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Asymptotics for some semilinear hyperbolic equations with non-autonomous damping

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ABSTRACT

Let V and H be Hilbert spaces such that $V \subset H \subset V'$ with dense and continuous injections. Consider a linear continuous operator $A: V \rightarrow V'$ which is assumed to be symmetric, monotone and semi-coercive. Given a function $f: V \rightarrow H$ and a map $\gamma \in W_{loc}^{1,1}(\mathbb{R}_+, \mathbb{R}_+)$ such that $\lim_{t \rightarrow +\infty} \gamma(t) = 0$, our purpose is to study the asymptotic behavior of the following semilinear hyperbolic equation

$$\frac{d^2 u}{dt^2}(t) + \gamma(t) \frac{du}{dt}(t) + Au(t) + f(u(t)) = 0, \quad t \geq 0. \quad (E)$$

The nonlinearity f is assumed to be monotone and conservative. Condition $\int_0^{+\infty} \gamma(t) dt = +\infty$ guarantees that some suitable energy function tends toward its minimum. The main contribution of this paper is to provide a general result of convergence for the trajectories of (E): if the quantity $\gamma(t)$ behaves as k/t^α , for some $\alpha \in]0, 1[$, $k > 0$ and t large enough, then $u(t)$ weakly converges in V toward an equilibrium as $t \rightarrow +\infty$. Strong convergence in V holds true under compactness or symmetry conditions. We also give estimates for the speed of convergence of the energy under some ellipticity-like conditions. The abstract results are applied to particular semilinear evolution problems at the end of the paper.

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1. Introduction

Throughout this paper, V stands for a real Hilbert space, whose scalar product and norm are respectively denoted by (\cdot, \cdot) and $\|\cdot\|$. Let H be another real Hilbert space with scalar product (\cdot, \cdot) and norm $|\cdot|$. Suppose that V is dense in H with continuous injection. By duality, the topological dual space H' of H is identified with a dense subspace of the topological dual V' of V . Identifying H with H' , we obtain $V \subset H \subset V'$, where each space is dense in the next one, each injection being continuous. We denote by $\langle \cdot, \cdot \rangle_{V', V}$ the duality pairing between V' and V . Let $a: V \times V \rightarrow \mathbb{R}$ be a continuous bilinear form satisfying

$$a(\cdot, \cdot) \text{ is symmetric, positive,} \quad (h_1)$$

$$\exists \lambda \geq 0, \mu > 0 \quad \text{such that} \quad \forall u \in V, \quad a(u, u) + \lambda |u|^2 \geq \mu \|u\|^2. \quad (h_2)$$

This last property is known as the semi-coercivity of the form a . We associate with $a(\cdot, \cdot)$ the continuous operator $A: V \rightarrow V'$ defined by $\langle Au, v \rangle_{V', V} = a(u, v)$ for all $u, v \in V$. We denote by $D(A)$ the domain of the operator A , i.e. $D(A) = \{v \in V; Av \in H\}$. Given a function $f: V \rightarrow H$ and a map $\gamma \in W_{loc}^{1,1}(\mathbb{R}_+, \mathbb{R}_+)$, we consider the following semilinear evolution equation of second-order in time

$$\frac{d^2 u}{dt^2}(t) + \gamma(t) \frac{du}{dt}(t) + Au(t) + f(u(t)) = 0, \quad t \geq 0. \quad (E)$$

The nonlinearity f is assumed to be conservative, i.e. derives from some potential $F \in C^1(V, \mathbb{R})$. The main purpose of the paper is to investigate the asymptotic behavior of the trajectories of (E) for a vanishing damping term, i.e. $\gamma(t) \rightarrow 0$ as $t \rightarrow +\infty$. It is clear that the decay properties of the map γ play a central role in the analysis. In particular, if the quantity $\gamma(t)$ tends to 0 too rapidly as $t \rightarrow +\infty$, convergence of the trajectories may fail. To motivate our study, let us show how it is connected to other questions of interest.

Case of a constant damping If $\gamma(t) \equiv \gamma$, existence and uniqueness are well known in the framework of damped wave equations. More precisely, if the map $f: V \rightarrow H$ is Lipschitz continuous on the bounded sets of V and if the map F satisfies suitable growth conditions, then for any $(u_0, v_0) \in D(A) \times V$, there exists a unique solution $u \in W_{loc}^{1,\infty}(\mathbb{R}_+, V) \cap W_{loc}^{2,\infty}(\mathbb{R}_+, H)$ of (E) such that $u(0) = u_0$ and $\frac{du}{dt}(0) = v_0$, see [12, Theorem II.3.2.1] or [20, Chapter IV, Theorem 4.1]. The trajectories of (E) are known to converge toward an equilibrium point $u_\infty \in \{v \in V, Av + f(v) = 0\}$ under assumptions like monotonicity or analyticity. In the case of a monotone map f , convergence is obtained for the weak topology of V and the main ingredient of the proof is the Opial lemma, cf. [3]. When the nonlinearity is analytic, convergence of the trajectories relies on the Łojasiewicz inequality, see [15,16] and the pioneering work [19] for parabolic problems.

Averaged heat equation With the same assumptions as above, consider the abstract heat equation

$$\frac{dv}{ds}(s) + Av(s) = 0, \quad s \geq 0. \quad (1.1)$$

It may be of interest to examine the case where the velocity $\frac{dv}{ds}(s)$ is proportional, not to the instantaneous vector $Av(s)$, but to some average taken over the interval $[0, s]$. The simplest such equation is

$$\frac{dv}{ds}(s) + \frac{1}{s} \int_0^s Av(\sigma) d\sigma = 0, \quad s > 0. \quad (1.2)$$

After multiplying this equality by s and differentiating, we obtain the following second-order in time equation

$$s \frac{d^2 v}{ds^2}(s) + \frac{dv}{ds}(s) + Av(s) = 0, \quad s > 0.$$

The change of variable $s = \frac{t^2}{4}$ allows to rewrite the above equation as

$$\frac{d^2 u}{dt^2}(t) + \frac{1}{t} \frac{du}{dt}(t) + Au(t) = 0, \quad t > 0,$$

where the map u is defined by $u(t) = v(\frac{t^2}{4})$ for every $t \geq 0$. This is exactly equation (E) with $\gamma(t) = \frac{1}{t}$ and $f \equiv 0$. Assuming that the injection $V \hookrightarrow H$ is compact, there exists a nondecreasing sequence $(\lambda_i)_{i \geq 1}$ of eigenvalues of A , along with a complete orthonormal basis of H , $(e_i)_{i \geq 1}$ consisting of the corresponding eigenvectors. Let $u(t) = \sum_{i=1}^{+\infty} u_i(t)e_i$ be the decomposition of the solution $u(t)$ on the basis of eigenfunctions. Every component u_i satisfies the following equation

$$\ddot{u}_i(t) + \frac{1}{t} \dot{u}_i(t) + \lambda_i u_i(t) = 0, \quad t > 0.$$

It ensues that each kernel component u_i , $i \in \{1, \dots, \dim(\ker A)\}$ verifies $u_i(t) = a_i \ln t + b_i$, for some $a_i, b_i \in \mathbb{R}$. In particular, it cannot converge as $t \rightarrow +\infty$, unless it is stationary. When the eigenvalue λ_i is positive, we let the reader check that

$$u_i(t) = a'_i J_0(\sqrt{\lambda_i}t) + b'_i Y_0(\sqrt{\lambda_i}t), \quad \text{for some } a'_i, b'_i \in \mathbb{R},$$

where J_0 and Y_0 denote respectively the zeroth Bessel functions of the first and second kind.¹ Recalling that

$$J_0(t) \sim \sqrt{\frac{2}{\pi t}} \cos\left(t - \frac{\pi}{4}\right) \quad \text{and} \quad Y_0(t) \sim \sqrt{\frac{2}{\pi t}} \sin\left(t - \frac{\pi}{4}\right) \quad \text{as } t \rightarrow +\infty,$$

we deduce that $u_i(t) \sim \frac{c_i}{\sqrt{t}} \cos(\sqrt{\lambda_i}t - \varphi_i)$ as $t \rightarrow +\infty$, for some $c_i, \varphi_i \in \mathbb{R}$. Coming back to the averaged heat equation (1.2), we then obtain for each component v_i

$$v_i(s) \sim \frac{c_i}{\sqrt{2}} s^{-\frac{1}{4}} \cos(2\sqrt{\lambda_i}s - \varphi_i) \quad \text{as } s \rightarrow +\infty.$$

It converges toward zero much more slowly than the corresponding component of the “pure” heat equation, equal to $v_i(0)e^{-\lambda_i s}$. The above discussion shows that the global behavior of (1.2) – or more generally (E) – differs considerably from the one of Eq. (1.1).

Heavy ball with asymptotically small friction Given a continuous map $\gamma: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and a potential $\Phi: H \rightarrow \mathbb{R}$ of class C^1 with a locally Lipschitz gradient, let us consider the following ordinary differential equation in the Hilbert space H

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla \Phi(x(t)) = 0, \quad t \geq 0. \quad (1.3)$$

¹ See [1,5] for standard references on Bessel equations.

When $\gamma(t) \equiv \gamma > 0$, the above equation is known under the terminology of “Heavy Ball with Friction” system, (HBF) for short. From a mechanical point of view, (HBF) corresponds to the equation describing the motion of a material point subjected to the conservative force $-\nabla\Phi(x)$ and the viscous friction force $-\gamma\dot{x}$. The (HBF) system can be studied in the classical framework of the theory of dissipative dynamical systems, cf. [11,13]. The trajectories of (HBF) are known to converge toward a critical point of Φ under various assumptions (see [2,4] for convex potentials and [14] for analytic ones). In the recent papers [8,9], it is considered the case of a vanishing damping $\gamma(t) \rightarrow 0$ as $t \rightarrow +\infty$. The corresponding equation is typically obtained from a first-order gradient system involving some memory aspects, see [7]. If the function Φ is convex and has a unique minimum \bar{x} , condition $\int_0^{+\infty} \gamma(t) dt = +\infty$ is sufficient to ensure (weak) convergence of the trajectories of (1.3) toward \bar{x} . When the function Φ has a continuum of equilibria, the more stringent condition $\int_0^{+\infty} e^{-\int_0^t \gamma(s) ds} dt < +\infty$ is necessary to obtain convergence of the trajectories. In the one-dimensional case, the slightly stronger condition $\int_0^{+\infty} e^{-\theta \int_0^t \gamma(s) ds} dt < +\infty$, for some $\theta \in]0, 1[$ is shown to be sufficient. In the higher-dimensional case, the general question of convergence is left open in [8,9]. The new techniques developed in the present paper allow to address this question and to fill partially the gap between necessary and sufficient conditions for convergence, see comments below.

Let us come back to Eq. (E) and precise now the framework of the paper. The nonlinearity f is assumed to be monotone and conservative, i.e. derives from some convex potential $F \in C^1(V, \mathbb{R})$. The set of equilibria $S = \{v \in V, Av + f(v) = 0\}$ is supposed to be nonempty. It is not our purpose to develop the well-posedness of Eq. (E) for given initial conditions. Throughout the paper, we assume the existence of a solution to Eq. (E) in the class

$$u \in W_{loc}^{1,1}(\mathbb{R}_+, V) \cap W_{loc}^{2,1}(\mathbb{R}_+, H). \quad (1.4)$$

We define the energy function \mathcal{E} along each trajectory by

$$\mathcal{E}(t) = \frac{1}{2} \left| \frac{du}{dt}(t) \right|^2 + \frac{1}{2} a(u(t), u(t)) + F(u(t)).$$

The major contribution of this paper is to provide a result of (weak) convergence in V for the trajectories of (E): if the quantity $\gamma(t)$ behaves as k/t^α , for some $\alpha \in]0, 1[$, $k > 0$ and t large enough, there exists an equilibrium $u_\infty \in S$ such that $u(t) \rightharpoonup u_\infty$ weakly in V as $t \rightarrow +\infty$. The exact statement is in fact slightly more general, see Theorem 3.13. The main ingredients of the proof are the Opial lemma along with accurate estimates of the energy decay, cf. Proposition 3.11. Strong convergence in V holds true under compactness or symmetry conditions. The technique of the proof is new and is also applicable to the differential equation (1.3).

The second contribution of the paper is to give sharp estimates for the speed of convergence of the energy $\mathcal{E}(t)$ as $t \rightarrow +\infty$. In the linear case ($f = 0$) and under some ellipticity-like condition, we obtain the following estimate

$$\mathcal{E}(t) \sim K e^{-\int_0^t \gamma(s) ds} \quad \text{as } t \rightarrow +\infty, \text{ for some } K > 0. \quad (1.5)$$

Notice that this estimate fails to be true if the trajectory is contained in $\ker A$, see Theorem 2.7 for a precise statement. In the nonlinear case, the same kind of estimate is obtained at a slightly lower degree of precision,² cf. Theorem 3.17.

Outline of the paper Section 2 is concerned with the linear hyperbolic equation (E_0) obtained by taking $f = 0$ in (E). We analyze the behavior of the trajectories by studying respectively their components with respect to the spaces $\ker A$ and $(\ker A)^\perp$. A sharp estimate of the energy decay is given

² In this case, a factor $\frac{2}{3}$ has to be introduced in the exponent of formula (1.5).

under some ellipticity-like condition. In Section 3, we deal with the general equation (E) by assuming that the nonlinearity f is monotone. It is shown in Section 3.1 that the energy $\mathcal{E}(t)$ vanishes as $t \rightarrow +\infty$, which allows to prove (weak) convergence of the trajectories in the case of a unique minimum. The general problem of convergence for a continuum of minima is treated in Section 3.2, which is the core of the paper. Additional results of strong convergence in V are given under some compactness or symmetry assumptions. Finally, the abstract results are applied to particular semilinear evolution problems in Section 4.

2. Linear hyperbolic equation

Let $a: V \times V \rightarrow \mathbb{R}$ be a continuous bilinear form satisfying (h_1) – (h_2) and let $A: V \rightarrow V'$ be the associate operator. Given a map $\gamma \in W_{loc}^{1,1}(\mathbb{R}_+, \mathbb{R}_+)$, we consider the following linear hyperbolic equation

$$\frac{d^2 u}{dt^2}(t) + \gamma(t) \frac{du}{dt}(t) + Au(t) = 0, \quad t \geq 0. \quad (E_0)$$

We assume the existence of a solution to Eq. (E_0) in the class (1.4). We define the energy function \mathcal{E} along each trajectory by

$$\mathcal{E}(t) = \frac{1}{2} \left| \frac{du}{dt}(t) \right|^2 + \frac{1}{2} a(u(t), u(t)).$$

We have $\mathcal{E} \in W_{loc}^{1,1}(\mathbb{R}_+)$ and

$$\begin{aligned} \dot{\mathcal{E}}(t) &= \left(\frac{d^2 u}{dt^2}(t), \frac{du}{dt}(t) \right) + \left\langle Au(t), \frac{du}{dt}(t) \right\rangle_{V', V} \\ &= -\gamma(t) \left| \frac{du}{dt}(t) \right|^2 \leq 0 \quad \text{a.e. on } \mathbb{R}_+, \end{aligned}$$

hence the function \mathcal{E} is a Lyapunov function for the system (E_0) . The purpose of this section is to establish results of convergence for the trajectory u , along with estimates of the energy decay. For every $t \geq 0$, we set $\hat{u}(t) = Pu(t)$, where P denotes the orthogonal projection onto the subspace³ $\ker A$ in the sense of H . Since $\hat{u}(t) \in \ker A$ for every $t \geq 0$, we have

$$\forall t \geq 0, \quad \frac{d^2 \hat{u}}{dt^2}(t) + \gamma(t) \frac{d\hat{u}}{dt}(t) = 0.$$

By integrating this equality twice, we find

$$\begin{aligned} \forall t \geq 0, \quad \hat{u}(t) &= \hat{u}(0) + \left(\int_0^t e^{-\int_0^s \gamma(\tau) d\tau} ds \right) \frac{d\hat{u}}{dt}(0) \\ &= Pu_0 + \left(\int_0^t e^{-\int_0^s \gamma(\tau) d\tau} ds \right) Pv_0. \end{aligned} \quad (2.1)$$

³ By using assumptions (h_1) – (h_2) , it is easy to check that $\ker A$ is closed in H . See also Remark 3.2.

If $Pv_0 \neq 0$, the above equality shows that the asymptotic behavior of the component \hat{u} is strongly related with the convergence of the integral $\int_0^{+\infty} e^{-\int_0^s \gamma(\tau) d\tau} ds$. The next proposition summarizes the different possible cases.

Proposition 2.1. Let us set $\bar{\omega} = \int_0^{+\infty} e^{-\int_0^s \gamma(\tau) d\tau} ds \in \mathbb{R}_+ \cup \{+\infty\}$.

- If $v_0 \in (\ker A)^\perp$, then $\hat{u}(t) = Pu_0$ for every $t \geq 0$.
- If $v_0 \notin (\ker A)^\perp$, then the solution \hat{u} converges if and only if $\bar{\omega} < +\infty$. More precisely, we have $\lim_{t \rightarrow +\infty} |\hat{u}(t)| = +\infty$ if $\bar{\omega} = +\infty$ while $\lim_{t \rightarrow +\infty} \hat{u}(t) = P(u_0 + \bar{\omega}v_0)$ if $\bar{\omega} < +\infty$.

Our purpose is now to evaluate the energy decay along each trajectory $u(\cdot)$. We start with a preliminary result corresponding to the case $\ker A = \{0\}$.

Lemma 2.2. Assume that the bilinear form $a(\cdot, \cdot)$ satisfies (h_1) – (h_2) and that

$$\exists \eta > 0, \forall u \in V, \quad a(u, u) \geq \eta |u|^2. \quad (2.2)$$

Let $\gamma \in W_{loc}^{1,1}(\mathbb{R}_+, \mathbb{R}_+)$ be a function such that $\lim_{t \rightarrow +\infty} \gamma(t) = 0$ and $\dot{\gamma} \in L^1(0, +\infty)$. Let u be a solution in the class (1.4) to Eq. (E_0) . Then, either the solution u is stationary, or there exists $K > 0$ such that

$$\mathcal{E}(t) \sim K e^{-\int_0^t \gamma(s) ds} \quad \text{as } t \rightarrow +\infty.$$

Proof. The main idea of the proof consists in using the function \mathcal{F} defined by⁴

$$\begin{aligned} \mathcal{F}(t) &= \frac{1}{2} \left| \frac{du}{dt}(t) \right|^2 + \frac{1}{2} a(u(t), u(t)) + \frac{\gamma(t)}{2} \left(\frac{du}{dt}(t), u(t) \right) \\ &= \mathcal{E}(t) + \frac{\gamma(t)}{2} \left(\frac{du}{dt}(t), u(t) \right). \end{aligned}$$

We have $\mathcal{F} \in W_{loc}^{1,1}(\mathbb{R}_+)$ and by differentiating the function \mathcal{F} , we find for almost every $t \geq 0$

$$\begin{aligned} \dot{\mathcal{F}}(t) &= \dot{\mathcal{E}}(t) + \frac{\dot{\gamma}(t)}{2} \left(\frac{du}{dt}(t), u(t) \right) + \frac{\gamma(t)}{2} \left(\frac{d^2u}{dt^2}(t), u(t) \right) + \frac{\gamma(t)}{2} \left| \frac{du}{dt}(t) \right|^2 \\ &= -\frac{\gamma(t)}{2} \left| \frac{du}{dt}(t) \right|^2 - \frac{\gamma(t)}{2} a(u(t), u(t)) + \left(\frac{\dot{\gamma}(t)}{2} - \frac{\gamma(t)^2}{2} \right) \left(\frac{du}{dt}(t), u(t) \right). \end{aligned}$$

Therefore we have

$$\dot{\mathcal{F}}(t) + \gamma(t) \mathcal{F}(t) = \frac{\dot{\gamma}(t)}{2} \left(\frac{du}{dt}(t), u(t) \right) \quad \text{a.e. on } \mathbb{R}_+. \quad (2.3)$$

Since $|\left(\frac{du}{dt}(t), u(t)\right)| \leq \frac{1}{2} \left| \frac{du}{dt}(t) \right|^2 + \frac{1}{2} |u(t)|^2$ and $a(u(t), u(t)) \geq \eta |u(t)|^2$ by assumption (2.2), we have

$$\left| \left(\frac{du}{dt}(t), u(t) \right) \right| \leq C \mathcal{E}(t), \quad \text{for some } C > 0. \quad (2.4)$$

⁴ The use of such an auxiliary function is classical, see for example [13, Lemma 3.2.6] in the case of an autonomous damping.

Recalling that $\lim_{t \rightarrow +\infty} \gamma(t) = 0$, the expression of \mathcal{F} shows that

$$\mathcal{F}(t) \sim \mathcal{E}(t) \quad \text{as } t \rightarrow +\infty. \quad (2.5)$$

We deduce from (2.3), (2.4) and (2.5) the existence of $D > 0$ and $t_0 \geq 0$ such that

$$|\dot{\mathcal{F}}(t) + \gamma(t)\mathcal{F}(t)| \leq D|\dot{\gamma}(t)|\mathcal{F}(t) \quad \text{a.e. on } [t_0, +\infty[.$$

Let us multiply each member of this inequality by $e^{\int_0^t \gamma(s) ds}$ and set $\mathcal{G}(t) = e^{\int_0^t \gamma(s) ds} \mathcal{F}(t)$. We obtain

$$|\dot{\mathcal{G}}(t)| \leq D|\dot{\gamma}(t)|\mathcal{G}(t) \quad \text{a.e. on } [t_0, +\infty[. \quad (2.6)$$

Observe that if $\mathcal{G}(t_1) = 0$ for some $t_1 \geq t_0$, then we have $\mathcal{F}(t_1) = 0$ and $\mathcal{E}(t_1) = 0$. Since the map \mathcal{E} is nonincreasing, we conclude that $\mathcal{E}(t) = 0$ for every $t \geq t_1$, i.e. the solution u is stationary. Now assume that $\mathcal{G}(t) > 0$ for every $t \geq t_0$ and divide each member of equality (2.6) by $\mathcal{G}(t)$. Since $\dot{\gamma} \in L^1(0, +\infty)$ by assumption, we deduce that

$$\left| \frac{d}{dt} \ln \mathcal{G} \right| (t) = \frac{|\dot{\mathcal{G}}(t)|}{\mathcal{G}(t)} \in L^1(0, +\infty).$$

It ensues that $\lim_{t \rightarrow +\infty} \ln \mathcal{G}(t)$ exists in \mathbb{R} . We deduce that $\lim_{t \rightarrow +\infty} e^{\int_0^t \gamma(s) ds} \mathcal{F}(t) = K > 0$. The conclusion immediately follows from estimate (2.5). \square

Remark 2.3. A result similar to Lemma 2.2 can be obtained by eliminating the first-order term in (E_0) via the change of variable $v(t) = e^{\frac{1}{2} \int_0^t \gamma(s) ds} u(t)$. The details are left to the reader.

Remark 2.4 (Case γ constant). Assuming that $\gamma(t) \equiv \gamma > 0$ and that $a(u, u) \geq \eta|u|^2$ for every $u \in V$, the estimate $\mathcal{E}(t) = O(e^{-\gamma t})$ remains true as $t \rightarrow +\infty$ if $\gamma < 2\eta^{1/2}$, see [13, Lemma 3.2.6]. However, it fails to be valid if $\gamma \geq 2\eta^{1/2}$, see [13, Proposition 3.2.5].

We now assume the following ellipticity-like condition

$$\forall u \in V, \quad a(u, u) \geq \eta|u - Pu|^2, \quad \text{for some } \eta > 0. \quad (2.7)$$

Remark 2.5. Under (h_2) , this condition is equivalent to the following one⁵

$$\forall u \in V, \quad a(u, u) \geq \eta' \|u - Pu\|^2, \quad \text{for some } \eta' > 0. \quad (2.8)$$

Indeed, assume that condition (2.7) is satisfied. Recalling that $Pu \in \ker A$, we deduce from (h_2) that

$$\forall u \in V, \quad a(u, u) + \lambda|u - Pu|^2 \geq \mu \|u - Pu\|^2.$$

It ensues that $(1 + \frac{\lambda}{\eta})a(u, u) \geq \mu \|u - Pu\|^2$ for every $u \in V$ and finally (2.8) is fulfilled with $\eta' = \frac{\eta\mu}{\eta + \lambda}$.

Remark 2.6. Suppose that the injection $V \hookrightarrow H$ is compact and that (h_1) – (h_2) hold true. The eigenvalues of A then define a nondecreasing sequence of nonnegative scalars tending to $+\infty$ and there exists an orthonormal basis of H consisting of the corresponding eigenvectors, see for example [17,20]. If η denotes the smallest eigenvalue of A greater than 0, it is clear that $a(u, u) \geq \eta|u|^2$ for every $u \in (\ker A)^\perp \cap V$ and therefore condition (2.7) holds true.

⁵ Condition (2.8) is used in [21, Section 4], where estimates of the energy decay are provided in the case of an autonomous damping.

The next result allows to estimate the energy decay under condition (2.7).

Theorem 2.7. Assume that the bilinear form $a(\cdot, \cdot)$ satisfies conditions (h_1) – (h_2) and (2.7). Let $\gamma \in W_{loc}^{1,1}(\mathbb{R}_+, \mathbb{R}_+)$ be a function such that $\lim_{t \rightarrow +\infty} \gamma(t) = 0$ and $\dot{\gamma} \in L^1(0, +\infty)$. Let u be a solution in the class (1.4) to Eq. (E₀). Then, either the trajectory is contained in $\ker A$, or there exists $K > 0$ such that

$$\mathcal{E}(t) \sim K e^{-\int_0^t \gamma(s) ds} \quad \text{as } t \rightarrow +\infty. \quad (2.9)$$

Proof. For every $t \geq 0$, we set $\widehat{u}(t) = Pu(t)$ and $\widetilde{u}(t) = u(t) - Pu(t)$. Since $\widehat{u}(t) \in \ker A$, $\frac{d\widehat{u}}{dt}(t) \in \ker A$ and $\frac{d\widetilde{u}}{dt}(t) \in (\ker A)^\perp$, we have for every $t \geq 0$

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{2} \left| \frac{d\widehat{u}}{dt}(t) + \frac{d\widetilde{u}}{dt}(t) \right|^2 + \frac{1}{2} a(\widehat{u}(t) + \widetilde{u}(t), \widehat{u}(t) + \widetilde{u}(t)) \\ &= \frac{1}{2} \left| \frac{d\widehat{u}}{dt}(t) \right|^2 + \frac{1}{2} \left| \frac{d\widetilde{u}}{dt}(t) \right|^2 + \frac{1}{2} a(\widetilde{u}(t), \widetilde{u}(t)). \end{aligned} \quad (2.10)$$

From equality (2.1), we deduce that for every $t \geq 0$

$$\left| \frac{d\widehat{u}}{dt}(t) \right|^2 = e^{-2 \int_0^t \gamma(s) ds} \left| \frac{d\widehat{u}}{dt}(0) \right|^2. \quad (2.11)$$

Let us now set $V_1 = (\ker A)^\perp \cap V$, $a_1 = a|_{V_1 \times V_1}$ and $A_1 = A|_{V_1}$. It is clear that \widetilde{u} is a solution of

$$\frac{d^2 \widetilde{u}}{dt^2}(t) + \gamma(t) \frac{d\widetilde{u}}{dt}(t) + A_1 \widetilde{u}(t) = 0.$$

On the other hand, condition (2.7) implies that $a_1(u, u) \geq \eta |u|^2$ for every $u \in V_1$. By applying Lemma 2.2 to the solution \widetilde{u} , we obtain that either the map \widetilde{u} is stationary or there exists $K_1 > 0$ such that

$$\frac{1}{2} \left| \frac{d\widetilde{u}}{dt}(t) \right|^2 + \frac{1}{2} a(\widetilde{u}(t), \widetilde{u}(t)) \sim K_1 e^{-\int_0^t \gamma(s) ds} \quad \text{as } t \rightarrow +\infty. \quad (2.12)$$

We now combine equalities (2.10), (2.11) with estimate (2.12). If $\int_0^{+\infty} \gamma(s) ds = +\infty$, we immediately obtain (2.9) with $K = K_1$. If $\int_0^{+\infty} \gamma(s) ds < +\infty$, then

$$\lim_{t \rightarrow +\infty} \mathcal{E}(t) = \frac{1}{2} e^{-2 \int_0^{+\infty} \gamma(s) ds} \left| \frac{d\widehat{u}}{dt}(0) \right|^2 + K_1 e^{-\int_0^{+\infty} \gamma(s) ds},$$

hence (2.9) is satisfied with $K = \frac{1}{2} e^{-\int_0^{+\infty} \gamma(s) ds} \left| \frac{d\widehat{u}}{dt}(0) \right|^2 + K_1$. \square

Remark 2.8. If the trajectory $u(\cdot)$ is contained in $\ker A$, estimate (2.9) is no more valid. In this case, we infer from equality (2.11) that $\mathcal{E}(t) = \frac{1}{2} e^{-2 \int_0^t \gamma(s) ds} \left| \frac{d\widehat{u}}{dt}(0) \right|^2$ for every $t \geq 0$.

Corollary 2.9. Under the hypotheses of Theorem 2.7, assume moreover that $\gamma \notin L^1(0, +\infty)$. Then we have $\lim_{t \rightarrow +\infty} \mathcal{E}(t) = 0$. If $\ker A = \{0\}$, then $u(t) \rightarrow 0$ strongly in V as $t \rightarrow +\infty$.

Proof. The first assertion is an immediate consequence of estimate (2.9), while the second one follows from

$$\forall t \geq 0, \quad \mathcal{E}(t) \geq \frac{1}{2} a(u(t), u(t)) \geq \frac{\eta'}{2} \|u(t)\|^2,$$

see inequality (2.8). \square

When $\ker A \neq \{0\}$, convergence of the trajectories is obtained under the following stronger assumption

$$\int_0^{+\infty} e^{-\frac{1}{2} \int_0^s \gamma(\tau) d\tau} ds < +\infty. \quad (2.13)$$

Corollary 2.10. *Under the hypotheses of Theorem 2.7, assume moreover that condition (2.13) is satisfied. Then, there exists $u_\infty \in \ker A$ such that $u(t) \rightarrow u_\infty$ strongly in V as $t \rightarrow +\infty$.*

Proof. First assume that the trajectory is contained in $\ker A$. Observing that $\bar{\omega} = \int_0^{+\infty} e^{-\int_0^s \gamma(\tau) d\tau} ds < +\infty$, we deduce from Proposition 2.1 that $u(t)$ converges strongly in H as $t \rightarrow +\infty$. If the trajectory is not contained in $\ker A$, we derive from estimate (2.9) that

$$\left| \frac{du}{dt}(t) \right| = O\left(e^{-\frac{1}{2} \int_0^t \gamma(s) ds}\right) \quad \text{as } t \rightarrow +\infty,$$

hence $\frac{du}{dt} \in L^1(\mathbb{R}_+, H)$ in view of condition (2.13). The trajectory u has a finite length, hence strongly converges in H toward some $u_\infty \in \ker A$. Using now the semi-coercivity condition (h_2) , we have

$$\begin{aligned} \mu \|u(t) - u_\infty\|^2 &\leq \lambda |u(t) - u_\infty|^2 + a(u(t) - u_\infty, u(t) - u_\infty) \\ &= \lambda |u(t) - u_\infty|^2 + a(u(t), u(t)). \end{aligned}$$

Since $\lim_{t \rightarrow +\infty} |u(t) - u_\infty| = 0$ and $\lim_{t \rightarrow +\infty} a(u(t), u(t)) = 0$ in view of Corollary 2.9, we conclude that $\lim_{t \rightarrow +\infty} \|u(t) - u_\infty\| = 0$. \square

Example 2.11. Suppose that there exist $\alpha, k > 0$ such that $\gamma(t) = \frac{k}{t^\alpha}$ for t large enough. If the bilinear form $a(\cdot, \cdot)$ satisfies conditions (h_1) – (h_2) and (2.7), we deduce from Theorem 2.7 and Corollary 2.10 that

- if $\alpha > 1$, then $\lim_{t \rightarrow +\infty} \mathcal{E}(t) > 0$;
- if $\alpha = 1$, then $\mathcal{E}(t) \sim \frac{K}{t^k}$ as $t \rightarrow +\infty$ and the trajectory $u(\cdot)$ strongly converges in V as soon as $k > 2$;
- if $\alpha \in (0, 1)$, then $\mathcal{E}(t) \sim K e^{-\frac{k}{1-\alpha} t^{1-\alpha}}$ as $t \rightarrow +\infty$ and the trajectory $u(\cdot)$ strongly converges in V for every $k > 0$.

Other results of convergence will be provided in the more general framework of semilinear equations.

3. Monotone conservative nonlinearity

The assumptions concerning the spaces V , H , the linear operator $A: V \rightarrow V'$ and the map $\gamma: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are the same as in Section 2. We consider the following semilinear hyperbolic equation

$$\frac{d^2 u}{dt^2}(t) + \gamma(t) \frac{du}{dt}(t) + Au(t) + f(u(t)) = 0, \quad t \geq 0. \quad (E)$$

We suppose that the nonlinearity $f : V \rightarrow H$ is conservative, i.e.

$$\exists F \in C^1(V, \mathbb{R}) \quad \text{such that} \quad \forall u, v \in V, \quad \langle F'(u), v \rangle_{V', V} = (f(u), v). \quad (k_1)$$

Moreover, we assume that the map f is monotone

$$\forall u, v \in V, \quad (f(u) - f(v), u - v) \geq 0, \quad (k_2)$$

which is equivalent to the convexity of the potential F . Defining $\Phi : V \rightarrow \mathbb{R}$ by

$$\Phi(v) = \frac{1}{2}a(v, v) + F(v),$$

we obtain a function of class C^1 whose first derivative is given by $\langle \Phi'(u), v \rangle_{V', V} = a(u, v) + (f(u), v)$, or equivalently $\Phi'(u) = Au + f(u)$. Moreover, Φ is convex, which amounts to

$$\forall u, v \in V, \quad a(u, v - u) + (f(u), v - u) \leq \Phi(v) - \Phi(u). \quad (3.1)$$

Consequently, minimum and stationary points of Φ coincide, i.e.

$$\operatorname{argmin} \Phi = \{v \in V \mid Av + f(v) = 0\}, \quad (3.2)$$

where $\operatorname{argmin} \Phi = \{v \in V \mid \Phi(v) = \inf \Phi\}$. We suppose that

$$S = \operatorname{argmin} \Phi \neq \emptyset. \quad (k_3)$$

It is clear in view of Eq. (E) that nothing is changed if some constant is added to the potential Φ . Without loss of generality, we will systematically assume that $\inf \Phi = 0$.

Remark 3.1. Assume that a is coercive, i.e. (h_2) holds with $\lambda = 0$. Then the map $u \mapsto a(u, u)$ is strongly convex and since the function F is convex, the map Φ is also strongly convex. This implies immediately that the set $\operatorname{argmin} \Phi$ is a singleton, hence the non-vacuity condition (k_3) holds true. Now assume that (h_2) holds with $\lambda > 0$. To overcome the lack of coercivity, suppose that there exist $\varepsilon > 0$ and $C \geq 0$ such that $F(u) \geq \varepsilon|u|^2 - C$ for every $u \in V$. Without loss of generality, we can assume that $\varepsilon \leq \frac{\lambda}{2}$. For every $u \in V$, we have

$$\begin{aligned} \Phi(u) &= \frac{1}{2}a(u, u) + F(u) \geq \frac{\varepsilon}{\lambda}a(u, u) + F(u) \\ &\geq \frac{\varepsilon\mu}{\lambda}\|u\|^2 - \varepsilon|u|^2 + \varepsilon|u|^2 - C \\ &= \frac{\varepsilon\mu}{\lambda}\|u\|^2 - C, \end{aligned}$$

which shows that $\lim_{\|u\| \rightarrow +\infty} \Phi(u) = +\infty$. Since the function Φ is convex and continuous, this classically implies condition (k_3) .

It is immediate to check that the set S is convex, closed in V and that $S \subset D(A)$.

Remark 3.2. Under assumption (h_2) , let us show that S is closed in H . Let (u_n) be a sequence in S such that $\lim_{n \rightarrow +\infty} u_n = \bar{u}$ strongly in H , for some $\bar{u} \in H$. Since the function F is convex, there exist $b, c \in \mathbb{R}$ such that, for all $u \in V$, $F(u) \geq -b|u| - c$. Therefore we have for all $u \in V$,

$$\frac{1}{2}a(u, u) \leq \Phi(u) + b|u| + c. \quad (3.3)$$

Recalling that $\Phi(u_n) = 0$ for every $n \in \mathbb{N}$, we deduce that $\frac{1}{2}a(u_n, u_n) \leq b|u_n| + c$, hence the sequence $(a(u_n, u_n))$ is bounded. From hypothesis (h_2) , we infer that the sequence (u_n) is bounded in V . It ensues that there exist $\hat{u} \in V$ and a subsequence (u_{n_k}) such that $\lim_{k \rightarrow +\infty} u_{n_k} = \hat{u}$ weakly in V . We immediately have $\hat{u} = \bar{u}$ and the weak lower semicontinuity of Φ implies that $\Phi(\bar{u}) \leq \liminf_{k \rightarrow +\infty} \Phi(u_{n_k}) = 0$, hence $\bar{u} \in S$.

Remark 3.3 (Case $f(0) = 0$). If $f(0) = 0$ then we have

$$S = \ker A \cap \{v \in V \mid f(v) = 0\} \neq \emptyset.$$

Indeed, if $w \in S$ then in particular $(Aw, w) + (f(w), w) = 0$, and by monotonicity of f we have $(f(w) - f(0), w) \geq 0$, hence $(Aw, w) = (f(w), w) = 0$ and therefore $Aw = 0$.

In the sequel, we assume the existence of a solution to Eq. (E) in the class (1.4). We define the energy function \mathcal{E} along each trajectory by

$$\mathcal{E}(t) = \frac{1}{2} \left| \frac{du}{dt}(t) \right|^2 + \Phi(u(t)).$$

We have $\mathcal{E} \in W_{loc}^{1,1}(\mathbb{R}_+)$ and

$$\begin{aligned} \dot{\mathcal{E}}(t) &= \left(\frac{d^2u}{dt^2}(t), \frac{du}{dt}(t) \right) + \left\langle Au(t) + f(u(t)), \frac{du}{dt}(t) \right\rangle_{V',V} \\ &= -\gamma(t) \left| \frac{du}{dt}(t) \right|^2 \leq 0 \quad \text{a.e. on } \mathbb{R}_+, \end{aligned}$$

hence the function \mathcal{E} is a Lyapunov function for Eq. (E). We deduce that for every $t \geq 0$

$$\frac{1}{2} \left| \frac{du}{dt}(t) \right|^2 \leq \mathcal{E}(t) \leq \mathcal{E}(0) \quad \text{and} \quad \Phi(u(t)) \leq \mathcal{E}(t) \leq \mathcal{E}(0). \quad (3.4)$$

In particular, we have $\frac{du}{dt} \in L^\infty(\mathbb{R}_+, H)$. In the sequel, we will consider solutions which are bounded in H , i.e. satisfying $u \in L^\infty(\mathbb{R}_+, H)$.

Remark 3.4. Under assumption (h_2) , it is easy to see that $u \in L^\infty(\mathbb{R}_+, H)$ implies $u \in L^\infty(\mathbb{R}_+, V)$. Indeed, let us assume that $\{u(t); t \geq 0\}$ is bounded in H . From inequality (3.3), we have $\frac{1}{2}a(u(t), u(t)) \leq \Phi(u(t)) + b|u(t)| + c$ for all $t \in \mathbb{R}_+$. Recalling that $\Phi(u(t)) \leq \mathcal{E}(0)$ in view of (3.4), we infer that $\{a(u(t), u(t)); t \geq 0\}$ is bounded. From hypothesis (h_2) , we conclude that $\{u(t); t \geq 0\}$ is bounded in V .

3.1. Summability of the energy. Case of a unique equilibrium

We now prove that the map $\gamma \mathcal{E}$ is summable over \mathbb{R}_+ and that $\lim_{t \rightarrow +\infty} \mathcal{E}(t) = 0$.

Proposition 3.5. Assume that the bilinear form $a(\cdot, \cdot)$ and the function f satisfy respectively hypotheses (h_1) – (h_2) and (k_1) – (k_3) . Let $\gamma \in W_{loc}^{1,1}(\mathbb{R}_+, \mathbb{R}_+)$ be a map such that $\dot{\gamma} \in L^1(0, +\infty)$. Let u be a solution in the class (1.4) to Eq. (E) and assume that $u \in L^\infty(\mathbb{R}_+, H)$. Then:

- (i) $\int_0^{+\infty} \gamma(t) \mathcal{E}(t) dt < +\infty$.
- (ii) If moreover $\gamma \notin L^1(0, +\infty)$, then $\lim_{t \rightarrow +\infty} \mathcal{E}(t) = 0$, hence

$$\lim_{t \rightarrow +\infty} \left| \frac{du}{dt}(t) \right| = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} \Phi(u(t)) = 0. \quad (3.5)$$

Proof. (i) The proof follows the same arguments as those of [8, Proposition 3.1]. Let us take $v \in S$ and define the function $p: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by $p(t) = \frac{1}{2}|u(t) - v|^2$. By differentiating, we find for every $t \geq 0$

$$\dot{p}(t) = \left(\frac{du}{dt}(t), u(t) - v \right).$$

Since $\frac{du}{dt} \in W_{loc}^{1,1}(\mathbb{R}_+, H)$ by assumption, it is immediate to check that $\dot{p} \in W_{loc}^{1,1}(\mathbb{R}_+)$. Hence the map \dot{p} is differentiable almost everywhere on \mathbb{R}_+ and we have

$$\ddot{p}(t) = \left(\frac{d^2u}{dt^2}(t), u(t) - v \right) + \left| \frac{du}{dt}(t) \right|^2 \quad \text{a.e. on } \mathbb{R}_+.$$

By combining the expressions of \dot{p} , \ddot{p} and by using the convexity of the function Φ , we obtain

$$\begin{aligned} \ddot{p}(t) + \gamma(t)\dot{p}(t) &= a(u(t), v - u(t)) + (f(u(t)), v - u(t)) + \left| \frac{du}{dt}(t) \right|^2 \\ &\leq -\Phi(u(t)) + \left| \frac{du}{dt}(t) \right|^2 \quad \text{a.e. on } \mathbb{R}_+. \end{aligned} \quad (3.6)$$

It follows that

$$\ddot{p}(t) + \gamma(t)\dot{p}(t) + \mathcal{E}(t) \leq \frac{3}{2} \left| \frac{du}{dt}(t) \right|^2 \quad \text{a.e. on } \mathbb{R}_+. \quad (3.7)$$

Let us multiply this inequality by $\gamma(t)$ and integrate on $[0, t]$. By using the fact that $\dot{\mathcal{E}}(t) = -\gamma(t) \left| \frac{du}{dt}(t) \right|^2$ almost everywhere on \mathbb{R}_+ , we derive that

$$\int_0^t \gamma(s) \mathcal{E}(s) ds \leq \frac{3}{2} (\mathcal{E}(0) - \mathcal{E}(t)) - \int_0^t \gamma(s) \ddot{p}(s) ds - \int_0^t \gamma(s)^2 \dot{p}(s) ds. \quad (3.8)$$

For the last two integrals, let us use a technique of integration by parts.

$$-\int_0^t \gamma(s) \ddot{p}(s) ds = \gamma(0)\dot{p}(0) - \gamma(t)\dot{p}(t) + \int_0^t \dot{\gamma}(s)\dot{p}(s) ds. \quad (3.9)$$

Recall that the map u is bounded in H by assumption. On the other hand, the map $\frac{du}{dt}$ is bounded in H , see (3.4). Hence we infer the existence of $M > 0$ such that $p(t) \leq M$ and $|\dot{p}(t)| \leq M$ for every $t \geq 0$. Therefore

$$-\int_0^t \gamma(s) \ddot{p}(s) ds \leq M\gamma(0) + M\gamma(t) + M \int_0^t |\dot{\gamma}(s)| ds.$$

Since $\dot{\gamma} \in L^1(0, +\infty)$ by assumption, the right-hand side is majorized by some $M' \geq 0$. On the other hand, we have

$$\begin{aligned} -\int_0^t \gamma(s)^2 \dot{p}(s) ds &= \gamma(0)^2 p(0) - \gamma(t)^2 p(t) + 2 \int_0^t \gamma(s) \dot{\gamma}(s) p(s) ds \\ &\leq M\gamma(0)^2 + 2M \int_0^t \gamma(s) |\dot{\gamma}(s)| ds. \end{aligned} \quad (3.10)$$

Using again the assumption $\dot{\gamma} \in L^1(0, +\infty)$, we obtain that the right-hand side is majorized by some $M'' \geq 0$. Coming back to inequality (3.8), we conclude that $\int_0^t \gamma(s) \mathcal{E}(s) ds \leq \frac{3}{2} \mathcal{E}(0) + M' + M''$ for every $t \geq 0$ and the expected estimate follows.

(ii) Let us argue by contradiction and assume that $\lim_{t \rightarrow +\infty} \mathcal{E}(t) = l > 0$. The map \mathcal{E} is nonincreasing, hence $\mathcal{E}(t) \geq l$ for every $t \geq 0$. Since $\gamma \notin L^1(0, +\infty)$, we deduce that

$$\int_0^{+\infty} \gamma(t) \mathcal{E}(t) dt \geq l \int_0^{+\infty} \gamma(t) dt = +\infty,$$

a contradiction with the result of (i). The last assertion is immediate. \square

In view of the previous result, we can prove weak convergence of the trajectories in the case of a unique equilibrium. The general case of multiple equilibria is more delicate and will be discussed in Section 3.2.

Corollary 3.6 (Case of a unique equilibrium). *Under the hypotheses of Proposition 3.5, assume moreover that $\operatorname{argmin} \Phi = \{\bar{u}\}$ for some $\bar{u} \in V$. Then the solution $u(t)$ weakly converges in V toward \bar{u} as $t \rightarrow +\infty$. Furthermore, if $u(t)$ strongly converges⁶ in H then it strongly converges in V .*

Proof. By assumption, the solution u is bounded in H . In view of hypothesis (h_2) and Remark 3.4, it is also bounded in V . Hence there exist $u_\infty \in V$ and a subsequence (t_n) tending to $+\infty$ such that $\lim_{n \rightarrow +\infty} u(t_n) = u_\infty$ weakly in V . Since Φ is convex and continuous for the strong topology of V , it is lower semicontinuous for the weak topology of V . Hence, we have $\Phi(u_\infty) \leq \liminf_{n \rightarrow +\infty} \Phi(u(t_n))$. From the second part of (3.5) we deduce that $\Phi(u_\infty) \leq 0$, i.e. $u_\infty \in \operatorname{argmin} \Phi = \{\bar{u}\}$. Hence \bar{u} is the unique limit point of the map $t \mapsto u(t)$ as $t \rightarrow +\infty$ for the weak topology of V . It ensues that $\lim_{t \rightarrow +\infty} u(t) = \bar{u}$ weakly in V . Let us now prove the second point. The argument is given in [3, pp. 548–549] but we recall it for the sake of completeness. From (h_2) , we have

⁶ This assumption is satisfied if the injection $V \hookrightarrow H$ is compact.

$$\begin{aligned}\mu \|u(t) - \bar{u}\|^2 &\leq \lambda |u(t) - \bar{u}|^2 + a(u(t) - \bar{u}, u(t) - \bar{u}) \\ &= \lambda |u(t) - \bar{u}|^2 + 2\Phi(u(t)) - 2F(u(t)) - 2a(u(t), \bar{u}) + a(\bar{u}, \bar{u}).\end{aligned}\quad (3.11)$$

Since $u(t) \rightarrow \bar{u}$ strongly in H and weakly in V , we have $\lim_{t \rightarrow +\infty} |u(t) - \bar{u}|^2 = 0$ and $\lim_{t \rightarrow +\infty} a(u(t), \bar{u}) = a(\bar{u}, \bar{u})$. On the other hand, by weak lower semicontinuity of the continuous convex function $F : V \rightarrow \mathbb{R}$, we infer that $\liminf_{t \rightarrow +\infty} F(u(t)) \geq F(\bar{u})$. Recalling finally property (3.5), we deduce from inequality (3.11) that

$$\mu \limsup_{t \rightarrow +\infty} \|u(t) - \bar{u}\|^2 \leq -2F(\bar{u}) - a(\bar{u}, \bar{u}) = 0.$$

We conclude that $u(t) \rightarrow \bar{u}$ strongly in V . \square

3.2. Convergence of the trajectories

3.2.1. Case of a non-vanishing damping

When the damping coefficient $\gamma(t)$ is constant, i.e. $\gamma(t) \equiv \gamma > 0$, the solutions of (E) weakly converge in V toward an equilibrium point, see [3]. We are going to show that this property still holds true if

$$\begin{cases} \lim_{t \rightarrow +\infty} \gamma(t) = \gamma_\infty > 0, \\ \dot{\gamma} \in L^1(0, +\infty). \end{cases} \quad (I_1)$$

Theorem 3.7. Assume that the bilinear form $a(\cdot, \cdot)$ and the function f satisfy respectively (h_1) – (h_2) and (k_1) – (k_3) . Let $\gamma \in W_{loc}^{1,1}(\mathbb{R}_+, \mathbb{R}_+)$ be a map satisfying (I_1) . Let u be a solution in the class (1.4) to Eq. (E). Then, there exists $u_\infty \in S$ such that $u(t) \rightharpoonup u_\infty$ weakly in V as $t \rightarrow +\infty$. Furthermore, if $u(t)$ strongly converges in H then it strongly converges in V .

Proof. Let $v \in S$ and define the map $p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by $p(t) = \frac{1}{2}|u(t) - v|^2$ as in the proof of Proposition 3.5. Inequality (3.6) implies that

$$\ddot{p}(t) + \gamma(t)\dot{p}(t) \leq \left| \frac{du}{dt}(t) \right|^2 \quad \text{a.e. on } \mathbb{R}_+.$$

Let us multiply each member of this inequality by $e^{\int_0^t \gamma(\tau) d\tau}$ and integrate on $[0, t]$. Recalling that $\dot{p} \in W_{loc}^{1,1}(\mathbb{R}_+)$, we obtain

$$\dot{p}(t) \leq e^{-\int_0^t \gamma(\tau) d\tau} \dot{p}(0) + e^{-\int_0^t \gamma(\tau) d\tau} \int_0^t e^{\int_0^s \gamma(\tau) d\tau} \left| \frac{du}{ds}(s) \right|^2 ds. \quad (3.12)$$

We now show that the right member of the above inequality is a summable function. Since $\lim_{t \rightarrow +\infty} \gamma(t) = \gamma_\infty > 0$, there exists $t_0 > 0$ such that $\gamma(t) \geq \gamma_\infty/2$ for every $t \geq t_0$. From Lemma 3.8 (i) below, we have

$$\int_0^{+\infty} e^{-\int_0^t \gamma(\tau) d\tau} dt < +\infty. \quad (3.13)$$

Lemma 3.8. *Let us assume that there exist $k > 0$ and $t_0 > 0$ such that $\gamma(t) \geq k$ for every $t \geq t_0$. Then we have*

- (i) $\int_0^{+\infty} e^{-\int_0^t \gamma(\tau) d\tau} dt < +\infty$;
- (ii) $\int_s^{+\infty} e^{-\int_0^t \gamma(\tau) d\tau} dt \leq \frac{1}{k} e^{-\int_0^s \gamma(\tau) d\tau}$ for s large enough.

Lemma 3.8 is a particular case of a more general result that will be proved next, see Lemma 3.14. Coming back to inequality (3.12), we find by applying Fubini theorem

$$\int_0^{+\infty} e^{-\int_0^t \gamma(\tau) d\tau} \int_0^t e^{\int_0^s \gamma(\tau) d\tau} \left| \frac{du}{ds}(s) \right|^2 ds dt = \int_0^{+\infty} \left| \frac{du}{ds}(s) \right|^2 e^{\int_0^s \gamma(\tau) d\tau} \int_s^{+\infty} e^{-\int_0^t \gamma(\tau) d\tau} dt ds. \quad (3.14)$$

From Lemma 3.8 (ii), we obtain

$$e^{\int_0^s \gamma(\tau) d\tau} \int_s^{+\infty} e^{-\int_0^t \gamma(\tau) d\tau} dt \leq \frac{2}{\gamma_\infty} \leq \frac{4}{\gamma_\infty^2} \gamma(s).$$

Recalling that $\dot{\mathcal{E}}(t) = -\gamma(t) \left| \frac{du}{dt}(t) \right|^2$, we have the estimate $\int_0^{+\infty} \gamma(s) \left| \frac{du}{ds}(s) \right|^2 ds < +\infty$. Hence we deduce from equality (3.14) that

$$\int_0^{+\infty} e^{-\int_0^t \gamma(\tau) d\tau} \int_0^t e^{\int_0^s \gamma(\tau) d\tau} \left| \frac{du}{ds}(s) \right|^2 ds dt < +\infty. \quad (3.15)$$

By combining inequality (3.12) with estimates (3.13) and (3.15), we infer that $[\dot{p}]_+ \in L^1(0, +\infty)$ and hence $\lim_{t \rightarrow +\infty} p(t)$ exists. In particular the map u is bounded in H . The end of the proof is the same as in [3, Theorem 3.1] but the arguments are given for the sake of completeness. Since $u \in L^\infty(\mathbb{R}_+, H)$, we deduce from hypothesis (h_2) and Remark 3.4 that $u \in L^\infty(\mathbb{R}_+, V)$. Let $\bar{u} \in V$ be a weak cluster point of $\{u(t); t \rightarrow +\infty\}$ for the weak topology of V . There exists a sequence $t_n \rightarrow +\infty$ such that $u(t_n) \rightharpoonup \bar{u}$ weakly in V as $n \rightarrow +\infty$. Since the function Φ is lower semicontinuous for the weak topology of V , we have⁷ in view of Proposition 3.5

$$\Phi(\bar{u}) \leq \liminf_{n \rightarrow +\infty} \Phi(u(t_n)) = \lim_{t \rightarrow +\infty} \Phi(u(t)) = 0,$$

which implies that $\bar{u} \in S$. Let us prove that $\{u(t); t \rightarrow +\infty\}$ has a unique cluster point for the weak topology in V . We apply the following argument due to Opial [18]. Let $\bar{u}_1, \bar{u}_2 \in S$ be two cluster points of $\{u(t); t \rightarrow +\infty\}$ for the weak topology of V . According to the first part of the proof, we can assert that $\lim_{t \rightarrow +\infty} |u(t) - \bar{u}_i|^2 = l_i$ exists for each $i = 1, 2$. Moreover there exists a sequence $t_n \rightarrow +\infty$ such that $u(t_n) \rightharpoonup \bar{u}_1$ weakly in V as $n \rightarrow +\infty$. Since the injection $V \hookrightarrow H$ is continuous, $u(t_n) \rightharpoonup \bar{u}_1$ weakly in H as $n \rightarrow +\infty$. From the equality

$$|u(t) - \bar{u}_1|^2 - |u(t) - \bar{u}_2|^2 = |\bar{u}_1 - \bar{u}_2|^2 + 2(\bar{u}_1 - \bar{u}_2, \bar{u}_2 - u(t)),$$

we infer that $l_1 - l_2 = -|\bar{u}_1 - \bar{u}_2|^2$. On the other hand, if we take $t_m \rightarrow +\infty$ such that $u(t_m) \rightharpoonup \bar{u}_2$ weakly in V as $m \rightarrow +\infty$, we find $l_1 - l_2 = |\bar{u}_1 - \bar{u}_2|^2$. As a consequence, $|\bar{u}_1 - \bar{u}_2|^2 = 0$. This establishes the uniqueness of the cluster points of $\{u(t); t \rightarrow +\infty\}$ for the weak topology of V . Hence $u(t) \rightharpoonup u_\infty$ weakly in V as $t \rightarrow +\infty$ for some $u_\infty \in V$.

⁷ Observe that Proposition 3.5 applies rightfully since we have proved that $u \in L^\infty(\mathbb{R}_+, H)$.

For the second point, the reader is referred to the corresponding argument in the proof of Corollary 3.6. \square

An interesting situation ensuring strong convergence in V is the case where the nonlinearity satisfies the symmetry property $F(-u) = F(u)$ for all $u \in V$.

Theorem 3.9. *Under the hypotheses of Theorem 3.7, assume moreover that the function F is even, i.e. $F(-u) = F(u)$ for all $u \in V$. Then there exists $u_\infty \in S$ such that $u(t) \rightarrow u_\infty$ strongly in V .*

Proof. The argument was originated by Bruck, see [6, Theorem 5]. It has been adapted to the framework of second-order in time equations, see for example [2, Theorem 2.4 (i)] or [3, Remark 3.2] in the case of a constant damping parameter γ . Let us fix $t_0 > 0$ and define the map $q : [0, t_0] \rightarrow \mathbb{R}$ by

$$q(t) = |u(t)|^2 - |u(t_0)|^2 - \frac{1}{2}|u(t) - u(t_0)|^2.$$

A first differentiation gives for all $t \in [0, t_0]$

$$\dot{q}(t) = \left(\frac{du}{dt}(t), u(t) + u(t_0) \right).$$

Since $\frac{du}{dt} \in W_{loc}^{1,1}(\mathbb{R}_+, H)$ by assumption, it is immediate to check that the map \dot{q} is absolutely continuous, hence differentiable almost everywhere on $[0, t_0]$ and we have

$$\ddot{q}(t) = \left(\frac{d^2u}{dt^2}(t), u(t) + u(t_0) \right) + \left| \frac{du}{dt}(t) \right|^2 \quad \text{a.e. on } [0, t_0].$$

By combining the expressions of \dot{q} , \ddot{q} , we obtain for almost every $t \in [0, t_0]$

$$\begin{aligned} \ddot{q}(t) + \gamma(t)\dot{q}(t) &= -a(u(t), u(t) + u(t_0)) - (f(u(t)), u(t) + u(t_0)) + \left| \frac{du}{dt}(t) \right|^2 \\ &= -\langle \Phi'(u(t)), u(t) + u(t_0) \rangle_{V', V} + \left| \frac{du}{dt}(t) \right|^2. \end{aligned} \quad (3.16)$$

Since the function Φ is convex and even, we have for all $u, v \in V$

$$\Phi(v) - \Phi(u) = \Phi(-v) - \Phi(u) \geq -\langle \Phi'(u), v + u \rangle_{V', V}.$$

Hence inequality (3.16) gives

$$\ddot{q}(t) + \gamma(t)\dot{q}(t) \leq \Phi(u(t_0)) - \Phi(u(t)) + \left| \frac{du}{dt}(t) \right|^2 \quad \text{a.e. on } [0, t_0]. \quad (3.17)$$

Recalling that the energy function $\mathcal{E}(t)$ is nonincreasing, we have $\frac{1}{2}|\frac{du}{dt}(t)|^2 + \Phi(u(t)) \geq \frac{1}{2}|\frac{du}{dt}(t_0)|^2 + \Phi(u(t_0))$ for every $t \in [0, t_0]$. Therefore

$$\forall t \in [0, t_0], \quad \Phi(u(t_0)) - \Phi(u(t)) \leq \frac{1}{2} \left| \frac{du}{dt}(t) \right|^2.$$

Using inequality (3.17), we deduce that

$$\ddot{q}(t) + \gamma(t)\dot{q}(t) \leq \frac{3}{2} \left| \frac{du}{dt}(t) \right|^2 \quad \text{a.e. on } [0, t_0].$$

Let us multiply each member of this inequality by $e^{\int_0^t \gamma(\tau) d\tau}$ and integrate on $[0, t]$. Since the map \dot{q} is absolutely continuous, we find

$$\dot{q}(t) \leq e^{-\int_0^t \gamma(\tau) d\tau} \dot{q}(0) + \frac{3}{2} e^{-\int_0^t \gamma(\tau) d\tau} \int_0^t e^{\int_0^s \gamma(\tau) d\tau} \left| \frac{du}{ds}(s) \right|^2 ds.$$

Let us integrate this inequality on $[t, t_0]$, we obtain

$$-q(t) \leq \dot{q}(0) \int_t^{t_0} e^{-\int_0^s \gamma(\tau) d\tau} ds + \frac{3}{2} (h(t_0) - h(t)),$$

where we have set

$$h(t) = \int_0^t e^{-\int_0^s \gamma(\tau) d\tau} \int_0^s e^{\int_0^\sigma \gamma(\tau) d\tau} \left| \frac{du}{d\tau}(\sigma) \right|^2 d\sigma ds.$$

We deduce from the previous inequality that

$$\frac{1}{2} |u(t) - u(t_0)|^2 \leq |u(t)|^2 - |u(t_0)|^2 + \dot{q}(0) \int_t^{t_0} e^{-\int_0^s \gamma(\tau) d\tau} ds + \frac{3}{2} (h(t_0) - h(t)). \quad (3.18)$$

In the proof of Theorem 3.7, we showed that $\lim_{t \rightarrow +\infty} |u(t) - v|^2$ exists for all $v \in \operatorname{argmin} \Phi$. Since Φ is convex and even, we have $0 \in \operatorname{argmin} \Phi$, hence $\lim_{t \rightarrow +\infty} |u(t)|^2$ exists. On the other hand, the integral $\int_0^{+\infty} e^{-\int_0^s \gamma(\tau) d\tau} ds$ is finite from (3.13), while $\lim_{t \rightarrow +\infty} h(t)$ exists in view of estimate (3.15). We then deduce from inequality (3.18) that $\{u(t); t \rightarrow +\infty\}$ is a Cauchy net in H hence strongly converges in H . It suffices to use the second part of Theorem 3.7 to obtain the strong convergence in V . \square

3.2.2. Case of a vanishing damping

It is assumed in this paragraph that the damping parameter $\gamma(t)$ vanishes as $t \rightarrow +\infty$. The trajectories of (E) are clearly more volatile in this framework. Our purpose is to obtain results of convergence for the trajectories, assuming that $\gamma(t)$ tends slowly enough toward 0. We are going to show that the convergence properties stated in the previous paragraph still hold true if the quantity $\gamma(t)$ behaves as k/t^α , for some $\alpha \in]0, 1[$, $k > 0$ and t large enough. The main step consists in establishing a refinement of Proposition 3.5 via sharp estimates for the energy decay. Let us start with a technical lemma that will be crucial in the sequel.

Lemma 3.10. Assume that the bilinear form $a(\cdot, \cdot)$ and the function f satisfy respectively hypotheses (h_1) – (h_2) and (k_1) – (k_3) . Let $\gamma \in W_{loc}^{1,1}(\mathbb{R}_+, \mathbb{R}_+)$ be a function such that $\lim_{t \rightarrow +\infty} \gamma(t) = 0$. Let u be a solution in the class (1.4) to Eq. (E) and assume that $u \in L^\infty(\mathbb{R}_+, H)$. We are given some $t_0 \geq 0$ along with a non-constant map $\lambda \in C^3([t_0, +\infty[, \mathbb{R})$ such that $\lambda(t) \geq 0$, $\dot{\lambda}(t) \geq 0$, $\ddot{\lambda}(t) \geq 0$ and $\ddot{\lambda}(t) \leq 0$ for every $t \geq t_0$. Assume that the

map $t \mapsto \dot{\lambda}(t)|\frac{du}{dt}(t)|$ is bounded, that $\int_{t_0}^{+\infty} \dot{\lambda}(t)|\dot{\gamma}(t)|dt < +\infty$ and that $\lambda(t)\gamma(t) \geq 2\dot{\lambda}(t)$ for every $t \geq t_0$. Then the following estimates hold true:

- (i) $\int_{t_0}^{+\infty} \dot{\lambda}(t)\mathcal{E}(t)dt < +\infty$.
- (ii) $\lim_{t \rightarrow +\infty} \lambda(t)\mathcal{E}(t) = 0$.
- (iii) $\int_{t_0}^{+\infty} \lambda(t)\gamma(t)|\frac{du}{dt}(t)|^2 dt < +\infty$.

Proof. Let us consider the map p defined by $p(t) = \frac{1}{2}|u(t) - v|^2$ for some $v \in S$, see the proof of Proposition 3.5. Recall that we have from inequality (3.7)

$$\mathcal{E}(t) \leq \frac{3}{2} \left| \frac{du}{dt}(t) \right|^2 - \ddot{p}(t) - \gamma(t)\dot{p}(t) \quad \text{a.e. on } \mathbb{R}_+. \quad (3.19)$$

Now define the map $\mathcal{E}_\lambda : [t_0, +\infty[\rightarrow \mathbb{R}_+$ by $\mathcal{E}_\lambda(t) = \lambda(t)\mathcal{E}(t)$. It is clear that $\mathcal{E}_\lambda \in W_{loc}^{1,1}([t_0, +\infty[)$. Since $\dot{\mathcal{E}}(t) = -\gamma(t)|\frac{du}{dt}(t)|^2$ for almost every $t \geq 0$, we have

$$\dot{\mathcal{E}}_\lambda(t) = \dot{\lambda}(t)\mathcal{E}(t) - \lambda(t)\gamma(t) \left| \frac{du}{dt}(t) \right|^2 \quad \text{a.e. on } [t_0, +\infty[. \quad (3.20)$$

From the assumption $\lambda(t)\gamma(t) \geq 2\dot{\lambda}(t)$ for every $t \geq t_0$, we deduce that

$$\dot{\lambda}(t) \left| \frac{du}{dt}(t) \right|^2 \leq \frac{1}{2}\dot{\lambda}(t)\mathcal{E}(t) - \frac{1}{2}\dot{\mathcal{E}}_\lambda(t) \quad \text{a.e. on } [t_0, +\infty[. \quad (3.21)$$

By combining inequalities (3.19) and (3.21), we infer that

$$\dot{\lambda}(t)\mathcal{E}(t) \leq -3\dot{\mathcal{E}}_\lambda(t) - 4\dot{\lambda}(t)[\ddot{p}(t) + \gamma(t)\dot{p}(t)] \quad \text{a.e. on } [t_0, +\infty[.$$

Let us integrate this inequality on $[t_0, t]$; we find

$$\int_{t_0}^t \dot{\lambda}(s)\mathcal{E}(s)ds \leq 3\mathcal{E}_\lambda(t_0) - 4 \int_{t_0}^t \dot{\lambda}(s)\ddot{p}(s)ds - 4 \int_{t_0}^t \dot{\lambda}(s)\gamma(s)\dot{p}(s)ds. \quad (3.22)$$

For the last two integrals, let us use a technique of integration by parts.

$$\begin{aligned} - \int_{t_0}^t \dot{\lambda}(s)\ddot{p}(s)ds &= -\dot{\lambda}(t)\dot{p}(t) + \dot{\lambda}(t_0)\dot{p}(t_0) + \int_{t_0}^t \ddot{\lambda}(s)\dot{p}(s)ds \\ &= -\dot{\lambda}(t)\dot{p}(t) + \dot{\lambda}(t_0)\dot{p}(t_0) + \ddot{\lambda}(t)p(t) - \ddot{\lambda}(t_0)p(t_0) - \int_{t_0}^t \ddot{\lambda}(s)p(s)ds. \end{aligned}$$

The map u is bounded in H by assumption, hence there exist $M, M' > 0$ such that $p(t) \leq M$ and $|\dot{p}(t)| \leq M'|\frac{du}{dt}(t)|$ for every $t \geq 0$. Therefore we deduce from the above equality that

$$- \int_{t_0}^t \dot{\lambda}(s)\ddot{p}(s)ds \leq M'\dot{\lambda}(t) \left| \frac{du}{dt}(t) \right| + M'\dot{\lambda}(t_0) \left| \frac{du}{dt}(t_0) \right| + M\ddot{\lambda}(t) + M \int_{t_0}^t |\ddot{\lambda}(s)|ds.$$

Recalling that $\ddot{\lambda}(t) \leq 0$ and that the map $t \mapsto \dot{\lambda}(t)|\frac{du}{dt}(t)|$ is bounded by some $M'' > 0$, we obtain

$$-\int_{t_0}^t \dot{\lambda}(s)\ddot{p}(s) ds \leq 2M'M'' + M\ddot{\lambda}(t) + M(\ddot{\lambda}(t_0) - \ddot{\lambda}(t)) = 2M'M'' + M\ddot{\lambda}(t_0). \quad (3.23)$$

On the other hand, we have

$$\begin{aligned} -\int_{t_0}^t \dot{\lambda}(s)\gamma(s)\dot{p}(s) ds &= -\dot{\lambda}(t)\gamma(t)p(t) + \dot{\lambda}(t_0)\gamma(t_0)p(t_0) + \int_{t_0}^t \ddot{\lambda}(s)\gamma(s)p(s) ds + \int_{t_0}^t \dot{\lambda}(s)\dot{\gamma}(s)p(s) ds \\ &\leq M\dot{\lambda}(t_0)\gamma(t_0) + M \int_{t_0}^t \ddot{\lambda}(s)\gamma(s) ds + M \int_{t_0}^t \dot{\lambda}(s)|\dot{\gamma}(s)| ds. \end{aligned} \quad (3.24)$$

Observe that

$$\begin{aligned} \int_{t_0}^t \ddot{\lambda}(s)\gamma(s) ds &= \dot{\lambda}(t)\gamma(t) - \dot{\lambda}(t_0)\gamma(t_0) - \int_{t_0}^t \dot{\lambda}(s)\dot{\gamma}(s) ds \\ &\leq \dot{\lambda}(t)\gamma(t) + \int_{t_0}^t \dot{\lambda}(s)|\dot{\gamma}(s)| ds. \end{aligned} \quad (3.25)$$

Since $\lim_{t \rightarrow +\infty} \gamma(t) = 0$, we have for every $t \geq t_0$

$$\dot{\lambda}(t)\gamma(t) = \dot{\lambda}(t) \int_t^{+\infty} -\dot{\gamma}(s) ds \leq \dot{\lambda}(t) \int_t^{+\infty} |\dot{\gamma}(s)| ds \leq \int_t^{+\infty} \dot{\lambda}(s)|\dot{\gamma}(s)| ds,$$

the last equality being a consequence of the fact that the map $\dot{\lambda}$ is nondecreasing. The finiteness of the integral $\int_t^{+\infty} \dot{\lambda}(s)|\dot{\gamma}(s)| ds$ is ensured by assumption. In view of (3.25), we deduce that

$$\int_{t_0}^t \ddot{\lambda}(s)\gamma(s) ds \leq \int_{t_0}^{+\infty} \dot{\lambda}(s)|\dot{\gamma}(s)| ds < +\infty.$$

Coming back to (3.24), we infer that

$$-\int_{t_0}^t \dot{\lambda}(s)\gamma(s)\dot{p}(s) ds \leq M\dot{\lambda}(t_0)\gamma(t_0) + 2M \int_{t_0}^{+\infty} \dot{\lambda}(s)|\dot{\gamma}(s)| ds < +\infty. \quad (3.26)$$

By combining inequalities (3.22), (3.23) and (3.26), we conclude that the quantity $\int_{t_0}^t \dot{\lambda}(s)\mathcal{E}(s) ds$ is uniformly majorized with respect to t , whence (i).

Let us now come back to Eq. (3.20). By taking the positive part of each member, we find $(\dot{\mathcal{E}}_\lambda)_+(t) \leq \dot{\lambda}(t)\mathcal{E}(t)$. This implies that $(\dot{\mathcal{E}}_\lambda)_+ \in L^1(0, +\infty)$ and therefore $l = \lim_{t \rightarrow +\infty} \lambda(t)\mathcal{E}(t)$ exists in \mathbb{R}_+ . We have to prove that $l = 0$. Let us argue by contradiction and assume that $l > 0$. Then $\mathcal{E}(t) \sim l/\lambda(t)$ for t

large enough. From (i), we deduce that $\int_{t_0}^{+\infty} \dot{\lambda}(s)/\lambda(s) ds < +\infty$, i.e. $\lim_{t \rightarrow +\infty} \ln \lambda(t) < +\infty$. Hence the nondecreasing convex map λ has a finite limit as $t \rightarrow +\infty$, which implies that it is constant. But it contradicts the assumption and we conclude that $l = 0$, which shows (ii).

By integrating equality (3.20) on $[t_0, t]$, we obtain

$$\int_{t_0}^t \lambda(s) \gamma(s) \left| \frac{du}{ds}(s) \right|^2 ds = \int_{t_0}^t \dot{\lambda}(s) \mathcal{E}(s) ds + \mathcal{E}_\lambda(t_0) - \mathcal{E}_\lambda(t) \leq \int_{t_0}^{+\infty} \dot{\lambda}(s) \mathcal{E}(s) ds + \mathcal{E}_\lambda(t_0) < +\infty.$$

Letting $t \rightarrow +\infty$, we immediately obtain (iii). \square

A repeated application of Lemma 3.10 allows to derive sharp estimates for the energy decay under some suitable conditions. These estimates will be the keystone for proving convergence of the trajectories.

Proposition 3.11. Assume that the bilinear form $a(\cdot, \cdot)$ and the function f satisfy respectively hypotheses (h_1) – (h_2) and (k_1) – (k_3) . Let $\gamma \in W_{loc}^{1,1}(\mathbb{R}_+, \mathbb{R}_+)$ be a function such that $\lim_{t \rightarrow +\infty} \gamma(t) = 0$. Assume that $\int_0^{+\infty} t^{1-(\frac{1}{2})^n} |\dot{\gamma}(t)| dt < +\infty$ for some $n \in \mathbb{N}$ and that there exists $t_0 > 0$ such that $\gamma(t) \geq \frac{4}{t}$ for every $t \geq t_0$. Let u be a solution in the class (1.4) to Eq. (E) and assume that $u \in L^\infty(\mathbb{R}_+, H)$. Then we have

- (i) $\int_0^{+\infty} t^{1-(\frac{1}{2})^n} \mathcal{E}(t) dt < +\infty$.
- (ii) $\lim_{t \rightarrow +\infty} t^{2-(\frac{1}{2})^n} \mathcal{E}(t) = 0$.
- (iii) $\int_0^{+\infty} t^{2-(\frac{1}{2})^n} \gamma(t) \left| \frac{du}{dt}(t) \right|^2 dt < +\infty$.

Proof. First we use Lemma 3.10 with the map λ_0 defined by $\lambda_0(t) = t$ for every $t \geq 0$. Let us verify that the assumptions of Lemma 3.10 are satisfied. Recall that the map $t \mapsto \left| \frac{du}{dt}(t) \right|$ is bounded, see (3.4). On the other hand, the finiteness of the integral $\int_0^{+\infty} |\dot{\gamma}(t)| dt$ is a consequence of the assumption $\int_0^{+\infty} t^{1-(\frac{1}{2})^n} |\dot{\gamma}(t)| dt < +\infty$. Finally, the assumption $\lambda_0(t) \gamma(t) \geq 2\dot{\lambda}_0(t)$ is trivially verified since $\gamma(t) \geq \frac{4}{t}$ for every $t \geq t_0$. Lemma 3.10 (ii) then shows that $\lim_{t \rightarrow +\infty} t \mathcal{E}(t) = 0$. Since $\mathcal{E}(t) \geq \frac{1}{2} \left| \frac{du}{dt}(t) \right|^2$, we deduce that $\lim_{t \rightarrow +\infty} t^{1/2} \left| \frac{du}{dt}(t) \right| = 0$. This suggests to apply Lemma 3.10 with the map λ_1 defined by $\lambda_1(t) = t^{3/2}$. The boundedness of the map $\lambda_1 \left| \frac{du}{dt} \right|$ is guaranteed by the previous step. The other assumptions of Lemma 3.10 are trivially satisfied. Lemma 3.10 (ii) then shows that $\lim_{t \rightarrow +\infty} t^{3/2} \mathcal{E}(t) = 0$, thus implying that $\lim_{t \rightarrow +\infty} t^{3/4} \left| \frac{du}{dt}(t) \right| = 0$. By using recursively Lemma 3.10, we let the reader check that $\lim_{t \rightarrow +\infty} t^{1-(\frac{1}{2})^n} \left| \frac{du}{dt}(t) \right| = 0$. Define the map λ_n by $\lambda_n(t) = t^{2-(\frac{1}{2})^n}$. The boundedness of the map $\lambda_n \left| \frac{du}{dt} \right|$ is implied by the previous step, while the integral $\int_0^{+\infty} \dot{\lambda}_n(t) |\dot{\gamma}(t)| dt$ is finite by assumption. Lemma 3.10 applied with the map λ_n yields conclusions (i), (ii) and (iii) of Proposition 3.11. \square

Given $n \in \mathbb{N}$, $k > 0$ and $t_0 > 0$, the following condition plays a central role in the sequel

$$\left\{ \begin{array}{l} \lim_{t \rightarrow +\infty} \gamma(t) = 0, \\ \int_0^{+\infty} t^{1-(\frac{1}{2})^n} |\dot{\gamma}(t)| dt < +\infty, \\ \forall t \geq t_0, \quad \gamma(t) \geq \frac{k}{t^{1-(\frac{1}{2})^{n+1}}}. \end{array} \right. \quad (I_2)$$

Hypothesis (I_2) automatically implies $\dot{\gamma} \in L^1(0, +\infty)$ together with $\gamma \notin L^1(0, +\infty)$.

Remark 3.12. Assume that the map $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is nonincreasing and that there exist $\alpha \in]0, 1[$, $k, k' > 0$ and $t_0 > 0$ such that

$$\forall t \geq t_0, \quad \frac{k}{t^\alpha} \leq \gamma(t) \leq \frac{k'}{t^\alpha}. \quad (3.27)$$

Let us show that condition (l_2) is satisfied if the integer $n \in \mathbb{N}$ is chosen such that⁸ $\alpha \in]1 - (\frac{1}{2})^n, 1 - (\frac{1}{2})^{n+1}]$. Since $\alpha \leq 1 - (\frac{1}{2})^{n+1}$, we have

$$\frac{k}{t^{1-(\frac{1}{2})^{n+1}}} \leq \frac{k}{t^\alpha} \leq \gamma(t),$$

and the third condition of (l_2) is proved. Recalling that $\dot{\gamma}(t) \leq 0$, an immediate integration by parts gives

$$\begin{aligned} \int_{t_0}^t s^{1-(\frac{1}{2})^n} |\dot{\gamma}(s)| ds &= - \int_{t_0}^t s^{1-(\frac{1}{2})^n} \dot{\gamma}(s) ds \\ &= t_0^{1-(\frac{1}{2})^n} \gamma(t_0) - t^{1-(\frac{1}{2})^n} \gamma(t) + \left(1 - \left(\frac{1}{2}\right)^n\right) \int_{t_0}^t \frac{\gamma(s)}{s^{(\frac{1}{2})^n}} ds. \end{aligned}$$

Since $0 \leq \gamma(t) \leq \frac{k'}{t^\alpha}$ for every $t \geq t_0$, we infer that

$$\int_{t_0}^t s^{1-(\frac{1}{2})^n} |\dot{\gamma}(s)| ds \leq t_0^{1-(\frac{1}{2})^n} \gamma(t_0) + k' \left(1 - \left(\frac{1}{2}\right)^n\right) \int_{t_0}^t \frac{ds}{s^{\alpha+(\frac{1}{2})^n}}.$$

From the choice of n , we have $\alpha + (\frac{1}{2})^n > 1$, hence the integral $\int_{t_0}^{+\infty} \frac{ds}{s^{\alpha+(\frac{1}{2})^n}}$ is convergent. In view of the above inequality, we conclude that $\int_{t_0}^{+\infty} s^{1-(\frac{1}{2})^n} |\dot{\gamma}(s)| ds < +\infty$. Notice that if $n = 0$, this condition reduces to $\int_0^{+\infty} |\dot{\gamma}(t)| dt < +\infty$, which is automatically satisfied since $\dot{\gamma} \leq 0$. It follows that if $\alpha \in]0, \frac{1}{2}]$, one may take $k' = +\infty$ in condition (3.27) (no required upper bound).

Let us now state the main result of this section.

Theorem 3.13. Assume that the bilinear form $a(\cdot, \cdot)$ and the function f satisfy respectively (h_1) – (h_2) and (k_1) – (k_3) . Let $\gamma \in W_{loc}^{1,1}(\mathbb{R}_+, \mathbb{R}_+)$ be a map satisfying (l_2) . Let u be a solution in the class (1.4) to Eq. (E) and assume that $u \in L^\infty(\mathbb{R}_+, H)$. Then, there exists $u_\infty \in S$ such that $u(t) \rightharpoonup u_\infty$ weakly in V as $t \rightarrow +\infty$. Furthermore, if $u(t)$ strongly converges⁹ in H then it strongly converges in V . Finally, if the potential function F is even, the convergence is strong in V .

Proof. The proof follows the same lines as the ones of Theorem 3.7. Given $v \in S$, we define the map $p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by $p(t) = \frac{1}{2}|u(t) - v|^2$. Recall that

⁸ Its explicit expression is given by $n = -\lceil \frac{\ln(1-\alpha)}{\ln 2} \rceil - 1$, where $\lceil x \rceil$ denotes the integer part of $x \in \mathbb{R}$.

⁹ This assumption is satisfied if the injection $V \hookrightarrow H$ is compact.

$$\dot{p}(t) \leq e^{-\int_0^t \gamma(\tau) d\tau} \dot{p}(0) + e^{-\int_0^t \gamma(\tau) d\tau} \int_0^t e^{\int_0^s \gamma(\tau) d\tau} \left| \frac{du}{ds}(s) \right|^2 ds \quad (3.28)$$

(see formula (3.12)). We have to show that the right member of the above inequality is a summable function. From Lemma 3.14 (i) below applied with $\theta = 1 - (\frac{1}{2})^{n+1}$, we have

$$\int_0^{+\infty} e^{-\int_0^t \gamma(\tau) d\tau} dt < +\infty. \quad (3.29)$$

Lemma 3.14. *Let us assume that there exist $\theta \in [0, 1]$, $k > 0$ and $t_0 > 0$ such that $\gamma(t) \geq \frac{k}{t^\theta}$ for every $t \geq t_0$. Then:*

- (i) $\int_0^{+\infty} e^{-\int_0^t \gamma(\tau) d\tau} dt < +\infty$.
- (ii) *For every $c > 1$, we have for s large enough*

$$\int_s^{+\infty} e^{-\int_0^t \gamma(\tau) d\tau} dt \leq \frac{c}{k} s^\theta e^{-\int_0^s \gamma(\tau) d\tau}. \quad (3.30)$$

If $\theta = 0$, one can take $c = 1$ in the above inequality.

The proof of Lemma 3.14 is postponed to Appendix A. On the other hand, by applying Fubini theorem, we find

$$\int_0^{+\infty} e^{-\int_0^t \gamma(\tau) d\tau} \int_0^t e^{\int_0^s \gamma(\tau) d\tau} \left| \frac{du}{ds}(s) \right|^2 ds dt = \int_0^{+\infty} \left| \frac{du}{ds}(s) \right|^2 e^{\int_0^s \gamma(\tau) d\tau} \int_s^{+\infty} e^{-\int_0^t \gamma(\tau) d\tau} dt ds. \quad (3.31)$$

From Lemma 3.14 (ii) applied with $\theta = 1 - (\frac{1}{2})^{n+1}$, we obtain

$$e^{\int_0^s \gamma(\tau) d\tau} \int_s^{+\infty} e^{-\int_0^t \gamma(\tau) d\tau} dt \leq \frac{2}{k} s^{1 - (\frac{1}{2})^{n+1}}.$$

Since $\gamma(s) \geq \frac{k}{s^{1 - (\frac{1}{2})^{n+1}}}$, we derive that

$$e^{\int_0^s \gamma(\tau) d\tau} \int_s^{+\infty} e^{-\int_0^t \gamma(\tau) d\tau} dt \leq \frac{2}{k^2} s^{2 - (\frac{1}{2})^n} \gamma(s).$$

From Proposition 3.11 (iii) we have $\int_0^{+\infty} s^{2 - (\frac{1}{2})^n} \gamma(s) \left| \frac{du}{ds}(s) \right|^2 ds < +\infty$, hence we deduce from equality (3.31) that

$$\int_0^{+\infty} e^{-\int_0^t \gamma(\tau) d\tau} \int_0^t e^{\int_0^s \gamma(\tau) d\tau} \left| \frac{du}{ds}(s) \right|^2 ds dt < +\infty. \quad (3.32)$$

By combining inequality (3.28) with estimates (3.29) and (3.32), we infer that $[\dot{p}]_+ \in L^1(0, +\infty)$ and hence $\lim_{t \rightarrow +\infty} p(t)$ exists. The end of the proof is the same as the one of Theorem 3.7. For the second point, the reader is referred to the corresponding argument in the proof of Corollary 3.6. Finally, if the potential function F is even, the arguments of the proof of Theorem 3.9 apply directly. Details are left to the reader. \square

Remark 3.15. The assumption $u \in L^\infty(\mathbb{R}_+, H)$ arises in the statement of Theorem 3.13, while it is useless in the framework of Theorem 3.7. In the proof of this last one, the existence of $\lim_{t \rightarrow +\infty} |u(t) - v|^2$ relies on the general estimate $\gamma |\frac{du}{dt}|^2 \in L^1(0, +\infty)$, and gives the boundedness of u as a by-product. By contrast, in Theorem 3.13 the existence of $\lim_{t \rightarrow +\infty} |u(t) - v|^2$ needs a sharper estimate (see Proposition 3.11 (iii)), which uses some boundedness assumption for the map u . The question to know if the assumption $u \in L^\infty(\mathbb{R}_+, H)$ is really necessary in Theorem 3.13 remains open.

In view of Remark 3.12, we obtain directly the following corollary of Theorem 3.13.

Corollary 3.16. Assume that the bilinear form $a(\cdot, \cdot)$ and the function f satisfy the same hypotheses as in Theorem 3.13. Let $\gamma \in W_{loc}^{1,1}(\mathbb{R}_+, \mathbb{R}_+)$ be a nonincreasing map and suppose that there exist $\alpha \in]0, 1[$, $k, k' > 0$ and $t_0 > 0$ such that¹⁰

$$\forall t \geq t_0, \quad \frac{k}{t^\alpha} \leq \gamma(t) \leq \frac{k'}{t^\alpha}.$$

Then we have the same conclusions as in Theorem 3.13.

3.3. Decay estimates for a strong set of minima

Recall that the set $S = \operatorname{argmin} \Phi$ is convex and closed in H , see Remark 3.2. Let us denote by P_S the projection operator onto the set S in the sense of H . In this paragraph, we assume that the function $\Phi : V \rightarrow \mathbb{R}$ satisfies¹¹

$$\exists \eta > 0 \quad \text{such that} \quad \forall u \in V, \quad \Phi(u) \geq \frac{\eta}{2} |u - P_S(u)|^2. \quad (3.33)$$

If $\gamma \notin L^1(0, +\infty)$, we know from Proposition 3.5 (ii) that $\lim_{t \rightarrow +\infty} \mathcal{E}(t) = 0$. Under assumption (3.33), we are able to evaluate the speed of convergence of $\mathcal{E}(t)$ as $t \rightarrow +\infty$.

Theorem 3.17. Assume that the bilinear form $a(\cdot, \cdot)$ and the function f satisfy respectively (h_1) – (h_2) and (k_1) – (k_3) . Let $\gamma \in W_{loc}^{1,1}(\mathbb{R}_+, \mathbb{R}_+)$ be a function satisfying $\lim_{t \rightarrow +\infty} \gamma(t) = 0$ and $\dot{\gamma}(t) = o(\gamma(t))$ as $t \rightarrow +\infty$. We suppose that the function $\Phi : V \rightarrow \mathbb{R}$ defined by $\Phi(u) = \frac{1}{2}a(u, u) + F(u)$ satisfies condition (3.33). Let u be a solution in the class (1.4) to Eq. (E). Then, for all $m \in]0, \frac{2}{3}[$, there exist $C > 0$ and $t_0 \geq 0$ such that

$$\forall t \geq t_0, \quad \mathcal{E}(t) \leq C e^{-m \int_0^t \gamma(s) ds}.$$

Proof. Define the map $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ by $\varphi(t) = \frac{1}{2}d_H^2(u(t), S)$, where $d_H(\cdot, S)$ stands for the distance function from the set S in the sense of H . By differentiating, we find for every $t \geq 0$

$$\dot{\varphi}(t) = \left(\frac{du}{dt}(t), u(t) - P_S(u(t)) \right). \quad (3.34)$$

¹⁰ This condition is satisfied if there exists $k'' > 0$ such that $\gamma(t) \sim \frac{k''}{t^\alpha}$ as $t \rightarrow +\infty$. On the other hand, one can take $k' = +\infty$ if $\alpha \in]0, \frac{1}{2}[$, see Remark 3.12.

¹¹ If $f = 0$, the set S coincides with $\ker A$ and we recover condition (2.7) of Section 2.

Since $\frac{du}{dt} \in W_{loc}^{1,1}(\mathbb{R}_+, H)$ by assumption, it is immediate to check that $\dot{\varphi} \in W_{loc}^{1,1}(\mathbb{R}_+)$, hence the map $\dot{\varphi}$ is differentiable almost everywhere on \mathbb{R}_+ . Consider now some $t > 0$ where the maps $\dot{\varphi}$ and $\frac{du}{dt}$ are both differentiable, and let us majorize the quantity $\ddot{\varphi}(t)$. For that purpose, we use a technique of differential quotient. For all $h \neq 0$, we have

$$\begin{aligned} \frac{1}{h}(\dot{\varphi}(t+h) - \dot{\varphi}(t)) &= \frac{1}{h} \left(\frac{du}{dt}(t), u(t+h) - P_S(u(t+h)) - u(t) + P_S(u(t)) \right) \\ &\quad + \frac{1}{h} \left(\frac{du}{dt}(t+h) - \frac{du}{dt}(t), u(t+h) - P_S(u(t+h)) \right). \end{aligned}$$

The monotonicity of P_S implies that

$$\begin{aligned} &-\frac{1}{h} \left(\frac{du}{dt}(t), P_S(u(t+h)) - P_S(u(t)) \right) \\ &\leq \frac{1}{h^2} \left(u(t+h) - u(t) - h \frac{du}{dt}(t), P_S(u(t+h)) - P_S(u(t)) \right). \end{aligned}$$

Hence we obtain

$$\begin{aligned} \frac{1}{h}(\dot{\varphi}(t+h) - \dot{\varphi}(t)) &\leq \frac{1}{h} \left(\frac{du}{dt}(t), u(t+h) - u(t) \right) \\ &\quad + \frac{1}{h^2} \left(u(t+h) - u(t) - h \frac{du}{dt}(t), P_S(u(t+h)) - P_S(u(t)) \right) \\ &\quad + \frac{1}{h} \left(\frac{du}{dt}(t+h) - \frac{du}{dt}(t), u(t+h) - P_S(u(t+h)) \right). \end{aligned}$$

Taking the limit as $h \rightarrow 0$, we derive that

$$\ddot{\varphi}(t) \leq \left| \frac{du}{dt}(t) \right|^2 + \left(\frac{d^2u}{dt^2}(t), u(t) - P_S(u(t)) \right). \quad (3.35)$$

By combining formulae (3.34) and (3.35), and using the convexity of the function Φ , we deduce that for almost every $t \in \mathbb{R}_+$

$$\begin{aligned} \ddot{\varphi}(t) + \gamma(t)\dot{\varphi}(t) &\leq \left| \frac{du}{dt}(t) \right|^2 + \left(\frac{d^2u}{dt^2}(t) + \gamma(t) \frac{du}{dt}(t), u(t) - P_S(u(t)) \right) \\ &= \left| \frac{du}{dt}(t) \right|^2 - a(u(t), u(t) - P_S(u(t))) - (f(u(t)), u(t) - P_S(u(t))) \\ &\leq \left| \frac{du}{dt}(t) \right|^2 - \Phi(u(t)) + \Phi(P_S(u(t))) = \left| \frac{du}{dt}(t) \right|^2 - \Phi(u(t)). \end{aligned}$$

It follows that

$$\ddot{\varphi}(t) + \gamma(t)\dot{\varphi}(t) + \mathcal{E}(t) \leq \frac{3}{2} \left| \frac{du}{dt}(t) \right|^2 \quad \text{a.e. on } \mathbb{R}_+.$$

Multiplying this formula by $\frac{2}{3}\gamma(t)$ and recalling that $\dot{\mathcal{E}}(t) = -\gamma(t)|\frac{du}{dt}(t)|^2$ for almost every $t \in \mathbb{R}_+$, we obtain

$$\frac{2}{3}\gamma(t)(\ddot{\varphi}(t) + \gamma(t)\dot{\varphi}(t)) + \dot{\mathcal{E}}(t) + \frac{2}{3}\gamma(t)\mathcal{E}(t) \leq 0 \quad \text{a.e. on } \mathbb{R}_+. \quad (3.36)$$

This suggests to define the function $\mathcal{F} : \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$\begin{aligned} \mathcal{F}(t) &= \Phi(u(t)) + \frac{1}{2} \left| \frac{du}{dt}(t) \right|^2 + \frac{2}{3}\gamma(t) \left(\frac{du}{dt}(t), u(t) - P_S(u(t)) \right) \\ &= \mathcal{E}(t) + \frac{2}{3}\gamma(t)\dot{\varphi}(t). \end{aligned} \quad (3.37)$$

In view of inequality (3.36), we immediately find

$$\dot{\mathcal{F}}(t) + \frac{2}{3}\gamma(t)\mathcal{F}(t) \leq \frac{2}{3} \left(\dot{\gamma}(t) - \frac{1}{3}\gamma(t)^2 \right) \left(\frac{du}{dt}(t), u(t) - P_S(u(t)) \right) \quad \text{a.e. on } \mathbb{R}_+. \quad (3.38)$$

Since $|\frac{du}{dt}(t), u(t) - P_S(u(t))| \leq \frac{1}{2}|\frac{du}{dt}(t)|^2 + \frac{1}{2}|u(t) - P_S(u(t))|^2$ and $\Phi(u(t)) \geq \frac{\eta}{2}|u(t) - P_S(u(t))|^2$ by assumption, we have

$$\left| \left(\frac{du}{dt}(t), u(t) - P_S(u(t)) \right) \right| \leq C\mathcal{E}(t), \quad \text{for some } C > 0. \quad (3.39)$$

Recalling that $\lim_{t \rightarrow +\infty} \gamma(t) = 0$, the expression of \mathcal{F} shows that

$$\mathcal{F}(t) \sim \mathcal{E}(t) \quad \text{as } t \rightarrow +\infty. \quad (3.40)$$

Let us fix some $m \in]0, \frac{2}{3}[$. Using the fact that $\dot{\gamma}(t) = o(\gamma(t))$ and $\gamma(t)^2 = o(\gamma(t))$ as $t \rightarrow +\infty$, we deduce from (3.38), (3.39) and (3.40) the existence of $t_0 \geq 0$ such that

$$\dot{\mathcal{F}}(t) + \frac{2}{3}\gamma(t)\mathcal{F}(t) \leq \left(\frac{2}{3} - m \right) \gamma(t)\mathcal{F}(t) \quad \text{a.e. on } [t_0, +\infty[,$$

hence $\dot{\mathcal{F}}(t) + m\gamma(t)\mathcal{F}(t) \leq 0$ for almost every $t \geq t_0$. Let us multiply by $e^{m \int_0^t \gamma(s) ds}$ and integrate on $[t_0, t]$. Since the function \mathcal{F} is absolutely continuous, we find $\mathcal{F}(t) \leq D e^{-m \int_0^t \gamma(s) ds}$, with $D = e^{m \int_0^{t_0} \gamma(s) ds} \mathcal{F}(t_0)$. Conclusion follows from estimate (3.40). \square

Remark 3.18. Under the hypotheses of Theorem 3.17, assume that there exists $k > 3$ such that $\gamma(t) \geq \frac{k}{t}$ for t large enough. Fix $m \in]\frac{2}{k}, \frac{2}{3}[$. From Theorem 3.17, there exist $C > 0$ and $t_0 \geq 0$ such that

$$\forall t \geq t_0, \quad \frac{1}{2} \left| \frac{du}{dt}(t) \right|^2 \leq \mathcal{E}(t) \leq \frac{C}{t^{mk}}.$$

Hence we have $|\frac{du}{dt}(t)| \leq \frac{(2C)^{1/2}}{t^{mk/2}}$ and since $mk > 2$, we deduce that $|\frac{du}{dt}| \in L^1(0, +\infty)$. The trajectory u has a finite length, therefore it strongly converges in H toward some $u_\infty \in S$.

4. Application to particular semilinear evolution problems

We suppose that Ω is a bounded open subset of \mathbb{R}^n with boundary $\partial\Omega$ sufficiently regular.

4.1. Hyperbolic problems of order two in space

Example 4.1. Given a map $\gamma: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and a function $f \in C^1(\mathbb{R})$, let us consider the following damped wave equation

$$\frac{d^2 u}{dt^2} + \gamma(t) \frac{du}{dt} - \Delta u + f(u) = 0 \quad \text{on } \Omega \times]0, +\infty[, \quad (4.1)$$

with Dirichlet boundary condition:

$$u = 0 \quad \text{on } \partial\Omega \times]0, +\infty[. \quad (4.2)$$

The functional setting of the evolution problem (4.1)–(4.2) is given by

$$H = L^2(\Omega), \quad V = H_0^1(\Omega) \quad \text{and} \quad a(u, v) = \int_{\Omega} \nabla u(x) \nabla v(x) dx.$$

Hypothesis (h_1) is trivially verified while hypothesis (h_2) is satisfied with $\lambda = 0$, since the bilinear form a is coercive. On the other hand, we assume that the function f satisfies the following properties:

- (i) There exist $C, \alpha \geq 0$ such that $(n-2)\alpha \leq 2$ and $|f'(r)| \leq C(1 + |r|^\alpha) \quad \forall r \in \mathbb{R}$.
- (ii) f is nondecreasing.

Define the function $F \in C^2(\mathbb{R})$ by $F(r) = \int_0^r f(s) ds$ for every $r \in \mathbb{R}$. For simplicity of notation, we write $F(u)$ for $\int_{\Omega} F(u(x)) dx$. Hypothesis (k_1) is a consequence of assumption (i) above, see for example [10, pp. 73–75]. The monotonicity hypothesis (k_2) is ensured by point (ii). Finally the coercivity of the bilinear form a implies that the equilibrium set is a singleton $\{\bar{u}\}$, see Remark 3.1. In particular, the non-vacuity condition (k_3) is satisfied. If the map $\gamma \in W_{loc}^{1,1}(\mathbb{R}_+, \mathbb{R}_+)$ is such that $\dot{\gamma} \in L^1(0, +\infty)$ and $\gamma \notin L^1(0, +\infty)$, we derive from Corollary 3.6 that $u(t) \rightharpoonup \bar{u}$ weakly in $H_0^1(\Omega)$ as $t \rightarrow +\infty$. Since the injection $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, the second part of Corollary 3.6 shows that the convergence is strong in $H_0^1(\Omega)$. On the other hand, the coercivity of a implies that condition (3.33) is fulfilled. If the map γ satisfies $\lim_{t \rightarrow +\infty} \gamma(t) = 0$ and $\dot{\gamma}(t) = o(\gamma(t))$ as $t \rightarrow +\infty$, Theorem 3.17 then shows that for every $m \in]0, \frac{2}{3}[$,

$$\frac{1}{2} \int_{\Omega} \left\{ \left| \frac{\partial u}{\partial t}(t, x) \right|^2 + |\nabla u(t, x)|^2 \right\} dx + \int_{\Omega} F(u(t, x)) dx = O(e^{-m \int_0^t \gamma(s) ds}) \quad \text{as } t \rightarrow +\infty.$$

Example 4.2. Let us consider the damped wave equation (4.1) with Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega \times]0, +\infty[$. The functional setting of the evolution problem is given by

$$H = L^2(\Omega), \quad V = H^1(\Omega) \quad \text{and} \quad a(u, v) = \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx.$$

The bilinear form a is semi-coercive, hypothesis (h_2) is satisfied with $\lambda = \mu = 1$. To overcome the lack of coercivity, assumptions (i)–(ii) above are supplemented with the following one:

- (iii) There exist $\varepsilon > 0$ and $D \geq 0$ such that $F(r) \geq \varepsilon r^2 - D$ for every $r \in \mathbb{R}$.

Assumption (iii) implies that condition (k_3) is verified, see Remark 3.1. Hypotheses (k_1) – (k_2) are fulfilled as in the previous example. If the map $\gamma \in W_{loc}^{1,1}(\mathbb{R}_+, \mathbb{R}_+)$ satisfies (l_1) (respectively (l_2)), we derive from Theorem 3.7 (respectively 3.13) that there exists a solution u_∞ of

$$\begin{cases} -\Delta u + f(u) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

such that $u(t) \rightharpoonup u_\infty$ weakly in $H^1(\Omega)$ as $t \rightarrow +\infty$. Since the injection $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, the second part of Theorem 3.7 (respectively 3.13) shows that the convergence is strong in $H^1(\Omega)$.

Example 4.3. Let us consider the following equation

$$\frac{d^2 u}{dt^2} + \gamma(t) \frac{du}{dt} - \Delta u - \lambda_1 u + f(u) = 0 \quad \text{on } \Omega \times]0, +\infty[, \quad (4.3)$$

with Dirichlet boundary condition. Here λ_1 stands for the smallest eigenvalue of the Laplacian–Dirichlet operator. The functional setting of the evolution problem is given by

$$H = L^2(\Omega), \quad V = H_0^1(\Omega) \quad \text{and} \quad a(u, v) = \int_{\Omega} [\nabla u(x) \cdot \nabla v(x) - \lambda_1 u(x)v(x)] dx.$$

It is immediate to check that (h_1) – (h_2) are satisfied. Under the above assumptions (i), (ii) and (iii), we obtain as previously that conditions (k_1) – (k_3) hold true. If the map $\gamma \in W_{loc}^{1,1}(\mathbb{R}_+, \mathbb{R}_+)$ satisfies (l_1) (respectively (l_2)), we derive from Theorem 3.7 (respectively 3.13) that there exists a solution u_∞ of

$$\begin{cases} -\Delta u - \lambda_1 u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

such that $u(t) \rightarrow u_\infty$ strongly in $H_0^1(\Omega)$ as $t \rightarrow +\infty$.

Example 4.4. The equation arising in the previous example can be generalized as follows

$$\frac{d^2 u}{dt^2} + \gamma(t) \frac{du}{dt} - \Delta u - \sum_{i=1}^{+\infty} \eta_i P_i u + f(u) = 0 \quad \text{on } \Omega \times]0, +\infty[,$$

see [21, Example 4.5]. We still assume Dirichlet boundary conditions. Let us explicit the notations: $(\lambda_i)_{i \geq 1}$ (respectively $(e_i)_{i \geq 1}$) is the sequence of eigenvalues (respectively eigenfunctions normalized in $L^2(\Omega)$) of $(-\Delta)$ in $H_0^1(\Omega)$. For each $i \geq 1$, P_i denotes the orthogonal projection on $\text{span}\{e_i\}$ in the sense of $L^2(\Omega)$. We assume that the nonnegative sequence $(\eta_i)_{i \geq 1}$ is bounded and that $\eta_i \leq \lambda_i$ for every $i \geq 1$. The functional setting of the evolution problem is given by

$$H = L^2(\Omega), \quad V = H_0^1(\Omega) \quad \text{and} \quad a(u, v) = \int_{\Omega} \nabla u(x) \cdot \nabla v(x) dx - \sum_{i=1}^{+\infty} \eta_i \int_{\Omega} P_i u(x) \cdot P_i v(x) dx.$$

It is easy to check that hypotheses (h_1) – (h_2) hold true. Under the additional assumptions (i), (ii) and (iii), we then obtain (k_1) – (k_3) . If the map $\gamma \in W_{loc}^{1,1}(\mathbb{R}_+, \mathbb{R}_+)$ satisfies (l_1) or (l_2) , we obtain as in the previous example the existence of an equilibrium u_∞ such that $u(t) \rightarrow u_\infty$ strongly in $H_0^1(\Omega)$ as $t \rightarrow +\infty$.

4.2. A higher-order example

Example 4.5. Let us consider the following equation

$$\frac{d^2 u}{dt^2} + \gamma(t) \frac{du}{dt} + \Delta^2 u + f(u) = 0 \quad \text{on } \Omega \times]0, +\infty[, \quad (4.4)$$

with the boundary condition:

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \times]0, +\infty[. \quad (4.5)$$

The functional setting of the evolution problem (4.4)–(4.5) is given by

$$H = L^2(\Omega), \quad V = \left\{ u \in H^2(\Omega), u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\} \quad \text{and} \quad a(u, v) = \int_{\Omega} \Delta u(x) \cdot \Delta v(x) dx.$$

Hypothesis (h_1) is trivially verified. Moreover, from the regularity results relative to the Laplacian–Dirichlet problem, there exists $\kappa > 0$ such that $\|u\|_{H^2(\Omega)} \leq \kappa \|\Delta u\|_{L^2(\Omega)}$. Hence condition (h_2) is satisfied with $\lambda = 0$, i.e. the bilinear form a is coercive. We assume that the function f satisfies assumption (ii) along with the following variant of (i):

(i') There exist $C, \alpha \geq 0$ such that $(n-4)\alpha \leq 4$ and $|f'(r)| \leq C(1 + |r|^\alpha) \quad \forall r \in \mathbb{R}$.

By using Sobolev's imbedding theorem, we let the reader check that hypothesis (k_1) is a consequence of assumption (i') above. The monotonicity hypothesis (k_2) is ensured by (ii). Finally in view of Remark 3.1, the coercivity of the bilinear form a implies that the equilibrium set is a singleton $\{\bar{u}\}$ and in particular (k_3) holds true. If the map $\gamma \in W_{loc}^{1,1}(\mathbb{R}_+, \mathbb{R}_+)$ is such that $\dot{\gamma} \in L^1(0, +\infty)$ and $\gamma \notin L^1(0, +\infty)$, we derive from Corollary 3.6 that $u(t) \rightarrow \bar{u}$ strongly in $H^2(\Omega)$ as $t \rightarrow +\infty$. On the other hand, the coercivity of a implies that condition (3.33) is fulfilled. If the map γ is such that $\lim_{t \rightarrow +\infty} \gamma(t) = 0$ and $\dot{\gamma}(t) = o(\gamma(t))$ as $t \rightarrow +\infty$, Theorem 3.17 then shows that for every $m \in]0, \frac{2}{3}[$,

$$\frac{1}{2} \int_{\Omega} \left\{ \left| \frac{\partial u}{\partial t}(t, x) \right|^2 + |\Delta u(t, x)|^2 \right\} dx + \int_{\Omega} F(u(t, x)) dx = O\left(e^{-m \int_0^t \gamma(s) ds}\right) \quad \text{as } t \rightarrow +\infty.$$

Appendix A

Proof of Lemma 3.14. (i) From the assumption $\gamma(t) \geq \frac{k}{t^\theta}$, we deduce the existence of $\alpha \in \mathbb{R}$ such that $\int_0^t \gamma(\tau) d\tau \geq \frac{k}{1-\theta} t^{1-\theta} + \alpha$ for every $t \geq t_0$. Therefore, we have

$$\int_0^{+\infty} e^{-\int_0^t \gamma(\tau) d\tau} dt \leq e^{-\alpha} \int_0^{+\infty} e^{-\frac{k}{1-\theta} t^{1-\theta}} dt < +\infty.$$

(ii) By using the assumption $\gamma(t) \geq \frac{k}{t^\theta}$, we find

$$\int_s^{+\infty} e^{-\int_0^t \gamma(\tau) d\tau} dt \leq \frac{1}{k} \int_s^{+\infty} t^\theta \gamma(t) e^{-\int_0^t \gamma(\tau) d\tau} dt. \quad (\text{A.1})$$

An integration by parts in the right-hand side then yields

$$\int_s^{+\infty} t^\theta \gamma(t) e^{-\int_0^t \gamma(\tau) d\tau} dt = \left[-t^\theta e^{-\int_0^t \gamma(\tau) d\tau} \right]_s^{+\infty} + \theta \int_s^{+\infty} t^{\theta-1} e^{-\int_0^t \gamma(\tau) d\tau} dt. \quad (\text{A.2})$$

Remark that $t^\theta e^{-\int_0^t \gamma(\tau) d\tau} \leq e^{-\alpha} t^\theta e^{-\frac{k}{1-\theta} t^{1-\theta}}$, hence $\lim_{t \rightarrow +\infty} t^\theta e^{-\int_0^t \gamma(\tau) d\tau} = 0$. Therefore, we deduce from (A.1) and (A.2) that

$$\int_s^{+\infty} e^{-\int_0^t \gamma(\tau) d\tau} dt \leq \frac{1}{k} s^\theta e^{-\int_0^s \gamma(\tau) d\tau} + \frac{\theta}{k} \int_s^{+\infty} t^{\theta-1} e^{-\int_0^t \gamma(\tau) d\tau} dt.$$

If $\theta = 0$, formula (3.30) is proved with $c = 1$. Now assume that $\theta \in]0, 1[$ and take $c > 1$. The right term in the above inequality is clearly negligible with respect to the left one, hence $\frac{\theta}{k} \int_s^{+\infty} t^{\theta-1} e^{-\int_0^t \gamma(\tau) d\tau} dt \leq (1 - \frac{1}{c}) \int_s^{+\infty} e^{-\int_0^t \gamma(\tau) d\tau} dt$ for s large enough. Formula (3.30) follows immediately. \square

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