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A note on the fundamental solution for the Tricomi-type equation in the hyperbolic domain

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Abstract

Barros-Neto and Gelfand (Duke Math. J. 98 (3) (1999) 465; Duke Math. J. 117 (2) (2003) 561) constructed for the Tricomi operator $y\partial_x^2 + \partial_y^2$ on the plane the fundamental solutions with the supports in the regions related to the geometry of the characteristics of the Tricomi operator. We give for the Tricomi-type operator $\partial_t^2 - t^m \Delta_x$ a fundamental solution relative to an arbitrary point of \mathbb{R}^{n+1} with the support in the region $t \geq 0$, where the operator is hyperbolic. Our key observation is that the *fundamental solution for the Tricomi-type operator can be written like an integral of the distributions generated by the fundamental solution of the Cauchy problem for the wave equation*. The application of that fundamental solution to the $L^p - L^q$ estimate for the forced Tricomi-type equation is given as well.

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0. Introduction

Recently in [1,2] Barros-Neto and Gelfand constructed the fundamental solutions for the Tricomi operator \mathcal{T} ,

$$\mathcal{T}u = yu_{xx} + u_{yy}. \quad (0.1)$$

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They have obtained explicit solutions in the sense of distributions of the equation

$$\mathcal{T}E = \delta(x - x_0, y - y_0), \tag{0.2}$$

where $\delta(x - x_0, y - y_0)$ is the Dirac function at (x_0, y_0) , an arbitrary point in the plane. A solution E of (0.2) is said to be a *fundamental solution relative to point* (x_0, y_0) . In the first of cited papers [2] the authors emphasize as physically meaningful fundamental solutions two of them with the support in $\overline{D_I}$ and $\overline{D_{II}}$, while in the second paper they suggest a fundamental solution with the support in the closure of the complement of the region D_I . From now on we will focus our attention on the fundamental solution with the support in the closure $\overline{D_I(x_0; y_0)}$ of the region

$$D_I(x_0; y_0) := \{(x, y) \in \mathbb{R}^2; 3|x - x_0| < 2(y - y_0)^{3/2}\}.$$

Barros-Neto and Cardoso [3] considered similar problem for the generalized Tricomi operator

$$\mathcal{T}u = y\Delta_x + u_{yy}, \tag{0.3}$$

where $\Delta_x = \sum_{j=1}^n \frac{\partial^2}{\partial_j^2}$ is the Laplace operator. To construct fundamental solution relative to an arbitrary point $(a, 0)$ on the hyperplane $y = 0$ in \mathbb{R}^{n+1} , they use the Fourier transform with respect to the variable x . There are many articles and books which employ the Fourier integral operators to construct a parametrix and fundamental solutions to the Cauchy problem (see, e.g. [19]). But as it is mentioned in [3], if $n > 1$ in the construction of the fundamental solution for the operator using that approach, technical difficulties in evaluating Fourier transforms involving Bessel functions do occur. In [3] the authors partially circumvent these difficulties by calculating integrals of the type

$$I_\varepsilon(a, b) = \int_0^\infty e^{-\varepsilon t} t^{-\lambda} J_\mu(at) J_\mu(bt) dt$$

with $b = 0$. Those integrals allow authors to obtain the fundamental solution relative to point $(x_0, 0)$ only.

Thus there was a gap related to the case when $y_0 < 0, n > 1$. In this note we fill up that gap and develop a tool for the investigation of the nonlinear Tricomi-type equations.

In 1923, Tricomi [17] initiated the work on boundary value problems for linear partial differential operator of mixed type (0.1) and related equations of variable type. Then, in 1945 Frankl [9] drew attention to the fact that the Tricomi problem was closely connected to the study of gas flows with nearly sonic speeds. Namely Tricomi equation describes the transition from subsonic flow (elliptic region) to supersonic

flow (hyperbolic region) [5]. That initiated an extremely intensive study of the different problems for the Tricomi equation as well as for other equations with the characteristics of variable multiplicity. There is a long history of finding fundamental solution for such operators even in the higher dimensional cases. That is impossible to give in short note a complete bibliography and we refer only to [2,7,8,10,12,15,18].

In this note we consider a slightly generalized operator, sometimes called also the Gellerstedt operator,

$$\mathcal{T}u := u_{tt} - t^m \Delta_x u, \quad (0.4)$$

with $m \in \mathbb{N}$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and Δ_x the Laplace operator in \mathbb{R}^n . The well-posedness of the Cauchy problem for (0.4) in the hyperbolic domain $t > 0$ and in the different functional spaces is exhaustively investigated. The existence of the different fundamental solutions for the Cauchy problem is established (see, e.g. [19] for the bibliography). The parametrix in the form of the Fourier integral operators with the amplitude functions represented by the Bessel functions is constructed in [20].

Unfortunately, we must admit that the results of all above-mentioned papers and books are not suitable enough for deriving the so-called $L^p - L^q$ estimates for the equations with the right-hand side force function. On the other hand to study the local and global existence in the Cauchy problem for the semilinear equations of the form

$$u_{tt} - t^m \Delta u = f(u),$$

$L^p - L^q$ estimates are very useful. In fact, they are the main tool in establishing existence theorems for the semilinear wave equation (see, e.g. [16]). The well-known Duhamel's principle allows to obtain the above-mentioned $L^p - L^q$ estimates for the nonhomogeneous wave equation by reduction to the Cauchy problem for the homogeneous wave equation and, consequently, to the corresponding $L^p - L^q$ -decay estimates for the last one (see, e.g. [6]). For the operator (0.4) with variable coefficient the Duhamel's principle does not work. In the present paper we suggest some integral transformation that serves for the nonhomogeneous equation involving the operator (0.4) in the left-hand side. This integral transformation is as good as the Duhamel's principle for the wave equation. According to our knowledge this transformation is novel.

The classical works on the Tricomi ($m = 1$) and Gellerstedt ($m = 2k + 1$) equations (see, e.g. [7,8,18]) appeal to the singular Cauchy problem for the Euler–Poisson–Darboux equation,

$$\Delta u = u_{tt} + \frac{c}{t} u_t, \quad c \in \mathbb{C}$$

and to the Asgeirsson mean value theorem to handle high-dimensional case. Our approach is free of an equation with the singularities and seems to us more immediate.

Recently the semilinear Tricomi equations became the focus of interest of many authors (see, also [11,13,14]), and the creation of a tool for the investigation of the

local and global solvability in the Cauchy problem for these equations appears to be a worthwhile undertaking.

Therefore, our goal in this note is an explicit construction of the fundamental solution relative to an arbitrary point $(x_0; t_0)$, $t_0 \geq 0$, with the support in the closure $\overline{D_1(x_0; t_0)}$ of the region $D_1(x_0; t_0) := \{(x, t) \in \mathbb{R}^{n+1}; (m + 2)|x - x_0| < 2(t - t_0)^{m/2+1}\}$. We will show that such fundamental solution is “an integral” of the one-parameter family of the distributions generated by the fundamental solution $E^{we}(x, t; x_0)$ of the Cauchy problem for the wave equation, that is the solution of the problem

$$E^{we}_{tt} - \Delta E^{we} = 0, \quad E^{we}(x, 0) = \delta(x - x_0), \quad E^{we}_t(x, 0) = 0.$$

The existence of the operator transforming solutions of the Cauchy problem for the *wave equation* into the solutions of the Cauchy problem for the *nonhomogeneous Tricomi equation* we will call “time-speed transformation principle”. As a particular case ($m = 0$) it includes also “in-two-steps” Duhamel’s principle. Roughly speaking the time-speed transformation principle assists to make time-dependent speed of propagation equal to a constant one.

We give in this note some application to the Cauchy problem for the linear equation

$$u_{tt} - t^m \Delta u = f(x, t). \tag{0.5}$$

As a consequence we conclude that the strong Huygens’ principle does not hold for any dimension n and for every $m > 0$. For $m = 1$ that is proved in [1–3]. Then we derive $L^p - L^q$ estimates for solutions of (0.5) with a support in the upper half-space. Applications to the nonlinear problems will be given in forthcoming papers.

1. Fundamental solutions: main results

To motivate our approach we recall the following well-known feature of the string equation and wave equation. The function

$$u(x, t) = \frac{1}{2} \int_0^t d\tau \int_{-t+\tau}^{t-\tau} f(x + z, \tau) dz \tag{1.1}$$

solves the Cauchy problem for the nonhomogeneous string equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = f(x, t), \quad u(x, 0) = 0, \quad u_t(x, 0) = 0.$$

If we plug $f(x, t) = \delta(x - x_0)\delta(t - t_0)$ in (1.1), where $0 \leq t_0 < t$, then we get the fundamental solution $E = E(x, t; x_0, t_0)$ for the string operator,

$$\frac{\partial^2}{\partial t^2} E - \frac{\partial^2}{\partial x^2} E = \delta(x - x_0)\delta(t - t_0),$$

with the support in the closure $\overline{D_1(x_0; t_0)}$ of the cone $D_1(x_0; t_0) := \{(x, t); |x - x_0| < t - t_0\}$. In fact

$$E(x, t; x_0, t_0) = \begin{cases} 1/2 & \text{if } (x, t) \in \overline{D_1(x_0; t_0)}; \\ 0 & \text{otherwise.} \end{cases}$$

To extract from this well-known fundamental solution the key observation and to adjust that to our purpose we note here that for $t \geq t_0$ it can be written in the following way:

$$E(x, t; x_0, t_0) = t \int_0^{1-(t_0/t)} \frac{1}{2} \{ \delta(x - x_0 + zt) + \delta(x - x_0 - zt) \} dz,$$

where one-parameter family of the distributions $\frac{1}{2} \{ \delta(x - x_0 + zt) + \delta(x - x_0 - zt) \}$, $t > 0$, is generated by the fundamental solution $E^{\text{string}} = \frac{1}{2} \{ \delta(x - x_0 + y) + \delta(x - x_0 - y) \}$ of the Cauchy problem for the string equation,

$$E_{yy}^{\text{string}} - E_{xx}^{\text{string}} = 0, \quad E^{\text{string}}(x, 0) = \delta(x - x_0), \quad E_y^{\text{string}}(x, 0) = 0.$$

It turns out that such relation between the *fundamental solution to the operator* and the *fundamental solution to the Cauchy problem* exists also for the wave equation. This elementary integral relation will serve as a guide to build a bridge between the *fundamental solution to the Tricomi-type operator* and the *fundamental solution to the Cauchy problem for the wave equation*. Such integral transformation with more sophisticated kernel will be given in Theorem 1.2 by formula (1.10). That is an essence of our *key observation*.

So then we turn to the wave equation and set in $D_1(x_0, t_0)$, $t > t_0 \geq 0$, analogously to the one-dimensional case,

$$E_1(x, t; x_0, t_0) = t \int_0^{1-(t_0/t)} E^{\text{we}}(x, rt; x_0) dr,$$

where if n is odd, then

$$E^{\text{we}}(x, t; x_0) := \frac{1}{\omega_{n-1} 1 \cdot 3 \cdot 5 \cdots (n-2)} \frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \frac{1}{t} \delta(|x - x_0| - t), \quad (1.2)$$

while for n even,

$$E^{\text{we}}(x, t; x_0) := \frac{2}{\omega_{n-1} 1 \cdot 3 \cdot 5 \cdots (n-1)} \frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \frac{1}{\sqrt{t^2 - |y|^2}} \chi_{B_t(x)}. \quad (1.3)$$

Here $\chi_{B_t(x)}$ denotes the characteristic function of the ball $B_t(x) := \{x \in \mathbb{R}^n; |x| \leq t\}$. Constant ω_{n-1} is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. The distribution $\delta(|x - y| - t)$ is defined by

$$\langle \delta(|x - \cdot| - t), f(\cdot) \rangle = \int_{|y|=t} f(x + y) dy \quad \text{for all } f \in C_0^\infty(\mathbb{R}^n). \quad (1.4)$$

It can be easily verified that *distribution $E_1(x, t; x_0, t_0)$ is a fundamental solution to the wave operator with the support in $D_1(x_0, t_0)$.*

In this section we give the fundamental solution to the Tricomi-type operator

$$T := \frac{\partial^2}{\partial t^2} - t^{2k} \Delta, \quad (1.5)$$

where $2k$ is an integer number, and $k \geq 1/2$. The *fundamental solution E of the operator T relative to the point (x_0, t_0)* is a distribution $E \in \mathcal{D}'(\mathbb{R}^{n+1})$ such that

$$TE = \delta(x - x_0, t - t_0), \quad \frac{\partial^2 E}{\partial t^2} - t^{2k} \Delta E = \delta(x - x_0, t - t_0). \quad (1.6)$$

Here $\delta(x - x_0, t - t_0)$ is the Dirac function at (x_0, t_0) . We look for the fundamental solution with a support in the “forward cone” $D_1(x_0, t_0)$, $t_0 \geq 0$, defined as follows

$$D_1(x_0, t_0) := \left\{ (x, t) \in \mathbb{R}^{n+1}; |x - x_0| < \frac{1}{k+1} (t^{k+1} - t_0^{k+1}) \right\}.$$

First we consider one-dimensional case $n = 1$. Define for $t_0 \geq 0$ in the domain $D_1(0, t_0)$ a function

$$E(x, t; 0, t_0) := (x + \phi(t) + \phi(t_0))^{-\gamma} (-x + \phi(t_0) + \phi(t))^{-\gamma} F(\gamma, \gamma; 1; \zeta), \quad (1.7)$$

where $F(\gamma, \gamma; 1; \zeta)$ is the hypergeometric function (see, e.g. [4]), while

$$\zeta = \frac{(x + \phi(t) - \phi(t_0))(x - \phi(t) + \phi(t_0))}{(x + \phi(t) + \phi(t_0))(x - \phi(t) - \phi(t_0))}, \quad \phi(t) := \frac{t^{k+1}}{k+1}, \quad \gamma := \frac{k}{2} \phi(1). \quad (1.8)$$

Let $E(x, t; 0, b)$ be a function defined by (1.7) and (1.8), and define

$$E_I(x, t; 0, t_0) = \begin{cases} c_k E(x, t; 0, t_0) & \text{in } D_I(0, t_0), \\ 0 & \text{elsewhere.} \end{cases} \tag{1.9}$$

Here $c_k = (k + 1)^{-\frac{k}{k+1}} 2^{-\frac{1}{k+1}}$. Since function $E = E(x, t; 0, t_0)$ is smooth in $D_I(0, t_0)$ and bounded on the boundary of $D_I(0, t_0)$, it follows that $E_I(x, t; 0, t_0)$ is a locally integrable function and defines a distribution whose support is in the closure $\overline{D_I(0, t_0)}$ of $D_I(0, t_0)$. The next theorem generalizes Theorem 3.1 [2] (See also [12, Proposition 69].) and gives our first result.

Theorem 1.1. *The distribution $E_I(x, t; 0, t_0)$ is a fundamental solution for the operator T relative to point $(0, t_0)$.*

Note that for $t > t_0 \geq 0$ one can rewrite formally that fundamental solution as follows:

$$\begin{aligned} & E_I(x, t; x_0, t_0) \\ &= 2c_k t \phi(1)^{\phi(1)} \int_0^{1-(t_0/t)^{k+1}} dr \left(\left(\frac{t_0}{t} \right)^{k+1} + r + 1 \right)^{-\gamma} \left(\left(\frac{t_0}{t} \right)^{k+1} - r + 1 \right)^{-\gamma} \\ & \quad \times F \left(\gamma, \gamma; 1; \frac{(-r + 1 - (t_0/t)^{k+1})(-r - 1 + (t_0/t)^{k+1})}{(-r + 1 + (t_0/t)^{k+1})(-r - 1 - (t_0/t)^{k+1})} \right) E^{\text{string}}(x, \phi(t)r; x_0), \end{aligned}$$

where the distribution $E^{\text{string}}(x, t; x_0) := \frac{1}{2} \{ \delta(x - x_0 + t) + \delta(x - x_0 - t) \}$ coincides with the fundamental solution $E^{\text{string}}(x, t; x_0)$ of the Cauchy problem for the string equation, $E_{tt}^{\text{string}} - E_{xx}^{\text{string}} = 0$, $E^{\text{string}}(x, 0) = \delta(x - x_0)$, $E_t^{\text{string}}(x, 0) = 0$. Thus, in the new writing we have the one-parameter family $E^{\text{string}}(x, \phi(t)r; x_0)$, parameter $t \in [t_0, \infty)$, generated by the fundamental solution E^{string} .

Now we are going to show that such reduction of the fundamental solution for the *Tricomi-type operator* to the fundamental solution for the Cauchy problem for *wave equation* is possible for an arbitrary dimension n .

Next we construct the fundamental solution with the support in the forward ‘‘cone’’ $D_I(x_0, t_0) = \{(x, t) \mid |x - x_0| \leq (t^{k+1} - t_0^{k+1})/(k + 1)\}$ for the operator (1.5) in \mathbb{R}^n , $x \in \mathbb{R}^n$, with odd n , $n = 2m + 1$, $m \in \mathbb{N}$. Namely we set in $D_I(x_0, t_0)$, $t > t_0 \geq 0$,

$$\begin{aligned} & E_I(x, t; x_0, t_0) \\ &= 2c_k t \phi(1)^{\phi(1)} \int_0^{1-(t_0/t)^{k+1}} dr \left(\left(\frac{t_0}{t} \right)^{k+1} + r + 1 \right)^{-\gamma} \left(\left(\frac{t_0}{t} \right)^{k+1} - r + 1 \right)^{-\gamma} \end{aligned}$$

$$\begin{aligned} & \times F\left(\gamma, \gamma; 1; \frac{(-r + 1 - (t_0/t)^{k+1})(-r - 1 + (t_0/t)^{k+1})}{(-r + 1 + (t_0/t)^{k+1})(-r - 1 - (t_0/t)^{k+1})}\right) \\ & \times E^{\text{we}}(x, \phi(t)r; x_0) \end{aligned} \tag{1.10}$$

while

$$E_I(x, t; x_0, t_0) = 0 \quad \text{if } (x, t) \notin D_I(x_0, t_0). \tag{1.11}$$

Here the distribution

$$E^{\text{we}}(x, r; x_0) := \frac{1}{c_0^{(n)}} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{\frac{n-3}{2}} r^{n-2} \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \delta(x - x_0 + ry) dS_y \tag{1.12}$$

coincides with the fundamental solution $E^{\text{we}}(x, t; x_0)$ of the Cauchy problem for the wave equation,

$$E_{tt}^{\text{we}} - \Delta E^{\text{we}} = 0, \quad E^{\text{we}}(x, 0) = \delta(x - x_0), \quad E_t^{\text{we}}(x, 0) = 0, \tag{1.13}$$

represented also by (1.2). Here $c_0^{(n)} = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (n-2)$. Denoting by $\langle E_I(x, t; x_0, t_0), \psi(x_0) \rangle$ the value of the distribution $E_I(x, t; x_0, t_0)$ on the test function $\psi \in C_0^\infty(\mathbb{R}^n)$, we give a meaning to (1.10) as follows:

$$\begin{aligned} & \langle E_I(x, t; x_0, t_0), \psi(x_0) \rangle \\ & = 2c_k t \phi(1)^{\phi(1)} \int_0^{1-(t_0/t)^{k+1}} dr \left(\left(\frac{t_0}{t}\right)^{k+1} + r + 1 \right)^{-\gamma} \left(\left(\frac{t_0}{t}\right)^{k+1} - r + 1 \right)^{-\gamma} \\ & \times F\left(\gamma, \gamma; 1; \frac{(-r + 1 - (t_0/t)^{k+1})(-r - 1 + (t_0/t)^{k+1})}{(-r + 1 + (t_0/t)^{k+1})(-r - 1 - (t_0/t)^{k+1})}\right) \\ & \times \langle E^{\text{we}}(x, \phi(t)r; x_0), \psi(x_0) \rangle. \end{aligned} \tag{1.14}$$

Then spatial translation implies $E^{\text{we}}(x, t; x_0) = E^{\text{we}}(x - x_0, t; 0)$ and $E_I(x, t; x_0, t_0) = E_I(x - x_0, t; 0, t_0)$. In particular, in (1.14) one can replace $\langle E^{\text{we}}(x, \phi(t)r; x_0), \psi(x_0) \rangle$ with $\langle E^{\text{we}}(x - x_0, \phi(t)r; 0), \psi(x_0) \rangle$.

Theorem 1.2. *Let n be odd, $n = 2m + 1$, $m \in \mathbb{N}$. Then the distribution $E_I(x, t; 0, t_0)$ defined by (1.10), (1.11), (1.12), and (1.13) is a fundamental solution for the operator T in $x \in \mathbb{R}^n$ relative to point (x_0, t_0) .*

Let n be even, $n = 2m$, $m \in \mathbb{N}$. Then the distribution $E_1(x, t; 0, t_0)$ defined by (1.10), (1.11), with (1.3), and (1.13) is a fundamental solution for the operator T in $x \in \mathbb{R}^n$ relative to point (x_0, t_0) .

We give a direct proof of Theorem 1.1 in the next section. The fundamental solution from Theorem 1.1 is used to get a representation of the solution to the Cauchy problem described by Theorem 3.1. Then we give another proof of that representation. Some details of that second proof set up a base for the proof of Theorem 3.4.

To prove Theorem 1.2 we first establish a representation of the solution to the Cauchy problem for the nonhomogeneous equation with the homogeneous initial data (Theorem 3.4). Then we plug $f(x, t) = \delta(t - t_0)\delta(x - x_0)$ in that representation and obtain statements of Theorem 1.2.

2. Proof of Theorem 1.1

In the characteristic coordinates l and m ,

$$l = x + \phi(t), \quad m = x - \phi(t) \tag{2.1}$$

the operator T reads

$$\begin{aligned} \frac{\partial^2}{\partial t^2} - t^{2k} \frac{\partial^2}{\partial x^2} &= -2^{\frac{2}{k+1}} (k+1)^{\frac{2k}{k+1}} (l-m)^{\frac{2k}{k+1}} \\ &\times \left\{ \frac{\partial^2}{\partial l \partial m} - \frac{k}{2(k+1)(l-m)} \left(\frac{\partial}{\partial l} - \frac{\partial}{\partial m} \right) \right\}. \end{aligned}$$

Consider point $(x, t) = (0, b)$, then two backward characteristics meet the x line at the points $x = a$ and $x = -a$, $a := \phi(b)$. Note that the point $(l, m) = (\phi(b), -\phi(b))$ represents point $(0, b)$ in characteristic coordinates. The following lemma is a generalization of (2.2)[2] (see also [12, Chapter 9]), where the case with $k = 1/2$ is considered.

Lemma 2.1. *The function*

$$E(l, m; a, b) = (l - b)^{-\gamma} (a - m)^{-\gamma} F \left(\gamma, \gamma; 1; \frac{(l - a)(m - b)}{(l - b)(m - a)} \right) \tag{2.2}$$

solves the equation

$$\left\{ \frac{\partial^2}{\partial l \partial m} - \frac{k}{2(k+1)(l-m)} \left(\frac{\partial}{\partial l} - \frac{\partial}{\partial m} \right) \right\} E(l, m; a, b) = 0. \tag{2.3}$$

Proof. Indeed, after simple calculation we obtain

$$\begin{aligned} & \left\{ \frac{\partial^2}{\partial l \partial m} - \frac{k}{2(k+1)(l-m)} \left(\frac{\partial}{\partial l} - \frac{\partial}{\partial m} \right) \right\} E(l, m; a, b) \\ &= \frac{1}{m-l} (a-b)(-b+l)^{-1-\gamma} (a-m)^{-1-\gamma} \\ & \quad \times \left\{ z(1-z)F''(\gamma, \gamma; 1; z) + \left(1 - \frac{2k+1}{k+1}z \right) F'(\gamma, \gamma; 1; z) - \gamma^2 F(\gamma, \gamma; 1; z) \right\}, \end{aligned}$$

where $z = \frac{(l-a)(m-b)}{(l-b)(m-a)}$. Hence (2.3) holds. The lemma is proved. \square

According to the next proposition the function $R(l, m; a, b)$ defined by

$$R(l, m; a, b) := (l-m)^{\frac{k}{k+1}} E(l, m; a, b) \tag{2.4}$$

is the Riemann function of the *reduced hyperbolic* form T_h , of the operator T ,

$$T_h := \frac{\partial^2}{\partial l \partial m} - \frac{k}{2(k+1)(l-m)} \left(\frac{\partial}{\partial l} - \frac{\partial}{\partial m} \right),$$

relative to the point (a, b) . To formulate and to prove that proposition we consider the formally adjoint operator

$$T_h^* := \frac{\partial^2}{\partial l \partial m} + \frac{k}{2(k+1)(l-m)} \left(\frac{\partial}{\partial l} - \frac{\partial}{\partial m} \right) - \frac{k}{(k+1)(l-m)^2}$$

and the following lemma, which is a generalization of (2.4) [2], where the case with $k = 1/2$ is considered.

Lemma 2.2. *If v is a solution of the equation $T_h^*v = 0$, then $u = (l-m)^{-c}v$ with $c = k/(k+1)$ is a solution to $T_h u = 0$, and vice versa.*

Proof. Indeed, direct calculations lead to

$$T_h^*v = (l-m)^c \left[\frac{\partial^2}{\partial m \partial l} u - \frac{k}{2(k+1)(l-m)} \left(\frac{\partial}{\partial l} u - \frac{\partial}{\partial m} u \right) \right]$$

provided that $c = k/(k+1)$. Lemma is proved. \square

Proposition 2.3. *The function $R(l, m; a, b)$ is the unique solution of the equation $T^*v = 0$ that satisfies the following conditions:*

- (i) $R_l = \frac{k}{2(k+1)(l-m)} R$ along the line $m = b$;
- (ii) $R_m = -\frac{k}{2(k+1)(l-m)} R$ along the line $l = a$;
- (iii) $R(a, b; a, b) = 1$.

Proof. First of all the equation $T^*R = 0$ is satisfied due to Lemma 2.1 and Lemma 2.2. Then along the line $m = b$ we have

$$R(l, b; a, b) = (l - b)^\gamma (a - b)^{-\gamma} F(\gamma, \gamma; 1; 0) = \left(\frac{l - b}{a - b}\right)^\gamma = \exp\left(\int_a^l \frac{\gamma}{(t - b)} dt\right).$$

Hence (i) holds. In the similar way we verify the remaining statements. \square

Proof of Theorem 1.1. In fact the proof is an almost verbatim repetition of the proof of Theorem 3.1 of [2], therefore we omit almost all details and keep only the main steps and formulas. Note that the operator T is formally self-adjoint, $T = T^*$. We must show that

$$\langle E_I, T\varphi \rangle = \varphi(0, b) \quad \text{for every } \varphi \in C_0^\infty(\mathbb{R}_+^2).$$

Since $E(x, t; 0, b)$ is locally integrable in \mathbb{R}^2 , this is equivalent to showing that

$$\int \int_{\mathbb{R}_+^2} E_I(x, t; 0, b) T\varphi(x, t) dx dt = \varphi(0, b) \quad \text{for every } \varphi \in C_0^\infty(\mathbb{R}_+^2). \tag{2.5}$$

In the mean time $2^{-\phi(1)}(k+1)^{-\frac{k}{k+1}}(l-m)^{-\frac{k}{k+1}}$ is the Jacobian of the transformation (2.1). Hence the integral in the left-hand side of (2.5) is equal to

$$\begin{aligned} & -2^{\frac{2}{k+1}}(k+1)^{\frac{2k}{k+1}} \int_{-\infty}^{-l_0} \int_{l_0}^{\infty} c_k E(l, m; l_0, -l_0) \frac{1}{2^{\phi(1)}(k+1)^{\frac{k}{k+1}}(l-m)^{\frac{k}{k+1}}} dl dm \\ & \times \left\{ (l-m)^{\frac{2k}{k+1}} \left[\frac{\partial^2}{\partial l \partial m} - \frac{k}{2(k+1)(l-m)} \left(\frac{\partial}{\partial l} - \frac{\partial}{\partial m} \right) \right] \varphi \right\}. \end{aligned}$$

Here $c_k = \phi(1)^{k\phi(1)}$. Then using Riemann function we write

$$\begin{aligned} & \int \int_{\mathbb{R}_+^2} E_1(x, t; 0, b) T \varphi(x, t) dx dt \\ &= - \int_{-\infty}^{-l_0} \int_{l_0}^{\infty} R(l, m; l_0, -l_0) \left\{ \frac{\partial^2}{\partial l \partial m} - \frac{k}{2(k+1)(l-m)} \left(\frac{\partial}{\partial l} - \frac{\partial}{\partial m} \right) \right\} \varphi dl dm. \end{aligned}$$

Integrating by parts we obtain (2.5) and this completes the proof. \square

3. Application to the Cauchy problem

Consider now the Cauchy problem for the equation

$$\frac{\partial^2 u}{\partial t^2} - t^{2k} \frac{\partial^2 u}{\partial x^2} = f(x, t), \quad t > 0, \quad x \in \mathbb{R}, \tag{3.1}$$

with vanishing initial data,

$$u(x, 0) = u_t(x, 0) = 0. \tag{3.2}$$

For every $(x, t) \in D_1(0, b)$ one has $a - \phi(t) \leq x \leq -a + \phi(t)$, so that

$$\begin{aligned} E(x, t; 0, b) &= (\phi(b) + x + \phi(t))^{-\gamma} (\phi(b) - x + \phi(t))^{-\gamma} \\ &\times F \left(\gamma, \gamma; 1; \frac{(x + \phi(t) - \phi(b))(x - \phi(t) + \phi(b))}{(x + \phi(t) + \phi(b))(x - \phi(t) - \phi(b))} \right). \end{aligned}$$

The coefficient of the Tricomi equation is independent of x , therefore $E_1(x, t; y, b) = E_1(x - y, t; 0, b)$. Using the fundamental solution from Theorem 1.1 one can write the convolution

$$\begin{aligned} u(x, t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_1(x, t; y, b) f(y, b) db dy \\ &= \int_0^t db \int_{-\infty}^{\infty} E_1(x - y, t; 0, b) f(y, b) dy \end{aligned}$$

since $\text{supp } f \subset \{t \geq 0\}$. Then according to the definition of the function E_1 we obtain

Theorem 3.1. *Assume that the function f is continuous along with its all second-order derivatives, and that for every fixed t it has a compact support, $\text{supp } f(\cdot, t) \subset \mathbb{R}$.*

Then the function defined by the integral representation

$$\begin{aligned}
 u(x, t) = & c_k \int_0^t db \int_{x-(\phi(t)-\phi(b))}^{x+\phi(t)-\phi(b)} dy f(y, b) \\
 & \times (x - y + \phi(t) + \phi(b))^{-\gamma} (\phi(b) - (x - y) + \phi(t))^{-\gamma} \\
 & \times F\left(\gamma, \gamma; 1; \frac{(x - y + \phi(t) - \phi(b))(x - y - \phi(t) + \phi(b))}{(x - y + \phi(t) + \phi(b))(x - y - \phi(t) - \phi(b))}\right) \quad (3.3)
 \end{aligned}$$

is a C^2 -solution to the Cauchy problem for Eq. (3.1) with vanishing initial data, (3.2).

The following corollary is a manifestation of the time-speed transformation principle. Indeed, it implies the existence of an operator transforming the solutions of the Cauchy problem for the string equation to the solutions of the Cauchy problem for the nonhomogeneous Tricomi equation. As a particular case ($k = 0$) it includes also “in-two-steps” Duhamel’s principle, but unlike the last one, it reduces the equation with the time-dependent speed of propagation to the one with the speed of propagation independent of time.

Corollary 3.2. *The solution $u(t, x)$ of the Cauchy problem (3.1)–(3.2) can be represented as follows:*

$$\begin{aligned}
 u(x, t) = & 2c_k \int_0^1 db \int_0^{1-b^{k+1}} ds v(x, \phi(t)s; tb)(b^{k+1} + 1 - s)^{-\gamma}(b^{k+1} + 1 + s)^{-\gamma} \\
 & \times t^2 \phi(1)^{\phi(1)} F\left(\gamma, \gamma; 1; \frac{(-s + 1 - b^{k+1})(-s - 1 + b^{k+1})}{(-s + 1 + b^{k+1})(-s - 1 - b^{k+1})}\right), \quad (3.4)
 \end{aligned}$$

where the functions $v(x, t; \tau) := \frac{1}{2}(f(x + t, \tau) + f(x - t, \tau))$, $\tau \in [0, \infty)$, form a one-parameter family of solutions to the Cauchy problem for the string equation

$$v_{tt} - v_{xx} = 0, \quad v(x, 0; \tau) = f(x, \tau), \quad v_t(x, 0; \tau) = 0.$$

The next corollary solves the problem with the initial data. Namely, we set $f(x, t) = \delta(t)\varphi(x)$ and obtain the following statement.

Corollary 3.3. *The solution $u(t, x)$ of the Cauchy problem*

$$u_{tt} - t^{2k}u_{xx} = 0, \quad u(x, 0) = 0, \quad u_t(x, 0) = \varphi(x),$$

can be represented as follows:

$$u(x, t) = tc_k \phi(1)^{\phi(1)} F(\gamma, \gamma; 1; 1) \int_0^1 \{\varphi(x - \phi(t)s) + \varphi(x + \phi(t)s)\}(1 - s^2)^{-\gamma} ds.$$

In the last formula the function $\frac{1}{2}\{\varphi(x - \phi(t)s) + \varphi(x + \phi(t)s)\}$ coincides with the solution $v(x, t)$ to the Cauchy problem for the string equation, $v_{tt} - v_{xx} = 0$, $v(x, 0) = \varphi(x)$, $v_t(x, 0) = 0$, taken at the point $(x, \phi(t)s)$, that is with $v(x, \phi(t)s)$.

Now we consider the case $x \in \mathbb{R}^n$, $n \geq 2$.

Theorem 3.4. *The classical solution $u = u(x, t)$ of the Cauchy problem*

$$u_{tt} - t^{2k} \Delta u = f(x, t), \quad u(x, 0) = 0, \quad u_t(x, 0) = 0 \tag{3.5}$$

with $n = 2m + 1$, $m \in \mathbb{N}$, $x \in \mathbb{R}^n$, and $f \in C_{x,t}^{(n+3)/2,2}$ is given by the following formula:

$$\begin{aligned} u(x, t) &= 2c_k \int_0^t db \int_0^{\phi(t)-\phi(b)} dr_1 \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{\frac{n-3}{2}} \right. \\ &\quad \times \left. \frac{r^{n-2}}{\omega_{n-1} c_0^{(n)}} \int_{S^{n-1}} f(x + ry, b) dS_y \right)_{r=r_1} \\ &\quad \times (r_1 + \phi(t) + \phi(b))^{-\gamma} (\phi(b) - r_1 + \phi(t))^{-\gamma} \\ &\quad \times F \left(\gamma, \gamma; 1; \frac{(-r_1 + \phi(t) - \phi(b))(-r_1 - \phi(t) + \phi(b))}{(-r_1 + \phi(t) + \phi(b))(-r_1 - \phi(t) - \phi(b))} \right), \end{aligned} \tag{3.6}$$

where $c_0^{(n)} = 1 \cdot 3 \cdot \dots \cdot (n - 2)$.

If n is even, $n = 2m$, $m \in \mathbb{N}$, and $f \in C_{x,t}^{n/2+2,2}$, then the classical solution $u = u(x, t)$ of the Cauchy problem (3.5) can be represented as follows:

$$\begin{aligned} u(x, t) &= 2c_k \int_0^t db \int_0^{\phi(t)-\phi(b)} dr_1 \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{\frac{n-2}{2}} \right. \\ &\quad \times \left. \frac{2r^{n-2}}{\omega_{n-1} c_0^{(n)}} \int_{B_1^n(0)} \frac{f(x + ry, b)}{\sqrt{1 - |y|^2}} dV_y \right)_{r=r_1} \\ &\quad \times (r_1 + \phi(t) + \phi(b))^{-\gamma} (\phi(b) - r_1 + \phi(t))^{-\gamma} \\ &\quad \times F \left(\gamma, \gamma; 1; \frac{(-r_1 + \phi(t) - \phi(b))(-r_1 - \phi(t) + \phi(b))}{(-r_1 + \phi(t) + \phi(b))(-r_1 - \phi(t) - \phi(b))} \right). \end{aligned} \tag{3.7}$$

Here $B_1^n(0) := \{|y| \leq 1\}$ is the unit ball in \mathbb{R}^n , while $c_0^{(n)} = 1 \cdot 3 \cdot \dots \cdot (n - 1)$.

Corollary 3.5. *The solution $u = u(x, t)$ of the Cauchy problem*

$$u_{tt} - t^{2k} \Delta u = 0, \quad u(x, 0) = 0, \quad u_t(x, 0) = \varphi(x)$$

($n = 2m + 1, m \in \mathbb{N}$) can be represented as

$$u(x, t) = t2c_k \phi(1)^{\phi(1)} F(\gamma, \gamma; 1; 1) \int_0^1 (1 - s^2)^{-\gamma} v(x, s\phi(t)) ds.$$

Here the function $v(x, s\phi(t)) := (\frac{1}{c_0^{(m)}} \frac{\partial}{\partial r} ((\frac{1}{r} \frac{\partial}{\partial r})^{\frac{n-3}{2}} r^{n-2} \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \varphi(x+ry) dS_y))_{r=s\phi(t)}$ coincides with the value $v(s\phi(t), x)$ of the solution $v(t, x)$ of the Cauchy problem $v_{tt} - \Delta v = 0, v(x, 0) = \varphi(x), v_t(x, 0) = 0$.

Corollary 3.6. *If $k \neq 0$ then the strong Huygens’ principle does not hold.*

4. Proof of Theorem 3.1

First we note that the function $u(x, t) = t^2/2$ is the unique solution to the Cauchy problem (3.1)–(3.2) with the force function $f(x, t) \equiv 1$. Therefore in the next lemma we give a representation for that particular solution, which is helpful to handle the more general case.

Lemma 4.1. *One has*

$$\begin{aligned} \frac{1}{2}t^2 &= c_k \int_0^t db \int_{x-(\phi(t)-\phi(b))}^{x+\phi(t)-\phi(b)} dy \\ &\times (x - y + \phi(t) + \phi(b))^{-\gamma} (\phi(b) - (x - y) + \phi(t))^{-\gamma} \\ &\times F\left(\gamma, \gamma; 1; \frac{(x - y + \phi(t) - \phi(b))(x - y - \phi(t) + \phi(b))}{(x - y + \phi(t) + \phi(b))(x - y - \phi(t) - \phi(b))}\right). \end{aligned} \tag{4.1}$$

Proof. First we prove the convergence of the integral. The argument

$$z(x - y, t, b) = \frac{(x - y + \phi(t) - \phi(b))(x - y - \phi(t) + \phi(b))}{(x - y + \phi(t) + \phi(b))(x - y - \phi(t) - \phi(b))}$$

of the hypergeometric function is nonnegative for the prescribed values of variables. Moreover,

$$0 \leq z(x - y, t, b) \leq \frac{(t^{k+1} - b^{k+1})^2}{(t^{k+1} + b^{k+1})^2} \leq 1.$$

The integrand is nonnegative (see, e.g., the hypergeometric series (1) of Section 2.8, v.1 [4]) and is less than

$$C_{F,k}(x - y + \phi(t) + \phi(b))^{-\gamma}(\phi(b) - x + y + \phi(t))^{-\gamma}$$

uniformly for all $0 \leq b \leq t$ and all $x \in \mathbb{R}$, such that $x - \phi(t) + \phi(b) \leq y \leq x + \phi(t) - \phi(b)$. Next we use

$$\begin{aligned} & \int_{x - (\phi(t) - \phi(b))}^{x + \phi(t) - \phi(b)} dy (x - y + \phi(t) + \phi(b))^{-\gamma} (\phi(b) - x + y + \phi(t))^{-\gamma} \\ &= \phi(1)^{\phi(1)} \int_{-(t^{k+1} - b^{k+1})}^{t^{k+1} - b^{k+1}} (b^{k+1} + t^{k+1} + z)^{-\gamma} (b^{k+1} + t^{k+1} - z)^{-\gamma} dz. \end{aligned}$$

By means of representation (7) of Section 2.12, v.1 [4],

$$F(a, b; c; z) = \frac{2\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^{\pi/2} \frac{(\sin t)^{2b-1} (\cos t)^{2c-2b-1}}{(1 - z \sin^2 t)^a} dt,$$

the last integral can be evaluated and estimated as follows:

$$\begin{aligned} & 2\phi(1)^{\phi(1)} (t^{k+1} - b^{k+1})(t^{k+1} + b^{k+1})^{-\frac{k}{k+1}} F\left(\frac{1}{2}, \gamma; \frac{3}{2}; \frac{(t^{k+1} - b^{k+1})^2}{(t^{k+1} + b^{k+1})^2}\right) \\ & \leq C'_{F,k} \phi(1)^{\phi(1)} (t^{k+1} - b^{k+1})(t^{k+1} + b^{k+1})^{-\frac{k}{k+1}}. \end{aligned}$$

Hence the right-hand side of (4.1) is less than

$$\int_0^t C_{F,k} C'_{F,k} \phi(1)^{\phi(1)} (t^{k+1} - b^{k+1})(t^{k+1} + b^{k+1})^{-\frac{k}{k+1}} db = C(F, k)t^2.$$

Further we note that the function of the right-hand side of (4.1) is independent of x and according to Theorem 1.1 solves the equation with the right-hand side $f(t, x) = 1$. The uniqueness in the Cauchy problem implies $u(t, x) = a + bt + ct^2$ so that $a = b = 0$, $c = 1/2$. This completes the proof of the lemma. \square

Proof of Theorem 3.1. It follows from Theorem 1.1 and Lemma 4.1 that the integral of the right-hand side of (3.3) defines a continuous function $u = u(x, t)$, which solves Eq. (3.1) and such that

$$|u(x, t)| \leq \frac{1}{2} t^2 \max_{b \in [0, t], y \in [x - \phi(t), x + \phi(t)]} |f(y, b)|.$$

Eq. (3.1) is partially hypoelliptic in the direction of time, so that

$$u \in C^\infty([0, T]; \mathcal{D}'(\mathbb{R}_x)).$$

Further for every $\varphi \in C_0^\infty(\mathbb{R}_x)$ the function $v(t) := \langle u(\cdot, t), \varphi(\cdot) \rangle$ belongs to $C^2([0, T])$ and solves the equation

$$v_{tt} - t^{2k} \langle u(\cdot, t), \Delta \varphi(\cdot) \rangle = \langle f(\cdot, t), \varphi(\cdot) \rangle.$$

Hence $v(0) = v_t(0) = 0$ implies $u(0) = u_t(0) = 0$ in $\mathcal{D}'(\mathbb{R}_x)$. The theorem is proved. \square

Proof of Corollary 3.2. We derive from Theorem 3.1

$$\begin{aligned} u(x, t) &= c_k t \int_0^1 db \int_{-\phi(t)(1-b^{k+1})}^{\phi(t)(1-b^{k+1})} dy f(x + y, tb) \\ &\quad \times (-y + \phi(t) + \phi(t)b^{k+1})^{-\gamma} (\phi(t)b^{k+1} + y + \phi(t))^{-\gamma} \\ &\quad \times F\left(\gamma, \gamma; 1; \frac{(-y + \phi(t) - \phi(t)b^{k+1})(-y - \phi(t) + \phi(t)b^{k+1})}{(-y + \phi(t) + \phi(t)b^{k+1})(-y - \phi(t) - \phi(t)b^{k+1})}\right), \end{aligned}$$

which can be easily transformed into (3.4). The corollary is proved. \square

Proof of Corollary 3.3. If we plug $f(x, t) = \delta(t)\varphi(x)$ in (3.3), then we can rewrite this solution as follows:

$$\begin{aligned} u(x, t) &= c_k F(\gamma, \gamma; 1) \left\{ \int_{-\phi(t)}^0 dy \varphi(x + y) (-y + \phi(t))^{-\gamma} (y + \phi(t))^{-\gamma} \right. \\ &\quad \left. + \int_0^{\phi(t)} dy \varphi(x + y) (-y + \phi(t))^{-\gamma} (y + \phi(t))^{-\gamma} \right\}. \end{aligned}$$

That completes the proof of Corollary 3.3. \square

Remark 4.2. If we denote $y = x + \phi(t)(2s - 1)$, then the representation given by Corollary 3.3 can be reduced to the fractional derivatives, [15, (4.7) Chapter V].

5. Proofs of Theorem 3.4 and Theorem 1.2

We consider the case $x \in \mathbb{R}^n$, where $n = 2m + 1$. First for the given function $u = u(x, t)$ we define the spherical means of u about point x :

$$I_u(x, r, t) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} u(x + ry, t) dS_y,$$

where ω_{n-1} denotes the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Then we define an operator Ω_r by

$$\Omega_r(u)(x, t) := \left(\frac{1}{r} \frac{\partial}{\partial r}\right)^{m-1} r^{2m-1} I_u(x, r, t).$$

One can show that there are constants $c_j^{(n)}$, $j = 0, \dots, m - 1$, where $n = 2m + 1$, with $c_0^{(n)} = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (n - 2)$, such that

$$\left(\frac{1}{r} \frac{\partial}{\partial r}\right)^{m-1} r^{2m-1} \varphi(r) = r \sum_{j=0}^{m-1} c_j^{(n)} r^j \frac{\partial^j}{\partial r^j} \varphi(r).$$

One can recover the functions according to

$$\begin{aligned} u(x, t) &= \lim_{r \rightarrow 0} I_u(x, r, t) = \lim_{r \rightarrow 0} \frac{1}{c_0^{(n)} r} \Omega_r(u)(x, t), \\ u(x, 0) &= \lim_{r \rightarrow 0} \frac{1}{c_0^{(n)} r} \Omega_r(u)(x, 0), \quad u_t(x, 0) = \lim_{r \rightarrow 0} \frac{1}{c_0^{(n)} r} \Omega_r(\partial_t u)(x, 0), \\ f(x, t) &= \lim_{r \rightarrow 0} I_f(x, r, t) = \lim_{r \rightarrow 0} \frac{1}{c_0^{(n)} r} \Omega_r(f)(x, t), \\ f(x, 0) &= \lim_{r \rightarrow 0} \frac{1}{c_0^{(n)} r} \Omega_r(f)(x, 0), \quad f_t(x, 0) = \lim_{r \rightarrow 0} \frac{1}{c_0^{(n)} r} \Omega_r(\partial_t f)(x, 0). \end{aligned}$$

It is well known that $\Delta_x \Omega_r h = \frac{\partial^2}{\partial r^2} \Omega_r h$ for every function $h \in C^2(\mathbb{R}^n)$. Therefore we arrive at the following mixed problem for the function $v(x, r, t) := \Omega_r(u)(x, r, t)$:

$$\begin{aligned} v_{tt}(x, r, t) - t^{2k} v_{rr}(x, r, t) &= F(x, r, t) \quad \text{for all } t \geq 0, r \geq 0, x \in \mathbb{R}^n, \\ v(x, 0, t) &= 0 \quad \text{for all } t \geq 0, x \in \mathbb{R}^n, \\ v(x, r, 0) &= 0, \quad v_t(x, r, 0) = 0 \quad \text{for all } r \geq 0, x \in \mathbb{R}^n, \\ F(x, r, t) &:= \Omega_r(f)(x, t), \quad F(x, 0, t) = 0 \quad \text{for all } x \in \mathbb{R}^n. \end{aligned}$$

It must be noted here that the spherical mean I_u defined for $r > 0$ has an extension as even function for $r < 0$ and hence $\Omega_r(u)$ has a natural extension as an odd function. That allows replacing the mixed problem with the Cauchy problem. Namely, let functions \tilde{v} and \tilde{F} be the continuations of the functions v and F , respectively, by

$$\tilde{v}(x, r, t) = \begin{cases} v(x, r, t), & \text{if } r \geq 0, \\ -v(x, -r, t), & \text{if } r \leq 0, \end{cases} \quad \tilde{F}(x, r, t) = \begin{cases} F(x, r, t), & \text{if } r \geq 0, \\ -F(x, -r, t), & \text{if } r \leq 0. \end{cases}$$

Then \tilde{v} solves the Cauchy problem

$$\begin{aligned} \tilde{v}_{tt}(x, r, t) - t^{2k} \tilde{v}_{rr}(x, r, t) &= \tilde{F}(x, r, t) \quad \text{for all } t \geq 0, \quad r \in \mathbb{R}, \quad x \in \mathbb{R}^n, \\ \tilde{v}(x, r, 0) &= 0, \quad \tilde{v}_t(x, r, 0) = 0 \quad \text{for all } r \in \mathbb{R}, \quad x \in \mathbb{R}^n. \end{aligned}$$

Hence according to Theorem 3.1 one has the representation

$$\begin{aligned} \tilde{v}(x, r, t) &= c_k \int_0^t db \int_{r - (\phi(t) - \phi(b))}^{r + \phi(t) - \phi(b)} dr_1 \tilde{F}(x, r_1, b) \\ &\quad \times (r - r_1 + \phi(t) + \phi(b))^{-\gamma} (\phi(b) - (r - r_1) + \phi(t))^{-\gamma} \\ &\quad \times F\left(\gamma, \gamma; 1; \frac{(r - r_1 + \phi(t) - \phi(b))(r - r_1 - \phi(t) + \phi(b))}{(r - r_1 + \phi(t) + \phi(b))(r - r_1 - \phi(t) - \phi(b))}\right). \end{aligned}$$

Since $u(x, t) = \lim_{r \rightarrow 0} (\tilde{v}(x, r, t) / (c_0^{(n)} r))$, we consider a case with $r < t$ in the above representation to obtain:

$$\begin{aligned} u(x, t) &= c_k \frac{1}{c_0^{(n)}} \int_0^t db \int_0^{\phi(t) - \phi(b)} dr_1 \lim_{r \rightarrow 0} \frac{1}{r} \{ \tilde{F}(x, r - r_1, b) + \tilde{F}(x, r + r_1, b) \} \\ &\quad \times (r_1 + \phi(t) + \phi(b))^{-\gamma} (\phi(b) - r_1 + \phi(t))^{-\gamma} \\ &\quad \times F\left(\gamma, \gamma; 1; \frac{(-r_1 + \phi(t) - \phi(b))(-r_1 - \phi(t) + \phi(b))}{(-r_1 + \phi(t) + \phi(b))(-r_1 - \phi(t) - \phi(b))}\right). \end{aligned}$$

Then by definition of the function \tilde{F} we replace $\lim_{r \rightarrow 0} \frac{1}{r} \{ \tilde{F}(x, r - r_1, b) + \tilde{F}(x, r + r_1, b) \}$ with $2 \left(\frac{\partial}{\partial r} F(x, r, b) \right)_{r=r_1}$ in the last formula. The definitions of $F(x, r, t)$ and of the operator Ω_r yield:

$$\begin{aligned} u(x, t) &= 2c_k \frac{1}{c_0^{(n)}} \int_0^t db \int_0^{\phi(t) - \phi(b)} dr_1 \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{m-1} r^{2m-1} I_f(x, r, t) \right)_{r=r_1} \\ &\quad \times (r_1 + \phi(t) + \phi(b))^{-\gamma} (\phi(b) - r_1 + \phi(t))^{-\gamma} \\ &\quad \times F\left(\gamma, \gamma; 1; \frac{(-r_1 + \phi(t) - \phi(b))(-r_1 - \phi(t) + \phi(b))}{(-r_1 + \phi(t) + \phi(b))(-r_1 - \phi(t) - \phi(b))}\right), \end{aligned}$$

where \mathbb{R}^n , $n = 2m + 1$, $m \in \mathbb{N}$. Thus the solution to the Cauchy problem is given by (3.6). We employ the method of descent to complete the proof for the case with even n , $n = 2m$, $m \in \mathbb{N}$. Theorem 3.4 is proved. \square

Proof of Corollary 3.5. For $f(x, t) = \delta(t)\varphi(x)$ according to Theorem 3.4 we have

$$u(x, t) = 2 \frac{c_k}{c_0^{(n)}} F(\gamma, \gamma; 1; 1) \int_0^{\phi(t)} dr_1 \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{\frac{n-3}{2}} \frac{r^{n-2}}{\omega_{n-1}} \int_{S^{n-1}} \varphi(x + ry) dS_y \right)_{r=r_1} \times (r_1 + \phi(t))^{-\gamma} (-r_1 + \phi(t))^{-\gamma}.$$

The change of variable completes the proof of the corollary. \square

Proof of Theorem 1.2. The set $f(x, t) = \delta(x - x_0)\delta(t - t_0)$ in (3.6):

$$E(x, t; x_0, t_0) = 2c_k \int_0^{\phi(t)-\phi(t_0)} dr (r + \phi(t) + \phi(t_0))^{-\gamma} (\phi(t_0) - r + \phi(t))^{-\gamma} \times F\left(\gamma, \gamma; 1; \frac{(-r + \phi(t) - \phi(t_0))(-r - \phi(t) + \phi(t_0))}{(-r + \phi(t) + \phi(t_0))(-r - \phi(t) - \phi(t_0))}\right) \times E^{we}(x, r; x_0).$$

The evident transformations of the last representation lead to (1.10). \square

6. Application to $L^p - L^q$ estimates

The estimates for the solutions of the nonhomogeneous wave equation are generally obtained by the use of Duhamel’s principle (see, e.g. [6,16]). For the Tricomi-type equation we use the representation of the solutions given by the theorems of Section 3. First we consider the one-dimensional case.

Theorem 6.1. *For every function $f \in C^2(\mathbb{R} \times [0, \infty))$ such that $f(\cdot, t) \in C_0^\infty(\mathbb{R}_x)$ for arbitrary $t \in [0, \infty)$, the solution $u = u(x, t)$ to the Cauchy problem (3.1),(3.2) satisfies*

$$\|u(\cdot, t)\|_{L^q(\mathbb{R}_x)} \leq C_{k,p,\rho} \int_0^t (t^{k+1} - b^{k+1})^{\frac{1}{p}} (t^{k+1} + b^{k+1})^{-\frac{k}{k+1}} \|f(\cdot, b)\|_{L^p(\mathbb{R}_x)} db$$

with p, q , such that $1 < p < \rho'$, $1/q = 1/p - 1/\rho'$, $1/\rho + 1/\rho' = 1$.

Proof. From

$$u(x, t) = \int_0^t db \int_{-\infty}^{\infty} E_1(x - y, t; 0, b) f(y, b) dy$$

due to Young’s inequality we have

$$\|u(x, t)\|_{L^q(\mathbb{R}_x)} \leq c_k \int_0^t db \left(\int_{-\phi(t)-\phi(b)}^{\phi(t)-\phi(b)} |E(x, t; 0, b)|^\rho dx \right)^{1/\rho} \|f(x, b)\|_{L^p(\mathbb{R}_x)}.$$

Consider now

$$\begin{aligned} & \int_{-\phi(t)-\phi(b)}^{\phi(t)-\phi(b)} |E(x, t; 0, b)|^\rho dx \\ &= \int_{-\phi(t)-\phi(b)}^{\phi(t)-\phi(b)} (x + \phi(t) + \phi(b))^{-\frac{k\rho}{2(k+1)}} (-x + \phi(t) + \phi(b))^{-\frac{k\rho}{2(k+1)}} \\ & \quad \times F\left(\gamma, \gamma; 1; \frac{(x + \phi(t) - \phi(b))(x - \phi(t) + \phi(b))}{(x + \phi(t) + \phi(b))(x - \phi(t) - \phi(b))}\right)^\rho dx. \end{aligned}$$

Estimating the hypergeometric function we easily obtain that the right-hand side is less than or equal to

$$C \int_{-\phi(t)-\phi(b)}^{\phi(t)-\phi(b)} (\phi(b) + x + \phi(t))^{-\frac{k\rho}{2(k+1)}} (\phi(b) - x + \phi(t))^{-\frac{k\rho}{2(k+1)}} dx,$$

which in turn is (see the integral representation for $F(a, b; c; z)$ used in the proof of Lemma 4.1)

$$C_{k,p} (t^{k+1} - b^{k+1})(t^{k+1} + b^{k+1})^{-\frac{k\rho}{k+1}} F\left(\frac{1}{2}, \frac{k\rho}{2(k+1)}; \frac{3}{2}; \frac{(t^{k+1} - b^{k+1})^2}{(t^{k+1} + b^{k+1})^2}\right)$$

with some constant $C_{k,p}$. We conclude

$$\int_{-\phi(t)-\phi(b)}^{\phi(t)-\phi(b)} |E(x, t; 0, b)|^\rho dx \leq C_{k,p} (t^{k+1} - b^{k+1})(t^{k+1} + b^{k+1})^{-\frac{k\rho}{k+1}}.$$

Thus the theorem is proved. \square

In some applications to the semilinear problems the space of the force functions $f = f(x, t)$ is endowed with the norm $\max_\tau(\tau^{-\mu} \|f(\cdot, \tau)\|_{L^p(\mathbb{R}_x)})$, $\mu \geq 0$, therefore we give here an estimate for the solutions in that norm.

Corollary 6.2. *Suppose that the function $f \in C^2(\mathbb{R} \times [0, \infty))$ is such that $f(x, t) \in C_0^\infty(\mathbb{R}_x)$ for every $t \in [0, \infty)$, and that with some $\mu > -1$,*

$$t^{-\mu} \| f(\cdot, t) \|_{L^p(\mathbb{R}_x)} \leq \text{const} \quad \text{for all } t.$$

Then the solution $u = u(x, t)$ of the Cauchy problem (3.1),(3.2) satisfies

$$\| u(\cdot, t) \|_{L^q(\mathbb{R}_x)} \leq C t^{\mu-k+1+(k+1)\frac{1}{\rho}} \max_{0 \leq \tau \leq t} (\| f(\cdot, \tau) \|_{L^p(\mathbb{R}_x)} \tau^{-\mu}) \quad \text{for all } t,$$

with p, q , such that $1 < p < \rho', 1/q = 1/p - 1/\rho', 1/\rho + 1/\rho' = 1$.

Proof. Indeed, according to Theorem 6.1 we have

$$\begin{aligned} \| u(\cdot, t) \|_{L^q(\mathbb{R}_x)} &\leq C \max_{0 \leq \tau \leq t} (\tau^{-\mu} \| f(\cdot, \tau) \|_{L^p(\mathbb{R})}) \\ &\times \int_0^t b^\mu (t^{k+1} - b^{k+1})^{\frac{1}{\rho}} (t^{k+1} + b^{k+1})^{-\frac{k}{k+1}} db, \end{aligned}$$

where the integral is a positively homogeneous function of order $\mu - k + 1 + (k + 1)\frac{1}{\rho}$ of variable t . \square

To consider the high-dimensional case we start with some corollary from the well-known results on $L^p - L^q$ estimates.

Lemma 6.3. *For $\varphi \in C_0^\infty(\mathbb{R}^n)$ the functions*

$$\left(\frac{1}{c_0^{(n)}} \frac{\partial}{\partial r} \left(\left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{\frac{n-3}{2}} r^{n-2} \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \varphi(x + ry) dS_y \right) \right)_{r=s\phi(t)}$$

and

$$\left(\frac{2}{c_0^{(n)}} \frac{\partial}{\partial r} \left(\left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{\frac{n-2}{2}} \frac{r^{n-2}}{\omega_{n-1}} \int_{B_1^n(0)} \frac{1}{\sqrt{1 - |y|^2}} \varphi(x + ry) dV_y \right) \right)_{r=s\phi(t)}$$

coincide with the value $v(x, s\phi(t))$ of the solution $v(x, t)$ of the Cauchy problem $v_{tt} - \Delta v = 0, v(x, 0) = \varphi(x), v_t(x, 0) = 0$ for odd $n, n = 2m + 1$, and even n ,

$n = 2m + 2$, respectively. They satisfy the inequality

$$\|v(\cdot, s\phi(t))\|_{L^q(\mathbb{R}_x^n)} \leq C s^{-n(\frac{1}{p}-\frac{1}{q})} t^{-n(k+1)(\frac{1}{p}-\frac{1}{q})} \|\phi\|_{L^p(\mathbb{R}_x^n)} \quad \text{for all } s, t \in (0, \infty)$$

provided that $1 < p \leq 2$, $1/p + 1/q = 1$.

Proof. It follows from the results of [6]. \square

Theorem 6.4. For the solution $u = u(x, t)$ of the Cauchy problem ($n > 1$)

$$u_{tt} - t^{2k} \Delta u = f(x, t), \quad u(x, 0) = 0, \quad u_t(x, 0) = 0 \tag{6.1}$$

with the function $f \in C^2([0, \infty) \times \mathbb{R}^n)$ such that $f(t, \cdot) \in C_0^\infty(\mathbb{R}_x^n)$ for every $t \in [0, \infty)$, the following estimate holds

$$\begin{aligned} \|u(\cdot, t)\|_{L^q(\mathbb{R}_x^n)} &\leq C t^{2-n(k+1)(\frac{1}{p}-\frac{1}{q})} \int_0^1 \|f(\cdot, tb)\|_{L^p(\mathbb{R}_x^n)} db \int_0^{1-b^{k+1}} s^{-n(\frac{1}{p}-\frac{1}{q})} \\ &\quad \times (b^{k+1} + 1 + s)^{-\gamma} (b^{k+1} + 1 - s)^{-\gamma} ds, \end{aligned}$$

provided that $n(\frac{1}{p} - \frac{1}{q}) < 1$, $1 < p \leq 2$, $1/p + 1/q = 1$.

Proof. We give a proof for the odd n only, since the proof for even n is very similar. According to Theorem 3.4 for $n = 2m + 1$, $x \in \mathbb{R}^n$, the classical solution $u = u(x, t)$ to the Cauchy problem with $f \in C_{x,t}^{(n+3)/2,2}$ is given by the formula (3.6), which can be rewritten as follows:

$$\begin{aligned} u(x, t) &= t^2 2c_k \phi(1)^{\phi(1)} \int_0^1 db \int_0^{1-b^{k+1}} ds (s + 1 + b^{k+1})^{-\gamma} (b^{k+1} - s + 1)^{-\gamma} \\ &\quad \times \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{\frac{n-3}{2}} \frac{r^{n-2}}{\omega_{n-1} c_0^{(n)}} \int_{S^{n-1}} f(x + ry, tb) dS_y \right)_{r=s\phi(t)} \\ &\quad \times F \left(\gamma, \gamma; 1; \frac{(-s + 1 - b^{k+1})(-s - 1 + b^{k+1})}{(-s + 1 + b^{k+1})(-s - 1 - b^{k+1})} \right), \end{aligned}$$

where $c_0^{(n)} = 1 \cdot 3 \cdot \dots \cdot (n - 2)$. To estimate its norm we write

$$\begin{aligned} \|u(\cdot, t)\|_{L^q(\mathbb{R}_x^n)} &\leq t^2 2c_k \phi(1)^{\phi(1)} \int_0^1 db \int_0^{1-b^{k+1}} ds (s + 1 + b^{k+1})^{-\gamma} (b^{k+1} - s + 1)^{-\gamma} \\ &\quad \left\| \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right)^{\frac{n-3}{2}} \frac{r^{n-2}}{\omega_{n-1} c_0^{(n)}} \int_{S^{n-1}} f(x + ry, tb) dS_y \right) \right\|_{r=s\phi(t)} \Big\|_{L^q(\mathbb{R}_x^n)} \\ &\quad \times F \left(\gamma, \gamma; 1; \frac{(-s + 1 - b^{k+1})(-s - 1 + b^{k+1})}{(-s + 1 + b^{k+1})(-s - 1 - b^{k+1})} \right). \end{aligned}$$

An application of Lemma 6.3 gives

$$\begin{aligned} \|u(\cdot, t)\|_{L^q(\mathbb{R}_x^n)} &\leq C t^{2-n(k+1)(\frac{1}{p}-\frac{1}{q})} \int_0^1 \|f(x, tb)\|_{L^p(\mathbb{R}_x^n)} db \\ &\quad \times \int_0^{1-b^{k+1}} ds s^{-n(\frac{1}{p}-\frac{1}{q})} (s + 1 + b^{k+1})^{-\gamma} (b^{k+1} - s + 1)^{-\gamma} \\ &\quad \times F \left(\gamma, \gamma; 1; \frac{(-s + 1 - b^{k+1})(-s - 1 + b^{k+1})}{(-s + 1 + b^{k+1})(-s - 1 - b^{k+1})} \right), \end{aligned}$$

since $n(\frac{1}{p} - \frac{1}{q}) < 1$. The theorem is proved. \square

Corollary 6.5. *If we assume that the function $f \in C^2([0, \infty) \times \mathbb{R}^n)$ is such that $f(\cdot, t) \in C_0^\infty(\mathbb{R}_x^n)$ for every $t \in [0, \infty)$, and with some $\mu > -1$,*

$$t^{-\mu} \|f(\cdot, t)\|_{L^p(\mathbb{R}^n)} \leq \text{const} \quad \text{for all } t,$$

then the solution $u = u(x, t)$ of the Cauchy problem (6.1) satisfies

$$\|u(\cdot, t)\|_{L^q(\mathbb{R}_x^n)} \leq C t^{\mu+2-n(k+1)(\frac{1}{p}-\frac{1}{q})} \max_{0 \leq \tau \leq t} \left(\|f(\cdot, \tau)\|_{L^p(\mathbb{R}_x^n)} \tau^{-\mu} \right) \quad \text{for all } t.$$

Proof. Indeed, according to the theorem

$$\begin{aligned} &\|u(\cdot, t)\|_{L^q(\mathbb{R}_x^n)} \\ &\leq C t^{1-n(k+1)(\frac{1}{p}-\frac{1}{q})} \max_{0 \leq \tau \leq t} \left(\|f(\cdot, \tau)\|_{L^p(\mathbb{R}_x^n)} \tau^{-\mu} \right) \int_0^t b^\mu db \end{aligned}$$

$$\begin{aligned}
& \times \int_0^1 \int_0^{1-(b/t)^{k+1}} s^{-n(\frac{1}{p}-\frac{1}{q})} ((b/t)^{k+1} + 1 + s)^{-\gamma} ((b/t)^{k+1} + 1 - s)^{-\gamma} ds \\
& \leq C t^{\mu+2-n(k+1)(\frac{1}{p}-\frac{1}{q})} \max_{0 \leq \tau \leq t} (\|f(\cdot, \tau)\|_{L^p(\mathbb{R}_x^n)} \tau^{-\mu}) \int_0^1 b^\mu db \\
& \quad \times \int_0^{1-b^{k+1}} s^{-n(\frac{1}{p}-\frac{1}{q})} (b^{k+1} + 1 + s)^{-\gamma} (b^{k+1} + 1 - s)^{-\gamma} ds,
\end{aligned}$$

which completes the proof of the corollary. \square

In conclusion we note that we did not intend to minimize the regularity hypothesis on the function f needed in Theorems 6.1, 6.4, and Corollaries 6.2, 6.5. That minimization is crucial for the weak solutions of the nonlinear equations and it will be done in a forthcoming paper.

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Further reading

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