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Global well-posedness and scattering for the focusing energy-critical nonlinear Schrödinger equations of fourth order in the radial case

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ABSTRACT

We consider the focusing energy-critical nonlinear Schrödinger equation of fourth order $iu_t + \Delta^2 u = |u|^{\frac{8}{d-4}}u$, $d \geqslant 5$. We prove that if a maximal-lifespan radial solution $u: I \times \mathbb{R}^d \to \mathbb{C}$ obeys $\sup_{t\in I} \|\Delta u(t)\|_2 < \|\Delta W\|_2$, then it is global and scatters both forward and backward in time. Here W denotes the ground state, which is a stationary solution of the equation. In particular, if a solution has both energy and kinetic energy less than those of the ground state W at some point in time, then the solution is global and scatters.

1. Introduction

Fourth-order Schrödinger equations have been introduced by Karpman [12] and Karpman, Shagalov [13] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Such fourth-order Schrödinger equa-

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tions are written as

$$i\partial_t u + \Delta^2 u + \varepsilon \Delta u + f\big(|u|^2\big)u = 0, \tag{1}$$

where $\varepsilon = \pm 1$ or $\varepsilon = 0$, and $u : I \times \mathbb{R}^d \to \mathbb{C}$ is a complex-valued function. In this paper, we will investigate the focusing energy-critical case when $\varepsilon = 0$, namely,

$$\begin{cases} iu_t + \Delta^2 u = |u|^{\frac{8}{d-4}} u, & \text{in } \mathbb{R}^d \times \mathbb{R},\ d \geqslant 5, \\ u(0) = u_0(x), & \text{in } \mathbb{R}^d. \end{cases} \tag{2}$$

The name 'energy-critical' refers to the fact that the scaling symmetry

$$u(t,x) \mapsto u_\lambda(t,x) := \lambda^{\frac{d-4}{2}} u\big(\lambda^4 t, \lambda x\big)$$

leaves both the equation and the energy invariant. The energy of a solution is defined by

$$E\big(u(t)\big) = \frac{1}{2}\int\limits_{\mathbb{R}^d} \big|\Delta u(t,x)\big|^2\,dx - \frac{d-4}{2d}\int\limits_{\mathbb{R}^d} \big|u(t,x)\big|^{\frac{2d}{d-4}}\,dx$$

and is conserved under the flow. We refer to the Laplacian term in the formula above as the kinetic energy and to the second term as the potential energy.

Definition 1.1 *(Solutions).* Let $d \geqslant 5$. A function $u : I \times \mathbb{R}^d \to \mathbb{C}$ on a non-empty time interval $I \subset \mathbb{R}$ is a solution to (2) if it lies in the class $C_t^0 \dot{H}_x^2(K \times \mathbb{R}^d) \cap L_{t,x}^{\frac{2(d+4)}{d-4}}(K \times \mathbb{R}^d)$ for all compact $K \subset I$, and obeys the Duhamel formula

$$u(t) = e^{i(t-t_0)\Delta^2} u(t_0) - i\int\limits_{t_0}^{t} e^{i(t-\tau)\Delta^2} F\big(u(\tau)\big)\,d\tau$$

for all $t, t_0 \in I$, where $F(u) = |u|^{\frac{8}{d-4}} u$. We refer to I as the lifespan of u. We say that u is a maximal-lifespan solution if the solution cannot be extended to any strictly larger interval. We say that u is a global solution if $I = \mathbb{R}$.

We define the scattering size of a solution to (2) on a time interval I by

$$S_I(u) := \int\limits_I \int\limits_{\mathbb{R}^d} \big|u(t,x)\big|^{\frac{2(d+4)}{d-4}}\,dx\,dt.$$

If $I = \mathbb{R}$, we write $S_{\mathbb{R}}(u) = S(u)$.

Associated to the notion of solution is a corresponding notion of blowup, which precisely corresponds to the impossibility of continuing the solution.

Definition 1.2 *(Blowup).* We say that a solution u to (2) blows up forward in time if there exists a time $t_1 \in I$ such that

$$S_{[t_1, \sup(I))}(u) = \infty$$

and that u blows up backward in time if there exists a time $t_1 \in I$ such that

$$S_{(\inf(I),t_1]}(u) = \infty.$$

Sharp dispersive estimates for the biharmonic Schrödinger operator in (1), namely for the linear group associated to $i\partial_t + \Delta^2 \pm \Delta$, have recently been obtained in Ben-Artzi, Koch, and Saut [2], while specific nonlinear fourth-order Schrödinger equations as in (1) have been recently discussed in Fibich, Ilan, and Papanicolaou [8], Guo and Wang [9], Hao, Hsiao, and Wang [10,11], Miao and Zhang [22] and Segata [26]. In [23], B. Pausader established the global well-posedness in the defocusing subcritical case, namely, $f(u) = |u|^{p-1}u$ with $1 < p < 1 + \frac{8}{d-4}$. Moreover, he established the global well-posedness and scattering for radial data in the defocusing critical case, namely, $p = 1 + \frac{8}{d-4}$, where a very important Strichartz estimate was established.

Lemma 1.3 *(Strichartz estimates). (See [23].) If (q, r) is such that $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, where $2 \leqslant q, r \leqslant \infty$ and $(q, r, d) \neq (2, \infty, 2)$. Let u be the solution of*

$$\begin{cases} iu_t + \Delta^2 u = h, \\ u(t_0) \in \dot{H}^2(\mathbb{R}^d). \end{cases} \tag{3}$$

Then we have

$$\|\Delta u\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} \lesssim \|\Delta u_0\|_2 + \|\nabla h\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}.$$

The key feature of such lemma is that the spacetime norm of the second derivative of u is estimated using only one derivative of the forcing term. In fact, this is the consequence of smoothing effect for all higher-order nonlinear Schrödinger equations, see Proposition 2 in [22]. This is in sharp contrast with the classical second-order nonlinear Schrödinger equations, where the estimate like (3) does not hold true as it would violate Galilean invariance. Moreover, local well-posedness and stability were established.

Theorem 1.4 *(Local well-posedness). (See [23].) Let $d \geqslant 5$. Given $u_0 \in \dot{H}_x^2(\mathbb{R}^d)$ and $t_0 \in \mathbb{R}$, there exists a unique maximal-lifespan solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to (2) with initial data $u(t_0) = u_0$. This solution also has the following properties:*

- *(Local existence) I is an open neighborhood of t_0.*
- *(Energy conservation) The energy of u is conserved, that is, $E(u(t)) = E(u_0)$ for all $t \in I$.*
- *(Continuous dependence) If $u_0^{(n)}$ is a sequence converging to u_0 in $\dot{H}_x^2(\mathbb{R}^d)$ and $u^{(n)} : I_n \times \mathbb{R}^d \to \mathbb{C}$ are the associated maximal-lifespan solutions, then $u^{(n)}$ converge locally uniformly to u, that is, on every compact interval $K \subset I$, and $K \subset I_n$ for all sufficiently large n, u_n converges strongly to u in $C_t^0 \dot{H}_x^2(K \times \mathbb{R}^d) \cap L_{t,x}^{\frac{2(d+4)}{d-4}}(K \times \mathbb{R}^d)$ as $n \to \infty$.*
- *(Blowup criterion) If $\sup(I)$ is finite, then u blows up forward in time; if $\inf(I)$ is finite, then u blows up backward in time.*
- *(Scattering) If $\sup(I) = \infty$ and u does not blow up forward in time, then u scatters forward in time, that is, there exists a unique $u_+ \in \dot{H}_x^2(\mathbb{R}^d)$ such that*

$$\lim_{t\to\infty} \left\| u(t) - e^{it\Delta^2} u_+ \right\|_{\dot{H}_x^2(\mathbb{R}^d)} = 0. \tag{4}$$

Conversely, given $u_+ \in \dot{H}_x^2(\mathbb{R}^d)$ there is a unique solution to (2) in a neighborhood of infinity so that (4) holds.

- (*Small data global existence*) *If* $\|\Delta u_0\|_2$ *is sufficiently small* (*depending on* d), *then* u *is a global solution which does not blow up either forward or backward in time. Indeed, in this case* $S_{\mathbb{R}}(u) \lesssim \|\Delta u_0\|_2^{\frac{2(d+4)}{d-4}}$.

Theorem 1.5 *(Stability). (See [23].) Let* $d \geqslant 5$. *Let* $I \subset \mathbb{R}$ *be a compact time interval such that* $0 \in I$, *and* $\tilde{u}$ *be an approximate solution of* (2) *in the sense that*

$$i\partial_t \tilde{u} + \Delta^2 \tilde{u} - |\tilde{u}|^{\frac{8}{d-4}} \tilde{u} = e$$

for some e *with* $\nabla e \in L_t^2 L_x^{\frac{2d}{d+2}}(I \times \mathbb{R}^d)$. *Assume that*

$$\|\tilde{u}\|_{L_{t,x}^{\frac{2(d+4)}{d-4}}(I\times\mathbb{R}^d)} < +\infty \quad \text{and} \quad \|\tilde{u}\|_{L_t^\infty \dot{H}_x^2(I\times\mathbb{R}^d)} < +\infty.$$

For any $\Lambda > 0$ *there exists* $\delta_0 > 0$ *such that if*

$$\|\nabla e\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)} \leqslant \delta$$

and if $u_0 \in \dot{H}_x^2(\mathbb{R}^d)$ *satisfies*

$$\big\|\tilde{u}(0) - u_0\big\|_{\dot{H}^2} \leqslant \Lambda \quad \text{and} \quad \big\|\nabla e^{it\Delta^2}\big(\tilde{u}(0) - u_0\big)\big\|_{L_t^{\frac{2(d+4)}{d-4}} L_x^{\frac{2d(d+4)}{d^2-2d+8}}(I\times\mathbb{R}^d)} \leqslant \delta$$

for some $\delta \in (0, \delta_0)$, *then there exists a solution* $u \in C(I, \dot{H}^2)$ *of* (2) *such that* $u(0) = u_0$. *Moreover,*

$$\|u\|_{L_{t,x}^{\frac{2(d+4)}{d-4}}(I\times\mathbb{R}^d)} < \infty.$$

Let

$$W(x) = W(x,t) = \left(\frac{(d(d-4)(d^2-4))^{\frac{1}{4}}}{1+|x|^2}\right)^{\frac{d-4}{2}}$$

be a stationary solution of (2). That is $W \geqslant 0$ solves the nonlinear elliptic equation

$$\Delta^2 W = |W|^{\frac{8}{d-4}} W. \tag{5}$$

Analogous to the nonlinear Schrödinger equations of the second order, we have

Conjecture 1.6. *Let* $d \geqslant 5$ *and let* $u : I \times \mathbb{R}^d \to \mathbb{C}$ *be a solution to* (2) *and* W *is the stationary solution of this equation. If*

$$E_* := \sup_{t\in I} \big\|\Delta u(t)\big\|_2 < \|\Delta W\|_2, \tag{6}$$

then

$$\int_I \int_{\mathbb{R}^d} \big|u(t,x)\big|^{\frac{2(d+4)}{d-4}}\, dx\, dt \leqslant C(E_*) < \infty.$$

Naturally, we will apply the ideas and techniques which come from the study of classical focusing nonlinear Schrödinger equations to fourth-order nonlinear Schrödinger equations. For energy-critical nonlinear Schrödinger equations

$$\begin{cases} iu_t + \Delta u = \lambda |u|^{\frac{4}{d-2}} u, & \text{in } \mathbb{R}^d \times \mathbb{R}, \\ u(0) = u_0(x) \in \dot{H}^1_x(\mathbb{R}^d), \end{cases} \tag{7}$$

the local well-posedness and global well-posedness for small data were established by T. Cazenave and F.B. Weissler [4] regardless of the sign of λ. There have been a lot of works devoted to obtaining the global well-posedness and scattering for large data in defocusing case $\lambda = 1$, see [3,5,25,27,29], etc.

However, the global well-posedness and scattering for large data in focusing case $\lambda = -1$ remains not completely solved. In [14], C.E. Kenig and F. Merle introduced an efficient approach to deal with the focusing energy-critical nonlinear Schrödinger equations, where they obtained global well-posedness and scattering for radial data with energy and kinetic energy less than those of ground state in the focusing case in dimensions $3 \leqslant d \leqslant 5$. They reduced matters to a rigidity theorem using a concentration compactness theorem, with the aid of localized Virial identity. The radiality enters only at one point in the proof of the rigidity theorem because of the difficulty in controlling the motion of spatial center of global solutions. Moreover, their result is sharp because the ground state itself is the solutions of (7) but it does not scatter. One of the main ingredients in their arguments is proved by S. Keraani in [17], namely, the fact that every sequence of solutions to the linear Schrödinger equation, with bounded data in $\dot{H}^1(\mathbb{R}^d)$ ($d \geqslant 3$) can be written, up to subsequence, as an almost orthogonal sum of sequences of the type $\lambda_n^{-\frac{d-2}{2}} V((t-t_n)/\lambda_n^2, (x-x_n)/\lambda_n)$, where V is a solution of the linear Schrödinger equation, with a small remainder term in Strichartz norm. Earlier steps in this direction include [1].

R. Killip and M. Visan [20] extended C.E. Kenig and F. Merle's result to nonradial case in $d \geqslant 5$. The method is to reduce minimal kinetic energy blowup solutions to almost periodic solutions modulo symmetries, which match one of the three scenarios: finite-time blowup, low-to-high frequency cascade and soliton. Then the aim is to eliminate such solutions. The finite-time blowup solutions can be precluded using the method in [14]. For the other two types of solutions, R. Killip and M. Visan proved that they admit additional regularities, namely, they belong to $L_t^\infty \dot{H}_x^{-\epsilon}$ for some $\epsilon > 0$. In particular, they are in L_x^2. Similar ideas have appeared in [18] and [19] in order to deal with mass-critical nonlinear Schrödinger equations. But different from before, a remarkable difficulty comes from the minimal kinetic energy blowup solution because the kinetic energy, unlike the energy, is not conserved. Related arguments (for the cubic NLS in three spatial dimensions) appeared in [16]. Now the low-to-high frequency cascade can be precluded by negative regularity and the conservation of mass. To preclude the soliton, one need to control of motion of spatial center of the soliton solution. The method comes from [6] and [15] with the aid of negative regularity. The fist step is to note that a minimal kinetic energy blowup solution with finite mass must have zero momentum. A second ingredient is a compactness property of the orbit of $\{u(t)\}$ in L_x^2. Finally the soliton-like solution is precluded by using a truncated Virial identity. Note that the negative regularity in [20] cannot be obtained in dimensions three and four because the dispersion is too weak. Indeed the method of [14] and [20] can be applied to defocusing case without much difficulty.

In this paper, we will verify Conjecture 1.6 in radial case, namely,

Theorem 1.7 *(Spacetime bounds). Let $d \geqslant 5$ and let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a radial solution to* (2)*. If*

$$E_* := \sup_{t \in I} \left\| \Delta u(t) \right\|_2 < \| \Delta W \|_2,$$

then

$$\int_I \int_{\mathbb{R}^d} \left|u(t,x)\right|^{\frac{2(d+4)}{d-4}} dx\,dt \leqslant C(E_*) < \infty.$$

A more effective criterion for global well-posedness (depending directly on u_0) can be obtained using an energy-trapping argument in Section 3 (the corresponding argument for nonlinear Schrödinger equations is in [14]).

Corollary 1.8. *Let $d \geqslant 5$ and let $u_0 \in \dot{H}^2_x(\mathbb{R}^d)$ be a radial function and such that $\|\Delta u_0\|_2 < \|\Delta W\|_2$ and $E(u_0) < E(W)$. Then the corresponding solution u to* (2) *is global and moreover,*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left|u(t,x)\right|^{\frac{2(d+4)}{d-4}} dt\,dx < \infty.$$

In this paper, we establish the corresponding result of the theorem in [14] on the setting of nonlinear Schrödinger equations of fourth order. For later use in [21], the arguments here are direct "fourth order" analogue of [20], including [14], [17] and [18]. First, we will do a lot of ground work including establishing concentration compactness principle and the energy-trapping of the ground state. Next, we reduce the failure of Conjecture 1.6 to almost periodic solutions, where we will rely heavily on Theorem 1.5. To show that such almost periodic solutions match one of the three scenarios, we analyze the properties of the almost periodic solutions such as quasi-uniqueness of N, compactness of almost periodic solutions, etc. (see Section 4). Because we are considering the minimal kinetic blowup solution, the assumption (6) plays an important role, which is used in the proof of quasi-uniqueness of N. Finally, we established localized Virial identity and precluded all the possibility of the three scenarios. Note that the radiality enters only in Section 8, so all the conclusions in Sections 3–7 remain true for general solutions. Moreover, the method here applies equally well to defocusing nonlinear Schrödinger equations of fourth order. Because no Galilean transformation is available for (2), it seems difficult to remove the radial assumption as in [20] even in high dimensions. But we can remove the radial assumption in the defocusing case in dimensions $d \geqslant 9$, see [21].

After the paper was finished and submitted, we learned that B. Pausader has obtained independently the similar result in [24], where the author proved the same result with (6) replaced by $E(u) < E(W)$.

The rest of the paper is organized as follows: In Section 2, we introduce some notations. The energy-trapping of the ground state is given in Section 3. In Section 4, we define almost periodic solutions and list their properties. The concentration compactness principle is proved in Section 5. In Section 6, we reduce the failure of Conjecture 1.6 to the existence of almost periodic solutions and in Section 7, we prove that such solutions must admit one of three scenarios, namely, we set up three enemies. Finally, we preclude all the scenarios in Section 8.

2. Notations

We introduce some notations. If X, Y are nonnegative quantities, we use $X \lesssim Y$ or $X = O(Y)$ to denote the estimate $X \leqslant CY$ for some C which may depend on the energy $E(u)$ and $X \sim Y$ to denote the estimate $X \lesssim Y \lesssim X$. Sometimes we write $X \sim_{c,C,u} Y$ to mean the implicit constant depends on c, C and $E(u)$. We use $X \ll Y$ to mean $X \leqslant cY$ for some small constant c which is again allowed to depend on $E(u)$. We write $L^q_t L^r_x$ to denote the Banach space with norm

$$\|u\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} \left|u(t,x)\right|^r dx\right)^{q/r} dt\right)^{1/q},$$

with the usual modifications when q or r is equal to infinity, or when the domain $\mathbb{R}\times\mathbb{R}^d$ is replaced by spacetime slab such as $I\times\mathbb{R}^d$. When $q=r$ we abbreviate $L_t^qL_x^q$ as $L_{t,x}^q$.

We use $C\gg 1$ to denote various large finite constants. and $0<c\ll 1$ to denote various small constants.

The Fourier transform on $\mathbb{R}^d$ is defined by

$$\hat{f}(\xi):=(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}f(x)\,dx,$$

giving rise to the fractional differentiation operators $|\nabla|^s$, defined by

$$\widehat{|\nabla|^s f}(\xi):=|\xi|^s\hat{f}(\xi).$$

These define the homogeneous Sobolev norms

$$\|f\|_{\dot{H}_x^s}:=\big\||\nabla|^s f\big\|_{L_x^2(\mathbb{R}^d)}.$$

Let $e^{it\Delta^2}$ be the free fourth-order Schrödinger propagator given by

$$\widehat{e^{it\Delta^2}f}(\xi)=e^{it|\xi|^4}\hat{f}(\xi).$$

We recall some basic facts in Littlewood–Paley theory. Let $\varphi(\xi)$ be a radial bump function supported in the ball $\{\xi\in\mathbb{R}^d\colon\ |\xi|\leqslant\frac{11}{10}\}$ and equal to 1 on the ball $\{\xi\in\mathbb{R}^d\colon\ |\xi|\leqslant 1\}$. For each number $N>0$, we define the Fourier multipliers

$$\begin{aligned}\widehat{P_{\leqslant N}f}(\xi)&:=\varphi(\xi/N)\hat{f}(\xi),\\ \widehat{P_{\geqslant N}f}(\xi)&:=\big(1-\varphi(\xi/N)\big)\hat{f}(\xi),\\ \widehat{P_N f}(\xi)&:=\big(\varphi(\xi/N)-\varphi(2\xi/N)\big)\hat{f}(\xi)\end{aligned}$$

and similarly $P_{<N}$ and $P_{\geqslant N}$. We also define

$$P_{M<\cdot\leqslant N}:=P_{\leqslant N}-P_{\leqslant M}=\sum_{M<N'\leqslant N}P_{N'}$$

whenever $M<N$. We will usually use these multipliers when M and N are dyadic numbers; in particular, all summations over N or M are understood to be over dyadic numbers. Nevertheless, it will occasionally be convenient to allow M and N to not be a power of 2. Note that P_N is not truly a projection; to get around this, we will occasionally need to use fattened Littlewood–Paley operators:

$$\tilde{P}_N:=P_{N/2}+P_N+P_{2N}. \tag{8}$$

They obey $P_N\tilde{P}_N=\tilde{P}_NP_N=P_N$.

As all Fourier multipliers, the Littlewood–Paley operators commute with the propagator $e^{it\Delta^2}$, as well as with differential operators such as $i\partial_t+\Delta^2$. We will use basic properties of these operators many times, including

Lemma 2.1 *(Bernstein estimates). For $1 \leqslant p \leqslant q \leqslant \infty$,*

$$\big\| |\nabla|^{\pm s} P_N f \big\|_{L_x^p(\mathbb{R}^d)} \sim N^{\pm s} \| P_N f \|_{L_x^p(\mathbb{R}^d)},$$

$$\| P_{\leqslant N} f \|_{L_x^q(\mathbb{R}^d)} \lesssim N^{\frac{d}{p} - \frac{d}{q}} \| P_{\leqslant N} f \|_{L_x^p(\mathbb{R}^d)},$$

$$\| P_N f \|_{L_x^q(\mathbb{R}^d)} \lesssim N^{\frac{d}{p} - \frac{d}{q}} \| P_N f \|_{L_x^p(\mathbb{R}^d)}.$$

3. Ground state

Let

$$W(x) = W(x,t) = \left(\frac{(d(d-4)(d^2-4))^{\frac{1}{4}}}{1+|x|^2} \right)^{\frac{d-4}{2}}$$

be a stationary solution of (2). That is $W \geqslant 0$ solves the nonlinear elliptic equation

$$\Delta^2 W = |W|^{\frac{8}{d-4}} W, \tag{9}$$

then by the invariances of the equation, for $\theta_0 \in [-\pi, \pi]$, $\lambda_0 > 0$, $x_0 \in \mathbb{R}^d$,

$$W_{\theta_0, x_0, \lambda_0} = e^{i\theta_0} \lambda_0^{\frac{d-4}{2}} W\big(\lambda_0(x - x_0)\big)$$

is still a solution. By the work of [7], we have the following characterization of W:

$$\forall u \in \dot{H}^2, \quad \|u\|_{L^{2^\#}} \leqslant C_d \|\Delta u\|_{L^2}; \tag{10}$$

moreover, if $u \neq 0$ is such that

$$\|u\|_{L^{2^\#}} = C_d \|\Delta u\|_{L^2}, \tag{11}$$

then there exist $(\theta_0, \lambda_0, x_0)$ such that $u = W_{\theta_0, x_0, \lambda_0}$, where C_d is the best constant of the Sobolev inequality in dimension d and $2^\# = \frac{2d}{d-4}$.

Eq. (9) gives $\int |\Delta W|^2 = \int |W|^{2^\#}$. Also, (11) yields $C_d^2 \int |\Delta W|^2 = (\int |W|^{2^\#})^{\frac{d-4}{d}}$, an easy computation shows that

$$\int |\Delta W|^2 = C_d^{-d/2} \quad \text{and} \quad E(W) = \frac{2}{d} C_d^{-d/2}.$$

Lemma 3.1. *Assume that*

$$\|\Delta u\|_2 < \|\Delta W\|_2$$

and $E(u) \leqslant (1-\delta_0) E(W)$ where $\delta_0 > 0$. Then there exists $\bar{\delta} = \bar{\delta}(\delta_0, d)$ such that

$$\int |\Delta u|^2 \leqslant (1 - \bar{\delta}) \int |\Delta W|^2, \tag{12}$$

$$\int \big(|\Delta u|^2 - |u|^{2^\#} \big) \geqslant \bar{\delta} \int |\Delta u|^2, \tag{13}$$

$$E(u) \geqslant 0. \tag{14}$$

Proof. Consider the function $f(y)=\frac{1}{2}y-\frac{d-4}{2d}C_d^{2^\#}y^{\frac{2^\#}{2}}$ and let $\bar{y}=\|\Delta u\|_2^2$. From (10), we have

$$f(\bar{y})\leqslant E(u)\leqslant(1-\delta_0)E(W)=(1-\delta_0)\frac{2}{dC_d^{d/2}}.$$

Note that $f(0)=0$, $f'(y)=\frac{1}{2}-\frac{C_d^{2^\#}}{2}y^{\frac{2^\#}{2}-1}$, so $f'(y)=0$ if and only if $y=y_C$, where $y_C=C_d^{-d/2}=\int|\Delta W|^2$. Note also that $f(y_C)=E(W)$. But since $0<\bar{y}<y_C$ and $f(\bar{y})\leqslant(1-\delta_0)f(y_C)$ and f is nonnegative and strictly increasing between 0 and y_C, $f''(y_C)\neq 0$, we have $0<f(\bar{y})$ and $\bar{y}\leqslant(1-\bar{\delta})\int|\Delta W|^2$. Thus (12) and (14) hold.

To show (13), consider the function $g(y)=y-C_d^{2^\#}y^{\frac{d}{d-4}}$. Because of (10), we have that $\int(|\Delta u|^2-|u|^{2^\#})\geqslant g(\bar{y})$. Note that $g(y)=0$ if and only if $y=0$ or $y=y_C$ and that $g'(0)=1$, $g'(y_C)=-\frac{4}{d-4}$. We then have, for $0<y<y_C$, $g(y)\geqslant C\min\{y,y_C-y\}$, so (13) follows from $0\leqslant\bar{y}<(1-\bar{\delta})y_C$ which is given by (12). □

By energy conservation, Lemma 3.1 and a continuity argument, we have

Theorem 3.2 *(Energy trapping). Let u be a solution of* (2) *with initial data u_0 such that*

$$\int|\Delta u_0|^2<\int|\Delta W|^2\quad\text{and}\quad E(u_0)<(1-\delta_0)E(W).$$

Let $I\ni 0$ be the maximal interval of existence. Let $\bar{\delta}=\bar{\delta}(\delta_0,d)$ be as in Lemma 3.1, *then for each $t\in I$, we have*

$$\int\left|\Delta u(t)\right|^2\leqslant(1-\bar{\delta})\int|\Delta W|^2,\tag{15}$$

$$\int\left(\left|\Delta u(t)\right|^2-\left|u(t)\right|^{2^\#}\right)\geqslant\bar{\delta}\int\left|\Delta u(t)\right|^2,\tag{16}$$

$$E\big(u(t)\big)\geqslant 0.\tag{17}$$

Proof. See [14]. □

Corollary 3.3. *Let u, u_0 be as in Theorem* 3.2. *Then for all $t\in I$ we have $E(u(t))\simeq\int|\Delta u(t)|^2\simeq\int|\Delta u_0|^2$, with comparability constants which depend only on δ_0.*

Proof. $E(u(t))\leqslant\int|\Delta u|^2\,dx$. But by (16) we have

$$\begin{aligned}E(u)&\geqslant\left(\frac{1}{2}-\frac{1}{2^\#}\right)\int\left|\Delta u(t)\right|^2dx+\frac{1}{2^\#}\int\left(\left|\Delta u(t)\right|^2-\left|u(t)\right|^{2^\#}\right)dx\\&\geqslant C_{\bar{\delta}}\int\left|\Delta u(t)\right|^2dx,\end{aligned}$$

so the first equivalence follows. For the second one, note that $E(u(t))=E(u_0)\simeq\int|\Delta u_0|^2\,dx$, by the first equivalence when $t=0$. □

4. Almost periodic solutions

Definition 4.1 *(Symmetry group).* For any phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, position $x_0 \in \mathbb{R}^d$ and scaling parameter $\lambda > 0$, we define the unitary transformation $g_{\theta,x_0,\lambda} : \dot{H}^2(\mathbb{R}^d) \to \dot{H}^2(\mathbb{R}^d)$ by the formula

$$[g_{\theta,x_0,\lambda} f](x) := \lambda^{-\frac{d-4}{2}} e^{i\theta} f\big(\lambda^{-1}(x-x_0)\big).$$

We let G be the collection of such transformations. If $u : I \times \mathbb{R}^d \to \mathbb{C}$ is a function, we define $T_{g_{\theta,x_0,\lambda}} u : \lambda^4 I \times \mathbb{R}^d \to \mathbb{C}$ where $\lambda^4 I := \{\lambda^4 t:\ t \in I\}$ by the formula

$$[T_{g_{\theta,x_0,\lambda}} u](t,x) := \lambda^{-\frac{d-4}{2}} e^{i\theta} u\big(\lambda^{-4}t, \lambda^{-1}(x-x_0)\big).$$

We also let $G_{\mathrm{rad}} \subset G$ denote the collection of transformations in G which preserve spherical symmetry, or more explicitly,

$$G_{\mathrm{rad}} := \{g_{\theta,0,\lambda}:\ \theta \in \mathbb{R}/2\pi\mathbb{Z};\ \lambda > 0\}.$$

Definition 4.2 *(Enlarged group).* For any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, position $x_0 \in \mathbb{R}^d$, scaling parameter $\lambda > 0$ and time t_0, we define the unitary transformation $g_{\theta,x_0,\lambda,t_0} : \dot{H}^2_x(\mathbb{R}^d) \to \dot{H}^2_x(\mathbb{R}^d)$ by the formula

$$g_{\theta,x_0,\lambda,t_0} = g_{\theta,x_0,\lambda} e^{it_0\Delta^2}.$$

Let G' be the collection of such transformations. We also let G' act on global spacetime functions $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ by defining

$$(T_{g_{\theta,x_0,\lambda,t_0}} u)(t,x) := \frac{1}{\lambda^{\frac{d-4}{2}}} e^{i\theta} \big(e^{it_0\Delta^2} u\big)\left(\frac{t}{\lambda^4}, \frac{x-x_0}{\lambda}\right).$$

Given any two sequences g_n, g_n' in G', we say that g_n and g_n' are asymptotically orthogonal if $(g_n)^{-1} g_n'$ diverges to infinity in G. If we write explicitly

$$g_n = g_{\theta_n,x_n,\lambda_n,t_n}, \qquad g_n' = g_{\theta_n',x_n',\lambda_n',t_n'},$$

then the asymptotic orthogonality is equivalent to

$$\lim_{n\to\infty}\left(\frac{\lambda_n}{\lambda_n'} + \frac{\lambda_n'}{\lambda_n} + \frac{|t_n\lambda_n^4 - t_n'(\lambda_n')^4|}{\lambda_n^2\lambda_n'^2} + \frac{|x_n - x_n'|^2}{\lambda_n\lambda_n'}\right) = +\infty.$$

Remark 4.3. If g_n, $g_n' \in G'$ are asymptotically orthogonal, then

$$\lim_{n\to\infty}\langle \Delta g_n f, \Delta g_n' f'\rangle_{L^2_x(\mathbb{R}^d)} = 0 \quad \text{for all } f, f' \in \dot{H}^2_x\big(\mathbb{R}^d\big). \tag{18}$$

A variant of this is that if $v, v' \in L_{t,x}^{\frac{2(d+4)}{d-4}}(\mathbb{R} \times \mathbb{R}^d)$, then

$$\lim_{n\to\infty}\big\| |T_{g_n} v|^{1-\theta} |T_{g_n'} v'|^{\theta}\big\|_{L^q_t L^r_x} = 0 \tag{19}$$

for any $0 < \theta < 1$ and admissible pair (q,r) $(q < \infty)$, that is, $\frac{4}{q} + \frac{d}{r} = \frac{d}{2} - 2$.

Definition 4.4 *(Almost periodic solutions).* Let $d \geqslant 5$. A solution u to (2) with lifespan I is said to be almost periodic modulo G if there exist functions $N : I \to \mathbb{R}^+$, $x : I \to \mathbb{R}^d$ and $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t \in I$, and $\eta > 0$,

$$\int\limits_{|x-x(t)| \geqslant C(\eta)/N(t)} \left|\Delta u(t,x)\right|^2 dx \leqslant \eta \tag{20}$$

and

$$\int\limits_{|\xi| \geqslant C(\eta)N(t)} |\xi|^4 \left|\hat{u}(t,\xi)\right|^2 d\xi \leqslant \eta. \tag{21}$$

We refer to the function N as the frequency scale function for the solution u, x the spatial center function, and to C as the compactness modulus function.

By Ascoli–Arzela theorem, almost periodicity modulo G means that the quotient orbit $\{Gu(t): t \in I\}$ is a precompact set of $G \setminus \dot{H}^2$, where $G \setminus \dot{H}^2$ is the moduli space of G-orbits $Gf := \{gf: g \in G\}$ of $\dot{H}^2(\mathbb{R}^d)$. Moreover, a family of functions is precompact in $\dot{H}^2_x$ if and only if it is norm-bounded and there exists a compactness modulus function C so that

$$\int\limits_{|x| \geqslant C(\eta)} \left|\Delta f(x)\right|^2 dx + \int\limits_{|\xi| \geqslant C(\eta)} |\xi|^4 \left|\hat{f}(\xi)\right|^2 d\xi \leqslant \eta$$

for all functions f in the family. By Sobolev embedding, any solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ that is almost periodic modulo G must also satisfy

$$\int\limits_{|x-x(t)| \geqslant C(\eta)/N(t)} \left|u(t,x)\right|^{\frac{2d}{d-4}} dx \leqslant \eta.$$

Lemma 4.5 *(Quasi-uniqueness of N). Let u be a non-zero solution to* (2) *satisfying* (6) *with lifespan I that is almost periodic modulo G with frequency scale function $N : I \to \mathbb{R}^+$ and compactness modulus function $C : \mathbb{R}^+ \to \mathbb{R}^+$ and also almost periodic modulo G with frequency scale function $N' : I \to \mathbb{R}^+$ and compactness modulus function $C' : \mathbb{R}^+ \to \mathbb{R}^+$. Then we have*

$$N(t) \sim_{u,C,C'} N'(t)$$

for all $t \in I$.

Proof. It suffices to prove $N'(t) \lesssim_{u,C,C'} N(t)$, for all $t \in I$. Otherwise, there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} N(t_n)/N'(t_n) = 0$. For any $\eta > 0$, by Definition 4.4, we have

$$\int\limits_{|x-x'(t_n)| \geqslant C'(\eta)/N'(t_n)} \left|\Delta u(t_n,x)\right|^2 dx \leqslant \eta$$

and

$$\int\limits_{|\xi| \geqslant C(\eta)N(t_n)} |\xi|^4 \left|\hat{u}(t_n,\xi)\right|^2 d\xi \leqslant \eta. \tag{22}$$

Let $u(t_n,x) = u_1(t_n,x) + u_2(t_n,x)$, where $u_1(t_n,x) = u(t_n,x)1_{|x-x'(t_n)|\geqslant C'(\eta)/N'(t_n)}$, $u_2(t_n,x) = u(t_n,x)1_{|x-x'(t_n)|<C'(\eta)/N'(t_n)}$. Then by Plancherel's theorem, we have

$$\int_{\mathbb{R}^d} |\xi|^4 \big|\hat{u}_1(t_n,\xi)\big|^2\, d\xi \lesssim \eta, \tag{23}$$

while from Cauchy–Schwartz we have

$$\sup_{\xi\in\mathbb{R}^d} |\xi|^4 \big|\hat{u}_2(t_n,\xi)\big|^2 \lesssim_{\eta,C'} \big\|\Delta u(t_n)\big\|_2^2 N'(t_n)^{-d}.$$

Integrating the last inequality over the ball $|\xi| \leqslant C(\eta)N(t_n)$, we get

$$\begin{aligned}\int_{|\xi|\leqslant C(\eta)N(t)} |\xi|^4 \big|\hat{u}(t_n,\xi)\big|^2\, d\xi &\lesssim \int_{\mathbb{R}^d} |\xi|^4 \big|\hat{u}_1(t_n,\xi)\big|^2\, d\xi + \int_{|\xi|\leqslant C(\eta)N(t)} |\xi|^4 \big|\hat{u}_2(t_n,\xi)\big|^2\, d\xi \\ &\lesssim \eta + O_{\eta,C,C'}\big(\big\|\Delta u(t_n)\big\|_2^2 N(t_n)^d N'(t_n)^{-d}\big).\end{aligned}$$

This, combined with (22), (6) and Corollary 3.3, yields that

$$\begin{aligned}\int \big|\Delta u_0(x)\big|^2\, dx \sim \int_{\mathbb{R}^d} |\xi|^4 \big|\hat{u}(t_n,\xi)\big|^2\, d\xi &\lesssim \eta + O_{\eta,C,C'}\big(\big\|\Delta u(t_n)\big\|_2^2 N(t_n)^d N'(t_n)^{-d}\big) \\ &\lesssim \eta + O_{\eta,C,C'}\big(\|\Delta W\|_2^2 N(t_n)^d N'(t_n)^{-d}\big).\end{aligned}$$

Since $\lim_{n\to\infty} N(t_n)/N'(t_n) = 0$, we have

$$\int \big|\Delta u_0(x)\big|^2\, dx \lesssim \eta.$$

By the arbitrary of η, we get

$$\int \big|\Delta u_0(x)\big|^2\, dx = 0.$$

Thus, $u_0 \equiv 0$ and by mass conservation, $u(t) \equiv 0$ for all $t \in \mathbb{R}$. This contradicts that u is non-zero. $\square$

Lemma 4.6 *(Quasi-continuous dependence of N on u). Let $u^{(n)}$ be a sequence of solutions to* (2) *with lifespan $I^{(n)}$ satisfying* (6)*, which are almost periodic modulo scaling with frequency scale functions $N^{(n)} : I^{(n)} \to \mathbb{R}^+$ and compactness modulus function $C : \mathbb{R}^+ \to \mathbb{R}^+$, independent of n. Suppose that $u^{(n)}$ converge locally uniformly to a non-zero solution u to* (2) *with lifespan I. Then u is almost periodic modulo scaling with a frequency scale function $N : I \to \mathbb{R}^+$ and compactness modulus function C. Furthermore, we have*

$$N(t) \sim_{u,C} \liminf_{n\to\infty} N^{(n)}(t) \sim_{u,C} \limsup_{n\to\infty} N^{(n)}(t) \tag{24}$$

for all $t \in I$. Finally, if all $u^{(n)}$ are spherically symmetric, then u is also.

Proof. We first show that

$$0 < \liminf_{n\to\infty} N^{(n)}(t) \leqslant \limsup_{n\to\infty} N^{(n)}(t) < +\infty \tag{25}$$

for all $t \in I$. Indeed, if one of these inequalities failed for some t, then (by passing to a subsequence if necessary) $N^{(n)}(t)$ would converge to zero or infinity as $n \to \infty$. Thus by Definition 4.4, $u^{(n)}(t)$ would converge weakly to zero, hence by the local uniform convergence, would converge strongly to zero. But this contradicts the hypothesis that u is not identically zero. This establishes (25).

From (25), we see that for each $t \in I$ the sequence $N^{(n)}(t)$ has at least one limit point $N(t)$. Thus, using the local uniform convergence we easily verify that u is almost periodic modulo scaling with frequency scale function N and compactness modulus function C. It is also clear that if all $u^{(n)}$ are spherically symmetric, then u is also.

It remains to establish (24), which we prove by contradiction. Suppose it fails. Then given any $A = A_u$, there exists a $t \in I$ for which $N^{(n)}(t)$ has at least two limit points which are separated by a ratio of at least A, and so u has two frequency scale functions with compactness modulus function C which are separated by this ratio. But this contradicts Lemma 4.5 for A large enough depending on u. Hence (24) holds. □

Definition 4.7 *(Normalized solution).* Let u be a solution to (2), which is almost periodic modulo G with frequency scale function N, position center function x. We say that u is normalized if the lifespan I contains zero and

$$N(0) = 1, \qquad x(0) = 0.$$

More generally, we can define the normalization of a solution u at time t_0 in its lifespan I to be

$$u^{[t_0]} := T_{g_{0,-x(t_0)N(t_0),N(t_0)}}\big(u(\cdot + t_0)\big). \tag{26}$$

Observe that $u^{[t_0]}$ is a normalized solution which is almost periodic modulo G and has lifespan

$$I^{[t_0]} := \big\{s \in \mathbb{R}\colon\ t_0 + s/N(t_0)^4 \in I\big\}.$$

It has frequency scale function

$$N_{u^{[t_0]}}(t) = \frac{N(t_0 + tN(t_0)^{-4})}{N(t_0)}$$

and position center function

$$x_{u^{[t_0]}}(t) = N(t_0)\big[x\big(t_0 + tN(t_0)^{-4}\big) - x(t_0)\big].$$

Lemma 4.8 *(Compactness of almost periodic solutions). Let $u^{(n)}$ be a sequence of normalized maximal-lifespan solutions to* (2) *satisfying* (6) *with lifespan $I^{(n)} \ni 0$, which are almost periodic modulo G with frequency scale functions $N^{(n)} : I^{(n)} \to \mathbb{R}^+$ and a uniform compactness modulus function $C : \mathbb{R}^+ \to \mathbb{R}^+$. Assume that we also have a uniform energy bound*

$$0 < \inf_n E\big(u^{(n)}\big) \leqslant \sup_n E\big(u^{(n)}\big) < \infty.$$

Then after passing to a subsequence if necessary, there exists a non-zero maximal-lifespan solution u to (2) *with lifespan $I \ni 0$ that is almost periodic modulo G, such that $u^{(n)}$ converge locally uniformly to u. Moreover, if all $u^{(n)}$ are spherically symmetric and almost periodic modulo G_{rad}, then u is also.*

Proof. By hypothesis and Definition 4.4, we see that for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\int_{|x|\geqslant R} \left|\Delta u^{(n)}(0,x)\right|^2 dx \leqslant \varepsilon$$

and

$$\int_{|\xi|\geqslant R} |\xi|^4 \left|\widehat{u^{(n)}}(0,\xi)\right|^2 d\xi \leqslant \varepsilon$$

for all n. Since $\sup_n E(u^{(n)}) < \infty$, we have $\sup_n \|\Delta u^{(n)}(0)\|_2^2 < \infty$. By the Ascoli–Arzela theorem, we see that the sequence $u^{(n)}(0)$ is precompact in the strong topology of $\dot{H}^2(\mathbb{R}^d)$. Thus passing to a subsequence if necessary, we can find $u_0 \in \dot{H}^2(\mathbb{R}^d)$ such that $u^{(n)}(0)$ converge strongly to u_0 in $\dot{H}^2(\mathbb{R}^d)$. Since $0 < \inf_n E(u^{(n)})$, we see that u_0 is not identically zero.

Now let u be the maximal Cauchy development of u_0 from time 0, with lifespan I. By Theorem 1.4, $u^{(n)}$ converges locally uniformly to u. The remaining claims now follow from Lemma 4.6. $\square$

Corollary 4.9 *(Local constancy of N). Let u be a non-zero maximal-lifespan solution to* (2) *satisfying* (6) *with lifespan I that is almost periodic modulo G with frequency scale function $N : I \to \mathbb{R}^+$. Then there exists a small number δ, depending on u, such that for every $t_0 \in I$ we have*

$$\left[t_0 - \delta N(t_0)^{-4}, t_0 + \delta N(t_0)^{-4}\right] \subset I \tag{27}$$

and

$$N(t) \sim_u N(t_0) \tag{28}$$

whenever $|t - t_0| \leqslant \delta N(t_0)^{-4}$.

Proof. Let us establish (27) first. We argue by contradiction. Assume that (27) failed. Then there exist sequences $t_n \in I$ and $\delta_n \to 0$ such that $t_n + \delta_n N(t_n)^{-4} \notin I$ for all n. Define the normalization $u^{[t_n]}$ of u from time t_n by (24). Then $u^{[t_n]}$ are maximal-lifespan normalized solutions whose lifespan $I^{[t_n]}$ contain 0 but not δ_n; they are also almost periodic modulo G with frequency scale functions

$$N^{[t_n]}(s) := N\big(t_n + sN(t_n)^{-4}\big)/N(t_n)$$

and the same compactness modulus function C as u. Applying Lemma 4.8 (and passing to a subsequence if necessary), we conclude that $u^{[t_n]}$ converge locally uniformly to a maximal-lifespan solution v with some lifespan $J \ni 0$. By Theorem 1.4, J is open and so contains δ_n for all sufficiently large n. This contradicts the local uniform convergence as, by hypothesis, δ_n does not belong to $I^{[t_n]}$. Hence (27) holds.

We now show (28). Again, we argue by contradiction, shrinking δ if necessary. Assume (28) failed no matter how small one select δ. Then one can find sequences t_n, $t_n' \in I$ such that $s_n := (t_n' - t_n)N(t_n)^4 \to 0$ but $N(t_n')/N(t_n)$ converge to either zero or infinity. If we define $u^{[t_n]}$ and $N^{[t_n]}$ as before and apply Lemma 4.8, we see once again that $u^{[t_n]}$ converge locally uniformly to maximal solution with some open interval $J \ni 0$. But then $N^{[t_n]}(s_n)$ converge to either zero or infinity and thus by Definition 4.4, $u^{[t_n]}(s_n)$ are converging weakly to zero. On the other hand, since s_n converge to zero and $u^{[t_n]}$ are locally uniformly convergent to $v \in C^0_{t,loc}\dot{H}^2_x(J \times \mathbb{R}^d)$, we may conclude that $u^{[t_n]}(s_n)$ converge strongly to $v(0)$ in $\dot{H}^2_x(\mathbb{R}^d)$. Thus $\|v(0)\|_{\dot{H}^2_x(\mathbb{R}^d)} = 0$. So $v(0) \equiv 0$. Since $E(u^{[t_n]}) = E(u)$, we see that u vanishes. Thus (28) holds. $\square$

As a direct consequence of Corollary 4.9, we have

Corollary 4.10 *(Blowup criterion). Let u be a non-zero maximal-lifespan solution to* (2) *satisfying* (6) *with lifespan I that is almost periodic modulo G with frequency scale function $N : I \to \mathbb{R}^+$. If T is any finite endpoint of I, then*

$$\lim_{t\to T} N(t) = \infty.$$

Lemma 4.11 *(Local quasi-boundedness of N). Let u be a non-zero solution to* (2) *with lifespan I that is almost periodic modulo G with frequency scale function $N : I \to \mathbb{R}^+$. If K is any compact subset of I, then*

$$0 < \inf_{t\in K} N(t) \leqslant \sup_{t\in K} N(t) < \infty.$$

Proof. We only prove the first inequality; the argument for the last is similar.

We argue by contradiction. Suppose that the first inequality fails. Then, there exists a sequence $t_n \in K$ such that $\lim_{n\to\infty} N(t_n) = 0$ and hence by Definition 4.4, $u(t_n)$ converge weakly to zero. Since K is compact, we can assume t_n converge to a limit $t_0 \in K$. As $u \in C_t^0 \dot{H}_x^2(K \times \mathbb{R}^d)$, we see that $u(t_n)$ converge strongly to $u(t_0)$. Thus $u(t_0)$ must be zero, contradicting the hypothesis. □

5. Concentration compactness

Theorem 5.1 *(Linear profile decomposition). Fix $d \geqslant 5$ and $\{u_n\}_{n\geqslant 1}$ be a sequence of functions bounded in $\dot{H}_x^2(\mathbb{R}^d)$. Then after passing to a subsequence if necessary, there exist a sequence of functions $\{\phi^j\}_{j\geqslant 1} \subset \dot{H}_x^2(\mathbb{R}^d)$, group elements $g_n^j \in G$ and times $t_n^j \in \mathbb{R}$ such that we have the decomposition*

$$u_n = \sum_{j=1}^{J} g_n^j e^{it_n^j \Delta^2} \phi^j + w_n^J \tag{29}$$

for all $J > 1$; here $w_n^J \in \dot{H}_x^2(\mathbb{R}^d)$ obey

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left\| e^{it\Delta^2} w_n^J \right\|_{L_{t,x}^{\frac{2(d+4)}{d-4}}(\mathbb{R}\times\mathbb{R}^d)} = 0. \tag{30}$$

Moreover, for any $j \neq j'$,

$$\lim_{n\to\infty} \left(\frac{\lambda_n}{\lambda_n'} + \frac{\lambda_n'}{\lambda_n} + \frac{|t_n\lambda_n^4 - t_n'(\lambda_n')^4|}{\lambda_n^2\lambda_n'^2} + \frac{|x_n - x_n'|^2}{\lambda_n\lambda_n'} \right) = +\infty. \tag{31}$$

Furthermore, for any $J \geqslant 1$ we have the kinetic energy decoupling property

$$\lim_{n\to\infty} \left[\|\Delta u_n\|_2^2 - \sum_{j=1}^{J} \left\| \Delta\phi^j \right\|_2^2 - \left\| \Delta w_n^J \right\|_2^2 \right] = 0. \tag{32}$$

Remark 5.2. In fact, for any (q, r) $(q \neq 2)$ such that $\frac{4}{q} + \frac{d}{r} = \frac{d}{2} - 2$, we have by Hölder's inequality,

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left\| e^{it\Delta^2} w_n^J \right\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} = 0. \tag{33}$$

Moreover, by interpolation we have

$$\lim_{J\to\infty}\limsup_{n\to\infty}\left\|\nabla e^{it\Delta^2}w_n^J\right\|_{L_t^aL_x^b(I\times\mathbb{R}^d)}=0, \tag{34}$$

where $\frac{4}{a}+\frac{d}{b}=\frac{d}{2}-1$, $a\neq 2$.

The proof of Theorem 5.1 is very similar to Theorem 1.6 of [17]. We need only establish the following

Lemma 5.3. *Fix $d\geqslant 5$. For every $f\in\dot H_x^2(\mathbb{R}^d)$, we have*

$$\|f\|_{L^{\frac{2d}{d-4}}}\leqslant C\|\Delta f\|_{L^2}^{\frac{d-4}{d}}\|\Delta f\|_{\dot B^0_{2,\infty}}^{\frac{4}{d}}.$$

Lemma 5.4. *Let $\{t^j\}$, $\{\lambda^j\}$, $\{x^j\}$ be sequences as in (31) and $V^j\in L_{t,x}^{\frac{2(d+4)}{d-4}}(\mathbb{R}\times\mathbb{R}^d)$ for every $j\geqslant 1$, then*

$$\lim_{n\to\infty}\left\|\sum_{j=1}^J\frac{1}{(\lambda_n^j)^{\frac{d-4}{2}}}V^j\left(\frac{\cdot-t_n^j}{(\lambda_n^j)^4},\frac{\cdot-x_n^j}{\lambda_n^j}\right)\right\|_{L_{t,x}^{\frac{2(d+4)}{d-4}}(\mathbb{R}\times\mathbb{R}^d)}^{\frac{2(d+4)}{d-4}}\leqslant\sum_{j=1}^J\left\|V^j\right\|_{L_{t,x}^{\frac{2(d+4)}{d-4}}(\mathbb{R}\times\mathbb{R}^d)}^{\frac{2(d+4)}{d-4}}. \tag{35}$$

Lemma 5.5. *For all $J\geqslant 1$ and all $1\leqslant j\leqslant J$, the sequence $e^{-it_n^j\Delta^2}[(g_n^j)^{-1}w_n^J]$ converges weakly to zero in $\dot H_x^2(\mathbb{R}^d)$ as $n\to\infty$. In particular, this implies the kinetic energy decoupling (32).*

Proof of Lemma 5.3. This is a direct adaption of the proof in Bahouri and Gerard [1]. For every $A>0$, we decompose $f=P_{>A}f+P_{\leqslant A}f$, then we have

$$\|P_{\leqslant A}f\|_{L^\infty}\leqslant\sum_{2^k\leqslant A}\|P_kf\|_{L^\infty}\lesssim\sum_{2^k\leqslant A}2^{k(\frac{d}{2}-2)}\|\Delta P_kf\|_{L^2}\lesssim A^{\frac{d}{2}-2}\|\Delta f\|_{\dot B^0_{2,\infty}}\triangleq\frac{\lambda}{2}.$$

Then $A(\lambda)=(\frac{\lambda}{2C\|\Delta f\|_{\dot B^0_{2,\infty}}})^{\frac{2}{d-4}}$ and

$$\begin{aligned}m\{|f|>\lambda\}&\leqslant m\left\{|P_{>A(\lambda)}f|>\frac{\lambda}{2}\right\}\\&\leqslant\frac{4}{\lambda^2}\|P_{>A(\lambda)}f\|_2^2\\&\leqslant\frac{4}{\lambda^2}A(\lambda)^{-4}\left\||\cdot|^2\widehat{P_{>A(\lambda)}f}(\cdot)\right\|_2^2.\end{aligned}$$

Therefore,

$$\|f\|_{L^{\frac{2d}{d-4}}}^{\frac{2d}{d-4}}=\frac{2d}{d-4}\int_0^{+\infty}\lambda^{\frac{d+4}{d-4}}m\{|f|>\lambda\}\,d\lambda$$

$$\lesssim \int_0^\infty \frac{4}{\lambda^2} A(\lambda)^{-4} \lambda^{\frac{d+4}{d-4}} \left(\int_{|\xi|>A(\lambda)} |\xi|^4 \left|\hat{f}(\xi)\right|^2 d\xi \right) d\lambda$$

$$\lesssim \|\Delta f\|_{\dot{B}^0_{2,\infty}}^{\frac{8}{d-4}} \|\Delta f\|_{L^2}^2.$$

This completes the proof of Lemma 5.3. □

Proof of Lemma 5.4. Let

$$V_n^j = \frac{1}{(\lambda_n^j)^{\frac{d-4}{2}}} V^j\left(\frac{\cdot - t_n^j}{(\lambda_n^j)^4}, \frac{\cdot - x_n^j}{\lambda_n^j} \right),$$

it suffices to prove that

$$\lim_{n\to\infty} \left\| \sum_{j=1}^J V_n^j \right\|_{L_{t,x}^{\frac{2(d+4)}{d-4}}(\mathbb{R}\times\mathbb{R}^d)}^{\frac{2(d+4)}{d-4}} \leqslant \sum_{j=1}^J \|V^j\|_{L_{t,x}^{\frac{2(d+4)}{d-4}}(\mathbb{R}\times\mathbb{R}^d)}^{\frac{2(d+4)}{d-4}}.$$

We denote the maximal integer less than a by $[a]$ and let $k(d) = [\frac{2(d+4)}{d-4}]$, then

$$\begin{aligned}
\left\| \sum_{j=1}^J V_n^j \right\|_{L_{t,x}^{\frac{2(d+4)}{d-4}}(\mathbb{R}\times\mathbb{R}^d)}^{\frac{2(d+4)}{d-4}} &= \int_{\mathbb{R}\times\mathbb{R}^d} \left| \sum_{j=1}^J V_n^j \right|^{k(d)} \left| \sum_{j=1}^J V_n^j \right|^{\frac{2(d+4)}{d-4} - k(d)} dt\, dx \\
&\leqslant \sum_{j=1}^J \int_{\mathbb{R}\times\mathbb{R}^d} \left|V_n^j\right|^{k(d)} \left| \sum_{j=1}^J V_n^j \right|^{\frac{2(d+4)}{d-4} - k(d)} dt\, dx \\
&\quad + \sum_{j=1}^J \sum_{j'\neq j} \sum_{j_3,\dots,j_{k(d)}} \int_{\mathbb{R}\times\mathbb{R}^d} \left|V_n^j V_n^{j'}\right| \left|V_n^{j_3} \cdots V_n^{j_{k(d)}}\right| \left| \sum_{j=1}^J V_n^j \right|^{\frac{2(d+4)}{d-4} - k(d)} dt\, dx \\
&= A + B.
\end{aligned}$$

We estimate A first:

$$A \leqslant \sum_{j=1}^J \int_{\mathbb{R}\times\mathbb{R}^d} \left|V_n^j\right|^{\frac{2(d+4)}{d-4}} dt\, dx + \sum_{j=1}^J \sum_{j\neq j'} \int_{\mathbb{R}\times\mathbb{R}^d} \left|V_n^j\right|^{k(d)} \left|V_n^{j'}\right|^{\frac{2(d+4)}{d-4} - k(d)} dt\, dx.$$

The second term can be written as

$$\begin{aligned}
&\sum_{j=1}^J \sum_{j\neq j'} \int_{\mathbb{R}\times\mathbb{R}^d} \left|V_n^j\right|^{2k(d) - \frac{2(d+4)}{d-4}} \left|V_n^j V_n^{j'}\right|^{\frac{2(d+4)}{d-4} - k(d)} \\
&\quad \lesssim \sum_{j=1}^J \sum_{j\neq j'} \left\|V_n^j\right\|_{L_{t,x}^{\frac{2(d+4)}{d-4}}(\mathbb{R}\times\mathbb{R}^d)}^{2k(d) - \frac{2(d+4)}{d-4}} \left\|V_n^j V_n^{j'}\right\|_{L_{t,x}^{\frac{d+4}{d-4}}(\mathbb{R}\times\mathbb{R}^d)}^{\frac{2(d+4)}{d-4} - k(d)}
\end{aligned}$$

$$= \sum_{j=1}^{J} \sum_{j \neq j'} \left\| V^j \right\|_{L_{t,x}^{\frac{2(d+4)}{d-4}}(\mathbb{R}\times\mathbb{R}^d)}^{2k(d)-\frac{2(d+4)}{d-4}} \left\| V_n^j V_n^{j'} \right\|_{L_{t,x}^{\frac{d+4}{d-4}}(\mathbb{R}\times\mathbb{R}^d)}^{\frac{2(d+4)}{d-4}-k(d)},$$

which is $o(1)$ as $n \to \infty$ by (19). Now we estimate B. By Hölder's inequality,

$$\begin{aligned}
&\int_{\mathbb{R}\times\mathbb{R}^d} \left|V_n^j V_n^{j'}\right| \left|V_n^{j_3} \cdots V_n^{J_{k(d)}}\right| \left|\sum_{j=1}^{J} V_n^j\right|^{\frac{2(d+4)}{d-4}-k(d)} dt\,dx \\
&\leqslant \sum_{l=1}^{J} \left\| V_n^j V_n^{j'} \right\|_{L_{t,x}^{\frac{d+4}{d-4}}(\mathbb{R}\times\mathbb{R}^d)} \left\| V_n^{j_3} \right\|_{L_{t,x}^{\frac{2(d+4)}{d-4}}(\mathbb{R}\times\mathbb{R}^d)} \cdots \left\| V_n^{j_{k(d)}} \right\|_{L_{t,x}^{\frac{2(d+4)}{d-4}}(\mathbb{R}\times\mathbb{R}^d)} \left\| V_n^{l} \right\|_{L_{t,x}^{\frac{2(d+4)}{d-4}}(\mathbb{R}\times\mathbb{R}^d)} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus we establish Lemma 5.4. $\square$

Proof of Lemma 5.5. Fix $J \geqslant 1$ and $1 \leqslant j \leqslant J$. Notice that $\{u_n\}_{n\geqslant 1}$ and ϕ^j are bounded in $\dot{H}_x^2(\mathbb{R}^d)$, by (29) we deduce that $\{e^{-it_n^j\Delta^2}[(g_n^j)^{-1} w_n^J]\}_{n\geqslant 1}$ is bounded in $\dot{H}_x^2(\mathbb{R}^d)$. Using Alaoglu's theorem (and passing to a subsequence if necessary), we obtain that $e^{-it_n^j\Delta^2}[(g_n^j)^{-1} w_n^J]$ converges weakly in $\dot{H}_x^2(\mathbb{R}^d)$ to some $\psi \in \dot{H}_x^2(\mathbb{R}^d)$. To prove this lemma, it suffices to show that $\psi \equiv 0$.

By weak convergence and (29),

$$\begin{aligned}
\|\psi\|_{\dot{H}_x^2(\mathbb{R}^d)} &= \lim_{n\to\infty} \left\langle \Delta e^{-it_n^j\Delta^2}\left[\left(g_n^j\right)^{-1} w_n^J\right], \Delta\psi \right\rangle \\
&= \lim_{n\to\infty} \left\langle \Delta e^{-it_n^j\Delta^2}\left[\left(g_n^j\right)^{-1}\left(\sum_{l=J+1}^{L} g_n^l e^{it_n^l\Delta^2}\varphi^l + w_n^L\right)\right], \Delta\psi \right\rangle \\
&= \sum_{l=J+1}^{L} \lim_{n\to\infty} \left\langle \Delta g_n^l e^{it_n^l\Delta^2}\varphi^l, \Delta g_n^j e^{it_n^j\Delta^2}\psi \right\rangle + \lim_{n\to\infty} \left\langle \Delta e^{-it_n^j\Delta^2}\left(g_n^j\right)^{-1} w_n^L, \Delta\psi \right\rangle
\end{aligned}$$

for all $L > J$. By (18),

$$\lim_{n\to\infty} \left\langle \Delta g_n^l e^{it_n^l\Delta^2}\varphi^l, \Delta g_n^j e^{it_n^j\Delta^2}\psi \right\rangle = 0$$

for all $L \geqslant l \geqslant J+1 > j$.

On the other hand, combining the fact that the family $\{e^{-it_n^j\Delta^2}[(g_n^j)^{-1} w_n^L]\}_{n,L\geqslant 1}$ is bounded in $\dot{H}_x^2(\mathbb{R}^d)$ with

$$\lim_{L\to\infty} \limsup_{n\to\infty} S_{\mathbb{R}}\left(e^{it\Delta^2} e^{-it_n^j\Delta^2}\left[\left(g_n^j\right)^{-1} w_n^L\right]\right) = \lim_{L\to\infty} \limsup_{n\to\infty} S_{\mathbb{R}}\left(e^{it\Delta^2} w_n^L\right) = 0,$$

we deduce that $e^{-it_n^j\Delta^2}(g_n^j)^{-1} w_n^L$ converges weakly to zero in $\dot{H}_x^2(\mathbb{R}^d)$ as $n, L \to \infty$. Thus for L sufficiently large,

$$\limsup_{n\to\infty} \left|\left\langle \Delta e^{-it_n^j\Delta^2}\left(g_n^j\right)^{-1} w_n^L, \Delta\psi \right\rangle\right| \leqslant \frac{1}{2}\|\psi\|_{\dot{H}_x^2(\mathbb{R}^d)}^2.$$

So we have $\psi \equiv 0$. This finishes the proof of Lemma 5.5. $\square$

6. Reduction to almost periodic solutions

Theorem 6.1. *Suppose $d \geqslant 5$ is such that Conjecture* 1.6 *failed. Then there exists a maximal-lifespan solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to* (2) *such that $\sup_{t\in I} \|\Delta u(t)\|_2 < \|\Delta W\|_2$, u is almost periodic modulo G, and u blows up both forward and backward in time. Moreover, u has minimal kinetic energy among all blowup solutions, that is*

$$\sup_{t\in I}\big\|\Delta u(t)\big\|_2 \leqslant \sup_{t\in J}\big\|\Delta v(t)\big\|_2$$

for all maximal-lifespan solution $v : J \times \mathbb{R}^d \to \mathbb{C}$ that blows up at least one time direction. If furthermore $d \geqslant 5$ and Conjecture 1.6 *failed for spherically symmetric data, then we can also ensure that u is spherically symmetric and almost periodic modulo G_{rad}.*

For any $0 \leqslant E_0 \leqslant \|\Delta W\|_2^2$, we define

$$L(E_0) := \sup\Big\{S(u):\ u : I \times \mathbb{R}^d \to \mathbb{C} \text{ such that } \sup_{t\in I}\big\|\Delta u(t)\big\|_2^2 \leqslant E_0\Big\},$$

where the supremum is taken over all solutions $u : I \times \mathbb{R}^d \to \mathbb{C}$ to (2) obeying $\sup_{t\in I} \|\Delta u(t)\|_2^2 \leqslant E_0$. Thus $L : [0, \|\Delta W\|_2^2] \to [0, \infty]$ is a non-decreasing function with $L(\|\Delta W\|_2^2) = \infty$. Moreover, from Theorem 1.4,

$$L(E_0) \lesssim_d E_0^{\frac{d+4}{d-4}} \quad \text{for } E_0 \leqslant \eta_0,$$

where $\eta_0 = \eta_0(d)$ is the threshold from the small data theory.

From Theorem 1.5, we see that L is continuous. Therefore, there must exist a unique critical kinetic energy E_c such that $L(E_0) < \infty$ for $E_0 < E_c$ and $L(E_0) = \infty$ for $E_0 \geqslant E_c$. In particular, if $u : I \times \mathbb{R}^d \to \mathbb{C}$ is a maximal-lifespan solution to (2) such that $\sup_{t\in I} \|\Delta u(t)\|_2^2 < E_c$, then u is global and

$$S(u) \leqslant L\Big(\sup_{t\in I}\big\|\Delta u(t)\big\|_2^2\Big).$$

Failure of Conjecture 1.6 is equivalent to the existence of $0 < E_c < \|\Delta W\|_2^2$.

Proposition 6.2 *(Palais–Smale condition modulo symmetries). Fix $d \geqslant 5$. Let $u_n : I_n \times \mathbb{R}^d \to \mathbb{C}$ be a sequence of solutions to* (2) *such that*

$$\limsup_{n\to\infty} \sup_{t\in I_n}\big\|\Delta u_n(t)\big\|_2^2 = E_c \tag{36}$$

and let $t_n \in I_n$ be a sequence of times such that

$$\lim_{n\to\infty} S_{\geqslant t_n}(u_n) = \lim_{n\to\infty} S_{\leqslant t_n}(u_n) = \infty.$$

Then the sequence $u_n(t_n)$ has a subsequence which converges in $\dot H_x^2(\mathbb{R}^d)$ modulo G.

Proof. By the time translation symmetry of (2), we may set $t_n = 0$ for all $n \geqslant 1$. Thus,

$$\lim_{n\to\infty} S_{\geqslant 0}(u_n) = \lim_{n\to\infty} S_{\leqslant 0}(u_n) = \infty. \tag{37}$$

Applying Theorem 5.1 to the sequence $u_n(0)$ (which is bounded in $\dot{H}_x^2$ by (36)) and passing to a subsequence if necessary, we obtain the decomposition

$$u_n(0) = \sum_{j=1}^{J} g_n^j e^{it_n^j \Delta^2} \phi^j + w_n^J.$$

Redefining the subsequence once for every j and using a diagonal argument, we may assume that for each j, the sequence $\{t_n^j\}_{n \geqslant 1}$ converges to some $t^j \in [-\infty, +\infty]$. If $t^j \in (-\infty, +\infty)$, then by replacing ϕ^j by $e^{it^j \Delta^2}\phi^j$ and $t_n^j - t^j$ by t_n^j, we may assume that $t^j = 0$. Moreover, absorbing the error $\sum_{1 \leqslant j \leqslant J:\ t^j = 0} g_n^j (e^{it_n^j \Delta^2}\phi^j - \phi^j)$ into the error term w_n^J, we may assume that $t_n^j \equiv 0$. Thus either $t_n^j \equiv 0$ or $t_n^j \to \pm\infty$.

We now define the nonlinear profiles $v^j : I^j \times \mathbb{R}^d \to \mathbb{C}$ associated to ϕ^j and t_n^j as follows:

- If $t_n^j \equiv 0$, then v^j is the maximal-lifespan solution to (2) with initial data $v^j(0) = \phi^j$.
- If $t_n^j \to +\infty$, then v^j is the maximal-lifespan solution to (2) that scatters forward in time to $e^{it\Delta^2}\phi^j$.
- If $t_n^j \to -\infty$, then v^j is the maximal-lifespan solution to (2) that scatters backward in time to $e^{it\Delta^2}\phi^j$.

For each $j, n \geqslant 1$, we define $v_n^j : I_n^j \times \mathbb{R}^d \to \mathbb{C}$ by

$$v_n^j(t) := T_{g_n^j}\big[v^j\big(\cdot + t_n^j\big)\big](t),$$

where $I_n^j := \{t \in \mathbb{R}:\ (\lambda_n^j)^{-4}t + t_n^j \in I^j\}$. Each v_n^j is a solution to (2) with initial data at time $t = 0$ given by $v_n^j(0) = g_n^j v^j(t_n^j)$ and maximal lifespan $I_n^j = (-T_{n,j}^-, T_{n,j}^+)$, where $-\infty \leqslant -T_{n,j}^- < 0 < T_{n,j}^+ \leqslant +\infty$.

By (32), there exists $J_0 \geqslant 1$ such that

$$\|\Delta\phi^j\|_2 \leqslant \eta_0 \quad \text{for all } j \geqslant J_0,$$

where $\eta_0 = \eta_0(d)$ is the threshold for the small data theory. Here, by Theorem 1.4 for all $n \geqslant 1$ and all $j \geqslant J_0$ the solutions v_n^j are global and moreover,

$$\sup_{t\in\mathbb{R}} \|\Delta v_n^j\|_2^2 + S_{\mathbb{R}}\big(v_n^j\big) \lesssim \|\Delta\phi^j\|_2^2. \tag{38}$$

Lemma 6.3 *(At least one bad profile). There exists* $1 \leqslant j_0 < J_0$ *such that*

$$\limsup_{n\to\infty} S_{[0, T_{n,j_0}^+)}\big(v_n^{j_0}\big) = \infty.$$

Proof. Assume for a contradiction that for all $1 \leqslant j < J_0$,

$$\limsup_{n\to\infty} S_{[0, T_{n,j}^+)}\big(v_n^j\big) < \infty, \tag{39}$$

which implies $T_{n,j}^+ = \infty$ for all $1 \leqslant j < J_0$ and all sufficiently large n. Moreover, subdividing $[0, +\infty)$ into intervals where the scattering size of v_n^j is small, applying the Strichartz inequality on each such interval, and then summing, we obtain

$$\limsup_{n\to\infty}\|v_n^j\|_{\dot S^2([0,\infty))}<\infty$$

for all $1\leqslant j<J_0$, where

$$\|f\|_{\dot S^2(I)}:=\sup_{(q,r):\ \frac{4}{q}+\frac{d}{r}=\frac{d}{2}}\|\Delta f\|_{L_t^qL_x^r(I\times\mathbb{R}^d)}.$$

By (36), (38), (39) and (32), we have

$$\sum_{j\geqslant 1}S_{[0,+\infty)}\big(v_n^j\big)\lesssim 1+\sum_{j\geqslant J_0}\|\Delta\phi^j\|_2^2\lesssim 1+E_c \tag{40}$$

for all n large enough. Now we define the approximation

$$u_n^J(t):=\sum_{j=1}^J v_n^j(t)+e^{it\Delta^2}w_n^J.$$

Note that

$$\begin{aligned}\big\|u_n^J(0)-u_n(0)\big\|_{\dot H_x^2(\mathbb{R}^d)}&\lesssim\Bigg\|\sum_{j=1}^J\big(g_n^jv^j\big(t_n^j\big)-g_n^je^{it_n^j\Delta^2}\phi^j\big)\Bigg\|_{\dot H_x^2(\mathbb{R}^d)}\\&\lesssim\sum_{j=1}^J\big\|v^j\big(t_n^j\big)-e^{it_n^j\Delta^2}\phi^j\big\|_{\dot H_x^2(\mathbb{R}^d)}\end{aligned}$$

and hence, by our choice of v^j,

$$\limsup_{n\to\infty}\big\|u_n^J(0)-u_n(0)\big\|_{\dot H_x^2(\mathbb{R}^d)}=0.$$

We now show that u_n^J does not blow up forward in time. Indeed, by (19), the fact that v_n^j does not blow up forward in time, Lemma 5.4, (30) and (40), we have

$$\begin{aligned}\lim_{J\to\infty}\limsup_{n\to\infty}S_{[0,\infty)}\big(u_n^J\big)&\lesssim\lim_{J\to\infty}\limsup_{n\to\infty}\Bigg(S_{[0,\infty)}\Bigg(\sum_{j=1}^J v_n^j\Bigg)+S_{[0,\infty)}\big(e^{it\Delta^2}w_n^J\big)\Bigg)\\&\lesssim\lim_{J\to\infty}\limsup_{n\to\infty}\sum_{j=1}^J S_{[0,\infty)}\big(v_n^j\big)\lesssim 1+E_c.\end{aligned}\tag{41}$$

Similarly, we can obtain that

$$\lim_{J\to\infty}\limsup_{n\to\infty}\big\|u_n^J\big\|_{\dot S^2([0,\infty))}\leqslant C(E_c)<\infty. \tag{42}$$

In order to apply Theorem 1.5, it suffices to show u_n^J asymptotically solves (2) in the sense that

$$\lim_{J\to\infty}\limsup_{n\to\infty}\big\|\nabla\big[\big(i\partial_t+\Delta^2\big)u_n^J-F\big(u_n^J\big)\big]\big\|_{L_t^2L_x^{\frac{2d}{d+2}}([0,\infty)\times\mathbb{R}^d)}=0, \tag{43}$$

which reduces to proving

$$\lim_{J\to\infty}\limsup_{n\to\infty}\left\|\nabla\left(\sum_{j=1}^{J}F\big(v_n^j\big)-F\left(\sum_{j=1}^{J}v_n^j\right)\right)\right\|_{L_t^2L_x^{\frac{2d}{d+2}}([0,\infty)\times\mathbb{R}^d)}=0 \tag{44}$$

and

$$\lim_{J\to\infty}\limsup_{n\to\infty}\big\|\nabla\big(F\big(u_n^J-e^{it\Delta^2}w_n^J\big)-F\big(u_n^J\big)\big)\big\|_{L_t^2L_x^{\frac{2d}{d+2}}([0,\infty)\times\mathbb{R}^d)}=0. \tag{45}$$

We first address (44). Note that we can write

$$\left|\nabla\left[\sum_{j=1}^{J}F\big(v_n^j\big)-F\left(\sum_{j=1}^{J}v_n^j\right)\right]\right|\lesssim_J\sum_{j\neq j'}\big|\nabla v_n^j\big|\big|v_n^{j'}\big|^{\frac{8}{d-4}}$$

for $d>12$ and

$$\left|\nabla\left[\sum_{j=1}^{J}F\big(v_n^j\big)-F\left(\sum_{j=1}^{J}v_n^j\right)\right]\right|\lesssim_J\sum_{j\neq j'}\Big(\big|\nabla v_n^j\big|\big|v_n^{j'}\big|^{\frac{8}{d-4}}+\big|v_n^j\big|\big|v_n^{j'}\big|^{\frac{12-d}{d-4}}\big|\nabla v_n^{j'}\big|\Big)$$

for $5\leqslant d\leqslant 12$. By the similar argument deriving (19), for any $j\neq j'$, we have

$$\limsup_{n\to\infty}\Big\|\big|v_n^{j'}\big|^{\frac{8}{d-4}}\big|\nabla v_n^j\big|\Big\|_{L_t^2L_x^{\frac{2d}{d+2}}([0,\infty)\times\mathbb{R}^d)}=0$$

for all $d\geqslant 5$ and

$$\limsup_{n\to\infty}\Big\|\big|v_n^j\big|\big|v_n^{j'}\big|^{\frac{12-d}{d-4}}\big|\nabla v_n^{j'}\big|\Big\|_{L_t^2L_x^{\frac{2d}{d+2}}([0,\infty)\times\mathbb{R}^d)}=0$$

for $5\leqslant d\leqslant 12$. Thus we have

$$\begin{aligned}&\limsup_{n\to\infty}\left\|\nabla\left[\sum_{j=1}^{J}F\big(v_n^j\big)-F\left(\sum_{j=1}^{J}v_n^j\right)\right]\right\|_{L_t^2L_x^{\frac{2d}{d+2}}([0,\infty)\times\mathbb{R}^d)}\\&\quad\lesssim_J\limsup_{n\to\infty}\sum_{j\neq j'}\Big\|\big|v_n^{j'}\big|^{\frac{8}{d-4}}\big|\nabla v_n^j\big|\Big\|_{L_t^2L_x^{\frac{2d}{d+2}}([0,\infty)\times\mathbb{R}^d)}=0\end{aligned}$$

and (44) follows.

We now consider (45). In dimensions $d\geqslant 12$, by Hölder and interpolation, we have

$$\begin{aligned}&\big\|\nabla\big(F\big(u_n^J-e^{it\Delta^2}w_n^J\big)-F\big(u_n^J\big)\big)\big\|_{L_t^2L_x^{\frac{2d}{d+2}}([0,\infty)\times\mathbb{R}^d)}\\&\quad\lesssim\big\|e^{it\Delta^2}w_n^J\big\|^{\frac{8}{d-4}}_{L_t^{\frac{8(d+4)}{3(d-4)}}L_x^{\frac{2d(d+4)}{(d+1)(d-4)}}([0,\infty)\times\mathbb{R}^d)}\big\|\nabla u_n^J\big\|_{L_{t,x}^{\frac{2(d+4)}{d-2}}([0,\infty)\times\mathbb{R}^d)}\\&\qquad+\big\|e^{it\Delta^2}w_n^J\big\|^{\frac{8}{d-4}}_{L_t^{\frac{8(d+4)}{3(d-4)}}L_x^{\frac{2d(d+4)}{(d+1)(d-4)}}([0,\infty)\times\mathbb{R}^d)}\big\|\nabla e^{it\Delta^2}w_n^J\big\|_{L_{t,x}^{\frac{2(d+4)}{d-2}}([0,\infty)\times\mathbb{R}^d)}\end{aligned}$$

$$+ \left\| u_n^J \right\|^{\frac{8}{d-4}}_{L_t^{\frac{8(d+4)}{3(d-4)}} L_x^{\frac{2d(d+4)}{(d+1)(d-4)}}([0,\infty)\times\mathbb{R}^d)} \left\| \nabla e^{it\Delta^2} w_n^J \right\|_{L_{t,x}^{\frac{2(d+4)}{d-2}}([0,\infty)\times\mathbb{R}^d)},$$

so (45) follows from (41), (33), (34) and the fact that $e^{it\Delta^2} w_n^J$ is bounded in $\dot{S}^2$. In dimensions $5 \leqslant d < 12$, one must add the term

$$\left\| u_n^J \right\|^{\frac{12-d}{d-4}}_{L_t^\infty L_x^{\frac{2d}{d-4}}([0,\infty)\times\mathbb{R}^d)} \left\| \nabla u_n^J \right\|_{L_t^{\frac{8}{3}} L_x^{\frac{2d}{d-5}}([0,\infty)\times\mathbb{R}^d)} \left\| e^{it\Delta^2} w_n^J \right\|_{L_t^8 L_x^{\frac{2d}{d-5}}([0,\infty)\times\mathbb{R}^d)},$$

which is acceptable, too.

We are now in a position to apply Theorem 1.5; invoking (41), we conclude that for n sufficiently large,

$$S_{[0,\infty)}(u_n) \lesssim 1 + E_c,$$

this contradicts (37). This finishes the proof of Lemma 6.3. $\square$

Let us return to the proof of Proposition 6.2 now. Rearranging the indices, we may assume that there exists $1 \leqslant J_1 < J_0$ such that

$$\limsup_{n\to\infty} S_{[0,T_{n,j}^+)}\bigl(v_n^j\bigr) = \infty \quad \text{for } 1 \leqslant j \leqslant J_1$$

and

$$\limsup_{n\to\infty} S_{[0,\infty)}\bigl(v_n^j\bigr) < \infty \quad \text{for } j > J_1. \tag{46}$$

Passing to a subsequence in n, we can guarantee that $S_{[0,T_{n,1}^+)}(v_n^1) \to \infty$.

For each $m, n \geqslant 1$ let us define an integer $j(m,n) \in \{1, \dots, J_1\}$ and an interval K_n^m of the form $[0,\tau)$ by

$$\sup_{1\leqslant j\leqslant J_1} S_{K_n^m}\bigl(v_n^j\bigr) = S_{K_n^m}\bigl(v_n^{j(m,n)}\bigr) = m. \tag{47}$$

By the pigeonhole principle, there is a $1 \leqslant j_1 \leqslant J_1$, so that for infinite many m, one has $j(m,n) = j_1$, for infinite many n. Note that the infinite set of n for which this holds may be m-dependent. By reordering the indices, we may assume that $j_1 = 1$. Then by the definition of the critical kinetic energy, we obtain

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \sup_{t\in K_n^m} \left\| \nabla v_n^1(t) \right\|_2^2 \geqslant E_c. \tag{48}$$

On the other hand, by virtue of (46) and (47), all v_n^j have finite scattering size on K_n^m for each $m \geqslant 1$. Thus, by the same argument used in Lemma 6.3, we see that for n and J sufficiently large, u_n^J is a good approximation to u_n on each K_n^m. More precisely,

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left\| u_n^J - u_n \right\|_{L_t^\infty \dot{H}^2(K_n^m\times\mathbb{R}^d)} = 0 \tag{49}$$

for each $m \geqslant 1$.

Lemma 6.4 *(Kinetic energy decoupling for u_n^J). For all $J \geqslant 1$ and $m \geqslant 1$,*

$$\limsup_{n\to\infty} \sup_{t\in K_n^m} \left| \| \Delta u_n^J \|_2^2 - \sum_{j=1}^{J} \| \Delta v_n^j(t) \|_2^2 - \| \Delta w_n^J \|_2^2 \right| = 0.$$

Proof. Fix $J \geqslant 1$ and $m \geqslant 1$. Then for all $t \in K_n^m$,

$$\| \Delta u_n^J \|_2^2 = \sum_{j=1}^{J} \| \Delta v_n^j \|_2^2 + \| \Delta w_n^J \|_2^2 + \sum_{j\neq j'} \langle \Delta v_n^j(t), \Delta v_n^{j'}(t) \rangle$$
$$+ \sum_{j=1}^{J} \left(\langle \Delta e^{it\Delta^2} w_n^J, \Delta v_n^j(t) \rangle + \langle \Delta v_n^j(t), \Delta e^{it\Delta^2} w_n^J \rangle \right).$$

It suffices to prove that for all sequences $t_n \in K_n^m$,

$$\lim_{n\to\infty} \langle \Delta v_n^j(t_n), \Delta v_n^{j'}(t_n) \rangle = 0 \tag{50}$$

and

$$\lim_{n\to\infty} \langle \Delta e^{it_n\Delta^2} w_n^J, \Delta v_n^j(t_n) \rangle = 0 \tag{51}$$

for all $1 \leqslant j, j' \leqslant J$ with $j \neq j'$. We will only demonstrate the latter, which requires Lemma 5.5; the former can be deduced in much the same manner using (31). By a change of variables,

$$\langle \Delta e^{it_n^j\Delta^2} w_n^J, \Delta v_n^j(t_n) \rangle = \left\langle \Delta e^{it_n(\lambda_n^j)^{-4}\Delta^2} \left[\left(g_n^j\right)^{-1} w_n^J \right], \Delta v^j \left(\frac{t_n}{(\lambda_n^j)^4} + t_n^j \right) \right\rangle. \tag{52}$$

As $t_n \in K_n^m \subset [0, T_{n,j}^+)$ for all $1 \leqslant j \leqslant J_1$, we have $t_n(\lambda_n^j)^{-4} + t_n^j \in I^j$ for all $j \geqslant 1$. Recall that I^j is the maximal lifespan of v^j; for $j > J_1$ we have $\mathbb{R}^+ \subset I^j$. By refining the sequence once for every j and using the standard diagonalization argument, we may assume $t_n(\lambda_n^j)^{-4} + t_n^j$ converges for every j.

Fix $1 \leqslant j \leqslant J$. If $t_n(\lambda_n^j)^{-4} + t_n^j$ converges to some point τ^j in the interior of I^j, then by the continuity of the flow, $v^j(t_n(\lambda_n^j)^{-4} + t_n^j)$ converges to $v^j(\tau^j)$ in $\dot H_x^2(\mathbb{R}^d)$. On the other hand, by (32),

$$\limsup_{n\to\infty} \left\| e^{it_n(\lambda_n^j)^{-4}\Delta^2} \left[\left(g_n^j\right)^{-1} w_n^J \right] \right\|_{\dot H_x^2(\mathbb{R}^d)} = \limsup_{n\to\infty} \| w_n^J \|_{\dot H_x^2(\mathbb{R}^d)} \lesssim E_c^{\frac{1}{2}}.$$

Combining this with (52), we obtain

$$\lim_{n\to\infty} \langle \Delta e^{it_n^j\Delta^2} w_n^J, \Delta v_n^j(t_n) \rangle = \lim_{n\to\infty} \langle \Delta e^{it_n(\lambda_n^j)^{-4}\Delta^2} \left[\left(g_n^j\right)^{-1} w_n^J \right], \Delta v^j\left(\tau^j\right) \rangle$$
$$= \lim_{n\to\infty} \langle \Delta e^{-it_n^j\Delta^2} \left[\left(g_n^j\right)^{-1} w_n^J \right], \Delta e^{-i\tau^j\Delta^2} v^j\left(\tau^j\right) \rangle.$$

Invoking Lemma 5.5, we deduce (51).

Consider the case when $t_n(\lambda_n^j)^{-4} + t_n^j$ converges to $\sup I^j$. Then we must have $\sup I^j = \infty$ and v^j scatters forward in time. In fact, this is clearly true if $t_n^j \to \infty$ as $n \to \infty$; in other cases, failure would imply

$$\limsup_{n\to\infty} S_{[0,t_n]}\big(v_n^j\big) = \limsup_{n\to\infty} S_{[t_n^j, t_n(\lambda_n^j)^{-4}+t_n^j]}\big(v^j\big) = \infty,$$

which contradicts $t_n \in K_n^m$. Therefore, there exists $\psi^j \in \dot{H}_x^2(\mathbb{R}^d)$ such that

$$\lim_{n\to\infty} \big\| v^j\big(t_n\big(\lambda_n^j\big)^{-4} + t_n^j\big) - e^{i(t_n(\lambda_n^j)^{-4}+t_n^j)\Delta^2}\psi^j \big\|_{\dot{H}_x^2(\mathbb{R}^d)} = 0.$$

Together with (52), this yields

$$\lim_{n\to\infty} \big\langle \Delta e^{it_n^j\Delta^2} w_n^J, \Delta v_n^j(t_n)\big\rangle = \lim_{n\to\infty} \big\langle \Delta e^{-it_n^j\Delta^2}\big[\big(g_n^j\big)^{-1} w_n^J\big], \Delta\psi^j\big\rangle,$$

which by Lemma 5.5 implies (51).

Finally, we consider the case when $t_n(\lambda_n^j)^{-4} + t_n^j$ converges to $\inf I^j$. Since $t_n(\lambda_n^j)^{-4} \geqslant 0$ and $\inf I^j < \infty$ for all $j \geqslant 1$ we see that t_n^j does not converge to $+\infty$. Moreover, if $t_n^j \equiv 0$, then $\inf I^j < 0$; as $t_n(\lambda_n^j)^{-4} \geqslant 0$, we see that t_n^j cannot be identically zero. This leaves $t_n^j \to -\infty$ as $n \to \infty$. Thus $\inf I^j = -\infty$ and v^j scatters backward in time to $e^{it\Delta^2}\phi^j$. We obtain

$$\lim_{n\to\infty} \big\| v^j\big(t_n\big(\lambda_n^j\big)^{-4} + t_n^j\big) - e^{i(t_n(\lambda_n^j)^{-4}+t_n^j)\Delta^2}\phi^j \big\|_{\dot{H}_x^2(\mathbb{R}^d)} = 0,$$

which by (51) implies

$$\lim_{n\to\infty} \big\langle \Delta e^{it_n^j\Delta^2} w_n^J, \Delta v_n^j(t_n)\big\rangle = \lim_{n\to\infty} \big\langle \Delta e^{-it_n^j\Delta^2}\big[\big(g_n^j\big)^{-1} w_n^J\big], \Delta\phi^j\big\rangle.$$

Invoking Lemma 5.5 once again, we derive (51). This finishes the proof of Lemma 6.4. $\square$

Thus by (36), (49) and Lemma 6.4, we have

$$E_c \geqslant \limsup_{n\to\infty} \sup_{t\in K_n^m} \big\| \Delta u_n(t)\big\|_2^2 = \lim_{J\to\infty} \limsup_{n\to\infty} \left\{ \big\|\Delta w_n^J\big\|_2^2 + \sup_{t\in K_n^m} \sum_{j=1}^{J} \big\|\Delta v_n^j(t)\big\|_2^2 \right\}.$$

Invoking (48), this implies $J = 1$, $v_n^j \equiv 0$ for all $j \geqslant 2$ and $w_n := w_n^1$ converges to zero strongly in $\dot{H}_x^2$. In other words,

$$u_n(0) = g_n e^{i\tau_n\Delta^2}\phi + w_n \tag{53}$$

for some $g_n \in G$, $\tau_n \in \mathbb{R}$ and some functions ϕ, $w_n \in \dot{H}_x^2(\mathbb{R}^d)$ with $w_n \to 0$ strongly in $\dot{H}_x^2(\mathbb{R}^d)$. Moreover, the sequence $\tau_n \equiv 0$ or $\tau_n \to \pm\infty$.

If $\tau_n \equiv 0$, (53) immediately implies that $u_n(0)$ converges modulo G to ϕ, which proves Proposition 6.2 in this case.

Finally, we will show that this is the only possible case, that is, τ_n cannot converge to either ∞ or $-\infty$. We argue by contradiction. Assume that τ_n converges to $+\infty$, the proof in the negative time direction is essentially the same. By the Strichartz inequality, $S_{\mathbb{R}}(e^{it\Delta^2}\phi) < \infty$. Thus we have

$$\lim_{n\to\infty} S_{\geqslant 0}\big(e^{it\Delta^2} e^{i\tau_n\Delta^2}\phi\big) = 0.$$

Since the action of G preserves linear solutions and the scattering size, this implies

$$\lim_{n\to\infty} S_{\geqslant 0}\big(e^{it\Delta^2} g_n e^{i\tau_n\Delta^2}\phi\big) = 0.$$

Combining this with (53) and $w_n \to 0$ in $\dot{H}_x^2$, we conclude

$$\lim_{n\to\infty} S_{\geqslant 0}\big(e^{it\Delta^2} u_n(0)\big) = 0.$$

An application of Lemma 1.5 yields

$$\lim_{n\to\infty} S_{\geqslant 0}(u_n) = 0,$$

which contradicts (36). □

Proof of Theorem 6.1. Suppose $d \geqslant 5$ is such that Conjecture 1.6 failed. Then the critical kinetic energy E_c must obey $E_c < \|\Delta W\|_2^2$. By the definition of the critical kinetic energy, we can find a sequence $u_n : I_n \times \mathbb{R}^d \to \mathbb{C}$ of solutions to (2) with I_n compact,

$$\sup_{n\geqslant 1}\sup_{t\in I_n}\big\|\Delta u_n(t)\big\|_2^2 = E_c \quad \text{and} \quad \lim_{n\to\infty} S_{I_n}(u_n) = \infty. \tag{54}$$

Let $t_n \in I_n$ be such that $S_{\geqslant t_n}(u_n) = S_{\leqslant t_n}(u_n)$. Then

$$\lim_{n\to\infty} S_{\geqslant t_n}(u_n) = \lim_{n\to\infty} S_{\leqslant t_n}(u_n) = \infty. \tag{55}$$

Using the time translation, we may take all $t_n = 0$.

Applying Proposition 6.2 and passing to a subsequence if necessary, we can find $g_n \in G$ and a function $u_0 \in \dot{H}_x^2(\mathbb{R}^d)$ such that $g_n u_n(0) \to u_0$ strongly in $\dot{H}_x^2(\mathbb{R}^d)$. By applying the group action T_{g_n} to the solution u_n we may take all the g_n to be identity. Thus $u_n(0)$ converges strongly to u_0 in $\dot{H}_x^2(\mathbb{R}^d)$.

Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be the maximal-lifespan solution to (2) with initial data $u(0) = u_0$. As $u_n(0) \to u_0$ in $\dot{H}_x^2(\mathbb{R}^d)$, Theorem 1.5 shows that $I \subseteq \liminf I_n$ and

$$\lim_{n\to\infty} \|u_n - u\|_{L_t^\infty \dot{H}_x^2(K\times\mathbb{R}^d)} = 0 \quad \text{for all compact } K \subset I.$$

Thus by (54),

$$\sup_{t\in I}\big\|\Delta u(t)\big\|_2^2 \leqslant E_c. \tag{56}$$

Next we prove that u blows up both forward and backward in time. Indeed, if u does not blow up forward in time, then $[0,\infty) \subset I$ and $S_{\geqslant 0}(u) < \infty$. By Theorem 1.5, this implies $S_{\geqslant 0}(u_n) < \infty$ for sufficiently large n, which contradicts (55). A similar argument proves that u blows up backward in time.

Therefore, by our definition of E_c, $\sup_{t\in I}\|\Delta u(t)\|_2^2 \geqslant E_c$. Combining this with (56), we obtain

$$\sup_{t\in I}\big\|\Delta u(t)\big\|_2^2 = E_c.$$

It remains to show that u is almost periodic modulo G. Consider an arbitrary sequence $\tau_n \in I$. As u blows up in both time directions

$$S_{\geqslant \tau_n}(u) = S_{\leqslant \tau_n}(u) = \infty.$$

Applying Proposition 6.2, we conclude that $u(\tau_n)$ admits a convergent subsequence in $\dot{H}_x^2(\mathbb{R}^d)$ modulo G. Thus the orbit $\{Gu(t)\colon\ t \in I\}$ is precompact in $G \setminus \dot{H}_x^2(\mathbb{R}^d)$.

A direct analogue of Theorem 7.3 in [28] shows that if u_n is a sequence of bounded radial functions in $\dot{H}^2(\mathbb{R}^d)$, then there exists a family of radial functions φ^j, $j = 1, 2, \ldots,$ in $\dot{H}^2(\mathbb{R}^d)$ and group elements $G_n^{(j)} \in G'_{\mathrm{rad}}$ for $j, n = 1, 2, \ldots,$ such that we have decomposition (29) for all $l = 1, 2, \ldots,$ where $w_n^l \in \dot{H}^2$ is radial and obeys (30). Moreover g_n^j, $g_n^{(j')}$ are asymptotically orthogonal in the sense of (31) for any $j \neq j'$ and for $l \geqslant 1$, we have the energy decoupling property (32). This concludes the proof of Theorem 6.1. □

7. Three enemies

Theorem 7.1 *(Three special scenarios for blowup). Fix $d \geqslant 5$ and suppose that Conjecture* 1.6 *fails for this choice of d. Then there exists a minimal kinetic energy, maximal-lifespan solution $u : I \times \mathbb{R}^d \to \mathbb{C}$, which is almost periodic modulo G, $S_I(u) = \infty$, and obeys $\sup_{t \in I} \|\Delta u\|_2 < \|\Delta W\|_2$. If furthermore $d \geqslant 5$ and Conjecture* 1.6 *failed for spherically symmetric data, then u may be chosen to be spherically symmetric and almost periodic modulo G_{rad}.*

With or without spherical symmetry, we can also ensure that the lifespan I and the frequency scale function $N : I \to \mathbb{R}^+$ match one of the following three scenarios:

I (Finite time blowup) *We have that either $|\inf I| < \infty$ or $\sup I < \infty$.*

II (Soliton-like solution) *We have $I = \mathbb{R}$ and*

$$N(t) = 1 \quad \textit{for all } t \in \mathbb{R}.$$

III (Low-to-high frequency cascade) *We have $I = \mathbb{R}$ and*

$$\inf_{t \in \mathbb{R}} N(t) \geqslant 1, \quad \textit{and} \quad \limsup_{t \to \infty} N(t) = \infty.$$

Proof. The proof is a straightforward adaptation of the similar proof in Killip, Tao and Visan [18,20]. Let $v : J \times \mathbb{R}^d \to \mathbb{C}$ denote a minimal kinetic energy blowup solution whose existence is guaranteed by Theorem 6.1. We denote the frequency scale function of v by $N_v(t)$ and spatial center function of v by $x_v(t)$. For any $T \geqslant 0$, define the quantity

$$\operatorname{osc}(T) = \inf_{t_0 \in J} \frac{\sup\{N_v(t)\colon\ t \in J \text{ and } |t - t_0| \leqslant T N_v(t_0)^{-4}\}}{\inf\{N_v(t)\colon\ t \in J \text{ and } |t - t_0| \leqslant T N_v(t_0)^{-4}\}}. \tag{57}$$

Case I. $\lim_{T \to \infty} \operatorname{osc}(T) < \infty$.

In this case, we can find a finite number $A = A_v$, a sequence t_n of times in J, and a sequence $T_n \to \infty$ such that

$$\frac{\sup\{N_v(t)\colon\ t \in J \text{ and } |t - t_n| \leqslant T_n N_v(t_n)^{-4}\}}{\inf\{N_v(t)\colon\ t \in J \text{ and } |t - t_n| \leqslant T_n N_v(t_n)^{-4}\}} < A$$

for all n. Note that this, together with Corollary 4.9, implies that

$$\left[t_n - T_n/N_v(t_n)^4, t_n + T_n/N_v(t_n)^4\right] \subset J$$

and

$$N_v(t) \sim_v N_v(t_n)$$

for all t in this interval.

Let $v^{[t_n]}$ be the normalization of v at times t_n as in (26), then $v^{[t_n]}$ is a maximal-lifespan normalized solution with lifespan

$$J_n := \left\{s \in \mathbb{R}\colon\ t_n + \frac{1}{N_v(t_n)^4} s \in J\right\} \supset [-T_n, T_n]$$

and energy $E(v)$. It is almost periodic modulo G with frequency scale function

$$N_{v^{[t_n]}}(s) := \frac{1}{N_v(t_n)} N_v\left(t_n + \frac{1}{N_v(t_n)^4} s\right)$$

and compactness modulus function C. In particular, we see that

$$N_{v^{[t_n]}}(s) \sim_v 1 \tag{58}$$

for all $s \in [-T_n, T_n]$.

We now apply Lemma 4.8 and conclude (passing to a subsequence if necessary) that $v^{[t_n]}$ converge locally uniformly to a maximal-lifespan solution u with energy $E(v)$ defined on an open interval I containing 0 and which is almost periodic modulo G. As $T_n \to \infty$, Lemma 4.6 and (58) imply that the frequency scale function $N : I \to \mathbb{R}^+$ of v satisfies

$$0 < \inf_{t \in I} N(t) \leqslant \sup_{t \in I} N(t) < \infty.$$

In particular, by Corollary 4.10, I has no finite endpoints and hence $I = \mathbb{R}$. By modifying C by a bounded amount we may now normalize $N(t) \equiv 1$. Thus we have constructed a soliton-like solution in the sense of Theorem 7.1.

When $\operatorname{osc}(T)$ is unbounded, we must seek a solution belonging to one of the remaining two scenarios. We introduce the quantity

$$a(t_0) = \frac{N_v(t_0)}{\sup\{N_v(t)\colon\ t \leqslant t_0\}} + \frac{N_v(t_0)}{\sup\{N_v(t)\colon\ t \geqslant t_0\}}.$$

Case II. $\lim_{T\to\infty} \operatorname{osc}(T) = \infty$ and $\inf_{t_0 \in J} a(t_0) = 0$.

As $\inf_{t_0 \in J} a(t_0) = 0$, there exists a sequence of times $t_n \in J$ such that $a(t_n) \to 0$ as $n \to \infty$. By the definition of a, we can also find times $t_n^- < t_n < t_n^+$ with $t_n^-, t_n^+ \in J$ such that

$$\frac{N_v(t_n^-)}{N_v(t_n)} \to +\infty \quad \text{and} \quad \frac{N_v(t_n^+)}{N_v(t_n)} \to +\infty.$$

Next we choose times $t_n' \in (t_n^-, t_n^+)$ so that

$$N_v\left(t_n'\right) \leqslant 2 \inf\left\{N(t)\colon\ t \in \left[t_n^-, t_n^+\right]\right\}.$$

In particular,

$$N_v(t_n') \sim \inf_{t_n^- \leqslant t \leqslant t_n^+} N_v(t),$$

which allows us to deduce that

$$\frac{N_v(t_n^-)}{N_v(t_n')} \to +\infty \quad \text{and} \quad \frac{N_v(t_n^+)}{N_v(t_n')} \to +\infty.$$

We define the rescaled and translated times $s_n^- < 0 < s_n^+$ by

$$s_n^{\pm} := N_v(t_n')^4 (t_n^{\pm} - t_n')$$

and the normalizations $v^{[t_n']}$ at times t_n' by (26). These are normalized maximal-lifespan solutions with lifespans containing $[s_n^-, s_n^+]$, which are almost periodic modulo G with frequency scale functions

$$N_{v^{[t_n']}}(s) := \frac{1}{N_v(t_n')} N_v\left(t_n' + \frac{1}{N_v(t_n')^4} s\right). \tag{59}$$

By the way we choose t_n', we see that

$$N_{v^{[t_n']}}(s) \gtrsim 1 \tag{60}$$

for all $s_n^- \leqslant s \leqslant s_n^+$. Moreover,

$$N_{v^{[t_n']}}(s_n^{\pm}) \to \infty \quad \text{as } n \to \infty \tag{61}$$

for either choice of sign.

We now apply Lemma 4.8 and conclude (passing to subsequence if necessary) that $v^{[t_n']}$ converge locally uniformly to a maximal-lifespan solution u of energy $E(v)$ defined on an open interval I containing 0, which is almost periodic modulo G.

Let N be a frequency scale function for u. From Lemma 4.11 we see that $N(t)$ is bounded from above on any compact set $K \subset I$. From this, Lemmas 4.5 and 4.6, we see that $N_{v^{[t_n']}}(t)$ is also bounded from above, uniformly in $t \in K$, for all sufficiently large n (depending on K). As a consequence of this and (61), we see that s_n^- and s_n^+ cannot be any limit points in K; thus $K \subset [s_n^-, s_n^+]$ for all sufficiently large n. Therefore $s_n^{\pm}$ converge to the endpoints of I. If $\sup(I) < +\infty$ or $|\inf(I)| < +\infty$, then u blows up in finite time. Otherwise, $I = \mathbb{R}$. In this case, we need to show that

$$\limsup_{t \to -\infty} N(t) = \limsup_{t \to +\infty} N(t) = \infty.$$

By time reversal symmetry, it suffices to establish that $\lim_{t \to +\infty} N(t) = \infty$. By (60) and Lemma 4.6, we conclude that

$$\inf_{t \in \mathbb{R}} N(t) \gtrsim 1.$$

Suppose $\lim_{t \to +\infty} N(t) < \infty$, then $N(t) \sim_u 1$ for all $t \geqslant 0$. We conclude from Lemma 4.6 that for every $m \geqslant 1$, there exists an n_m such that

$$N_{v^{[t_{n_m}']}}(t) \sim_u 1$$

for all $0 \leqslant t \leqslant m$. But by (57) and (59) this implies $\operatorname{osc}(\frac{\varepsilon m}{2}) \lesssim 1$ for all m and some $\varepsilon = \varepsilon(u) > 0$ independent of m. Note that ε is chosen as a lower bound on the quantities $N(t''_{n_m})^4/N(t'_{n_m})^4$ where $t''_{n_m} = t'_{n_m} + \frac{m}{2}N(t'_{n_m})^{-4}$. This contradicts the hypothesis $\lim_{T\to\infty}\operatorname{osc}(T) = \infty$ and so settles Case II.

Case III. $\lim_{T\to\infty}\operatorname{osc}(T) = \infty$ and $\inf_{t_0\in J} a(t_0) > 0$.

Let $\varepsilon = \varepsilon(v) > 0$ be such that $\inf_{t_0\in J} a(t_0) \geqslant 2\varepsilon$. We call a time t_0 future-spreading if $N(t) \leqslant \varepsilon^{-1}N(t_0)$ for all $t \geqslant t_0$; we call a time t_0 past-spreading if $N(t) \leqslant \varepsilon^{-1}N(t_0)$ for all $t \leqslant t_0$. Note that every $t_0 \in J$ is future-spreading, past-spreading or possibly both.

We will show that either all sufficiently late times are future-spreading or that all sufficiently early times are past-spreading. We only show the first half because the other half is similar. If this were false, there would be a future-spreading time t_0 and a sequence of past-spreading times t_n that converges to $\sup(J)$. For sufficiently large n, we have $t_n \geqslant t_0$. Since $N_v(t_0) \leqslant \varepsilon^{-1}N_v(t_n)$ and $N_v(t_n) \leqslant \varepsilon^{-1}N_v(t_0)$ we see that

$$N_v(t_n) \sim_v N_v(t_0)$$

for all such n. For any $t_0 < t < t_n$, we know that t is either past-spreading or future-spreading; thus we have either $N_v(t_0) \leqslant \varepsilon^{-1}N_v(t)$ or $N_v(t_n) \leqslant \varepsilon^{-1}N_v(t)$. Also, since t_0 is future-spreading $N_v(t) \leqslant \varepsilon^{-1}N_v(t_0)$ and t_n is past-spreading, $N_v(t) \leqslant \varepsilon^{-1}N_v(t_n)$, we conclude that

$$N_v(t) \sim_v N_v(t_0)$$

for all $t_0 < t < t_n$; since t_n converges to $\sup(J)$, this claim in fact holds for all $t_0 < t < \sup(J)$. From Corollary 4.10 we see that v does not blow up forward in finite time, that is, $\sup(J) = \infty$. This implies that $\lim_{T\to\infty}\operatorname{osc}(T) < \infty$, a contradiction. We may now assume that future-spreading occurs for all sufficiently late times; more precisely, we can find $t_0 \in J$ such that all times $t \geqslant t_0$ are future-spreading.

Choose T so that $\operatorname{osc}(T) > 2\varepsilon^{-1}$. We will now recursively construct an increasing sequence of times $\{t_n\}_{n=1}^{\infty}$ so that

$$0 \leqslant t_{n+1} - t_n \leqslant 8TN_v(t_n)^{-4} \quad \text{and} \quad N_v(t_{n+1}) \leqslant \frac{1}{2}N_v(t_n).$$

Given t_n, set $t'_n := t_n + 16TN_v(t_n)^{-4}$. If $N_v(t'_n) \leqslant \frac{1}{2}N_v(t_n)$ we choose $t_{n+1} = t'_n$ and the properties set out above follow immediately. If $N_v(t'_n) > \frac{1}{2}N_v(t_n)$, then

$$J_n := \left[t'_n - TN_v\left(t'_n\right)^{-4}, t'_n + TN_v\left(t'_n\right)^{-4}\right] \subseteq \left[t_n, t_n + 8TN_v(t_n)^{-4}\right]. \tag{62}$$

As t_n is future-spreading, this allows us to conclude that $N_v(t) \leqslant \varepsilon^{-1}N_v(t_n)$ on J_n, but then by the way T is chosen, we may find $t_{n+1} \in J_n$ so that $N_v(t_{n+1}) \leqslant N_v(t_n)$. Having obtained a sequence of times obeying (62), we may conclude that any subsequential limit u of $v^{[t_n]}$ is a finite-time blowup solution. To elaborate, set $s_n = (t_0 - t_n)N_v(t_n)^4$ and note that $N_{v^{[t_n]}}(s_n) \geqslant 2^n$. However s_n is a bounded sequence; indeed,

$$|s_n| = N(t_n)^4\sum_{k=0}^{n-1}[t_{k+1} - t_k] \leqslant 8T\sum_{k=0}^{n-1}\frac{N(t_n)^4}{N(t_k)^4} \leqslant 8T\sum_{k=0}^{n-1}2^{-(n-k)} \leqslant 8T.$$

In this way, we see that the solution u must blow up at some time $-8T \leqslant t < 0$.

This completes the proof of Theorem 7.1. □

8. Kill the enemies

Theorem 8.1 *(No finite-time blowup). Let d $\geqslant 5$. Then there are no maximal-lifespan radial solutions $u : I \times \mathbb{R}^d \to \mathbb{C}$ to (2) that are almost periodic modulo G_{rad}, obey*

$$S_I(u) = \infty, \tag{63}$$

$$\sup_{t\in I} \left\| \Delta u(t) \right\|_{L^2} < \| \Delta W \|_{L^2}$$

and are such that either $|\inf I| < \infty$ or $\sup I < \infty$.

Proof. Suppose for a contradiction that there existed such a solution u. Without loss of generality, we may assume that $\sup I < \infty$. Then by Corollary 4.10,

$$\liminf_{t \nearrow \sup I} N(t) = \infty. \tag{64}$$

We now show that

$$\limsup_{t \nearrow \sup I} \int_{|x| \leqslant R} \left| u(t,x) \right|^2 dx = 0 \quad \text{for all } R > 0. \tag{65}$$

In fact, let $u(t,x) = N(t)^{\frac{d-4}{2}} v(N(t)x, t)$, then

$$\begin{aligned} \int_{|x|\leqslant R} \left| u(t,x) \right|^2 dx &= N(t)^{-4} \int_{|x|\leqslant RN(t)} \left| v(x,t) \right|^2 dx \\ &= N(t)^{-4} \int_{|x|\leqslant \epsilon RN(t)} \left| v(x,t) \right|^2 dx + N(t)^{-4} \int_{\substack{|x|\leqslant RN(t) \\ |x| > \epsilon RN(t)}} \left| v(x,t) \right|^2 dx \\ &\doteq A + B. \end{aligned}$$

By Hölder's inequality, we have

$$A \leqslant \left(N(t)\right)^{-4} \left(\epsilon RN(t)\right)^4 \|v\|_{L^{2^\#}}^2 \leqslant (\epsilon R)^4 \|\Delta W\|_2^2.$$

A can be acceptable if we take ϵ to be sufficiently small. By Hölder, (64) and the fact that u is almost periodic, we have

$$B \leqslant R^4 \|v\|_{L^{2^\#}(|x| > \epsilon RN(t))}^2 \to 0 \quad \text{as } t \to \sup I.$$

Thus (65) is proved.

For $t \in I$, define

$$M_R(t) := \int_{\mathbb{R}^d} \phi\left(\frac{|x|}{R}\right) \left| u(t,x) \right|^2 dx,$$

where ϕ is a smooth, radial function such that $\phi(r) = 1$ for $r \leqslant 1$ and $\phi(r) = 0$ for $r \geqslant 2$. By (65),

$$\limsup_{t \nearrow \sup I} M_R(t) = 0 \quad \text{for all } R > 0. \tag{66}$$

On the other hand,

$$\partial_t M_R(t) = -2\,\mathrm{Im} \int \Delta\left(\phi\left(\frac{|x|}{R}\right)\right) \bar{u}\, \Delta u \, dx - 2\,\mathrm{Im} \int \nabla\left(\phi\left(\frac{|x|}{R}\right)\right) \cdot \nabla \bar{u}\, \Delta u \, dx.$$

So by Hölder and Hardy's inequality, we have

$$\begin{aligned} \left|\partial_t M_R(t)\right| &\lesssim \int\limits_{|x| \sim R} \frac{|u||\Delta u|}{R^2}\, dx + \int\limits_{|x| \sim R} \frac{|\nabla u||\Delta u|}{R} \\ &\lesssim \left\| \frac{u}{|x|^2} \right\|_2 \|\Delta u\|_2 + \left\| \frac{|\nabla u|}{|x|} \right\|_2 \|\Delta u\|_2 \\ &\lesssim \|\Delta u\|_2^2 \lesssim \|\Delta W\|_2^2. \end{aligned}$$

Thus,

$$M_R(t_1) = M_R(t_2) + \int\limits_{t_2}^{t_1} \partial_t M_R(t)\, dt \lesssim M_R(t_2) + |t_1 - t_2| \|\Delta W\|_2^2$$

for all $t_1, t_2 \in I$ and $R > 0$. Let $t_2 \nearrow \sup I$ and invoking (66), we have

$$M_R(t_1) \lesssim |\sup I - t_1| \|\Delta W\|_2^2.$$

Now letting $R \to \infty$ and using the conservation of mass, we obtain $u_0 \in L_x^2(\mathbb{R}^d)$. Finally, letting $t_1 \nearrow \sup I$, we deduce $u_0 = 0$. Thus $u \equiv 0$, contradicting (63). □

Theorem 8.2 *(Absence of cascades and solitons). Let $d \geqslant 5$. There are no global radial solutions to* (2) *that are low-to-high cascades or solitons in the sense of Theorem* 7.1.

Proof. We will show that no global radial solutions that are almost periodic modulo G_{rad} with the frequency scale function $N(t) \geqslant 1$ for all $t \in \mathbb{R}$.

By the almost periodicity of u and Hardy's inequality, we have that for any $\epsilon > 0$, there exists $R(\epsilon) > 0$ such that for all $t \in [0, +\infty)$,

$$\int\limits_{|x| > R(\epsilon)} \left(|\Delta u|^2 + \frac{|\nabla u|^2}{|x|^2} + \frac{|u|^2}{|x|^4} \right) dx \leqslant \epsilon. \tag{67}$$

On the other hand, (13) and Corollary 3.3 yields

$$4 \int |\Delta u|^2 - 4 \int |u|^{2^\#} \geqslant \tilde{C}_{\delta_0} \int |\Delta u_0|^2.$$

This and (67) with $\epsilon = \epsilon_0 \int |\Delta u_0|^2$, implies that there exists $R_0 > 0$ such that for all $t \in [0, \infty)$, we have

$$4 \int\limits_{|x| \leqslant R_0} |\Delta u|^2 - 4 \int\limits_{|x| \leqslant R_0} |u|^{2^\#} \geqslant C_{\delta_0} \int |\Delta u_0|^2. \tag{68}$$

Lemma 8.3. *Define*

$$z_R(t) = \operatorname{Im} \int x\phi\left(\frac{|x|}{R}\right) \cdot \nabla \bar{u} u \, dx, \tag{69}$$

where $\phi(r)$ is a smooth function with $\phi(r) = 1$ when $r \leqslant 1$ and $\phi(r) = 0$ when $r \geqslant 2$. Then

$$z_R'(t) \geqslant 4 \int\limits_{|x| \leqslant R} \left(|\Delta u|^2 - |u|^{2^\#}\right) dx - O\left(\int\limits_{R \leqslant |x| \leqslant 2R} \left(\frac{|u||\Delta u|}{R^2} + \frac{|\nabla u||\Delta u|}{R} + |\Delta u|^2 \right) dx \right). \tag{70}$$

Thus, by (67) and (68), we get

$$z_R'(t) \geqslant C_{d,\delta_0} \int |\Delta u_0|^2$$

for R large.

Integrating in t, we have

$$z_R(t) - z_R(0) \geqslant t C_{d,\delta_0} \int |\Delta u_0|^2.$$

But by the definition of $z_R(t)$, we have

$$\left| z_R(t) - z_R(0) \right| \leqslant 2R^4 \|\Delta W\|_2^2,$$

which is a contradiction for t large. $\square$

Proof of Lemma 8.3. We will compute $\partial_t z_R(t)$:

$$\begin{aligned}
\partial_t z_R(t) &= \operatorname{Im} \int x\phi\left(\frac{|x|}{R}\right) \cdot \nabla \bar{u}_t u \, dx + \operatorname{Im} \int x\phi\left(\frac{|x|}{R}\right) \cdot \nabla \bar{u} u_t \, dx \\
&= 2 \operatorname{Im} \int x\phi\left(\frac{|x|}{R}\right) \cdot \nabla \bar{u} u_t \, dx + \operatorname{Im} \int \nabla \cdot \left(x\phi\left(\frac{|x|}{R}\right)\right) \bar{u} u_t \\
&\doteq A + B.
\end{aligned}$$

A and B can be computed as follows:

$$\begin{aligned}
A = {} & (4-d) \int \phi\left(\frac{|x|}{R}\right)\left(|\Delta u|^2 - |u|^{2^\#}\right) dx \\
& + 2 \operatorname{Re} \int \phi'\left(\frac{|x|}{R}\right) \frac{(3-d) x \cdot \nabla \bar{u} \Delta u}{R|x|} \, dx + 2 \int \phi'\left(\frac{|x|}{R}\right) |\Delta u|^2 \, dx \\
& + 2 \operatorname{Re} \int \phi''\left(\frac{|x|}{R}\right) \frac{x \cdot \nabla \bar{u} \Delta u}{R^2} \, dx + \frac{d-4}{d} \int \phi'\left(\frac{|x|}{R}\right) \frac{|x||u|^{2^\#}}{R} \, dx, \\
B = {} & d \int \phi\left(\frac{|x|}{R}\right)\left(|\Delta u|^2 - |u|^{2^\#}\right) dx + 2d \operatorname{Re} \int \frac{x \cdot \nabla \bar{u} \Delta u}{R|x|} \, dx \\
& + d \operatorname{Re} \int \left(\phi''\left(\frac{|x|}{R}\right) \frac{\bar{u}}{R^2} + \phi'\left(\frac{|x|}{R}\right) \frac{(d-1) x \cdot \nabla \bar{u}}{R|x|} \right) \Delta u \, dx + \int \phi'\left(\frac{|x|}{R}\right) \frac{|x||\Delta u|^2}{R} \, dx
\end{aligned}$$

$$+\operatorname{Re}\int\left(\phi'\left(\frac{|x|}{R}\right)\frac{d-1}{R|x|}+\phi''\left(\frac{|x|}{R}\right)\frac{d+1}{R^2}+\phi'''\left(\frac{|x|}{R}\right)\frac{|x|}{R^3}\right)\bar{u}\,\Delta u\,dx$$
$$+2\operatorname{Re}\int\left(\phi'\left(\frac{|x|}{R}\right)\frac{1}{R|x|}+\phi''\left(\frac{|x|}{R}\right)\frac{1}{R^2}\right)x\cdot\nabla\bar{u}\,\Delta u\,dx.$$

Since ϕ', ϕ'' and ϕ''' are supported in $\{x\colon\ R\leqslant|x|\leqslant 2R\}$, (70) follows. $\square$

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