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Diffusion–dispersion limits for multidimensional scalar conservation laws with source terms

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ABSTRACT

In this paper we consider conservation laws with diffusion and dispersion terms. We study the convergence for approximation applied to conservation laws with source terms. The proof is based on the Hwang and Tzavaras's new approach [Seok Hwang, Athanasios E. Tzavaras, Kinetic decomposition of approximate solutions to conservation laws: Application to relaxation and diffusion–dispersion approximations, Comm. Partial Differential Equations 27 (5–6) (2002) 1229–1254] and the kinetic formulation developed by Lions, Perthame, and Tadmor [P.-L. Lions, B. Perthame, E. Tadmor, A kinetic formulation of multidimensional scalar conservation laws and related equations, J. Amer. Math. Soc. 7 (1) (1994) 169–191].

1. Introduction

This article is concerned with the following Cauchy problem

$$\partial_t u + \operatorname{div}_x A(u) = \epsilon \Delta_x u + \delta \sum_{j=1}^{d} \partial_{x_j x_j x_j} u + g^{\epsilon}(x) s(u), \qquad u(0,x) = u_0^{\epsilon,\delta}, \tag{1}$$

where $x \in \mathbb{R}^d$, $t \geqslant 0$ and the flux function is assumed to be regular, $A \in C^2(\mathbb{R})$ and the source term is defined as follows:

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- The source term is bounded and non-increasing in u,

$$s \in \mathcal{C}^1(\mathbb{R}), \quad s'(u) \leqslant 0, \quad 0 \leqslant g(x) \leqslant G.$$

- We also assume that the function s' is bounded below, namely,

$$\exists \lambda > 0 \quad \text{such that} \quad -\lambda \leqslant s'(u).$$

- $g^\epsilon(x) = g *_x \varphi_\epsilon(x)$, where φ is a mollified function and $g \in L^1(\mathbb{R}^d)$.

The existence of solutions for Eqs. (1) is well known and we now investigate the convergence for solutions of Eq. (1) toward a weak solution of the following equation:

$$\partial_t u + \operatorname{div}_x A(u) = g(x)s(u), \qquad u(0,x) = u_0, \quad x \in \mathbb{R}^d, \ t \geqslant 0, \tag{2}$$

such that the solution u^ϵ of Eq. (1) converges toward an entropy solution of Eq. (2) if

$$\delta = O\big(\epsilon^2\big), \tag{3}$$

and toward the Kruzkov's unique solution of Eq. (2) under the condition

$$\delta = o\big(\epsilon^2\big). \tag{4}$$

For the homogeneous conservation laws, there have been many recent studies concerning the convergence for solutions of Eq. (1). Lax and Levermore [9–11] showed that the solution u_δ of the Korteweg–de Vries equation

$$\partial_t u + u\partial_x u + \delta \partial_{xxx} u = 0 \tag{5}$$

does not converge to a solution of the following Burgers equation

$$\partial_t u + u\partial_x u = 0. \tag{6}$$

On the other hand, following Schonbek's work [18], we can obtain the strong convergence of uniform L^p_{loc} bounded approximation solutions to the following:

$$\partial_t u + \partial_x A(u) = \epsilon \partial_x u + \delta \partial_{xxx} u \tag{7}$$

and it gives an important contribution of another method of compensated compactness in the L^p setting for $p > 1$. In Kondo and LeFloch [5], they have developed Schonbek's work with using Diperna's uniqueness theorem for measure value solutions. As another approach, Hwang and Tzavaras [4] used the kinetic formulation with the velocity averaging lemma to study the convergence of approximate solutions of multidimensional scalar conservation laws and, of course, the flux verifies the nonlinearity condition (8). There are many related works about those problems [1–3,12,19]. In this paper the main contribution is to study convergence for solution of Eqs. (1) toward nonhomogeneous conservation laws and to converge to the unique solution of Eq. (2) in $L^p(\mathbb{R}^d)$ for $1 < p < 2$ under the condition $\delta = o(\epsilon^2)$. The main tool is based on the kinetic formulation developed by Lions et al. [13] and the strong trace results (see Karlsen and Kwon [7] for the nonhomogeneous case and Kwon, Panov, and Vasseur [8,14,15,20] for the homogeneous cases) for scalar conservation laws. From now on we restrict the flux A verifying the following:

$$\mathcal{L}\big(\big\{\xi \mid \tau + \zeta \cdot A'(\xi) = 0\big\}\big) = 0, \quad \text{for every } (\tau,\xi) \neq (0,0), \tag{8}$$

where $\mathcal{L}$ is the Lebesgue measure. We now state the main theorem.

Theorem 1.1. *Assume that there exists $C > 0$ independent of ϵ such that*

$$\|u_0^\epsilon\|_{L^2(\mathbb{R}^d)} + \sqrt{\epsilon}\|\nabla u_0^\epsilon\|_{L^2(\mathbb{R}^d)} \leqslant C. \tag{9}$$

Consider the flux functions A_j to be globally Lipschitz for all $j = 1, 2, \ldots, d$ and the flux A satisfying the nonlinearity condition (8)*. Then*:

- *If $\delta = O(\epsilon^2)$, then the solution u^ϵ converges toward a weak solution of* (2) *in $L^p_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ for all $1 < p < 2$.*
- *If $\delta = o(\epsilon^2)$, then the limit is the unique Kruzkov's entropy solution.*

Remark 1.1. In this paper we will follow Hwang and Tzavaras's framework [4] to show the strong convergence and the uniqueness proof for $\delta = o(\epsilon^2)$ is based on Perthame's work [16].

Remark 1.2. The strong trace result in Kwon and Karlsen [7] also plays an essential role in the uniqueness proof for the case of $\delta = o(\epsilon^2)$.

This paper is organized as follows. In Section 2 we deduce some uniform bounds. Section 3 is devoted to showing the strong convergence for each case: $\delta = O(\epsilon^2)$ and $\delta = o(\epsilon^2)$. In Section 4 we introduce the strong trace result and provide the uniqueness proof for the case of $\delta = o(\epsilon^2)$.

2. Uniform bounds

In this section we are going to deduce some uniform bounds.

Theorem 2.1. *Assume that the flux functions A_j are globally Lipschitz for each $j = 1, 2, \ldots, d$ and the initial data u_0^ϵ verifies the bounds* (9)*. Consider a solution u^ϵ verifying* (1)*. Then, the following uniform bounds hold*:

$$u^\epsilon(t,x) \in_b L^\infty\big((0,T); L^2\big(\mathbb{R}^d\big)\big), \tag{10}$$

$$2\epsilon \sum_{j=1}^d \big(u^\epsilon_{x_j}(t,x)\big)^2 \in_b L^1\big((0,T)\times\mathbb{R}^d\big), \tag{11}$$

$$\epsilon^3 \sum_{j=1}^d \big(\partial_{x_j x_j} u^\epsilon(t,x)\big)^2 \in_b L^1\big((0,T)\times\mathbb{R}^d\big) \tag{12}$$

for sufficiently small $\epsilon > 0$ where $v \in_b Y$ means that v is uniformly bounded in a Banach space Y.

Proof. Let $u_\epsilon = u$ for simplicity and let F be any smooth function $F : \mathbb{R} \to \mathbb{R}$. Multiplying Eq. (1) by $F'(u)$ yields the following equality:

$$\begin{aligned}\partial_t F(u) + \operatorname{div}_x Q(u) &= \epsilon \sum_{j=1}^d \partial_{x_j}\big(F'(u)\partial_{x_j} u\big) - \epsilon F''(u)|\nabla_x u|^2 + \delta \sum_{j=1}^d \partial_{x_j}\big(F'(u)\partial_{x_j x_j} u\big) \\ &\quad - \delta F''(u) \sum_{j=1}^d (\partial_{x_j} u)(\partial_{x_j x_j} u) + g^\epsilon(x) s(u) F'(u),\end{aligned} \tag{13}$$

where $Q' = F'A'$. We next integrate (13) for x and obtain

$$\partial_t \int_{\mathbb{R}^d} F(u)\,dx + \epsilon \int_{\mathbb{R}^d} F''(u)|\nabla_x u|^2\,dx$$
$$= \frac{\delta}{2}\int_{\mathbb{R}^d} F'''(u)\sum_{j=1}^d (\partial_{x_j} u)^3\,dx + \int_{\mathbb{R}^d} g^\epsilon(x)s(u)F'(u)\,dx. \tag{14}$$

Integrating (14) with respect to t yields the following equality:

$$\int_{\mathbb{R}^d} F\big(u(t)\big)\,dx + \epsilon \int_0^t \int_{\mathbb{R}^d} F''(u)|\nabla_x u|^2\,dx\,dt$$
$$= \int_{\mathbb{R}^d} F(u_0)\,dx + \frac{\delta}{2}\int_0^t\int_{\mathbb{R}^d} F'''(u)\sum_{j=1}^d (\partial_{x_j} u)^3\,dx\,dt + \int_0^t\int_{\mathbb{R}^d} g^\epsilon(x)s(u)F'(u)\,dx\,dt. \tag{15}$$

Taking $F(u) = u^2$ in (15), it follows that

$$\int_{\mathbb{R}^d} u(t)^2\,dx + 2\epsilon\int_0^t\int_{\mathbb{R}^d} |\nabla_x u|^2\,dx\,dt = \int_{\mathbb{R}^d} u_0^2\,dx + 2\int_0^t\int_{\mathbb{R}^d} g^\epsilon(x)s(u)u\,dx\,dt. \tag{16}$$

We recall $g(x) \geqslant 0$ and $s'(u) \leqslant 0$ for a.e. $x \in \mathbb{R}^d$. Then we see that

$$\int_{\mathbb{R}^d} u(t)^2\,dx + 2\epsilon\int_0^t\int_{\mathbb{R}^d} |\nabla_x u|^2\,dx + 2\int_0^t\int_{\mathbb{R}^d} g^\epsilon(x)\big|s(u)-s(0)\big||u|\,dx\,dt$$
$$\leqslant \int_{\mathbb{R}^d} u_0^2\,dx + 2\int_0^t\int_{\mathbb{R}^d} |g^\epsilon(x)\big|s(0)\big||u|\,dx\,dt$$
$$\leqslant \int_{\mathbb{R}^d} u_0^2\,dx + \int_0^t\int_{\mathbb{R}^d} u(t)^2\,dx\,dt + G\big|s(0)\big|^2\big\|g^\epsilon(x)\big\|_{L^1(\mathbb{R}^d)}. \tag{17}$$

Gronwall's inequality implies that

$$\int_{\mathbb{R}^d} u(t)^2\,dx + 2\epsilon\int_0^t\int_{\mathbb{R}^d} |\nabla_x u|^2\,dx\,dt + 2\int_0^t\int_{\mathbb{R}^d} g^\epsilon(x)\big|s(u)-s(0)\big||u|\,dx\,dt \leqslant C(T) \tag{18}$$

for all $t \in (0,T)$ which shows (10) and (11).

To estimate (12), we first differentiate Eq. (1) for variables x_k and multiply by $\partial_{x_k} u$. Then it follows that

$$\partial_t\left(\frac{1}{2}|\partial_{x_k} u|^2\right) + \sum_{j=1}^d \left(\partial_{x_j}\big(|\partial_{x_k} u|^2 A_j'(u)\big) - \partial_{x_k} u\,\partial^2_{x_j x_k} u\,A_j'(u)\right)$$

$$= \epsilon \sum_{j=1}^{d} \bigl(\partial_{x_j}(\partial_{x_k} u \partial_{x_k x_j} u) - |\partial_{x_k x_j} u|^2\bigr) + \delta \sum_{j=1}^{d} \partial_{x_j}\left(\partial_{x_k} u \partial_{x_k x_j x_j} u - \frac{1}{2}|\partial_{x_k x_j} u|^2\right)$$
$$+ \partial_{x_k}\bigl(g^{\epsilon}(x) s(u)\bigr)\partial_{x_k} u. \tag{19}$$

Integrating (19) with respect to variables (t, x), we see that

$$\int_{\mathbb{R}^d} |\partial_{x_j} u|^2 \,dx + 2\epsilon \sum_{j=1}^{d} \int_0^t \int_{\mathbb{R}^d} |\partial_{x_j x_k} u|^2 \,dx\,dt$$
$$\leqslant \int_{\mathbb{R}^d} |\partial_{x_k} u_0|^2 \,dx\,dt + \sum_{j=1}^{d} \int_0^t \int_{\mathbb{R}^d} 2\|A_j'\|_\infty \bigl|\partial^2_{x_j x_k} u\bigr| |\partial_{x_k} u| \,dx\,dt$$
$$+ \int_0^t \int_{\mathbb{R}^d} \partial_{x_k}\bigl(g^{\epsilon}(x) s(u)\bigr)\partial_{x_k} u \,dx\,dt$$
$$\leqslant \int_{\mathbb{R}^d} |\partial_{x_k} u_0|^2 \,dx\,dt + \frac{1}{\epsilon} M \int_0^t \int_{\mathbb{R}^d} |\partial_{x_k} u|^2 \,dx\,dt$$
$$+ \epsilon \sum_{j=1}^{d} \int_0^t \int_{\mathbb{R}^d} |\partial_{x_j x_k} u|^2 \,dx\,dt - \int_0^t \int_{\mathbb{R}^d} g^{\epsilon}(x) s(u) \partial_{x_k x_k} u \,dt\,dx$$
$$\leqslant \int_{\mathbb{R}^d} |\partial_{x_k} u_0|^2 \,dx\,dt + \frac{1}{\epsilon} M \int_0^t \int_{\mathbb{R}^d} |\partial_{x_k} u|^2 \,dx\,dt$$
$$+ \epsilon \sum_{j=1}^{d} \int_0^t \int_{\mathbb{R}^d} |\partial_{x_j x_k} u|^2 \,dx\,dt + \int_0^t \int_{\mathbb{R}^d} \bigl|g^{\epsilon}(x)\bigr| \bigl(\bigl|s(u) - s(0)\bigr| + \bigl|s(0)\bigr|\bigr) |\partial_{x_k x_k} u| \,dx\,dt$$
$$\leqslant \int_{\mathbb{R}^d} |\partial_{x_k} u_0|^2 \,dx\,dt + \frac{1}{\epsilon} M \int_0^t \int_{\mathbb{R}^d} |\partial_{x_k} u|^2 \,dx\,dt$$
$$+ \epsilon \sum_{j=1}^{d} \int_0^t \int_{\mathbb{R}^d} |\partial_{x_j x_k} u|^2 \,dx\,dt + G\lambda \int_0^t \int_{\mathbb{R}^d} |u| |\partial_{x_k x_k} u| \,dx\,dt$$
$$+ \bigl|s(0)\bigr| \int_0^t \int_{\mathbb{R}^d} \bigl|g^{\epsilon}(x)\bigr| |\partial_{x_k x_k} u| \,dt\,dx$$
$$\leqslant \int_{\mathbb{R}^d} |\partial_{x_k} u_0|^2 \,dx\,dt + \frac{1}{\epsilon} M \int_0^t \int_{\mathbb{R}^d} |\partial_{x_k} u|^2 \,dx\,dt$$
$$+ \epsilon \sum_{j=1}^{d} \int_0^t \int_{\mathbb{R}^d} |\partial_{x_j x_k} u|^2 \,dx\,dt + \frac{G\lambda}{2} \int_0^t \int_{\mathbb{R}^d} \frac{1}{\epsilon^2} |u|^2$$

$$
\begin{aligned}
&+\epsilon^2\sum_{k=1}^{d}|\partial_{x_kx_k}u|^2\,dt\,dx+\frac{|s(0)|}{2}\int_0^t\int_{\mathbb{R}^d}\frac{1}{\epsilon^2}\big|g^\epsilon(x)\big|^2\,dt\,dx\\
&+\int_0^t\int_{\mathbb{R}^d}\epsilon^2\sum_{j=1}^{d}|\partial_{x_jx_k}u|^2\,dt\,dx\\
\leqslant{}&\int_{\mathbb{R}^d}|\partial_{x_k}u_0|^2\,dx\,dt+\frac{1}{\epsilon}M\int_0^t\int_{\mathbb{R}^d}|\partial_{x_k}u|^2\,dx\,dt\\
&+\epsilon\sum_{j=1}^{d}\int_0^t\int_{\mathbb{R}^d}|\partial_{x_jx_k}u|^2\,dx\,dt+C\int_0^t\int_{\mathbb{R}^d}\frac{1}{\epsilon^2}\big(|u|^2+\big|g^\epsilon\big|\big)\,dt\,dx\\
&+\frac{\epsilon}{2}\int_0^t\int_{\mathbb{R}^d}\sum_{j=1}^{d}|\partial_{x_jx_k}u|^2\,dt\,dx. \qquad (20)
\end{aligned}
$$

Notice that

$$\epsilon^2<\frac{\epsilon}{2}$$

for sufficiently small ϵ. Thus, the above inequalities (9) and (20) deduce

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\epsilon^2|\partial_{x_k}u|^2\,dx+\sum_{j=1}^{d}\int_0^t\int_{\mathbb{R}^d}\frac{\epsilon^3}{2}|\partial_{x_jx_k}u|^2\,dx\,dt\\
&\quad\leqslant\int_{\mathbb{R}^d}\epsilon^2|\partial_{x_k}u_0|^2\,dx+C\int_0^t\int_{\mathbb{R}^d}\big(|u|^2+\big|g^\epsilon\big|\big)\,dt\,dx+M\int_0^t\int_{\mathbb{R}^d}\epsilon|\partial_{x_k}u|^2\,dx\,dt\leqslant C(T) \qquad (21)
\end{aligned}
$$

for all $t\in(0,T)$. The proof is complete. □

3. Convergence results

In this section we first show convergence of solutions and then uniqueness for $\delta=o(\epsilon^2)$. Let $u^{\epsilon,\delta}:=u^\epsilon$ (we only deal with $\delta=O(\epsilon^2)$ and $\delta=o(\epsilon^2)$). We now begin with Eq. (13) and define a function χ by

$$\chi(v,\xi)=\begin{cases}\mathbf{1}_{\{0\leqslant\xi\leqslant v\}} & \text{if } v\geqslant 0,\\ -\mathbf{1}_{\{v\leqslant\xi\leqslant 0\}} & \text{if } v<0.\end{cases}$$

Let us denote f^ϵ by $f^\epsilon(t,x,\xi)=\chi(u^\epsilon,\xi)$ where u^ϵ is a solution of Eq. (1). For any $\Phi\in C_c^\infty(\mathbb{R}^+\times\mathbb{R}^d)$, $F\in C_c^\infty(\mathbb{R})$, one has the following:

$$
\begin{aligned}
&\int\big\{f^\epsilon(t,x,\xi)\partial_t\Phi(t,x)+A(\xi)\cdot\nabla_x\Phi(t,x)f^\epsilon(t,x,\xi)\big\}F'(\xi)\,d\xi\,dt\,dx\\
&\quad=\int\sum_{j=1}^{d}\big(\epsilon\partial_{x_j}u^\epsilon+\delta\partial^2_{x_jx_j}u^\epsilon\big)F'\big(u^\epsilon\big)\partial_{x_j}\Phi(t,x)\,dt\,dx
\end{aligned}
$$

$$+\int F''(u^\epsilon)\left(\epsilon|\nabla_x u^\epsilon|^2+\delta\sum_{j=1}^d(\partial_{x_j}u^\epsilon)(\partial^2_{x_jx_j}u^\epsilon)\right)\Phi(t,x)\,dt\,dx$$
$$-\int g^\epsilon(x)s(u^\epsilon)F'(u^\epsilon)\Phi(t,x)\,dt\,dx \tag{22}$$

which implies that

$$\begin{aligned}\partial_t f^\epsilon+A'(\xi)\cdot\nabla_x f^\epsilon &= \sum_{j=1}^d \partial_{x_j}\big[\big(\epsilon\partial_{x_j}u^\epsilon+\delta\partial^2_{x_jx_j}u^\epsilon\big)\big(\delta(u^\epsilon-\xi)\big)\big]\\ &\quad+\partial_\xi\left[\left(\epsilon|\nabla_x u^\epsilon|^2+\delta\sum_{j=1}^d(\partial_{x_j}u^\epsilon)(\partial^2_{x_jx_j}u^\epsilon)\right)\big(\delta(u^\epsilon-\xi)\big)\right]\\ &\quad+g^\epsilon(x)s(u^\epsilon)\delta(u^\epsilon-\xi)\\ &=:\sum_{j=1}^d\partial_{x_j}\Gamma_j^\epsilon+\partial_\xi\Lambda_1^\epsilon+\Lambda_2^\epsilon \quad \text{in } \mathcal{D}'.\end{aligned}\tag{23}$$

Observe the following equality on Λ_2^ϵ: for any $\sigma\in C_c^\infty(\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R})$,

$$\begin{aligned}\big|\langle\Lambda_2^\epsilon,\sigma\rangle\big| &\leqslant \|\sigma\|_\infty\int|g^\epsilon(x)||s(u^\epsilon)|\,dt\,dx\\ &\leqslant C(T)\|\sigma\|_\infty\left(\int|g^\epsilon(x)|\,dx+\int|u^\epsilon|^2\,dt\,dx\right)\\ &\leqslant C'(T)\|\sigma\|_\infty.\end{aligned}\tag{24}$$

Thus, we show that the measure Λ_2^ϵ is bounded thanks to (24). Indeed we may use Sobolev injection to represent the following:

$$\Lambda_2^\epsilon(t,x,\xi)=\operatorname{div}_{(t,x,\xi)}\lambda_2^\epsilon(t,x,\xi),\tag{25}$$

where $\lambda_2^\epsilon(t,x,\xi)$ is compact in $L^q(\mathbb{R}^{d+2})$ for some $q>1$. Combining (25) and the following Lemma 3.1 yield that:

$$\partial_t f^\epsilon+A'(\xi)\cdot\nabla_x f^\epsilon=\sum_{j=1}^d\partial_{x_j}\big(\bar\gamma_j^\epsilon+\partial_\xi\gamma_j^\epsilon\big)+\partial_\xi\operatorname{div}_{(t,x,\xi)}\lambda_1^\epsilon+\operatorname{div}_{(t,x,\xi)}\lambda_2^\epsilon,\tag{26}$$

where $\bar\gamma_j^\epsilon,\gamma_j^\epsilon\to 0$ in L^2 for $j=1,2,\dots,d$ and $\lambda_i^\epsilon\in W^{1,q}_{\mathrm{loc}}$ for $i=1,2$.

Lemma 3.1. *(See [4].) Consider Γ_j^ϵ and Λ_1^ϵ given in* (23)*. Then, we have the following*:

$$\Gamma_j^\epsilon=\bar\gamma_j^\epsilon+\partial_\xi\gamma_j^\epsilon,\quad \textit{and}\quad \Lambda_1^\epsilon=\operatorname{div}_{(t,x,\xi)}\lambda_1^\epsilon,$$

where $\bar\gamma_j^\epsilon,\gamma_j^\epsilon\to 0$ in L^2 for $j=1,2,\dots,d$ and $\lambda_i^\epsilon\in W^{1,q}_{\mathrm{loc}}$ for $i=1,2$.

We are now able to prove the strong convergence of f^ϵ which is based on the averaging lemma. We use the following theorem, which is a particular case of the version of Perthame and Souganidis (see [17]):

Theorem 3.1. *Let N be an integer, f_n bounded in $L^\infty(\mathbb{R}^{N+1})$ and $\{h_n^1, h_n^2\}$ be relatively compact in $[L^p(\mathbb{R}^{N+1})]^{2N}$ with $1 < p < +\infty$ solutions of the transport equation*:

$$\partial_t f_n + A'(\xi) \cdot \nabla_x f_n = \partial_\xi \big(\nabla_{(t,x)} \cdot h_n^1\big) + \nabla_{(t,x)} \cdot h_n^2,$$

where $A' \in [C^2(\mathbb{R})]^N$ verifies the non-degeneracy condition (8)*. Let $\phi \in \mathcal{D}(\mathbb{R})$, then the average $u_n^\phi(y) = \int_{\mathbb{R}} \phi(\xi) f_n(y,\xi)\, d\xi$ is relatively compact in $L^p(\mathbb{R}^N)$.*

Therefore, we see that up to subsequence $\int_{\mathbb{R}} \phi(\xi) f_n(y,\xi)\, d\xi$ converges in $L^p(\mathbb{R}^N)$ for $1 < p < \infty$ and hence $u^\epsilon = \int f_n\, d\xi$ converges to $u = \int f\, d\xi$ in $L^p_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^d)$ for $1 < p < 2$ thanks to the uniqueness of the limit. Indeed, as shown in Hwang and Tzavaras [4] we obtain the following kinetic equation:

$$\partial_t f + A'(\xi) \cdot \nabla_x f + g(x)s(\xi)\big(\partial_\xi f - \delta(\xi)\big) = \partial_\xi m \quad \text{in } \mathcal{D}' \tag{27}$$

for some $m \in \mathcal{M}((0,T) \times \mathbb{R}^d \times (-L, L))$. We now need to show that the above equation (27) holds on the domain $(0,\infty) \times \mathbb{R}^d$. Let us denote $v(t,x) = u(t+T,x)$ where u is a solution of Eq. (1) and then v is obviously a solution of Eq. (1) with initial data $v_0(x) := v(0,x) = u(T,x)$. Then the uniform bounds (10) and (11) in Theorem 2.1 verify the assumption (9) and thus we can show the above argument for the solution v which provides that: there exists u on $(T,2T) \times \mathbb{R}^d$ such that u^ϵ converges strongly to u and u is a solution of Eq. (2) on $(T,2T) \times \mathbb{R}^d$. In conclusion, using a standard diagonalization process, we obtain the strong convergence and (27) on the $\mathbb{R}^+ \times \mathbb{R}^d$. To complete the proof of Theorem 1.1, we need to show the uniqueness of solution u to (2) for $\delta = o(\epsilon^2)$. The proof will be provided in the following section.

4. Uniqueness proof for $\delta = o(\epsilon^2)$

We first introduce the strong trace result [7] for Eq. (2) near the boundary $\{t = 0\}$ which plays an important role in the uniqueness proof of (2) and (27).

Theorem 4.1. *Let the flux function A lie in $C^2(\mathbb{R})$ and satisfy* (8)*. Consider any function $u \in L^\infty(\mathbb{R} \times \mathbb{R}^d)$ which verifies* (2) *and* (27) *in $\mathbb{R} \times \mathbb{R}^d$. Then, there exists $u^\tau \in L^\infty(\mathbb{R}^d)$ such that for every compact set $K \Subset \Omega$*:

$$\operatorname*{ess\,lim}_{s \to 0} \int_K \big|u(s,x) - u^\tau(x)\big|\, dx = 0. \tag{28}$$

In particular, the trace u^τ is unique and for any function $F \in C^0(\mathbb{R})$, $F(u)$ has also a strong trace $F(u)^\tau$ and

$$\big[F(u)\big]^\tau = F\big(u^\tau\big).$$

From Theorem 4.1, we are able to show the uniqueness of (2) and (27).

Proposition 4.1. *Assume that the measure m in* (27) *is nonnegative. Let u and v be solutions of* (2) *and* (27) *with initial conditions $u(0,x) = u_0(x) \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $v(0,x) = v_0(x) \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, respectively. Then, for a.e. $t \in \mathbb{R}^+$,*

$$\int_{B_R} \big|u(t,x) - v(t,x)\big|\, dx \leqslant \int_{B_R} \big|u_0(x) - v_0(x)\big|\, dx, \tag{29}$$

where $B_R = \{x \in \mathbb{R}^d \mid \|x\| \leqslant R\}$ for any $R > 0$.

Remark 4.1. The measure m given in (27) is nonnegative for $\delta = o(\epsilon^2)$, but we do not know whether it is nonnegative or not for $\delta = O(\epsilon^2)$ (see Hwang and Tzavaras [4]). Thus, we cannot verify the uniqueness of Eqs. (2) and (27) for $\delta = O(\epsilon^2)$.

Remark 4.2. We are able to see Kwon [6] in which the same method is used for the initial boundary value problem for multidimensional scalar conservation laws.

Proof of Proposition 4.1. We first need to regularize the kinetic equation (27) with respect to the variables (t,x) by convolution of mollified functions. This method was first initiated by Perthame [16] for the uniqueness proof of an initial value problem. Let u and v be solutions of (2) and (27). We set two χ functions f_1 and f_2 corresponding to solutions u and v, respectively by $f_1(t,x,\xi) = \chi(\xi;u(t,x))$ and $f_2(t,x,\xi) = \chi(\xi;v(t,x))$. We recall kinetic equations (27) for f_1 and f_2 respectively. From (27), there exist $m_1, m_2 \in \mathcal{M}^+(\mathbb{R}^+ \times \mathbb{R}^d \times (-L,L))$ such that

$$\begin{aligned}
&\partial_t f_1 + a(\xi)\cdot\nabla_x f_1 + g(x)s(\xi)\big(\partial_\xi f_1 - \delta(\xi)\big) = \partial_\xi m_1,\\
&\partial_t f_2 + a(\xi)\cdot\nabla_x f_2 + g(x)s(\xi)\big(\partial_\xi f_2 - \delta(\xi)\big) = \partial_\xi m_2
\end{aligned} \tag{30}$$

for χ functions f_1, f_2 respectively.

We now want to show the following inequality:

$$\int_{B_R}\int_{-L}^{L} \partial_t\big|f_1(t,x,\xi) - f_2(t,x,\xi)\big|^2 - g(x)s'(\xi)\big|f_1(t,x,\xi) - f_2(t,x,\xi)\big|^2\,d\xi\,dx \leqslant 0 \tag{31}$$

for a.e. $t \in \mathbb{R}$. We need first to regularize f_1 and f_2 with respect to variable (t,x). We set $\epsilon = (\epsilon_1,\epsilon_2)$ and define ϕ_ϵ by

$$\phi_\epsilon(t,x) = \frac{1}{\epsilon_1}\phi_1\left(\frac{t}{\epsilon_1}\right)\frac{1}{\epsilon_2^d}\phi_2\left(\frac{x}{\epsilon_2}\right),$$

where $\phi_1 \in C_c^\infty(\mathbb{R})$ and $\phi_2 \in C_c^\infty(\mathbb{R}^d)$ verifying $\phi_j \geqslant 0$, $\int \phi_j = 1$, $j = 1,2$, $\operatorname{supp}(\phi_1) \subset (0,1)$. We next mention some convenient notations:

- $f_1^\epsilon(t,x,\xi) = f_1(\cdot,\cdot,\xi) *_{(t,x)} \phi_\epsilon(t,x)$, $f_2^\epsilon(t,x,\xi) = f_2(\cdot,\cdot,\xi) *_{(t,x)} \phi_\epsilon(t,x)$,
- $m_1^\epsilon(t,x,\xi) = m_1(\cdot,\cdot,\xi) *_{(t,x)} \phi_\epsilon(t,x)$, $m_2^\epsilon(t,x,\xi) = m_2(\cdot,\cdot,\xi) *_{(t,x)} \phi_\epsilon(t,x)$,

where $*_{(t,x)}$ means convolution in (t,x) and we extend f_1, f_2, m_1, m_2 to $\mathbb{R}^{d+1}$ by putting 0 on $(\mathbb{R}^+ \times \mathbb{R}^d)^c$. The following lemma is devoted to controlling the part of entropy defect measures m_1, m_2 of u, v respectively and the proof is provided in Perthame [16].

Lemma 4.1. *Let m_1 and m_2 be nonnegative measures given in Eq.* (27)*. Then, the following holds*:

$$\lim_{\epsilon\to 0}\int_{-L}^{L} m_1^\epsilon(\cdot,\cdot,\xi)\delta_{(\xi=u)} * \phi_\epsilon + m_2^\epsilon(\cdot,\cdot,\xi)\delta_{(\xi=v)} * \phi_\epsilon\,d\xi = 0 \tag{32}$$

in $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^d)$.

Proof of (31). Consider a regular mollified function ϕ_ϵ as defined above. Let us take the convolution of two kinetic equations (30). Then we subtract these two equations obtained above and multiply them by $f_1^\epsilon - f_2^\epsilon$, which yields

$$\int_{\mathbb{R}^d}\int_{-L}^{L} \partial_t\left|f_1^\epsilon(t,x,\xi)-f_2^\epsilon(t,x,\xi)\right|^2+2\left[\left(g(x)s(\xi)\partial_\xi(f_1-f_2)\right)*_{(t,x)}\phi_\epsilon\right]\left(f_1^\epsilon-f_2^\epsilon\right)d\xi\,dx$$
$$=2\int_{\mathbb{R}^d}\int_{-L}^{L}\partial_\xi\left(m_1^\epsilon(t,x,\xi)-m_2^\epsilon(t,x,\xi)\right)\left(f_1^\epsilon(t,x,\xi)-f_2^\epsilon(t,x,\xi)\right)d\xi\,dx \tag{33}$$

for a.e. $t>0$. We need to deal with the part of defect measure. From Lemma 4.1, we obtain

$$\begin{aligned}
&\lim_{\epsilon\to0}\int_{\mathbb{R}^d}\int_{-L}^{L}\partial_\xi\left(m_1^\epsilon(\cdot,\cdot,\xi)-m_2^\epsilon(\cdot,\cdot,\xi)\right)\left(f_1^\epsilon(\cdot,\cdot,\xi)-f_2^\epsilon(\cdot,\cdot,\xi)\right)d\xi\,dx\\
&\quad=-\lim_{\epsilon\to0}\int_{\mathbb{R}^d}\int_{-L}^{L}\left(m_1^\epsilon(\cdot,\cdot,\xi)-m_2^\epsilon(\cdot,\cdot,\xi)\right)\partial_\xi\left(f_1^\epsilon(\cdot,\cdot,\xi)-f_2^\epsilon(\cdot,\cdot,\xi)\right)d\xi\,dx\\
&\quad=-\lim_{\epsilon\to0}\int_{\mathbb{R}^d}\int_{-L}^{L}m_1^\epsilon(\cdot,\cdot,\xi)(\delta_{(\xi=v)}*\phi_\epsilon)+m_2^\epsilon(\cdot,\cdot,\xi)(\delta_{(\xi=u)}*\phi_\epsilon)\,d\xi\,dx\\
&\quad\leqslant 0
\end{aligned}\tag{34}$$

for a.e. $t>0$. Thus, we have the following inequality:

$$\int_{\mathbb{R}^d}\int_{-L}^{L}\partial_t\left|f_1(t,x,\xi)-f_2(t,x,\xi)\right|^2d\xi\,dx+\int_{\mathbb{R}^d}\int_{-L}^{L}-g(x)s'(\xi)\left|f_1(t,x,\xi)-f_2(t,x,\xi)\right|^2d\xi\,dx\leqslant0 \tag{35}$$

for all $t\in\mathbb{R}^+$, which provides

$$\int_{\mathbb{R}^d}\int_{-L}^{L}\partial_t\left|f_1(t,x,\xi)-f_2(t,x,\xi)\right|^2d\xi\,dx\leqslant0 \tag{36}$$

thanks to $g(x)\geqslant0$ and $s'(\xi)\geqslant0$ for a.e. $(x,\xi)\in\mathbb{R}^d\times(-L,L)$. Integrating (36) for variable t gives

$$\int_{B_R}\int_{-L}^{L}\left|f_1(t,x,\xi)-f_2(t,x,\xi)\right|^2d\xi\,dx\leqslant\int_{B_R}\int_{-L}^{L}\left|f_1(s,x,\xi)-f_2(s,x,\xi)\right|^2d\xi\,dx$$

for $0<s<t$.

Since $u_0^\epsilon\rightharpoonup u_0$ and $u_\epsilon\to u$ in L^1_{loc}, we are easily able to see that $u(t,\cdot)$ weakly converges to u_0 as t tends to 0 and thus Theorem 4.1 yields that

$$\int_{B_R}\left|u(t,x)-v(t,x)\right|dx\leqslant\int_{B_R}\left|u_0(x)-v_0(x)\right|dx$$

for all $t\in\mathbb{R}^+$ and $R>0$. Indeed, $|f_1(t,x,\xi)-f_2(t,x,\xi)|^2=|f_1(t,x,\xi)-f_2(t,x,\xi)|$. The proof is complete. $\square$

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