Available online at www.sciencedirect.com

ScienceDirect

Journal of Differential Equations 294 (2021) 88–117

Journal of Differential Equations

www.elsevier.com/locate/jde

The Cahn–Hilliard equation with a nonlinear source term

Alain Miranville [a,b,*]

[a] Henan Normal University, School of Mathematics and Information Science, Xinxiang, Henan, China

[b] Laboratoire I3M et Laboratoire de Mathématiques et Applications, Université de Poitiers, UMR CNRS 7348, Equipe DACTIM-MIS, Site du Futuroscope - Téléport 2, 11 Boulevard Marie et Pierre Curie - Bâtiment H3 - TSA 61125, 86073 Poitiers Cedex 9, France

Received 25 August 2020; revised 22 March 2021; accepted 20 May 2021

Available online 26 May 2021

Abstract

Our aim in this paper is to prove the existence of solutions to the Cahn–Hilliard equation with a general nonlinear source term. An essential difficulty is to obtain a global in time solution. Indeed, due to the presence of the source term, one cannot exclude the possibility of blow up in finite time when considering regular nonlinear terms and when considering an approximated scheme. Considering instead logarithmic nonlinear terms, we give sufficient conditions on the source term which ensure the existence of a global in time weak solution. These conditions are satisfied by several important models and applications which can be found in the literature.

MSC: 35K55; 35B45

Keywords: Cahn–Hilliard equation; Nonlinear source term; Logarithmic nonlinear term; Existence of solutions; Applications

* Correspondence to: Henan Normal University, School of Mathematics and Information Science, Xinxiang, Henan, China.

E-mail address: Alain.Miranville@math.univ-poitiers.fr.

https://doi.org/10.1016/j.jde.2021.05.045

0022-0396/

1. Introduction

The Cahn–Hilliard equation was proposed in [6,7] in order to describe phase separation processes in binary alloys. Since then, it has been thoroughly studied. It would simply be impossible to cite all papers related to the mathematical and numerical analysis of the equation and we instead refer the interested readers to the recent reviews [35,36].

What is also remarkable with the Cahn–Hilliard equation is that it (or some of its variants) has been used in many other applications, such as dealloying in corrosion processes (see [15]), population dynamics (see [13]), tumor growth (see [2,25]), bacterial films (see [26]), thin films (see [39]), chemistry (see [42]), image processing (see [4,8,14]) and even astronomy in the rings of Saturn (see [40]) and ecology (surprisingly, the clustering of mussels can be perfectly well described by the Cahn–Hilliard equation; see [29] and even Youtube for videos).

We consider in this article the following more general equation:

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + g(x,u) = 0. \tag{1.1}$$

This equation appears in several important of the aforementioned applications (see also below for specific examples). In particular, when $g \equiv 0$, we recover the original Cahn–Hilliard equation.

Equation (1.1), endowed with Dirichlet type boundary conditions, was studied, for rather general source terms g, in [16,17,32,33]. Now, for most applications of interest, Neumann boundary conditions are the relevant ones, as one wants to take advantage of separation and clustering effects. In that case, however, due to the fact that, contrary to the original Cahn–Hilliard equation, one no longer has the conservation of the spatial average of the order parameter u, the existence of global in time solutions becomes a challenging problem (see [5,9,11,17,34,35]). Even worse, it was observed in [11,17] (see also [35]) that one can have blow up in finite time when considering regular nonlinear terms f (typically, the usual cubic nonlinear term $f(s) = s^3 - s$), which is problematic in view of applications. Nevertheless, it was proved in [31,34,35] that, in some particular cases, considering instead a logarithmic nonlinear term (which is relevant, as such a nonlinear term is the thermodynamically relevant one for the original Cahn–Hilliard equation), one can prove the existence of global in time solutions.

Our aim in this paper is to give sufficient conditions on the nonlinear source term g that ensure the existence of global in time weak solutions, when considering Neumann boundary conditions and logarithmic nonlinear terms f. One key step is to give proper approximations of the singular nonlinear term f which allow to derive the necessary a priori estimates to pass to the limit. In particular, these sufficient conditions are satisfied for most of the source terms g appearing in the literature.

This paper is organized as follows. In the first section, we set the problem and state the main result, namely, the existence of global in time weak solutions. Then, in Section 3, we prove our main result. Finally, in Section 4, we give and discuss several applications.

2. Setting of the problem and main result

We consider the following initial and boundary value problem, in a bounded and regular domain $\Omega \subset \mathbb{R}^n$, $n = 1$, 2 or 3, with boundary Γ:

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + g(x,u) = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma, \tag{2.2}$$

$$u|_{t=0} = u_0. \tag{2.3}$$

First, concerning the (nonlinear) source term $g : \Omega \times \mathbb{R} \to \mathbb{R}$, we make the following assumptions:

$$g(., s) \text{ is measurable on } \Omega,\ \forall s \in \mathbb{R}, \tag{2.4}$$

$$g(x, .) \text{ is continuous on } \mathbb{R},\ \text{for a.a. } x \in \Omega, \tag{2.5}$$

$$|g(x,s)| \le c(1 + |s|^{2p+2}),\ \text{for a.a. } x \in \Omega,\ \forall s \in \mathbb{R},\ p \in \mathbb{N} \cup \{0\}, \tag{2.6}$$

$$\exists \alpha,\ \beta \in \mathbb{R},\ \alpha < \beta,\ \text{and } \lambda > 0 \text{ such that } g(x,s) = \lambda s + \tilde{g}(x,s),\ \text{with} \tag{2.7}$$

$$\lambda\alpha \le -\tilde{g}(x,s) \le \lambda\beta,\ \text{for a.a. } x \in \Omega,\ \forall s \in [\alpha, \beta].$$

Remark 2.1. This significantly improves the assumptions made in [34], Remark 4.7, namely, $g(x,s) = g(s)$ is of class $\mathcal{C}^2$, with g'' bounded, and

$$g(s) = \lambda s + \tilde{g}(s),\ \lambda > 0,\ \frac{\kappa}{\lambda} \le 1,$$

where $\kappa = \max_{s \in [-1,1]} |\tilde{g}(s)|$ (here, $\alpha = -1$, $\beta = 1$). These assumptions, as well as the proof given in [34], only allow to handle (some) at most quadratic source terms $g(s)$.

Next, as far as the nonlinear term f is concerned, we take

$$f(s) = c_0(\frac{\alpha+\beta}{2} - s) + c_1 \ln(\frac{s-\alpha}{\beta - s}),\ s \in (\alpha, \beta),\ c_1 < \frac{1}{4}c_0(\beta - \alpha),$$

where α and β are the constants in (2.7). Note that the condition $c_1 < \frac{1}{4}c_0(\beta - \alpha)$ is made to ensure that the potential

$$F(s) = \int_{\frac{\alpha+\beta}{2}}^{s} f(\xi)\, d\xi$$

$$= -\frac{c_0}{2}(\frac{\alpha+\beta}{2} - s)^2 + c_1((s-\alpha)\ln(\frac{2(s-\alpha)}{\beta-\alpha}) + (\beta - s)\ln(\frac{2(\beta-s)}{\beta-\alpha}))$$

has a double-well structure and that phase separation processes can occur. These functions satisfy

$$f' \ge -c_0,\ F \ge -c_2,\ c_2 \ge 0, \tag{2.8}$$

$$f(s)(s-m) \ge c_3(m)(|f(s)| + F(s)) - c_4(m), \tag{2.9}$$

$$s,\ m \in (\alpha, \beta),\ c_3(m) > 0,\ c_4(m) \ge 0,$$

where c_3 and c_4 depend continuously on m. This can be proved as in [35], after a proper rescaling.

Our aim in this paper is to prove the following.

Theorem 2.2. *We assume that* $u_0 \in H^1(\Omega)$*,* $\alpha < u_0(x) < \beta$ *a.e. and* $\langle u_0 \rangle \in (\alpha, \beta)$*. Then, there exists at least one weak solution* u *to* (2.1)-(2.3) *such that,* $\forall T > 0$*,*

$$u \in L^\infty(0, T; H^1(\Omega)) \cap \mathcal{C}([0, T]; H^{-1}(\Omega)) \cap L^2(0, T; H^2(\Omega)),$$

$$\frac{\partial u}{\partial t} \in L^2(0, T; H^{-1}(\Omega)),$$

$$\alpha < u(x, t) < \beta \text{ a.e.}$$

Remark 2.3. Uniqueness and further regularity are open problems for a general source term g (see Remark 3.5 below).

2.1. Notation

We set, for $v \in L^1(\Omega)$,

$$\langle v \rangle = \frac{1}{\mathrm{Vol}(\Omega)} \int_\Omega v(x)\, dx$$

and, for $v \in H^{-1}(\Omega) = H^1(\Omega)'$,

$$\langle v \rangle = \frac{1}{\mathrm{Vol}(\Omega)} \langle v, 1 \rangle_{H^{-1}(\Omega), H^1(\Omega)},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product. Furthermore, we set, whenever it makes sense,

$$\overline{v} = v - \langle v \rangle.$$

We denote by $((\cdot, \cdot))$ the usual L^2-scalar product, with associated norm $\| \cdot \|$. We also set $\| \cdot \|_{-1} = \|(-\Delta)^{-\frac{1}{2}} \cdot \|$, where $(-\Delta)^{-1}$ denotes the inverse of the minus Laplace operator associated with Neumann boundary conditions and acting on functions with null spatial average; $\| \cdot \|_{-1}$ is a norm on $\{v \in H^{-1}(\Omega),\ \langle v \rangle = 0\}$ which is equivalent to the usual H^{-1}-norm. More generally, we denote by $\| \cdot \|_X$ the norm on the Banach space X.

We note that

$$v \mapsto (\|\overline{v}\|_{-1}^2 + \langle v \rangle^2)^{\frac{1}{2}},\ v \mapsto (\|\overline{v}\|^2 + \langle v \rangle^2)^{\frac{1}{2}},$$

$$v \mapsto (\|\nabla v\|^2 + \langle v \rangle^2)^{\frac{1}{2}} \text{ and } v \mapsto (\|\Delta v\|^2 + \langle v \rangle^2)^{\frac{1}{2}}$$

are norms on $H^{-1}(\Omega)$, $L^2(\Omega)$, $H^1(\Omega)$ and $H^2(\Omega)$, respectively, which are equivalent to the usual norms on these spaces.

Throughout this paper, the same letters c and c' denote (nonnegative or positive) constants which may vary from line to line, or even in a same line.

3. Proof of Theorem 2.2

3.1. First step: approximated problems

We set

$$F(s) = -\frac{c_0}{2}(\frac{\alpha+\beta}{2} - s)^2 + F_1(s).$$

Following [19], we introduce the functions $F_{1,N}$ of class $\mathcal{C}^{4p+4}$ defined on the whole real line by, for $N \in \mathbb{N}$:

$$F_{1,N}^{(4p+4)}(s) = \begin{cases} F_1^{(4p+4)}(\beta - \frac{1}{N}), \ s > \beta - \frac{1}{N}, \\ F_1^{(4p+4)}(s), \ s \in [\alpha + \frac{1}{N}, \beta - \frac{1}{N}], \\ F_1^{(4p+4)}(\alpha + \frac{1}{N}), \ s < \alpha + \frac{1}{N}, \end{cases}$$

$$F_{1,N}^{(k)}(\frac{\alpha+\beta}{2}) = F_1^{(k)}(\frac{\alpha+\beta}{2}), \ k = 0, \ \cdots, \ 4p+3,$$

yielding

$$F_{1,N}(s) = \begin{cases} \sum_{k=0}^{4p+4} \frac{1}{k!} F_1^{(k)}(\beta - \frac{1}{N})(s - \beta + \frac{1}{N})^k, \ s > \beta - \frac{1}{N}, \\ F_1(s), \ s \in [\alpha + \frac{1}{N}, \beta - \frac{1}{N}], \\ \sum_{k=0}^{4p+4} \frac{1}{k!} F_1^{(k)}(\alpha + \frac{1}{N})(s - \alpha - \frac{1}{N})^k, \ s < \alpha + \frac{1}{N}. \end{cases}$$

Here, for $k \in \mathbb{N} \cup \{0\}$, $F^{(k)}$ denotes the kth order derivative of F, being understood that $F^{(0)} = F$ (also recall that p is defined in (2.6)). We then set $F_N(s) = -\frac{c_0}{2}(\frac{\alpha+\beta}{2} - s)^2 + F_{1,N}(s)$, $f_{1,N} = F'_{1,N}$ and $f_N = F'_N$.

The following hold, for N large enough (see [19]):

$$f'_N \geq -c_0, \tag{3.1}$$

$$F_N(s) \geq c_5 s^{4p+4} - c_6, \ s \in \mathbb{R}, \ c_5 > 0, \ c_6 \geq 0, \tag{3.2}$$

where the constants c_5 and c_6 are independent of N.

Furthermore, we have the following.

Proposition 3.1. *The following holds, for N large enough:*

$$f_N(s)(s-m) \geq c_7(m)(|f_N(s)| + F_N(s)) - c_8(m), \tag{3.3}$$

$$s \in \mathbb{R}, \ m \in (\alpha, \beta), \ c_7(m) > 0, \ c_8(m) \geq 0,$$

where the constants c_7 and c_8 depend continuously on m and are independent of N.

Proof. First, note that, for symmetry reasons, it suffices to take $s \geq \frac{\alpha+\beta}{2}$. Furthermore, for $s \in [\frac{\alpha+\beta}{2}, \beta - \frac{1}{N}]$, then (3.3) follows from (2.9). Here, N is taken large enough so that $\beta - \frac{1}{N} \geq \frac{\alpha+\beta}{2}$.

We assume from now on that $s \geq \beta - \frac{1}{N}$ and fix $m \in (\alpha, \beta)$.

Note that

$$f_N(s) = c_0(\frac{\alpha+\beta}{2} - s) + f_{1,N}(s)$$
$$= c_0(\frac{\alpha+\beta}{2} - s) + \sum_{k=0}^{4p+3} \frac{1}{k!} f_1^{(k)}(\beta - \frac{1}{N})(s - \beta + \frac{1}{N})^k,$$

where $f_1 = F_1'$. Furthermore, we can take N large enough so that $f_1^{(k)}(\beta - \frac{1}{N}) \geq 0$, $k = 0, \cdots, 4p+3$, since $\lim_{s\to\beta^-} f_1^{(k)}(s) = +\infty$, $k = 0, \cdots, 4p+3$. This yields that

$$f_N(s) \geq c_0(\frac{\alpha+\beta}{2} - s) + f_1(\beta - \frac{1}{N}) + f_1'(\beta - \frac{1}{N})(s - \beta + \frac{1}{N})$$
$$= (f_1'(\beta - \frac{1}{N}) - c_0)(s - \beta + \frac{1}{N}) + f_1(\beta - \frac{1}{N}) + c_0(\frac{\alpha+\beta}{2} - \beta + \frac{1}{N})$$
$$\geq (f_1'(\beta - \frac{1}{N}) - c_0)(s - \beta + \frac{1}{N}) + f_1(\beta - \frac{1}{N}) + c_0\frac{\alpha-\beta}{2}.$$

Choosing N large enough so that $f_1'(\beta - \frac{1}{N}) \geq c_0$ and $f_1(\beta - \frac{1}{N}) \geq c_0\frac{\beta-\alpha}{2}$, we see that $f_N \geq 0$. We can also take N large enough so that $s - m \geq 0$.

Next, note that, since $m < \beta$ and $s \geq \beta - \frac{1}{N}$, we can choose, for N large enough, a constant c_m such that $s - m - c_m \geq 0$. Indeed, we can see that $s - m - c_m \geq \beta - m - c_m - \frac{1}{N}$.

Writing

$$f_N(s)(s-m) - c_m f_N(s) = f_N(s)(s - m - c_m) \geq 0,$$

it follows that

$$f_N(s)(s-m) \geq c_m f_N(s) = c_m|f_N(s)|.$$

Finally, we note that, similarly,

$$(f_N(s)(s-m) - F_N(s))' = f_N'(s)(s-m) \geq 0,$$

for N large enough, which yields

$$f_N(s)(s-m) \geq F_N(s) + f(\beta - \frac{1}{N})(\beta - \frac{1}{N} - m) - F(\beta - \frac{1}{N}) \geq F_N(s) - F(\beta - \frac{1}{N}),$$

and we conclude, noting that F is bounded on (α, β). □

We finally have the following coercivity property on f_N.

Proposition 3.2. *The following holds, for N large enough:*

$$(f_N(s) - f_N(m))(s-m) \geq c_9(s-m)^{4p+4} - c_{10}, \tag{3.4}$$

$$s,\ m \in \mathbb{R},\ c_9 > 0,\ c_{10} \geq 0,$$

where the constants c_9 and c_{10} are independent of s, m and N.

Proof. Set

$$\varphi_N(s) = (f_N(s) - f_N(m))(s-m),$$

so that

$$\varphi_N(s) = (s-m)\int_m^s f_N'(\xi)\, d\xi. \tag{3.5}$$

Proceeding exactly as in the proof of (3.2) in [19], we can see that, taking N large enough,

$$f_N'(s) \geq cs^{4p+2} - c',\ s \in \mathbb{R},\ c > 0,\ c' \geq 0, \tag{3.6}$$

where the constants c and c' are independent of N.

Let us assume that $s \geq m$ (the case $s \leq m$ can be treated in a similar way). Then, it follows from (3.5)-(3.6) that

$$\int_m^s f_N'(\xi)\, d\xi \geq \frac{c}{4p+3}(s^{4p+3} - m^{4p+3}) - c'(s-m)$$

and

$$\varphi_N(s) \geq \frac{c}{4p+3}(s^{4p+3} - m^{4p+3})(s-m) - c'(s-m)^2.$$

Noting finally that (see, e.g., [11], Remark 2.11)

$$(s^{4p+3} - m^{4p+3})(s-m) \geq c(s-m)^{4p+4},\ s,\ m \in \mathbb{R},\ c > 0,$$

we deduce that

$$\varphi_N(s) \geq c(s-m)^{4p+4} - c'(s-m)^2$$

and (3.4) follows, employing Young's inequality. □

Having this, we introduce the following approximated problems, for $N \in \mathbb{N}$:

$$\frac{\partial u_N}{\partial t} + \Delta^2 u_N - \Delta f_N(u_N) + g(x, u_N) = 0, \tag{3.7}$$

$$\frac{\partial u_N}{\partial \nu} = \frac{\partial \Delta u_N}{\partial \nu} = 0 \text{ on } \Gamma, \tag{3.8}$$

$$u_N|_{t=0} = u_0. \tag{3.9}$$

Actually, we consider the following equivalent weaker formulation:

$$(-\Delta)^{-1}\frac{\partial \overline{u}_N}{\partial t} - \Delta \overline{u}_N + \overline{f_N(u_N)} + (-\Delta)^{-1}\overline{g(x, u_N)} = 0, \tag{3.10}$$

$$\frac{d\langle u_N\rangle}{dt} + \langle g(x, u_N)\rangle = 0, \tag{3.11}$$

$$\frac{\partial \overline{u}_N}{\partial \nu} = 0 \text{ on } \Gamma, \tag{3.12}$$

$$\overline{u}_N|_{t=0} = \overline{u}_0, \ \langle u_N\rangle|_{t=0} = \langle u_0\rangle, \tag{3.13}$$

recalling that $u_N = \overline{u}_N + \langle u_N\rangle$.

The existence of a local in time solution to (3.10)-(3.13) is based on a standard Galerkin scheme and the a priori estimates below.

More precisely, one can associate with (3.10)-(3.13) the following variational formulation, for $T > 0$ given:

Find $(\overline{u}_N, \langle u_N\rangle) : [0, T] \to V \times \mathbb{R}$, $V = \{v \in H^1(\Omega),\ \langle v\rangle = 0\}$, such that

$$\frac{d}{dt}(((-\Delta)^{-1}\overline{u}_N, v)) + ((\nabla \overline{u}_N, \nabla v)) + ((f_N(u_N), v)) + (((-\Delta)^{-1}\overline{g(x, u_N)}, v)) \tag{3.14}$$
$$= 0 \text{ in } \mathcal{D}'(0, T),\ \forall v \in V,$$

$$\frac{d\langle u_N\rangle}{dt} + \langle g(x, u_N)\rangle = 0 \text{ in } \mathcal{D}'(0, T), \tag{3.15}$$

$$\overline{u}_N|_{t=0} = \overline{u}_0 \in V,\ \langle u_N\rangle|_{t=0} = \langle u_0\rangle \in \mathbb{R}, \tag{3.16}$$

where $u_N = \overline{u}_N + \langle u_N\rangle$ and $\mathcal{D}'$ denotes the space of distributions. Let then $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvectors of the minus Laplace operator associated with Neumann boundary conditions and acting on functions with vanishing spatial average and $v_1,\ v_2,\ \cdots$ be associated eigenvectors such that the v_j's form an orthonormal in $\{v \in L^2(\Omega),\ \langle v\rangle = 0\}$ and orthogonal in V basis (see, e.g., [35] for details). We set

$$V_m = \mathrm{Span}(v_1, \cdots, v_m),\ m \in \mathbb{N}.$$

Noting that $H^1(\Omega) = \mathbb{R} \oplus V$, we also set $\lambda_0 = 0$ and $v_0 = \frac{1}{\mathrm{Vol}(\Omega)^{\frac{1}{2}}}$ to obtain an orthonormal in $L^2(\Omega)$ and orthogonal in $H^1(\Omega)$ basis. Having this, we introduce the following approximated problems, for $m \in \mathbb{N}$:

Find $(\overline{u}_{N,m}, \langle u_{N,m}\rangle) : [0, T] \to V_m \times \mathbb{R}$, $\overline{u}_{N,m}(x, t) = \sum_{j=1}^m d_{N,j,m}(t) v_j(x)$, $\langle u_{N,m}\rangle = \frac{1}{\mathrm{Vol}(\Omega)^{\frac{1}{2}}} d_{N,0,m}$, such that

$$\frac{d}{dt}(((-\Delta)^{-1}\overline{u}_{N,m}, v)) + ((\nabla\overline{u}_{N,m}, \nabla v)) + ((f_N(u_{N,m}), v)) \tag{3.17}$$
$$+ (((-\Delta)^{-1}\overline{g(x, u_{N,m})}, v)) = 0 \text{ in } \mathcal{D}'(0, T), \ \forall v \in V_m,$$

$$\frac{d\langle u_{N,m}\rangle}{dt} + \langle g(x, u_{N,m})\rangle = 0 \text{ in } \mathcal{D}'(0, T), \tag{3.18}$$

$$\overline{u}_{N,m}|_{t=0} = P_m(\overline{u}_0), \ \langle u_{N,m}\rangle|_{t=0} = \langle u_0\rangle, \tag{3.19}$$

where $u_{N,m} = \sum_{j=0}^{m} d_{N,j,m} v_j$ and P_m denotes the orthogonal projector onto V_m, with respect to the $L^2(\Omega)$-scalar product.

Proving the local in time existence of a solution to (3.17)-(3.19) easily follows from the Cauchy–Caratheodory theorem. However, what is important is to prove that this local in time solution is defined on a time interval which is independent of N (and m), which necessitates uniform a priori estimates. We will derive these estimates based on the original problem (3.10)-(3.13). These can be justified within the approximated problems (3.17)-(3.19), passing to the weak lower limit $m \to +\infty$ at the end of the procedure.

3.2. *Second step: uniform a priori estimates*

In what follows, all constants are independent of the approximating parameter N.

First a priori estimate:

Let us multiply (3.10) by $-\Delta\overline{u}_N$ to obtain, integrating over Ω and by parts,

$$\frac{1}{2}\frac{d}{dt}\|\overline{u}_N\|^2 + \|\Delta\overline{u}_N\|^2 - ((f_N(u_N), \Delta u_N)) + ((g(x, u_N), \overline{u}_N)) = 0. \tag{3.20}$$

Remark 3.3. This estimate is formal. Actually, we should work at the approximated level and write it for $u_{N,m}$, by taking $v = v_j$ in (3.17), multiplying the resulting equation by $\lambda_j d_{N,j,m}$ and summing over $j = 1, \cdots, m$. We can note that all constants below are also independent of m at the approximated level. All the other estimates can be justified in a similar way (see also Remark 3.4 below). Finally, as already mentioned, we would pass to the weak lower limit $m \to +\infty$ at the end of the procedure, in particular, after employing Gronwall's lemma, to obtain the desired regularity and uniform estimates on u_N. That said, we continue to work formally, for simplicity.

Note that, owing to (3.1) and a proper interpolation inequality and taking N large enough,

$$-((f_N(u_N), \Delta u_N)) = ((f_N'(u_N)\nabla\overline{u}_N, \nabla\overline{u}_N)) \geq -c_0\|\nabla\overline{u}_N\|^2$$
$$\geq -c\|\overline{u}_N\|\|\Delta\overline{u}_N\| \geq -\frac{1}{2}\|\Delta\overline{u}_N\|^2 - c\|\overline{u}_N\|^2, \tag{3.21}$$

owing also to standard elliptic regularity results. Furthermore, it follows from (2.6) and Young's inequality that

$$\|g(x, u_N)\|^2 \leq c(\|u_N\|^{4p+4}_{L^{4p+4}(\Omega)} + 1), \tag{3.22}$$

which yields

$$|((g(x,u_N),\overline{u}_N))| \le c_\varepsilon \|\overline{u}_N\|^2 + \varepsilon(\|u_N\|^{4p+4}_{L^{4p+4}(\Omega)} + 1), \ \forall \varepsilon > 0. \tag{3.23}$$

It follows from (3.20)-(3.23) that

$$\frac{d}{dt}\|\overline{u}_N\|^2 + \|\Delta\overline{u}_N\|^2 \le c_\varepsilon(\|\overline{u}_N\|^2 + \langle u_N\rangle^2) + \varepsilon(\|u_N\|^{4p+4}_{L^{4p+4}(\Omega)} + 1), \ \forall \varepsilon > 0. \tag{3.24}$$

Next, we multiply (3.11) by $\langle u_N\rangle$ to find

$$\frac{1}{2}\frac{d}{dt}\langle u_N\rangle^2 \le |\langle g(x,u_N)\rangle||\langle u_N\rangle| \le c\|g(x,u_N)\|_{L^1(\Omega)}|\langle u_N\rangle|,$$

which yields, proceeding as above and employing Young's inequality,

$$\frac{d}{dt}\langle u_N\rangle^2 \le c_\varepsilon(\|\overline{u}_N\|^2 + \langle u_N\rangle^2) + \varepsilon(\|u_N\|^{4p+4}_{L^{4p+4}(\Omega)} + 1), \ \forall \varepsilon > 0. \tag{3.25}$$

Summing (3.24) and (3.25), we have

$$\begin{aligned}&\frac{d}{dt}(\|\overline{u}_N\| + \langle u_N\rangle^2) + \|\Delta\overline{u}_N\|^2 \\ &\quad \le c_\varepsilon(\|\overline{u}_N\|^2 + \langle u_N\rangle^2) + \varepsilon(\|u_N\|^{4p+4}_{L^{4p+4}(\Omega)} + 1), \ \forall \varepsilon > 0.\end{aligned} \tag{3.26}$$

We now multiply (3.10) by $\overline{u}_N$ and obtain

$$\frac{1}{2}\frac{d}{dt}\|\overline{u}_N\|^2_{-1} + \|\nabla\overline{u}_N\|^2 + ((f_N(u_N),\overline{u}_N)) + (((-\Delta)^{-1}\overline{g(x,u_N)},\overline{u}_N)) = 0. \tag{3.27}$$

Note that, owing to (3.4) (with $s = u_N$ and $m = \langle u_N\rangle$) and taking N large enough,

$$\begin{aligned}((f_N(u_N),\overline{u}_N)) &= ((f_N(u_N) - f_N(\langle u_N\rangle),\overline{u}_N)) \\ &\ge c\|\overline{u}_N\|^{4p+4}_{L^{4p+4}(\Omega)} - c' \ge c\|u_N\|^{4p+4}_{L^{4p+4}(\Omega)} - c'(\langle u_N\rangle^{4p+4} + 1),\end{aligned} \tag{3.28}$$

where we have also employed Young's inequality. Furthermore,

$$\begin{aligned}|(((-\Delta)^{-1}\overline{g(x,u_N)},\overline{u}_N))| &\le c\|g(x,u_N)\|\|\overline{u}_N\| \\ &\le c_\varepsilon(\|\overline{u}_N\|^2 + \langle u_N\rangle^2) + \varepsilon(\|u_N\|^{4p+4}_{L^{4p+4}(\Omega)} + 1), \ \forall \varepsilon > 0.\end{aligned} \tag{3.29}$$

It thus follows from (3.27)-(3.29) that

$$\begin{aligned}&\frac{d}{dt}\|\overline{u}_N\|^2_{-1} + c(\|\nabla\overline{u}_N\|^2 + \|u_N\|^{4p+4}_{L^{4p+4}(\Omega)}) \\ &\quad \le c_\varepsilon(\|\overline{u}_N\|^2 + \langle u_N\rangle^2)^{2p+2} + \varepsilon(\|u_N\|^{4p+4}_{L^{4p+4}(\Omega)} + 1), \ c > 0, \ \forall \varepsilon > 0.\end{aligned} \tag{3.30}$$

Summing (3.26) and (3.30), we find, choosing ε small enough,

$$\frac{d}{dt}(\|\overline{u}_N\|_{-1}^2+\|\overline{u}_N\|^2+\langle u_N\rangle^2) \\ +c(\|u_N\|_{H^2(\Omega)}^2+\|u_N\|_{L^{4p+4}(\Omega)}^{4p+4}) \le c'((\|\overline{u}_N\|^2+\langle u_N\rangle^2)^{2p+2}+1),\ c>0. \tag{3.31}$$

In particular, it follows from (3.31) that we have a differential inequality of the form

$$y' \le c(y^{2p+2}+1),\ y=\|\overline{u}_N\|_{-1}^2+\|\overline{u}_N\|^2+\langle u_N\rangle^2.$$

Considering the ODE

$$z'=c(z^{2p+2}+1),\ z(0)=y(0),$$

we deduce from the comparison principle that there exists $T_0=T_0(\|u_0\|)>0$ such that

$$\|u_N\| \le c,\ t\in[0,T_0].$$

This, together with (3.31), yields uniform (with respect to N) estimates on u_N in $L^\infty(0,T_0;L^2(\Omega))$, $L^2(0,T_0;H^2(\Omega))$ and $L^{4p+4}(0,T_0;L^{4p+4}(\Omega))$.

Second a priori estimate:

We assume that $t\in[0,T_0]$.

It follows from (3.11), (3.22) and the continuous embedding $L^2(\Omega)\subset L^1(\Omega)$ that

$$\left|\frac{d\langle u_N\rangle}{dt}\right| \le c\|g(x,u_N)\| \le c(\|u_N\|_{L^{4p+4}(\Omega)}^{4p+4}+1)^{\frac{1}{2}},$$

so that

$$\langle u_0\rangle - c\int_0^t(\|u_N\|_{L^{4p+4}(\Omega)}^{4p+4}+1)^{\frac{1}{2}}\,ds \le \langle u_N(t)\rangle \\ \le \langle u_0\rangle + c\int_0^t(\|u_N\|_{L^{4p+4}(\Omega)}^{4p+4}+1)^{\frac{1}{2}}\,ds. \tag{3.32}$$

Note that, employing Cauchy–Schwarz's inequality,

$$\int_0^t(\|u_N\|_{L^{4p+4}(\Omega)}^{4p+4}+1)^{\frac{1}{2}}\,ds \le \sqrt{t}\left(\int_0^t(\|u_N\|_{L^{4p+4}(\Omega)}^{4p+4}+1)\,ds\right)^{\frac{1}{2}},$$

so that, owing to (3.31),

$$\int_0^t (\|u_N\|_{L^{4p+4}(\Omega)}^{4p+4} + 1)^{\frac{1}{2}}\, ds \le ct + c'\sqrt{t}\|u_0\|. \tag{3.33}$$

It thus follows from (3.32)-(3.33) that

$$\langle u_0\rangle - ct - c'\sqrt{t}\|u_0\| \le \langle u_N(t)\rangle \le \langle u_0\rangle + ct + c'\sqrt{t}\|u_0\|. \tag{3.34}$$

Let us now fix $\delta > 0$ such that

$$\alpha + 2\delta \le \langle u_0\rangle \le \beta - 2\delta. \tag{3.35}$$

It follows from (3.34)-(3.35) that there exists $T_1 = T_1(\alpha, \beta, \delta) > 0$, $T_1 \le T_0$, such that

$$\alpha + \delta \le \langle u_N(t)\rangle \le \beta - \delta,\ t \in [0, T_1]. \tag{3.36}$$

We assume from now on that $t \in [0, T_1]$.

Third a priori estimate:

We again multiply (3.10) by $\overline{u}_N$ and now note that, owing to (3.3) (with $s = u_N$ and $m = \langle u_N\rangle$) and (3.36) and taking N large enough,

$$((f_N(u_N), \overline{u}_N)) \ge c(\|f_N(u_N)\|_{L^1(\Omega)} + \int_\Omega F_N(u_N)\, dx) - c', \tag{3.37}$$

where the constants c and c' depend on δ. Then, note that it follows from (2.6), (3.2) and (3.22) that, taking N large enough,

$$\|g(x, u_N)\|^2 \le c(\int_\Omega F_N(u_N)\, dx + c'), \tag{3.38}$$

where c' is such that

$$\int_\Omega F_N(u_N)\, dx + c' \ge 0.$$

Thus, proceeding as in the First a priori estimate, we obtain the differential inequality

$$\frac{d}{dt}(\|\overline{u}_N\|_{-1}^2 + \|\overline{u}_N\|^2 + \langle u_N\rangle^2) \tag{3.39}$$
$$+ c(\|u_N\|_{H^2(\Omega)}^2 + \|f_N(u_N)\|_{L^1(\Omega)} + \int_\Omega F_N(u_N)\, dx) \le c'(\|\overline{u}_N\|^2 + \langle u_N\rangle^2 + 1),\ c > 0.$$

In particular, it follows from (3.39) that we have a uniform estimate on $f_N(u_N)$ in $L^1(0, T_1; L^1(\Omega))$.

Fourth a priori estimate:

Let us multiply (3.10) by $\frac{\partial \overline{u}_N}{\partial t}$ to find

$$\frac{1}{2}\frac{d}{dt}(\|\nabla \overline{u}_N\|^2 + 2\int_\Omega F_N(u_N)\,dx) + \|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}^2 \tag{3.40}$$
$$-((f_N(u_N), \langle \frac{\partial u_N}{\partial t}\rangle)) + (((-\Delta)^{-1}\overline{g(x,u_N)}, \frac{\partial \overline{u}_N}{\partial t})) = 0.$$

Note that, owing to (3.38) and taking N large enough,

$$|((f_N(u_N), \langle \frac{\partial u_N}{\partial t}\rangle))| = |((f_N(u_N), \langle g(x,u_N)\rangle))| \tag{3.41}$$
$$\leq c\|f_N(u_N)\|_{L^1(\Omega)}\|g(x,u_N)\|_{L^1(\Omega)} \leq c\|f_N(u_N)\|_{L^1(\Omega)}\|g(x,u_N)\|$$
$$\leq c\|f_N(u_N)\|_{L^1(\Omega)}(\int_\Omega F_N(u_N)\,dx + c').$$

Furthermore,

$$|(((-\Delta)^{-1}\overline{g(x,u_N)}, \frac{\partial \overline{u}_N}{\partial t}))| = |((g(x,u_N), (-\Delta)^{-1}\frac{\partial \overline{u}_N}{\partial t}))| \tag{3.42}$$
$$\leq c\|g(x,u_N)\|\|\frac{\partial \overline{u}_N}{\partial t}\|_{-1} \leq c(\int_\Omega F_N(u_N)\,dx + c') + \frac{1}{2}\|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}^2.$$

It thus follows from (3.40)-(3.42) that

$$\frac{d}{dt}(\|\nabla \overline{u}_N\|^2 + 2\int_\Omega F_N(u_N)\,dx) + \|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}^2 \tag{3.43}$$
$$\leq c(\|f_N(u_N)\|_{L^1(\Omega)} + 1)(\|\nabla \overline{u}_N\|^2 + 2\int_\Omega F_N(u_N)\,dx + c').$$

Recall that it follows from the previous a priori estimates that $\|f_N(u_N)\|_{L^1(\Omega)}$ belongs to $L^1(0,T_1)$. We thus deduce, employing Gronwall's lemma and in view of the properties of F_N, that u_N is uniformly bounded in $L^\infty(0,T_1;H^1(\Omega))$ and $L^\infty(0,T_1;L^{4p+4}(\Omega))$ and that $\frac{\partial u_N}{\partial t}$ is uniformly bounded in $L^2(0,T_1;H^{-1}(\Omega))$ (recall also that $|\langle \frac{\partial u_N}{\partial t}\rangle| \leq c\|g(x,u_N)\|$ and can be estimated).

Fifth a priori estimate:

It follows from (3.10) and (3.38) and taking N large enough that

$$\|\overline{f_N(u_N)}\|^2 \leq c(\|u_N\|_{H^2(\Omega)}^2 + \|\frac{\partial \overline{u}_N}{\partial t}\|_{-1}^2 + \|g(x,u_N)\|^2)$$

and

$$\|\overline{f_N(u_N)}\|^2 \leq c(\|u_N\|^2_{H^2(\Omega)} + \|\frac{\partial \overline{u}_N}{\partial t}\|^2_{-1} + \int_\Omega F_N(u_N)\,dx + c'). \tag{3.44}$$

Remark 3.4. Note that, owing to the third and fourth a priori estimates, we have enough regularity to pass to the limit in the Galerkin approximations (3.17)-(3.19), employing standard Aubin–Lions compactness results. Therefore, (3.10) holds a.e., so that the above estimates can be derived directly from (3.10) and we do not need to work on the Galerkin approximations.

Next, employing once more (3.3), with $s = u_N$ and $m = \langle u_N \rangle$, we see that, for N large enough,

$$\begin{aligned} |\langle f_N(u_N)\rangle| &\leq c|((f_N(u_N), \overline{u}_N))| + c' = c|((\overline{f_N(u_N)}, u_N))| + c' \\ &\leq c\|\overline{f_N(u_N)}\|\|u_N\| + c', \end{aligned}$$

owing to (3.36), so that

$$|\langle f_N(u_N)\rangle| \leq c(\|\overline{f_N(u_N)}\| + 1). \tag{3.45}$$

We finally deduce from (3.44)-(3.45) and the previous a priori estimates that $f_N(u_N)$ is uniformly bounded in $L^2(0, T_1; L^2(\Omega))$.

3.3. Third step: passage to the limit and local in time existence

It follows from the a priori estimates and standard Aubin–Lions compactness results that there exists a function u such that, at least for a subsequence that we do not relabel,

$$u_N \to u \text{ in } L^\infty(0, T_1; H^1(\Omega)) \text{ weak star and in } L^2(0, T_1; H^2(\Omega)) \text{ weakly,}$$

$$u_N \to u \text{ in } L^2(0, T_1; L^2(\Omega)) \text{ and a.e.,}$$

$$\frac{\partial u_N}{\partial t} \to \frac{\partial u}{\partial t} \text{ in } L^2(0, T_1; H^{-1}(\Omega)) \text{ weakly,}$$

as $N \to +\infty$.

The passage to the limit in the linear terms, having in mind the variational formulation (3.14)-(3.16), is straightforward.

Next, since $f_N(u_N)$ is uniformly bounded in $L^1(0, T_1; L^1(\Omega))$, we can prove in a standard way (i.e., exactly as in the case of the usual Cahn–Hilliard equation; see [35] for details) that $\alpha < u(x,t) < \beta$ a.e. This, in turn, allows to prove that $f_N(u_N)$ converges to $f(u)$ a.e., owing to the explicit expression of f_N. Finally, since $f_N(u_N)$ is uniformly bounded in $L^2(0, T_1, L^2(\Omega))$, it follows that $f_N(u_N)$ converges to $f(u)$ in $L^2(0, T_1; L^2(\Omega))$ weakly, which is sufficient to pass to the limit in the nonlinear term $f_N(u_N)$.

We can pass to the limit in the nonlinear source term $g(x, u_N)$ in a similar way, owing to (2.5) and noting that it follows from (3.22) and the above a priori estimates that $g(x, u_N)$ is uniformly bounded in $L^\infty(0, T_1; L^2(\Omega)) \subset L^2(0, T_1; L^2(\Omega))$.

We thus conclude on the existence of a local in time weak solution to the problem.

3.4. Fourth step: global in time existence

Let $T^\star$ be the maximal existence time of a local in time solution and assume that $T^\star < +\infty$. Then, one has, immediately,

$$\|u(t)\| \le \| \max(|\alpha|, |\beta|)\| = \mathrm{Vol}(\Omega)^{\frac{1}{2}} \max(|\alpha|, |\beta|),\ t \in [0, T^\star), \tag{3.46}$$

since $u \in (\alpha, \beta)$ a.e., meaning that the upper bound on $\|u\|$ is independent of $T^\star$.

However, in order to extend the solution, we need to make sure that

$$\langle u(t)\rangle \in [\alpha + \delta, \beta - \delta],\ t \in [0, T^\star),$$

for some $\delta \in (\alpha, \beta)$ (having this, we can repeat all estimates above, for the limit solution, and see that they hold for $t \in [0, T^\star)$).

To do so, note that

$$\frac{d\langle u\rangle}{dt} + \langle g(x, u)\rangle = 0,$$

which yields

$$\frac{d\langle u\rangle}{dt} + \lambda\langle u\rangle = -\langle \tilde{g}(x, u)\rangle,$$

so that, employing Gronwall's lemma,

$$\langle u(t)\rangle = \langle u_0\rangle e^{-\lambda t} - e^{-\lambda t} \int_0^t e^{\lambda s} \langle \tilde{g}(x, u)\rangle\, ds.$$

We thus deduce from (2.7) that

$$\langle u_0\rangle e^{-\lambda t} + \alpha(1 - e^{-\lambda t}) \le \langle u(t)\rangle \le \langle u_0\rangle e^{-\lambda t} + \beta(1 - e^{-\lambda t}),\ t \in [0, T^\star). \tag{3.47}$$

It immediately follows from (3.47) that

$$\langle u(t)\rangle \in [\alpha + \delta, \beta - \delta],\ t \in [0, T^\star),\ \delta > 0. \tag{3.48}$$

Indeed, setting

$$\varphi(s) = \langle u_0\rangle e^{-\lambda s} + \alpha(1 - e^{-\lambda s}),$$

it is easy to see that φ takes values in such an interval, noting that φ is monotone decreasing. We proceed in a similar way for the right-hand side.

Having this, we can extend the solution by continuity, taking $u(T^\star)$ as initial datum, leading to a contradiction.

Remark 3.5. Let us assume that $g(x,s) = g(s)$ is of class $\mathcal{C}^1$ and monotone increasing, with g' bounded and $g(s) f(s) \ge -c$, $c \ge 0$ (see [31] for a nontrivial example; see also below). Then, we can prove that

$$\frac{\partial u}{\partial t} \in L^\infty(r, T; H^{-1}(\Omega)) \cap L^2(r, T; H^1(\Omega)),$$

$\forall r < T$, $r > 0$ and $T > 0$ given. Indeed, following [22], rewrite the problem in the equivalent form

$$\frac{\partial u}{\partial t} + g(u) = \Delta \mu, \tag{3.49}$$

$$\mu = -\Delta u + f(u), \tag{3.50}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \mu}{\partial \nu} = 0 \text{ on } \Gamma. \tag{3.51}$$

The estimates below are formal, but they can be justified within a Galerkin scheme and by considering approximated problems/solutions as above. First, note that it follows from (3.50) that

$$\langle \mu \rangle = \langle f(u) \rangle,$$

so that, owing to the regularity obtained above, $\mu \in L^2(0, T; H^1(\Omega))$, since

$$\overline{\mu} = -(-\Delta)^{-1} \frac{\partial \overline{u}}{\partial t} - (-\Delta)^{-1} \overline{g(u)}. \tag{3.52}$$

Next, let us multiply (3.49) by $\frac{\partial \mu}{\partial t}$ to have

$$((\frac{\partial u}{\partial t}, \frac{\partial \mu}{\partial t})) = -\frac{1}{2} \frac{d}{dt} \|\nabla \mu\|^2 - ((g(u), \frac{\partial \mu}{\partial t})). \tag{3.53}$$

Let us then differentiate (3.50) with respect to time to obtain

$$\frac{\partial \mu}{\partial t} = -\Delta \frac{\partial u}{\partial t} + f'(u) \frac{\partial u}{\partial t}. \tag{3.54}$$

Multiply (3.54) by $\frac{\partial u}{\partial t}$ to find

$$((\frac{\partial u}{\partial t}, \frac{\partial \mu}{\partial t})) = \|\nabla \frac{\partial u}{\partial t}\|^2 + ((f'(u) \frac{\partial u}{\partial t}, \frac{\partial u}{\partial t})) \ge \|\nabla \frac{\partial u}{\partial t}\|^2 - c_0 \|\frac{\partial u}{\partial t}\|^2, \tag{3.55}$$

owing to (2.8). Combine (3.53) and (3.55) to have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \mu\|^2 + \|\nabla \frac{\partial u}{\partial t}\|^2 + ((g(u), \frac{\partial \mu}{\partial t})) &\le c_0 \|\frac{\partial u}{\partial t}\|^2 \\ &\le \frac{1}{2} \|\nabla \frac{\partial u}{\partial t}\|^2 + c(\|\frac{\partial \overline{u}}{\partial t}\|_{-1}^2 + \langle \frac{\partial u}{\partial t} \rangle^2), \end{aligned} \tag{3.56}$$

owing to a proper interpolation inequality. Now, note that

$$((g(u), \frac{\partial \mu}{\partial t})) = \frac{d}{dt}((g(u), \mu)) - ((g'(u)\frac{\partial u}{\partial t}, \mu)). \tag{3.57}$$

Let us then combine (3.56)-(3.57) to obtain

$$\begin{aligned}\frac{d}{dt}(\frac{1}{2}\|\nabla\mu\|^2 + ((g(u),\mu))) + \|\nabla\frac{\partial u}{\partial t}\|^2 &\le c\|\frac{\partial u}{\partial t}\|^2_{H^{-1}(\Omega)} + c'\|\frac{\partial u}{\partial t}\|\|\mu\| \\ &\le \frac{1}{2}\|\nabla\frac{\partial u}{\partial t}\|^2 + c(\|\frac{\partial u}{\partial t}\|^2_{H^{-1}(\Omega)} + \|\mu\|^2),\end{aligned}$$

recalling that g' is bounded, so that

$$\frac{d}{dt}(\frac{1}{2}\|\nabla\mu\|^2 + ((g(u),\mu))) + \frac{1}{2}\|\nabla\frac{\partial u}{\partial t}\|^2 \le c(\|\frac{\partial u}{\partial t}\|^2_{H^{-1}(\Omega)} + \|\mu\|^2). \tag{3.58}$$

Set finally

$$\Lambda = \frac{1}{2}\|\nabla\mu\|^2 + ((g(u),\mu)).$$

Note that, since $g' \ge 0$,

$$\begin{aligned}((g(u),\mu)) = ((g(u), -\Delta u + f(u))) &= ((g'(u)\nabla u, \nabla u)) + ((g(u), f(u)) \\ &\ge ((g(u), f(u))).\end{aligned}$$

Therefore,

$$\Lambda \ge \frac{1}{2}\|\nabla\mu\|^2 - c,\ c \ge 0,$$

and an application of the uniform Gronwall's lemma yields that

$$\overline{\mu} \in L^\infty(r, T_1; H^1(\Omega)),$$

for some $T_1 > 0$ (as in the Second a priori estimate above), $r \in (0, T_1)$ given. Actually, if we look carefully at the a priori estimates derived above, we can see that the final time T_1 and constants only depend on α, β and $\delta > 0$ as in (3.35). But then, it follows from the Fourth step that we can, without loss of generality, consider δ depending only on α, β and the final time T; we also assume, without loss of generality, that 2δ and δ in (3.35) and (3.48), respectively, are the same, making a further iteration if needed. Therefore, taking $u(T_1)$ as initial datum, $\langle u(T_1)\rangle$ satisfies (3.35) for the same $\delta > 0$ as the one taken for u_0 and we can extend the solution to $[r, 2T_1]$, repeating all the above estimates on $[T_1, 2T_1]$ and now employing Gronwall's lemma. Note however that $\nabla\mu$ does not a priori satisfy any continuity property (the same holds for Λ). Nevertheless, we can still, without loss of generality, keep the same interval, reducing it a bit if necessary. Indeed, we can write, e.g.,

$$\int_{\frac{3T_1}{4}}^{T_1} \|\nabla\mu\|^2\,dt \le \frac{T_1}{4}\|\nabla\mu\|^2_{L^\infty(\frac{3T_1}{4},T_1;L^2(\Omega)^n)},$$

which yields that there exists $t_\star \in (\frac{3T_1}{4}, T_1)$ such that

$$\|\nabla\mu(t_\star)\|^2 \le \|\nabla\mu\|^2_{L^\infty(\frac{3T_1}{4},T_1;L^2(\Omega)^n)} \le \|\nabla\mu\|_{L^\infty(r,T_1;L^2(\Omega)^n)}$$

(we thus assume, without loss of generality, that $t_\star = T_1$). Proceeding recursively, we extend the solution and the estimates to $[r, T]$. The result finally follows from (3.52). Having this, we can obtain further regularity results and prove stronger separation properties of u from the singular points α and β in one and two space dimensions, yielding also some uniqueness results (see [22, 31,35] for more details). Note however that the assumptions made on g here are very restrictive, although the assumption $g' \ge 0$ can be relaxed to $g' \ge -c$, $c \ge 0$, by combining (3.58) with (3.43) (written for u).

4. Applications

In the examples below, assumptions (2.4)-(2.6) are clearly satisfied and there remains to check (2.7).

4.1. The Cahn–Hilliard–Oono equation

This corresponds to the simplest situation, namely, g is linear (see [38]),

$$g(x,s) = g(s) = \gamma s,\ \gamma > 0,\ \alpha = -1,\ \beta = 1.$$

In that case, $\tilde{g} = 0$, so that we have the existence of global in time solutions. Actually, here, one can recover all results known for the original Cahn–Hilliard equation (corresponding to $\gamma = 0$), namely, uniqueness, additional regularity, (strict) separation from the pure states, etc. In particular, note that the condition $g(s)f(s) \ge -c$, $c \ge 0$, follows from (2.9). We refer the interested reader to [22,35] for more details.

Remark 4.1. We can more generally take

$$g(x,s) = g(s) = \gamma(s-c),\ c \in (-1,1),\ \gamma > 0.$$

Moreover, the case $c = \langle u_0\rangle$ corresponds to the Ohta–Kawasaki equation which appears in separation processes for block copolymers (see [37]). In that particular case, we recover the conservation of mass, $\langle u(t)\rangle = \langle u_0\rangle$, $t \ge 0$. Writing then the Ohta–Kawasaki equation in the equivalent form

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + \gamma\overline{u} = 0, \tag{4.1}$$

we obtain, multiplying (4.1) by $\frac{\partial u}{\partial t}$ (note that $\langle\frac{\partial u}{\partial t}\rangle = 0$),

$$\frac{d}{dt}(\|\nabla u\|^2+\gamma\|\overline{u}\|_{-1}^2+2\int\limits_\Omega F(u)\,dx)+\|\frac{\partial u}{\partial t}\|_{-1}^2=0. \tag{4.2}$$

This corresponds to the energy dissipation and the Ohta–Kawasaki model actually has a variational structure. More precisely, it is an $H^{-1}(\Omega)$-gradient flow for the nonlocal total free energy

$$\Psi_{\mathrm{OK}}=\int\limits_\Omega(\frac{1}{2}|\nabla u|^2+\frac{\gamma}{2}|(-\Delta)^{-\frac{1}{2}}(u-\langle u_0\rangle)|^2+F(u))\,dx \tag{4.3}$$

(by comparison, the total free energy associated with the original Cahn–Hilliard equation, known as the Ginzburg–Landau free energy, reads $\Psi_{\mathrm{GL}}=\int_\Omega(\frac{1}{2}|\nabla u|^2+F(u))\,dx$). Here, besides the absolute value, $|\cdot|$ also denotes the usual Euclidean norm, with associated scalar product $\cdot$. This energy dissipation, together with the conservation of mass, significantly simplifies the mathematical analysis of the problem, also when compared to the Cahn–Hilliard–Oono equation. In particular, this allows to directly prove the existence and uniqueness of global in time solutions, proceeding as in the case of the Cahn–Hilliard equation. We can also prove, proceeding again as in the case of the Cahn–Hilliard equation, further regularity and the strict separation from the pure states ± 1 in one and two space dimensions. We refer the interested reader to [35] for the mathematical analysis of the Cahn–Hilliard equation; the changes, for the Ohta–Kawasaki model, are minor.

Remark 4.2. The Cahn–Hilliard–Oono equation also has a variational structure. Indeed, set

$$\Psi_{\mathrm{CHO}}=\int\limits_\Omega(\frac{1}{2}|\nabla u|^2+F(u)+\int\limits_\Omega u(y)k(x,y)u(x)\,dy)\,dx, \tag{4.4}$$

where, in three space dimensions, one takes

$$k(x,y)=\frac{\gamma}{4\pi|x-y|}. \tag{4.5}$$

Writing then, as in the derivation of the classical Cahn–Hilliard equation,

$$\frac{\partial u}{\partial t}=\Delta\partial_u\Psi_{\mathrm{CHO}}, \tag{4.6}$$

where ∂_u denotes a variational derivative, we find the Cahn–Hilliard–Oono equation, noting that $-\frac{1}{4\pi|x-y|}$ is the Green function associated with the Laplace operator. Indeed, considering a small variation, we have

$$\begin{aligned}\delta\Psi_{\mathrm{CHO}}&=\int\limits_\Omega(-\nabla u\cdot\nabla\delta u+f(u)\delta u+\int\limits_\Omega k(x,y)u(y)\delta u(x)\,dy)\,dx\\&=\int\limits_\Omega(-\Delta u+f(u)+\int\limits_\Omega k(x,y)u(y)\,dy)\delta u(x)\,dx\end{aligned}$$

so that

$$\partial_u \Psi_{\mathrm{CHO}} = -\Delta u + f(u) + \int_\Omega k(x, y)u(y)\, dy.$$

Noting that the Laplacian corresponds to the x-variable, we see that

$$\Delta\partial_u \Psi_{\mathrm{CHO}} = -\Delta^2 u + \Delta f(u) + \int_\Omega \Delta k(x, y)u(y)\, dy.$$

Finally, by definition of Green's function and denoting by d_i the Dirac mass at 0 (this is of course formal, since the Dirac mass is not a function),

$$\int_\Omega \Delta(-k(x, y))u(y)\, dy = \gamma \int_\Omega d_i(x - y)u(y)\, dy = \gamma u(x),$$

which yields

$$\Delta\partial_u \Psi_{\mathrm{CHO}} = -\Delta^2 u + \Delta f(u) - \gamma u,$$

from which the Cahn–Hilliard–Oono equation follows. We can note that such a nonlocal free energy (without the term $\frac{1}{2}|\nabla u|^2$) is considered in the literature related with nonlocal Cahn–Hilliard models (see, e.g., [3,20,21] and references therein) and are delicate to study from a mathematical point of view. In particular, this variational structure would not simplify the mathematical analysis of the Cahn–Hilliard–Oono equation. Indeed, we again stress that the above considerations and computations are formal. Furthermore, the Laplace operator considered in (4.6) is not associated with Neumann boundary conditions and (4.6) does not yield the conservation of mass, as expected here.

4.2. The Cahn–Hilliard equation in binary image inpainting

We consider the following equation, proposed in [4] in view of applications to binary (i.e., black and white) image inpainting:

$$\frac{\partial u}{\partial t} + \varepsilon\Delta^2 u - \frac{1}{\varepsilon}\Delta f(u) + \lambda_0 \chi_{\Omega\setminus D}(x)(s - h(x)) = 0,\ \lambda_0,\ \varepsilon > 0.$$

Here, h is a given (damaged) image, $h(x) \in [0, 1]$ a.e., χ is the indicator function and $D \subset \Omega$ is the inpainting (i.e., damaged/missing) region. Furthermore, ε is related to the interface thickness. In this context, the additional term

$$g(x, s) = \lambda_0 \chi_{\Omega\setminus D}(x)(s - h(x))$$

is known as fidelity term and is added in order to keep the solution u close to the image outside the inpainting region. The idea in this model is to solve the equation up to steady state to obtain an inpainted (i.e., restored) version $u(x)$ of $h(x)$.

Well-posedness results, in the case of a cubic nonlinear term f, were obtained in [5,9].

Then, it was noted in [10] that logarithmic nonlinear terms f are also relevant from a numerical point of view, as they allow to have better inpainting results, as well as better convergence times.

Let us thus take $\alpha = 0$, $\beta = 1$ and

$$f(s) = -c_0(s - \frac{1}{2}) + c_1 \ln \frac{s}{1-s},\ c_0,\ c_1 > 0,\ c_1 < \frac{c_0}{4},\ s \in (0,1).$$

Then, write

$$g(x,s) = \lambda_0 s - \lambda_0 \chi_D(x) s - \lambda_0 \chi_{\Omega\setminus D}(x) h(x),$$

so that

$$\tilde{g}(x,s) = -\lambda_0 \chi_D(x) s - \lambda_0 \chi_{\Omega\setminus D}(x) h(x).$$

Noting that $\tilde{g}(x,s) = -\lambda_0 s$ when $x \in D$ and $\tilde{g}(x,s) = -\lambda_0 h(x)$ when $x \in \Omega\setminus D$, we see that $-\tilde{g}(x,s) \in [0, \lambda_0]$, for a.a. x, $\forall s \in [0,1]$. We thus deduce the existence of global in time weak solutions. Note that this improves the results in [10,35], where the condition $\int_{\Omega\setminus D} h(x)\,dx = 0$ is assumed.

Remark 4.3. Proceeding as in Remark 4.2, we can see that the inpainting model has a variational structure, considering the total free energy (for simplicity, we take $h \equiv 0$)

$$\Psi_{\mathrm{CHI}} = \int\limits_{\Omega} (\frac{1}{2}|\nabla u|^2 + F(u) + \int\limits_{\Omega\setminus D} u(y)k(x,y)u(x)\,dy)\,dx, \tag{4.7}$$

where, in three space dimensions,

$$k(x,y) = \frac{\lambda_0}{4\pi|x-y|}, \tag{4.8}$$

while, in two space dimensions,

$$k(x,y) = -\frac{\lambda_0}{2\pi} \ln|x-y|. \tag{4.9}$$

Again, having this variational structure does not simplify the mathematical analysis of the problem. Furthermore, obtaining a similar variational structure for more general, in particular, nonlinear, functions g would be more delicate.

4.3. The Cahn–Hilliard equation with a proliferation term

We take

$$g(x,s) = g(s) = s^2 + as,\ a \in \mathbb{R}.$$

In that case, one models (actually, after a proper rescaling) wound healing and the clustering of malignant brain tumor cells (for $a = -1$; see [25]) or the literal attractive interactions between adsorbed molecules which may induce a transition in the chemisorbed overlayer (see [42]).

The corresponding equation, for a cubic nonlinear term f, was studied in [11,16] (see also [35]). In particular, it was proved that one can have blow up in finite time, with an order parameter going to $-\infty$, which is not satisfactory, in view of applications.

The case of a logarithmic nonlinear term, for $a = -1$, was investigated in [34,35] and global in time existence was proved.

More generally, when $a < 0$, we can take $\alpha = 0$ and $\beta = \lambda = -a$. Writing $g(s) = -as + \tilde{g}(s)$, we have $-\tilde{g}(s) = -s^2 - 2as$ and it is easy to see that (2.7) is satisfied. Similarly, when $\alpha = 0$, we can take $\alpha = 0$ and $\beta = \lambda = 1$ and, when $a > 0$, we can tale $\alpha = 0$, $\beta = a$ and $\lambda = 2a$ (note that these choices are not optimal).

4.4. An application to tumor growth

We take (see [2])

$$g(x, s) = \frac{\lambda_d}{2}(1+s) - \lambda_g(1+s)^2(1-s)^2 - h(x),\ h \in L^\infty(\Omega),\ h \geq 0,\ \alpha = -1,\ \beta = 1.$$

Here, the positive constants λ_d and λg are the death and growth rates, respectively, and h can be related to some nutrient.

We then take

$$f(s) = -c_0 s + c_1 \ln \frac{1+s}{1-s},\ c_0,\ c_1 > 0,\ c_1 < \frac{c_0}{2},\ s \in (-1, 1),$$

and assume that $h(x) \in [0, h_\star]$ a.e. Writing

$$g(x, s) = \frac{\lambda_d}{2} s + \tilde{g}(x, s),\ \tilde{g}(x, s) = \frac{\lambda_d}{2} - \lambda_g(1+s)^2(1-s)^2 - h(x),$$

it is easy to show that $-\tilde{g}(x, s) \in [-\frac{\lambda_d}{2}, \lambda_g - \frac{\lambda_d}{2} + h_\star]$ for a.a. x, $\forall s \in [-1, 1]$. Therefore, one has the existence of global in time weak solutions when

$$\frac{\lambda_g + h_\star}{\lambda_d} \leq 1.$$

Remark 4.4. Let us take $h \equiv 0$ and write $g(x, s) = g(s) = \lambda s + \tilde{g}(s)$, $\lambda > 0$ given, and set $\varphi(s) = -\tilde{g}(s) = (\lambda - \frac{\lambda_d}{2})s - \frac{\lambda_d}{2} + \lambda_g(1+s)^2(1-s)^2$. Then, $\varphi(1) = \lambda - \lambda_d$, which yields that, necessarily, $\lambda \geq \frac{\lambda_d}{2}$. Furthermore, noting that $\varphi(s) \leq \lambda - \lambda_d + \lambda_g$, $s \in [-1, 1]$, we see that the above sufficient condition on the existence of global in time solutions is reasonable. Of course, this upper bound is not optimal, meaning that this condition can be improved, but not so much.

Remark 4.5. In the case of a regular nonlinear term f, say, $f(s) = s^3 - s$, we can prove the existence of a local in time (strong) solution. Let us indeed consider the initial and boundary value problem

$$(-\Delta)^{-1}\frac{\partial\overline{u}}{\partial t} - \Delta u + \overline{f(u)} + (-\Delta)^{-1}\overline{g(x,u)} = 0, \tag{4.10}$$

$$\frac{d\langle u\rangle}{dt} + \langle g(x,u)\rangle = 0, \tag{4.11}$$

$$\frac{\partial\overline{u}}{\partial\nu} = 0 \text{ on } \Gamma, \tag{4.12}$$

$$\overline{u}|_{t=0} = \overline{u}_0,\ \langle u\rangle|_{t=0} = \langle u_0\rangle. \tag{4.13}$$

Multiplying (4.10) by $-\Delta u$, we have (see the First a priori estimate)

$$\frac{d}{dt}\|\overline{u}\|^2 + \|\Delta u\|^2 \le c\|u\|^2 - 2((g(x,u),\overline{u})).$$

Writing, owing to the continuous embedding $H^1(\Omega) \subset L^5(\Omega)$ and employing Young's inequality,

$$\begin{aligned}|((g(x,u),\overline{u}))| &\le c(1+\|u\|)(1+\|u\|^5_{L^5(\Omega)})\\ &\le c(1+\|u\|^6_{H^1(\Omega)}),\end{aligned}$$

it follows that

$$\frac{d}{dt}\|\overline{u}\|^2 + \|\Delta u\|^2 \le c(1+\|u\|^6_{H^1(\Omega)}). \tag{4.14}$$

Multiplying (4.11) by $\langle u\rangle$, we obtain, proceeding in a similar way,

$$\frac{d}{dt}\langle u\rangle^2 + \langle u\rangle^2 \le c(1+\|u\|^5_{H^1(\Omega)}). \tag{4.15}$$

Multiplying now (4.10) by $\frac{\partial\overline{u}}{\partial t}$, we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|^2 + 2\int_\Omega F(u)\,dx) + \|\frac{\partial\overline{u}}{\partial t}\|^2_{-1}\\ &- ((f(u),\langle\frac{\partial u}{\partial t}\rangle)) + (((-\Delta)^{-1}\overline{g(x,u)},\frac{\partial\overline{u}}{\partial t})) = 0.\end{aligned}$$

Writing, employing Hölder's and Young's inequalities,

$$\begin{aligned}|(((-\Delta)^{-1}\overline{g(x,u)},\frac{\partial\overline{u}}{\partial t}))| &= |((g(x,u),(-\Delta)^{-1}\frac{\partial\overline{u}}{\partial t}))|\\ &\le c\int_\Omega (1+|u|^4)|(-\Delta)^{-1}\frac{\partial\overline{u}}{\partial t}|\,dx \le c(1+\|u\|^4_{L^5(\Omega)})\|(-\Delta)^{-1}\frac{\partial\overline{u}}{\partial t}\|_{L^5(\Omega)}\\ &\le c(1+\|u\|^4_{H^1(\Omega)})\|\frac{\partial\overline{u}}{\partial t}\|_{-1} \le \frac{1}{2}\|\frac{\partial\overline{u}}{\partial t}\|^2_{-1} + c(1+\|u\|^8_{H^1(\Omega)})\end{aligned}$$

and

$$|((f(u),\langle\frac{\partial u}{\partial t}\rangle))|\le c\|f(u)\|_{L^1(\Omega)}\|g(x,u)\|_{L^1(\Omega)}$$
$$\le c(\int_\Omega F(u)\,dx+c')^2,$$

where $F(s)=\frac{1}{4}s^4-\frac{1}{2}s^2$, we end up with the differential inequality

$$\frac{d}{dt}(\|\nabla u\|^2+2\int_\Omega F(u)\,dx)+\|\frac{\partial\overline{u}}{\partial t}\|_{-1}^2\le c(\|u\|_{H^1(\Omega)}^2+2\int_\Omega F(u)\,dx+c')^4. \tag{4.16}$$

We finally multiply (4.10) by $-\Delta^3 u$ and have

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2+\|\Delta^2 u\|^2+((\Delta f(u),\Delta^2 u))+((g(x,u),\Delta^2 u))=0.$$

Note that (see [35])

$$|((\Delta f(u),\Delta^2 u))|\le\frac{1}{4}\|\Delta^2 u\|^2+c(1+\|u\|_{H^1(\Omega)}^{14}).$$

Furthermore,

$$|((g(x,u),\Delta^2 u))|\le c\int_\Omega(1+|u|^4)|\Delta^2 u|\,dx$$
$$\le c(1+\|u\|_{L^\infty(\Omega)}^2)(1+\|u\|_{L^4(\Omega)}^2)\|\Delta^2 u\|.$$

Employing the Agmon inequality

$$\|u\|_{L^\infty(\Omega)}\le c\|u\|_{H^1(\Omega)}^{\frac{1}{2}}\|u\|_{H^2(\Omega)}^{\frac{1}{2}}$$

and the interpolation inequality

$$\|u\|_{H^2(\Omega)}\le c\|u\|_{H^1(\Omega)}^{\frac{2}{3}}\|u\|_{H^4(\Omega)}^{\frac{1}{3}},$$

we obtain

$$|((g(x,u),\Delta^2 u))|\le c(1+\|u\|_{H^1(\Omega)}^{\frac{5}{3}}\|u\|_{H^4(\Omega)}^{\frac{1}{3}})(1+\|u\|_{H^1(\Omega)}^2)\|u\|_{H^4(\Omega)}$$
$$\le c(1+\|u\|_{H^1(\Omega)}^2)(1+(1+\|u\|_{H^1(\Omega)}^{\frac{5}{3}})\|u\|_{H^4(\Omega)}^{\frac{4}{3}})$$
$$\le\frac{1}{4}\|\Delta^2 u\|^2+c(1+\|u\|_{H^1(\Omega)}^{11}).$$

We deduce from the above the differential inequality

$$\frac{d}{dt}\|\Delta u\|^2 + \|\Delta^2 u\|^2 \leq c(1 + \|u\|_{H^1(\Omega)}^{11}). \tag{4.17}$$

Combining (4.14)-(4.17), we find a differential inequality of the form

$$\frac{dE}{dt} + c(\|u\|_{H^4(\Omega)}^2 + \|\frac{\partial u}{\partial t}\|_{H^{-1}(\Omega)}^2) \leq c'E^7, \ c > 0, \tag{4.18}$$

where

$$E = \|\overline{u}\|^2 + \langle u\rangle^2 + \|\nabla u\|^2 + 2\int_\Omega F(u)\,dx + \|\Delta u\|^2 + c$$

satisfies

$$E \geq c\|u\|_{H^2(\Omega)}^2, \ c > 0.$$

We indeed deduce from (4.18) the existence of a local in time strong solution. However, one does not know whether this solution is global in time or whether it blows up in finite time, though numerical simulations suggest that one should have global in time solutions (see [2,35]). The global in time existence result obtained in this paper is thus the first one for this tumor growth model.

4.5. An application to metabolites concentrations in the brain

We consider the following equation, proposed in [31] to model metabolites concentrations (e.g., lactate) in the brain:

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + \frac{ku}{k' + u} = J, \ J, \ k, \ k' > 0, \ u \in (0, 1),$$

where

$$f(s) = -c_0(s - \frac{1}{2}) + c_1 \ln\frac{s}{1-s}, \ c_0, \ c_1 > 0, \ c_1 < \frac{c_0}{4}, \ s \in (0, 1).$$

We assume for simplicity that J is a constant. Actually, in order to prove the existence of solutions, one considers the slightly modified equation

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) + \frac{ku}{k' + |u|} = J,$$

so to avoid the nonlinear term $\frac{ku}{k'+u}$ to become singular when considering approximated problems (see [31]). However, since $u \geq 0$ a.e., we recover a solution to the original equation. Note that, for a regular (cubic) nonlinear term f, we cannot prove that the solution u remains nonnegative and deduce the existence of a (local in time) solution to the original problem. Indeed, we can construct counterexamples in which the solution instantaneously becomes negative (see [31]).

We take

$$g(x,s) = g(s) = \frac{ks}{k'+s} - J,\ \alpha = 0,\ \beta = 1.$$

Let us write

$$g(s) = \lambda s + \tilde{g}(s),\ \lambda \geq \frac{k}{k'}.$$

Then, the function

$$\varphi(s) = -\tilde{g}(s) = \lambda s - \frac{ks}{k'+s} + J$$

is monotone increasing, with $\varphi(0) = J \geq 0$ and $\varphi(1) = \lambda - \frac{k}{k'+1} + J$. Therefore, one has a global in time weak solution when $J \leq \frac{k}{k'+1}$, which corresponds to the assumption made in [31].

Let us now consider the ODE

$$y' + \frac{ky}{k'+y} = J,\ y(0) \in (0,1),$$

corresponding to spatially homogeneous solutions to our problem. Here, one easily proves the existence of global in time positive solutions. Furthermore, one has

$$(k'+y)y' = k'J + (J-k)y.$$

Therefore, when $J = k$, it follows that

$$k'y + \frac{y^2}{2} = k'Jt + c,$$

whereas, when $J > k$, then

$$\frac{y}{J-k} + \frac{1}{J-k}(k' - \frac{k'J}{J-k}) \ln(k'J + (J-k)y) = t + c,$$

where c depends on the initial condition. Thus, in both cases, we cannot have a global in time solution to our initial problem, since, otherwise $y(t)$ would tend to $+\infty$ as time goes to $+\infty$ and the logarithmic nonlinear term would not make sense.

Also note that, when $J < k$, then

$$\frac{y}{J-k} + \frac{1}{J-k}(k' - \frac{k'J}{J-k}) \ln|k'J + (J-k)y| = t + c,$$

so that y converges to the equilibrium $y_e = \frac{k'J}{k-J}$. Noting that $y_e \leq 1$ if and only if $J \leq \frac{k}{k'+1}$, this shows that the above condition is sharp.

From a biological point of view, this suggests that one has global in time solutions only when one remains in the viability domain of the cells. If we think, e.g., of the lactate concentration in glial cells, then, when $J > \frac{k}{k'+1}$, the cells are not able to manage the lactate surplus, leading to necrosis.

Let us further assume that $J < \frac{k}{k'+1}$. Then, we have

$$g(s) = \frac{k-J}{k'+s}(s - \frac{Jk'}{k-J}).$$

Noting that the condition $J < \frac{k}{k'+1}$ implies $k > J$ and $\frac{Jk'}{k-J} \in (0,1)$, it follows from (2.8)-(2.9) that

$$\begin{aligned} g(s)f(s) &\geq c\frac{k-J}{k'+s}F(s) - c'\frac{k-J}{k'+s} \\ &\geq -c\frac{k-J}{k'+s} \geq -c',\ c' \geq 0,\ s \in (0,1), \end{aligned}$$

and we can derive additional regularity, noting that g is monotone increasing, with g' bounded. Note that the formal calculations given in Remark 3.5 can be justified within a Galerkin scheme for the approximated problems and that, in that case, we would take $g(s) = \frac{ks}{k'+|s|} - J$ (this function g is still monotone increasing, with g' bounded). Therefore, when $s \geq 0$, we can prove as above that

$$g(s)f_N(s) \geq -c,\ c \geq 0,$$

owing to (3.3), whereas, when $s < 0$, we can choose N large enough so that $f_N \leq 0$ and thus see that

$$g(s)f_N(s) \geq 0.$$

Here, f_N is as defined in the previous section. This corrects several imprecisions and omissions in the proof given in [31].

4.6. A counterexample: a Cahn–Hilliard model in image segmentation

We consider the following equation, proposed in [43] in view of applications to image segmentation:

$$\frac{\partial u}{\partial t} + \varepsilon_1 \Delta^2 u - \frac{1}{\varepsilon_1}\Delta f(u) + \frac{\varepsilon_2 h(x)}{\varepsilon_2^2 + (u-\frac{1}{2})^2} = 0,$$

where

$$h(x) = \frac{1}{\pi}(\lambda_1(i(x)-c_1)^2 - \lambda_2(i(x)-c_2)^2).$$

Here, ε_1, ε_2, λ_1 and λ_2 are positive constants and i is a given image taking values in [0, 1].

The global in time well-posedness, for a cubic nonlinear term, was studied in [30].

Let us now take h constant, $h \neq 0$, and assume that f is a logarithmic nonlinear term as in the previous example. Then, set

$$g(x,s) = g(s) = \frac{\varepsilon_2 h}{\varepsilon_2^2 + (u - \frac{1}{2})^2},\ \alpha = 0,\ \beta = 1$$

and take $\lambda > 0$. Writing $g(s) = \lambda s + \tilde{g}(s)$, we see that $-\tilde{g}(0) = -\frac{\varepsilon_2 h}{\varepsilon_2^2 + \frac{1}{4}} < 0$ if $h > 0$, whereas $-\tilde{g}(1) = \lambda - \frac{\varepsilon_2 h}{\varepsilon_2^2 + \frac{1}{4}} > \lambda$ if $h < 0$. This shows that $-\tilde{g}$ cannot belong to $[0, \lambda]$ and that we cannot expect to have a global in time existence result in general when considering logarithmic nonlinear terms and our approach. Actually, here, we should only expect a local existence result in general. Note that this is consistent with the fact that, when h is a nonvanishing constant, then the spatially homogeneous solutions go to infinity as time goes to $+\infty$; actually, numerical simulations show that the same can hold when h is not a constant and changes sign (see [30]).

4.7. Concluding remarks

Remark 4.6. The study of stationary solutions for the above models is also of interest. For instance, this is important for the inpainting model, since the restored image is expected to be an equilibrium of the problem. This is studied in [5,18]. We can note that an open problem is to prove that single trajectories converge to an equilibrium, which is expected in the inpainting model. We also mention that the authors in [18] consider more general source terms g, in particular, for biological applications. The study of stationary solutions for the Ohta–Kawasaki model can be found in, e.g., [28,41] and references therein. Note that all results mentioned above are for regular nonlinear terms f. It would thus be interesting to also address the case of logarithmic nonlinear terms f.

Remark 4.7. Actually, the models considered here are all diffuse interface models, meaning that one should actually consider the equation

$$\frac{\partial u}{\partial t} + \varepsilon^2 \Delta^2 u - \Delta f(u) + g(x,u) = 0,\ \varepsilon > 0, \tag{4.19}$$

where ε is related with the (thin) interface thickness. In particular, it would be interesting to see how an interface evolves when ε is small and to study the sharp interface limit $\varepsilon \to 0^+$. In the case of the Cahn–Hilliard equation, $g \equiv 0$, it is known that an interface is driven by the mean curvature and tends to a sphere (see, e.g., [1,27] and references therein). The case of nonvanishing source terms g will be addressed elsewhere; see also [12,23,24] for the Ohta–Kawasaki model.

Acknowledgments

The author wishes to thank an anonymous referee for her/his careful reading of the paper and useful comments.

References

[1] N.D. Alikakos, P.W. Bates, X. Chen, Convergence of the Cahn-Hiliiard equation to the Hele-Shaw model, Arch. Ration. Mech. Anal. 128 (1994) 165–205.
[2] A.C. Aristotelous, O.A. Karakashian, S.M. Wise, Adaptive, second-order in time, primitive-variable discontinuous Galerkin schemes for a Cahn-Hilliard equation with a mass source, IMA J. Numer. Anal. 35 (2015) 1167–1198.
[3] P.W. Bates, J. Han, The Neumann boundary problem for a nonlocal Cahn-Hilliard equation, J. Differ. Equ. 212 (2005) 235–277.

[4] A. Bertozzi, S. Esedoglu, A. Gillette, Inpainting of binary images using the Cahn-Hilliard equation, IEEE Trans. Image Process. 16 (2007) 285–291.
[5] A. Bertozzi, S. Esedoglu, A. Gillette, Analysis of a two-scale Cahn-Hilliard model for binary image inpainting, Multiscale Model. Simul. 6 (2007) 913–936.
[6] J.W. Cahn, On spinodal decomposition, Acta Metall. 9 (1961) 795–801.
[7] J.W. Cahn, J.E. Hilliard, Free energy of a nonuniform system I. Interfacial free energy, J. Chem. Phys. 28 (1958) 258–267.
[8] V. Chalupecký, Numerical studies of Cahn-Hilliard equations and applications in image processing, in: Proceedings of Czech-Japanese Seminar in Applied Mathematics, August 4–7, 2004, Czech Technical University in Prague, 2004.
[9] L. Cherfils, H. Fakih, A. Miranville, Finite-dimensional attractors for the Bertozzi-Esedoglu-Gillette-Cahn-Hilliard equation in image inpainting, Inverse Probl. Imaging 9 (2015) 105–125.
[10] L. Cherfils, H. Fakih, A. Miranville, On the Bertozzi-Esedoglu-Gillette-Cahn-Hilliard equation with logarithmic nonlinear terms, SIAM J. Imaging Sci. 8 (2015) 1123–1140.
[11] L. Cherfils, A. Miranville, S. Zelik, On a generalized Cahn-Hilliard equation with biological applications, Discrete Contin. Dyn. Syst., Ser. B 19 (2014) 2013–2026.
[12] R. Choksi, M.A. Peletier, Small volume fraction limit of the diblock copolymer problem: I. Sharp interface functional, SIAM J. Math. Anal. 42 (2010) 1334–1370.
[13] D. Cohen, J.M. Murray, A generalized diffusion model for growth and dispersion in a population, J. Math. Biol. 12 (1981) 237–248.
[14] I.C. Dolcetta, S.F. Vita, Area-preserving curve-shortening flows: from phase separation to image processing, Interfaces Free Bound. 4 (2002) 325–343.
[15] J. Erlebacher, M.J. Aziz, A. Karma, N. Dimitrov, K. Sieradzki, Evolution of nanoporosity in dealloying, Nature 410 (2001) 450–453.
[16] H. Fakih, A Cahn-Hilliard equation with a proliferation term for biological and chemical applications, Asymptot. Anal. 94 (2015) 71–104.
[17] H. Fakih, Asymptotic behavior of a generalized Cahn-Hilliard equation with a mass source, Appl. Anal. 96 (2017) 324–348.
[18] H. Fakih, R. Mghames, N. Nasreddine, On the Cahn-Hilliard equation with mass source for biological applications, Commun. Pure Appl. Anal. 20 (2021) 495–510.
[19] S. Frigeri, M. Grasselli, Nonlocal Cahn-Hilliard-Navier-Stokes systems with singular potentials, Dyn. Partial Differ. Equ. 9 (2012) 273–304.
[20] H. Gajewski, K. Zacharias, On a nonlocal phase separation model, J. Math. Anal. Appl. 286 (2003) 11–31.
[21] C.G. Gal, A. Giorgini, M. Grasselli, The nonlocal Cahn-Hilliard equation with singular potential: well-posedness, regularity and strict separation property, J. Differ. Equ. 263 (2017) 5253–5297.
[22] A. Giorgini, M. Grasselli, A. Miranville, The Cahn-Hilliard-Oono equation with singular potential, Math. Models Methods Appl. Sci. 27 (2017) 2485–2510.
[23] D. Goldman, C.B. Muratov, S. Serfaty, The Γ-limit of the two-dimensional Ohta-Kawasaki energy. I. Droplet density, Arch. Ration. Mech. Anal. 210 (2013) 581–613.
[24] D. Goldman, C.B. Muratov, S. Serfaty, The Γ-limit of the two-dimensional Ohta-Kawasaki energy. Droplet arrangement via the renormalized energy, Arch. Ration. Mech. Anal. 212 (2014) 445–501.
[25] E. Khain, L.M. Sander, A generalized Cahn-Hilliard equation for biological applications, Phys. Rev. E 77 (2008) 051129.
[26] I. Klapper, J. Dockery, Role of cohesion in the material description of biofilms, Phys. Rev. E 74 (2006) 0319021.
[27] A.A. Lee, A. Münch, A. Süli, Sharp-interface limits of the Cahn-Hilliard equation with degenerate mobility, SIAM J. Math. Anal. 76 (2016) 433–456.
[28] J.-Ph. Lessard, E. Sander, T. Wanner, Rigorous continuation of bifurcation points in the diblock copolymer equation, J. Comput. Dyn. 4 (2017) 71–118.
[29] Q.-X. Liu, A. Doelman, V. Rottschäfer, M. de Jager, P.M.J. Herman, M. Rietkerk, J. van de Koppel, Phase separation explains a new class of self-organized spatial patterns in ecological systems, Proc. Natl. Acad. Sci. 110 (29) (2013) 11905–11910, https://doi.org/10.1073/pnas.1222339110.
[30] L. Li, L. Cherfils, A. Miranville, Ph. Rogeon, R. Guillevin, On a Cahn–Hilliard model for image segmentation, Math. Methods Appl. Sci. 44 (2021) 5753–5766.
[31] L. Li, A. Miranville, R. Guillevin, Cahn–Hilliard models for glial cells, Appl. Math. Optim. (2021), https://doi.org/10.1007/s00245-020-09696-x.
[32] A. Miranville, Asymptotic behaviour of a generalized Cahn-Hilliard equation with a proliferation term, Appl. Anal. 92 (2013) 1308–1321.

[33] A. Miranville, A generalized Cahn-Hilliard equation with logarithmic potentials, in: Viktor A. Sadovnichiy, Mikhail Z. Zgurovsky (Eds.), Continuous and Distributed Systems II, in: Studies in Systems, Decision and Control, vol. 30, Springer, 2015, pp. 147–158.
[34] A. Miranville, Existence of solutions to a Cahn-Hilliard type equation with a logarithmic nonlinear term, Mediterr. J. Math. 16 (2019) 1–18.
[35] A. Miranville, The Cahn-Hilliard Equation: Recent Advances and Applications, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 95, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2019.
[36] A. Novick-Cohen, The Cahn-Hilliard equation, in: C.M. Dafermos, M. Pokorny (Eds.), Handbook of Differential Equations, in: Evolutionary Partial Differential Equations, vol. 4, Elsevier, Amsterdam, 2008, pp. 201–228.
[37] T. Ohta, K. Kawasaki, Equilibrium morphology of block copolymer melts, Macromolecules 19 (1986) 2621–2632.
[38] Y. Oono, S. Puri, Computationally efficient modeling of ordering of quenched phases, Phys. Rev. Lett. 58 (1987) 836–839.
[39] A. Oron, S.H. Davis, S.G. Bankoff, Long-scale evolution of thin liquid films, Rev. Mod. Phys. 69 (1997) 931–980.
[40] S. Tremaine, On the origin of irregular structure in Saturn's rings, Astron. J. 125 (2003) 894–901.
[41] J.B. van den Berg, J.F. Williams, Rigorously computing symmetric stationary states of the Ohta-Kawasaki problem in three dimensions, SIAM J. Math. Anal. 51 (2019) 131–158.
[42] J. Verdasca, P. Borckmans, G. Dewel, Chemically frozen phase separation in an adsorbed layer, Phys. Rev. E 52 (1995) 4616–4619.
[43] W. Yang, Z. Huang, W. Zhu, Image segmentation using the Cahn-Hilliard equation, J. Sci. Comput. 79 (2019) 1057–1077.