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Global wave-front sets of Banach, Fréchet and modulation space types, and pseudo-differential operators

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ABSTRACT

We introduce global wave-front sets $\mathrm{WF}_{\mathcal{B}}(f)$, $f \in \mathscr{S}'(\mathbf{R}^d)$, with respect to suitable Banach or Fréchet spaces $\mathcal{B}$. An important special case is given by the modulation spaces $\mathcal{B} = M(\omega, \mathscr{B})$, where ω is an appropriate weight function and $\mathscr{B}$ is a translation invariant Banach function space. We show that the standard properties for known notions of wave-front set extend to $\mathrm{WF}_{\mathcal{B}}(f)$. In particular, we prove that micro-locality and micro-ellipticity hold for a class of globally defined pseudo-differential operators $\mathrm{Op}_t(a)$, acting continuously on the involved spaces.

0. Introduction

An important question for a linear operator T is whether T possesses "convenient" invertibility properties. For example, if $T = 1 - \Delta$, where Δ is the Laplace operator, and $f \in \mathscr{S}'(\mathbf{R}^d)$ fulfills

$$Tf = g, \tag{0.1}$$

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for some given $g \in \mathscr{S}(\mathbf{R}^d)$, then it follows by Fourier's inversion formula that $f \in \mathscr{S}(\mathbf{R}^d)$. Similar facts are true when $1 - \Delta$ is replaced by any partial differential operator whose symbol satisfies certain hypoellipticity conditions. Furthermore, if in addition the coefficients are constant, then f can easily be computed by a similar Fourier argument as for $1 - \Delta$.

In most situations, T fails to be (globally) hypoelliptic, and the involved functions or distributions f and g do not belong to $\mathscr{S}(\mathbf{R}^d)$. Therefore the target and image spaces of T need to be replaced by appropriate Banach or Fréchet spaces $\mathcal{B}$ and $\mathcal{C}$, respectively.

From now on we assume that T is a pseudo-differential operator with symbols belonging to an extended family of SG symbols. (See Sections 1, 2 and 5 for strict definitions.) We remark that the SG-calculus of pseudo-differential operators was introduced independently by C. Parenti and H.O. Cordes in the '70s, see, e.g., [2,3,23], and that several important global problems in science and technology can be formulated in terms of such operators. For example, any linear partial differential operator with constant coefficients, Klein–Gordon's equation, and Schrödinger equations for different atoms can be formulated within the framework of the SG-calculus.

The assumptions on $\mathcal{B}$ and $\mathcal{C}$ in (0.1) are few, and are specified in Section 1. For example $\mathcal{B}$ and $\mathcal{C}$ can be modulation spaces, a family of Banach spaces of functions and tempered distributions, introduced by H.G. Feichtinger in [7], and developed further and generalized by H.G. Feichtinger and K.H. Gröchenig in [10]. (See [10,12] for general facts and [9] for a modern approach to modulation spaces.) We remark that the family of modulation spaces is broad in the sense that it contains the Sobolev spaces H^2_s and the Sobolev–Kato spaces $H^2_{s,t}$. (Cf. Remark 1.6.) Since the union and intersection of the Sobolev–Kato spaces equals $\mathscr{S}'$ and $\mathscr{S}$, respectively, similar facts are true for modulation spaces.

We recall that T is continuous on $\mathscr{S}$ and on $\mathscr{S}'$, and if $\mathcal{B}$ is a modulation space, then it is not complicated to find the smallest modulation space $\mathcal{C}$ such that T is continuous from $\mathcal{B}$ to $\mathcal{C}$, in view of [13,28]. Furthermore, if in addition T satisfies an appropriate hypoellipticity condition and g in (0.1) belongs to $\mathcal{C}$, then it follows that f belongs to $\mathcal{B}$. Hence for such choices of T, $\mathcal{B}$ and $\mathcal{C}$ it is not complicated to obtain satisfactory answers on the following questions:

Q1. Is T extendable to a continuous operator from $\mathcal{B}$ to $\mathcal{C}$?
Q2. Let $g \in \mathcal{C}$ in (0.1). Is it true that $f \in \mathcal{B}$?

In Sections 2 and 5 we introduce global wave-front sets $\mathrm{WF}_{\mathcal{B}}(f)$ of the distribution f, with respect to the Banach or Fréchet space $\mathcal{B}$, and establish basic mapping properties of such wave-front sets. The set $\mathrm{WF}_{\mathcal{B}}(f)$ is the union of three components, $\mathrm{WF}^m_{\mathcal{B}}(f)$, $m = 1, 2, 3$. The first component, $\mathrm{WF}^1_{\mathcal{B}}(f)$, is the local component which agrees with $\mathrm{WF}_{\mathcal{B}}(f)$ in [4], and informs about singular points of f with respect to $\mathcal{B}$, and the directions where these singularities propagate. The components $\mathrm{WF}^m_{\mathcal{B}}(f)$, $m = 2, 3$, inform where at infinity, the growths and oscillations respectively for f are strong enough such that f fails to belong to $\mathcal{B}$. We note that if $\mathcal{B} = \mathscr{S}$, then the components $\mathrm{WF}^m_{\mathcal{B}}(f)$, $m = 1, 2, 3$, agree with $\mathrm{WF}^{\psi}_{\mathcal{S}}(f)$, $\mathrm{WF}^{e}_{\mathcal{S}}(f)$ and $\mathrm{WF}^{\psi e}_{\mathcal{S}}(f)$, respectively, in [6].

We establish mapping properties for the wave-front sets, and for example prove that if T is continuous on $\mathscr{S}'$ and restricts to a continuous map from $\mathcal{B}$ to $\mathcal{C}$, then

$$\mathrm{WF}_{\mathcal{C}}(g) \subseteq \mathrm{WF}_{\mathcal{B}}(f) \subseteq \mathrm{WF}_{\mathcal{C}}(g) \cup \mathrm{Char}(T), \quad g = Tf, \tag{0.2}$$

for the global wave-front sets, and similarly for their components. That is, $\mathrm{WF}_{\mathcal{C}}(g)$ is contained in $\mathrm{WF}_{\mathcal{B}}(f)$, and opposite inclusion is obtained by including $\mathrm{Char}(T)$, the set of characteristic points of T. In particular, by using the equivalence

$$\mathrm{WF}^1_{\mathcal{B}}(f) = \emptyset \quad \Longleftrightarrow \quad f \in \mathcal{B}_{\mathrm{loc}}, \tag{0.3}$$

proved in [4], in combination with

$$\mathrm{WF}_{\mathcal{B}}(f) = \emptyset \quad \Longleftrightarrow \quad f \in \mathcal{B}, \tag{0.4}$$

proved in Section 2, it follows that wave-front sets can be used to give detailed answers (locally and globally) on the questions Q1 and Q2, here above. Here (0.4) is proved in the case $\mathcal{B} = \mathscr{S}$ in [6]. Furthermore, we may use our results to obtain answers on the following questions:

Q3. Assume that f in (0.1) fails to belong to $\mathcal{B}$ in some way. In what ways might g fail to belong to $\mathcal{C}$?

Q4. Assume that g in (0.1) fails to belong to $\mathcal{C}$ in some way. In what ways might f fail to belong to $\mathcal{B}$?

It is of fundamental interests to obtain detailed answers on such questions. In Section 4 we present related examples, where it is shown how the results can be applied to establish properties of solutions to certain partial differential equations. Here we remark that the local components in (0.2) also include improvements concerning the set $\mathrm{Char}(T)$. In fact, let $\mathrm{Char}'(T)$ be the set of characteristic points of T as it is defined in [16, Chapter XVIII]. Then $\mathrm{Char}(T) \subseteq \mathrm{Char}'(T)$, where strict inclusion might occur. (Cf. Remark 2.4 in [4] and Remark 1.4 in [24].)

More generally, in order to get more detailed information on the links between f and g in (0.1), we introduce in Section 5 global wave-front sets with respect to *sequences* of appropriate Banach or Fréchet spaces, and prove that the usual properties (0.2)–(0.4) for such wave-front sets, still hold. For example, from these investigations, it follows that (0.2)–(0.4) holds when $\mathcal{B} = \mathcal{C}$ equals $\mathscr{S}(\mathbf{R}^d)$, $Q_0(\mathbf{R}^d)$ or $Q(\mathbf{R}^d)$, where

$$Q_0\big(\mathbf{R}^d\big) \equiv \big\{ f \in C^\infty\big(\mathbf{R}^d\big)\colon \big|\partial^\alpha f(x)\big| \lesssim \langle x\rangle^N \text{ for some } N \text{ and every } \alpha \in \mathbf{Z}^d \big\}$$

and

$$Q\big(\mathbf{R}^d\big) \equiv \big\{ f \in C^\infty\big(\mathbf{R}^d\big)\colon \big|\partial^\alpha f(x)\big| \lesssim \langle x\rangle^{N_\alpha},\ \text{where } N_\alpha \text{ depends on } \alpha \in \mathbf{Z}^d \big\}$$

(cf. Remark 5.10). Here and in what follows we write $A \lesssim B$ when $A \leqslant cB$ for a suitable constant $c > 0$. In particular, if $\mathrm{Char}(T) = \emptyset$ (for example when $T = 1 - \Delta$), $f, g \in \mathscr{S}'$ and (0.1) holds, then Tf belongs to Q_0, if and only if f belongs to Q_0. The same is true if Q_0 is replaced by Q or $\mathscr{S}$.

We remark that if $\mathcal{B}$ equals $\mathscr{S}$ or $H^2_{s,t}$, then $\mathrm{WF}_{\mathcal{B}}(f)$ agree with the wave-front sets of f with respect to $\mathscr{S}$ and $H^2_{s,t}$, respectively, given in [6] and [22], see also [17]. Consequently, we recover the micro-locality and micro-ellipticity properties that hold for wave-front sets of Sobolev type introduced by Hörmander [16], and classical wave-front sets with respect to smoothness (cf. Sections 8.1 and 8.2 in [15]), as well as for wave-front sets of Banach function type in [4], and wave-front sets with respect to $\mathscr{S}$ and $H^2_{s_1,s_2}$ in [6,22]. In particular, our approach links the "local" analysis carried out in [4,24], with the "global" analogue treated, e.g., in [6].

The paper is organized as follows. In Section 1 we recall the definition and basic properties of pseudo-differential operators, translation invariant Banach function spaces and modulation spaces. Here we also define three types of sets of characteristic points and show some properties for them. One of these characteristic sets coincides with the one defined in [4].

In Section 2 we define the global wave-front sets of $\mathcal{B}$-type $\mathrm{WF}_{\mathcal{B}}(f)$, and its components $\mathrm{WF}^m_{\mathcal{B}}(f)$. Furthermore, we prove that $\mathrm{WF}^1_{M(\omega,\mathscr{B})}(f)$ and $\mathrm{WF}^1_{\mathscr{F}\mathscr{B}_0(\omega)}(f)$ coincide with the wave-front sets defined in [4], when $M(\omega,\mathscr{B})$ is locally the same as the Fourier BF-space $\mathscr{F}\mathscr{B}_0(\omega)$. The remaining part of the section is devoted to the proof of a relation between the wave-front sets of $\mathcal{B}$-type and the sets of characteristic points. Here we also prove (0.4).

Sections 3 and 5 are devoted to mapping properties for pseudo-differential operators in the context of these wave-front sets. Especially we prove (0.2) for appropriate spaces. Here the most general situation is given in Section 5, involving wave-front sets with respect to sequences of spaces. Examples of applications of our results are given in Section 4 and at the end of Section 5. Finally, in Appendix A we prove some properties for the sets of characteristic points stated in Section 1.

1. Preliminaries

In what follows we let Γ denote an open cone in $\mathbf{R}^d \setminus 0$. (The vertices of the cones are always origin.) If $\xi \in \mathbf{R}^d \setminus 0$ is fixed, then an open cone which contains ξ is sometimes denoted by Γ_ξ.

1.1. Weight functions

Let ω and v be positive measurable functions on $\mathbf{R}^d$. Then ω is called v-moderate if

$$\omega(x+y) \lesssim \omega(x)v(y). \tag{1.1}$$

If v in (1.1) can be chosen as a polynomial, then ω is called a function or weight of *polynomial type*. We let $\mathscr{P}(\mathbf{R}^d)$ be the set of all polynomial type functions on $\mathbf{R}^d$. If $\omega(x,\xi) \in \mathscr{P}(\mathbf{R}^{2d})$ is constant with respect to the x-variable or the ξ-variable, then we sometimes write $\omega(\xi)$, respectively $\omega(x)$, instead of $\omega(x,\xi)$. In this case we consider ω as an element in $\mathscr{P}(\mathbf{R}^{2d})$ or in $\mathscr{P}(\mathbf{R}^d)$ depending on the situation. We say that v is submultiplicative if (1.1) holds for $\omega = v$. For convenience we assume that all submultiplicative weights are even, and we always let v and v_j stand for submultiplicative weights, if nothing else is stated.

Without loss of generality we may assume that every $\omega \in \mathscr{P}(\mathbf{R}^d)$ is smooth and satisfies the ellipticity condition $\partial^\alpha \omega/\omega \in L^\infty$. In fact, by Lemma 1.2 in [27] it follows that for each $\omega \in \mathscr{P}(\mathbf{R}^d)$, there is a smooth and elliptic $\omega_0 \in \mathscr{P}(\mathbf{R}^d)$ which is equivalent to ω in the sense

$$\omega \asymp \omega_0. \tag{1.2}$$

Here and in what follows we use the notation $A \asymp B$ when $A \lesssim B \lesssim A$.

We need some more conditions on the involved weights. More precisely let $r, \rho \geqslant 0$. Then $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ is the set of all $\omega(x,\xi)$ in $\mathscr{P}(\mathbf{R}^{2d}) \cap C^\infty(\mathbf{R}^{2d})$ such that

$$\langle x\rangle^{r|\alpha|}\langle\xi\rangle^{\rho|\beta|}\frac{\partial_x^\alpha\partial_\xi^\beta\omega(x,\xi)}{\omega(x,\xi)} \in L^\infty\big(\mathbf{R}^{2d}\big),$$

for every multi-indices α and β. Note that $\mathscr{P}_{r,\rho}$ is different here compared to [4], and that there are elements in $\mathscr{P}(\mathbf{R}^{2d})$ which have no equivalent elements in $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$. On the other hand, if $s, t \in \mathbf{R}$ and $r, \rho \in [0,1]$, then $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ contains all weights of the form $\omega(x,\xi) = \langle x\rangle^t\langle\xi\rangle^s$, which are one of the most common type of weights in the applications.

1.2. Translation invariant Banach function spaces

Next we define Banach function spaces (BF-spaces) and present some useful properties.

Definition 1.1. Let $\mathscr{B} \subseteq L^1_{\mathrm{loc}}(\mathbf{R}^d)$, and let $v \in \mathscr{P}(\mathbf{R}^d)$ be submultiplicative. Then $\mathscr{B}$ is called a (*translation*) *invariant BF-space on* $\mathbf{R}^d$ (with respect to v), if the following is true:

(1) $\mathscr{S}(\mathbf{R}^d) \subseteq \mathscr{B} \subseteq \mathscr{S}'(\mathbf{R}^d)$ (continuous embeddings);
(2) if $x \in \mathbf{R}^d$ and $f \in \mathscr{B}$, then $f(\cdot - x) \in \mathscr{B}$, and

$$\big\|f(\cdot - x)\big\|_{\mathscr{B}} \lesssim v(x)\|f\|_{\mathscr{B}}; \tag{1.3}$$

(3) if $f, g \in L^1_{\mathrm{loc}}(\mathbf{R}^d)$ satisfy $g \in \mathscr{B}$ and $|f(x)| \leqslant |g(x)|$ almost everywhere, then $f \in \mathscr{B}$ and

$$\|f\|_{\mathscr{B}} \lesssim \|g\|_{\mathscr{B}};$$

(4) if $f \in \mathscr{B}$ and $\varphi \in C_0^\infty(\mathbf{R}^d)$, then $f * \varphi \in \mathscr{B}$, and

$$\|f * \varphi\|_{\mathscr{B}} \leqslant \|\varphi\|_{L^1_{(v)}} \|f\|_{\mathscr{B}}. \tag{1.4}$$

The property (3) in Definition 1.1 is associated with solidity in function space theory. Consequently, the translation invariant BF-spaces here are also solid BF-spaces. Since C_0^∞ is dense in $L^1_{(v)}$, it follows from (1.4) that the convolution product extends uniquely to a continuous multiplication from $\mathscr{B} \times L^1_{(v)}(\mathbf{R}^d)$ to $\mathscr{B}$.

Assume that $\mathscr{B}$ is a translation invariant BF-space. If $f \in \mathscr{B}$ and $h \in L^\infty$, then it follows from (3) in Definition 1.1 that $f \cdot h \in \mathscr{B}$ and

$$\|f \cdot h\|_{\mathscr{B}} \lesssim \|f\|_{\mathscr{B}} \|h\|_{L^\infty}. \tag{1.5}$$

Also let $\omega \in \mathscr{P}(\mathbf{R}^d)$. Then the Fourier BF-space $\mathscr{F}\mathscr{B}(\omega)$ is the set of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that $\xi \mapsto \hat{f}(\xi)\omega(\xi)$ belongs to $\mathscr{B}$. Here and in what follows, $\mathscr{F}$ is the Fourier transform on $\mathscr{S}'(\mathbf{R}^d)$, which takes the form

$$(\mathscr{F} f)(\xi) = \hat{f}(\xi) \equiv (2\pi)^{-d/2} \int\limits_{\mathbf{R}^d} f(x) e^{-i\langle x, \xi \rangle}\, dx$$

when $f \in L^1(\mathbf{R}^d)$. It follows that $\mathscr{F}\mathscr{B}(\omega)$ is a Banach space under the norm

$$\|f\|_{\mathscr{F}\mathscr{B}(\omega)} \equiv \|\hat{f}\omega\|_{\mathscr{B}}. \tag{1.6}$$

Remark 1.2. Definition 1.1 is not the standard definition of Banach function spaces, and there are several approaches to the theory of such spaces. We refer to [8,21] and the references therein for more facts about such spaces.

Remark 1.3. In several situations it is convenient to permit an x dependency for the weight ω in the definition of Fourier BF-spaces. More precisely, for each $\omega(x, \xi) \in \mathscr{P}(\mathbf{R}^{2d})$ and each translation invariant BF-space $\mathscr{B}$ on $\mathbf{R}^d$, we let $\mathscr{F}\mathscr{B}(\omega)$ be the set of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that

$$\|f\|_{\mathscr{F}\mathscr{B}(\omega)} = \|f\|_{\mathscr{F}\mathscr{B}(\omega),x} \equiv \big\|\hat{f}\omega(x,\cdot)\big\|_{\mathscr{B}}$$

is finite. Since ω is v-moderate for some $v \in \mathscr{P}(\mathbf{R}^{2d})$ it follows that different choices of x give rise to equivalent norms. Therefore the condition $\|f\|_{\mathscr{F}\mathscr{B}(\omega)} < \infty$ is independent of x, and it follows that $\mathscr{F}\mathscr{B}(\omega)$ is independent of x although $\|\cdot\|_{\mathscr{F}\mathscr{B}(\omega)}$ might depend on x.

1.3. Modulation spaces

Let $\phi \in \mathscr{S}(\mathbf{R}^d)$. Then the *short-time Fourier transform* of $f \in \mathscr{S}(\mathbf{R}^d)$ with respect to (the window function) ϕ is defined by

$$V_\phi f(x, \xi) = (2\pi)^{-d/2} \int\limits_{\mathbf{R}^d} f(y)\overline{\phi(y-x)} e^{-i\langle y, \xi \rangle}\, dy. \tag{1.7}$$

More generally, the short-time Fourier transform of $f \in \mathscr{S}'(\mathbf{R}^d)$ with respect to $\phi \in \mathscr{S}'(\mathbf{R}^d)$ is defined by

$$(V_\phi f) = \mathscr{F}_2 F, \quad \text{where } F(x, y) = (f \otimes \overline{\phi})(y, y - x). \tag{1.7$'$}$$

Here $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x, y) \in \mathscr{S}'(\mathbf{R}^{2d})$ with respect to the y-variable. The definition (1.7)$'$ makes sense, since the mappings $\mathscr{F}_2$ and $F \mapsto S^*F = F \circ S$ with $S(x, y) = (y, y - x)$ are homeomorphisms on $\mathscr{S}'(\mathbf{R}^{2d})$. It is obvious that (1.7) and (1.7)$'$ agree when $f, \phi \in \mathscr{S}(\mathbf{R}^d)$. We refer to [11,12] for more facts about the short-time Fourier transform.

We mainly follow [9] when defining modulation spaces. Let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$ with respect to $v \in \mathscr{P}(\mathbf{R}^{2d})$, $\phi \in \mathscr{S}(\mathbf{R}^d) \setminus 0$ and let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ be such that ω is v-moderate. The *modulation space* $M(\omega, \mathscr{B})$ consists of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that $V_\phi f \cdot \omega \in \mathscr{B}$. We note that $M(\omega, \mathscr{B})$ is a Banach space with the norm

$$\|f\|_{M(\omega,\mathscr{B})} \equiv \|V_\phi f \omega\|_{\mathscr{B}} \tag{1.8}$$

(cf. [10]).

Remark 1.4. Assume that $p, q \in [1, \infty]$, and let $L_1^{p,q}(\mathbf{R}^{2d})$ and $L_2^{p,q}(\mathbf{R}^{2d})$ be the sets of all $F \in L^1_{\mathrm{loc}}(\mathbf{R}^{2d})$ such that

$$\|F\|_{L_1^{p,q}} \equiv \left(\int \left(\int \left| F(x, \xi) \right|^p dx \right)^{q/p} d\xi \right)^{1/q} < \infty$$

and

$$\|F\|_{L_2^{p,q}} \equiv \left(\int \left(\int \left| F(x, \xi) \right|^q d\xi \right)^{p/q} dx \right)^{1/p} < \infty.$$

Then $M(\omega, L_1^{p,q}(\mathbf{R}^{2d}))$ is equal to the *classical* modulation space $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, and $M(\omega, L_2^{p,q}(\mathbf{R}^{2d}))$ is equal to the space $W^{p,q}_{(\omega)}(\mathbf{R}^d)$, related to Wiener-amalgam spaces (cf. [7,9,10,12]).

For notational convenience we set $M^p_{(\omega)} = M^{p,p}_{(\omega)} = W^{p,p}_{(\omega)}$. Furthermore, if $\omega = 1$, then we write $M^{p,q}$, M^p and $W^{p,q}$ instead of $M^{p,q}_{(\omega)}$, $M^p_{(\omega)}$ and $W^{p,q}_{(\omega)}$ respectively.

In the following proposition we list some important properties for modulation spaces. We refer to [12] for the proof.

Proposition 1.5. *Let $\omega, v_0, v \in \mathscr{P}(\mathbf{R}^{2d})$ be such that v and v_0 are submultiplicative, and ω is v-moderate. Also let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$ with respect to v_0 and $f \in \mathscr{S}'(\mathbf{R}^d)$. Then the following is true*:

(1) *if $\phi \in M^1_{(v_0 v)}(\mathbf{R}^d) \setminus 0$, then $f \in M(\omega, \mathscr{B})$ if and only if $V_\phi f \cdot \omega \in \mathscr{B}$. Furthermore,* (1.8) *defines a norm on $M(\omega, \mathscr{B})$, and different choices of ϕ give rise to equivalent norms*;
(2) $\mathscr{S}(\mathbf{R}^d) \subseteq M^1_{(v_0 v)}(\mathbf{R}^d) \subseteq M(\omega, \mathscr{B}) \subseteq M^\infty_{(1/(v_0 v))}(\mathbf{R}^d) \subseteq \mathscr{S}'(\mathbf{R}^d)$.

Proposition 1.5(1) allows us to be rather vague about the choice of $\phi \in M^1_{(v_0 v)} \setminus 0$ in (1.8). For example, if $C > 0$ is a constant and S_0 is a subset of $\mathscr{S}'$, then $\|a\|_{M(\omega,\mathscr{B})} \leqslant C$ for every $a \in S_0$, means that the inequality holds for some choice of $\phi \in M^1_{(v_0 v)} \setminus 0$ and every $a \in S_0$. Evidently, for any other choice of $\phi \in M^1_{(v_0 v)} \setminus 0$, a similar inequality is true although C may have to be replaced by a larger constant, if necessary.

In what follows we let σ_s and $\sigma_{s,t}$ be the weights

$$\sigma_s(x, \xi) = \langle x, \xi \rangle^s \quad \text{and} \quad \sigma_{s,t}(x, \xi) = \langle x \rangle^t \langle \xi \rangle^s, \quad x, \xi \in \mathbf{R}^d. \tag{1.9}$$

Remark 1.6. Several important spaces agree with certain modulation spaces. In fact, let $s,t\in\mathbf{R}$. Then $M^2_{(\sigma_{s,t})}(\mathbf{R}^d)$ is equal to the weighted Sobolev space (or Sobolev–Kato space) $H^2_{s,t}(\mathbf{R}^d)$ in [6,22], the set of all $f\in\mathscr{S}'(\mathbf{R}^d)$ such that $\langle x\rangle^t\langle D\rangle^s f\in L^2$. In particular, $M^2_{(\sigma_{s,0})}$ and $M^2_{(\sigma_{0,t})}$ are equal to H^2_s and L^2_t, respectively.

Furthermore, $M^2_{(\sigma_s)}(\mathbf{R}^d)$ is equal to the Shubin–Sobolev space of order s. (Cf. e.g. [20].)

We recall that Fourier BF-spaces and modulation spaces are locally the same. In fact, let $\varphi\in\mathscr{S}(\mathbf{R}^d)\setminus 0$, $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, $\omega\in\mathscr{P}(\mathbf{R}^{2d})$ and set $\omega_0(\xi)=\omega(x_0,\xi)$ for some fixed $x_0\in\mathbf{R}^d$. Then

$$\mathscr{B}_0\equiv\bigl\{f\in\mathscr{S}'\bigl(\mathbf{R}^d\bigr);\ \varphi\otimes f\in\mathscr{B}\bigr\}\tag{1.10}$$

is a translation invariant BF-space on $\mathbf{R}^d$ under the norm $\|f\|_{\mathscr{B}_0}\equiv\|\varphi\otimes f\|_{\mathscr{B}}$. The space $\mathscr{B}_0$ is independent of $\varphi\in\mathscr{S}(\mathbf{R}^d)\setminus 0$, and different choices of φ give rise to equivalent norms. Furthermore

$$M(\omega,\mathscr{B})\cap\mathscr{E}'\bigl(\mathbf{R}^d\bigr)=\mathscr{F}\mathscr{B}_0(\omega_0)\cap\mathscr{E}'\bigl(\mathbf{R}^d\bigr)\tag{1.11}$$

(cf. [4,26]).

1.4. Pseudo-differential operators and symbol classes

Next we recall some facts in Chapter XVIII in [16] concerning pseudo-differential operators. Let $a\in\mathscr{S}(\mathbf{R}^{2d})$, and $t\in\mathbf{R}$ be fixed. Then the pseudo-differential operator $\operatorname{Op}_t(a)$ is the linear and continuous operator on $\mathscr{S}(\mathbf{R}^d)$ defined by the formula

$$\bigl(\operatorname{Op}_t(a)f\bigr)(x)=(2\pi)^{-d}\iint a\bigl((1-t)x+ty,\xi\bigr)f(y)e^{i\langle x-y,\xi\rangle}\,dy\,d\xi.\tag{1.12}$$

For general $a\in\mathscr{S}'(\mathbf{R}^{2d})$, the pseudo-differential operator $\operatorname{Op}_t(a)$ is defined as the continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ with distribution kernel

$$K_{t,a}(x,y)=(2\pi)^{-d/2}\bigl(\mathscr{F}_2^{-1}a\bigr)\bigl((1-t)x+ty,x-y\bigr).\tag{1.13}$$

This definition makes sense, since $\mathscr{F}_2$ and the map

$$F\mapsto F\circ S_t\quad\text{with } S_t(x,y)=\bigl((1-t)x+ty,x-y\bigr)$$

are homeomorphisms on $\mathscr{S}'(\mathbf{R}^{2d})$. We also note that the latter definition of $\operatorname{Op}_t(a)$ agrees with the operator in (1.12) when $a\in\mathscr{S}(\mathbf{R}^{2d})$.

If $t=0$, then $\operatorname{Op}_t(a)$ is the Kohn–Nirenberg representation $\operatorname{Op}(a)=a(x,D)$, and if $t=1/2$, then $\operatorname{Op}_t(a)$ is the Weyl quantization.

Let $a\in\mathscr{S}'(\mathbf{R}^{2d})$ and $s,t\in\mathbf{R}$. Then there is a unique $b\in\mathscr{S}'(\mathbf{R}^{2d})$ such that $\operatorname{Op}_s(a)=\operatorname{Op}_t(b)$. By straightforward applications of Fourier's inversion formula, it follows that

$$\operatorname{Op}_s(a)=\operatorname{Op}_t(b)\quad\Longleftrightarrow\quad b(x,\xi)=e^{i(t-s)\langle D_x,D_\xi\rangle}a(x,\xi)\tag{1.14}$$

(cf. Section 18.5 in [16]).

Next we discuss our symbol classes. Let $m,\mu,r,\rho\in\mathbf{R}$ be fixed. Then $\mathrm{SG}^{m,\mu}_{r,\rho}(\mathbf{R}^{2d})$ is the set of all $a\in C^\infty(\mathbf{R}^{2d})$ such that

$$\bigl|D_x^\alpha D_\xi^\beta a(x,\xi)\bigr|\lesssim\langle x\rangle^{m-r|\alpha|}\langle\xi\rangle^{\mu-\rho|\beta|},$$

for all multi-indices α and β. Usually we assume that $r,\rho\geqslant 0$ and $\rho+r>0$.

More generally, assume that $\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$. Then $\mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^{2d})$ consists of all $a \in C^\infty(\mathbf{R}^{2d})$ such that

$$\left|D_x^\alpha D_\xi^\beta a(x,\xi)\right| \lesssim \omega(x,\xi)\langle x\rangle^{-r|\alpha|}\langle\xi\rangle^{-\rho|\beta|}, \quad x,\xi \in \mathbf{R}^d, \tag{1.15}$$

for all multi-indices α and β. We note that

$$\mathrm{SG}^{(\omega)}_{r,\rho}\big(\mathbf{R}^{2d}\big) = S(\omega, g_{r,\rho}), \tag{1.16}$$

when $g = g_{r,\rho}$ is the Riemannian metric on $\mathbf{R}^{2d}$, defined by the formula

$$(g_{r,\rho})_{(y,\eta)}(x,\xi) = \langle y\rangle^{-2r}|x|^2 + \langle\eta\rangle^{-2\rho}|\xi|^2 \tag{1.17}$$

(cf. Sections 18.4–18.6 in [16]). Furthermore, $\mathrm{SG}^{(\omega)}_{r,\rho} = \mathrm{SG}^{m,\mu}_{r,\rho}$ when $\omega(x,\xi) = \langle x\rangle^m\langle\xi\rangle^\mu$.

The following result shows that pseudo-differential operators with symbols in $\mathrm{SG}^{(\omega)}_{r,\rho}$ behave well.

Proposition 1.7. *Let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, $s,t \in \mathbf{R}$, $r,\rho \geqslant 0$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ and let $a,b \in \mathscr{S}'(\mathbf{R}^{2d})$ be such that $\mathrm{Op}_s(a) = \mathrm{Op}_t(b)$. Then the following is true:*

(1) *$a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ if and only if $b \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$, and then*

$$a - b \in \mathrm{SG}^{(\omega_0/\sigma_{\rho,r})}_{r,\rho}\big(\mathbf{R}^{2d}\big);$$

(2) *if $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$, then $\mathrm{Op}_t(a)$ is continuous on $\mathscr{S}(\mathbf{R}^d)$ and extends uniquely to a continuous operator on $\mathscr{S}'(\mathbf{R}^d)$;*

(3) *if $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$, then $\mathrm{Op}_t(a)$ is continuous from $M(\omega,\mathscr{B})$ to $M(\omega/\omega_0,\mathscr{B})$;*

(4) *there exist $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ and $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ such that for every choice of $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ and every translation invariant BF-space $\mathscr{B}$ on $\mathbf{R}^{2d}$, the mappings*

$$\mathrm{Op}_t(a) : \mathscr{S}\big(\mathbf{R}^d\big) \to \mathscr{S}\big(\mathbf{R}^d\big), \qquad \mathrm{Op}_t(a) : \mathscr{S}'\big(\mathbf{R}^d\big) \to \mathscr{S}'\big(\mathbf{R}^d\big) \quad \textit{and}$$
$$\mathrm{Op}_t(a) : M(\omega,\mathscr{B}) \to M(\omega/\omega_0,\mathscr{B})$$

are continuous bijections with inverses $\mathrm{Op}_t(b)$.

Proof. From the assumptions it follows that $g_{r,\rho}$ in (1.17) is slowly varying, σ-temperate and satisfies $g_{r,\rho} \leqslant g^\sigma_{r,\rho}$, and that ω is $g_{r,\rho}$-continuous and $(\sigma, g_{r,\rho})$-temperate (see Sections 18.4–18.6 in [16] for definitions). The assertions (1) and (2) are now consequences of Proposition 18.5.10 and Theorem 18.6.2 in [16], and (1.16).

Finally, (3) and (4) follow immediately from [28, Theorem 3.2] and [13, Theorem 2.1]. The proof is complete. □

We remark that explicit bijections of the form in Proposition 1.7(4) can be found in [13, Section 3].

The following definition is motivated by Proposition 1.7(3) and (4).

Definition 1.8. Let $r,\rho \in [0,1]$, $t \in \mathbf{R}$, $\mathcal{B}$ be a topological vector space of distributions on $\mathbf{R}^d$ such that

$$\mathscr{S}\big(\mathbf{R}^d\big) \subseteq \mathcal{B} \subseteq \mathscr{S}'\big(\mathbf{R}^d\big)$$

with continuous embeddings. Then $\mathcal{B}$ is called SG-*admissible* (*with respect to* r, ρ *and* d) when $\mathrm{Op}_t(a)$ maps $\mathcal{B}$ continuously into itself, for every $a \in \mathrm{SG}^{0,0}_{r,\rho}$.

If $\mathcal{B}$ and $\mathcal{C}$ are SG-admissible with respect to r, ρ and d, and $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, then the pair $(\mathcal{B}, \mathcal{C})$ is called SG-*ordered* (*with respect to* ω_0), when the mappings

$$\mathrm{Op}_t(a) : \mathcal{B} \to \mathcal{C} \quad \text{and} \quad \mathrm{Op}_t(b) : \mathcal{C} \to \mathcal{B}$$

are continuous for every $a \in \mathrm{SG}_{r,\rho}^{(\omega_0)}(\mathbf{R}^{2d})$ and $b \in \mathrm{SG}_{r,\rho}^{(1/\omega_0)}(\mathbf{R}^{2d})$.

Remark 1.9. Let $\mathscr{B}$, t, r, ρ, ω and ω_0 be as in Proposition 1.7, and let $\mathcal{B}$ be SG-admissible with respect to r, ρ and d. Then there is a unique SG-admissible $\mathcal{C}$ such that $(\mathcal{B}, \mathcal{C})$ is an SG-ordered pair with respect to ω_0.

In fact, let a be as in Proposition 1.7(4). Then $\mathcal{C}$ is the image of $\mathcal{B}$ under $\mathrm{Op}_t(a)$. The details are left for the reader.

In particular, $\mathscr{S}(\mathbf{R}^d)$, $\mathscr{S}'(\mathbf{R}^d)$ and $M(\omega, \mathscr{B})$ are SG-admissible, and

$$\big(\mathscr{S}\big(\mathbf{R}^d\big), \mathscr{S}\big(\mathbf{R}^d\big)\big), \quad \big(\mathscr{S}'\big(\mathbf{R}^d\big), \mathscr{S}'\big(\mathbf{R}^d\big)\big) \quad \text{and} \quad \big(M(\omega, \mathscr{B}), M(\omega/\omega_0, \mathscr{B})\big)$$

are SG-ordered with respect to ω_0.

If $a \in \mathrm{SG}_{r,\rho}^{(\omega_0)}(\mathbf{R}^{2d})$, then

$$\big|a(x,\xi)\big| \lesssim \omega_0(x,\xi).$$

On the other hand, a is invertible, in the sense that $1/a$ is a symbol in $\mathrm{SG}_{r,\rho}^{(1/\omega_0)}(\mathbf{R}^{2d})$, if and only if

$$\omega_0(x,\xi) \lesssim \big|a(x,\xi)\big|. \tag{1.18}$$

A slightly relaxed condition appears when (1.18) holds for all points (x,ξ), outside a compact set $K \subseteq \mathbf{R}^{2d}$. In this case we say that a is *elliptic* (with respect to ω_0).

In the following we discuss more local invertibility conditions for symbols in $\mathrm{SG}_{r,\rho}^{(\omega_0)}(\mathbf{R}^{2d})$ in terms of sets of characteristic points of the involved symbols. We remark that our definition of such sets is slightly different compared to [16, Definition 18.1.5] and [6] in view of Remark 1.16 below.

Definition 1.10. Let $r, \rho \geqslant 0$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ and let $a \in \mathrm{SG}_{r,\rho}^{(\omega_0)}(\mathbf{R}^{2d})$.

(1) a is called *locally* or *type*-1 *invertible* with respect to ω_0 at the point $(x_0, \xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, if there exist a neighborhood X of x_0, an open conical neighborhood Γ of ξ_0 and a positive constant R such that (1.18) holds for $x \in X$, $\xi \in \Gamma$ and $|\xi| \geqslant R$.
(2) a is called *Fourier-locally* or *type*-2 *invertible* with respect to ω_0 at the point $(x_0, \xi_0) \in (\mathbf{R}^d \setminus 0) \times \mathbf{R}^d$, if there exist an open conical neighborhood Γ of x_0, a neighborhood X of ξ_0 and a positive constant R such that (1.18) holds for $x \in \Gamma$, $|x| \geqslant R$ and $\xi \in X$.
(3) a is called *oscillating* or *type*-3 *invertible* with respect to ω_0 at the point $(x_0, \xi_0) \in (\mathbf{R}^d \setminus 0) \times (\mathbf{R}^d \setminus 0)$, if there exist open conical neighborhoods Γ_1 of x_0 and Γ_2 of ξ_0, and a positive constant R such that (1.18) holds for $x \in \Gamma_1$, $|x| \geqslant R$, $\xi \in \Gamma_2$ and $|\xi| \geqslant R$.

If $m \in \{1,2,3\}$ and a is *not* type-m invertible with respect to ω_0 at (x_0, ξ_0), then (x_0, ξ_0) is called *type-m characteristic* for a with respect to ω_0. The set of type-m characteristic points for a with respect to ω_0 is denoted by $\mathrm{Char}^m_{(\omega_0)}(a)$.

The (*global*) *set of characteristic points* (the characteristic set), for a symbol $a \in \mathrm{SG}_{r,\rho}^{(\omega_0)}(\mathbf{R}^{2d})$ with respect to ω_0, is

$$\mathrm{Char}(a) = \mathrm{Char}_{(\omega_0)}(a) = \mathrm{Char}^1_{(\omega_0)}(a) \cup \mathrm{Char}^2_{(\omega_0)}(a) \cup \mathrm{Char}^3_{(\omega_0)}(a).$$

Remark 1.11. In the case $\omega_0 = 1$ we exclude the phrase "with respect to ω_0" in Definition 1.10. For example, $a \in \mathrm{SG}^{0,0}_{r,\rho}(\mathbf{R}^{2d})$ is *type*-1 *invertible* at $(x_0, \xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ if $(x_0, \xi_0) \notin \mathrm{Char}^1_{(\omega_0)}(a)$ with $\omega_0 = 1$. This means that there exist a neighborhood X of x_0, an open conical neighborhood Γ of ξ_0 and $R > 0$ such that (1.18) holds for $\omega_0 = 1$, $x \in X$ and $\xi \in \Gamma$ satisfies $|\xi| \geqslant R$.

In the next definition we introduce different classes of cutoff functions (see also Definition 1.9 in [4]).

Definition 1.12. Let $X \subseteq \mathbf{R}^d$ be open, $\Gamma \subseteq \mathbf{R}^d \setminus 0$ be an open cone, $x_0 \in X$ and let $\xi_0 \in \Gamma$.

(1) A smooth function φ on $\mathbf{R}^d$ is called a *cutoff* (*function*) with respect to x_0 and X, if $0 \leqslant \varphi \leqslant 1$, $\varphi \in C_0^\infty(X)$ and $\varphi = 1$ in an open neighborhood of x_0. The set of cutoffs with respect to x_0 and X is denoted by $\mathscr{C}_{x_0}(X)$ or $\mathscr{C}_{x_0}$.
(2) A smooth function ψ on $\mathbf{R}^d$ is called a *directional cutoff* (*function*) with respect to ξ_0 and Γ, if there is a constant $R > 0$ and open conical neighborhood $\Gamma_1 \subseteq \Gamma$ of ξ_0 such that the following is true:
   - $0 \leqslant \psi \leqslant 1$ and $\operatorname{supp} \psi \subseteq \Gamma$;
   - $\psi(t\xi) = \psi(\xi)$ when $t \geqslant 1$ and $|\xi| \geqslant R$;
   - $\psi(\xi) = 1$ when $\xi \in \Gamma_1$ and $|\xi| \geqslant R$.

   The set of directional cutoffs with respect to ξ_0 and Γ is denoted by $\mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$ or $\mathscr{C}^{\mathrm{dir}}_{\xi_0}$.

Remark 1.13. Let $X \subseteq \mathbf{R}^d$ be open and $\Gamma, \Gamma_1, \Gamma_2 \subseteq \mathbf{R}^d \setminus 0$ be open cones. Then the following is true:

(1) if $x_0 \in X$, $\xi_0 \in \Gamma$, $\varphi \in \mathscr{C}_{x_0}(X)$ and $\psi \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$, then $c_1 = \varphi \otimes \psi$ belongs to $\mathrm{SG}^{0,0}_{1,1}(\mathbf{R}^{2d})$, and is type-1 invertible at (x_0, ξ_0);
(2) if $x_0 \in \Gamma$, $\xi_0 \in X$, $\psi \in \mathscr{C}^{\mathrm{dir}}_{x_0}(\Gamma)$ and $\varphi \in \mathscr{C}_{\xi_0}(X)$, then $c_2 = \psi \otimes \varphi$ belongs to $\mathrm{SG}^{0,0}_{1,1}(\mathbf{R}^{2d})$, and is type-2 invertible at (x_0, ξ_0);
(3) if $x_0 \in \Gamma_1$, $\xi_0 \in \Gamma_2$, $\psi_1 \in \mathscr{C}^{\mathrm{dir}}_{x_0}(\Gamma_1)$ and $\psi_2 \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma_2)$, then $c_3 = \psi_1 \otimes \psi_2$ belongs to $\mathrm{SG}^{0,0}_{1,1}(\mathbf{R}^{2d})$, and is type-3 invertible at (x_0, ξ_0).

In the following proposition we show that $\mathrm{Op}_t(a)$ for $t \in \mathbf{R}$ satisfies convenient invertibility properties of the form

$$\mathrm{Op}_t(a)\,\mathrm{Op}_t(b) = \mathrm{Op}_t(c) + \mathrm{Op}_t(h), \tag{1.19}$$

outside the set of characteristic points for a symbol a. Here $\mathrm{Op}_t(b)$, $\mathrm{Op}_t(c)$ and $\mathrm{Op}_t(h)$ have the roles of "local inverse", "local identity" and smoothing operators respectively. From these propositions it also follows that our sets of characteristic points in Definition 1.10 are related to those in [6,16].

Before stating the results we let $\mathbb{I}_m$ and Ω_m, $m = 1, 2, 3$, be the sets

$$\mathbb{I}_1 \equiv [0,1] \times (0,1], \qquad \mathbb{I}_2 \equiv (0,1] \times [0,1], \qquad \mathbb{I}_3 \equiv (0,1] \times (0,1] = \mathbb{I}_1 \cap \mathbb{I}_2,$$

and

$$\begin{aligned} \Omega_1 = \mathbf{R}^d \times \big(\mathbf{R}^d \setminus 0\big), \qquad \Omega_2 = \big(\mathbf{R}^d \setminus 0\big) \times \mathbf{R}^d, \\ \Omega_3 = \big(\mathbf{R}^d \setminus 0\big) \times \big(\mathbf{R}^d \setminus 0\big), \end{aligned} \tag{1.20}$$

which will frequently appear.

Proposition 1.14. *Let $m \in \{1,2,3\}$, $(r,\rho) \in \mathbb{I}_m$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ and let $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$. Also let Ω_m be as in* (1.20)*, $(x_0, \xi_0) \in \Omega_m$, and let (r_0, ρ_0) be equal to $(r, 0)$, $(0, \rho)$ and (r, ρ) when m is equal to* 1, 2 *and* 3*, respectively. Then the following conditions are equivalent*:

(1) $(x_0,\xi_0)\notin \mathrm{Char}^m_{(\omega_0)}(a)$;
(2) *there is an element* $c\in \mathrm{SG}^{0,0}_{r,\rho}$ *which is type-m invertible at* (x_0,ξ_0), *and an element* $b\in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$ *such that* $ab=c$;
(3) (1.19) *holds for some* $c\in \mathrm{SG}^{0,0}_{r,\rho}$ *which is type-m invertible at* (x_0,ξ_0), *and some elements* $h\in \mathrm{SG}^{-r_0,-\rho_0}_{r,\rho}$ *and* $b\in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$;
(4) (1.19) *holds for some* $c_m\in \mathrm{SG}^{0,0}_{r,\rho}$ *in Remark* 1.13 *which is type-m invertible at* (x_0,ξ_0), *and some elements* h *and* $b\in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$, *where* $h\in \mathscr S$ *when* $m\in\{1,3\}$ *and* $h\in \mathrm{SG}^{-\infty,0}$ *when* $m=2$.
Furthermore, if $t=0$, *then the supports of* b *and* h *can be chosen to be contained in* $X\times \mathbf R^d$ *when* $m=1$, *in* $\Gamma\times\mathbf R^d$ *when* $m=2$, *and in* $\Gamma_1\times \mathbf R^d$ *when* $m=3$.

Proposition 1.14 for $m=1,2$ follows by the same arguments as in the proof of Proposition 2.3 in [4], and the case $m=3$ follows by similar arguments. For completeness we give a proof of Proposition 1.14 in the case $m=3$ in Appendix A.

As a consequence of Proposition 1.14, we can show that the sets of characteristic points are invariant under the choice of pseudo-differential calculus.

Proposition 1.15. *Let* $m\in\{1,2,3\}$, $(r,\rho)\in \mathbb I_m$, $\omega_0\in \mathscr P_{r,\rho}(\mathbf R^{2d})$, $s,t\in\mathbf R$ *and let* $a,b\in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf R^{2d})$ *be such that* $\mathrm{Op}_s(a)=\mathrm{Op}_t(b)$. *Then* $\mathrm{Char}^m_{(\omega_0)}(a)=\mathrm{Char}^m_{(\omega_0)}(b)$.

Proof. We may assume that $s=0$. We only consider the case $m=3$. The other cases follow by similar arguments and are left for the reader. By [16, Proposition 18.5.10], it follows that $b=a+h$, where $h\in \mathrm{SG}^{(\omega_0/\sigma_{\rho,r})}_{r,\rho}$. (Cf. (1.9).) Then for each $\varepsilon>0$ there is a constant $R>0$ such that $|h(x,\xi)|\leqslant \varepsilon\omega_0(x,\xi)$ when $|x|\geqslant R$ or $|\xi|\geqslant R$. This implies that (3) in Definition 1.10 is fulfilled for a, if and only if it is fulfilled for b. This gives the result. $\square$

Remark 1.16. Let $\omega_0(x,\xi)=\langle\xi\rangle^r$, $r\in\mathbf R$, and assume that $a\in \mathrm{SG}^{r,0}_{1,0}(\mathbf R^{2d})=\mathrm{SG}^{(\omega_0)}_{1,0}(\mathbf R^{2d})$ is polyhomogeneous with principal symbol $a_r\in \mathrm{SG}^{r,0}_{1,0}(\mathbf R^{2d})$ (cf. Definition 18.1.5 in [16]). Also let $\mathrm{Char}'(a)$ be the set of characteristic points of $\mathrm{Op}(a)$ in the classical sense (i.e., in the sense of Definition 18.1.25 in [16]). Then

$$\mathrm{Char}^1_{(\omega_0)}(a)\subseteq \mathrm{Char}'(a), \tag{1.21}$$

where strict inclusion might appear in view of Remark 1.4 and Example 3.9 in [24].

By similar arguments it follows that the sets $\mathrm{Char}^2_{(\omega_0)}(a)$ and $\mathrm{Char}^3_{(\omega_0)}(a)$ are contained in corresponding sets of characteristic points in [6].

2. Global wave-front sets

In this section we define global wave-front sets for temperate distributions with respect to Banach or Fréchet spaces and establish some properties. The basic ideas behind these definitions can be found in [6].

We start by introducing the complements of the wave-front sets. More precisely, let Ω_m, $m\in\{1,2,3\}$, be given by (1.20), $\mathcal B$ be a Banach or Fréchet space such that $\mathscr S(\mathbf R^d)\subseteq \mathcal B\subseteq \mathscr S'(\mathbf R^d)$, and let $f\in \mathscr S'(\mathbf R^d)$. Then the point $(x_0,\xi_0)\in\Omega_m$ is called *type-m* regular for f with respect to $\mathcal B$, if

$$\mathrm{Op}(c_m)f\in\mathcal B, \tag{2.1}$$

for some c_m in Remark 1.13. The set of all type-m regular points for f with respect to $\mathcal B$, is denoted by $\Theta^m_{\mathcal B}(f)$.

Definition 2.1. Let $m \in \{1,2,3\}$, Ω_m be as in (1.20), and let $\mathcal{B}$ be a Banach or Fréchet space such that $\mathscr{S}(\mathbf{R}^d) \subseteq \mathcal{B} \subset \mathscr{S}'(\mathbf{R}^d)$:

(1) the *type-m wave-front set* of $f \in \mathscr{S}'(\mathbf{R}^d)$ with respect to $\mathcal{B}$ is the complement of $\Theta^m_{\mathcal{B}}(f)$ in Ω_m, and is denoted by $\mathrm{WF}^m_{\mathcal{B}}(f)$;
(2) the *global wave-front set* $\mathrm{WF}_{\mathcal{B}}(f) \subseteq (\mathbf{R}^d \times \mathbf{R}^d) \setminus 0$ is the set

$$\mathrm{WF}_{\mathcal{B}}(f) \equiv \mathrm{WF}^1_{\mathcal{B}}(f) \cup \mathrm{WF}^2_{\mathcal{B}}(f) \cup \mathrm{WF}^3_{\mathcal{B}}(f).$$

The sets $\mathrm{WF}^1_{\mathcal{B}}(f)$, $\mathrm{WF}^2_{\mathcal{B}}(f)$ and $\mathrm{WF}^3_{\mathcal{B}}(f)$ in Definition 2.1, are also called the *local*, *Fourier-local* and *oscillating* wave-front set of f with respect to $\mathcal{B}$.

From now on we assume that $\mathcal{B}$ in Definition 2.1 is SG-admissible, and recall that Sobolev–Kato spaces and, more generally, modulation spaces, and $\mathscr{S}(\mathbf{R}^d)$ are SG-admissible. (Cf. Definition 1.8, and Remarks 1.6 and 1.9.)

Remark 2.2. In a similar way as in [6, Remark 2.3], we note that Definition 2.1 does not change if the condition (2.1) with c_m as in Remark 1.13, $m = 1,2,3$, is replaced by

$$\psi(D)(\varphi \cdot f) \notin \mathcal{B}, \qquad \varphi(D)(\psi \cdot f) \notin \mathcal{B}, \quad \text{and} \quad \psi_2(D)(\psi_1 \cdot f) \notin \mathcal{B},$$

respectively (when $\mathcal{B}$ is SG-admissible).

We only prove the assertion in the case $m = 1$, leaving the verification for the other cases to the reader. Let $c(x,\xi) = \varphi(x)\psi(\xi)$ where $\varphi \in \mathscr{C}_{x_0}(\mathbf{R}^d)$ and $\psi \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\mathbf{R}^d \setminus 0)$, and let $c_1 \in \mathrm{SG}^{0,0}_{r,\rho}$ be equal to 1 on $\operatorname{supp} c$. Then it follows from the symbolic calculus that

$$\mathrm{Op}(c_1)\,\mathrm{Op}(c) = \mathrm{Op}(c)\,\mathrm{Op}(c_1) \bmod \mathrm{Op}(\mathscr{S}) = \mathrm{Op}(c) \bmod \mathrm{Op}(\mathscr{S}). \tag{2.2}$$

A combination of (2.2) and the facts that each pseudo-differential operator with symbol in $\mathrm{SG}^{0,0}_{r,\rho}$ is continuous on $\mathcal{B}$ now shows that (1) in Definition 2.1 does not depend on the order we apply the operators. Here we have also used the fact that elements in $\mathrm{Op}(\mathscr{S}(\mathbf{R}^{2d}))$ map $\mathscr{S}'(\mathbf{R}^d)$ continuously into $\mathscr{S}(\mathbf{R}^d) \subseteq \mathcal{B}$.

Remark 2.3. Let $X \subseteq \mathbf{R}^d$ be open, $\mathcal{B} \subseteq \mathscr{D}'(X)$ with continuous embedding, and let $\mathcal{B}_{\mathrm{loc}}$ be the set of all $f \in \mathscr{D}'(X)$ such that $\varphi \cdot f \in \mathcal{B}$ for every $\varphi \in C_0^\infty(X)$. Then $\mathcal{B}$ is called local if $\mathcal{B} \subseteq \mathcal{B}_{\mathrm{loc}}$.

If $\mathcal{B} \subseteq \mathscr{D}'(X)$ and $f \in \mathscr{D}'(\mathbf{R}^d)$, then the local wave-front set $\mathrm{WF}^1_{\mathcal{B}}(f)$ of f with respect to $\mathcal{B}$ is defined as the set of all $(x_0,\xi_0) \in X \times (\mathbf{R}^d \setminus 0)$ such that $\psi(D)(\varphi \cdot f) \notin \mathcal{B}$ for every $\varphi \in \mathscr{C}_{x_0}(X)$ and every $\psi \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\mathbf{R}^d \setminus 0)$. By Remark 2.2 it follows that this definition agrees with Definition 2.1(1) when $\mathcal{B}$ is SG-admissible and $f \in \mathscr{S}'$.

Let $\mathcal{B}_1, \mathcal{B}_2 \subseteq \mathscr{D}'(X)$ be local such that $(\mathcal{B}_1)_{\mathrm{loc}} \subseteq (\mathcal{B}_2)_{\mathrm{loc}}$ and $\psi(D)$ is continuous from $\mathcal{B}_j \cap \mathscr{E}'$ to $\mathcal{B}_j$, $j = 1,2$, when $\psi \in \mathscr{C}^{\mathrm{dir}}$. Then $\mathrm{WF}^1_{\mathcal{B}_2}(f) \subseteq \mathrm{WF}^1_{\mathcal{B}_1}(f)$ when $f \in \mathscr{D}'(X)$, by Definition 2.1 and Remark 2.2.

Remark 2.4. Let $f \in \mathscr{S}'(\mathbf{R}^d)$, $x_0, \xi_0 \in \mathbf{R}^d \setminus 0$, $\psi_{j,1} \in \mathscr{C}^{\mathrm{dir}}_{x_0}(\mathbf{R}^d \setminus 0)$, $\psi_{j,2} \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\mathbf{R}^d \setminus 0)$ for $j = 1,2$ be such that $\psi_{1,k} = 1$ on $\operatorname{supp}\psi_{2,k}$ for $k = 1,2$. Also let $\mathcal{B}$ be SG-admissible with respect to $r, \rho \in [0,1]$ and d. If $\psi_{1,1} \cdot \psi_{1,2}(D) f \in \mathcal{B}$, then $\psi_{2,1} \cdot \psi_{2,2}(D) f \in \mathcal{B}$.

In fact, if $c_j = \psi_{j,1} \otimes \psi_{j,2}$, then $c_1 = 1$ on $\operatorname{supp} c_2$, and it follows from the symbolic calculus that for some $h \in \mathscr{S}$ we have

$$\psi_{2,1} \cdot \psi_{2,2}(D) f = \mathrm{Op}(c_2) f = \mathrm{Op}(c_2)\,\mathrm{Op}(c_1) f + \mathrm{Op}(h) f.$$

The assertion now follows from the fact that $\mathrm{Op}(c_2)$ and $\mathrm{Op}(h)$ are continuous on $\mathcal{B}$.

The next proposition gives an alternative definition of the global wave-front set in terms of intersection of sets of characteristic points described in Section 1.

Proposition 2.5. *Let $m \in \{1,2,3\}$, $(r,\rho) \in \mathbb{I}_m$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, $(\mathcal{B},\mathcal{C})$ be an SG-ordered pair with respect to ω_0, $f \in \mathscr{S}'(\mathbf{R}^d)$, and let*

$$S_{\omega_0,\mathcal{C}} = \big\{a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}\big(\mathbf{R}^{2d}\big);\ \mathrm{Op}(a)f \in \mathcal{C}\big\}.$$

Then

$$\mathrm{WF}^m_{\mathcal{B}}(f) = \bigcap_{a \in S_{\omega_0,\mathcal{C}}} \mathrm{Char}^m_{(\omega_0)}(a). \tag{2.3}$$

Proof. Let $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$ and $b_1 \in \mathrm{SG}^{(\omega_0)}_{r,\rho}$ be chosen such that Proposition 1.7(4) is fulfilled after a is replaced by b_1. By Remark 1.9 it follows that $\mathrm{Op}_t(b_1)$ is continuous and bijective from $\mathcal{B}$ onto $\mathcal{C}$, with inverse $\mathrm{Op}_t(b)$. Since

$$\mathrm{Op}(b)\,\mathrm{Op}(a) \in \mathrm{Op}\big(\mathrm{SG}^{(1)}_{r,\rho}\big) = \mathrm{Op}\big(\mathrm{SG}^{0,0}_{r,\rho}\big)$$

when $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}$, we may assume that $\omega_0 = 1$ and $\mathcal{C} = \mathcal{B}$.

In order to prove (2.3) for $m = 1$, we first assume that $(x_0,\xi_0) \notin \mathrm{WF}^1_{\mathcal{B}}(f)$. By Definition 2.1 there exist $\varphi \in \mathscr{C}_{x_0}$ and $\psi \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}$ such that $a(x,\xi) \equiv \varphi(x)\psi(\xi) \in \mathrm{SG}^{0,0}_{1,1}$ and $\mathrm{Op}(a)f \in \mathcal{B}$. Since (1.18) is fulfilled with $\omega_0 = 1$, it follows that a is type-1 invertible at (x_0,ξ_0). Hence $(x_0,\xi_0) \notin \mathrm{Char}^1_{(\omega_0)}(a)$, and we have proved that $\bigcap \mathrm{Char}^1_{(\omega_0)}(a) \subseteq \mathrm{WF}^1_{\mathcal{B}}(f)$.

It remains to prove the opposite inclusion. Let $a \in \mathrm{SG}^{0,0}_{r,\rho}$ be such that $(x_0,\xi_0) \notin \mathrm{Char}^1(a)$ and $\mathrm{Op}(a)f \in \mathcal{B}$. By Proposition 1.14, there are $\varphi \in \mathscr{C}_{x_0}$, $\psi \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}$, $b \in \mathrm{SG}^{0,0}_{r,\rho}$ and $h \in \mathscr{S}$ such that

$$\mathrm{Op}(\varphi \otimes \psi) = \mathrm{Op}(b)\,\mathrm{Op}(a) + \mathrm{Op}(h).$$

Since $\mathrm{Op}(a)f \in \mathcal{B}$, $\mathrm{Op}(b)$ is continuous on $\mathcal{B}$, and $\mathrm{Op}(h)$ maps $\mathscr{S}'$ into $\mathscr{S}$, it follows that $\varphi \cdot (\psi(D)f) = \mathrm{Op}(\varphi \otimes \psi)f \in \mathcal{B}$. Hence $(x_0,\xi_0) \notin \mathrm{WF}^1_{\mathcal{B}}(f)$. This proves (2.3). By similar arguments we also get (2.3) when m equals 2 or 3. The details are left for the reader, and the proof is complete. $\square$

The next result describes the relation between “regularity with respect to $\mathcal{B}$” of temperate distributions and global wave-front sets:

Theorem 2.6. *Let $\mathcal{B}$ be SG-admissible, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Then*

$$f \in \mathcal{B} \quad \Longleftrightarrow \quad \mathrm{WF}_{\mathcal{B}}(f) = \emptyset.$$

For the proof we need the following lemma:

Lemma 2.7. *Let $\mathcal{B}$ be SG-admissible. Then the following is true:*

(1) *if $\mathrm{WF}^1_{\mathcal{B}}(f) = \emptyset$ ($\mathrm{WF}^2_{\mathcal{B}}(f) = \emptyset$), then for each bounded open set $X \subseteq \mathbf{R}^d$, there exists a non-negative $a \in \mathrm{SG}^{0,0}_{1,1}$ such that $a \geqslant 1$ on $X \times \mathbf{R}^d$ ($\mathbf{R}^d \times X$) and $\mathrm{Op}(a)f \in \mathcal{B}$;*

(2) *if $\mathrm{WF}^3_{\mathcal{B}}(f) = \emptyset$, then for some bounded open sets $X_1, X_2 \subseteq \mathbf{R}^d$ such that $0 \in X_1$ and $0 \in X_2$, there exists a non-negative $a \in \mathrm{SG}^{0,0}_{1,1}$ such that $a \geqslant 1$ on $(\mathbf{R}^d \setminus X_1) \times (\mathbf{R}^d \setminus X_2)$ and $\mathrm{Op}(a)f \in \mathcal{B}$.*

Proof. We only prove (1) in the case of the local wave-front set. The other assertions follow by similar arguments and are left for the reader.

The condition $\mathrm{WF}^1_{\mathcal{B}}(f)=\emptyset$ implies that for each $(x,\xi)\in\mathbf{R}^d\times(\mathbf{R}^d\setminus 0)$, there are functions $\varphi_{x,1}\in\mathscr{C}_x$ and $\psi_{\xi,1}\in\mathscr{C}^{\mathrm{dir}}_\xi$ such that $\varphi_{x,1}\cdot\psi_{\xi,1}(D)f\in\mathcal{B}$. Now let $x_0\in\mathbf{R}^d$ be fixed, and recall that each closed cone in $\mathbf{R}^d\setminus 0$ corresponds to a compact set on the unit sphere. Hence, by compactness, it follows that for some $\varphi_{x_0,2}\in\mathscr{C}_{x_0}$, $\xi_1,\dots,\xi_N\in\mathbf{R}^d\setminus 0$ and some constant $R>0$, we have

$$\varphi_{x_0,2}\otimes\psi_1\in \mathrm{SG}^{0,0}_{1,1},\qquad \varphi_{x_0,2}\cdot\psi_1(D)f\in\mathcal{B},\quad\text{and}$$
$$\psi_1(\xi)\equiv\sum_{j=1}^N\psi_{\xi_j,1}(\xi)\geqslant 1,\quad \text{when } |\xi|\geqslant R.$$

Now choose non-negative $\varphi_3\in C_0^\infty(\mathbf{R}^d)$ such that $\varphi_3(\xi)=1$ when $|\xi|\leqslant R$. Then $\varphi_{x_0,2}\cdot\varphi_3(D)f\in C_0^\infty(\mathbf{R}^d)\subseteq\mathcal{B}$, since $\varphi_{x_0,2}\otimes\varphi_3\in C_0^\infty(\mathbf{R}^{2d})$. Hence, for some $\varphi_{x_0}\in\mathscr{C}_{x_0}$, open neighborhood $U=U_{x_0}$ of x_0 and some constant $C>0$, the element $a_{x_0}=C\varphi_{x_0}\otimes(\psi_1+\varphi_3)$ belongs to $\mathrm{SG}^{0,0}_{1,1}$ and is larger than 1 on $U\times\mathbf{R}^d$. Furthermore, $\mathrm{Op}(a_{x_0})f\in\mathcal{B}$. Summing up we have proved that for each $x\in\mathbf{R}^d$, there is an open neighborhood U_x of x and an element $a_x\in\mathrm{SG}^{0,0}_{1,1}$ such that $a_x\geqslant 1$ on U_x and $\mathrm{Op}(a_x)f\in\mathcal{B}$.

For each compact set K we may find finite numbers of $U_{x_1},\dots U_{x_N}$ which cover K. The result now follows if we choose

$$a=a_{x_1}+\dots+a_{x_N}.\qquad\square$$

Proof of Theorem 2.6. The right implication is obvious by Definition 2.1, since operators in $\mathrm{Op}(\mathrm{SG}^{0,0}_{r,\rho})$ are continuous on $\mathcal{B}$.

Assume that $\mathrm{WF}_{\mathcal{B}}(f)=\emptyset$. Then $\mathrm{WF}^m_{\mathcal{B}}(f)=\emptyset$, $m=1,2,3$. By Lemma 2.7(2), there is a non-negative element $a_3\in\mathrm{SG}^{0,0}_{1,1}$, bounded open sets X_1,X_2 such that $0\in X_1$, $0\in X_2$, $a_3\geqslant 1$ in $(\mathbf{R}^d\setminus X_1)\times(\mathbf{R}^d\setminus X_2)$ and $\mathrm{Op}(a_3)f\in\mathcal{B}$. Furthermore, by Lemma 2.7(1), there are non-negative elements $a_1,a_2\in\mathrm{SG}^{0,0}_{1,1}$ such that $a_1\geqslant 1$ in $X_1\times\mathbf{R}^d$, $a_2\geqslant 1$ in $\mathbf{R}^d\times X_2$, $\mathrm{Op}(a_1)f\in\mathcal{B}$ and $\mathrm{Op}(a_2)f\in\mathcal{B}$. Hence, if $a=a_1+a_2+a_3$, it follows that

$$a\in\mathrm{SG}^{0,0}_{1,1},\qquad \mathrm{Op}(a)f\in\mathcal{B}\quad\text{and}\quad a\geqslant 1. \tag{2.4}$$

In particular, a is elliptic in $\mathrm{SG}^{0,0}_{1,1}$, which implies that for some $b\in\mathrm{SG}^{0,0}_{1,1}$ and $h\in\mathscr{S}$ we have

$$\mathrm{Op}(b)\,\mathrm{Op}(a)=\mathrm{Id}+\mathrm{Op}(h)$$

(cf. the proof of Proposition 1.14 in Appendix A). Since $\mathrm{Op}(b)$ and $\mathrm{Op}(h)$ are continuous on $\mathcal{B}$ and $\mathscr{S}\subseteq\mathcal{B}$, (2.4) gives

$$f=\mathrm{Op}(b)\,\mathrm{Op}(a)f-\mathrm{Op}(h)f\in\mathcal{B},$$

and the assertion follows. The proof is complete. $\square$

We conclude the section by giving some remarks on wave-front sets of modulation space type. We start to consider mapping properties under Fourier transformation. Here it is convenient to let ω_T be the composition of the weight $\omega\in\mathscr{P}(\mathbf{R}^{2d})$ with the torsion $T(x,\xi)=(-\xi,x)$, and $\mathscr{B}_T$ denote the space of the pull-backs of the elements of the translation invariant BF-space $\mathscr{B}$ on $\mathbf{R}^{2d}$ with respect to T. That is,

$$\mathscr{B}_T=\{F\circ T;\ F\in\mathscr{B}\},\quad\text{and}\quad \omega_T=\omega\circ T,\quad\text{where } T(x,\xi)=(-\xi,x). \tag{2.5}$$

The first two equalities in the following proposition are related to Lemma 2.4 in [6].

Proposition 2.8. *Let $m, n \in \{1, 2, 3\}$ be such that n equals 2, 1 and 3, when m equals 1, 2 and 3, respectively. Also let $\mathscr{B}$ be a translation invariant BF-space on $\mathbf{R}^{2d}$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, and let T, $\mathscr{B}_T$ and ω_T be as in (2.5). If $f \in \mathscr{S}'(\mathbf{R}^{2d})$, then*

$$T\big(\mathrm{WF}^m_{M(\omega,\mathscr{B})}(f)\big) = \mathrm{WF}^n_{M(\omega_T,\mathscr{B}_T)}(\hat{f}).$$

Proof. By Fourier's inversion formula we have

$$\big|(V_\phi f)\circ T\big| = |V_{\hat{\phi}}\hat{f}|, \qquad \mathscr{F}\big(a\cdot\big(b(D)f\big)\big) = \check{a}(D)(b\cdot\hat{f}).$$

The result is now a straightforward consequence of these identities, Remark 2.2 and the definitions. The details are left for the reader. □

Next we consider wave-front sets with respect to Fourier BF-spaces, and make comparisons with wave-front sets of modulation space types. In fact, in Definition 2.1 we may choose $\mathcal{B}$ as the Fourier BF-space $\mathscr{F}\mathscr{B}_0(\omega)$, where $\mathscr{B}$ is a translation invariant BF-space on $\mathbf{R}^d$ and $\omega \in \mathscr{P}(\mathbf{R}^{2d})$. We remark that if $\mathscr{B}$ is a translation invariant BF-space on $\mathbf{R}^{2d}$ such that (1.10) holds, then (1.11) gives

$$\mathrm{WF}^1_{M(\omega,\mathscr{B})}(f) = \mathrm{WF}^1_{\mathscr{F}\mathscr{B}_0(\omega)}(f), \quad f \in \mathscr{S}'\big(\mathbf{R}^d\big). \tag{2.6}$$

The first type of wave-front sets with respect to general modulation space and Fourier BF-spaces were introduced in [4]. Here we recall these definitions and show that they agree with corresponding type-1 wave-front sets. Let $f \in \mathscr{S}'(\mathbf{R}^d)$, $\phi \in C_0^\infty(\mathbf{R}^d)$ and $\omega \in \mathscr{P}(\mathbf{R}^{2d})$. Also let χ_Γ be the characteristic function of Γ. Then $\mathrm{WF}'_{M(\omega,\mathscr{B})}(f)$ (denoted by $\mathrm{WF}_{M(\omega,\mathscr{B})}(f)$ in [4]) consists of all pairs $(x_0,\xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ such that

$$\big\|\big(V_\phi(\varphi f)\big)\cdot(1\otimes\chi_\Gamma)\cdot\omega\big\|_{\mathscr{B}} = +\infty$$

for every choice of open conical neighborhood Γ of ξ_0 and $\varphi \in \mathscr{C}_{x_0}$. The wave-front set $\mathrm{WF}'_{\mathscr{F}\mathscr{B}(\omega)}(f)$ (denoted by $\mathrm{WF}_{\mathscr{F}\mathscr{B}(\omega)}(f)$ in [4]) consists of all pairs $(x_0,\xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$ such that $|\varphi f|_{\mathscr{F}\mathscr{B}(\omega,\Gamma)} = +\infty$ for every choice of open conical neighborhood Γ of ξ_0 and $\varphi \in \mathscr{C}_{x_0}$. Here

$$|f|_{\mathscr{F}\mathscr{B}(\omega,\Gamma)} \equiv \|\hat{f}\omega\chi_\Gamma\|_{\mathscr{B}}.$$

Proposition 2.9. *Let $f \in \mathscr{S}'(\mathbf{R}^d)$, $\mathscr{B}$ be a translation invariant BF-space, $\mathscr{B}_0$ be defined by (1.10) and let $\omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then*

$$\mathrm{WF}^1_{M(\omega,\mathscr{B})}(f) = \mathrm{WF}'_{M(\omega,\mathscr{B})}(f) = \mathrm{WF}^1_{\mathscr{F}\mathscr{B}_0(\omega)}(f) = \mathrm{WF}'_{\mathscr{F}\mathscr{B}_0(\omega)}(f).$$

Proof. By Theorem 6.9 in [4] we have $\mathrm{WF}'_{M(\omega,\mathscr{B})}(f) = \mathrm{WF}'_{\mathscr{F}\mathscr{B}_0(\omega)}(f)$. Hence, in view of (2.6), it suffices to prove $\mathrm{WF}^1_{\mathscr{F}\mathscr{B}_0(\omega)}(f) = \mathrm{WF}'_{\mathscr{F}\mathscr{B}_0(\omega)}(f)$. By Remark 2.2 we have

$$\begin{aligned}
&(x_0,\xi_0) \notin \mathrm{WF}'_{\mathscr{F}\mathscr{B}_0(\omega)}(f) \\
&\iff \quad |\varphi_{x_0} f|_{\mathscr{F}\mathscr{B}_0(\omega,\Gamma)} < \infty \quad \text{for some } \varphi_{x_0} \in \mathscr{C}_{x_0} \text{ and } \Gamma = \Gamma_{\xi_0} \\
&\iff \quad \big\|\psi_{\xi_0}(D)(\varphi_{x_0} f)\big\|_{\mathscr{F}\mathscr{B}_0(\omega)} < \infty \quad \text{for some } \varphi_{x_0} \in \mathscr{C}_{x_0} \text{ and } \psi_{\xi_0} \in \mathscr{C}^{\mathrm{dir}}_{\xi_0} \\
&\iff \quad (x_0,\xi_0) \notin \mathrm{WF}^1_{\mathscr{F}\mathscr{B}_0(\omega)}(f).
\end{aligned}$$

This proves the result. □

3. Wave-front sets for pseudo-differential operators with smooth symbols

In this section we consider mapping properties for pseudo-differential operators with respect to global wave-front sets. More precisely, we prove that micro-locality and micro-ellipticity hold for pseudo-differential operators in $\operatorname{Op}(\mathrm{SG}^{(\omega_0)}_{r,\rho})$. We start with the following result:

Theorem 3.1. *Let $m \in \{1,2,3\}$, $(r,\rho) \in \mathbb{I}_m$, $t \in \mathbf{R}$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Moreover, let $(\mathcal{B},\mathcal{C})$ be an SG-ordered pair with respect to ω_0. Then*

$$\mathrm{WF}^m_{\mathcal{C}}\big(\operatorname{Op}_t(a)f\big) \subseteq \mathrm{WF}^m_{\mathcal{B}}(f) \subseteq \mathrm{WF}^m_{\mathcal{C}}\big(\operatorname{Op}_t(a)f\big) \cup \operatorname{Char}^m_{(\omega_0)}(a). \tag{3.1}$$

Proof. We only prove the case $m = 3$. The assertions for $m = 1$ and $m = 2$ follow by similar arguments and are left for the reader.

Assume that $(x_0,\xi_0) \notin \mathrm{WF}^3_{\mathcal{B}}(f)$. We shall prove that $(x_0,\xi_0) \notin \mathrm{WF}^3_{\mathcal{C}}(\operatorname{Op}(a)f)$. For some $\psi_{1,1} \in \mathscr{C}^{\mathrm{dir}}_{x_0}$ and $\psi_{1,2} \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}$ we have

$$\psi_{1,1} \cdot \psi_{1,2}(D)f \in \mathcal{B}, \tag{3.2}$$

in view of (1) in Definition 2.1. Let $\psi_{2,1} \in \mathscr{C}^{\mathrm{dir}}_{x_0}$ and $\psi_{2,2} \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}$ be such that $\psi_{1,j} = 1$ on $\operatorname{supp}\psi_{2,j}$, and set

$$c_1(x,\xi) = \psi_{1,1}(x)\psi_{1,2}(\xi) \quad \text{and} \quad c_2(x,\xi) = \psi_{2,1}(x)\psi_{2,2}(\xi).$$

Then $c_1, c_2 \in \mathrm{SG}^{0,0}_{r,\rho}$, and since $c_1 = 1$ on $\operatorname{supp} c_2$, and $\mathrm{SG}^{-\infty,-\infty}_{r,\rho} = \mathscr{S}$, it follows from the symbolic calculus that

$$\operatorname{Op}(c_2)\operatorname{Op}(a) = \operatorname{Op}(c_2)\operatorname{Op}(a)\operatorname{Op}(c_1) \mod \operatorname{Op}(\mathscr{S}). \tag{3.3}$$

Now we recall that the mappings

$$\operatorname{Op}(a) : \mathcal{B} \to \mathcal{C}, \qquad \operatorname{Op}(c_2) : \mathcal{C} \to \mathcal{C} \tag{3.4}$$

are continuous (cf. Proposition 1.7). A combination of (3.2)–(3.4) with the facts that $\operatorname{Op}(c_1) = \psi_{1,1} \cdot \psi_{1,2}(D)$ and that $\operatorname{Op}(h)$ maps $\mathscr{S}'$ into $\mathscr{S}$ gives

$$\psi_{2,1} \cdot \psi_{2,2}(D)\big(\operatorname{Op}(a)f\big) = \operatorname{Op}(c_2)\operatorname{Op}(a)f = \operatorname{Op}(c_2)\operatorname{Op}(a)\operatorname{Op}(c_1)f \mod \mathscr{S} \in \mathcal{C}.$$

This proves that $(x_0,\xi_0) \notin \mathrm{WF}^3_{\mathcal{C}}(\operatorname{Op}(a)f)$, and the first inclusion in (3.1) follows.

It remains to prove the second inclusion in (3.1). Assume that

$$(x_0,\xi_0) \notin \mathrm{WF}^3_{\mathcal{C}}\big(\operatorname{Op}(a)f\big) \cup \operatorname{Char}^3_{(\omega_0)}(a).$$

By Remark 2.4, there exist $\psi_{1,1} \in \mathscr{C}^{\mathrm{dir}}_{x_0}$ and $\psi_{1,2} \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}$, $b \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}$ and $h \in \mathscr{S}$ such that

$$\psi_{1,1} \cdot \psi_{1,2}(D)\big(\operatorname{Op}(a)f\big) \in \mathcal{C}$$

and (1.19) holds for $c = c_1 \equiv \psi_{1,1} \otimes \psi_{1,2}$. We claim that

$$\operatorname{Op}(c_2) = \operatorname{Op}(c_2)\operatorname{Op}(b)\operatorname{Op}(c_1)\operatorname{Op}(a) + \operatorname{Op}(h), \tag{3.5}$$

for some $h \in \mathscr{S}$, where $c_2 = \psi_{2,1} \otimes \psi_{2,2}$, and $\psi_{2,1} \in \mathscr{C}^{\mathrm{dir}}_{x_0}$ and $\psi_{2,2} \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}$ are such that $\psi_{1,j} = 1$ on $\operatorname{supp} \psi_{2,j}$ for $j = 1, 2$.

In fact, by combining (1.19) with the fact that $\mathrm{Op}(c_2)\,\mathrm{Op}(c_1) = \mathrm{Op}(c_2) \mod \mathrm{Op}(\mathscr{S})$, we get

$$\begin{aligned}\mathrm{Op}(c_2) &= \mathrm{Op}(c_2)\,\mathrm{Op}(b)\,\mathrm{Op}(a) \mod \mathrm{Op}(\mathscr{S})\\ &= \mathrm{Op}(c_2)\,\mathrm{Op}(c_1)\,\mathrm{Op}(b)\,\mathrm{Op}(a) \mod \mathrm{Op}(\mathscr{S})\\ &= \mathrm{Op}(c_2)\,\mathrm{Op}(b)\,\mathrm{Op}(c_1)\,\mathrm{Op}(a) \mod \mathrm{Op}(\mathscr{S}),\end{aligned}$$

and (3.5) follows. Here the last equality follows from the fact that

$$\mathrm{Op}(c_2)\big[\mathrm{Op}(b), \mathrm{Op}(c_1)\big] \in \mathrm{Op}(\mathscr{S}) \quad \text{and} \quad \mathrm{Op}(\mathscr{S})\,\mathrm{Op}(a) \subseteq \mathrm{Op}(\mathscr{S}),$$

when $c_1 = 1$ on $\operatorname{supp} c_2$, where $[\cdot,\cdot]$ denotes the commutator. A combination of Proposition 1.7, (3.5) and the fact that $\mathrm{Op}(c_1)(\mathrm{Op}(a)f) \in \mathcal{C}$ now shows that the mappings

$$\mathrm{Op}(b) : \mathcal{C} \to \mathcal{B}, \qquad \mathrm{Op}(c_2) : \mathcal{B} \to \mathcal{B}$$

and

$$\mathrm{Op}(h) : \mathscr{S}' \to \mathscr{S}$$

are continuous and that $\mathrm{Op}(c_2)f \in \mathcal{B}$. Hence, we have showed that $(x_0, \xi_0) \notin \mathrm{WF}^3_{\mathcal{B}}(f)$, and the proof is complete. □

Corollary 3.2. *Let $m \in \{1,2,3\}$, $r > 0$, $f \in \mathscr{S}'(\mathbf{R}^d)$ and $\varphi \in C^\infty(\mathbf{R}^d)$ be such that $\langle x\rangle^{r|\alpha|}\partial^\alpha \varphi(x) \in L^\infty(\mathbf{R}^d)$ for every α. Also let $\mathcal{B}$ be SG-admissible with respect to r, 1 and d. Then*

$$\mathrm{WF}^m_{\mathcal{B}}(\varphi f) \subseteq \mathrm{WF}^m_{\mathcal{B}}(f). \tag{3.6}$$

Proof. It follows from the assumptions that $a \equiv \varphi \otimes 1 \in \mathrm{SG}^{0,0}_{r,1}$. Hence, for $m \in \{1,2,3\}$, Theorem 3.1 gives

$$\mathrm{WF}^m_{\mathcal{B}}(\varphi f) = \mathrm{WF}^m_{\mathcal{B}}\big(\mathrm{Op}(a)f\big) \subseteq \mathrm{WF}^m_{\mathcal{B}}(f),$$

as claimed. □

Next we apply Theorem 3.1 on operators which are elliptic with respect to $\omega_0 \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$ when $0 < r, \rho \leqslant 1$. We recall that a and $\mathrm{Op}(a)$ are called SG-*elliptic* with respect to $\mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ or ω_0, if there is a compact set $K \subset \mathbf{R}^{2d}$ such that (1.18) holds when $(x,\xi) \notin K$. By (1.15) it follows that

$$\big|D^\alpha_x D^\beta_\xi a(x,\xi)\big| \lesssim \big|a(x,\xi)\big|\langle x\rangle^{-r|\alpha|}\langle \xi\rangle^{-\rho|\beta|}, \quad (x,\xi) \in \mathbf{R}^{2d} \setminus K,$$

for every multi-index α, when a is SG-elliptic (see, e.g., [16,1]).

It follows from Lemma 2.7 that $\mathrm{Char}_{(\omega_0)}(a) = \emptyset$ if and only if a is SG-elliptic with respect to ω_0. The following result is now an immediate consequence of Theorems 2.6 and 3.1.

Theorem 3.3. *Let $m \in \{1,2,3\}$, $(r,\rho) \in \mathbb{I}_m$, $t \in \mathbf{R}$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ be SG-elliptic with respect to ω_0 and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Moreover, let $(\mathcal{B},\mathcal{C})$ be an SG-ordered pair with respect to ω_0. Then*

$$\mathrm{WF}^m_{\mathcal{C}}\big(\mathrm{Op}_t(a)f\big) = \mathrm{WF}^m_{\mathcal{B}}(f).$$

Furthermore, if $g \in \mathcal{C}$, and $f \in \mathscr{S}'(\mathbf{R}^d)$ solves the equation

$$\mathrm{Op}_t(a)f = g, \tag{3.7}$$

then $f \in \mathcal{B}$.

4. Examples

In this section we show how the results in the previous sections can be applied to problems involving partial differential operators. In particular, we focus on the sets of characteristic points and some links on possible application in numerical analysis.

In the first examples we consider wave-front properties for hypoelliptic problems, especially for hypoelliptic partial differential operators with constant coefficients.

Example 4.1. Let $a(x,\xi)$ be the symbol of a linear partial differential operator on $\mathbf{R}^d$ with constant coefficients, which is hypoelliptic in the sense of [16]. Then $a(x,\xi) = a_2(\xi)$ for some a_2. Furthermore, a is type-m elliptic, $m = 1,3$, with respect to

$$\omega_0(x,\xi) = \omega_0(\xi) = \left(1 + \left|a_2(\xi)\right|^2\right)^{1/2}, \tag{4.1}$$

which belongs to $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, for some $r, \rho > 0$. In particular we may apply Theorem 3.3 on $\mathrm{Op}(a)$.

For the set of characteristic points of type-2 we have

$$\mathrm{Char}^2_{(\omega_0)}(a) \subseteq \left(\mathbf{R}^d \setminus 0\right) \times \left\{(0,0)\right\}.$$

Hence if $\mathcal{B} = M(\omega, \mathscr{B})$ is a modulation space, $\mathcal{C} = M(\omega/\omega_0, \mathscr{B})$ and $f \in \mathscr{S}'(\mathbf{R}^d)$, then

$$\mathrm{WF}^m_{\mathcal{C}}\left(\mathrm{Op}(a)f\right) = \mathrm{WF}^m_{\mathcal{B}}(f), \quad m = 1,3, \quad \text{and}$$
$$\mathrm{WF}^2_{\mathcal{C}}\left(\mathrm{Op}(a)f\right) \subseteq \mathrm{WF}^2_{\mathcal{B}}(f) \subseteq \mathrm{WF}^2_{\mathcal{C}}\left(\mathrm{Op}(a)f\right) \cup \left(\left(\mathbf{R}^d \setminus 0\right) \times \{0\}\right).$$

Now we apply the wave-front properties to obtain information for a parametrix E to $\mathrm{Op}(a)$. From Proposition 1.7(3) it follows that $\mathrm{Op}(a)$ maps $M^{p,q}_{(\omega_0)}$ to $M^{p,q}$. Since

$$M^{p,q} \cap \mathscr{E}' = \mathscr{F}L^q \cap \mathscr{E}'$$

and $\mathrm{Op}(a)E$ is locally in $\mathscr{F}L^\infty$, it follows that E is locally in $\mathscr{F}L^\infty_{(\omega_0)}$. This means that for each $\varphi \in C_0^\infty$, we have

$$\left|\mathscr{F}(\varphi E)(\xi)\right| \lesssim \omega_0(\xi)^{-1}. \tag{4.2}$$

An interesting case appears when $x \in \mathbf{R}^d$ is replaced by $(x,t) \in \mathbf{R}^{d+1}$, and $\mathrm{Op}(a)$ agrees with the heat operator

$$\partial_t - \Delta_x, \quad (x,t) \in \mathbf{R}^{d+1},$$

which is a classical example on a hypoelliptic operator. The symbol is given by $a(x,t,\xi,\tau) = |\xi|^2 + i\tau$. In this case, ω_0 takes the form

$$\omega_0 = \left(1 + |\xi|^4 + |\tau|^2\right)^{1/2}. \tag{4.1$'$}$$

Hence, (4.2) shows that if E is a fundamental solution to $\mathrm{Op}(a)$, then for each $\varphi \in C_0^\infty$, there is a constant C such that

$$\left|\mathscr{F}(\varphi E)(\xi,\tau)\right| \leqslant C\left(1+|\xi|^4+|\tau|^2\right)^{-1/2}.$$

This can easily be verified by numerical computations.

Next we show how a small perturbation of any hypoelliptic operator in the previous example makes that all the sets of characteristic points become empty.

Example 4.2. Let $0 < r, \rho \leqslant 1$, $a(x,\xi) = a_1(x,\xi) + a_2(\xi)$ be such that the following conditions are fulfilled:

(1) $|a(x,\xi)| \geqslant c$ for some constant $c > 0$ outside a compact set in $\mathbf{R}^{2d}$;
(2) $a_1 \in \mathrm{SG}^{0,0}_{r,\rho}(\mathbf{R}^{2d})$;
(3) a_2 is the symbol of a linear partial differential operator with constant coefficient which is hypoelliptic in the sense of [16].

Then a is elliptic with respect to $\omega_0(x,\xi)$ in (4.1). Hence we may again apply Theorem 3.3 on $\mathrm{Op}(a)$.

An interesting case concerns the modified heat operator $a_1(x,t) + \partial_t - \Delta_x$, where $(x,t) \in \mathbf{R}^{d+1}$, $a_1(x,t) \in C^\infty(\mathbf{R}^{d+1})$ and a_1 is equal to $c > 0$ outside a compact set in $\mathbf{R}^{d+1}$. The symbol of the operator is $a(x,t,\xi,\tau) = a_1(x,t) + |\xi|^2 + i\tau$. In this case, a is elliptic with respect to (4.1)′. Hence, if $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\mathscr{B}$ is a translation invariant BF-space on $\mathbf{R}^d$, and (3.7) holds for some $f \in \mathscr{S}'(\mathbf{R}^d)$ and $g \in M(\omega/\omega_0, \mathscr{B})$, then it follows from Theorem 3.3 that $f \in M(\omega, \mathscr{B})$.

In the next example we consider the fundamental solutions of the Schrödinger operator.

Example 4.3. The Schrödinger operator for a free particle has the form

$$i\partial_t - \Delta_x, \quad (x,t) \in \mathbf{R}^{d+1},$$

and the symbol is given by $a(x,t,\xi,\tau) = |\xi|^2 - \tau$. Let ω_0 be the same as in (4.1). Then, $a \in \mathrm{SG}^{(\omega_0)}_{1,1}$, and the sets of characteristic points are

$$\mathrm{Char}^1_{(\omega_0)}(a) = \left\{(x,t,\xi,\tau) \in \mathbf{R}^{d+1} \times \left(\mathbf{R}^{d+1} \setminus 0\right);\ \xi = 0,\ \tau > 0\right\},$$
$$\mathrm{Char}^2_{(\omega_0)}(a) = \left\{(x,t,\xi,\tau) \in \left(\mathbf{R}^{d+1} \setminus 0\right) \times \mathbf{R}^{d+1};\ \tau = |\xi|^2\right\},$$
$$\mathrm{Char}^3_{(\omega_0)}(a) = \left\{(x,t,\xi,\tau) \in \left(\mathbf{R}^{d+1} \setminus 0\right) \times \left(\mathbf{R}^{d+1} \setminus 0\right);\ \xi = 0,\ \tau > 0\right\}.$$

This implies that numerical computations can be rather easily performed as long as the frequency τ related to the t-variable is negative.

Let E denote the fundamental solution of $\mathrm{Op}(a)$. From Proposition 1.7(3) it follows that $\mathrm{Op}(a)$ maps $M^{p,q}_{(\omega_0)}$ to $M^{p,q}$. Now let Γ be a cone which does not hit the set $\{(0,\tau);\ \tau > 0\}$. Then the same arguments as in Example 4.1 show that

$$\left\|\mathscr{F}(\varphi E)\psi\omega_0\right\|_{L^\infty} < \infty$$

for every $\varphi \in C_0^\infty$ and $\psi \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$.

Remark 4.4. Here we note, once again, that the sets of characteristic points $\mathrm{Char}'(a)$, as defined in Section 18.1 in [16], for the operators in Examples 4.3 and 4.1, are strictly larger compared to the corresponding local components $\mathrm{Char}^1_{(\omega_0)}(a)$. In fact, if a is the same as in Example 4.3 or 4.1, then

$$\mathrm{Char}'(a) = \{(x,t,\xi,\tau) \in \mathbf{R}^{d+1} \times (\mathbf{R}^{d+1} \setminus 0);\ \xi = 0,\ \tau \neq 0\},$$

which strictly includes $\mathrm{Char}^1_{(\omega_0)}(a)$.

In the next example we consider an operator with polynomial coefficients, with different growth with respect to spatial axis. We show the associated *loss of decay* at infinity, in terms of wave-front sets with respect to $H^2_{s,t}$ spaces.

Example 4.5. Let $a(x,\xi)$ be the symbol on $\mathbf{R}^4$, given by

$$a(x,\xi) = (1 + x_1^2 + x_2^4)(1 + |\xi|^2), \quad x, \xi \in \mathbf{R}^2.$$

Also assume that $f, g \in \mathscr{S}'(\mathbf{R}^2)$ are chosen such that the equation $\mathrm{Op}(a) f = g$ is fulfilled. By letting $\omega_0(x,\xi) = a(x,\xi)$, it follows that a is elliptic with respect to ω_0, and that $a \in \mathrm{SG}^{(\omega_0)}_{1,1}(\mathbf{R}^4)$. If

$$H^2_{(\omega_0)}(\mathbf{R}^2) \equiv \{f \in \mathscr{S}'(\mathbf{R}^2);\ (1 + x_1^2 + x_2^4)(1 - \Delta) f \in L^2(\mathbf{R}^2)\} = M^2_{(\omega_0)}(\mathbf{R}^2),$$

then it is obvious that $f \in H^2_{(\omega_0)}$, if and only if $g \in L^2$, which is also confirmed by Theorem 3.3.

Furthermore, $a \in \mathrm{SG}^{4,2}_{1,1}(\mathbf{R}^4)$ is SG-hypoelliptic with *inverse order* $(2,2)$, in the sense of [2], since, for $|x| + |\xi| \geqslant R > 0$,

$$\langle x\rangle^2 \langle\xi\rangle^2 \lesssim a(x,\xi) \lesssim \langle x\rangle^4 \langle\xi\rangle^2.$$

We consider $\mathrm{Op}(a)$ when acting on the scale of weighted Sobolev spaces $H^2_{s,t}(\mathbf{R}^2)$, $s,t \in \mathbf{R}$, and investigate the solutions of the equation $\mathrm{Op}(a) f = g$ with $g \in H^2_{0,0}(\mathbf{R}^2) = L^2(\mathbf{R}^2)$. Let

$$c_1 = \psi_1 \otimes \varphi_1, \quad \text{where } \psi_1 \in \mathscr{C}^{\mathrm{dir}}_{(x_1,0)},\ \varphi_1 \in \mathscr{C}_{\xi_0},\ x_1 \neq 0,\ \xi_0 \in \mathbf{R}^2,$$

and

$$c_2 = \psi_2 \otimes \varphi_2, \quad \text{where } \psi_2 \in \mathscr{C}^{\mathrm{dir}}_{(0,x_2)},\ \varphi_2 \in \mathscr{C}_{\xi_0},\ x_2 \neq 0,\ \xi_0 \in \mathbf{R}^2.$$

Then there exist $b_j \in \mathrm{SG}^{-2j,-2}_{1,1}(\mathbf{R}^4)$ and $h_j \in \mathscr{S}(\mathbf{R}^4)$ for $j = 1,2$ such that $\mathrm{Op}(b_j)\,\mathrm{Op}(a) = \mathrm{Op}(c_j) + \mathrm{Op}(h_j)$. (In fact, $\mathrm{Op}(a)$ is invertible with inverse $\mathrm{Op}(b)$, where $b \in \mathrm{SG}^{(1/\omega_0)}_{1,1}(\mathbf{R}^4)$.) This implies that for some choices of f and g, there are *anisotropic loss of decays* for f, in the sense that for any $\varepsilon > 0$,

$$\begin{aligned}
((x_1,0),\xi) &\in \mathrm{WF}^2_{H^2_{2,2+\varepsilon}}(f) \setminus \mathrm{WF}^2_{H^2_{2,2}}(f), \quad x_1 \neq 0,\\
((0,x_2),\xi) &\in \mathrm{WF}^2_{H^2_{2,4+\varepsilon}}(f) \setminus \mathrm{WF}^2_{H^2_{2,4}}(f), \quad x_2 \neq 0,
\end{aligned} \tag{4.3}$$

and the same results hold for the corresponding $\mathrm{WF}^3_{H^2_{s,t}}(f)$ components, by choosing symbols

$$c_j = \psi_{1j} \otimes \psi_{2j}, \quad \text{where } \psi_{1j} \in \mathscr{C}^{\mathrm{dir}}_x,\ \psi_{2j} \in \mathscr{C}^{\mathrm{dir}}_\xi,\ j = 1,2,\ x,\xi \neq 0.$$

In particular, the first formula in (4.3) holds also with any $(x_1, x_2) \in \mathbf{R}^2$, $x_1 \neq 0$, in place of $(x_1, 0)$, $x_1 \neq 0$.

In the next example, we apply the theory to the propagation of electromagnetic waves in a wave-guide. The corresponding problem, concerning electromagnetic waves in a uniform cylindrical wave-guide, has been considered by Kristensson in [19] and by Khrennikov and Nilsson in [18], and will be viewed here in the context of pseudo-differential operators and wave-front sets.

Example 4.6. Let $S \subset \mathbf{R}^3$ be a conducting cylindrical tube along the x_3 axis with ∂S as its boundary surface. With S_0 we denote the cross section in the $x_1 x_2$-plane, assumed to be compact and with a smooth boundary.

For the propagation of the electromagnetic wave, we then have a scalar field $\phi = \phi(x,t) = \phi(x_1, x_2, x_3, t)$, $(x_1, x_2, x_3) \in S$, $t \in \mathbf{R}$, satisfying the wave equation in the wave-guide, i.e.,

$$\left(\Delta_x - \partial_t^2\right)\phi = 0.$$

As boundary conditions we may choose either the Neumann boundary condition, given by $\partial_N \phi = 0$ on ∂S, or the Dirichlet boundary condition, given by $\phi = 0$ on ∂S. Here ∂_N is the normal derivative. When solving such type of equation one can use a splitting technique, separating the pairs of variables (x_1, x_2) and (x_3, t). The solution can then be written as

$$\phi(x_1, x_2, x_3, t) = \sum_{n=1}^{\infty} v_n(x_1, x_2) w_n(x_3, t),$$

where $\{v_n(x, y)\}$, $n = 1, 2, \ldots,$ is a complete orthonormal system of eigenfunctions of

$$\left(\partial_{x_1}^2 + \partial_{x_2}^2 + m_n^2\right) v_n(x_1, x_2) = 0,$$

with $\partial_N v_n = 0$ or $v_n = 0$ on ∂S_0, with eigenvalues $-m_n^2$. We have that w_n satisfies the Klein–Gordon equation

$$\left(\partial_{x_3}^2 - \partial_t^2 - m_n^2\right) w_n(x_3, t) = 0. \tag{4.4}$$

In the (x_1, x_2) direction the problem can be solved using discretization and Fourier series, since the cross section is finite. We therefore consider Eq. (4.4). The operator on the left-hand side of (4.4) can be written as a pseudo-differential operator with symbol $a(x_3, t, \xi_3, \tau) = \tau^2 - \xi_3^2 - m_n^2$. Let $\omega_0(x_3, t, \xi_3, \tau) = (1 + \xi_3^2 + \tau^2)$. Then $a \in \mathrm{SG}_{1,1}^{(\omega_0)}$ has the following sets of characteristic points

$$\mathrm{Char}^1_{(\omega_0)}(a) = \left\{(x_3, t, \xi_3, \tau) \in \mathbf{R}^4;\ \tau = \pm\xi_3 \neq 0\right\},$$

$$\mathrm{Char}^2_{(\omega_0)}(a) = \left\{(x_3, t, \xi_3, \tau) \in \left(\mathbf{R}^2 \setminus 0\right) \times \mathbf{R}^2;\ \tau = \pm\sqrt{\xi_3^2 + m_n^2}\right\},$$

$$\mathrm{Char}^3_{(\omega_0)}(a) = \left\{(x_3, t, \xi_3, \tau) \in \left(\mathbf{R}^2 \setminus 0\right) \times \mathbf{R}^2;\ \tau = \pm\xi_3 \neq 0\right\}.$$

Now let Γ be a closed cone which does not contain any point from the set $\{(\xi_3, \tau);\ \tau = \pm\xi_3\}$. Furthermore let w_n be the solution of Eq. (4.4). Since the right-hand side in (4.4) is zero, it follows that for every x_0 there exist $\varphi \in \mathscr{C}_{x_0}$ and $\psi \in \mathscr{C}^{\mathrm{dir}}_{\xi_0}(\Gamma)$ such that $\psi \mathscr{F}(\varphi w_n) \in \mathscr{S}$.

5. Wave-front sets with respect to sequences of spaces

In this section we define wave-front sets based on sequences of admissible spaces, and discuss basic results. In the first part we consider sequences of spaces which are parameterized with one index, and prove that the mapping properties in Section 3 extend to wave-front sets with respect to sequences. Thereafter we discuss further extensions where we consider sequences of spaces which are parameterized with two indices. In the last part we give some examples on new types of wave-front sets that can be constructed, and show some consequences of our investigations. For example, here we introduce wave-front sets which are related to “classical wave-front sets” in the sense that they are wave-front sets with respect to classical spaces of smooth functions. In particular, we show that (a refinement of) the wave-front set of Schwartz type treated in [6] can be obtained as a wave-front set based on sequences of admissible spaces.

5.1. Wave-front sets with respect to sequences with one index parameter

Again we start by introducing the complements of the wave-front sets. More precisely, let J be an index set of integers, Ω_m, $m\in\{1,2,3\}$, be given by (1.20), $(\mathcal{B}_j)=(\mathcal{B}_j)_{j\in J}$ be a sequence of Banach or Fréchet spaces such that $\mathscr{S}(\mathbf{R}^d)\subseteq\mathcal{B}_j\subseteq\mathscr{S}'(\mathbf{R}^d)$, for every j, and let $f\in\mathscr{S}'(\mathbf{R}^d)$. Then the point $(x_0,\xi_0)\in\Omega_m$ is called *type*-$(m,\cup)$ regular (*type*-$(m,\cap)$ regular) for f with respect to $(\mathcal{B}_j)$, if

$$\operatorname{Op}(c_m)f\in\bigcap_j\mathcal{B}_j\qquad\Bigl(\operatorname{Op}(c_m)f\in\bigcup_j\mathcal{B}_j\Bigr),\tag{2.1$'$}$$

for some c_m in Remark 1.13. The set of all type-$(m,\cup)$ regular points (type-$(m,\cap)$ regular points) for f with respect to $(\mathcal{B}_j)$, is denoted by $\Theta^{m,\cup}_{(\mathcal{B}_j)}(f)$ ($\Theta^{m,\cap}_{(\mathcal{B}_j)}(f)$).

It is also desirable that right-hand sides of (2.1)$'$ should be a vector space, which is guaranteed by imposing that $(\mathcal{B}_j)$ should be *ordered*, i.e. $\mathcal{B}_j$ should be increasing or decreasing with respect to $j\in J$.

Definition 5.1. Let J be an index set of integers, $m\in\{1,2,3\}$, Ω_m be as in (1.20), and let $(\mathcal{B}_j)_{j\in J}$ be a sequence of Banach or Fréchet spaces such that $\mathscr{S}(\mathbf{R}^d)\subseteq\mathcal{B}_j\subset\mathscr{S}'(\mathbf{R}^d)$, for every j:

(1) the *type*-$(m,\cup)$ *wave-front set* (*type*-$(m,\cap)$ *wave-front set*) of $f\in\mathscr{S}'(\mathbf{R}^d)$ with respect to $(\mathcal{B}_j)$ is the complement of $\Theta^{m,\cup}_{(\mathcal{B}_j)}(f)$ ($\Theta^{m,\cap}_{(\mathcal{B}_j)}(f)$) in Ω_m, and is denoted by $\mathrm{WF}^{m,\cup}_{(\mathcal{B}_j)}(f)$ ($\mathrm{WF}^{m,\cap}_{(\mathcal{B}_j)}(f)$);
(2) the *global wave-front sets* $\mathrm{WF}^{\cup}_{(\mathcal{B}_j)}(f)\subseteq(\mathbf{R}^d\times\mathbf{R}^d)\setminus 0$ and $\mathrm{WF}^{\cap}_{(\mathcal{B}_j)}(f)\subseteq(\mathbf{R}^d\times\mathbf{R}^d)\setminus 0$, of $\cup$ and $\cap$ types, respectively, are the sets

$$\mathrm{WF}^{\cup}_{(\mathcal{B}_j)}(f)\equiv\mathrm{WF}^{1,\cup}_{(\mathcal{B}_j)}(f)\cup\mathrm{WF}^{2,\cup}_{(\mathcal{B}_j)}(f)\cup\mathrm{WF}^{3,\cup}_{(\mathcal{B}_j)}(f),$$
$$\mathrm{WF}^{\cap}_{(\mathcal{B}_j)}(f)\equiv\mathrm{WF}^{1,\cap}_{(\mathcal{B}_j)}(f)\cup\mathrm{WF}^{2,\cap}_{(\mathcal{B}_j)}(f)\cup\mathrm{WF}^{3,\cap}_{(\mathcal{B}_j)}(f).$$

Example 5.2. We can consider wave-front sets with respect to sequences of the form

$$(\mathcal{B}_j)\equiv(\mathcal{B}_j)_{j\in J},\quad\text{with }\mathcal{B}_j=M(\omega_j,\mathscr{B}_j),\tag{5.1}$$

where $\omega_j\in\mathscr{P}(\mathbf{R}^{2d})$, $\mathscr{B}_j$ is a translation invariant BF-space on $\mathbf{R}^d$, and j belongs to some index set J.

Remark 5.3. Let $p_j,q_j\in[1,\infty]$, $\mathscr{B}_j=L^{p_j,q_j}_1(\mathbf{R}^{2d})$, $\omega_j(x,\xi)=\langle x,\xi\rangle^{-j}$ and let $\mathcal{B}_j$ be as in (5.1) for $j\in J=\mathbf{N}_0$. Then it follows that $\mathrm{WF}^{m,\cup}_{(\mathcal{B}_j)}(f)$, $m=1,2,3$, in Definition 5.1 are equal to the wave-front sets $\mathrm{WF}^{\psi}(f)$, $\mathrm{WF}^{e}(f)$ and $\mathrm{WF}^{\psi e}(f)$ in [6], respectively. In particular, it follows that $\mathrm{WF}^{\cup}_{(\mathcal{B}_j)}(f)$ is equal to the global wave-front set $\mathrm{WF}_{\mathscr{S}}(f)$, which in [6] is denoted by $\mathrm{WF}_{\mathcal{S}}(f)$.

Remark 5.4. Evidently, if $\mathcal{B}_j = \mathcal{B}$ for every $j \in J$, then

$$\mathrm{WF}^{m,\cup}_{(\mathcal{B}_j)}(f) = \mathrm{WF}^{m,\cap}_{(\mathcal{B}_j)}(f) = \mathrm{WF}^{m}_{\mathcal{B}}(f), \quad m = 1, 2, 3.$$

In the following two results we make some basic remarks, which also motivates the notations for wave-front sets of sequence types.

Proposition 5.5. *Let $m \in \{1, 2, 3\}$, $\mathcal{B}_j$ be the same as in Definition 5.1, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Then*

$$\bigcup \mathrm{WF}^{m}_{\mathcal{B}_j}(f) \subseteq \mathrm{WF}^{m,\cup}_{(\mathcal{B}_j)}(f), \qquad \bigcup \mathrm{WF}_{\mathcal{B}_j}(f) \subseteq \mathrm{WF}^{\cup}_{(\mathcal{B}_j)}(f),$$

and

$$\bigcap \mathrm{WF}^{m}_{\mathcal{B}_j}(f) = \mathrm{WF}^{m,\cap}_{(\mathcal{B}_j)}(f), \qquad \bigcap \mathrm{WF}_{\mathcal{B}_j}(f) = \mathrm{WF}^{\cap}_{(\mathcal{B}_j)}(f).$$

Proposition 5.5 is a straightforward consequence of the definitions. We refer to Proposition 4.4 in [5] for the proof. The details are left for the reader. We remark that we may choose $\mathcal{B}_j$ and $f \in \mathscr{S}'(\mathbf{R}^d)$ such that equality is not attained in the first inclusion in Proposition 5.5 when $m = 1$ (cf. [25, Example 1.11]).

The following generalization of Theorem 2.6 shows that the global wave-front sets of sequence types describe the regularity of a tempered distribution with respect to intersections and unions of the involved spaces, provided the latter are SG-admissible.

Theorem 2.6′. *Let $\mathcal{B}_j$ be SG-admissible for every j, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Then*

$$f \in \bigcap \mathcal{B}_j \quad \Longleftrightarrow \quad \mathrm{WF}^{\cup}_{(\mathcal{B}_j)}(f) = \emptyset, \tag{5.2}$$

and, if in addition $(\mathcal{B}_j)$ is ordered,

$$f \in \bigcup \mathcal{B}_j \quad \Longleftrightarrow \quad \mathrm{WF}^{\cap}_{(\mathcal{B}_j)}(f) = \emptyset. \tag{5.3}$$

Proof. We only prove the first equivalence: the second one follows by a similar argument and is left for the reader.

We notice that $f \in \bigcap \mathcal{B}_j \Leftrightarrow \bigcup \mathrm{WF}_{\mathcal{B}_j}(f) = \emptyset$ is an immediate consequence of the definitions and Theorem 2.6. Therefore, in view of Proposition 5.5, the equivalence (5.2) follows if we prove that $f \in \bigcap \mathcal{B}_j$ implies $\mathrm{WF}^{\cup}_{(\mathcal{B}_j)}(f) = \emptyset$.

To this aim, assume that $f \in \bigcap \mathcal{B}_j$. Since every $\mathcal{B}_j$ is SG-admissible, and $g_1 \otimes g_2 \in \mathrm{SG}^{0,0}_{1,1}$ when g_1 and g_2 are any cutoff or directional cutoff, it follows that $g_1 \cdot g_2(D) f \in \mathcal{B}_j$ for every j. This implies that $\mathrm{WF}^{\cup}_{(\mathcal{B}_j)}(f) = \emptyset$, and the result follows. □

5.2. Wave-front sets with respect to sequences of spaces with two indices parameters

Next we shall consider wave-front sets with respect to sequences of spaces, parameterized with two indices, and start by introducing the complements of the wave-front sets. More precisely, let J be an index set of integers, Ω_m, $m \in \{1, 2, 3\}$, be given by (1.20), $(\mathcal{B}_{j,k}) = (\mathcal{B}_{j,k})_{j,k \in J}$ be a sequence of Banach or Fréchet spaces such that $\mathscr{S}(\mathbf{R}^d) \subseteq \mathcal{B}_{j,k} \subseteq \mathscr{S}'(\mathbf{R}^d)$, for every j, k, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Then the point $(x_0,\xi_0)\in\Omega_m$ is called *type-*$(m,\cup\cap)$ *regular* (*type-*$(m,\cap\cup)$ *regular*) for f with respect to $(\mathcal{B}_{j,k})$, if

$$\operatorname{Op}(c_m)f\in\bigcap_j\Big(\bigcup_k\mathcal{B}_{j,k}\Big)\qquad\Big(\operatorname{Op}(c_m)f\in\bigcup_j\Big(\bigcap_k\mathcal{B}_{j,k}\Big)\Big),\tag{2.1}''$$

for some c_m in Remark 1.13. The set of all type-$(m,\cup\cap)$ regular points (type-$(m,\cap\cup)$ regular points) for f with respect to $(\mathcal{B}_{j,k})$, is denoted by $\Theta^{m,\cup\cap}_{(\mathcal{B}_{j,k})}(f)$ ($\Theta^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f)$).

Also in here it is desirable that right-hand sides of $(2.1)''$ should be a vector space. For this reason, the sequence $(\mathcal{B}_{j,k})$ is called *ordered with respect to* j, if $\mathcal{B}_{j,k}$ increases with j for every k fixed, or decreases with j for every k fixed. The definition of ordered sequences with respect to k is defined in analogous way.

Definition 5.6. Let J be an index set, $m\in\{1,2,3\}$, Ω_m be as in (1.20), and let $(\mathcal{B}_{j,k})_{j,k\in J}$ be a sequence of Banach or Fréchet spaces such that $\mathscr{S}(\mathbf{R}^d)\subseteq\mathcal{B}_{j,k}\subset\mathscr{S}'(\mathbf{R}^d)$, for every j:

(1) the *type-*$(m,\cup\cap)$ *wave-front set* (*type-*$(m,\cap\cup)$ *wave-front set*) of $f\in\mathscr{S}'(\mathbf{R}^d)$ with respect to $(\mathcal{B}_{j,k})$ is the complement of $\Theta^{m,\cup\cap}_{(\mathcal{B}_{j,k})}(f)$ ($\Theta^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f)$) in Ω_m, and is denoted by $\mathrm{WF}^{m,\cup\cap}_{(\mathcal{B}_{j,k})}(f)$ ($\mathrm{WF}^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f)$);
(2) the *global wave-front sets* $\mathrm{WF}^{\cup\cap}_{(\mathcal{B}_{j,k})}(f)\subseteq(\mathbf{R}^d\times\mathbf{R}^d)\setminus 0$ and $\mathrm{WF}^{\cap\cup}_{(\mathcal{B}_{j,k})}(f)\subseteq(\mathbf{R}^d\times\mathbf{R}^d)\setminus 0$, of $\cup\cap$ and $\cap\cup$ types, respectively, are the sets

$$\mathrm{WF}^{\cup\cap}_{(\mathcal{B}_{j,k})}(f)\equiv\mathrm{WF}^{1,\cup\cap}_{(\mathcal{B}_{j,k})}(f)\cup\mathrm{WF}^{2,\cup\cap}_{(\mathcal{B}_{j,k})}(f)\cup\mathrm{WF}^{3,\cup\cap}_{(\mathcal{B}_{j,k})}(f),$$
$$\mathrm{WF}^{\cap\cup}_{(\mathcal{B}_{j,k})}(f)\equiv\mathrm{WF}^{1,\cap\cup}_{(\mathcal{B}_{j,k})}(f)\cup\mathrm{WF}^{2,\cap\cup}_{(\mathcal{B}_{j,k})}(f)\cup\mathrm{WF}^{3,\cap\cup}_{(\mathcal{B}_{j,k})}(f).$$

Remark 5.7. In analogy with Remark 5.4 we notice that if $\mathcal{B}_{j,k}=\mathcal{B}_j$ is independent of $k\in J$, then

$$\mathrm{WF}^{m,\cup\cap}_{(\mathcal{B}_{j,k})}(f)=\mathrm{WF}^{m,\cup}_{(\mathcal{B}_j)}(f),\qquad\mathrm{WF}^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f)=\mathrm{WF}^{m,\cap}_{(\mathcal{B}_j)}(f),\quad m=1,2,3.$$

Hence, the families of wave-front sets in Definition 5.6 contain the wave-front sets in Definition 5.1.

Remark 5.8. We observe that if $m\in\{1,2,3\}$, $\mathcal{B}_{j,k}$ is SG-admissible for every j,k, Ω_m is given by (1.20) and $f\in\mathscr{S}'(\mathbf{R}^d)$, then $\mathrm{WF}^{m,\cup\cap}_{(\mathcal{B}_{j,k})}(f)$ and $\mathrm{WF}^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f)$ are closed subsets of Ω_m.

By Remark 5.7, it follows that the next two results generalize Proposition 5.5 and Theorem 2.6′. The proofs are similar to the latter results, see Theorems 4.9 and 4.10 in [5]. Here $\mathrm{WF}^{m,\cup}_{(\mathcal{B}_{j,k})_k}(f)$ ($\mathrm{WF}^{m,\cap}_{(\mathcal{B}_{j,k})_k}(f)$) is the type-$(m,\cup)$ (type-$(m,\cap)$) wave-front set of f with respect to $(\mathcal{B}_{j,k})_{k\in J}$, where $j\in J$ is fixed.

Proposition 5.5′. *Let $m\in\{1,2,3\}$, $\mathcal{B}_{j,k}$ be the same as in Definition 5.6, and let $f\in\mathscr{S}'(\mathbf{R}^d)$. Then*

$$\bigcup_j\mathrm{WF}^{m,\cap}_{(\mathcal{B}_{j,k})_k}(f)\subseteq\mathrm{WF}^{m,\cup\cap}_{(\mathcal{B}_{j,k})}(f),\qquad\bigcup_j\mathrm{WF}^{\cap}_{(\mathcal{B}_{j,k})_k}(f)\subseteq\mathrm{WF}^{\cup\cap}_{(\mathcal{B}_{j,k})}(f),$$

and

$$\bigcap_j\mathrm{WF}^{m,\cup}_{(\mathcal{B}_{j,k})_k}(f)=\mathrm{WF}^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f),\qquad\bigcap_j\mathrm{WF}^{\cup}_{(\mathcal{B}_{j,k})_k}(f)=\mathrm{WF}^{\cap\cup}_{(\mathcal{B}_{j,k})}(f).$$

Theorem 2.6″. *Let $\mathcal{B}_{j,k}$ be SG-admissible for every j and k, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Then*

$$f \in \bigcap_j \Big(\bigcup_k \mathcal{B}_{j,k} \Big) \iff \mathrm{WF}^{\cup\cap}_{(\mathcal{B}_{j,k})}(f) = \emptyset, \tag{5.2$'$}$$

provided $(\mathcal{B}_{j,k})$ is ordered with respect to k, and, if instead $(\mathcal{B}_{j,k})$ is ordered with respect to j,

$$f \in \bigcup_j \Big(\bigcap_k \mathcal{B}_{j,k} \Big) \iff \mathrm{WF}^{\cap\cup}_{(\mathcal{B}_{j,k})}(f) = \emptyset. \tag{5.3$'$}$$

Remark 5.9. We have that (5.2) is equivalent to $\bigcup \mathrm{WF}_{\mathcal{B}_j}(f) = \emptyset$, as observed in the proof of Theorem 2.6′, while (5.3) is equivalent to $\bigcap \mathrm{WF}_{\mathcal{B}_j}(f) = \emptyset$. That is, the inclusions in Proposition 5.5 turn into equalities when the left-hand sides are empty, and the same holds for the inclusions in Proposition 5.5′. Moreover, (5.2)′ is equivalent to $\bigcup_j (\bigcap_k \mathrm{WF}_{\mathcal{B}_{j,k}}(f)) = \emptyset$, and (5.3)′ implies $\bigcap_j (\bigcup_k \mathrm{WF}_{\mathcal{B}_{j,k}}(f)) = \emptyset$.

From now on we assume that the involved sequence spaces, $(\mathcal{B}_{j,k})$, are ordered with respect to k when wave-front sets of the form $\mathrm{WF}^{m,\cup\cap}_{(\mathcal{B}_{j,k})}(f)$ are involved, and ordered with respect to j when wave-front sets of the form $\mathrm{WF}^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f)$ are involved.

Remark 5.10. In a similar way as in Remark 5.3, we may construct wave-front sets with respect to the spaces $Q_0(\mathbf{R}^d)$ and $Q(\mathbf{R}^d)$ (see the introduction for the definition of these spaces).

In fact, let

$$p_{j,k}, q_{j,k} \in [1, \infty], \qquad \mathscr{B}_{j,k} = L_1^{p_{j,k}, q_{j,k}}\big(\mathbf{R}^{2d}\big), \qquad \omega_{j,k}(x, \xi) = \langle x \rangle^{-j} \langle \xi \rangle^k,$$
$$\mathcal{B}_{j,k} = M(\omega_{j,k}, \mathscr{B}_{j,k}), \qquad \mathcal{C}_{j,k} = \mathcal{B}_{k,j} \quad \text{when } j, k \in J = \mathbf{N}_0.$$

By similar arguments as in [14, Remark 2.18] it follows that

$$Q_0\big(\mathbf{R}^d\big) = \bigcup_j \Big(\bigcap_k \mathcal{B}_{j,k} \Big), \qquad Q\big(\mathbf{R}^d\big) = \bigcap_j \Big(\bigcup_k \mathcal{C}_{j,k} \Big).$$

Now we define the components of the wave-front sets with respect to Q_0 and Q as

$$\mathrm{WF}^m_{Q_0}(f) = \mathrm{WF}^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f), \qquad \mathrm{WF}^m_{Q}(f) = \mathrm{WF}^{m,\cup\cap}_{(\mathcal{C}_{j,k})}(f), \quad m = 1, 2, 3,$$

when $f \in \mathscr{S}'(\mathbf{R}^d)$. By Theorem 2.6″ it follows that (0.4) holds when $\mathcal{B} = Q_0(\mathbf{R}^d)$ or $\mathcal{B} = Q(\mathbf{R}^d)$.

We also note that Remark 2.3 gives that

$$\mathrm{WF}^1_{\mathscr{S}}(f) = \mathrm{WF}^1_{Q_0}(f) = \mathrm{WF}^1_{Q}(f) = \mathrm{WF}^1_{C^\infty}(f)$$

agrees with the classical wave-front set of f (cf. Section 8.1 in [16]).

5.3. Mapping properties for pseudo-differential operators

Next we consider mapping properties for pseudo-differential operators on wave-front sets of sequence types. The following result follows immediately from the definitions, Theorem 3.1 and its proofs.

Theorem 3.1′. *Let $m \in \{1,2,3\}$, $(r,\rho) \in \mathbb{I}_m$, $t \in \mathbf{R}$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Moreover, let $(\mathcal{B}_{j,k}, \mathcal{C}_{j,k})$ be an SG-ordered pair with respect to ω_0 for every $j,k \in J$. Then*

$$\begin{aligned} \mathrm{WF}^{m,\cup\cap}_{(\mathcal{C}_{j,k})}\big(\mathrm{Op}_t(a)f\big) &\subseteq \mathrm{WF}^{m,\cup\cap}_{(\mathcal{B}_{j,k})}(f) \\ &\subseteq \mathrm{WF}^{m,\cup\cap}_{(\mathcal{C}_{j,k})}\big(\mathrm{Op}_t(a)f\big) \cup \mathrm{Char}^m_{(\omega_0)}(a) \end{aligned} \tag{3.1′}$$

and

$$\begin{aligned} \mathrm{WF}^{m,\cap\cup}_{(\mathcal{C}_{j,k})}\big(\mathrm{Op}_t(a)f\big) &\subseteq \mathrm{WF}^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f) \\ &\subseteq \mathrm{WF}^{m,\cap\cup}_{(\mathcal{C}_{j,k})}\big(\mathrm{Op}_t(a)f\big) \cup \mathrm{Char}^m_{(\omega_0)}(a). \end{aligned} \tag{3.1″}$$

We note that several properties that are valid for the wave-front sets of modulation space types also hold for wave-front sets in the present section. The following generalization of Theorem 3.3 is an immediate consequence of Theorem 3.1, since $\mathrm{Char}_{(\omega_0)}(a) = \emptyset$, when a is SG-elliptic with respect to ω_0.

Theorem 3.3′. *Let $m \in \{1,2,3\}$, $(r,\rho) \in \mathbb{I}_3$, $t \in \mathbf{R}$, $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ and let $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ be SG-elliptic with respect to ω_0 and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Moreover, let $(\mathcal{B}_{j,k}, \mathcal{C}_{j,k})$ be an SG-ordered pair with respect to ω_0 for every $j,k \in J$. Then*

$$\mathrm{WF}^{m,\cup\cap}_{(\mathcal{C}_{j,k})}\big(\mathrm{Op}_t(a)f\big) = \mathrm{WF}^{m,\cup\cap}_{(\mathcal{B}_{j,k})}(f),$$

$$\mathrm{WF}^{m,\cap\cup}_{(\mathcal{C}_{j,k})}\big(\mathrm{Op}_t(a)f\big) = \mathrm{WF}^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f).$$

Next we list some consequences of the previous results. We are especially focused on mapping and wave-front properties in the framework of the spaces $\mathscr{S}$, Q_0 and Q.

We start with the following result, where the first part is a slight extension of [6, Theorem 1.1].

Proposition 5.11. *Let $m \in \{1,2,3\}$, $(r,\rho) \in \mathbb{I}_m$, $t \in \mathbf{R}$, and let $\omega_0 \in \mathscr{P}_{\rho,\delta}(\mathbf{R}^{2d})$. Also let $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$, $f \in \mathscr{S}'(\mathbf{R}^d)$, and $\mathcal{B}$ be equal to $\mathscr{S}(\mathbf{R}^d)$, $Q_0(\mathbf{R}^d)$ or $Q(\mathbf{R}^d)$. Then*

$$\mathrm{WF}^m_{\mathcal{B}}\big(\mathrm{Op}_t(a)f\big) \subseteq \mathrm{WF}^m_{\mathcal{B}}(f) \subseteq \mathrm{WF}^m_{\mathcal{B}}\big(\mathrm{Op}_t(a)f\big) \cup \mathrm{Char}^m_{(\omega_0)}(a).$$

Proof. The result is an immediate consequence of Remark 5.3, Theorems 2.6′ and 3.3′. □

A combination of the previous result and Theorems 2.6 and 3.3′ gives the following.

Proposition 5.12. *Let $m \in \{1,2,3\}$, $(r,\rho) \in \mathbb{I}_3$, $t \in \mathbf{R}$ and $\omega_0 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$. Also let $a \in \mathrm{SG}^{(\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ be SG-elliptic with respect to ω_0, $f \in \mathscr{S}'(\mathbf{R}^d)$, and $\mathcal{B}$ be equal to $\mathscr{S}(\mathbf{R}^d)$, $Q_0(\mathbf{R}^d)$ or $Q(\mathbf{R}^d)$. Then*

$$\mathrm{WF}^m_{\mathcal{B}}\big(\mathrm{Op}_t(a)f\big) = \mathrm{WF}^m_{\mathcal{B}}(f).$$

In particular, if $f, g \in \mathscr{S}'(\mathbf{R}^d)$ *satisfy* $\mathrm{Op}_t(a)f = g$, *then* $f \in \mathcal{B}$, *if and only if* $g \in \mathcal{B}$.

Example 5.13. Let r, ρ, a and ω_0 be the same as in Example 4.2. Then the conclusions in Proposition 5.12 hold for $\mathrm{Op}(a)$.

Example 5.14. Let $x = (x_1, x_2) \in \mathbf{R}^d$ and $\xi = (\xi_1, \xi_2) \in \mathbf{R}^d$, where $x_1, \xi_1 \in V_1$ and $x_2, \xi_2 \in V_2$ and $V_1 \oplus V_2 = \mathbf{R}^d$. We may consider wave-front sets with respect to a space of distributions which behave like $\mathscr{S}$ in the x_1-variable and like $\mathscr{S}'$ in the x_2-variable. More precisely, let

$$\omega_{j,0}(x_1, \xi_1) = \langle x_1, \xi_1\rangle^j, \qquad \omega_{0,k}(x_2, \xi_2) = \langle x_2, \xi_2\rangle^{-k} \quad \text{and}$$
$$\omega_{j,k}(x, \xi) = \langle x_1, \xi_1\rangle^j \langle x_2, \xi_2\rangle^{-k}.$$

Then

$$\bigcap_{j \geqslant 0} M^{p,q}_{(\omega_{j,0})}(V_1) = \mathscr{S}(V_1) \quad \text{and} \quad \bigcup_{k \geqslant 0} M^{p,q}_{(\omega_{0,k})}(V_2) = \mathscr{S}'(V_2).$$

Hence, we may interpret the set

$$\mathcal{B} = \bigcap_{j \geqslant 0}\left(\bigcup_{k \geqslant 0} M^{p,q}_{(\omega_{j,k})}(\mathbf{R}^d)\right)$$

as $\mathscr{S}(V_1; \mathscr{S}'(V_2))$, the set of all tempered distributions which behaves like $\mathscr{S}$ in the x_1-variable, and like $\mathscr{S}'$ in the x_2-variable.

The wave-front set with respect to $\mathcal{B}$ is the wave-front set of $\cup\cap$-type with respect to the sequence of modulation spaces $(M^{p,q}_{(\omega_{j,k})}(\mathbf{R}^d))_{j,k \geqslant 0}$. In particular, all the mapping properties (e.g. Theorems 3.1′ and 3.3′) hold for such wave-front sets.

The set of characteristic points which we considered so far is defined in terms of certain elliptic conditions for the given symbol class. In what follows we give examples on how the results in the present and previous section can be extended by replacing these elliptic types of set of characteristic points with hypoelliptic ones. For this reason, let $r, \rho \in [0, 1]$, and let $\omega_1, \omega_2 \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ be such that

$$\langle x\rangle^{-\delta r}\langle \xi\rangle^{-\delta\rho}\omega_2(x, \xi) \lesssim \omega_1(x, \xi) \lesssim C\omega_2(x, \xi), \tag{5.4}$$

for some constant $\delta < 1$. Then $a \in \mathrm{SG}^{(\omega_2)}_{r,\rho}(\mathbf{R}^{2d})$ is called SG-hypoelliptic with respect to (ω_2, ω_1), if there is a constant $c > 0$ such that

$$c\omega_1(x, \xi) \leqslant \big|a(x, \xi)\big| \tag{5.5}$$

outside a compact set in $\mathbf{R}^{2d}$.

Definition 5.15. The symbol a is called *locally* or *type*-1 hypoelliptic at $(x_0, \xi_0) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, if (5.5) holds when $x \in X$, $\xi \in \Gamma$ and $|\xi| > R$, for some neighborhood X of x_0, some conical neighborhood Γ of ξ_0 and some constant $R > 0$. The set of characteristic points of type-1 for a with respect to (ω_2, ω_1) is denoted by

$$\mathrm{Char}^1_{(\omega_2,\omega_1)}(a), \tag{5.6}$$

and consists of all $(x_0,\xi_0)\in \mathbf{R}^d\times(\mathbf{R}^d\setminus 0)$, where a fails to be type-1 hypoelliptic. The sets of characteristic points

$$\mathrm{Char}^2_{(\omega_2,\omega_1)}(a)\subseteq \big(\mathbf{R}^d\setminus 0\big)\times \mathbf{R}^d \quad\text{and}\quad \mathrm{Char}^3_{(\omega_2,\omega_1)}(a)\subseteq \big(\mathbf{R}^d\setminus 0\big)\times \big(\mathbf{R}^d\setminus 0\big) \tag{5.7}$$

are defined in analogous ways.

We observe that SG-hypoellipticity of a is equivalent to the fact that

$$\mathrm{Char}^m_{(\omega_2,\omega_1)}(a),\quad m=1,2,3,$$

are empty.

Now assume that $\omega_1,\omega_2\in\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ fulfill (5.4) for some $\delta<1$. Then the same arguments as in the proof of Proposition 1.14 in Appendix A, show that Proposition 1.14 holds for $\omega_0=\omega_2$, after the assumption $b\in \mathrm{SG}^{(1/\omega_0)}_{r,\rho}(\mathbf{R}^{2d})$ has been replaced by $b\in \mathrm{SG}^{(1/\omega_1)}_{r,\rho}(\mathbf{R}^{2d})$, and the sets of characteristic points have been replaced by the corresponding ones in (5.6) and (5.7).

The following extension of Theorem 3.1′ now follows by similar arguments as in the proof of Theorem 3.1. The details are left for the reader.

Theorem 3.1″. *Let $m\in\{1,2,3\}$, $(r,\rho)\in\mathbb{I}_m$, $t\in\mathbf{R}$, $\omega_1,\omega_2\in\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ be such that (5.4) holds for some $\delta<1$, $a\in \mathrm{SG}^{(\omega_2)}_{r,\rho}(\mathbf{R}^{2d})$ and let $f\in\mathscr{S}'(\mathbf{R}^d)$. Moreover, let $(\mathcal{B}_{j,k},\mathcal{C}_{j,k})$ and $(\mathcal{B}_{j,k},\mathcal{D}_{j,k})$ be SG-ordered pairs with respect to ω_2 and ω_1 respectively for every $j,k\in J$. Then*

$$\begin{aligned}\mathrm{WF}^{m,\cup\cap}_{(\mathcal{C}_{j,k})}\big(\mathrm{Op}_t(a)f\big)&\subseteq \mathrm{WF}^{m,\cup\cap}_{(\mathcal{B}_{j,k})}(f)\\ &\subseteq \mathrm{WF}^{m,\cup\cap}_{(\mathcal{D}_{j,k})}\big(\mathrm{Op}_t(a)f\big)\cup \mathrm{Char}^m_{(\omega_2,\omega_1)}(a),\end{aligned} \tag{3.1″}$$

and

$$\begin{aligned}\mathrm{WF}^{m,\cap\cup}_{(\mathcal{C}_{j,k})}\big(\mathrm{Op}_t(a)f\big)&\subseteq \mathrm{WF}^{m,\cap\cup}_{(\mathcal{B}_{j,k})}(f)\\ &\subseteq \mathrm{WF}^{m,\cap\cup}_{(\mathcal{D}_{j,k})}\big(\mathrm{Op}_t(a)f\big)\cup \mathrm{Char}^m_{(\omega_2,\omega_1)}(a).\end{aligned} \tag{3.1‴}$$

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Appendix A

In this appendix we give a proof of Proposition 1.14 for $m=3$.

Proof of Proposition 1.14 for $m=3$. We may assume that $t=0$ in view of Proposition 1.7. The equivalence between (1) and (2) follows easily by letting $b(x,\xi)=\psi_1(x)\psi_2(\xi)/a(x,\xi)$, for appropriate $\psi_1\in\mathscr{C}^{\mathrm{dir}}_{x_0}$ and $\psi_2\in\mathscr{C}^{\mathrm{dir}}_{\xi_0}$.

(4) ⇒ (3) is obvious in view of Remark 1.13. Assume that (3) holds. We claim that

$$\big|a(x,\xi)b(x,\xi)\big| \geqslant 1/2 \tag{A.1}$$

holds when

$$(x,\xi) \in \Gamma_1 \times \Gamma_2, \quad |x| \geqslant R,\ |\xi| \geqslant R, \tag{A.2}$$

for some choice of conical neighborhoods Γ_1 and Γ_2 of x_0 and ξ_0, respectively, and some $R > 0$. In fact, the assumptions imply that $ab = c + h_1$ for some $h_1 \in \mathrm{SG}^{-r,0}_{r,\rho} + \mathrm{SG}^{0,-\rho}_{r,\rho}$. By choosing R large enough and Γ_1 and Γ_2 sufficiently small conical neighborhoods of x_0 and ξ_0, respectively, it follows that $c(x,\xi) = 1$ and $|h(x,\xi)| \leqslant 1/2$ when (A.2) holds. This gives (A.1). Since $|b| \leqslant C/\omega$, it follows that (1.18) is fulfilled, and (1) follows.

It remains to prove that (2) implies (4). Assume therefore that (2) is true. Let $\psi_{1,k} \in \mathscr{C}^{\mathrm{dir}}_{x_0}(\Gamma_1)$ and $\psi_{2,k} \in C^{\mathrm{dir}}_{\xi_0}(\Gamma_2)$ for $k = 1,\dots,4$, be chosen such that

$$b_1(x,\xi) \equiv \psi_{1,1}(x)\psi_{2,1}(\xi)/a(x,\xi) \in \mathrm{SG}^{(1/\omega_0)}_{r,\rho},$$

and $\psi_{j,k} = 1$ on $\operatorname{supp}\psi_{j,k+1}$. If $c_1 = \psi_{1,1} \otimes \psi_{2,1} \in \mathrm{SG}^{0,0}_{r,\rho}$, then it follows that

$$\mathrm{Op}(b_j)\,\mathrm{Op}(a) = \mathrm{Op}(c_j) + \mathrm{Op}(h_j) \tag{A.3}$$

holds for $j = 1$ and some $h_1 \in \mathrm{SG}^{-r,-\rho}_{r,\rho}$.

For $j \geqslant 2$ we now define $\tilde b_j \in \mathrm{SG}^{(1/\omega)}_{r,\rho}$ by the Neumann series

$$\mathrm{Op}(\tilde b_j) = \sum_{k=0}^{j-1} (-1)^k\,\mathrm{Op}(\tilde r_k),$$

where $\mathrm{Op}(\tilde r_k) = \mathrm{Op}(h_1)^k\,\mathrm{Op}(b_1) \in \mathrm{Op}(\mathrm{SG}^{(\sigma_{-k\rho,-k\rho}/\omega_0)}_{r,\rho})$. Then (A.3) gives

$$\begin{aligned}
\mathrm{Op}(\tilde b_j)\,\mathrm{Op}(a) &= \sum_{k=0}^{j-1} (-1)^k\,\mathrm{Op}(h_1)^k\,\mathrm{Op}(b_1)\,\mathrm{Op}(a)\\
&= \sum_{k=0}^{j-1} (-1)^k\,\mathrm{Op}(h_1)^k\big(\mathrm{Op}(c_1) + \mathrm{Op}(h_1)\big)\\
&= \mathrm{Op}(c_1) + \mathrm{Op}(\tilde h_{1,j}) + \mathrm{Op}(\tilde h_{2,j}),
\end{aligned} \tag{A.4}$$

where

$$\mathrm{Op}(\tilde h_{1,j}) = (-1)^j\,\mathrm{Op}(h_1)^j \in \mathrm{Op}\big(\mathrm{SG}^{-jr,-j\rho}_{r,\rho}\big) \tag{A.5}$$

and

$$\mathrm{Op}(\tilde h_{2,j}) = -\sum_{k=1}^{j-1} (-1)^k\,\mathrm{Op}(h_1)^k\,\mathrm{Op}(1 - c_1) \in \mathrm{Op}\big(\mathrm{SG}^{0,0}_{r,\rho}\big).$$

By asymptotic expansions it follows that

$$\operatorname{Op}(\tilde{h}_{2,j}) = -\sum_{k=1}^{j-1}(-1)^k \operatorname{Op}(1-c_1)\operatorname{Op}(h_1)^k + \operatorname{Op}(\tilde{h}_{3,j}) + \operatorname{Op}(\tilde{h}_{4,j}), \tag{A.6}$$

for some $\tilde{h}_{3,j} \in \mathrm{SG}_{r,\rho}^{-r,-\rho}$ which is equal to zero on $\operatorname{supp} c_1$ and $\tilde{h}_{4,j} \in \mathrm{SG}_{r,\rho}^{-jr,-j\rho}$. Now let c, b_j and r_k be defined by the formulae

$$c(x,\xi) = \psi_{1,3}(x)\psi_{2,3}(\xi), \qquad \operatorname{Op}(b_j) = \operatorname{Op}(c)\operatorname{Op}(\tilde{b}_j) \in \operatorname{Op}\bigl(\mathrm{SG}_{r,\rho}^{(1/\omega_0)}\bigr),$$
$$\operatorname{Op}(r_k) = \operatorname{Op}(c)\operatorname{Op}(\tilde{r}_k) \in \operatorname{Op}\bigl(\mathrm{SG}_{r,\rho}^{(\sigma_{-k\rho,-k\rho/\omega_0})}\bigr).$$

Then

$$\operatorname{Op}(b_j) = \sum_{k=0}^{j-1}(-1)^k \operatorname{Op}(r_k)$$

and (A.4)–(A.6) give

$$\begin{aligned}\operatorname{Op}(b_j)\operatorname{Op}(a) &= \operatorname{Op}(c)\operatorname{Op}(c_1) + \operatorname{Op}(c)\operatorname{Op}(\tilde{h}_{1,j}) \\ &\quad - \sum_{k=1}^{j-1}(-1)^k \operatorname{Op}(c)\operatorname{Op}(1-c_1)\operatorname{Op}(h_1)^k + \operatorname{Op}(c)\operatorname{Op}(\tilde{h}_{3,j}) + \operatorname{Op}(c)\operatorname{Op}(\tilde{h}_{4,j}).\end{aligned}$$

Since $c_1 = 1$ and $\tilde{h}_{3,j} = 0$ on $\operatorname{supp} c$, and every element of $\operatorname{Op}(\mathrm{SG}_{r,\rho}^{-\infty,-\infty})$ maps continuously $\mathscr{S}'$ to $\mathscr{S}$, we find

$$\operatorname{Op}(c)\operatorname{Op}(c_1) = \operatorname{Op}(c) \mod \operatorname{Op}(\mathscr{S}),$$
$$\operatorname{Op}(c)\operatorname{Op}(\tilde{h}_{1,j}) \in \operatorname{Op}\bigl(\mathrm{SG}_{r,\rho}^{-jr,-j\rho}\bigr),$$
$$\sum_{k=1}^{j-1}(-1)^k \operatorname{Op}(c)\operatorname{Op}(1-c_1)\operatorname{Op}(h_1)^k \in \operatorname{Op}(\mathscr{S}),$$
$$\operatorname{Op}(c)\operatorname{Op}(\tilde{h}_{3,j}) \in \operatorname{Op}(\mathscr{S}),$$

and

$$\operatorname{Op}(c)\operatorname{Op}(\tilde{h}_{4,j}) \in \operatorname{Op}\bigl(\mathrm{SG}_{r,\rho}^{-jr,-j\rho}\bigr).$$

Hence, (A.3) follows for $c_j = c$ and some $h_j \in \mathrm{SG}_{r,\rho}^{-jr,-j\rho}$. By choosing $b \in \mathrm{SG}_{r,\rho}^{(1/\omega)}$ such that

$$b \sim \sum r_k,$$

the argument above shows that $\operatorname{Op}(b)\operatorname{Op}(a) = \operatorname{Op}(c) + \operatorname{Op}(h)$, with $h \in \mathrm{SG}_{r,\rho}^{-\infty,-\infty} = \mathscr{S}$, and (4) follows. The proof is complete. $\square$

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