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Maximum modulus estimate for the solution of the Stokes equations

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ABSTRACT

A maximum modulus estimate for the nonstationary Stokes equations in C^2 domain is found. The singular part and regular part of Poisson kernel are analyzed. The singular part consists of the gradient of single layer potential and the gradient of composite potential defined on only normal component of the boundary data. Furthermore, the normal velocity near the boundary is bounded if the boundary data is bounded. If the normal component of the boundary data is Dini-continuous and the tangential component of the boundary data is bounded, then the maximum modulus of velocity is bounded in whole domain.

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1. Introduction

A maximum modulus estimate of the nonstationary Stokes equations is presented. In the case of the stationary flow, Maremonti and Russo [4] obtained a quasi-maximum principle and Varnhorn [10] showed a maximum modulus theorem for $C^{1,\alpha}$ domain:

$$\max_{x\in\Omega}|u| \leqslant C(\Omega)\max_{x\in\partial\Omega}|u|,$$

where u is a solution to the stationary Stokes equations in domain Ω. We also note that Mazya and Rossmann [5] considered the maximum modulus estimate for the stationary Navier–Stokes equations in polygonal domain. In a canonical domain like ball, Kratz [3] found the best constant $C(\Omega)$ such that

$$\max_{x\in B_1}|u| \leqslant \frac{1}{2}n(n+1)\max_{x\in\partial B_1}|u|,$$

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where B_1 is the unit ball in $\mathbf{R}^n$. For a more general domain like Lipschitz in $\mathbf{R}^3$, Shen [7] obtained a maximum modulus estimate and the higher dimension problem is still unresolved.

The maximum modulus estimate of the nonstationary problem is heavily entangled with the structural form of Poisson kernel and the solvability of the boundary value problem is essential. As a classical result, Solonnikov [8] solved the initial–boundary problem in C^2 domain for the isotropic Sobolev spaces and later he [9] extended the solvability to the anisotropic Sobolev spaces. The L^2 solvability for the Lipschitz domain was obtained by Shen [6] for any dimension and Choe and Kozono [1] considered the case for the mixed norm potential spaces.

To be more specific, we state the nonstationary Stokes equations:

$$\begin{aligned} &u_t - \nu\Delta u + \nabla p = 0 && \text{in } \Omega \times (0,T),\\ &div\, u = 0 && \text{in } \Omega \times (0,T),\\ &u|_{t=0} = 0 && \text{in } \Omega,\\ &u|_{\partial\Omega\times(0,T)} = g && \text{on } \partial\Omega \times (0,T), \end{aligned} \tag{1.1}$$

where Ω is C^2 bounded connected domain in $\mathbf{R}^n$ and $0 < T < \infty$ and ν is the viscosity which we assume 1. In addition, we assume the boundary data g satisfies the compatibility condition:

$$\int_{\partial\Omega} g \cdot N \, d\sigma = 0$$

for almost all t, where N is the outward unit normal vector on the boundary. Since nontrivial initial data can be treated by solving homogeneous boundary value problems, we consider only the initial–boundary value problems with zero initial data.

Contrary to the stationary case, the quasi-maximum principle fails, namely, there is an unbounded solution whose boundary data is bounded. Heuristically speaking, at the boundary point where the normal component of boundary data has a jump discontinuity along an $(n-2)$-dimensional surface on the boundary passing to it, the tangential component of the velocity blows up in the neighborhood of it. So we cannot expect the quasi-maximum modulus theorem like the stationary case.

In this paper, we only consider the case that the space dimension is greater than or equal to 3. Dimension 2 case follows exactly the same path with logarithmic kernels. Denote E for the fundamental solution to Laplace equation and Γ for the fundamental solution to heat equation with unit conductivity. For a given boundary point $y \in \partial\Omega$, $N(y)$ is the outward unit normal vector at y. We define the $(n-1)$-dimensional convolution

$$\mathbf{S}(f)(x) = \int_{\partial\Omega} E(x-y) f(y)\, d\sigma(y)$$

for real-value function $f : \mathbf{R}^n \to \mathbf{R}$ which is just the single layer potential of f on $\partial\Omega$. We need a composite kernel. We define a composite kernel function $\kappa(x,t)$ on $\Omega \times (0,T)$ by

$$\kappa(x,t) = \int_{\partial\Omega} \frac{\partial\Gamma}{\partial N(y)}(x-y,t) E(y)\, d\sigma(y)$$

and a surface potential $\mathbf{T}$ for f by

$$\mathbf{T}(f)(x,t) = 4\int_0^t \int_{\partial\Omega} \kappa(x-y, t-s) f(y,s)\, d\sigma(y)\, ds,$$

for real-value function $f : \mathbf{R}^{n+1} \to \mathbf{R}$. We state our main theorem: For given $x \in \Omega$, $\bar{x}$ is the nearest point of x on $\partial\Omega$ such that $dist(x, \partial\Omega) = |\bar{x} - x|$ and for a vector valued function $v(x)$, we define the normal component and tangential component to the nearest point $\bar{x}$ by

$$v_N(x) = \big(v(x) \cdot N(\bar{x})\big)N(\bar{x}) \quad \text{and} \quad v_T(x) = v(x) - \big(v(x) \cdot N(\bar{x})\big)N(\bar{x}).$$

Theorem 1.1. *Suppose that the domain Ω is bounded C^2 and u is a solution to (1.1) for bounded boundary data g. The normal component of the velocity u_N is bounded and there is also a constant $C(\Omega)$ such that*

$$\max_{(x,t)\in\Omega\times(0,T)} \big|u_N(x,t)\big| \leqslant C(\Omega) \max_{(y,t)\in\partial\Omega\times(0,T)} \big|g(y,t)\big|.$$

Furthermore, the tangential component of the velocity u satisfies

$$\max_{(x,t)\in\Omega\times(0,T)} \big|u_T(x,t) - \nabla\mathbf{S}(g\cdot N)_T(x,t) - \nabla\mathbf{T}(g\cdot N)_T(x,t)\big| \leqslant C(\Omega) \max_{(y,t)\in\partial\Omega\times(0,T)} \big|g(y,t)\big|.$$

Define the modulus of continuity of f at x by $\omega(f)(r,x) = \sup_{y\in B_r(x)\cap\Omega} |f(y) - f(x)|$ and we say f is Dini-continuous in Ω if

$$\|f\|_{Dini,\Omega} = \sup_{x\in\Omega} \int_0^{r_0} \omega(f)(r,x)\,\frac{dr}{r} < \infty$$

for an $r_0 > 0$. From a direct computation, we have $\nabla\mathbf{S}(f)$ and $\nabla\mathbf{T}(f)$ are bounded if f is Dini-continuous on $\partial\Omega$ and we obtain a maximum modulus estimate:

Corollary 1.2. *Suppose that the domain Ω is bounded C^2 and u is a solution to (1.1). Suppose g is bounded on $\partial\Omega$ and the normal component g_N is Dini-continuous. Then, there is a constant $C(\Omega)$ depending only on Ω such that*

$$\max_{(x,t)\in\Omega\times(0,T)} \big|u(x,t)\big| \leqslant C(\Omega)\Big(\max_{(y,t)\in\partial\Omega\times(0,T)} \big|g(y,t)\big| + \max_{t\in(0,T)} \big\|g_N(\cdot,t)\big\|_{Dini,\partial\Omega}\Big).$$

As a separate interest, we obtain an improved L^2 theory like Lemma 4.1. When the $L^2(\partial\Omega)$ norm of the boundary data is bounded in time, then $\|u(\cdot,t)\|_{L^2(\Omega)}$ is bounded in time. Consequently, the local boundedness holds too (see Corollary 4.2).

2. Kernels on half-plane

To study Eq. (1.1), we consider the case of $\Omega = \mathbf{R}^n_+ = \{(x', x_n) \in \mathbf{R}^n \mid x' \in \mathbf{R}^{n-1},\ 0 < x_n < \infty\}$ and for the notational simplicity we set $D_{x_i} = \frac{\partial}{\partial x_i}$ and double indices mean summation up to n. For notation, we denote $x = (x', x_n)$, that is, $x' = (x_1, x_2, \ldots, x_{n-1})$. Indeed, the symbol $'$ means the coordinate up to $n-1$ and ω_n is the surface area of the unit sphere in $\mathbf{R}^n$.

We let Γ be the fundamental solution to the heat equation such that

$$\Gamma(x,t) = \begin{cases} \frac{1}{\sqrt{2\pi t}^n} e^{-\frac{|x|^2}{2t}}, & t > 0, \\ 0, & t \leqslant 0 \end{cases}$$

and H be the Newtonian potential of Γ such that

$$H(x,t) = \int_{\mathbf{R}^n} \Gamma(y,t)E(x-y)\,dy.$$

The Stokes fundamental matrix (F,γ) for $\mathbf{R}^n$, $n\geqslant 3$, is

$$F_{ij}(x,t)=\delta_{ij}\Gamma(x,t)+\frac{1}{(n-2)\omega_n}D_{x_i}D_{x_j}H(x,t),$$
$$\gamma_i=\frac{\delta(t)}{\omega_n}\frac{x_i}{|x|^n},$$

where $\delta(t)$ is the Dirac delta function and δ_{ij} is the Kronecker delta function.

Let (G,θ) be a Green's matrix of Stokes equations for the half-space $\mathbf{R}^n_+$, that is,

$$G_{ij}(x,y,t,s)=F_{ij}(x-y,t-s)-v^{(x,t)}(y,s),$$
$$\theta_i(x,y,t,s)=\gamma_i(x-y,t-s)-\lambda_i^{(x,t)}(y,s),$$

where $(v^{(x,t)},\lambda^{(x,t)})$ is a solution of (1.1) with boundary data $F(x-y,t-s)$ for fixed $(x,t)\in\mathbf{R}^n_+\times(0,T)$. Then, the Poisson kernel (K,π) for the half-space is defined by

$$\begin{aligned}K_{ij}\big(x'-y',x_n,t\big)&=\frac{\partial G_{ij}(x,y,t)}{\partial y_n}\bigg|_{y_n=0}-\delta_{jn}\theta_i(x,y,t)|_{y_n=0}\\&=-2\delta_{ij}D_{x_n}\Gamma\big(x'-y',x_n,t\big)+4L_{ij}\big(x'-y',x_n,t\big)-\delta_{jn}\delta(t)D_{x_i}E\big(x'-y',x_n\big),\\\pi_j(x'-y',x_n,t)&=-2\delta(t)D_{x_j}D_{x_n}E\big(x'-y',x_n\big)+4D_{x_n}D_{x_n}A\big(x'-y',x_n,t\big)\\&\quad+4D_tD_{x_j}A(x'-y',x_n,t),\end{aligned}$$

where we defined that

$$L_{ij}(x,t)=D_{x_j}\int_0^{x_n}\int_{\mathbf{R}^{n-1}}D_{z_n}\Gamma(z,t)D_{x_i}E(x-z)\,dz,$$
$$A(x,t)=\int_{\mathbf{R}^{n-1}}\Gamma\big(z',0,t\big)E\big(x'-z',x_n\big)\,dz'.$$

L_{ij} and A satisfy the estimates

$$\big|D_{x_n}^{l_0}D_{x'}^{k_0}D_t^{m_0}L_{ij}(x,t)\big|\leqslant\frac{c}{t^{m_0+\frac{1}{2}}(|x|^2+t)^{\frac{1}{2}n+\frac{1}{2}k_0}(x_n^2+t)^{\frac{1}{2}l_0}},\tag{2.1}$$
$$\big|D_x^jD_t^mA(x,t)\big|\leqslant\frac{c}{t^{m+\frac{1}{2}}(|x|^2+t)^{\frac{n-2+|j|}{2}}},\tag{2.2}$$

where $1\leqslant i\leqslant n$ and $1\leqslant j\leqslant n-1$ (see [2] and [8]). The estimates (2.1) of L_{ij} and the estimate of the fundamental solution to heat equation Γ imply that

$$\big|D_{x_n}^{l_0}D_{x'}^{k_0}D_t^{m_0}K_{ij}(x,t)\big|\leqslant\frac{c}{t^{m_0+\frac{1}{2}}(|x|^2+t)^{\frac{1}{2}n+\frac{1}{2}k_0}(x_n^2+t)^{\frac{1}{2}l_0}}.\tag{2.3}$$

The solution (u,p) of the Stokes system (1.1) in $\Omega=\mathbf{R}^n_+$ with boundary data g is expressed by

$$u^i(x,t)=\sum_{j=1}^{n}\int_0^t\int_{\mathbf{R}^{n-1}} K_{ij}(x'-y',x_n,t-s)g_j(y',s)\,dy'\,ds,$$

$$p(x,t)=\sum_{j=1}^{n}\int_0^t\int_{\mathbf{R}^{n-1}} \pi_j(x'-y',x_n,t-s)g_j(y',s)\,dy'\,ds. \tag{2.4}$$

We have relations among L and A such that

$$\sum_{1\leqslant i\leqslant n} L_{ii}=-2D_{x_n}\Gamma,\quad L_{in}=L_{ni}+B_{in}, \tag{2.5}$$

where $B_{in}(x,t)=\int_{\mathbf{R}^{n-1}} D_{x_n}\Gamma(x'-y',x_n,t)D_{y_i}E(y',0)\,dy'=\frac{\partial}{x_i}\kappa(x,t)$ if $i\neq n$ and $B_{nn}=0$.

For further computation, we introduce Lebesgue spaces and Sobolev spaces:

$$L^p(\Omega)=\left\{f;\int_\Omega |f|^p\,dx<\infty\right\},\qquad W^{1,p}(\Omega)=\left\{f;\int_\Omega |f|^p+|\nabla f|^p\,dx<\infty\right\},$$

$$L^p\big(0,T;W^{1,p}(\Omega)\big)=\left\{f;\int_0^T\int_\Omega |f|^p+|\nabla f|^p\,dx\,dt<\infty\right\}.$$

3. Maximum modulus estimate in the half-space

In this section, we consider the maximum modulus estimate in the half-space. The normal derivative $D_{x_n}\Gamma$ has uniformly bounded L^1 norm with respect to x_n on $\partial\mathbf{R}^n_+\times(0,T)$ (see (3.9)) and hence we focus only on the kernel function L_{ij}. By introducing a composite kernel κ we are able to identify the singular kernels. The following lemma is a key stone for the maximum modulus estimate.

Lemma 3.1. *Let $1\leqslant i\leqslant n$ and $1\leqslant j\leqslant n-1$. Then*

$$\int_0^\infty\int_{\mathbf{R}^{n-1}} \big|L_{ij}(x',x_n,t)\big|\,dx'\,dt<C, \tag{3.1}$$

where $C>0$ is independent of $x_n>0$ and hence it follows that

$$\int_0^\infty\int_{\mathbf{R}^{n-1}} \big|L_{in}(x',x_n,t)-B_{in}(x',x_n,t)\big|\,dx'\,dt<C, \tag{3.2}$$

where $C>0$ is independent of $x_n>0$.

The maximum modulus theorem for the half-space follows from Lemma 3.1.

Theorem 3.2. *Let $g=(g_1,g_2,\dots,g_n)\in L^\infty(\partial\mathbf{R}^n_+\times(0,T))$ and (u,p) be represented by (2.4). Then,*

$$\big\|u_T-\nabla\mathbf{S}_T(g_n)-\nabla\mathbf{T}_T(g_n)\big\|_{L^\infty(\mathbf{R}^n_+\times(0,T))}\leqslant C\|g\|_{L^\infty(\partial\mathbf{R}^n_+\times(0,T))} \tag{3.3}$$

for some $C > 0$. Furthermore, the normal component of the velocity u is bounded and there is also a constant C such that

$$\max_{(x,t)\in\mathbf{R}^n_+\times(0,T)}\left|u_n(x,t)\right| \leqslant C \max_{(y,t)\in\partial\mathbf{R}^n_+\times(0,T)}\left|g(y,t)\right|.$$

To show the L^1 boundedness of L_{ij}, we note that

$$L_{ij}(x,t) = 2^{\frac{3}{2}}\pi^{\frac{1}{2}}\int_0^{x_n} t^{-\frac{3}{2}} y_n e^{-\frac{|y_n|^2}{t}} \int_{\mathbf{R}^{n-1}} D_{y_j}\Gamma'\big(y',t\big)D_{y_i}E\big(x'-y',x_n-y_n\big)\,dy'\,dy_n, \tag{3.4}$$

where Γ' is Gaussian kernel in $\mathbf{R}^{n-1}$.

Lemma 3.3. *For $1 \leqslant j \leqslant n-1$, we get*

$$\left|\int_{|x'-y'|\leqslant\frac{1}{2}|x'|} D_{y_j}\Gamma'\big(y',t\big)D_{y_n}E\big(x'-y',y_n\big)\,dy'\right| \leqslant Ct^{-\frac{n-1}{2}}e^{-c\frac{|x'|^2}{t}}\left|x'\right|^{-1} + Ct^{-\frac{n}{2}-\frac{1}{2}}\left|x'\right|e^{-c\frac{|x'|^2}{t}},$$

$$\left|\int_{\frac{1}{2}|x'|\leqslant|y'|\leqslant 2|x'|,\,|x'-y'|\geqslant\frac{1}{2}|x'|} D_{y_j}\Gamma'\big(y',t\big)D_{y_n}E\big(x'-y',y_n\big)\,dy'\right| \leqslant Ct^{-\frac{n}{2}-\frac{1}{2}}\left|x'\right|e^{-c\frac{|x'|^2}{t}},$$

$$\left|\int_{|y'|\leqslant\frac{1}{2}|x'|} D_{y_j}\Gamma'\big(y',t\big)D_{y_n}E\big(x'-y',y_n\big)\,dy'\right| \leqslant C\left|x'\right|^{-n}\int_{|y'|\leqslant\frac{1}{2}\frac{|x'|}{\sqrt{t}}} \left|y'\right|^2 e^{-c|y'|^2}\,dy',$$

$$\left|\int_{|y'|\geqslant 2|x'|} D_{y_j}\Gamma'\big(y',t\big)D_{y_n}E\big(x'-y',y_n\big)\,dy'\right| \leqslant Ct^{-\frac{n}{2}}\int_{\frac{2|x'|}{\sqrt{t}}\leqslant|y'|} \left|y'\right|^{-n+2}e^{-c|y'|^2}\,dy', \tag{3.5}$$

where $C > 0$ and $c > 0$ are independent of x', y_n and t.

Proof. Using integration by parts, we get

$$\begin{aligned}
&\int_{|x'-y'|\leqslant\frac{1}{2}|x'|} D_{y_j}\Gamma'\big(y',t\big)D_{x_n}E\big(x'-y',y_n\big)\,dy' \\
&\quad= \int_{|x'-y'|=\frac{1}{2}|x'|} \frac{x_j-y_j}{|x'-y'|}\Gamma'\big(y',t\big)D_{y_n}E\big(x'-y',y_n\big)\,\sigma\big(dy'\big) \\
&\qquad- \int_{|x'-y'|\leqslant\frac{1}{2}|x'|} \Gamma'\big(y',t\big)D_{y_j}D_{y_n}E\big(x'-y',y_n\big)\,dy'.
\end{aligned} \tag{3.6}$$

For y' with $|x'-y'| = \frac{1}{2}|x'|$, we get $|\Gamma'(y',t)| \leqslant Ct^{-\frac{n-1}{2}}e^{-c\frac{|x'|^2}{t}}$ and $|D_{y_n}E(x'-y',y_n)| \leqslant C\frac{1}{(|x'|^2+y_n^2)^{\frac{n-1}{2}}}$. Here, the first term of the right hand side in (3.6) is dominated by

$$\int\limits_{|x'-y'|=\frac{1}{2}|x'|} \left|\Gamma'(y',t)\right|\left|D_{y_n}E(x'-y',y_n)\right|\sigma(dy') \leqslant Ct^{-\frac{n-1}{2}}e^{-c\frac{|x'|^2}{t}}\frac{|x'|^{n-2}}{(|x'|^2+y_n^2)^{\frac{n-1}{2}}}$$

$$\leqslant Ct^{-\frac{n-1}{2}}e^{-c\frac{|x'|^2}{t}}\left|x'\right|^{-1}. \tag{3.7}$$

Since $\int_{|x'-y'|\leqslant\frac{1}{2}|x'|} D_{y_j}D_{x_n}E(x'-y',y_n)\,dy'=0$, using the Mean-Value Theorem, the second term of the right hand side of (3.6) satisfies

$$\left|\int\limits_{|x'-y'|\leqslant\frac{1}{2}|x'|}\left(\Gamma'(y',t)-\Gamma'(x',t)\right)D_{y_j}D_{x_n}E(x'-y',y_n)\,dy'\right|$$

$$\leqslant C\left|x'\right|t^{-\frac{n-1}{2}-1}e^{-c\frac{|x'|^2}{t}}\int\limits_{|x'-y'|\leqslant\frac{1}{2}|x'|}\frac{|x'-y'|^2y_n}{(|x'-y'|^2+y_n^2)^{\frac{n}{2}+1}}\,dy'$$

$$\leqslant C\left|x'\right|t^{-\frac{n-1}{2}-1}e^{-c\frac{|x'|^2}{t}}\int\limits_{\mathbf{R}^{n-1}}\frac{1}{(|y'|^2+1)^{\frac{n}{2}}}\,dy'. \tag{3.8}$$

By (3.6)–(3.8), we obtain $(3.5)_1$.

For $(3.5)_2$, note that for y' satisfying $\frac{1}{2}|x'|\leqslant|y'|\leqslant 2|x'|$ we have $|x'-y'|\geqslant\frac{1}{2}|x'|$. We have $|D_{y_j}\Gamma'(y',t)|\leqslant Ct^{-\frac{n-1}{2}-1}|x'|e^{-c\frac{|x'|^2}{t}}$ and $D_{y_n}E(x'-y',y_n)\leqslant C|x'|^{-\frac{n-1}{2}}$, and thus we get

$$\left|\int\limits_{\frac{1}{2}|x'|\leqslant|y'|\leqslant2|x'|,\,|x'-y'|\geqslant\frac{1}{2}|x'|}D_{y_j}\Gamma'(y',t)D_{y_n}E(x'-y',y_n)\,dy'\right|\leqslant Ct^{-\frac{n}{2}-\frac{1}{2}}\left|x'\right|e^{-c\frac{|x'|^2}{t}}.$$

Hence, we obtain $(3.5)_2$.

Since $\int_{|y'|\leqslant\frac{1}{2}|x'|}D_{y_j}\Gamma'(y',t)\,dy=0$, using the Mean-Value Theorem, $(3.5)_3$ is proved by

$$\left|\int\limits_{|y'|\leqslant\frac{1}{2}|x'|}D_{y_j}\Gamma'(y',t)\left(D_{x_n}E(x'-y',y_n)-D_{x_n}E(x',y_n)\right)dy'\right|$$

$$\leqslant C\left(\left|x'\right|^2+y_n\right)^{-\frac{n}{2}}\int\limits_{|y'|\leqslant\frac{1}{2}|x'|}t^{-\frac{n+1}{2}}\left|y'\right|^2e^{-c\frac{|y'|^2}{t}}\,dy'$$

$$\leqslant C\left(\left|x'\right|^2+y_n\right)^{-\frac{n}{2}}\int\limits_{|y'|\leqslant\frac{1}{2}\frac{|x'|}{\sqrt{t}}}\left|y'\right|^2e^{-c|y|^2}\,dy'.$$

Finally, $(3.5)_4$ follows by

$$\left|\int\limits_{|y'|\geqslant2|x'|}D_{y_j}\Gamma'(y',t)D_{y_n}E(x'-y',y_n)\,dy'\right|\leqslant Ct^{-\frac{n+1}{2}}\int\limits_{2|x'|\leqslant|y'|}\left|y'\right|^{-n+2}e^{-c\frac{|y'|^2}{t}}\,dy'$$

$$=Ct^{-\frac{n}{2}}\int\limits_{\frac{2|x'|}{\sqrt{t}}\leqslant|y'|}\left|y'\right|^{-n+2}e^{-c|y'|^2}\,dy'. \qquad \square$$

Following a similar proof to Lemma 3.3, we get the following lemma.

Lemma 3.4. *For $1 \leqslant i, j \leqslant n-1$, we get*

$$\left| \int\limits_{|x'-y'| \leqslant \frac{1}{2}|x'|} D_{y_j} \Gamma'(y',t) D_{y_i} E(x'-y', y_n)\, dy' \right| \leqslant C t^{-\frac{n}{2}-\frac{1}{2}} |x'| e^{-c\frac{|x'|^2}{t}},$$

$$\left| \int\limits_{\frac{1}{2}|x'| \leqslant |y'| \leqslant 2|x'|, |x'-y'| \geqslant \frac{1}{2}|x'|} D_{y_j} \Gamma'(y',t) D_{y_i} E(x'-y', y_n)\, dy' \right| \leqslant C t^{-\frac{n}{2}-\frac{1}{2}} |x'| e^{-c\frac{|x'|^2}{t}},$$

$$\left| \int\limits_{|y| \leqslant \frac{1}{2}|x'|} D_{y_j} \Gamma'(y',t) D_{y_i} E(x'-y', y_n)\, dy' \right| \leqslant C |x'|^{-n} \int\limits_{|y'| \leqslant C\frac{1}{2}\frac{|x'|}{\sqrt{t}}} |y'|^2 e^{-c|y'|^2}\, dy',$$

$$\left| \int\limits_{|y'| \geqslant 2|x'|} D_{y_j} \Gamma'(y',t) D_{y_i} E(x'-y', y_n)\, dy' \right| \leqslant C t^{-\frac{n}{2}} \int\limits_{\frac{2|x'|}{\sqrt{t}} \leqslant |y'|} |y'|^{-n+2} e^{-c|y'|^2}\, dy'.$$

Proof of Lemma 3.1. Note that for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n-1$

$$\int\limits_0^T \int\limits_{\mathbf{R}^{n-1}} |K_{ij}(x', x_n, t)|\, dx'\, dt \leqslant \int\limits_0^T \int\limits_{\mathbf{R}^{n-1}} |D_{x_n} \Gamma(x', x_n, t)|\, dx'\, dt + \int\limits_0^T \int\limits_{\mathbf{R}^{n-1}} |L_{ij}(x', x_n, t)|\, dx'\, dt.$$

Here, using change of variables ($\frac{x_n^2}{t} = s$), we get

$$\begin{aligned}
\int\limits_0^T \int\limits_{\mathbf{R}^{n-1}} |D_{x_n} \Gamma(x', x_n, t)|\, dx'\, dt &= C \int\limits_0^T t^{-\frac{3}{2}} x_n e^{-\frac{x_n^2}{t}} \int\limits_{\mathbf{R}^{n-1}} t^{-\frac{n-1}{2}} e^{-\frac{|x'|^2}{2t}}\, dx'\, dt \\
&= C x_n \int\limits_0^T t^{-\frac{3}{2}} e^{-\frac{x_n^2}{t}} \int\limits_{\mathbf{R}^{n-1}} e^{-\frac{1}{2}|x'|^2}\, dx'\, dt \\
&= C x_n \int\limits_0^T \left(\frac{x_n^2}{s}\right)^{-\frac{3}{2}} x_n^2 s^{-2} e^{-\frac{1}{2}s}\, ds. \qquad (3.9)
\end{aligned}$$

Hence, to prove Lemma 3.1, it is sufficient to show $\int_0^T \int_{\mathbf{R}^{n-1}} |L_{ij}(x', x_n, t)|\, dx'\, dt < \infty$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n-1$.

By $(2.1)_1$, for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n-1$, we get

$$\begin{aligned}
\int\limits_0^{x_n^2} \int\limits_{\mathbf{R}^{n-1}} |L_{ij}(x', x_n, t)|\, dx'\, dt &\leqslant C \int\limits_0^{x_n^2} \int\limits_{\mathbf{R}^{n-1}} t^{-\frac{1}{2}} \left(|x'|^2 + x_n^2 + t\right)^{-\frac{n}{2}}\, dx'\, dt \\
&\leqslant C \int\limits_0^{x_n^2} t^{-\frac{1}{2}} \left(x_n^2 + t\right)^{-\frac{1}{2}}\, dt = C. \qquad (3.10)
\end{aligned}$$

To calculate $\int_{x_n^2}^T \int_{\mathbf{R}^{n-1}} |L_{ij}(x', x_n, t)|\,dx'\,dt$, we may assume $x_n^2 \leqslant T$. By the representation (3.4), and Lemma 3.3 and Lemma 3.4, we have

$$\begin{aligned}
&\int_{x_n^2}^{T} \int_{\mathbf{R}^{n-1}} \left|L_{ij}(x', x_n, t)\right| dx'\,dt \\
&\leqslant C \int_{x_n^2}^{T} \int_{\mathbf{R}^{n-1}} \int_0^{x_n} t^{-\frac{3}{2}} y_n e^{-\frac{y_n^2}{2t}} \left(t^{-\frac{n-1}{2}} e^{-c\frac{|x'|^2}{t}} \left|x'\right|^{-1} + t^{-\frac{n}{2}-\frac{1}{2}} \left|x'\right| e^{-c\frac{|x'|^2}{t}} \right. \\
&\quad \left. + \left|x'\right|^{-n} \int_{|y'| \leqslant \frac{1}{2}\frac{|x'|}{\sqrt{t}}} \left|y'\right|^2 e^{-c|y'|^2}\,dy' + t^{-\frac{n}{2}} \int_{\frac{2|x'|}{\sqrt{t}} \leqslant |y'|} \left|y'\right|^{-n+2} e^{-c|y'|^2}\,dy' \right) dy_n\,dx'\,dt \\
&= I + II + III + IV, \qquad (3.11)
\end{aligned}$$

where

$$\begin{aligned}
I &= \int_{x_n^2}^{T} \int_{\mathbf{R}^{n-1}} \int_0^{x_n} t^{-\frac{3}{2}} y_n e^{-\frac{y_n^2}{2t}} t^{-\frac{n-1}{2}} e^{-c\frac{|x'|^2}{t}} \left|x'\right|^{-1} dy_n\,dx'\,dt, \\
II &= \int_{x_n^2}^{T} \int_{\mathbf{R}^{n-1}} \int_0^{x_n} t^{-\frac{3}{2}} y_n e^{-\frac{y_n^2}{2t}} t^{-\frac{n}{2}-\frac{1}{2}} \left|x'\right| e^{-c\frac{|x'|^2}{t}}\,dy_n\,dx'\,dt, \\
III &= \int_{x_n^2}^{T} \int_{\mathbf{R}^{n-1}} \int_0^{x_n} t^{-\frac{3}{2}} y_n e^{-\frac{y_n^2}{2t}} \left|x'\right|^{-n} \int_{|y'| \leqslant \frac{1}{2}\frac{|x'|}{\sqrt{t}}} \left|y'\right|^2 e^{-c|y'|^2}\,dy'\,dy_n\,dx'\,dt, \\
IV &= \int_{x_n^2}^{T} \int_{\mathbf{R}^{n-1}} \int_0^{x_n} t^{-\frac{3}{2}} y_n e^{-\frac{y_n^2}{2t}} t^{-\frac{n}{2}} \int_{\frac{2|x'|}{\sqrt{t}} \leqslant |y'|} \left|y'\right|^{-n+2} e^{-|y'|^2}\,dy'\,dy_n\,dx'\,dt.
\end{aligned}$$

Using change of variables twice, we have

$$\begin{aligned}
I &= \int_{x_n^2}^{T} t^{-\frac{n}{2}-1} \int_{\mathbf{R}^{n-1}} e^{-c\frac{|x'|^2}{2t}} \left|x'\right|^{-1} t \int_0^{\frac{x_n}{\sqrt{t}}} y_n e^{-\frac{1}{2}y_n^2}\,dy_n\,dx'\,dt \\
&\leqslant C \int_{x_n^2}^{T} t^{-\frac{n}{2}-1} \int_{\mathbf{R}^{n-1}} e^{-c\frac{|x'|^2}{t}} \left|x'\right|^{-1} t \left(\frac{x_n}{\sqrt{t}}\right)^2 dx'\,dt \\
&= C \int_{x_n^2}^{T} x_n^2 t^{-2}\,dt \\
&= C \qquad (3.12)
\end{aligned}$$

and

$$\begin{aligned} II &= \int_{x_n^2}^{T} t^{-\frac{n}{2}-2} \int_{\mathbf{R}^{n-1}} e^{-c\frac{|x'|^2}{t}} |x| t \int_0^{\frac{x_n}{\sqrt{t}}} y_n e^{-\frac{1}{2}y_n^2}\, dy_n\, dx'\, dt \\ &\leqslant C \int_{x_n^2}^{T} t^{-\frac{n}{2}-2} \int_{\mathbf{R}^{n-1}} e^{-c\frac{|x'|^2}{t}} |x'| t \left(\frac{x_n}{\sqrt{t}}\right)^2 dx'\, dt \\ &= C \int_{x_n^2}^{T} x_n^2 t^{-2}\, dt \\ &= C. \end{aligned} \tag{3.13}$$

We divide III into two parts $III = III_1 + III_2$, where

$$III_1 = \int_{x_n^2}^{T} \int_{|x'|\leqslant\sqrt{t}} \int_0^{x_n} t^{-\frac{3}{2}} y_n e^{-\frac{y_n^2}{2t}} |x'|^{-n} \int_{|y'|\leqslant\frac{1}{2}\frac{|x'|}{\sqrt{t}}} |y'|^2 e^{-c|y'|^2}\, dy'\, dy_n\, dx'\, dt,$$

$$III_2 = \int_{x_n^2}^{T} \int_{|x'|\geqslant\sqrt{t}} \int_0^{x_n} t^{-\frac{3}{2}} y_n e^{-\frac{y_n^2}{2t}} |x'|^{-n} \int_{|y'|\leqslant\frac{1}{2}\frac{|x'|}{\sqrt{t}}} |y'|^2 e^{-c|y'|^2}\, dy'\, dy_n\, dx'\, dt.$$

Here,

$$\begin{aligned} III_1 &\leqslant C \int_{x_n^2}^{T} t^{-\frac{3}{2}} \int_{|x'|\leqslant\sqrt{t}} |x'|^{-n} \left(\frac{|x'|}{\sqrt{t}}\right)^{n+1} \int_0^{x_n} y_n e^{-\frac{y_n^2}{2t}}\, dy_n\, dx'\, dt \\ &= C \int_{x_n^2}^{T} x_n^2 t^{-\frac{n}{2}-2} \int_{|x'|\leqslant\sqrt{t}} |x'|\, dx'\, dt \\ &\leqslant C \int_{x_n^2}^{T} x_n^2 t^{-2}\, dt \\ &= C, \\ III_2 &\leqslant C \int_{x_n^2}^{T} t^{-\frac{3}{2}} \int_{|x'|\geqslant\sqrt{t}} |x'|^{-n} \int_0^{x_n} y_n e^{-\frac{y_n^2}{2t}}\, dy_n\, dx'\, dt \\ &\leqslant C \int_{x_n^2}^{T} t^{-\frac{3}{2}} x_n^2 \int_{|x'|\geqslant\sqrt{t}} |x'|^{-n}\, dx'\, dt \end{aligned}$$

$$\leqslant C\int_{x_n^2}^{T} t^{-2}x_n^2\,dt$$
$$= C.$$

Hence, we get

$$III < C. \tag{3.14}$$

Similarly, we divide IV into two parts

$$IV = \int_{x_n^2}^{T}\int_{|x'|\leqslant\sqrt{t}}\int_{0}^{x_n} + \int_{x_n^2}^{T}\int_{\sqrt{t}\leqslant|x'|}\int_{0}^{x_n}$$
$$= IV_1 + IV_2.$$

Here, with straightforward integrations

$$IV_1 \leqslant C\int_{x_n^2}^{T} t^{-\frac{n}{2}-\frac{3}{2}}\int_{|x'|\leqslant\sqrt{t}}\int_{0}^{x_n} y_n e^{-\frac{y_n^2}{2t}}\,dy_n\,dx'\,dt$$
$$\leqslant C\int_{x_n^2}^{T} t^{-\frac{n}{2}-\frac{3}{2}}x_n^2\int_{|x'|\leqslant\sqrt{t}} dx'\,dt$$
$$\leqslant C\int_{x_n^2}^{T} t^{-2}x_n^2\,dt$$
$$= C$$

and

$$IV_2 \leqslant C\int_{x_n^2}^{T} t^{-\frac{n}{2}-\frac{3}{2}}\int_{\sqrt{t}\leqslant|x'|}\int_{0}^{x_n} y_n e^{-\frac{y_n^2}{2t}}\int_{\frac{2|x'|}{\sqrt{t}}\leqslant|y'|}\left|y'\right|^{-n+2}e^{-c|y'|^2}\,dy'\,dy_n\,dx'\,dt$$
$$\leqslant C\int_{x_n^2}^{T} t^{-\frac{n}{2}-\frac{3}{2}}x_n^2\int_{\sqrt{t}\leqslant|x'|}\int_{\frac{2|x'|}{\sqrt{t}}\leqslant|y'|}\left|y'\right|^{-n+2}e^{-c|y'|^2}\,dy'\,dx'\,dt$$
$$= C\int_{x_n^2}^{T} t^{-2}x_n^2\int_{1\leqslant|x'|}\int_{|x'|\leqslant|y'|}\left|y'\right|^{-n+2}e^{-c|y'|^2}\,dy'\,dx'\,dt$$
$$\leqslant C.$$

Hence, we get

$$IV \leqslant C. \tag{3.15}$$

Therefore, from (3.10)–(3.15), we prove

$$\int_0^T \int_{\mathbf{R}^{n-1}} \left|L_{ij}(x', x_n, t)\right| dx'\, dt \leqslant C \tag{3.16}$$

for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n-1$, where C is independent of x_n. With (3.9), this implies (3.1).

By the second identity of (2.5) and (3.16), we prove (3.2) for the case $i \neq n$, and by the first identity of (2.5) and (3.16), we prove (3.2) for the case $i = n$. This ends the proof of Lemma 3.1. □

Proof of Theorem 3.2. We begin the proof of Theorem 3.2 by the representation (2.4) of u such that

$$\begin{aligned} u^i(x,t) &= \sum_{j=1}^{n} \int_0^t \int_{\mathbf{R}^{n-1}} K_{ij}(x'-y', x_n, t-s) g_j(y', s)\, dy'\, ds \\ &= \sum_{j=1}^{n-1} \int_0^t \int_{\mathbf{R}^{n-1}} K_{ij}(x'-y', x_n, t-s) g_j(y', s)\, dy'\, ds \\ &\quad + \int_0^t \int_{\mathbf{R}^{n-1}} K_{in}(x'-y', x_n, t-s) g_n(y', s)\, dy'\, ds \end{aligned}$$

and the last potential for g_n is written as

$$\begin{aligned} \int_0^t \int_{\mathbf{R}^{n-1}} K_{in}(x'-y', x_n, t-s) g_n(y', s)\, dy'\, ds &= -2\delta_{in} \int_0^t \int_{\mathbf{R}^{n-1}} D_{x_n} \Gamma(x'-y', x_n, t-s) g_n(y', s)\, dy'\, ds \\ &\quad + 4 \int_0^t \int_{\mathbf{R}^{n-1}} L_{in}(x'-y', x_n, t-s) g_n(y', s)\, dy'\, ds \\ &\quad - \frac{\partial}{\partial x_i} \int_0^t \int_{\mathbf{R}^{n-1}} E(x'-y', x_n) g_n(y', s)\, dy'\, ds. \end{aligned}$$

Since $L_{in} = L_{ni} + B_{in}$, we have

$$\begin{aligned} \int_0^t \int_{\mathbf{R}^{n-1}} L_{in}(x'-y', x_n, t-s) g_n(y', s)\, dy'\, ds &= \int_0^t \int_{\mathbf{R}^{n-1}} L_{ni}(x'-y', x_n, t-s) g_n(y', s)\, dy'\, ds \\ &\quad + \frac{\partial}{\partial x_i} \int_0^t \int_{\mathbf{R}^{n-1}} \kappa(x'-y', x_n, t-s) g_n(y', s)\, dy'\, ds \end{aligned}$$

for $1 \leqslant i \leqslant n-1$, where we defined the composite kernel function $\kappa(x,t)$ on $\mathbf{R}^n_+ \times (0,T)$ by

$$\kappa(x,t) = \int_{\mathbf{R}^{n-1}} \frac{\partial}{\partial x_n} \Gamma\big(x' - z', x_n, t\big) E\big(z', 0\big)\, dz'.$$

Define the surface potential $\mathbf{T}(g_n)$ by

$$\mathbf{T}(g_n)(x,t) = 4 \int_0^t \int_{\mathbf{R}^{n-1}} \kappa\big(x' - y', x_n, t-s\big) g_n\big(y', s\big)\, dy'\, ds. \tag{3.17}$$

Moreover, we have that

$$\frac{\partial}{\partial x_i} \int_{\mathbf{R}^{n-1}} E\big(x' - y', x_n\big) g_n\big(y', s\big)\, dy' = \frac{\partial}{\partial x_i} \mathbf{S}(g_n).$$

Therefore we conclude that the tangential part, which is associated with L_{ij}, satisfies

$$\big|u_T(x,t) - \nabla \mathbf{S}_T(g_n)(x,t) - \nabla \mathbf{T}_T(g_n)(x,t)\big| \leqslant C \|g\|_{L^\infty(\mathbf{R}^{n-1} \times (0,T))}$$

for all $(x,t) \in \mathbf{R}^n_+ \times (0,T)$.

The normal velocity u_n behaves even better. First, we know that $\frac{\partial}{\partial x_n} \mathbf{S}(g_n)$ is the Poisson kernel expression of the solution for the Laplace equation in the half-space and satisfies the maximum principle. In the case $i = n$, we have a relation from (2.5)

$$L_{nn} = - \sum_{1 \leqslant i \leqslant n-1} L_{ii} - 2 D_{x_n} \Gamma$$

which has a bounded L^1 norm on the lateral surface. This concludes the maximum modulus estimate of u_n. □

4. Maximum modulus estimate in C^2 domain

We denote Green's matrix for the domain Ω by $(G^\Omega, \theta^\Omega)$ and for a given point $x \in \Omega$ we let $\bar{x} \in \partial\Omega$ satisfy $|x - \bar{x}| = dist(x, \partial\Omega)$. The interior L^∞ bound estimate can be shown by the layer potential method in [6] and we consider separately the case that the generic point x is close enough to $\partial\Omega$.

Indeed, to see the interior boundedness, we need to show the boundedness of the double layer potential in $L^\infty(0,T; L^2(\partial\Omega))$. Since the boundary data is bounded, we can represent the solution by the double layer potential in [6] from L^2 theory such that

$$\begin{aligned} u_i(x,t) &= \int_0^t \int_{\partial\Omega} \frac{\partial F_{ij}}{\partial N(y)}(x-y, t-s) h_j(y,s)\, d\sigma(y)\, ds - \int_{\partial\Omega} \frac{y_i - x_i}{\omega_n |y-x|^n} h(y,t) \cdot N(y)\, d\sigma(y) \\ &= (\mathbf{K}h)_i(x,t) \end{aligned} \tag{4.1}$$

and

$$g = -\frac{1}{2} h + \mathbf{K}h = \left(-\frac{1}{2}\mathbf{I} + \mathbf{K}\right) h \tag{4.2}$$

in the sense of $L^2(\partial\Omega\times(0,T))$ for an $h\in L^2(\partial\Omega\times(0,T))$ (see Theorem 2.3.6 and Theorem 5.1.2 in [6]). Furthermore $-\frac{1}{2}\mathbf{I}+\mathbf{K}$ is invertible on $L^2_\sigma(\partial\Omega\times(0,T))$, where the subscript σ means solenoidal. From the representation, we have a continuity lemma in time of the density function h in (4.1).

Lemma 4.1. *The inverse of the double layer potential $-\frac{1}{2}\mathbf{I}+\mathbf{K}$ is bounded in time as an operator of $L^2(\partial\Omega)$ and there is a constant $\epsilon>0$ such that for $0\leqslant t_1<t_2\leqslant T$,*

$$\begin{aligned}\big\|h(\cdot,t_2)\big\|_{L^2(\partial\Omega)}&\leqslant C_\epsilon\big\|g(\cdot,t_2)-g(\cdot,t_1)\big\|_{L^2(\partial\Omega)}+C_\epsilon(t_2-t_1)^{\frac{1}{2}-\epsilon}\|h\|_{L^\infty(t_1,t_2;L^2(\partial\Omega))}\\&\quad+C_\epsilon\|h\|_{L^\infty(0,t_1;L^2(\partial\Omega))}.\end{aligned}$$

By an iteration there is C such that

$$\left\|\left(-\frac{1}{2}\mathbf{I}+\mathbf{K}\right)^{-1}g\right\|_{L^\infty(0,T;L^2(\partial\Omega))}\leqslant C\|g\|_{L^\infty(0,T;L^2(\partial\Omega))}.$$

Proof. We assume the boundary data $g\in L^\infty(0,T;L^2(\partial\Omega))$ and after arranging the singular integrals in the double layer potential expression we have

$$\begin{aligned}g_i(x,t_2)-g_i(x,t_1)&=-\frac{1}{2}\big(h_i(x,t_2)-h_i(x,t_1)\big)\\&\quad-\int_{\partial\Omega}\frac{y_i-x_i}{\omega_n|y-x|^n}\big(h(y,t_1)-h(y,t_2)\big)\cdot N(y)\,d\sigma(y)\\&\quad+\int_{t_1}^{t_2}\int_{\partial\Omega}\frac{\partial F_{ij}}{\partial N(y)}(x-y,t_2-s)h_j(y,s)\,d\sigma(y)\,ds\\&\quad+\int_{0}^{t_1}\int_{\partial\Omega}\left(\frac{\partial F_{ij}}{\partial N(y)}(x-y,t_2-s)-\frac{\partial F_{ij}}{\partial N(y)}(x-y,t_1-s)\right)h_j(y,s)\,d\sigma(y)\,ds\\&=\left(-\frac{1}{2}\mathbf{I}+\mathbf{H}\right)\big(h(\cdot,t_2)-h(\cdot,t_1)\big)+\mathbf{E}_1h+\mathbf{E}_2h\end{aligned}$$

for almost all $0\leqslant t_1<t_2\leqslant T$ and $x\in\partial\Omega$.

We claim $-\frac{1}{2}\mathbf{I}+\mathbf{H}:L^2(\partial\Omega)\to L^2(\partial\Omega)$ is invertible and

$$\left\|\left(-\frac{1}{2}\mathbf{I}+\mathbf{H}\right)^{-1}e\right\|_{L^2(\partial\Omega)}\leqslant C\|e\|_{L^2(\partial\Omega)}$$

for a constant C. First of all, if we set $e=(-\frac{1}{2}\mathbf{I}+\mathbf{H})f$ and consider the normal components, then we have

$$e_N=\left(-\frac{1}{2}\mathbf{I}+N\cdot\mathbf{H}\right)f_N,$$

where $N \cdot \mathbf{H}$ is the standard double layer potential operator of Laplace equation and $-\frac{1}{2}\mathbf{I} + N \cdot \mathbf{H}$ is invertible. So given vector valued function $e \in L^2(\partial\Omega)$, there is a scalar valued function $w \in L^2(\Omega)$ satisfying

$$e_N = \left(-\frac{1}{2}\mathbf{I} + N \cdot \mathbf{H}\right) w$$

with $\|w\|_{L^2(\partial\Omega)} \leqslant C\|e\|_{L^2(\partial\Omega)}$. Here w is the normal component of f and the tangential component v of f is obtained by

$$v = -2(e - e_N N) - 2\big(\mathbf{H}w - (N \cdot \mathbf{H})wN\big).$$

Therefore we get

$$f = v + wN$$

and f satisfies

$$\|f\|_{L^2(\partial\Omega)} \leqslant C\|e\|_{L^2(\partial\Omega)}.$$

It remains the estimate $\mathbf{E}_1 h$ and $\mathbf{E}_2 h$. Since Ω is C^2 domain, in the case of Gaussian kernel, there is C such that for all $(x, y, t) \in \partial\Omega \times \partial\Omega \times (0, T)$

$$\left|\frac{\partial}{\partial N(y)} F(x - y, t)\right| \leqslant C\big|\nabla_y F(x - y, t)\big||x - y|.$$

Applying Lemma 3.3 and Lemma 3.4, we get

$$\begin{aligned}\big|\nabla_y F(x - y, t)\big| &\leqslant C\big|\nabla_y \Gamma(x - y, t)\big| + C|x - y|^3 t^{-\frac{n}{2}-2} e^{-c\frac{|x|^2}{t}} \\ &\quad + C|x - y|^{-n+1} t^{-1} \int\limits_{|z| \leqslant \frac{|x-y|}{2\sqrt{t}}} |z|^2 e^{-c|z|^2}\, dz + Ct^{-\frac{n}{2}-\frac{1}{2}} \int\limits_{|z| \geqslant \frac{|x-y|}{2\sqrt{t}}} |z|^{-n+3} e^{-c|z|^2}\, dz.\end{aligned}$$

Therefore we get from Minkowski inequality and Young's convolution inequality

$$\begin{aligned}&\left(\int\limits_{\partial\Omega}\left(\int\limits_{t_1}^{t_2}\int\limits_{\partial\Omega} |x - y|\big|\nabla_y \Gamma(x - y, t_2 - s)\big|\big|h(y, s)\big|\, d\sigma(y)\, ds\right)^2 d\sigma(x)\right)^{\frac{1}{2}} \\ &\leqslant C\left(\int\limits_{\partial\Omega}\left|\int\limits_{t_1}^{t_2} \frac{1}{\sqrt{t_2 - s}} \frac{1}{\sqrt{t_2 - s}^{n-1}} \int\limits_{\partial\Omega} \frac{|x - y|^2}{t_2 - s} e^{-\frac{|x-y|^2}{2(t_2-s)}} \big|h(y, s)\big|\, d\sigma(y)\, ds\right|^2 d\sigma(x)\right)^{\frac{1}{2}} \\ &\leqslant C\int\limits_{t_1}^{t_2} \frac{1}{\sqrt{t_2 - s}}\left(\int\limits_{\partial\Omega}\left|\frac{1}{\sqrt{t_2 - s}^{n-1}} \int\limits_{\partial\Omega} \frac{|x - y|^2}{t_2 - s} e^{-\frac{|x-y|^2}{2(t_2-s)}} \big|h(y, s)\big|\, d\sigma(y)\right|^2 d\sigma(x)\right)^{\frac{1}{2}} ds \\ &\leqslant C\int\limits_{t_1}^{t_2} \frac{1}{\sqrt{t_2 - s}} \big\|h(\cdot, s)\big\|_{L^2(\partial\Omega)}\, ds \\ &\leqslant C\sqrt{t_2 - t_1}\|h\|_{L^\infty(t_1, t_2; L^2(\partial\Omega))}.\end{aligned}$$

Using a similar calculation, we get

$$\left\|\mathbf{E}_2 h(\cdot,t_1)\right\|_{L^2(\partial\Omega)} \leqslant C_\epsilon (t_2-t_1)^{\frac{1}{2}-\epsilon} \|h\|_{L^\infty(t_1,t_2;L^2(\partial\Omega))}.$$

By the same token, assuming $\|h\|_{L^\infty(0,t_1;L^2(\Omega))}$ is bounded, we have that

$$\left\|\mathbf{E}_2 h(\cdot,t_1)\right\|_{L^2(\partial\Omega)} \leqslant C\|h\|_{L^\infty(0,t_1;L^2(\partial\Omega))}. \qquad \square$$

We let the generic point x be away from the boundary, say $dist(x,\partial\Omega)=r_0>0$. Since the kernel of the double layer is bounded by $\frac{C}{r_0^{n-1-\epsilon}}$ for each $\epsilon>0$ and the density function h of g for the double layer potential is bounded in $L^\infty(0,T;L^2(\partial\Omega))$, the interior L^∞ estimate follows.

Corollary 4.2. *Suppose the boundary data $g\in L^\infty(0,T;L^2(\partial\Omega))$. If $dist(x,\partial\Omega)\geqslant r_0>0$, $\epsilon>0$ and $t<T$, then there is C such that*

$$\left|u(x,t)\right| \leqslant \frac{C}{r_0^{n-1-\epsilon}} \|g\|_{L^\infty(0,T;L^2(\partial\Omega))}.$$

Now we start the boundary estimate. Since Stokes equations is translation and rotation invariant, we assume that $\bar{x}=0$ and $x=(0,x_n)$, $x_n>0$. If x is close enough to $\partial\Omega$, there is a ball $B_r(0)$ centered at origin and C^2 function $\Phi:\mathbf{R}^{n-1}\to\mathbf{R}$ such that $\Omega\cap B_r(0)=\{x_n>\Phi(x')\}\cap B_r(0)$. Furthermore, Φ satisfies

$$\left|\Phi(x')\right|\leqslant C\left|x'\right|^2, \qquad \left|\nabla'\Phi(x')\right|\leqslant C\left|x'\right|, \qquad \left|\nabla'\nabla'\Phi(x')\right|\leqslant C \tag{4.3}$$

for $x'\in B_r'(0)$ and the outward unit normal vector $N(x',\Phi(x'))$ at $(x',\Phi(x'))\in\partial\Omega\cap B_r(0)$ is

$$N\left(x',\Phi(x')\right)=\frac{1}{\sqrt{1+|\nabla'\Phi(x')|^2}}\left(\nabla'\Phi(x'),-1\right).$$

We define a transform $\mu:\Omega\cap B_r(0)\to\mathbf{R}^n_+$ such that

$$\mu(y)=\mu\left(y',y_n\right)=\left(y',y_n-\Phi\left(y'\right)\right)$$

and note that $\mu^{-1}(y',y_n)=(y',y_n+\Phi(y'))$. Since our generic point x is $(0,x_n)$, we have $\mu(x)=x$. Hence Green's matrix G on the half-space can be transformed to a function μG on Ω such that

$$\mu G(x,y,t)=G\left(\mu(x),\mu(y),t\right)=G\left(x',x_n-\Phi\left(x'\right),y',y_n-\Phi\left(y'\right),t\right)$$

and satisfies the zero boundary condition

$$\mu G\left(x,y',\Phi\left(y'\right),t\right)=0.$$

Moreover, the transformed Green's matrix $(\mu G,\mu\theta)$ satisfies perturbed Stokes equations in $\Omega\times(0,T)$

$$\begin{aligned}\frac{\partial}{\partial t}(\mu G)_{ij}-\Delta_y(\mu G)_{ij}+\frac{\partial}{\partial y_j}(\mu\theta)_i &= \delta_{ij}\delta(x-y)\delta(t)+D_{y_n}(\mu G)_{ij}\Delta'\Phi+2D_{y_k}D_{y_n}(\mu G)_{ij}D_{y_k}\Phi\\ &\quad -D_{y_n}D_{y_n}(\mu G)_{ij}D_{y_k}\Phi D_{y_k}\Phi-D_{y_n}(\mu\theta)_iD_{y_j}\Phi\\ &=\delta_{ij}\delta(x-y)\delta(t)+R(x,y,t)\end{aligned}$$

and the solenoidal condition

$$D_{y_j}(\mu G)_{ij} = -D_{y_n}(\mu G)_{ij} D_{y_j}\Phi = S_i(x, y, t).$$

Therefore, if we let the perturbation $(J, \eta) = (G^{\Omega} - \mu G, \theta^{\Omega} - \mu\theta)$, then (J, η) satisfies the perturbation equations:

$$\frac{\partial}{\partial t} J_{ij}(x, y, t) - \Delta_y J_{ij}(x, y, t) + \frac{\partial}{\partial y_j}\eta_j(x, y, t) = R_{ij}(x, y, t), \tag{4.4}$$

$$D_{y_j} J_{ij}(x, y, t) = S_i(x, y, t), \tag{4.5}$$

where R is

$$\begin{aligned} R_{ij} &= -D_{y_n}(\mu G)_{ij}\Delta'_y\Phi - 2D_{y_k}D_{y_n}(\mu G)_{ij}D_{y_k}\Phi \\ &\quad + D_{y_n}D_{y_n}(\mu G)_{ij}D_{y_k}\Phi D_{y_k}\Phi + D_{y_n}(\mu\theta)_i D_{y_j}\Phi \\ &= I + II + III + IV. \end{aligned}$$

We have already discussed the boundedness of velocity u in the interior by double layer potential in L^2 theory, we begin to prove the boundedness near the boundary.

The plan to get L^1 bound of the perturbation J of Poisson kernel on $\partial\Omega \times (0, T)$ relies on the $L^p(0, T; W^{2,p}(\Omega))$ estimate and the trace theorem for it. Recall that the Poisson kernel is a derivative of Green's matrix and that is the reason that we need $L^p(0, T; W^{2,p}(\Omega))$ Sobolev type estimate. Therefore, we need to estimate the L^p norm of R in (4.4), $L^p(0, T; W^{1,p}(\Omega))$ norm of S and $L^p(0, T; W^{-1,p}(\Omega))$ norm of S_t in $(B_1 \cap \Omega) \times (0, T)$ in (4.5), where $W^{-1,p}(\Omega)$ is the dual space of $W^{1,p}(\Omega)$.

Since Green's matrix G is associated with the Gaussian kernel and the composite kernel H, we estimate their derivatives first. We have

$$\begin{aligned} \left|D_{y_n}\Gamma(x - y, t)\right| &\leqslant \frac{C}{\sqrt{t}^n}\frac{|y - x|}{t}e^{-\frac{|y'|^2 + |y_n - x_n|^2}{2t}}, \\ \left|D_{y_n}\Gamma\big(x - y^*, t\big)\right| &\leqslant \frac{C}{\sqrt{t}^n}\frac{|y^* - x|}{t}e^{-\frac{|y'|^2 + |y_n + x_n|^2}{2t}}, \\ \left|D_{y_k}D_{y_n}\Gamma(x - y, t)\right| &\leqslant \frac{C}{\sqrt{t}^n}\frac{|y - x|^2}{t^2}e^{-\frac{|y'|^2 + |y_n - x_n|^2}{2t}}, \\ \left|D_{y_k}D_{y_n}\Gamma\big(x - y^*, t\big)\right| &\leqslant \frac{C}{\sqrt{t}^n}\frac{|y^* - x|^2}{t^2}e^{-\frac{|y'|^2 + |y_n + x_n|^2}{2t}}. \end{aligned}$$

Since $|D_{y_k}\Phi(y')| \leqslant C|y'|$, $|\Delta'\Phi(y')| \leqslant C$ and $x' = 0$, we get

$$\left|D_{y_n}\Gamma(x - y, t)\Delta'\Phi\big(y'\big)\right| \leqslant \frac{C}{\sqrt{t}^{n+1}}\frac{|y - x|}{\sqrt{t}}e^{-\frac{|y'|^2 + |y_n - x_n|^2}{2t}} \in L^p\big((\Omega \cap B_r) \times (0, T)\big),$$

$$\left|D_{y_k}D_{y_n}\Gamma(x - y, t)\nabla'\Phi\big(y'\big)\right| \leqslant \frac{C}{\sqrt{t}^{n+1}}\frac{|y - x|^3}{\sqrt{t}^3}e^{-\frac{|y'|^2 + |y_n - x_n|^2}{2t}} \in L^p\big((\Omega \cap B_r) \times (0, T)\big)$$

as a function of y for all $p \in [1, \frac{n+2}{n+1})$. In the same way, we have

$$D_{y_n}\Gamma\big(x - y^*, t\big)\Delta'\Phi\big(y'\big), \quad D_{y_k}D_{y_n}\Gamma(x - y, t)\nabla'\Phi\big(y'\big) \in L^p\big((\Omega \cap B_r) \times (0, T)\big)$$

as a function of y for all $p \in [1, \frac{n+2}{n+1})$.

Applying (2.1), we have

$$\left| D_{y_n} D_{x_j} \int_0^{x_n} \int_{\mathbf{R}^{n-1}} D_{x_i} E(x-z) \Gamma\big(z-y^*,t\big)\, dz \right| \leqslant \frac{C}{t^{\frac{1}{2}} (|x'-y'|^2 + |y_n+x_n|^2 + t)^{\frac{1}{2}n}},$$

$$\left| D_{y_k} D_{y_n} D_{x_j} \int_0^{x_n} \int_{\mathbf{R}^{n-1}} D_{x_i} E(x-z) \Gamma\big(z-y^*,t\big)\, dz \right| \leqslant \frac{C}{t^{\frac{1}{2}} (|x'-y'|^2 + |y_n+x_n|^2 + t)^{\frac{1}{2}(n+1)}}.$$

Hence, we have for $p \in [1, \frac{n+2}{n+1})$

$$\begin{aligned}
&\left| D_{y_n} D_{x_j} \int_0^{x_n} \int_{\mathbf{R}^{n-1}} D_{x_i} E(x-z) \Gamma\big(z-y^*,t\big)\, dz\, \Delta' \Phi\big(y'\big) \right| \\
&\quad \leqslant \frac{C}{\sqrt{t}^{n+1}} \frac{1}{\sqrt{(|\frac{|y^*-x|^2}{t} + 1)}^n} \in L^p\big((\Omega \cap B_r) \times (0,T)\big), \\
&\left| D_{y_k} D_{y_n} D_{x_j} \int_0^{x_n} \int_{\mathbf{R}^{n-1}} D_{x_i} E(x-z) \Gamma\big(z-y^*,t\big)\, dz\, \nabla' \Phi\big(y'\big) \right| \\
&\quad \leqslant \frac{C}{\sqrt{t}^{n+1}} \frac{1}{\sqrt{(|\frac{|y^*-x|^2}{t} + 1)}^n} \in L^p\big((\Omega \cap B_r) \times (0,T)\big).
\end{aligned}$$

Although there is a transformation μ of domain, these estimates imply that *I*, *II* and *III* are in $L^p((\Omega \cap B_r) \times (0,T))$ as a function of y for all $p \in [1, \frac{n+2}{n+1})$.

It remains to get L^p estimate of the pressure kernel θ. For each fixed time t, we have

$$\begin{aligned}
|IV| &\leqslant C\big|y'\big| \left| \int_{\mathbf{R}^{n-1}} D_{x_i} E\big(x'-z', x_n\big) \frac{y_n}{t} \frac{1}{\sqrt{t}^n} e^{-\frac{|z'-y'|^2+y_n^2}{2t}}\, dz' \right| \\
&\quad + C\big|y'\big| \left| D_{y_n} D_{y_n} \int_{\mathbf{R}^{n-1}} \frac{1}{\sqrt{|x'-z'|^2 + x_n^2}^{n-2}} \frac{1}{\sqrt{t}^n} e^{-\frac{|z'-y'|^2+y_n^2}{2t}}\, dz' \right|. \qquad (4.6)
\end{aligned}$$

The first term on the right is L^p for $p \in [1, \frac{n+2}{n+1})$ by Young's convolution inequality since the kernel $\frac{x_n}{\sqrt{|z'|^2+x_n^2}^n}$ has bounded $L^1(\mathbf{R}^{n-1})$ estimate as a function of z' independent of x_n. For the second term, we recall the following proposition by Solonnikov (Proposition 2.3 in [9]):

Lemma 4.3. *Let $M(x,t)$ be a function defined for $x \in \mathbf{R}^n_+$ and $t > 0$ and having the properties*

$$M\big(\lambda x, \lambda^2 t\big) = \lambda^m M(x,t), \quad \lambda > 0,$$

$$\big|D_x^k D_t^s M(x,t)\big| \leqslant C t^{\frac{m-k-2s}{2}} \exp\left(-\frac{|x|^2}{2t}\right).$$

Then the integral

$$J(x, y_n, t) = \int_{\mathbf{R}^{n-1}} E(y) M\big(x' - y', x_n, t\big)\, dy'$$

satisfies the conditions

$$J\big(\lambda x, \lambda y_n, \lambda^2 t\big) = \lambda^{m+1} J(x, y_n, t),$$

$$\big|D_x^k D_{y_n}^l D_t^s J(x, y_n, t)\big| \leqslant C t^{\frac{m+n-1-2s-k_n}{2}} \big(\big|x'\big|^2 + (x_n + y_n)^2 + t\big)^{-\frac{|k'|+l+n-2}{2}} e^{-\frac{x_n^2}{2t}},$$

where $k = (k_1, \ldots, k_n)$ *and* $|k'| = k_1 + \cdots + k_{n-1}$.

See Proposition 2.3 in [9].

So, we find the second term of (4.6) without the transformation μ is bounded by

$$\begin{aligned} &C \frac{|y'|}{\sqrt{t}} \left(\left| \frac{y'}{\sqrt{t}} \right|^2 + \left(\frac{x_n + y_n}{\sqrt{t}} \right)^2 + 1 \right)^{-\frac{n}{2}} \frac{1}{\sqrt{t}^n} e^{-\frac{y_n^2}{2t}} \\ &\leqslant C \left(\frac{r}{\sqrt{t}} + 1 \right) \frac{|y'|}{\sqrt{t}} \left(\left| \frac{y'}{t} \right|^2 + \left(\frac{x_n + y_n}{t} \right)^2 + 1 \right)^{-\frac{n-1}{2}} \frac{1}{\sqrt{t}^n} e^{-\frac{y_n^2}{2t}} \end{aligned}$$

which is in $L^p((\Omega \cap B_r) \times (0, T))$ for $p \in (1, \frac{n+2}{n+1})$.

This concludes that *IV* is in L^p and R in (4.4) is in L^p for all $p \in (1, \frac{n+2}{n+1})$ after adjustment of the domain transformation μ.

To get $L^p(0, T; W^{1,p}(\Omega \cap B_r))$ bound of S in (4.5) we follow a similar program to R. Indeed, we have

$$\nabla_y S = -\nabla_y D_{y_n} (\mu G)_{ij} D_{y_j} \Phi\big(y'\big) - D_{y_n} (\mu G)_{ij} \nabla_y D_{y_j} \Phi\big(y'\big).$$

The terms in the right hand side have already been considered in the estimates of *I*, *II* and *III* of R except $D_{y_n} D_{y_n} (\mu G)_{ij} D_{y_j} \Phi(y')$. But, $D_{y_n} D_{y_n} (\mu G)_{ij} = \sum_{1 \leqslant j \leqslant n-1} D_{y_k} D_{y_n} (\mu G_{ij})$ and hence $D_{y_n} D_{y_n} (\mu G)_{ij} D_{y_j} \Phi(y')$ has the form of *II*. Therefore, we get

$$\|S\|_{L^p(0,T;W^{1,p}(\Omega \cap B_r))} < C \quad \text{independent of } x.$$

It remains to find $L^p(0, T; W^{-1,p}(\Omega \cap B_r))$ estimate of $D_t S$. Since S is defined as

$$S_i(x, y, t) = -D_{y_n} (\mu G)_{ij} D_{y_j} \Phi\big(y'\big)$$

and Φ is independent of y_n, $L^p(0, T; W^{-1,p}(\Omega \cap B_r))$ norm of S_t is bounded by

$$C \int_0^T \int_{\Omega \cap B_r} \big| D_t (\mu G) \nabla' \Phi\big(y'\big) \big|^p \, dy \, dt$$

for a constant C. By disregarding Φ, we have

$$D_t G_{ij} = \delta_{ij}\big(D_t\Gamma(x-y,t) - D_t\Gamma\big(x-y^*,t\big)\big) + 4(1-\delta_{jn})D_{x_j}\int_0^{x_n}\int_{\mathbf{R}^{n-1}} D_{x_i}E(x-z)D_t\Gamma\big(z-y^*,t\big)\,dz$$

and

$$|D_t G_{ij}| \leqslant C\bigg(\frac{1}{\sqrt{t}^{n+2}} + \frac{|x-y|^2}{\sqrt{t}^{n+4}}\bigg)e^{-\frac{|x-y|^2}{2t}} + C\bigg(\frac{1}{\sqrt{t}^{n+2}} + \frac{|x-y^*|^2}{\sqrt{t}^{n+4}}\bigg)e^{-\frac{|x-y^*|^2}{2t}} + C\bigg|D_{x_j}\int_0^{x_n}\int_{\mathbf{R}^{n-1}} D_{x_i}E(x-z)D_t\Gamma\big(z-y^*,t\big)\,dz\bigg|$$

and Proposition 2.5 in [9], we have

$$\bigg|D_{x_j}\int_0^{x_n}\int_{\mathbf{R}^{n-1}} D_{x_i}E(x-z)D_t\Gamma\big(z-y^*,t\big)\,dz\,\nabla'\Phi\big(y'\big)\bigg| \leqslant \frac{C|y'|}{t(|x-y^*|^2+t)^{\frac{n}{2}}}e^{-\frac{y_n^2}{2t}}.$$

This implies that $D_t G_{ij}\nabla'\Phi(y') \in L^p((\Omega\cap B_r)\times(0,T))$ for all $p \in [1, \frac{n+2}{n+1})$.

Since the estimates of R and S hold only in a small ball near boundary, we need a localization. For the localization, we take a cut off function ϕ such that $\phi = 1$ in B_r and $\phi = 0$ in the complement of B_{2r} and we consider $(\phi J, \phi\eta)$ as a solution to the inhomogeneous Stokes equations. We delete the generic point x in the various expressions. Therefore by Theorem 3.1 in [9], we obtain the following lemma for the perturbation (J,η).

Lemma 4.4. *There is a constant C depending on r and Ω such that*

$$\|J\|_{L^p(0,T;W^{2,p}(\Omega\cap B_r))} + \|\eta\|_{L^p(0,T;W^{1,p}(\Omega\cap B_r))} \leqslant C\big(1 + \big\|G^\Omega\big\|_{L^p(0,T;W^{1,p}(\Omega\cap(B_{2r}\setminus B_r)))} + \big\|\theta^\Omega\big\|_{L^p(\Omega\cap(B_{2r}\setminus B_r)\times(0,T))}\big)$$

for all $p \in (1, \frac{n+2}{n+1})$.

By the trace theorem in $W^{1,p}(\Omega\cap B_r)$, the following lemma also holds.

Lemma 4.5. *There is a constant C depending on r, Ω such that*

$$\|\nabla J\|_{L^p(0,T;W^{1-\frac{1}{p},p}(\partial\Omega\cap B_r))} + \|\eta\|_{L^p(0,T;W^{1-\frac{1}{p},p}(\partial\Omega\cap B_r))} \leqslant C\big(1 + \big\|G^\Omega\big\|_{L^p(0,T;W^{1,p}(\Omega\cap(B_{2r}\setminus B_r)))} + \big\|\theta^\Omega\big\|_{L^p(\Omega\cap(B_{2r}\setminus B_r)\times(0,T))}\big)$$

for all $p \in (1, \frac{n+2}{n+1})$.

The generic point x is in B_r and hence Green's matrix $(G^\Omega, \theta^\Omega)$ has no singularity in the complement of B_r as a function of (y,t). Therefore we have that for all $p \in [1,\infty]$

$$\big\|G^\Omega\big\|_{L^p(0,T;W^{1,p}(\Omega\cap(B_{2r}\setminus B_r)))} + \big\|\theta^\Omega\big\|_{L^p(\Omega\cap(B_{2r}\setminus B_r)\times(0,T))} \leqslant C$$

for a constant C depending only on p, r and Ω.

Now we prove our main theorem. The Poisson kernel $K^\Omega(x,y,t)$ satisfies

$$K^\Omega(x,y,t)=\frac{\partial}{\partial N(y)}G^\Omega(x,y,t)-\theta^\Omega(x,y,t)N(y),\quad \text{for all } (x,y,t)\in\Omega\times\partial\Omega\times(0,T).$$

We have that

$$G^\Omega=\mu G+J,\qquad \theta^\Omega=\mu\theta+\eta$$

and from Lemma 4.3 we know that ∇J and η have bounded $L^1((\partial\Omega\cap B_r)\times(0,T))$ norms independent of x since $L^p(0,T;W^{1-\frac{1}{p},p}(\partial\Omega\cap B_r))$ for $p\in(1,\frac{n+2}{n+1})$ is embedded in $L^1((\partial\Omega\cap B_r)\times(0,T))$. So we need to consider only $\frac{\partial}{\partial N(y)}\mu G(x,y,t)$ and $\mu\theta(x,y,t)$. The $L^1((\partial\Omega\cap B_r)\times(0,T))$ bound of $\mu\theta(x,y,t)$ as a function of y' for the generic point $x=(0,x_n)$ can be obtained by Lemma 4.1 after considering coordinate transform μ. From the definition of the transformation of μ and the local representation of the boundary $\partial\Omega$, we have that for $y=(y',y_n)=(y',\Phi(y'))\in\partial\Omega\cap B_r$

$$\begin{aligned}\frac{\partial}{\partial N(y)}G(x,y,t)&=\frac{1}{\sqrt{1+|\nabla'\Phi(y')|^2}}\nabla'_{y'}G\big(x,y',y_n-\Phi(y'),t\big)\cdot\nabla'_{y'}\Phi\big(y'\big)\\&\quad-\frac{1}{\sqrt{1+|\nabla'\Phi(y')|^2}}\frac{\partial}{\partial y_n}G\big(x,y',y_n-\Phi(y'),t\big)\\&=\frac{1}{\sqrt{1+|\nabla'\Phi(y')|^2}}\nabla'_{y'}G\big(x,y',0,t\big)\cdot\nabla'_{y'}\Phi\big(y'\big)\\&\quad-\frac{1}{\sqrt{1+|\nabla'\Phi(y')|^2}}\frac{\partial}{\partial y_n}G\big(x,y',0,t\big).\end{aligned}$$

Furthermore, we have already proved that the $L_1((\partial\Omega\cap B_r)\times(0,T))$ norm estimate of the first term

$$\int_0^T\int_{\partial\Omega\cap B_r}\left|\frac{1}{\sqrt{1+|\nabla'\Phi(y')|^2}}\nabla'_{y'}G\big(x,y',0,t\big)\cdot\nabla'_{y'}\Phi\big(y'\big)\right|dy'\,dt\leqslant C$$

for some C independent of x since $|\nabla'\Phi(y')|\leqslant c|y'|$.

By the expression of Poisson kernel K we have

$$\begin{aligned}-\frac{\partial}{\partial y_n}G_{ij}\big(x,y',0,t\big)&=K_{ij}\big(x'-y',x_n,t\big)+\delta_{jn}\eta_i\big(x'-y',0,t\big)\\&=-2\delta_{ij}D_{x_n}\Gamma\big(x'-y',x_n,t\big)+4\big(L_{ij}\big(x'-y',x_n,t\big)-\delta_{jn}B_{in}\big(x'-y',x_n,t\big)\big)\\&\quad+4\delta_{jn}B_{in}\big(x'-y',x_n,t\big)-\delta_{jn}\delta(t)D_{x_i}E\big(x'-y',x_n\big).\end{aligned}$$

We know already that $-2\delta_{ij}D_{x_n}\Gamma(x'-y',x_n,t)+4(L_{ij}(x'-y',x_n,t)-\delta_{jn}B_{in}(x'-y',x_n,t))$ has L^1 bounded norm as a function of (y',t). Therefore in the solution expression of u we can write

$$\int_0^t\int_{\partial\Omega\cap B_r}\delta_{jn}\delta(t-s)D_{x_i}E\big(x'-y',x_n\big)g_j(y,s)\,d\sigma_y=\frac{\partial}{\partial x_i}\int_{\partial\Omega\cap B_r}E\big(x'-y',x_n\big)g_n(y,t)\,d\sigma_y$$

and

$$4\int_0^t \int_{\partial\Omega\cap B_r} \delta_{jn} B_{in}\big(x'-y',x_n,t-s\big)g_j(y,s)\,d\sigma_y = \frac{\partial}{\partial x_i}\mathbf{T}(g_n)(x,t).$$

If we denote $e_n=(0,1)$ which is the coordinate vector for y_n, we have that the component of boundary data g_n is

$$g_n = g\cdot N(y) + g\cdot\big(e_n - N(y)\big) \quad \text{for } y\in\partial\Omega\cap B_r$$

where $g=(g_1,g_2,\dots,g_n)$ is the boundary data and hence we have

$$\begin{aligned}
&\nabla_x \int_{\partial\Omega\cap B_r} E\big(x'-y',x_n\big)g_n(y,t)\,d\sigma_y\\
&= \nabla_x \mathbf{S}\big(g\cdot NX^{\partial\Omega\cap B_r}\big)\\
&\quad + \nabla_x \int_{\partial\Omega\cap B_r} \big(E\big(x'-y',x_n\big) - E\big(x'-y',x_n-\Phi\big(y'\big)\big)\big)g(y,t)\cdot\big(e_n-N(y)\big)\,d\sigma_y\\
&\quad + \nabla_x \int_{\partial\Omega\cap B_r} E\big(x'-y',x_n\big)g(y,t)\cdot\big(e_n-N(y)\big)\,d\sigma_y,
\end{aligned}$$

where $\mathbf{S}$ is the single layer potential operator and X is the characteristic function. Since $x'=0$ and $\Phi(y')\leqslant C|y'|^2$,

$$\int_{\partial\Omega\cap B_r} \big|\nabla_x\big(E\big(x'-y',x_n\big) - E\big(x'-y',x_n-\Phi\big(y'\big)\big)\big)\big|\,d\sigma_y \leqslant C.$$

Then, by observing that

$$\big|e_n - N(y)\big| \leqslant C\big|y'\big|$$

we find that $\nabla_x E(x'-y',x_n)\cdot(e_n-N(y))$ has bounded L^1 norm as a function of y' and we have that

$$\sup_{x\in\Omega\cap B_r}\left|\nabla_x \int_{\partial\Omega\cap B_r} E\big(x'-y',x_n\big)g(y,t)\cdot\big(e_n-N(y)\big)\,d\sigma_y\right| \leqslant C\|g\|_{L_\infty(\partial\Omega\times(0,T))}.$$

Similarly we find that $\nabla\kappa(x'-y',x_n,t)(e_n-N(y))$ has a bounded L^1 norm as a function of (y',t) and we have that

$$\begin{aligned}
&\sup_{(x,t)\in\Omega\cap B_r\times(0,T)}\left|\int_0^t\int_{\partial\Omega\cap B_r} \nabla\kappa\big(x'-y',x_n,t-s\big)g(y,s)\cdot\big(e_n-N(y)\big)\,d\sigma_y\,ds\right|\\
&\quad\leqslant C\|g\|_{L^\infty(\partial\Omega\times(0,T))}.
\end{aligned}$$

With the interior L^∞ estimate, localization with the small balls B_r and the preceding kernel estimates of L^1 bound, we prove our main Theorem 1.1.

For Corollary 1.2, we observe that

$$\left|\nabla_x \mathbf{S}\left(g \cdot NX^{\partial\Omega\cap B_r}\right)\right| \leqslant C \int\limits_{\Omega\cap B_r} \frac{1}{|y'|^{n-1}} \left|g\left(y', \Phi\left(y'\right), t\right) \cdot N\left(y', \Phi\left(y'\right)\right)\right| dy',$$

where $x = (0, x_n)$. Similarly, we also have

$$\left|\nabla_x \mathbf{T}\left(g \cdot NX^{\partial\Omega\cap B_r}\right)\right| \leqslant C \int\limits_0^T \int\limits_{\Omega\cap B_r} \frac{1}{|y'|^{n-1}} \left|g\left(y', \Phi\left(y'\right), t\right) \cdot N\left(y', \Phi\left(y'\right)\right)\right| dy'\, dt.$$

The boundedness follows from the Dini-continuity of $g \cdot N$. □

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