamma} \frac{P_n(\sqrt{s+1}R)}{h_n(s, \mu, R)} \frac{e^{st}}{\phi_n(s)} ds \right| = C \left| \frac{1}{2\pi i} \int_{-\delta(n^2+1)-i\infty}^{-\delta(n^2+1)+i\infty} \frac{P_n(\sqrt{s+1}R)}{s + c(n) + k_n(s)} \frac{e^{st}}{\phi_n(s)} ds \right| \\ &\leq C e^{-\delta(n^2+1)t} \int_{-\infty}^{\infty} \frac{d\tau}{(|\tau| + n^3 + 1)(|\tau|^{1/2} + n + 1)} \\ &\leq C(n+1)^{-\frac{4}{3}} e^{-\delta(n^2+1)t}, \end{aligned}$$

which proves (5.9). \square

Furthermore, by Lemma 5.2, we have the next result.

Lemma 5.3. *If*

$$\int_0^{\infty} e^{2\delta_1 t} |b(t)|^2 dt \leq A, \quad 0 < \delta_1 < \delta,$$

then for all $n \neq 1$,

$$\int_0^\infty e^{2\delta_1 t} |b * E_{jn}|^2 dt \leq C A(n+1)^{-6}, \quad j = 1, 2. \quad (5.10)$$

The proof of this lemma is the same as in [17, Lemma 4.2].

From Lemma 5.3 and (3.33)–(3.35), we immediately obtain the following estimates of the last two terms on the right-hand side of (5.7).

Lemma 5.4. For $\delta_1 > 0$,

$$|\varepsilon|(n+1)^7 \int_0^\infty e^{2\delta_1 t} \left| \left(b_{n,m}^1 - \frac{n}{R} b_{n,m}^3 \right) * E_{1n} \right|^2 dt \leq C (B_{n,m}^1 + B_{n,m}^3), \quad (5.11)$$

$$|\varepsilon|(n+1)^7 \int_0^\infty e^{2\delta_1 t} |b_{n,m}^2 * E_{2n}|^2 dt \leq C B_{n,m}^2, \quad (5.12)$$

where C is independent of n .

Next, we shall establish the estimate of $\rho_{f,n,m}$ on the right-hand side of (5.7) as follows:

Lemma 5.5. For $\delta_1 > 0$,

$$|\varepsilon|(n+1)^8 \int_0^\infty e^{2\delta_1 t} |\rho_{f,n,m}|^2 dt \leq C (F_{n,m}^1 + F_{n,m}^2), \quad (5.13)$$

where the constant C is independent of n .

Proof. We begin with the term

$$L_1 \equiv \frac{1}{2\pi i} \int_\Gamma \frac{1}{\mu h_n(s, \mu, R)} \frac{\mu}{s+1} \frac{\partial \xi_{2,n,m}}{\partial r} \Big|_{r=R} e^{st} ds,$$

which appears in the definition of $\rho_{f,n,m}$. It follows from (3.49) that $\xi_{2,n,m}$ is the Laplace transform of the solution $\Psi_{2,n,m}$ of

$$\begin{aligned} \partial_t \Psi_{2,n,m} - \Delta \Psi_{2,n,m} + \left(\frac{n(n+1)}{r^2} + 1 \right) \Psi_{2,n,m} &= f_{n,m}^1 \quad \text{in } B_R, \quad t > 0, \\ \left(\frac{\partial \Psi_{2,n,m}}{\partial r} + \beta \Psi_{2,n,m} \right) \Big|_{r=R} &= 0, \quad \Psi_{2,n,m}|_{t=0} = 0. \end{aligned}$$

Since the Laplace transform of $\mu e^{-t} * \Psi_{2,n,m}$ is $\frac{\mu}{s+1} \xi_{2,n,m}$, then we can write

$$L_1 = E_{1n} * \frac{\partial}{\partial r} (\mu e^{-t} * \Psi_{2,n,m})|_{r=R}.$$

By parabolic estimates and (3.31), as in the proof of [27, (91) of Lemma 5.3], we obtain

$$|\varepsilon| \int_0^\infty e^{2\delta_1 t} (n+1)^2 \left| \frac{\partial \Psi_{2,n,m}(R, t)}{\partial r} \right|^2 dt \leq C F_{n,m}^1. \quad (5.14)$$

Let $b \equiv \mu e^{-t} * \frac{\partial \Psi_{2,n,m}}{\partial r} \Big|_{r=R}$, then, by changing the order of integration, we have

$$\begin{aligned} |\varepsilon| \int_0^\infty e^{2\delta_1 t} |b|^2 dt &= |\varepsilon| \mu^2 \int_0^\infty e^{2\delta_1 t} \left| \int_0^t \frac{\partial \Psi_{2,n,m}(R, t-\tau)}{\partial r} e^{-\tau} d\tau \right|^2 dt \\ &= |\varepsilon| \mu^2 \int_0^\infty e^{2\delta_1 t} \left| \int_0^t \frac{\partial \Psi_{2,n,m}(R, t-\tau)}{\partial r} e^{-\frac{\tau}{2} + \frac{\delta_1 \tau}{2}} e^{-\frac{\tau}{2} + \frac{\delta_1 \tau}{2}} e^{-\delta_1 \tau} d\tau \right|^2 dt \\ &\leq |\varepsilon| \mu^2 \int_0^\infty e^{2\delta_1 t} \left\{ \int_0^t \left| \frac{\partial \Psi_{2,n,m}(R, t-\tau)}{\partial r} \right|^2 e^{-\tau + \delta_1 \tau} e^{-2\delta_1 \tau} d\tau \int_0^t e^{-\tau + \delta_1 \tau} d\tau \right\} dt \\ &\leq C |\varepsilon| \mu^2 \int_0^\infty \int_0^t \left| \frac{\partial \Psi_{2,n,m}(R, t-\tau)}{\partial r} \right|^2 e^{2\delta_1(t-\tau)} e^{-\tau + \delta_1 \tau} d\tau dt \\ &\leq C |\varepsilon| \mu^2 \int_0^\infty \int_\tau^\infty \left| \frac{\partial \Psi_{2,n,m}(R, t-\tau)}{\partial r} \right|^2 e^{2\delta_1(t-\tau)} e^{-\tau + \delta_1 \tau} dt d\tau \\ &= C |\varepsilon| \mu^2 \int_0^\infty \int_0^\infty \left| \frac{\partial \Psi_{2,n,m}(R, t)}{\partial r} \right|^2 e^{2\delta_1 t} e^{-\tau + \delta_1 \tau} dt d\tau \\ &\leq C F_{n,m}^1 (n+1)^{-2}, \end{aligned}$$

where we have used (5.14). Hence, it follows from Lemma 5.3 that

$$|\varepsilon| \int_0^\infty e^{2\delta_1 t} |L_1|^2 dt = |\varepsilon| \int_0^\infty e^{2\delta_1 t} |E_{1n} * b|^2 dt \leq C F_{n,m}^1 (n+1)^{-8}. \quad (5.15)$$

Now we consider the term

$$L_2 \equiv \frac{1}{2\pi i} \int_\Gamma \frac{1}{\mu h_n(s, \mu, R)} \frac{\partial \phi_{2,n,m}}{\partial r} \Big|_{r=R} e^{st} ds.$$

By (3.55), it is easily seen that $\phi_{2,n,m}$ is the Laplace transform of the solution $\Phi_{2,n,m}$ of

$$-\Delta \Phi_{2,n,m} + \frac{n(n+1)}{r^2} \Phi_{2,n,m} = f_{n,m}^2 + \mu e^{-t} * f_{n,m}^1 \quad \text{in } B_R, \quad t > 0, \quad (5.16)$$

$$\left(\frac{\partial \Phi_{2,n,m}}{\partial r} + \beta \Phi_{2,n,m} \right) \Big|_{r=R} = 0, \quad (5.17)$$

then L_2 can be rewritten as

$$L_2 = E_{1n} * \frac{\partial \Phi_{2,n,m}(R, t)}{\partial r}.$$

From [27, Lemma 5.2], it follows that

$$(n+1)^2 \left| \frac{\partial \Phi_{2,n,m}(R, t)}{\partial r} \right|^2 \leq C \left[\|f_{n,m}^2(\cdot, t)\|_{L^2(B_R)}^2 + \|e^{-t} * f_{n,m}^1(\cdot, t)\|_{L^2(B_R)}^2 \right].$$

By the same argument as before, we get

$$|\varepsilon| \int_0^\infty e^{2\delta_1 t} \|e^{-t} * f_{n,m}^1(\cdot, t)\|_{L^2(B_R)}^2 dt \leq C F_{n,m}^1,$$

so that

$$|\varepsilon| \int_0^\infty e^{2\delta_1 t} \left| \frac{\partial \Phi_{2,n,m}(R, t)}{\partial r} \right|^2 dt \leq C(F_{n,m}^1 + F_{n,m}^2)(n+1)^{-2}.$$

Furthermore, it follows from Lemma 5.3 that

$$|\varepsilon| \int_0^\infty e^{2\delta_1 t} |L_2|^2 dt = |\varepsilon| \int_0^\infty e^{2\delta_1 t} \left| E_{1n} * \frac{\partial \Phi_{2,n,m}(R, t)}{\partial r} \right|^2 dt \leq C(F_{n,m}^1 + F_{n,m}^2)(n+1)^{-8}.$$

Combining this estimate with (5.15), we derive (5.13). Hence, the proof is complete. \square

By Lemmas 5.4 and 5.5, we obtain an estimate for (5.7) as follows:

Lemma 5.6. For all $n \neq 1$ and $|m| \leq n$:

$$(n+1)^7 \int_0^\infty e^{2\delta_1 t} |\rho_{n,m}(t) - \rho_{M,n,m}(t)|^2 dt \leq C|\varepsilon|(F_{n,m}^1 + F_{n,m}^2 + B_{n,m}^1 + B_{n,m}^2 + B_{n,m}^3), \quad (5.18)$$

and therefore also

$$\int_0^\infty e^{2\delta_1 t} \|\rho - \rho_M - \sum_m (\rho_{1,m} - \rho_{M,1,m}) Y_{1,m}\|_{H^{7/2}(\partial B_R)}^2 dt \leq C|\varepsilon|. \quad (5.19)$$

As in the proof of [17], we derive the estimates for w , q and ρ_t :

$$\int_0^\infty e^{2\delta_1 t} \|w - w_M - \sum_m (w_{1,m} - w_{M,1,m}) Y_{1,m}\|_{H^2(B_R)}^2 dt \leq C|\varepsilon|, \quad (5.20)$$

$$\int_0^\infty e^{2\delta_1 t} \|q - q_M - \sum_m (q_{1,m} - q_{M,1,m}) Y_{1,m}\|_{H^2(B_R)}^2 dt \leq C|\varepsilon|, \quad (5.21)$$

$$\int_0^\infty e^{2\delta_1 t} \left\| \frac{\partial}{\partial t} (\rho - \rho_M - \sum_m (\rho_{1,m} - \rho_{M,1,m}) Y_{1,m}) \right\|_{H^{1/2}(\partial B_R)}^2 dt \leq C|\varepsilon|. \quad (5.22)$$

5.2. Case 2: $n = 1$

As stated in Section 4, a translation of the origin does not change the equations (3.14)–(3.18), but change the initial data, so that in the new coordinate system, by (4.4), $\widehat{\rho}_{1,m}(s)$ is changed into

$$\begin{aligned} \widehat{\rho}_{1,m}(s) = & \frac{1 + \beta R}{\beta R} \frac{\beta + R P_0(R)}{\beta R P_0(R)} \frac{1}{\mu h_1(s, \mu, R)} \\ & \cdot \left\{ \frac{\beta R}{1 + \beta R} \widetilde{\rho}_{0,1,m} + \frac{\mu}{s+1} \frac{\partial \xi_{1,1,m}}{\partial r} \Big|_{r=R} - \frac{\partial \phi_{1,1,m}}{\partial r} \Big|_{r=R} \right. \\ & + \varepsilon \left[\frac{\mu}{s+1} \frac{\partial \xi_{2,1,m}}{\partial r} \Big|_{r=R} - \frac{\partial \phi_{2,1,m}}{\partial r} \Big|_{r=R} + \frac{\beta R}{1 + \beta R} \widehat{b}_{1,m}^1 \right. \\ & \left. \left. + \frac{\beta R^2}{1 + \beta R} \mu \widehat{b}_{1,m}^2 \frac{1}{(s+1)R + (\frac{1}{R} + \beta)/P_1(\sqrt{s+1}R)} - \frac{\beta}{1 + \beta R} \widehat{b}_{1,m}^3 \right] \right\}, \end{aligned} \quad (5.23)$$

or

$$\begin{aligned} \widehat{\rho}_{1,m}(s) = & M_{1,m}(s) + \frac{1 + \beta R}{\beta R} \frac{\beta + R P_0(R)}{\beta R P_0(R)} \frac{\varepsilon}{\mu h_1(s, \mu, R)} \\ & \cdot \left\{ \frac{\mu}{s+1} \frac{\partial \xi_{2,1,m}}{\partial r} \Big|_{r=R} - \frac{\partial \phi_{2,1,m}}{\partial r} \Big|_{r=R} + \frac{\beta R}{1 + \beta R} \widehat{b}_{1,m}^1 \right. \\ & \left. + \frac{\beta R^2}{1 + \beta R} \mu \widehat{b}_{1,m}^2 \frac{1}{(s+1)R + (\frac{1}{R} + \beta)/P_1(\sqrt{s+1}R)} - \frac{\beta}{1 + \beta R} \widehat{b}_{1,m}^3 \right\}, \end{aligned} \quad (5.24)$$

where

$$\begin{aligned} M_{1,m}(s) = & \frac{1 + \beta R}{\beta R} \frac{\beta + R P_0(R)}{\beta R P_0(R)} \frac{1}{\mu h_1(s, \mu, R)} \\ & \cdot \left\{ \frac{\beta R}{1 + \beta R} \widetilde{\rho}_{0,1,m} + \frac{\mu}{s+1} \frac{\partial \xi_{1,1,m}}{\partial r} \Big|_{r=R} - \frac{\partial \phi_{1,1,m}}{\partial r} \Big|_{r=R} \right\}, \end{aligned} \quad (5.25)$$

and the singularity of $1/h_1(s)$ at $s = 0$ is canceled by the expression in the brace in (5.23) at $s = 0$, guaranteed by results from section 4. Furthermore, we can take the inverse Laplace transform in (5.23) and move the contour Γ to $\operatorname{Re} s = -\delta$, thus obtaining the formula

$$\rho_{1,m}(t) = \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} \widehat{\rho}_{1,m}(s) e^{st} ds. \quad (5.26)$$

In order to estimate $\rho_{1,m}(t)$ in (5.26), we break up $\widehat{\rho}_{1,m}(s)$ given by the right-hand side of (5.24) into four terms, and each term has the form

$$\frac{A(s)}{h_1(s)}, \quad (5.27)$$

then we shall estimate the inverse Laplace transform separately for each term. However, since $s = 0$ is a simple root of $h_1(s)$ for each term (5.27), we therefore rewrite (5.27) in the form

$$\left(\frac{s}{s+1} \frac{1}{h_1(s)}\right) A(s) + \left(\frac{s}{s+1} \frac{1}{h_1(s)}\right) B(s), \quad B(s) = \frac{A(s)}{s}. \quad (5.28)$$

Introduce new functions

$$E_{1*} = \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{s}{s+1} \frac{e^{st}}{\mu h_1(s)} ds,$$

$$E_{2*} = \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{s}{s+1} \frac{1}{\mu h_1(s)} \frac{P_1(\sqrt{s+1}R)}{(s+1)R P_1(\sqrt{s+1}R) + \frac{1}{R} + \beta} e^{st} ds.$$

Since the singularity of $1/h_1(s)$ at $s = 0$ is canceled by the factor $s/(s+1)$ at $s = 0$ and $s/(s+1)$ goes to 1 as $|s| \rightarrow \infty$, by the same method as in the proof of Lemma 5.2 we immediately obtain the following estimates

$$|E_{1*}| = \left| \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{s}{s+1} \frac{e^{st}}{\mu h_1(s)} ds \right| \leq C e^{-\delta t}, \quad (5.29)$$

$$|E_{2*}| = \left| \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{s}{s+1} \frac{e^{st}}{\mu h_1(s)} \frac{P_1(\sqrt{s+1}R)}{(s+1)R P_1(\sqrt{s+1}R) + \frac{1}{R} + \beta} e^{st} ds \right| \leq C e^{-\delta t}. \quad (5.30)$$

Then we can write $\widehat{\rho}_{1,m}(s)$ in the form as

$$\begin{aligned} \widehat{\rho}_{1,m}(s) &= \widehat{\rho}_{M,1,m}(s) + \frac{1 + \beta R}{\beta R} \frac{\beta + R P_0(R)}{\beta R P_0(R)} \varepsilon \widehat{\rho}_{f,1,m}(s) \\ &\quad + \frac{\beta + R P_0(R)}{\beta R^2 P_0(R)} \varepsilon (R \widehat{b}_{1,m}^1(s) - \widehat{b}_{1,m}^3(s)) \widehat{E}_{1*}(s) \left(1 + \frac{1}{s}\right) \end{aligned} \quad (5.31)$$

$$+ \frac{\beta + RP_0(R)}{\beta P_0(R)} \varepsilon \widehat{b}_{1,m}^2(s) \widehat{E}_{2*}(s) \left(1 + \frac{1}{s}\right).$$

For the first term $\widehat{\rho}_{M,1,m}(s)$, we have

$$|\rho_{M,1,m}(t)| = \left| \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} \widehat{\rho}_{M,1,m}(s) e^{st} ds \right| = \widetilde{\rho}_{M,1,m}(t) + c_m(\varepsilon), \quad (5.32)$$

where $|c_m(\varepsilon)| \leq C|\varepsilon|$, and by the linear stability result [27, Theorem 1.2], $|\widetilde{\rho}_{M,1,m}(t)| \leq C e^{-\delta t}$ holds.

We then apply the previous lemmas and [17, Lemmas 5.2-5.4] to estimate the inverse Laplace transform of the last three terms in (5.31). For example, for the term

$$|\varepsilon| \widehat{b}_{1,m}^2(s) \widehat{E}_{2*}(s) \left(1 + \frac{1}{s}\right) = |\varepsilon| \widehat{b}_{1,m}^2(s) \widehat{E}_{2*}(s) + |\varepsilon| \widehat{b}_{1,m}^2(s) \widehat{E}_{2*}(s) \frac{1}{s},$$

since the inverse Laplace transform of $\widehat{b}_{1,m}^2 \widehat{E}_{2*}$ is $b_{1,m}^2 * E_{2*}$, for the first term on the right-hand side we can use the same argument as in the proof of Lemma 5.3 to establish

$$|\varepsilon| \int_0^\infty e^{2\delta_1 t} |b_{1,m}^2 * E_{2*}|^2 dt \leq C B_{1,m}^2, \quad (5.33)$$

while for the last term on the right-hand side, by [17, Lemmas 5.3 and 5.4] and (5.33), we have

$$\begin{aligned} |\varepsilon| \int_0^\infty e^{2\delta_1 t} \left| \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} \widehat{b}_{1,m}^2 \widehat{E}_{2*} \frac{e^{st}}{s} ds \right|^2 dt &= |\varepsilon| \int_0^\infty e^{2\delta_1 t} \left| \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} \widehat{b_{1,m}^2 * E_{2*}} \frac{e^{st}}{s} ds \right|^2 dt \\ &= |\varepsilon| \int_0^\infty e^{2\delta_1 t} \left| \int_t^\infty b_{1,m}^2 * E_{2*} d\tau \right|^2 dt \\ &\leq \frac{1}{\delta_1^2} \int_0^\infty e^{2\delta_1 t} |b_{1,m}^2 * E_{2*}|^2 dt \\ &\leq C B_{1,m}^2. \end{aligned}$$

Together with the above analysis, the main result for the case $n = 1$ is established.

Lemma 5.7. *For a positive constant C , we get*

$$\int_0^\infty e^{2\delta_1 t} |\rho_{1,m}(t) - \rho_{M,1,m}(t)|^2 dt \leq C|\varepsilon|. \quad (5.34)$$

In particular,

$$\int_0^\infty e^{2\delta_1 t} |\rho_{1,m}(t)|^2 dt \leq C. \quad (5.35)$$

6. Stability for $\mu < \mu^*$

We shall establish the asymptotic stability of the radially symmetric stationary solution for (3.1)–(3.5) by using two fixed point theorem. In Section 4, by the first fixed point theorem (Theorem 4.1) we determine the new center $\varepsilon a^*(\varepsilon)$. And as in the proof of [17, Lemma 7.1], we obtain that after performing a translation $x \rightarrow x + \varepsilon a^*(\varepsilon)$ on the initial data, there exists a unique global solution (w, q, ρ) of (3.14)–(3.18), satisfying

$$\|w\|_{C^{2+2\alpha/3, 1+\alpha/3}(B_R \times [0, \infty))} \leq C, \quad (6.1)$$

$$\|q\|_{C^{2+\alpha, \alpha/3}(B_R \times [0, \infty))} \leq C, \quad (6.2)$$

$$\|\rho, D_x \rho\|_{C^{3+\alpha, 1+\alpha/3}(\partial B_R \times [0, \infty))} \leq C. \quad (6.3)$$

We now proceed with a second fixed point argument.

Introduce the space X of functions $\Phi = (f^1, f^2, b^1, b^2, b^3)$ with norm $\|\Phi\|$ defined by the maximum of the left-hand sides of (3.20)–(3.29) with $\sqrt{|\varepsilon|}$ dropped, and set

$$X_1 = \{\Phi \in X : \sqrt{|\varepsilon|} \|\Phi\| \leq 1\},$$

then we define a new function $\tilde{\Phi} \equiv S\Phi = (\tilde{f}^1, \tilde{f}^2, \tilde{b}^1, \tilde{b}^2, \tilde{b}^3)$, where

$$\begin{aligned} \tilde{f}^1 &= -A_\varepsilon^1 w + A_\varepsilon w, & \tilde{f}^2 &= -A_\varepsilon w, \\ \tilde{b}^1 &= B_\varepsilon^1, & \tilde{b}^2 &= B_\varepsilon^2, & \tilde{b}^3 &= B_\varepsilon^3. \end{aligned}$$

Again as in the proof of [17], we obtain that S maps X_1 into itself and S is a contraction mapping so that S has a unique fixed point.

By the above argument, we can now state the main result of this paper.

Theorem 6.1. *Let $\mu < \mu^*$. If $|\varepsilon|$ is sufficiently small, then there exists a unique global solution (σ, p, r) of the problem (1.1)–(1.5) with the initial data (1.14) satisfying (2.10), and there exists a new center $\varepsilon a^*(\varepsilon)$, where $a^*(\varepsilon)$ is a bounded function of ε , such that*

$$\partial\Omega(t) \rightarrow \{|x - \varepsilon a^*(\varepsilon)| = R\}$$

exponentially fast as $t \rightarrow \infty$.

Remark 6.1. After the translation of the origin and the Hanzawa transformation, the global solution (σ, p, r) in the new variables (r, θ, φ) has the form:

$$\begin{aligned} \sigma(r, \theta, \varphi, t) &= \sigma_S(r) + \varepsilon w(r, \theta, \varphi, t), \\ p(r, \theta, \varphi, t) &= p_S(r) + \varepsilon q(r, \theta, \varphi, t), \\ \partial\Omega(t) : r &= R + \varepsilon \rho(\theta, \varphi, t), \end{aligned}$$

where w , q , ρ satisfy (6.1)–(6.3).

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