Gradient estimates for Dirichlet parabolic problems in unbounded domains

S. Fornaro[a], G. Metafune[a,*], E. Priola[b,1]

[a]*Dipartimento di Matematica "Ennio De Giorgi", Università di Lecce, C.P.193, 73100, Lecce, Italy*
[b]*Dipartimento di Matematica, Università di Torino, via Carlo Alberto 10, 10123, Torino, Italy*

Abstract

We consider autonomous parabolic Dirichlet problems in a regular unbounded open set $\Omega \subset \mathbf{R}^N$ involving second-order operator A with (possibly) unbounded coefficients. We determine new conditions on the coefficients of A yielding global gradient estimates for the bounded classical solution.
© 2004 Elsevier Inc. All rights reserved.

MSC: 35P05; 35J70; 35K65

Keywords: Gradient estimates; Unbounded coefficients; Maximum principles

1. Introduction and notation

In this paper we consider the following Dirichlet parabolic problem:

$$\begin{cases} u_t(t, x) - Au(t, x) = 0, & t \in (0, T), \ x \in \Omega, \\ u(t, \xi) = 0, & t \in (0, T), \ \xi \in \partial\Omega, \\ u(0, x) = f(x), & x \in \Omega, \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: simona.fornaro@unile.it (S. Fornaro), giorgio.metafune@unile.it (G. Metafune), priola@dm.unito.it (E. Priola).

where f is continuous and bounded in Ω. Here Ω is an unbounded smooth connected open set in $\mathbf{R}^N$ and A is a second-order elliptic operator, with (possibly) unbounded regular coefficients, i.e.,

$$A = \sum_{i,j=1}^{N} q_{ij} D_{ij} + \sum_{i,j=1}^{N} F_i D_i - V = \text{Tr}(QD^2) + F \cdot D - V. \tag{1.2}$$

Our aim is to prove global gradient estimates for the bounded classical solution u to (1.1), i.e.,

$$\|Du(t,\cdot)\|_\infty \leqslant \frac{C}{\sqrt{t}} \|f\|_\infty, \quad t \in (0,T), \tag{1.3}$$

where $\|f\|_\infty$ is the sup norm of f in Ω. Estimate (1.3) is classical when the coefficients of A are *bounded* on Ω and Ω is a bounded or unbounded open set with sufficiently smooth boundary, see e.g. [12, Chapter 3]. On the other hand in the recent literature the interest towards elliptic operators with unbounded coefficients is growing up, see for instance [4,6,5,13,21,2,16,18,14,23,24] and the references therein. The main motivation comes from well-known connections with stochastic differential equations, see e.g. [8,22].

As far as local gradient estimates for (1.1) are concerned, we mention [21], which establishes them in the Riemannian setting, and [5,20] for the case when Ω is an open subset of a Hilbert space and A is an Ornstein–Uhlenbeck operator. Recently, in [17], see also [18], it was stressed the problem to establish global gradient estimates as a preliminary step towards global Schauder estimates for Dirichlet elliptic problems involving unbounded coefficients in unbounded domains. Surprisingly enough, a counterexample in [24] shows that (1.3) fails in general even when Ω is a half plane. Moreover in [24], see also [11], connections between estimates (1.3) and some isoperimetric inequalities are investigated. We also mention [19] and [1], where gradient estimates are proved for Neumann parabolic problems, under convexity assumptions on Ω.

Let us explain our main assumptions to obtain (1.3) in the particular where $A = \Delta + F \cdot D$ and Ω is an unbounded open set with uniformly C^2-boundary. The dissipativity condition on F, see (1.5) below, is quite natural since a one-dimensional counterexample to gradient estimates is constructed in [1] when it fails. Observe also that, if $F = D\Phi$, then (1.5) is a concavity assumption on Φ. On the other hand, condition (1.6) seems to be quite new and, roughly speaking, it means that the component of the drift F along the inner normal is bounded from above in a neighborhood of $\partial\Omega$. Even though its connection with gradient estimates is not evident from an analytic point of view, its necessity is clear if one considers the Markov process governed by the operator A under Dirichlet boundary conditions. In fact the solution $u(t,x)$ to (1.1) corresponding to $f = 1$ represents the probability that the process starting from $x \in \Omega$ at time $t = 0$ is not absorbed by the boundary up to time t. If the (inner) normal component of F is unbounded from above in a neighborhood of $\partial\Omega$, one expects that $u(t,x) \to 1$ as $|x| \to \infty$ along the boundary. Since $u(t,\xi) = 0$ for $\xi \in \partial\Omega$, it follows that $u(t,\cdot)$ is

even not uniformly continuous, see Example 6.1 where this heuristic argument is made rigorous. Finally, we point out that the growth assumption (1.8), even though not very restrictive, seems to be a technical one in order to use our methods, see the proof of Theorem 1.3.

We use mainly analytic tools and we do not need any convexity assumption on Ω. Moreover we stress that our operator A may contain a potential term V which is difficult to treat by probabilistic methods.

In Section 2, we prove existence and uniqueness of classical bounded solutions u to (1.1). To this purpose we use both classical Schauder estimates and a nonstandard maximum principle for discontinuous solutions to (1.1), see Theorem A.2. In Section 3, by means of the distance function from the boundary of Ω, we prove some a priori estimates for Du. This is done in two steps. First, by comparison with certain one-dimensional operators one obtains boundary estimates for Du and then, using Bernstein's method, one shows that the same estimates hold in the whole Ω. However, the method works (and gives (1.3) with the right dependence of all constants involved), if one already knows that Du is bounded up to the boundary of Ω for positive t, see Proposition 3.3. To circumvent this difficulty, we subtract to the operator A a potential εW, where W is big enough to dominate the growth of F and, following ideas in [3,4,14], we show that the perturbed operator $A_\varepsilon = A - \varepsilon W$ generates an analytic semigroup in $L^p(\Omega)$ and characterize its domain. Choosing a large p and using Sobolev embedding, it follows that the bounded classical solution u_ε of problem (1.1) with A_ε instead of A and a smooth f has a bounded gradient in $[0, T] \times \Omega$. Therefore Proposition 3.3 applies and gives (1.3) for u_ε with a constant C independent of ε. An approximation argument then completes the proof. This program is carried out in Sections 4 and 5. In Section 6 we present the announced counterexample.

Let us collect our first hypotheses on Ω and the coefficients of A. We denote by B_R the euclidean ball with center at the origin and radius R.

Hypothesis 1.1. (i) Ω is a connected open subset of $\mathbf{R}^N$ with uniformly $C^{2+\alpha}$-boundary for some $0 < \alpha < 1$, see Appendix B.

(ii) $q_{ij}, F_i, V \in C^{1+\alpha}(\Omega \cap B_R)$ for every $i, j = 1, \ldots, N$ and $R > 0$; moreover $V \geqslant 0$ in Ω.

(iii) $q_{ij} = q_{ji} \in C^1_b(\Omega)$, and there exists $v > 0$ such that $\sum_{i,j=1}^N q_{ij}(x)\xi_i\xi_j \geqslant v|\xi|^2$, for every $x \in \Omega$ and $\xi \in \mathbf{R}^N$.

(iv) There exist a positive function $\phi \in C^2([0, T] \times \overline{\Omega})$ and $\lambda_0 > 0$ such that

$$\lim_{|x| \to +\infty, \, x \in \overline{\Omega}} \phi(t, x) = +\infty, \quad \text{uniformly in } [0, T], \quad (D_t - A + \lambda_0)\phi \geqslant 0.$$

The *Lyapunov* map ϕ introduced in assumption (iv) will be used to prove maximum principles, see Appendix A. Moreover condition (i) ensures that the *distance function*

$$r(x) = dist(x, \partial\Omega), \quad x \in \overline{\Omega} \tag{1.4}$$

is a C^2-function with bounded second-order derivatives in Ω_δ, for some $\delta > 0$, where we set

$$\Omega_\delta = \{x \in \overline{\Omega} : dist(x, \partial\Omega) < \delta\},$$

see [7, Lemma 14.16] and also Appendix B (note that (i) implies that the principal curvatures of $\partial\Omega$, when $\partial\Omega$ is considered as an hypersurface, are bounded). Our main result will be proved assuming also the conditions listed below.

$$\sum_{i,j=1}^{N} D_i F_j(x)\xi_i\xi_j \leqslant (sV(x)+k)|\xi|^2, \quad x \in \Omega, \ \xi \in \mathbf{R}^N, \tag{1.5}$$

$$\sum_{i,j=1}^{N} q_{ij}(x)D_{ij}r(x) + \sum_{i=1}^{N} F_i(x)\, D_i r(x) \leqslant M, \quad x \in \Omega_\delta \ (\text{for some } \delta > 0), \tag{1.6}$$

$$|DV(x)| \leqslant \beta(1+V(x)), \quad x \in \Omega, \tag{1.7}$$

$$|F(x)| \leqslant c_1 e^{c_2|x|}, \quad x \in \Omega, \tag{1.8}$$

for some constants $k, M, \beta, c_1, c_2 \in \mathbf{R}, \ s < 1/2$.

Observe that, since $q_{ij} \in C_b^1(\Omega)$ and Ω is uniformly C^2, (1.6) is only a condition on the component of F along the inner normal to $\partial\Omega$ in a neighborhood of $\partial\Omega$.

To specify the dependence of some constants we also introduce the quantity

$$h = \sup_{x \in \Omega} \left(\sum_{i,j=1}^{N} |Dq_{ij}(x)|^2 \right)^{1/2} \tag{1.9}$$

which is finite, since $q_{ij} \in C_b^1(\Omega)$.

Remark 1.2. Observe that assumption (iv) of Hypothesis 1.1 follows from the positivity of V and the boundedness of q_{ij}, when condition (1.5) holds with $s = 0$. In fact (1.5) implies, by differentiating the function $t \to F(tx) \cdot x$, that $F(x) \cdot x \leqslant F(0) \cdot x + k|x|^2$, hence the function $\phi(x) = 1 + |x|^2$ satisfies (iv), for a suitable λ_0.

We will prove the following theorem.

Theorem 1.3. *There exists a constant C depending only on $v, k, s, h, N, M, \beta, \delta, T$ such that the bounded classical solution u of* (1.1) *satisfies*

$$\|Du(t, \cdot)\|_\infty \leqslant \frac{C}{\sqrt{t}}\|f\|_\infty, \quad t \in (0, T), \quad f \in C_b(\Omega).$$

Notation: We define $Q = (0, T) \times \Omega$, $\partial' Q = (0, T) \times \partial\Omega \cup \{0\} \times \overline{\Omega}$ and $\partial_{tx} Q = \{0\} \times \partial\Omega$.

We denote by $D_t u$, $D_i u$ the partial derivatives with respect to the variables t and x_i, respectively, of a function u depending on $(t, x) \in \mathbf{R} \times \mathbf{R}^N$. Similarly, $D_{ij} u$ stands for $D_{x_i x_j} u$. The space gradient of u and its Hessian matrix with respect to the space variables are simply denoted by Du, $D^2 u$.

For $0 < \alpha < 1$ and $k \in \mathbf{N}$, $C^{k+\alpha}(\Omega)$ denotes the usual Hölder space of k-times continuously differentiable functions such that all the derivatives up to order k are bounded and those of order k are α-Hölder continuous in Ω (or equivalently in $\overline{\Omega}$). If $a < b$, $C^{1+\alpha/2, 2+\alpha}((a, b) \times \Omega)$ is the classical parabolic Hölder space, i.e. the space of functions $u = u(t, x)$ which are continuous and bounded in $(a, b) \times \Omega$ together with their first-order time derivatives and first and second-order space derivatives and such that $D_t u$ and $D_{ij} u$ are α-Hölder continuous in $(a, b) \times \Omega$ (or equivalently in $[a, b] \times \overline{\Omega}$) with respect to the parabolic distance $d((t, x), (s, y)) = |t - s|^{1/2} + |x - y|$. By $C^{1,2}((a, b) \times \Omega)$ we mean the space of functions $u(t, x)$ which are continuous in $(a, b) \times \Omega$ with their indicated derivatives (not necessarily bounded). Finally, $C_b^k(\Omega)$ indicates the Banach space of all k-times continuously differentiable functions in Ω, bounded together with their derivatives up to the order k. When $k = 0$ we simply write $C_b(\Omega)$. $C_0^\infty(\Omega)$ is the space of C^∞-functions with compact support in Ω. The symbols $\|\cdot\|_p$, $\|\cdot\|_\infty$ denote the L^p-norm and the sup-norm, respectively. The support of a function ϕ is denoted by supp ϕ.

A function u is called a *bounded classical solution* of (1.1) if $u \in C^{1,2}(Q)$, u is continuous in $\overline{Q} \setminus \partial_{tx} Q$, bounded in Q and solves (1.1).

2. Existence and uniqueness

As a preliminary step, let us show that (1.1) has a unique bounded classical solution.

Proposition 2.1. *Assume Hypothesis* 1.1. *If* $f \in C^{2+\alpha}(\Omega)$ *has compact support in* Ω, *then problem* (1.1) *has a unique bounded solution* u *which belongs to* $C^{1+\alpha/2, 2+\alpha}((0, T) \times (\Omega \cap B_R))$ *for every* $R > 0$. *Moreover,* $\|u\|_\infty \leqslant \|f\|_\infty$ *and* $u \geqslant 0$ *if* $f \geqslant 0$. *Finally,* Du *belongs to* $C^{1+\alpha/2, 2+\alpha}((\varepsilon, T) \times \Omega'))$ *for every* $\varepsilon > 0$ *and* Ω' *open bounded set with* $dist(\Omega', \mathbf{R}^N \setminus \Omega) > 0$. *In particular,* $Du \in C^{1,2}(Q)$.

Proof. Uniqueness is immediate consequence of a classical maximum principle, see Proposition A.1.

To prove the existence part, we consider a sequence of uniformly elliptic operators with coefficients in $C^\alpha(\Omega)$,

$$A^n = \sum_{i,j=1}^N q_{ij} D_{ij} + \sum_{i=1}^N F_i^n D_i - V^n u,$$

such that $F_i^n = F_i$, $V^n = V$ in $\Omega \cap B_n$, $V^n \geqslant 0$ and let $u_n \in C^{1+\alpha/2, 2+\alpha}(Q)$ be the solution of (1.1), with A^n instead of A (see e.g. [10, Theorem IV.5.2]). The classical

maximum principle yields $\|u_n\|_\infty \leqslant \|f\|_\infty$. Let us fix $R > 0$ and observe that, since Ω is unbounded and connected, $dist(\Omega \setminus B_{R+1}, \Omega \cap B_R) > 0$. Since $A^n = A^m = A$ in $\Omega \cap B_{R+1}$ for $n, m > R + 1$, by the local Schauder estimates [10, Theorem IV.10.1], there exists a constant C such that

$$\|u_n - u_m\|_{C^{1+\alpha/2,2+\alpha}((0,T)\times(\Omega\cap B_R))} \leqslant C\|u_n - u_m\|_{C((0,T)\times(\Omega\cap B_{R+1}))} \leqslant 2C\|f\|_\infty.$$

Therefore (u_n) is relatively compact in $C^{1,2}([0, T] \times (\overline{\Omega \cap B_R}))$. Considering an increasing sequence of balls and using a diagonal procedure we can extract a subsequence (u_{n_k}) convergent to a function $u \in C^{1+\alpha/2,2+\alpha}((0, T) \times (\Omega \cap B_R))$ for every $R > 0$ which solves (1.1) and satisfies $\|u\|_\infty \leqslant \|f\|_\infty$. By the maximum principle, $u \geqslant 0$, whenever $f \geqslant 0$.

In order to prove the last part of the statement it is sufficient to apply [9, Theorem 8.12.1] directly to the operator $D_t - A$. $\quad\square$

We now introduce linear operators $(P_t)_{t \geqslant 0}$ via the formula $(P_t f)(x) = u(t, x)$ for $f \in C^{2+\alpha}(\Omega)$, with compact support in Ω, where u is the solution of (1.1) given by the above proposition. Each operator P_t is positive and contractive with respect to the sup-norm, by the above proposition.

Now we consider the case where f is only continuous and bounded in Ω and extend the above maps $(P_t)_{t \geqslant 0}$ to a semigroup in $C_b(\Omega)$.

Proposition 2.2. *Assume Hypothesis* 1.1. *If f belongs to* $C_b(\Omega)$, *then problem* (1.1) *has a unique bounded classical solution u. Moreover,* $u(t, x) \to f(x)$ *as* $t \to 0$, *uniformly on compact sets of* Ω.

Proof. Uniqueness is an immediate consequence of a nonstandard maximum principle, see Theorem A.2. To show existence, we consider a sequence $(f_n) \in C_0^\infty(\Omega)$ convergent to f uniformly on compact subsets of Ω and such that $\|f_n\|_\infty \leqslant \|f\|_\infty$. Let $u_n \in C^{1+\alpha/2,2+\alpha}((0, T) \times (\Omega \cap B_R))$, for every $R > 0$, be the solution of (1.1) with f_n instead of f, given by the previous proposition. Let us fix $\varepsilon > 0$. By the Schauder estimates [10, Theorem IV.10.1], as in the proof of Proposition 2.1, we get a constant C such that

$$\|u_n - u_m\|_{C^{1+\alpha/2,2+\alpha}((\varepsilon,T)\times(\Omega\cap B_R))} \leqslant C\|u_n - u_m\|_{C((0,T)\times(\Omega\cap B_{R+1}))} \leqslant 2C\|f\|_\infty$$

and then, by a compactness argument, we can extract a subsequence (u_{n_k}) convergent to a function $u \in C^{1+\alpha/2,2+\alpha}((\varepsilon, T) \times (\Omega \cap B_R))$ for every $\varepsilon, R > 0$ which solves the equation $u_t - Au = 0$ in Q and such that $u(t, x) = 0$ for $t \in (0, T)$, $x \in \partial\Omega$. In the following, we write $u = P_t f$, for $f \in C_b(\Omega)$.

It remains to show that $u(t, x) \to f(x)$ as $t \to 0$, uniformly on compact sets of Ω.

Assume first that $f \in C_0(\Omega)$, i.e. f vanishes on $\partial\Omega$ and at infinity. Then we can choose (f_n) as above in such a way that $\|f_n - f\|_\infty \to 0$. The maximum principle implies that (u_n) is a Cauchy sequence in $C([0, T] \times \overline{\Omega})$, hence $u_n \to u$ uniformly in $\overline{Q}$ and $u(0, x) = f(x)$ for every $x \in \overline{\Omega}$.

Let $K \subset \Omega$ be a compact set and $\eta \in C_0(\Omega)$, $0 \leqslant \eta \leqslant 1$, be such that $\eta = 1$ in K. Then $P_t \eta \to \eta$ as $t \to 0$, uniformly in Ω, hence $P_t \eta \to 1$ uniformly in K and, since $0 \leqslant P_t(1-\eta) \leqslant 1 - P_t \eta$, we get $P_t(1-\eta) \to 0$ uniformly in K. For $f \in C_b(\Omega)$, writing $P_t f = P_t(\eta f) + P_t((1-\eta)f)$ and observing that $P_t(\eta f) \to \eta f$ uniformly in Ω and that $P_t((1-\eta)f) \to 0$ uniformly in K we obtain that $P_t f \to f$, uniformly in K. $\square$

Corollary 2.3. *The family* $(P_t)_{t \geqslant 0}$ *is a semigroup in* $C_b(\Omega)$.

Proof. The semigroup law $P_{t+s} = P_t P_s$ is immediate consequence of the uniqueness statement in Proposition 2.2. $\square$

Observe that the semigroup $(P_t)_{t \geqslant 0}$ is not strongly continuous. In fact $P_t f \to f$ as $t \to 0$, only uniformly on compact subsets of Ω. However, $P_t f \to f$ uniformly in Ω for every $f \in C_0(\Omega)$.

3. Some a priori estimates

In the following proposition we prove a preliminary boundary gradient estimate for bounded solutions of problem (1.1). We need the following lemma on gradient estimates for certain one-dimensional operators.

Lemma 3.1. *Let* $\delta > 0$ *and* $g : [0, +\infty) \times [0, \delta] \to \mathbf{R}$ *be the solution to*

$$
\begin{cases}
g_t(t,r) = v g_{rr}(t,r) + M g_r(t,r), & t > 0, \ r \in (0, \delta), \\
g(t,0) = 0, \ g(t,\delta) = 1, & t > 0, \\
g(0,r) = 1, & r \in (0, \delta).
\end{cases}
\tag{3.1}
$$

Then $g_r \geqslant 0$, $g_{rr} \leqslant 0$ *and for any* $T > 0$ *there exists* $c_T > 0$ *such that*

$$
0 \leqslant g(t,r) \leqslant \frac{c_T}{\sqrt{t}} r, \quad 0 < t \leqslant T, \ r \in (0, \delta).
$$

Proof. We define the operator $(B, D(B))$ in $C([0, \delta])$ by

$$
Bu = vu'' + Mu' \qquad D(B) = \{u \in C^2([0, \delta]) : u(0) = 0, (Bu)(\delta) = 0\}.
$$

Let us show that $(B, D(B))$ generates an analytic semigroup S_t of positive contractions in $C([0, \delta])$ (note that S_t is not strongly continuous since the domain $D(B)$ is not dense in $C([0, \delta])$).

Let $D = \{u \in C^2([0, \delta]) : u(0) = u(\delta) = 0\}$. Then (B, D) generates an analytic semigroup $(T_t)_{t \geqslant 0}$ in $C([0, \delta])$. Set $\psi(r) = a \int_0^r e^{-Ms/v} ds$. Then $B\psi = 0$, $\psi(0) = 0$ and $\psi(\delta) = 1$, if a is suitably chosen. It is easily seen that $S_t f = T_t(f - f(\delta)\psi) + f(\delta)\psi$ is the analytic semigroup generated by $(B, D(B))$ in $C([0, \delta])$. Since the regularity properties of $S_t f$ coincide with those of $T_t f$, it follows that $u(t,r) = S_t f(r)$ is a C^∞

function for $t > 0$, continuous at the points $(0, r)$, with $0 < r < \delta$. The maximum principle, see Theorem A.2, now yields positivity and contractivity of S_t.

We can prove the stated properties of g. Since $g = S_t 1$ we have $0 \leqslant g \leqslant 1$. Moreover $g(t + s, \cdot) = S_{t+s} 1 = S_t S_s 1 \leqslant S_t 1 = g(t, \cdot)$, hence g is decreasing with respect to t and $g_t \leqslant 0$. To prove that $g_r \geqslant 0$ we write

$$g_t = \nu \left(g_{rr} + \frac{M}{\nu} g_r \right) = \nu e^{-\frac{M}{\nu} r} \frac{d}{dr} \left(e^{\frac{M}{\nu} r} g_r \right) \leqslant 0,$$

$r \in (0, \delta)$. Then $e^{\frac{M}{\nu} r} g_r$ is decreasing. Since $g(t, \delta) = 1$ and $0 \leqslant g \leqslant 1$, we have $g_r(t, \delta) \geqslant 0$, hence $g_r \geqslant 0$. Now the identity $g_t = \nu g_{rr} + M g_r$ yields $g_{rr} \leqslant 0$.

Since $(S_t)_{t \geqslant 0}$ is analytic, for $0 < t \leqslant T$ we have $\|D^2 g(t, \cdot)\| \leqslant c_T t^{-1}$, hence $\|D g(t, \cdot)\| \leqslant c_T t^{-1/2}$ and the inequality $g(t, r) \leqslant c_T t^{-1/2} r$ follows, since $g(t, 0) = 0$. □

Proposition 3.2. *Assume Hypothesis* 1.1 *and* (1.6). *Then there exists* $\gamma = \gamma(\nu, M, \delta, T)$ *such that every bounded classical solution* u *of* (1.1), *differentiable with respect to the space variables on* $]0, T[\times \overline{\Omega}$, *satisfies the estimate*

$$|Du(t, \xi)| \leqslant \frac{\gamma}{\sqrt{t}} \|f\|_\infty, \quad t \in (0, T), \ \xi \in \partial\Omega. \tag{3.2}$$

Proof. For each $x \in \Omega_\delta$ let $\xi(x)$ be the unique point in $\partial\Omega$ satisfying $|x - \xi| = r(x)$. Note that

$$x = \xi(x) + \nu(\xi(x)) r(x),$$

where $\nu(\xi)$ is the unit inner normal to $\partial\Omega$ at $\xi \in \partial\Omega$. Recall also that $Dr(x) = \nu(\xi(x))$, $x \in \Omega_\delta$. See Appendix B for these properties of the distance function r. To proceed we remark that, since $u = 0$ on $\partial\Omega$,

$$Du(t, \xi) = \partial_\nu u(t, \xi), \quad \xi \in \partial\Omega, \ t > 0.$$

In order to prove the claim it is enough to show that

$$|w(t, x)| = w(t, x) \leqslant \frac{\gamma}{\sqrt{t}} r(x), \quad t \in (0, T), \ x \in \Omega_\delta, \tag{3.3}$$

where w is the solution to (1.1), corresponding to $f = 1$, and γ depends only on the stated parameters. Indeed, in the general case it is sufficient to observe that, for $x = \xi + r(x)\nu(\xi)$, $\xi \in \partial\Omega$ fixed,

$$\begin{aligned} |P_t f(x) - P_t f(\xi)| &= |P_t f(x)| \leqslant P_t |f|(x) \leqslant \|f\|_\infty P_t 1(x) = \|f\|_\infty w(t, x) \\ &\leqslant \frac{\gamma}{\sqrt{t}} r(x) \|f\|_\infty, \end{aligned}$$

and (3.2) follows easily dividing by r and letting $r \to 0$. To prove (3.3) we compare w with an auxiliary function z, using Theorem A.2. Let

$$z(t, x) = g(t, r(x)), \quad x \in \Omega_\delta,$$

where $g : [0, +\infty) \times [0, \delta] \to \mathbf{R}$ is the solution to (3.1). Now Lemma 3.1 yields

$$|z(t, x)| = g(t, r(x)) \leqslant \frac{\gamma}{\sqrt{t}} r(x), \quad 0 < t < T, \; x \in \Omega_\delta.$$

Thus we have only to prove that

$$w(t, x) \leqslant z(t, x), \quad x \in \Omega_\delta, \; t \in (0, T). \tag{3.4}$$

To verify (3.4), we consider $v = z - w$ in the cylinder $Q_\delta = (0, T) \times \Omega_\delta$. It is clear that v belongs to $C^{1,2}(Q_\delta)$, is continuous in $\overline{Q}_\delta \setminus \partial_{tx} Q_\delta$, bounded on Q_δ and nonnegative on $\partial' Q_\delta \setminus \partial_{tx} Q_\delta$. Moreover

$$v_t - Av = z_t - Az = g_t - vg_{rr} - Mg_r$$

$$+ \left(vg_{rr} + Mg_r - g_{rr} \sum_{i,j=1}^{N} q_{ij} D_i r D_j r - g_r F \cdot Dr - g_r \sum_{i,j=1}^{N} q_{ij} D_{ij} r + Vz \right)$$

$$= g_{rr} \left(v - \sum_{i,j=1}^{N} q_{ij} D_i r D_j r \right) + g_r \left(M - \sum_{i,j=1}^{N} q_{ij} D_{ij} r - F \cdot Dr \right) + Vz \geqslant 0,$$

since $z, g_r \geqslant 0$, $g_{rr} \leqslant 0$. The maximum principle Theorem A.2 now implies (3.4) and concludes the proof. $\square$

The following proposition is an a priori estimate on Du, where u is the bounded classical solution of (1.1). Its importance relies on pointing out the dependence of the constant C below.

Proposition 3.3. *Assume Hypothesis* 1.1, (1.5) *and* (1.7). *Then there exists a constant C depending on $v, h, k, s, \beta, T, \gamma$ with the following property. Every bounded classical solution u of* (1.1) *such that*

(i) *Du belongs to $C^{1,2}(Q)$,*
(ii) *$\sqrt{t}|Du|$ is continuous in $\overline{Q} \setminus \partial_{tx} Q$, bounded in Q and verifies $\lim_{t\to 0} \sqrt{t}|Du(t, x)| = 0$, $x \in \Omega$,*
(iii) *u satisfies* (3.2)

fulfills the estimate

$$\|Du(t, \cdot)\|_\infty \leqslant \frac{C}{\sqrt{t}} \|f\|_\infty, \quad t \in (0, T). \tag{3.5}$$

Proof. Changing V to $V + 1$ (hence u to $e^{-t}u$) we may assume that $|DV| \leqslant \beta V$. We use Bernstein's method and define the function

$$v(t, x) = u^2(t, x) + a\, t\, |Du(t, x)|^2, \quad t \in (0, T), \ x \in \Omega,$$

where $a > 0$ is a parameter to be chosen later. Then we have $v \in C^{1,2}(Q)$, v is continuous in $\overline{Q} \setminus \partial_{tx} Q$, bounded in Q and $v(0, x) = f^2(x)$. We claim that for a suitable value of $a > 0$, depending on v, h, k, s, β, T we have

$$v_t(t, x) - Av(t, x) \leqslant 0, \quad 0 < t < T, \ x \in \Omega. \tag{3.6}$$

This, by Theorem A.2, implies that

$$v(t, x) \leqslant \sup_{x \in \Omega} |v(0, x)| + \sup_{\xi \in \partial\Omega, \, t \in (0,T)} at |Du(t, \xi)|^2 \leqslant (1 + a\gamma^2) \|f\|_\infty^2,$$

$0 < t \leqslant T, \ x \in \Omega$, and (3.5) follows with $C = (a^{-1} + \gamma^2)^{1/2}$.

   To verify inequality (3.6), note that, by a straightforward computation, v satisfies the equation

$$v_t - Av = a|Du|^2 - 2 \sum_{i,j=1}^N q_{ij}\, D_i u\, D_j u + g_1 + g_2,$$

where

$$g_1 = at\left(2 \sum_{i,j=1}^N D_i F_j D_i u D_j u - 2u Du \cdot DV - V|Du|^2\right) - Vu^2,$$

$$g_2 = 2at\left(\sum_{i,j,k=1}^N D_k q_{ij} D_k u D_{ij} u - \sum_{i,j,k=1}^N q_{ij} D_{ik} u D_{jk} u\right).$$

Using the assumptions one has, for all $\varepsilon > 0$, $x \in \Omega$, $t \in (0, T)$,

$$
\begin{aligned}
v_t - Av \ &\leqslant \ (a - 2v + 2akt + at(2s-1)V)|Du|^2 \\
&\quad + 2at(h|Du||D^2u| + \beta V|u||Du| - v|D^2u|^2) - Vu^2 \\
&\leqslant \ (a - 2v + 2akt + at(2s-1)V)|Du|^2 \\
&\quad + at(h\varepsilon^{-1}|Du|^2 + h\varepsilon|D^2u|^2 \\
&\quad + \beta\varepsilon^{-1}Vu^2 + \beta\varepsilon V|Du|^2 - 2v|D^2u|^2) - Vu^2,
\end{aligned}
$$

where $|D^2u|^2 = \sum_{i,j=1}^N |D_{ij}u|^2$. Since $2s < 1$, choosing ε and a small enough we get immediately (3.6). $\square$

4. An auxiliary problem

In this section, we keep Hypothesis 1.1 and condition (1.7) and write our operator in divergence form

$$A = A_0 + \sum_{i=1}^{N} G_i D_i - V,$$

where $A_0 = \sum_{i,j=1}^{N} D_i(q_{ij} D_j)$ and $G_i = F_i - \sum_{j=1}^{N} D_j q_{ij}$.

Moreover, we assume that the potential V and the drift G satisfy the inequality

$$|G(x)| \leqslant \sigma V(x)^{1/2} + c_\sigma, \qquad x \in \Omega, \tag{4.1}$$

for some $\sigma > 0$ and show generation of an analytic semigroup in $L^p(\Omega)$, for $\sigma < \min\{2v(p-1), 2\}$. We follow the ideas of [3,4,14] where the situation $\Omega = \mathbf{R}^N$ is considered.

For simplicity, we assume throughout this section that $2 \leqslant p < \infty$. Observe that, since $q_{ij} \in C_b^1(\Omega)$, condition (4.1) holds equivalently for F or G with the same constant σ, possibly with a different choice of c_σ.

We endow A with the domain

$$D_p = \{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : Vu \in L^p(\Omega)\}$$

which is a Banach space when endowed with the norm

$$\|u\|_{D_p} = \|u\|_{W^{2,p}(\Omega)} + \|Vu\|_{L^p(\Omega)}$$

and remark that the set

$$D = \{u \in C^\infty(\overline{\Omega}) : u_{|\partial\Omega} = 0, \operatorname{supp} u \text{ compact in } \overline{\Omega}\}$$

is dense in D_p.

We need the following interpolative lemma which is analogous to [14, Proposition 2.3].

Lemma 4.1. *Assume Hypothesis* 1.1 *and that condition* (1.7) *hold. Then there exists C depending on N, p, β and the coefficients (q_{ij}) such that for every $0 < \varepsilon < 1$ and $u \in D_p$, $2 \leqslant p < \infty$, the following inequality holds*:

$$\|V^{1/2} Du\|_p \leqslant \varepsilon \|A_0 u\|_p + C\varepsilon^{-1}(\|u\|_p + \|Vu\|_p).$$

Proof. It suffices to establish the inequality above for functions $u \in D$. Moreover, changing V with $V + 1$, we may assume that $|DV| \leqslant \beta V \leqslant \beta V^{3/2}$.

Integrating by parts and using the fact that $u = 0$ on $\partial\Omega$ and $p \geqslant 2$ we have

$$
\begin{aligned}
\int_\Omega V^{\frac{p}{2}} |D_k u|^p &= \int_\Omega V^{\frac{p}{2}} |D_k u|^{p-2} D_k u D_k u \\
&= -\frac{p}{2} \int_\Omega V^{\frac{p}{2}-1} D_k V u |D_k u|^{p-2} D_k u - (p-1) \int_\Omega V^{\frac{p}{2}} u |D_k u|^{p-2} D_{kk} u \\
&\leqslant \frac{\beta p}{2} \int_\Omega |u| |D_k u|^{p-1} V^{\frac{p-1}{2}} V + (p-1) \int_\Omega V^{\frac{p-2}{2}} |D_k u|^{p-2} V |u| |D_{kk} u| \\
&\leqslant \frac{\beta p}{2} \Big(\int_\Omega V^{\frac{p}{2}} |D_k u|^p \Big)^{1-1/p} \Big(\int_\Omega V^p |u|^p \Big)^{1/p} \\
&\quad + (p-1) \Big(\int_\Omega V^{\frac{p}{2}} |D_k u|^p \Big)^{1-2/p} \Big(\int_\Omega V^p |u|^p \Big)^{1/p} \Big(\int_\Omega |D_{kk} u|^p \Big)^{1/p}.
\end{aligned}
$$

Setting $x = \|V^{1/2} D_k u\|_p$, $y = \|Vu\|_p$, $z = \|D_{kk} u\|_p$ we have obtained $x^2 \leqslant (\beta p)/2xy + (p-1)yz$, hence

$$
x \leqslant \frac{\beta p}{2} y + \sqrt{(p-1)yz} \leqslant C\varepsilon^{-1} y + \varepsilon z
$$

for $\varepsilon < 1$, with C depending on β, p and the statement follows with $\|D^2 u\|_p$ instead of $\|A_0 u\|_p$. To complete the proof it suffices to use the closedness of A_0 on $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. □

Proposition 4.2. *Assume Hypothesis* 1.1, *condition* (1.7) *and suppose that* (4.1) *holds with σ satisfying $\sigma < \min\{2v(p-1), 2\}$. Then (A, D_p) is closed in $L^p(\Omega)$, $2 \leqslant p < \infty$. Moreover, there is a constant λ_0 depending on c_σ with the following property: for every $\lambda > \lambda_0$ there exist C_1, C_2 depending only on $\lambda, N, p, \beta, \sigma, c_\sigma$ and the coefficients (q_{ij}), such that for every $u \in D_p$*

$$
\|u\|_{D_p} \leqslant C_1 \|\lambda u - Au\|_p \leqslant C_2 \|u\|_{D_p}.
$$

Finally, if $c_\sigma = 0$, then $\lambda_0 = 0$ and the inequality $\lambda\|u\|_p \leqslant \|(\lambda - A)u\|_p$ holds.

Proof. By density we may assume that $u \in D$. The right-hand side of the above inequality follows immediately from Lemma 4.1, since $|G| \leqslant \sigma V^{1/2} + c_\sigma$.

Changing V with $V + \omega$ for a suitable large ω, we may assume that $c_\sigma = 0$ and that $|DV| \leqslant \beta V$.

Let us multiply the identity $f = \lambda u - Au$ by $u|u|^{p-2}$. Integrating over Ω we get, since $u = 0$ on $\partial\Omega$,

$$
\int_\Omega (\lambda + V)|u|^p + (p-1) \int_\Omega \sum_{i,j=1}^N q_{ij} |u|^{p-2} D_i u D_j u \leqslant \|f\|_p \|u\|_p^{p-1}
$$

$$
+ \sigma \int_\Omega V^{1/2} |Du| |u|^{p-1}.
$$

The last term can be estimated with

$$\sigma\left(\int_\Omega V|u|^p\right)^{1/2}\left(\int_\Omega |u|^{p-2}|Du|^2\right)^{1/2}\leqslant \frac{\sigma}{2}\left(\int_\Omega V|u|^p+|u|^{p-2}|Du|^2\right).$$

Since $\sigma<\min\{2v(p-1),2\}$ we easily obtain, for $\lambda>0$, $\lambda\|u\|_p\leqslant\|f\|_p$. To estimate Vu we observe that

$$\int_\Omega (A_0u)V^{p-1}u|u|^{p-2}=-\sum_{i,j=1}^N\int_\Omega q_{ij}D_iuD_j\left(V^{p-1}u|u|^{p-2}\right)$$

$$=-(p-1)\int_\Omega\sum_{i,j=1}^N q_{ij}V^{p-1}|u|^{p-2}D_iuD_ju$$

$$-(p-1)\int_\Omega\sum_{i,j=1}^N q_{ij}V^{p-2}u|u|^{p-2}D_iuD_jV.$$

Multiplying the identity $\lambda u-Au=f$ by $V^{p-1}u|u|^{p-2}$ and integrating over Ω we obtain

$$\int_\Omega (\lambda V^{p-1}+V^p)|u|^p+v(p-1)\int_\Omega V^{p-1}|u|^{p-2}|Du|^2$$

$$\leqslant\int_\Omega (\lambda V^{p-1}+V^p)|u|^p+(p-1)\int_\Omega V^{p-1}|u|^{p-2}q(Du,Du)$$

$$=-(p-1)\int_\Omega V^{p-2}u|u|^{p-2}q(Du,DV)+\int_\Omega V^{p-1}u|u|^{p-2}G\cdot Du$$

$$+\int_\Omega fV^{p-1}u|u|^{p-2},$$

where $q(Du,DV)=\sum_{i,j=1}^N q_{ij}D_iuD_jV$ and similarly for $q(Du,Du)$. Next, observe that

$$\left|\int_\Omega V^{p-1}u|u|^{p-2}G\cdot Du\right|\leqslant \sigma\int_\Omega V^{p-1/2}|u|^{p-1}|Du|$$

$$\leqslant\sigma\left(\int_\Omega V^{p-1}|u|^{p-2}|Du|^2\right)^{1/2}\left(\int_\Omega V^p|u|^p\right)^{1/2}$$

$$\leqslant\frac{\sigma}{2}\left(\int_\Omega V^{p-1}|u|^{p-2}|Du|^2+\int_\Omega V^p|u|^p\right)$$

and that, for a suitable K depending only on $\|q_{ij}\|_\infty$,

$$
\begin{aligned}
\int_\Omega |u|^{p-1} V^{p-2} |q(Du, DV)| \;&\leqslant\; K \int_\Omega |u|^{p-1} V^{p-2} |Du||DV| \\
&\leqslant\; K\beta \left(\int_\Omega V^{p-1} |u|^{p-2} |Du|^2 \right)^{1/2} \left(\int_\Omega |u|^p V^{p-1} \right)^{1/2} \\
&\leqslant\; K\beta\varepsilon \left(\int_\Omega V^{p-1} |u|^{p-2} |Du|^2 + \int_\Omega V^p |u|^p \right) \\
&\quad + C_\varepsilon \int_\Omega |u|^p.
\end{aligned}
$$

In the last inequality we have used the inequality $t^{p-1} \leqslant \varepsilon t^p + C_\varepsilon$.

Since $\sigma < \min\{2v(p-1), 2\}$, taking a small ε one concludes that $\|Vu\|_p \leqslant C\|f\|_p$, with C as in the statement.

We now use Lemma 4.1 to estimate the second-order derivatives of u. We have

$$
\begin{aligned}
\|G \cdot Du\|_p \;&\leqslant\; \sigma \|V^{1/2} Du\|_p \leqslant \sigma(\varepsilon \|A_0 u\|_p + C\varepsilon^{-1} \|u\|_p + C\varepsilon^{-1} \|Vu\|_p) \\
&\leqslant\; \sigma(\varepsilon\|f\|_p + \varepsilon\|G \cdot Du\|_p + \varepsilon\|Vu\|_p + \varepsilon\lambda\|u\|_p \\
&\quad + C\varepsilon^{-1}\|u\|_p + C\varepsilon^{-1}\|Vu\|_p)
\end{aligned}
$$

hence, taking a small ε, $\|G \cdot Du\|_p \leqslant C\|f\|_p$ and $\|A_0 u\|_p \leqslant C\|f\|_p$, by difference. Using the closedness of A_0 on $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ given by the Calderon–Zygmund estimates, we get $\|D^2 u\|_p \leqslant C\|f\|_p$, with C as in the statement. $\quad\square$

Proposition 4.3. *Assume Hypothesis* 1.1, *condition* (1.7) *and suppose that* (4.1) *holds with σ satisfying $\sigma < \min\{2v(p-1), 2\}$. Then (A, D_p) generates a semigroup in $L^p(\Omega)$, $2 \leqslant p < \infty$.*

Proof. As in the proof of Proposition 4.2, we may assume that $c_\sigma = 0$, $|DV| \leqslant \beta V$, so that $\lambda\|u\|_p \leqslant \|\lambda u - Au\|_p$ for $\lambda > 0$. By the Lumer–Phillips theorem it suffices to show $\lambda - A$ is surjective for $\lambda > 0$.

Setting for $\varepsilon > 0$

$$
V_\varepsilon = \frac{V}{1 + \varepsilon V}, \qquad G_\varepsilon = \frac{G}{\sqrt{1 + \varepsilon V}},
$$

it is immediate to check that $V_\varepsilon, G_\varepsilon$ satisfy

$$
|DV_\varepsilon| \leqslant \beta V_\varepsilon, \qquad |G_\varepsilon| \leqslant \sigma V_\varepsilon^{1/2}.
$$

Since V_ε, G_ε are bounded, the operator $A_\varepsilon = A_0 + G_\varepsilon \cdot D - V_\varepsilon$ with domain $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ generates an analytic semigroup in $L^p(\Omega)$ see [12, Theorem 3.1.3], which is contractive by Proposition 4.2.

Given $f \in L^p(\Omega)$, let $u_\varepsilon \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $(\lambda - A_\varepsilon)u_\varepsilon = f$. By Proposition 4.2, $\|u_\varepsilon\|_{2,p}$, $\|V_\varepsilon u_\varepsilon\|_p \leqslant C\|f\|_p$ with C independent of ε. By weak compactness we find $\varepsilon_n \to 0$ such that (u_{ε_n}) converges weakly to a function u in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and strongly in $W_{\mathrm{loc}}^{1,p}(\Omega)$. Moreover we may assume that $(u_{\varepsilon_n}) \to u$ a.e. in Ω. By Fatou's lemma $\|Vu\|_p \leqslant C\|f\|_p$, hence $u \in D_p$ and it is easy to check that $(\lambda - A)u = f$. $\square$

Let us show that the above semigroup is analytic.

Theorem 4.4. *Assume Hypothesis* 1.1, *condition* (1.7) *and suppose that* (4.1) *holds with* σ *satisfying* $\sigma < \min\{2v(p-1), 2\}$. *Then* (A, D_p) *generates an analytic semigroup in* $L^p(\Omega)$, $2 \leqslant p < \infty$.

Proof. We keep the same notation of the proof of Proposition 4.2. We may assume that $c_\sigma = 0$. Let $u \in D$ and set $u^* := \overline{u}|u|^{p-2}$. Integrating by parts, since $u = 0$ on $\partial\Omega$, a lengthy but straightforward computation yields

$$-Re\left(\int_\Omega (Au)u^*\right) = (p-1)\int_\Omega |u|^{p-4}q(Re(\overline{u}Du), Re(\overline{u}Du))$$

$$+ \int_\Omega |u|^{p-4}q(Im(\overline{u}Du), Im(\overline{u}Du)) - \int_\Omega G \cdot Re(\overline{u}Du)|u|^{p-2} + \int_\Omega V|u|^p$$

and

$$\left|Im\int_\Omega (Au)u^*\right| \leqslant (p-2)\int_\Omega |u|^{p-4}q(Re(\overline{u}Du), Im(\overline{u}Du)) + \int_\Omega |G||u|^{p-2}|Im(\overline{u}Du)|.$$

Condition (4.1) implies

$$\int_\Omega |G||u|^{p-2}|Im(\overline{u}Du)| \leqslant \sigma \int_\Omega V^{\frac{1}{2}}|Im(\overline{u}Du)||u|^{\frac{p}{2}}|u|^{\frac{p-4}{2}}$$

$$\leqslant \sigma\left(\int_\Omega V|u|^p\right)^{\frac{1}{2}}\left(\int_\Omega |u|^{p-4}|Im(\overline{u}Du)|^2\right)^{\frac{1}{2}}$$

$$\leqslant \frac{\sigma}{\sqrt{v}}\left(\int_\Omega V|u|^p\right)^{\frac{1}{2}}\left(\int_\Omega |u|^{p-4}q(Im(\overline{u}Du), Im(\overline{u}Du))\right)^{\frac{1}{2}}$$

and

$$\int_\Omega |G||u|^{p-2}|Re(\overline{u}Du)| \leqslant \sigma \int_\Omega V^{\frac{1}{2}}|Re(\overline{u}Du)||u|^{\frac{p}{2}}|u|^{\frac{p-4}{2}}$$

$$\leqslant \sigma \left(\int_\Omega V|u|^p \right)^{\frac{1}{2}} \left(\int_\Omega |u|^{p-4} |Re(\overline{u} Du)|^2 \right)^{\frac{1}{2}}$$

$$\leqslant \frac{\sigma}{\sqrt{\nu}} \left(\int_\Omega V|u|^p \right)^{\frac{1}{2}} \left(\int_\Omega |u|^{p-4} q(Re(\overline{u} Du), Re(\overline{u} Du)) \right)^{\frac{1}{2}}.$$

If we put $B^2 := \int_\Omega |u|^{p-4} q(Re(\overline{u} Du), Re(\overline{u} Du))$, $C^2 := \int_\Omega |u|^{p-4} q(Im(\overline{u} Du), Im(\overline{u} Du))$, and $D^2 := \int_\Omega V|u|^p$, then we deduce from the previous estimates

$$-Re\left(\int_\Omega (Au)u^* \right) \geqslant \left(p - 1 - \frac{\sigma}{2\nu} \right) B^2 + C^2 + \left(1 - \frac{\sigma}{2} \right) D^2.$$

Therefore,

$$\left| Im\left(\int_\Omega (Au)u^* \right) \right| \leqslant (p-2)BC + \frac{\sigma}{\sqrt{\nu}} CD$$

and one can find $\kappa > 0$ such that

$$\left| Im\left(\int_\Omega (Au)u^* \right) \right| \leqslant \kappa \left[-Re\left(\int_\Omega (Au)u^* \right) \right]$$

for every $u \in D$ and, by density, for every $u \in D_p$. Since we already know that (A, D_p) generates a semigroup, by [15, Theorem 3.9, Chapter I] the proof is complete. $\square$

Remark 4.5. Observe that all the results proved until now, in this section (but not the next lemma), hold assuming less local regularity on the coefficients. For example $q_{ij} \in C_b^1(\Omega)$, $F \in L_{loc}^\infty(\Omega)$, $V \in C^1(\Omega)$ suffice. Moreover, the existence of the Lyapunov function ϕ is not necessary.

We call $(T_t)_{t \geqslant 0}$ the semigroup generated by A in $L^p(\Omega)$. For the proof of our main result we need some regularity results of the function $u(t, x) = (T_t f)(x)$.

Lemma 4.6. *Assume that the conditions of Theorem 4.4 hold for a fixed $p > N+1$ and let $f \in C_0^\infty(\Omega)$. Then the function $u(t, x) = (T_t f)(x)$ is the bounded classical solution of problem* (1.1) *and therefore has the regularity properties stated in Proposition* 2.1. *Moreover, Du is continuous and bounded in $\overline{Q}$.*

Proof. Since $f \in D_p$, the function $t \to T_t f$ is continuous from $[0, T]$ to $W^{2,p}(\Omega)$ and Sobolev embedding implies that u, Du are bounded and continuous in $\overline{Q}$. To complete the proof, we have to show that $u \in C^{1,2}(Q)$.

Let us fix $\varepsilon > 0$ and open bounded sets Ω_1, Ω_2 such that $\overline{\Omega}_1 \subset \Omega_2$ and $\overline{\Omega}_2 \subset \Omega$. Since $(T_t)_{t \geqslant 0}$ is analytic, u is continuously differentiable from $[\varepsilon, T]$ to $W^{2,p}(\Omega)$ and

Sobolev embedding yields $u_t \in C(Q)$. Set

$$\kappa = \sup_{\varepsilon \leqslant t \leqslant T} \left(\|u(t, \cdot)\|_{W^{2,p}(\Omega)} + \|u_t(t, \cdot)\|_{W^{2,p}(\Omega)} \right).$$

For every fixed $t \in [\varepsilon, T]$ the function $u(t, \cdot)$ belongs to $W^{2,p}(\Omega)$ and solves the equation

$$\sum_{i,j=1}^{N} q_{ij} D_{ij} u = -F \cdot Du + Vu - u_t$$

in Ω. Since the right-hand side belongs to $W_{\text{loc}}^{1,p}(\Omega)$ it follows that $u(t, \cdot) \in W_{\text{loc}}^{3,p}(\Omega)$ and that, for a suitable c depending on Ω_1, Ω_2 and the coefficients of A,

$$\sup_{\varepsilon \leqslant t \leqslant T} \|u(t, \cdot)\|_{W^{3,p}(\Omega_1)} \leqslant c\kappa,$$

see [7, Theorem 9.19]. We have thus proved that for every $i, j = 1, \ldots, N$, $D_t D_{ij} u$, $D D_{ij} u \in L^p([\varepsilon, T] \times \Omega_1)$. By Sobolev embedding, since $p > N + 1$, $D_{ij} u \in C(Q)$ and the proof is complete. $\quad\square$

5. Proof of Theorem 1.3

For $\varepsilon > 0$ let $V_\varepsilon(x) = \varepsilon \exp\{4c_2\sqrt{1 + |x|^2}\}$. Then $|DV_\varepsilon| \leqslant 4c_2 V_\varepsilon$ and for every $\sigma > 0$ there exists $c_\sigma > 0$ (depending on ε) such that $|F| \leqslant \sigma(V + V_\varepsilon)^{1/2} + c_\sigma$. Define $A_\varepsilon = A - V_\varepsilon$ and note that the hypotheses of Theorem 4.4 are satisfied.

Fix $p > N + 1$, $f \in C_0^\infty(\Omega)$ and let u_ε be the semigroup solution of (1.1) with A_ε instead of A, given by Theorem 4.4. By Lemma 4.6 the function u_ε is the bounded solution of the above problem and Du_ε is continuous and bounded in $\overline{Q}$. By Proposition 3.2 we deduce that $|Du_\varepsilon(t, \xi)| \leqslant (\gamma/\sqrt{t})\|f\|_\infty$, $\xi \in \partial\Omega$, with γ depending on v, M, δ, T and independent of ε.

Since u_ε satisfies the hypotheses of Proposition 3.3, we deduce that

$$\|Du_\varepsilon(t, \cdot)\|_\infty \leqslant (C/\sqrt{t})\|f\|_\infty,$$

with C as in the statement.

Observe that $\|u_\varepsilon\|_\infty \leqslant \|f\|_\infty$. Let us fix $R > 0$ and note that the C^α-norm of the coefficients of A_ε is bounded, uniformly with respect to $\varepsilon < 1$, in $\Omega \cap B_{R+1}$. By the local Schauder estimates [10, Theorem IV.10.1] applied to the operator $D_t - A_\varepsilon$, there exists a constant C, independent of $\varepsilon < 1$, such that

$$\begin{aligned}
\|u_\varepsilon\|_{C^{1+\alpha/2, 2+\alpha}((0,T) \times (\Omega \cap B_R))} &\leqslant C \left(\|u_\varepsilon\|_{C((0,T) \times (\Omega \cap B_{R+1}))} \right) + \|f\|_{C^{2+\alpha}(\Omega \cap B_{R+1})} \\
&\leqslant 2C\|f\|_{C^{2+\alpha}(\Omega)}.
\end{aligned}$$

By a standard compactness argument we conclude that a subsequence (u_{ε_n}) converges in $C^{1,2}([0, T] \times \overline{(\Omega \cap B_R)})$ for every R to a function u which is the bounded classical solution of (1.1) and satisfies $\|Du(t, \cdot)\|_\infty \leqslant (C/\sqrt{t})\|f\|_\infty$.

Finally, to treat the general case of $f \in C_b(\Omega)$ we consider a sequence $(f_n) \subset C_0^\infty(\Omega)$ convergent to f uniformly on compact subsets of Ω and such that $\|f_n\|_\infty \leqslant \|f\|_\infty$. Let u_n be the bounded classical solution of (1.1) relative to f_n. Then $\|Du_n(t, \cdot)\|_\infty \leqslant (C/\sqrt{t})\|f\|_\infty$, by the previous step. Since $(u_n) \to u$ in $C^{1,2}(Q)$, see the proof of Proposition 2.2, the estimate for Du follows. $\square$

6. Examples and applications

We first show that gradient estimates fail, in general, if condition (1.6) is not satisfied. We refer the reader to [1, Example 5.6] for an operator defined on the whole space, for which condition (1.5) is violated and gradient estimates fail. The following result refines and generalizes an example in [24].

Example 6.1. We consider the following Dirichlet problem in $\Omega = \mathbf{R}_+^2 = \{(x, y) \in \mathbf{R}^2, x > 0\}$

$$\begin{cases} u_t(t, x, y) = u_{xx}(t, x, y) + u_{yy}(t, x, y) + g(y)u_x(t, x, y), & t > 0, \ (x, y) \in \Omega, \\ u(t, 0, y) = 0, & t > 0, \ y \in \mathbf{R}, \\ u(0, x, y) = 1, & (x, y) \in \Omega, \end{cases}$$

where $g \in C^2(\mathbf{R})$ and

$$\lim_{y \to +\infty} g(y) = +\infty.$$

Observe that (1.6) fails. However, Proposition 2.2 yields existence and uniqueness of a bounded solution u. Let us show that, for $t > 0$, $u(t, \cdot)$ is *not* uniformly continuous in Ω. To this end, it is enough to show that, for every $t, x > 0$,

$$\sup_{y > 0} u(t, x, y) = 1. \tag{6.1}$$

Fix $n > 0$ and take c_n such that $g(y) \geqslant n$ for $y \geqslant c_n$. Define $R_n = (0, +\infty) \times (c_n, +\infty)$ and consider $v = v_n$ which solves

$$\begin{cases} v_t(t, x, y) = v_{xx}(t, x, y) + v_{yy}(t, x, y) + nv_x(t, x, y), & t > 0, \ (x, y) \in R_n, \\ v(t, z) = 0, & t > 0, \ z \in \partial R_n, \\ v(0, x, y) = 1, & (x, y) \in R_n, \end{cases}$$

We prove that for $t, x > 0$

(i) $\displaystyle\lim_{n \to \infty} \sup_{y > c_n} v_n(t, x, y) = 1;$ (ii) $u(t, x, y) \geqslant v_n(t, x, y).$

Clearly (i) and (ii) give (6.1). Let us verify (i). Note that $v_n(t, x, y) = a_n(t, x)b_n(t, y)$, where $a = a_n$, $b = b_n$ solve, respectively,

$$
\begin{cases}
a_t(t, x) = a_{xx}(t, x) + na_x(t, x), & t > 0, \\
a(t, 0) = 0, & t > 0, \\
a(0, x) = 1, & x > 0,
\end{cases}
\qquad
\begin{cases}
b_t(t, y) = b_{yy}(t, y), & t > 0, \\
b(t, c_n) = 0, & t > 0, \\
b(0, y) = 1, & y > c_n.
\end{cases}
$$

To find an explicit formula for a_n, we first remark that $a_n(t, x) = a_1(n^2 t, nx)$. Then, setting $v(t, x) = e^{x/2}e^{\frac{1}{4}t}a_1(t, x)$, v solves

$$
\begin{cases}
v_t(t, x) = v_{xx}(t, x), & t > 0, \ x > 0, \\
v(t, 0) = 0, & t > 0, \\
v(0, x) = e^{x/2}, & x > 0;
\end{cases}
$$

By a reflection argument we get easily an explicit expression for v and finally we obtain for any $t > 0$, $y \geqslant c_n$, $x \geqslant 0$,

$$
a_n(t, x) = \frac{e^{\frac{-n^2 t}{4}}}{n\sqrt{4\pi t}} \int_0^{+\infty} \left(e^{-\frac{|nx-z|^2}{4n^2 t}} - e^{-\frac{|nx+z|^2}{4n^2 t}} \right) e^{\frac{z-nx}{2}} dz, \quad b_n(t, y) = \int_0^{y-c_n} \frac{e^{\frac{-z^2}{4t}}}{\sqrt{\pi t}} dz.
$$

To check that (i) holds we write

$$
a_n(t, x) = A_n^1(t, x) - A_n^2(t, x),
$$

$$
A_n^1(t, x) = \frac{e^{\frac{-n^2 t}{4}}}{n\sqrt{4\pi t}} \int_0^{+\infty} \left(e^{-\frac{|nx-z|^2}{4n^2 t}} \right) e^{\frac{z-nx}{2}} dz,
$$

$$
A_n^2(t, x) = \frac{e^{\frac{-n^2 t}{4}}}{n\sqrt{4\pi t}} \int_0^{+\infty} \left(e^{-\frac{|nx+z|^2}{4n^2 t}} \right) e^{\frac{z-nx}{2}} dz.
$$

Let us consider A_n^1. By a change of variables we obtain

$$
A_n^1(t, x) = \frac{1}{\sqrt{\pi}} \int_{-\frac{x+n}{2\sqrt{t}}}^{+\infty} e^{-s^2} ds,
$$

which is increasing in x and converges to 1 as $n \to +\infty$. In a similar way we get that $A_n^2(t, x)$ is decreasing in x and converges to 0 as $n \to +\infty$. Then (i) easily follows.

To prove (ii) we use Theorem A.2. Set $w = u - v_n$ in $(0, T) \times R_n$. We have $w(0, x, y) = 0$, $(x, y) \in R_n$. Moreover $w(t, z) \geqslant 0$, $z \in \partial R_n$, $t > 0$. To conclude it suffices to verify that

$$
w_t(t, x, y) \geqslant \Delta w(t, x, y) + g(y)w_x(t, x, y), \quad t > 0, \ (x, y) \in R_n. \tag{6.2}
$$

Since $w_t = \Delta w + g(y)w_x + [g(y) - n](v_n)_x$, $g(y) \geqslant n$, for $y \geqslant c_n$ and $(v_n)_x(t, x, y) = (a_n)_x(t, x)b_n(t, y) \geqslant 0$, $t > 0$, $(x, y) \in R_n$, as verified above, (6.2) follows and the proof is complete.

For instance, we can take, in the above example, $g(y) = \sqrt{1 + y^2}$. On the other hand, if $g(y) = -\sqrt{1 + y^2}$ then all the conditions of Theorem 1.3 hold and gradient estimates hold.

Remark 6.2. We point out that our main result can be used to prove some boundary gradient estimates for solutions of Dirichlet elliptic problems, involving the operator A. Indeed if $\phi \in C_b(\Omega) \cap C^2(\Omega)$ solves

$$\begin{cases} A\phi(x) = 0, & x \in \Omega, \\ \phi(\xi) = 0, & \xi \in \partial\Omega, \end{cases} \tag{6.3}$$

then ϕ is the bounded classical solution of (1.1) with $f = \phi$. Thus, under the assumptions of Theorem 1.3, we get

$$\sup_{x \in \overline{\Omega}} |D\phi(x)| \leqslant C\|\phi\|_\infty.$$

This extends some classical boundary gradient estimates concerning linear and nonlinear second elliptic operators, involving bounded coefficients, see for instance [7, Section 14].

Remark 6.3. Theorem 1.3 has also some applications to isoperimetric inequalities, see [11,24].

Acknowledgments

Appendix A. Two maximum principles

We prove the following maximum principle for unbounded domains, using the Lyapunov function ϕ introduced in Hypothesis 1.1. The proof is similar to the one given in [13], see also [9, Chapter 8]. We recall that $Q = (0, T) \times \Omega$.

Proposition A.1. *Assume Hypothesis* 1.1. *Let* $u \in C(\overline{Q}) \cap C^{1,2}(Q)$ *be a bounded function satisfying*

$$\begin{cases} u_t(t, x) \leqslant Au(t, x), & 0 < t \leqslant T, \ x \in \Omega, \\ u(t, x) \leqslant 0, & 0 < t \leqslant T, \ x \in \partial\Omega, \\ u(0, x) \leqslant 0 & x \in \Omega, \end{cases}$$

Then $u \leqslant 0$.

Proof. Set $v = e^{-\lambda_0 t} u$. We prove that $v \leqslant 0$ in Q. To this end define

$$v_n(t, x) = v(t, x) - \frac{\phi(t, x)}{n}, \quad (t, x) \in Q.$$

It is easy to see that $(D_t - A + \lambda_0)v_n \leqslant 0$ in Q. Moreover $v_n(0, x) \leqslant 0$, $v_n(t, \xi) \leqslant 0$, $t > 0$, $x \in \Omega$, $\xi \in \partial\Omega$.

The function v_n has a maximum over $\overline{Q}$ in (t_n, x_n). Assume, by contradiction, that this maximum is positive. Then $0 < t_n \leqslant T$ and $x_n \in \Omega$ and therefore $(D_t - A + \lambda_0)v_n(t_n, x_n) > 0$, which is absurd. It follows that $v_n \leqslant 0$ and, letting $n \to \infty$, we infer that $v \leqslant 0$. □

Next we present a maximum principle for discontinuous solutions to parabolic problems. The result is suggested in [9] and involves special domains.

Theorem A.2. *Assume Hypothesis* 1.1. *Let Ω be an open subset of $\mathbf{R}^N$, $g_i : \overline{\Omega} \to \mathbf{R}$, $i = 1, \ldots, n$, be C^2-functions. Suppose that*

$$\Omega = \{x : g_i(x) > 0, \ i = 1, \ldots, n\}, \quad |Dg_i| \geqslant 1 \ on \ \Gamma_i = \partial\Omega \cap \{g_i = 0\}.$$

Let $u \in C^{1,2}(Q)$, u continuous on $\overline{Q} \setminus \partial_{tx} Q$, bounded on Q. If $u_t \leqslant Au$ in Q and $u \leqslant 0$ in $\partial' Q \setminus \partial_{tx} Q$, then $u \leqslant 0$ in Q.

Finally, if $u_t = Au$, $|u(t, \xi)| \leqslant K$ for $t > 0$, $\xi \in \partial\Omega$ and $|u(0, x)| \leqslant K$, $x \in \Omega$, then $\|u\|_\infty \leqslant K$.

Proof. The proof is given into two steps.

Step 1: *We assume in addition that Ω is bounded.* In this case the functions g_i are bounded in Ω together with their derivatives up to the second order. A long but straightforward computation shows that the functions

$$\psi_i(t, x) = \frac{1}{t^{\varepsilon v}} \exp\left(\lambda t - \frac{\varepsilon g_i^2(x)}{t}\right) \tag{A.1}$$

verify, for $\varepsilon > 0$ small enough and λ large enough, $(D_t - A)\psi_i \geqslant 0$, $i = 1, \ldots, n$, in $(0, \infty) \times \Omega$.

Let $M = \sup_Q u = \sup_{\overline{Q} \setminus \partial_{tx} Q} u > 0$ (otherwise the proof is finished). Let $\gamma > 0$ and define

$$u_\gamma(t, x) = u(t, x) - M\gamma^{\varepsilon v} \sum_{i=1}^{n} \frac{1}{(t + \gamma)^{\varepsilon v}} \exp\left(\lambda(t + \gamma) - \frac{\varepsilon g_i^2(x)}{t + \gamma}\right),$$

where ε and λ are given in (A.1). Clearly $(D_t - A)u_\gamma \leqslant 0$. Take $\eta > 0$ such that $\lambda\gamma - \frac{\varepsilon\eta}{\gamma} > 0$ and consider

$$I_\eta = \{x \in \overline{\Omega} : \exists i = i(x) = 1, \ldots, n : g_i^2(x) \leqslant \eta\}.$$

For each $x \in I_\eta$, one has

$$\gamma^{\varepsilon v} \sum_{i=1}^{n} \frac{1}{\gamma^{\varepsilon v}} \exp\left(\lambda \gamma - \frac{\varepsilon g_i^2(x)}{\gamma}\right) \geqslant \exp\left(\lambda \gamma - \frac{\varepsilon \eta}{\gamma}\right) > 1.$$

By continuity, there exists $\delta > 0$ such that for any $(t, x) \in [0, \delta] \times I_\eta$,

$$\gamma^{\varepsilon v} \sum_{i=1}^{n} \frac{1}{(t+\gamma)^{\varepsilon v}} \exp\left(\lambda(t+\gamma) - \frac{\varepsilon g_i^2(x)}{t+\gamma}\right) > 1.$$

It follows that $u_\gamma \leqslant M - M = 0$ in $([0, \delta] \times I_\eta) \setminus \partial_{tx} Q$.

Since $u(0, x) \leqslant 0$, $x \in \Omega \setminus I_\eta$, we have $u_\gamma(0, x) < 0$, $x \in \Omega \setminus I_\eta$ as well. Because Ω is bounded, by continuity we obtain $u_\gamma(t, x) \leqslant 0$, $(t, x) \in [0, \delta] \times \Omega \setminus I_\eta$, for some $\delta > 0$.

We have obtained that $u_\gamma \leqslant 0$ in $([0, \delta] \times \overline{\Omega}) \setminus \partial_{tx} Q$. Applying the classical maximum principle in $[\delta, T] \times \overline{\Omega}$, we get that $u_\gamma \leqslant 0$ in $\overline{Q} \setminus \partial_{tx} Q$. Letting $\gamma \to 0^+$, we infer the claim.

Step 2: *We consider a possibly unbounded Ω.* Here we will use the Lyapunov function ϕ. Set $v = e^{-\lambda_0 t} u$ and observe that $v_t - Av + \lambda_0 v \leqslant 0$. We prove that $v \leqslant 0$ in Q. Fix $R > 1$ and consider

$$\Omega_R = \Omega \cap B_R = \{g_i > 0\} \cap \{R^2 - |x|^2 > 0\}, \quad Q_R = (0, T) \times \Omega_R.$$

Note that Ω_R satisfies the same geometric assumptions of Ω if one adds to the set $\{g_1, \ldots, g_n\}$ the function $g_0(x) = R^2 - |x|^2$. Let $C_R = \inf_{[0,T] \times (\partial B_R \cap \Omega)} \phi$. Remark that $C_R \to \infty$ as $R \to \infty$. Define

$$v_R(t, x) = v(t, x) - \|v\|_\infty \frac{\phi(t, x)}{C_R}, \quad (t, x) \in Q_R.$$

It is easy to see that $(D_t - A + \lambda_0) v_R \leqslant 0$ in Q_R. Moreover $v_R(0, x) \leqslant 0$, $x \in \Omega_R$.

If $t \in (0, T)$, then $v_R(t, x) \leqslant 0$ for $x \in \partial B_R \cap \Omega$, since $\frac{\phi}{C_R} \geqslant 1$. Moreover $v_R(t, x) \leqslant 0$ for $x \in \partial \Omega$, $t \in (0, T)$. This shows that $v_R \leqslant 0$ on the parabolic boundary of Q_R.

Applying Step 1 to the operator $\tilde{A} = A - \lambda_0$ in Ω_R, we get $v_R \leqslant 0$, in Q_R, that is

$$v(t, x) \leqslant \|v\|_\infty \frac{\phi(t, x)}{C_R}.$$

Letting $R \to \infty$, we get the claim.

The last statement easily follows considering the functions $\pm u - K$. $\quad \square$

Observe that the above theorem covers also the case of certain non smooth domains, whose boundaries can be described by a finite number of functions g_i as in the statement, see e.g. Example 6.1.

Let us show that uniformly C^2 domains are covered by Theorem A.2.

Corollary A.3. . *Theorem* A.2 *holds for uniformly C^2-domains.*

Proof. It suffices to show that there exists a C^2-function $g : \overline{\Omega} \to \mathbf{R}$ such that $g > 0$ in Ω, $|Dg| \geqslant 1$ in $\partial\Omega = \{g = 0\}$. Let r be the distance function from $\partial\Omega$. Then $r \in C^2(\Omega_\delta)$ for some $\delta > 0$ and $|Dr| = 1$ on $\partial\Omega$. Let moreover η be a smooth function such that $0 \leqslant \eta \leqslant 1$, $\eta = 1$ in $\Omega_{\delta/2}$, $\eta = 0$ outside Ω_δ. It is easy to check that $g = \eta r + 1 - \eta$ satisfies the claim. $\square$

Appendix B. Some properties of the distance function

In this appendix, we collect some regularity results of the distance function $r(x) = dist(x, \partial\Omega)$, when $\partial\Omega$ is the boundary of a smooth open subset Ω of $\mathbf{R}^N$. These results are well-known in the case where Ω is bounded (see e.g. [7, Section 14.6]), but most of them may be extended, without much effort, to the unbounded case, as it is shown below.

First we define open sets with uniformly $C^{2+\alpha}$ boundaries, for $0 \leqslant \alpha < 1$.

Definition B.1. Let Ω be an open subset of $\mathbf{R}^N$. We say that $\partial\Omega$ is uniformly of class $C^{2+\alpha}$ if there exist a covering of $\partial\Omega$, at most countable, $\{U_j\}_{j \in \mathbf{N}}$, and a sequence of diffeomorphisms $\phi_j : \overline{U}_j \to \overline{B}_1$ of class $C^{2+\alpha}$ such that

$$\phi_j(U_j \cap \Omega) = \{y \in B_1 \mid y_N > 0\},$$

$$\phi_j(U_j \cap \partial\Omega) = \{y \in B_1 \mid y_N = 0\}$$

and the following properties are satisfied:

(i) there exists $k \in \mathbf{N}$ such that $\bigcap_{j \in J} U_j = \emptyset$, if $|J| > k$;
(ii) there exists $0 < \varepsilon < 1$ such that $\{x \in \Omega \mid r(x) < \varepsilon\} \subseteq \bigcup_{j \in \mathbf{N}} V_j$, where $V_j = \phi_j^{-1}(B_{1/2})$;
(iii) there exists $C > 0$ such that

$$\sup_{j \in \mathbf{N}} \sum_{0 \leqslant |\beta| \leqslant 2+\alpha} \|D^\beta \phi_j\|_\infty + \|D^\beta \phi_j^{-1}\|_\infty \leqslant C.$$

Now we show that such a set Ω satisfies a *uniform interior sphere condition*, i.e. at each point $y_0 \in \partial\Omega$ there exists a ball B_{y_0} depending on y_0, contained in Ω and such that $\overline{B}_{y_0} \cap \partial\Omega = \{y_0\}$; moreover the radii of these balls are bounded from below by a positive constant.

Proposition B.2. *If $\partial\Omega$ is uniformly of class C^2, then it satisfies a uniform interior sphere condition.*

Proof. Using condition (iii) and taking into account that ϕ_j is a diffeomorphism from $\overline{U}_j$ into $\overline{B}_1$, it is easy to see that if $y \in V_j$ and $|x - y| < 1/(2C)$, then $x \in U_j$.

Let $y_0 \in \partial\Omega$ and let $v(y_0)$ denote the unit inward normal vector to $\partial\Omega$ at y_0. For $0 \leqslant t < 1/(2C)$ the point $x = y_0 + tv(y_0)$ belongs to U_j and ($\phi_j^{(N)}$ denotes the Nth component of ϕ_j)

$$\phi_j^{(N)}(x) = t D\phi_j^{(N)}(y_0) \cdot v(y_0) + R(t)$$

with $|R(t)| \leqslant Ct^2/2$. Since $\phi_j^{(N)} = 0$ on $U_j \cap \partial\Omega$, then $D\phi_j^{(N)}(y_0) = kv(y_0)$, with $k \geqslant C^{-1}$, by (iii). This yields $\phi_j^{(N)}(x) \geqslant tC^{-1} - Ct^2/2 > 0$ for $0 < t < 2/C^3 := \delta$.

Thus, we have proved that

$$y + tv(y) \in \Omega, \qquad y \in \partial\Omega, \ t \in]0, \delta[.$$

Now, let $y \in \partial\Omega$ and set $B = B(z, \delta/2)$, where $z = y + v(y)\delta/2$. Then, it is easy to see that $B \subset \Omega$ and $y \in \partial B$. If y is not the unique point in $\partial\Omega \cap \partial B$, then it suffices to replace the above ball with that of radius $\delta/4$, centered at $z = y + v(y)\delta/4$. $\square$

We are now ready to prove the properties of the distance function used in this paper.

Proposition B.3. *Assume that $\partial\Omega$ is uniformly of class C^2 and let δ be a positive constant such that at each point of $\partial\Omega$ there exists a ball which satisfies the interior sphere condition at y_0 with radius greater or equal to δ. Then*

(a) *for every $x \in \Omega_\delta = \{y \in \overline{\Omega} \,|\, r(y) < \delta\}$ there exists a unique $\xi = \xi(x) \in \partial\Omega$ such that $|x - \xi| = r(x)$;*
(b) $r \in C_b^2(\Omega_\delta)$;
(c) $Dr(x) = v(\xi(x))$, *for every $x \in \Omega_\delta$.*

Proof. (a) The existence part is obvious. For the uniqueness assertion, let $x \in \Omega_\delta$ and $y \in \partial\Omega$ such that $r(x) = |x - y|$. From Proposition B.2 there exists a ball $B = B(z, \rho)$ such that $B \subset \Omega$ and $\overline{B} \cap \partial\Omega = \{y\}$. Moreover from the definition of δ, $x \in B$. It is easy to see that x and z lie on the normal direction $v(y)$ and that the balls $B(x, r(x))$ and $B(z, \rho)$ are tangent at y. Then $B(x, r(x))$ still verifies the interior sphere condition at y. It follows that for every $\overline{y} \in \partial\Omega \setminus \{y\}$, one has $\overline{y} \notin B(x, r(x))$, so that y is actually the unique point such that $|x - y| = r(x)$.

The proof of the last two assertions relies on the first statement and the implicit function theorem and it is completely similar to that of the case Ω bounded. We refer to [7, section 14.6]. $\square$

References

[1] M. Bertoldi, S. Fornaro, Gradient estimates in parabolic problems with unbounded coefficients, Preprint del Dipartimento di Matematica, Universitá di Pharma, no. 316, 2003, Studia Math, to appear.

[2] S. Cerrai, Second-Order PDE's in Finite and Infinite Dimension, Lecture Notes in Mathematics, Vol. 1762, Springer, Berlin, 2001.

[3] P. Cannarsa, V. Vespri, Generation of analytic semigroups by elliptic operators with unbounded coefficients, SIAM J. Math. Anal. 18 (1987) 857–872.

[4] P. Cannarsa, V. Vespri, Generation of analytic semigroups in the L^p-topology by elliptic operators in $\mathbf{R}^N$, Israel J. Math. 61 (1988) 235–255.

[5] G. Da Prato, B. Goldys, J. Zabczyk, Ornstein–Uhlenbeck semigroups in open sets of Hilbert spaces, C. R. Acad. Sci. Paris, Serie I 325 (1997) 433–438.

[6] G. Da Prato, A. Lunardi, On the Ornstein–Uhlenbeck operator in spaces of continuous functions, J. Funct. Anal. 131 (1995) 94–114.

[7] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd Ed., Springer, Berlin, 1983.

[8] R.Z. Has'minskii, Stochastic stability of differential equations, Sijthoff and Noordhoff, Alphen aan den Rijn-Germantown, 1980.

[9] N.V. Krylov, Lectures on elliptic and parabolic equations in Hölder spaces, American Mathematical Society, Providence, RI, 1996.

[10] O.A. Ladyzhenskaja, V.A. Solonnikov, N.N. Ural'ceva, Linear and quasilinear equations of parabolic type, Nauka, Moskow, 1967 English transl.: American Mathematical Society, Providence, RI, 1968 (in Russian).

[11] M. Ledoux, A simple analytic proof of an inequality by P. Buser, Proc. Amer. Math. Soc., 121 (3) (1994) 951–959.

[12] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, Basel, 1995.

[13] A. Lunardi, Schauder theorems for linear elliptic and parabolic problems with unbounded coefficients in $\mathbf{R}^N$, Studia Math. 128 (2) (1998) 171–198.

[14] G. Metafune, J. Prüss, A. Rhandi, R. Schnaubelt, L^p-regularity for a elliptic operators with unbounded coefficients, Preprint of the Institute of Analysis, Martin-Luther University, Halle-Wittenberg, no. 21, 2002.

[15] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, Berlin, 1983.

[16] E. Priola, A counterexample to Schauder estimates for elliptic operators with unbounded coefficients, Atti dell'Accademia Nazionale dei Lincei di Roma, Classe di Scienze Fisiche, Matematiche e Naturali 12 (9) (2001) 15–25.

[17] E. Priola, Dirichlet problems in a half space of a Hilbert space, Infinite Dimensional Anal. Quantum Probab. Related Topics 2 (5) (2002) 257–291.

[18] E. Priola, On a Dirichlet problem involving an Ornstein–Uhlenbeck operator, Potential Anal. 18 (2003) 251–287.

[19] Z. Qian, A gradient estimate on a manifold with convex boundary, Proc. Roy. Soc. Edinburgh Sect. A 127 (1997) 171–179.

[20] A. Talarczyk, Dirichlet problem for parabolic equations on Hilbert spaces, Studia Math. 141 (2) (2000) 109–142.

[21] A. Thalmaier, F.Y. Wang, Gradient estimates for harmonic functions on regular domains in Riemannian manifolds, J. Funct. Anal. 155 (1998) 109–124.

[22] S.R.S. Varadhan, Lectures on Diffusion Problems and Partial Differential Equations, Tata Institute of Fundamental Research, Bombay, 1980.

[23] F.Y. Wang, On estimation of the logarithmic Sobolev constant and gradient estimates of heat semigroups, Probab. Theory Related Fields 108 (1997) 87–101.

[24] F.Y. Wang, Gradient estimates of Dirichlet Heat Semigroups and Application to Isoperimetric Inequalities, Ann. Probab. 32 (2004) 424–440.