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Exponential convergence towards stationary states for the 1D porous medium equation with fractional pressure

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Abstract

We analyse the asymptotic behaviour of solutions to the one dimensional fractional version of the porous medium equation introduced by Caffarelli and Vázquez [1,2], where the pressure is obtained as a Riesz potential associated with the density. We take advantage of the displacement convexity of the Riesz potential in one dimension to show a functional inequality involving the entropy, entropy dissipation, and the Euclidean transport distance. An argument by approximation shows that this functional inequality is enough to deduce the exponential convergence of solutions in self-similar variables to the unique steady states.

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1. Introduction

In this work, we analyse the long-time asymptotics of the nonlinear nonlocal equation

$$\rho_t = \nabla \cdot \big(\rho\big(\nabla(-\Delta)^{-s}\rho + \lambda x\big)\big), \quad \lambda > 0,\ x \in \mathbb{R}^d, \tag{1.1}$$

obtained from the fractional version of the porous medium equation introduced by Caffarelli and Vázquez [1,2]

$$u_\tau = \nabla \cdot (u\nabla p), \quad p = (-\Delta)^{-s}u, \tag{1.2}$$

by passing to self-similar variables. Indeed, by adding the Fokker–Planck confining term $\nabla \cdot (xu)$, solutions to (1.1) will characterize the long-time asymptotic behaviour of solutions to (1.2). This connection will be further explained below.

The fractional porous medium equation (1.2) can be viewed as a continuity equation, $u_\tau + \nabla \cdot (u\mathbf{V}) = 0$, for a density or concentration $u(\tau, y)$ with velocity $\mathbf{V} = -\nabla p$, where the velocity potential or pressure p is related to u by the inverse of a fractional Laplacian operator $p = (-\Delta)^{-s}u$, $0 < s < 1$. The standard porous medium equation is recovered for $s = 0$. We assume that the unknown $u(\tau, y)$, representing a density or concentration, is defined for $y \in \mathbb{R}^d$ and $\tau > 0$ and supply initial data $u(y, 0) = u_0(y)$, a nonnegative mass distribution in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. We also point out that the pressure can be represented as

$$p = (-\Delta)^{-s}u = \mathcal{K} * u,$$

with the singular convolution kernel

$$\mathcal{K}(y) = c_{d,s}|y|^{2s-d}, \quad c_{d,s} = \frac{s2^{-2s}\Gamma(d/2 - s)}{\pi^{d/2}\Gamma(1 + s)}, \tag{1.3}$$

and $0 < s < \min(1, d/2)$, called the Riesz potential of u as in the standard textbooks [3,4]. This representation also makes sense for $s = d/2$ with the logarithm kernel $\mathcal{K}(y) = -2^{1-d}\pi^{-d/2}\Gamma(d/2)^{-1}\log|y|$ (see [5,6] in one dimension) and for $1/2 < s < 1$ in one dimension with the negative coefficient $c_{1,s}$ and the positive exponent $2s - 1$ in $\mathcal{K}(y)$. As a result, the kernel $\mathcal{K}(y)$ does not necessarily decay to zero at infinity in the last two cases, but the magnitude of the gradient $\nabla\mathcal{K}(y)$ does. When the kernel $\mathcal{K}(y)$ is replaced by a less singular radially symmetric function, the same equation appears in granular flow [7–10] and biological swarming [11–13].

To describe the long-time behaviour of solutions to (1.2), we study the transformed equation (1.1) by defining

$$\rho(t, x) := (1 + \tau)^\alpha u(\tau, y), \tag{1.4}$$

with the similarity variables $x = y(1 + \tau)^{-\beta}$ and $t = \log(1 + \tau)$. The exponents α and β can be determined from dimensional analysis and the mass conservation [14], which are given by

$$\alpha = d/(d + 2 - 2s), \qquad \beta = 1/(d + 2 - 2s). \tag{1.5}$$

In this way, the rescaled density $\rho(t, x)$ satisfies (1.1) with $\lambda = \beta = 1/(d + 2 - 2s)$. We will keep $\lambda > 0$ arbitrary in (1.1) as a parameter to characterize the convexity of the energy defined below and the convergence rate to the steady state later on. As a result, the long-time behaviour of the original density $u(\tau, y)$ is completely specified if we establish the convergence of $\rho(t, x)$ to the steady state $\rho_\infty(x)$ of (1.1) with $\lambda = \beta$.

The existence and uniqueness of the steady state ρ_∞ of (1.1) for each given mass was initially characterized by an obstacle problem in [2], and then the explicit expression of ρ_∞ was obtained by Biler, Imbert and Karch [15,16], for yet more general nonlinear dependence of the pressure $p = (-\Delta)^{-s} u^{m-1}$, $m > 1$. In case $m = 2$ of our interest here, the self-similar solution of (1.2) is given by

$$u(\tau, y) = (1+\tau)^{-d/(d+2-2s)} \rho_\infty\big(y(1+\tau)^{-1/(d+2-2s)}\big),$$

with the self-similar profile

$$\rho_\infty(x) = K_{d,s}\big(R^2 - |x|^2\big)_+^{1-s}$$

and the prefactor

$$K_{d,s} = \frac{2^{2s-1}\Gamma(d/2+1)}{\Gamma(2-s)\Gamma(d/2+1-s)}\lambda.$$

The radius of the support R is determined by the total conserved mass M, that is,

$$M = \int_{\mathbb{R}^d} u(\tau, y)\,dy = \frac{2^{2s}\pi^{d/2}\Gamma(d/2+1)\lambda}{(d+2-2s)\Gamma(d/2+1-s)^2} R^{d+2-2s}. \tag{1.6}$$

After these preliminary discussion, we concentrate on the convergence of $\rho(t, x)$ to the steady state $\rho_\infty(x)$ in the rest of the paper.

Let us point out that the fractional porous medium equation (1.1) can be viewed as a particular case of the aggregation equation [10,12,17] written as

$$\rho_t = \nabla \cdot \big(\rho(\nabla \mathcal{K} * \rho + \nabla V)\big), \quad x \in \mathbb{R}^d, \tag{1.7}$$

where $V(x) = \frac{\lambda}{2}|x|^2$ and $\mathcal{K}(x) = c_{d,s}|x|^{2s-d}$, $0 < s < 1$.

During the past fifteen years, several important techniques [18–20,10,21,22] have been developed for the convergence of linear or nonlinear Fokker–Planck equations to their steady states with sharp rate. These techniques can also be employed to prove the convergence of solutions of (1.1) to ρ_∞, by realizing that the free energy $\mathcal{E}(\rho)$ defined as

$$\begin{aligned}\mathcal{E}(\rho) &= \frac{1}{2}\int_{\mathbb{R}^d} \big\{(-\Delta)^{-s}\rho(x) + \lambda|x|^2\big\}\rho(x)\,dx \\ &= \frac{c_{d,s}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\rho(x)\rho(y)}{|x-y|^{d-2s}}\,dy\,dx + \lambda\int_{\mathbb{R}^d}\frac{|x|^2}{2}\rho(x)\,dx,\end{aligned} \tag{1.8}$$

is a Lyapunov functional for $0 < s < \min(1, d/2)$. One can similarly define the Lyapunov functional for $1/2 \le s < 1$ in one dimension, assuming that ρ satisfies a growth condition at infinity, namely $\rho \log|x| \in L^1(\mathbb{R})$ if $s = 1/2$ and $\rho|x|^{2s-1} \in L^1(\mathbb{R})$ if $1/2 < s < 1$. In fact, (1.1) is a gradient flow of the free energy functional (1.8) with respect to the Euclidean transport distance in the metric space of probability measures [22,23].

The basic properties of the energy $\mathcal{E}(\rho)$ and its dissipation $\mathcal{I}(\rho)$ defined below, together with the long-time asymptotics of solutions to (1.1), are already derived in [2]. More precisely, along the evolution governed by (1.1), one can obtain the formal relation $d\mathcal{E}(\rho)/dt = -\mathcal{I}(\rho)$, where we denote by $\mathcal{I}(\rho)$ the entropy production or entropy dissipation of $\mathcal{E}$ given by

$$\mathcal{I}(\rho) = \int_{\mathbb{R}^d} \rho|\nabla\xi|^2\,dx, \quad \text{with } \xi = \frac{\delta\mathcal{E}}{\delta\rho} = (-\Delta)^{-s}\rho + \frac{\lambda}{2}|x|^2.$$

Using this relation, the solution of (1.1) is shown to converge towards ρ_∞ in [2], but no rate is obtained. To be more precise, they show that solutions of the fractional porous medium equation (1.1) satisfy the energy inequality $\mathcal{E}(\rho(t,\cdot)) + \int_0^t \mathcal{I}(\rho(\tau,\cdot))d\tau \le \mathcal{E}(\rho(0,\cdot))$ that is enough to conclude the converge of $\rho(t,x)$ to the steady state $\rho_\infty(x)$.

In this work, we will focus on obtaining the sharp convergence rate for the solutions of the Cauchy problem for (1.1) towards the equilibrium ρ_∞, for all $0 < s < 1$ in one dimension, although many of the calculations are presented in general dimensions. In the particular case of $s = 1/2$ in one dimension, the kernel is given by the logarithmic potential and it was treated in [5], see also [6] for related functional inequalities. In fact, it is shown in [5] that the energy $\mathcal{E}(\rho)$ is displacement convex, which cannot be derived directly from the criteria given in the seminal paper by McCann [24]. We will take advantage of these techniques in [5] to prove certain functional inequalities, in particular the HWI inequalities as introduced in [25] (also obtained in [6] for the logarithmic case $s = 1/2$). This displacement convexity and related inequalities are then used to show the convergence towards equilibrium in one dimension, through the exponential decay of the transport distances and the relative energy, for general $s \in (0, 1)$.

Finally, we point out that the problem of sharp convergence rates in several space dimensions is still open. Moreover, it could be interesting to prove or disprove analogous functional inequalities involving nonlocal operators in several space dimensions corresponding to the ones established here in one dimension; see more comments at the end of Section 2. New techniques or inequalities have to be developed. Showing asymptotic convergence when the confining term $\nabla \cdot (\lambda x\rho)$ is replace by the general drift $\nabla \cdot (\rho\nabla V)$ is another interesting problem, see [26,10].

The organization of this work is as follows. We first remind the reader in Section 2 about the basics of the entropy/entropy dissipation method, together with the main functional inequality that we will prove in one dimension. In fact, we follow closely the strategy developed for nonlinear diffusion equations in [27,28,19,20,29,10] to reduce to the proof of a Log-Sobolev type inequality. This inequality is then proved in Section 3 as a consequence of the HWI inequality which crucially uses the displacement convexity. Finally, Section 4 is devoted to obtain the rate of convergence towards equilibrium of the solutions to (1.1) by an approximation method using the construction of solutions in [1].

2. Transport inequalities in dimension 1

In this section, we derive several inequalities originated from optimal transportation theory that will be used in the next section to show the exponential convergence of the relative entropy in one dimension.

Before starting the technical computations we are going to use to prove the transport inequalities, let us discuss a bit more on the equilibrium solution ρ_∞. It was recently proved in [26, Theorem 1.2] that $\mathcal{E}$ restricted to $\mathcal{P}(\mathbb{R}^d)$ is strictly convex in the classic sense for $0 < s < \min(1, d/2)$, and it has a unique compactly supported minimizer ρ_∞ characterized by

$$(-\Delta)^{-s}\rho_\infty(x) + \lambda\frac{|x^2|}{2} = C_*, \quad \forall x \in \operatorname{supp}(\rho_\infty), \tag{2.1a}$$

$$(-\Delta)^{-s}\rho_\infty(x) + \lambda\frac{|x^2|}{2} \geq C_*, \quad \text{a.e. } \mathbb{R}^d, \tag{2.1b}$$

for some constant C_* determined by the total mass. This formulation is equivalent to the obstacle problem in [2], for the rescaled pressure $P = (-\Delta)^{-s}\rho$ and the quadratic obstacle $\Phi(x) = C_* - \frac{\lambda}{2}|x|^2$. Using the following relation (see [15,16])

$$\begin{aligned}(-\Delta)^{-s}\big(R^2 - |x|^2\big)_+^{1-s} &= \frac{2^{-2s}\Gamma(2-s)\Gamma(d/2-s)}{\Gamma(d/2)}\left(R^2 - \frac{d-2s}{d}|x|^2\right) \\ &= \frac{\lambda}{2K_{d,s}}\left(\frac{d}{d-2s}R^2 - |x|^2\right), \quad \text{for all } |x| \leq R,\end{aligned} \tag{2.2}$$

it is easy to verify that $\rho_\infty = K_{d,s}(R^2 - |x|^2)_+^{1-s}$ is indeed the minimizer for $\mathcal{E}$ for $0 < s < \min(1, d/2)$. Similar computations can be done in the range $1/2 \leq s < 1$, see [5,17] for instance.

Now, we can consider the difference $\mathcal{E}(\rho|\rho_\infty) := \mathcal{E}(\rho) - \mathcal{E}(\rho_\infty)$ as a measure of the distance of ρ to the equilibrium state ρ_∞.

We know from Section 1 that the following relation holds for sufficiently smooth solutions ρ to (1.1)

$$\frac{d}{dt}\big(\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty)\big) = -\mathcal{I}(\rho).$$

Hence, once we have the following inequality for a sufficiently large class of functions

$$\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty) \leq \frac{1}{2\lambda}\mathcal{I}(\rho), \tag{2.3}$$

we can prove the exponential convergence of $\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty)$ to zero with exponential rate 2λ (but not necessarily the exponential convergence of $\mathcal{I}(\rho)$), by integrating

$$\frac{d}{dt}\big(\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty)\big) = -\mathcal{I}(\rho) \leq -2\lambda\big(\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty)\big)$$

in time. The inequality (2.3) is usually called, in the context of optimal transport, Log-Sobolev inequality in the linear diffusion case or generalized Log-Sobolev inequalities otherwise. We will revisit (2.3) in the next section by investigating the displacement convexity of the energy $\mathcal{E}(\rho)$. In particular, it becomes the logarithmic Sobolev inequality [30] for linear Fokker–Planck equation [28,31,32], and a special family of Gagliardo–Nirenberg inequalities for nonlinear Fokker–Planck equations with porous medium type diffusion [20,19,29].

Thus for the rest of this section, we shall prove a generalization of (2.3) and use it in the following section to obtain the desired decay for $\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty)$.

Let $\mathcal{P}_2(\mathbb{R})$ be the set of probability measures with second order moments, and $\mathcal{P}_{2,ac}(\mathbb{R})$ be the subset of probability measures of $\mathcal{P}_2(\mathbb{R})$ that are absolutely continuous with respect to the Lebesgue measure. Besides $\mathcal{E}(\rho)$ and $\mathcal{I}(\rho)$ introduced earlier, we also need the following versions of the energy and energy dissipation of a measure $\rho \in \mathcal{P}_{2,ac}(\mathbb{R})$:

$$\mathcal{E}_\varepsilon(\rho) := \mathcal{E}(\rho) + \varepsilon \int_{\mathbb{R}} \rho \log \rho,$$

$$\mathcal{I}_\varepsilon(\rho) := \int_{\mathbb{R}} \left|\partial_x(-\partial_{xx})^{-s}\rho(x) + \lambda x + \varepsilon \partial_x \log \rho(x)\right|^2 d\rho(x),$$

which are associated with the regularized equation (3.2) in the next section. Throughout the rest of the paper we shall commit an abuse of notation and identify every absolutely continuous measure with its density. So we shall write $d\rho(x)$ and $\rho(x)\,dx$ meaning the same thing.

We use optimal transport techniques to prove the Log-Sobolev, the Talagrand, and the HWI inequalities for the energy $\mathcal{E}_\varepsilon$ for smooth probability measures $\rho \in \mathcal{P}_{2,ac}(\mathbb{R})$. We shall focus on the so-called HWI inequality that generalizes certain elementary inequalities for convex functions on $\mathbb{R}^d$ with Euclidean distance replaced by the Wasserstein distance on $\mathcal{P}_2(\mathbb{R})$ (the space of probability measures with finite second moment). The Wasserstein distance on $\mathcal{P}_2(\mathbb{R})$ is defined for any $\rho_1, \rho_2 \in \mathcal{P}_2(\mathbb{R})$ by

$$W_2(\rho_1, \rho_2) := \left(\inf_{\pi \in \Pi(\rho_1,\rho_2)} \int_{\mathbb{R}\times\mathbb{R}} |x-y|^2 \, d\pi(x,y) \right)^{\frac{1}{2}},$$

where $\Pi(\rho_1, \rho_2)$ be the set of all nonnegative Radon measures on $\mathbb{R} \times \mathbb{R}$ with marginals (projections) ρ_1 and ρ_2. The HWI inequality is called so because it was first established in [25] for the relative Kullback information (denoted by H), the Wasserstein distance W_2 and the relative Fisher information (also denoted by I).

Before stating the main results, let us briefly review the following facts about the Wasserstein distance and the weak convergence in $\mathcal{P}_2(\mathbb{R})$ that shall be used in the proofs.

i) We say that a sequence $(\rho_n)_{n\in\mathbb{N}} \subseteq \mathcal{P}_2(\mathbb{R})$ weakly converges to $\rho \in \mathcal{P}(\mathbb{R})$ (denoted as $\rho_n \rightharpoonup \rho$), if

$$\lim_{n\to\infty} \int \varphi(x)\, d\rho_n(x) = \int \varphi(x)\, d\rho(x),$$

for all $\varphi \in C_b(\mathbb{R})$, the space of bounded and continuous functions.

ii) The pair $(\mathcal{P}_2(\mathbb{R}), W_2)$ is a complete metric space and the convergence under the distance W_2 is stronger than the convergence in the weak sense. In fact, the following three facts are equivalent for any $(\rho_n)_{n\in\mathbb{N}} \subseteq \mathcal{P}_2(\mathbb{R})$ and $\rho \in \mathcal{P}(\mathbb{R})$:
- $W_2(\rho_n, \rho) \to 0$ as $n \to +\infty$;
- $\rho_n \rightharpoonup \rho$ and

$$\lim_{n\to\infty} \int x^2\, d\rho_n(x) = \int x^2\, d\rho(x); \tag{2.4}$$

- $\rho_n \rightharpoonup \rho$ and

$$\lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} x^2 \, d\rho_n(x) = 0.$$

iii) Given $\rho_1, \rho_2 \in \mathcal{P}_2(\mathbb{R})$ with ρ_1 absolutely continuous with respect to the Lebesgue measure, there exists a Borel map $\theta : \mathbb{R} \to \mathbb{R}$ such that $\theta\#\rho_1 = \rho_2$, i.e.,

$$\int_{\mathbb{R}} \varphi(x)\, d\rho_2(x) = \int_{\mathbb{R}} \varphi\big(\theta(x)\big)\, d\rho_1(x), \quad \text{for every bounded Borel function } \varphi,$$

and θ also satisfies

$$W_2(\rho_1, \rho_2) = \left(\int_{\mathbb{R}} \big|x - \theta(x)\big|^2 \, d\rho_1(x) \right)^{\frac{1}{2}},$$

It is well known that the optimal map θ is nondecreasing on $\mathbb{R}$ and increasing on $\operatorname{supp}(\rho_1)$. In fact, θ can be written in terms of the cumulative distribution functions F_1 and F_2 of ρ_1 and ρ_2 respectively, that is $\theta(x) = F_2^{-1} \circ F_1(x)$, see [21, Chap 2].

For a detailed proof of the above results and generalizations, the reader may check the standard references [22] and [21]. Now, let us begin with the following technical lemma about the derivative of the Riesz potential.

Lemma 2.1. *Let* $0 < s \leq 1$ *and* $\rho \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C^\alpha(\mathbb{R})$ *with* $\alpha > \max(1 - 2s, 0)$. *Then* $(-\Delta)^{-s}\rho \in C^1(\mathbb{R})$ *and for any* $x \in \mathbb{R}$,

$$\partial_x(-\partial_{xx})^{-s}\rho(x) = -c_{1,s}(1-2s) \int_{\mathbb{R}} \frac{x-y}{|x-y|^{3-2s}} \big(\rho(y) - \rho(x)\big)\, dy, \quad \textit{if } s \in (0, 1/2]$$

or

$$\partial_x(-\partial_{xx})^{-s}\rho(x) = -c_{1,s}(1-2s) \int_{\mathbb{R}} \frac{x-y}{|x-y|^{3-2s}} \rho(y)\, dy, \quad \textit{if } s \in (1/2, 1].$$

Proof. Firstly, let us assume that $s \in (0, 1/2)$. To simplify the notation, we write $k_s(x) := c_{1,s}|x|^{2s-1}$. Hence, we note that under the hypothesis on ρ, we have that

$$\mathbf{u}_s(x) := -c_{1,s}(1-2s) \int_{\mathbb{R}} \frac{(x-y)}{|x-y|^{3-2s}} \big(\rho(y) - \rho(x)\big)\, dy = k_s' * \big(\rho - \rho(x)\big)$$

is well defined for all $x \in \mathbb{R}$.

Now, let $\eta \in C^1(\mathbb{R})$ be a radial function such that $0 \leq \eta \leq 1$, $\eta(x) = 0$ if $|x| \leq 1$, $\eta(x) = 1$ if $|x| \geq 2$ and $|\eta'| \leq 2$. Define $\eta_\varepsilon(x) := \eta(\varepsilon^{-1}x)$ and

$$p(x) := (-\partial_{xx})^{-s}\rho(x) = k_s * \rho(x),$$
$$p_\varepsilon(x) := (k_s\eta_\varepsilon) * \rho(x).$$

Since ρ is bounded, we have that $p \to p_\varepsilon$ uniformly on $\mathbb{R}$ as

$$\begin{aligned}\big|p(x) - p_\varepsilon(x)\big| &\le \int\limits_{|x-y|\le 2\varepsilon} k_s(x-y)\big(1-\eta_\varepsilon(x-y)\big)\rho(y)\,dy \\ &\le \|\rho\|_\infty \int\limits_{|y|\le 2\varepsilon} \frac{1}{|y|^{1-2s}}\,dy = C\|\rho\|_\infty \varepsilon^{2s}\end{aligned}$$

for all $x \in \mathbb{R}$, where C depends on s.

By the smoothness of $k_s\eta_\varepsilon$ we know that $p_\varepsilon \in C^1$ and $p'_\varepsilon(x) = (k_{d,s}\eta_\varepsilon)' * \rho(x)$, and since $k_s\eta_\varepsilon$ is radial, we can write

$$p'_\varepsilon(x) = \int\limits_{\mathbb{R}} (k_s\eta_\varepsilon)'(x-y)\big(\rho(y)-\rho(x)\big)\,dy.$$

Therefore,

$$\begin{aligned}&\big|\mathbf{u}_s(x) - p'_\varepsilon(x)\big| \\ &\quad = \Bigg|\int\limits_{|x-y|\le 2\varepsilon} \big(k_s(1-\eta_\varepsilon)\big)'(x-y)\big(\rho(y)-\rho(x)\big)\,dy\Bigg| \\ &\quad \le \int\limits_{|x-y|\le 2\varepsilon} \Big(\big|k'_s(x-y)\big|\big|1-\eta_\varepsilon(x-y)\big| + k_s(x-y)\big|\eta'_\varepsilon(x-y)\big|\Big)\big|\rho(y)-\rho(x)\big|\,dy \\ &\quad \le \int\limits_{|x-y|\le 2\varepsilon} \left(\frac{c_{1,s}(1-2s)}{|x-y|^{2-2s}} + \frac{2}{\varepsilon}\frac{c_{1,s}}{|x-y|^{1-2s}}\right)\big|\rho(y)-\rho(x)\big|\,dy \\ &\quad \le C \int\limits_{|x-y|\le 2\varepsilon} \left(\frac{1}{|x-y|^{2-2s-\alpha}} + \frac{1}{\varepsilon}\frac{1}{|x-y|^{1-2s-\alpha}}\right)dy \\ &\quad \le C_1\varepsilon^{\alpha+2s-1},\end{aligned} \tag{2.5}$$

where the constant C_1 only depends on s, α and on the Hölder constant of ρ. Thus, we also have that p'_ε converges uniformly to $\mathbf{u}_s$ as $\varepsilon \to 0$, and therefore $p' = \mathbf{u}_s$.

Now, if $s \in (1/2, 1]$, we only need to adapt the argument in formula (2.5) for the function

$$\mathbf{u}_s(x) := -c_{1,s}(1-2s)\int\limits_{\mathbb{R}} \frac{x-y}{|x-y|^{3-2s}}\rho(y)\,dy = k'_s * \rho$$

and using that $p'_\varepsilon = (k_s\eta_\varepsilon)' * \rho$ in the following way

$$\begin{aligned}\left|\mathbf{u}_s(x)-p'_\varepsilon(x)\right| &= C\|\rho\|_\infty \int\limits_{|x-y|\le 2\varepsilon} \left(\frac{1}{|x-y|^{2-2s}}+\frac{1}{\varepsilon}\frac{1}{|x-y|^{1-2s}}\right)dy \\ &= C_2\varepsilon^{2s-1},\end{aligned}$$

where the constant C_2 only depends on s and on the L^∞ norm of ρ.

Finally, if $s=1/2$ we have that

$$(-\partial_{xx})^{-\frac{1}{2}}\rho(x)=c_{1,\frac{1}{2}}\int\limits_{\mathbb{R}}\log|x-y|\rho(y)\,dy$$

and

$$\mathbf{u}_{\frac{1}{2}}(x)=-c_{1,\frac{1}{2}}\int\limits_{\mathbb{R}}\frac{(x-y)}{|x-y|^2}\big(\rho(y)-\rho(x)\big)\,dy.$$

Arguing as above for $k_{\frac{1}{2}}(x):=c_{1,\frac{1}{2}}\log|x|$ we arrive at the following estimates:

$$\left|p(x)-p_\varepsilon(x)\right|\le\|\rho\|_\infty\int\limits_{|y|\le 2\varepsilon}\left|\log|y|\right|dy=C\|\rho\|_\infty\varepsilon\big(|\log 2\varepsilon|+1\big)$$

and

$$\begin{aligned}\left|\mathbf{u}_{\frac{1}{2}}(x)-p'_\varepsilon(x)\right| &\le C\int\limits_{|x-y|\le 2\varepsilon}\left(\frac{1}{|x-y|}+\frac{1}{\varepsilon}\left|\log|x-y|\right|\right)\left|\rho(y)-\rho(x)\right|dy \\ &\le C\int\limits_{|x-y|\le 2\varepsilon}\left(\frac{1}{|x-y|^{1-\alpha}}+\frac{1}{\varepsilon}|x-y|^\alpha\left|\log|x-y|\right|\right)dy \\ &\le C\varepsilon^\alpha\big(1+\varepsilon+\varepsilon|\log 2\varepsilon|\big).\end{aligned}$$

Therefore, since all these estimates are uniform in x, we conclude that the lemma is true for all $s\in(0,1]$. □

Remark 2.2. With this expression for the derivative of $(-\partial_{xx})^{-s}\rho$ for $s<\frac{1}{2}$, we obtain the following equality that shall be used in the next proposition:

$$\begin{aligned}\frac{\partial_x(-\partial_{xx})^{-s}\rho(x)}{c_{1,s}(2s-1)} &= \lim_{r\to 0}\int\limits_{|x-y|\ge r}\frac{x-y}{|x-y|^{3-2s}}\big(\rho(y)-\rho(x)\big)\,dy \\ &= \lim_{r\to 0}\int\limits_{|x-y|\ge r}\frac{x-y}{|x-y|^{3-2s}}\rho(y)\,dy-\lim_{r\to 0}\rho(x)\int\limits_{|x-y|\ge r}\frac{x-y}{|x-y|^{3-2s}}\,dy \\ &= \lim_{r\to 0}\int\limits_{|x-y|\ge r}\frac{x-y}{|x-y|^{3-2s}}\rho(y)\,dy,\end{aligned}$$

where we only used the fact that k_s is radial and k_s' is integrable at the infinity. For $s > \frac{1}{2}$, the expression is valid without taking the limit, as the kernel is locally integrable.

The next proposition shows that the HWI inequality holds for $\mathcal{E}$ and $\mathcal{E}_\varepsilon$ at least for a class of bounded and Hölder continuous functions on $\mathbb{R}$. The proof follows the arguments given in [6] where the same inequality is proved for the case of the logarithmic interaction and strongly relies on the fact that the optimal transport map w.r.t. the Wasserstein distance is a monotone nondecreasing function on $\mathbb{R}$. We point out that the convexity of the confinement due to the drift measured by $\lambda > 0$ appears explicitly in the inequalities as in [10].

Theorem 2.3. *Let $s \in (0,1]$, $\lambda \in \mathbb{R}$, $\rho \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C^\alpha(\mathbb{R})$ nonnegative where $\alpha > \max(1-2s, 0)$ and with $\int \rho = 1$, and ρ_∞ the minimum point of $\mathcal{E}$ on $\mathcal{P}^2(\mathbb{R})$. Then*

$$\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty) \le \sqrt{\mathcal{I}(\rho)}\, W_2(\rho, \rho_\infty) - \frac{\lambda}{2} W_2^2(\rho, \rho_\infty).$$

Proof. For $s = 1/2$ this result was proven at [6]. So, let us suppose that $s \in (0, 1/2)$ and, to simplify, let us denote $K\rho(x) = \partial_x(-\partial_{xx})^{-s}\rho(x)$. Since ρ is absolutely continuous with respect to the Lebesgue measure, there exists a nondecreasing transport map θ such that $\theta\#\rho = \rho_\infty$.

Then, let us write

$$\sqrt{\mathcal{I}(\rho)}\, W_2(\rho, \rho_\infty) - \frac{\lambda}{2} W_2^2(\rho, \rho_\infty) - \mathcal{E}(\rho) + \mathcal{E}(\rho_\infty) = T_1 + T_2 + T_3$$

where

$$\begin{aligned}
T_1 &:= \left(\int \left|K\rho(x) + \lambda x\right|^2 d\rho(x)\right)^{1/2} \left(\int \left|x - \theta(x)\right|^2 d\rho(x)\right)^{1/2} \\
&\quad - \int \big(K\rho(x) + \lambda x\big)\big(x - \theta(x)\big)\, d\rho(x), \\
T_2 &:= \int \left\{\lambda x\big(x - \theta(x)\big) - \frac{\lambda}{2} x^2 + \frac{\lambda}{2}\theta(x)^2 - \frac{\lambda}{2}\left|x - \theta(x)\right|^2\right\} d\rho(x), \\
T_3 &:= \frac{c_{1,s}}{2} \int \frac{d\rho(x)\, d\rho(y)}{|\theta(x) - \theta(y)|^{1-2s}} - \frac{c_{1,s}}{2} \int \frac{d\rho(x)\, d\rho(y)}{|x - y|^{1-2s}} \\
&\quad - \int K\rho(x)\big(\theta(x) - x\big)\, d\rho(x),
\end{aligned}$$

where we added and subtracted several terms. This allows us to show that $T_1 \ge 0$ by the Cauchy–Schwarz inequality and $T_2 = 0$ for all $\lambda \in \mathbb{R}$. Now, for T_3 let us call $k_s(x) = c_{1,s}|x|^{2s-1}$. Then, by Remark 2.2

$$K\rho(x) = \lim_{r \to 0} \int\limits_{|y-x| \ge r} k_s'(x - y)\, d\rho(y)$$

and, since $k_s'(x) = -k_s'(-x)$, we can write

$$\begin{aligned}
&\int K\rho(x)\big(\theta(x)-x\big)\,d\rho(x)\\
&\quad=\lim_{r\to 0}\int\limits_{|y-x|\geq r}\big(\theta(x)-x\big)k_s'(x-y)\,d\rho(y)\,d\rho(x)\\
&\quad=\frac{1}{2}\lim_{r\to 0}\int\limits_{|y-x|\geq r}\big(\theta(x)-\theta(y)-x+y\big)k_s'(x-y)\,d\rho(y)\,d\rho(x).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
c_{1,s}\int\frac{d\rho(x)\,d\rho(y)}{|x-y|^{1-2s}}&=\lim_{r\to 0}\int\limits_{|y-x|\geq r}k_s(x-y)\,d\rho(x)\,d\rho(y),\\
c_{1,s}\int\frac{d\rho(x)\,d\rho(y)}{|\theta(x)-\theta(y)|^{1-2s}}&=\lim_{r\to 0}\int\limits_{|y-x|\geq r}k_s\big(\theta(x)-\theta(y)\big)\,d\rho(x)\,d\rho(y)
\end{aligned}$$

and then,

$$T_3=\lim_{r\to 0}\frac{1}{2}\int\big\{k_s\big(\theta(x)-\theta(y)\big)-k_s(x-y)-k_s'(x-y)\big(\theta(x)-\theta(y)-x+y\big)\big\}\,d\rho(x)\,d\rho(y).$$

The integrand is nonnegative by the convexity of k_s on the positive semi-axis and by the monotonicity of θ, so $T_3\geq 0$ as well.

If $s\in(1/2,1]$, we still have $k_s(x)=c_{1,s}|x|^{2s-1}$ convex because $c_{1,s}$ is negative in this range. Thus, the previous computations still apply. □

Remarks. 1) It is known that, if the HWI inequality holds for some $\lambda>0$, then the Log-Sobolev inequality also holds. One just needs to maximize the right-hand side for $W_2\geq 0$ or use the Young's inequality for $(\lambda^{-\frac{1}{2}}\sqrt{\mathcal{I}})(\lambda^{\frac{1}{2}}W_2)$. Then we have that

$$\mathcal{E}(\rho)-\mathcal{E}(\rho_\infty)\leq\frac{1}{2\lambda}\mathcal{I}(\rho), \tag{2.6}$$

for all ρ satisfying the assumptions of the theorem above.

2) Note that in the proof of Theorem 2.3 we did not use the fact that ρ_∞ is the minimum of $\mathcal{E}$, only the fact that $\mathcal{E}(\rho_\infty)<\infty$. In fact, the same inequality holds for any ρ_0 in the place of ρ_∞, and also with ρ_∞ in the place of ρ, since ρ_∞ is absolutely continuous with respect to the Lebesgue measure, which allows the existence of the map θ by the item (iii) from page 742. Therefore, if we exchange ρ and ρ_∞ in the HWI we obtain the fractional version of the so-called Talagrand inequality or transportation cost inequality

$$W_2(\rho,\rho_\infty)\leq\sqrt{\frac{2}{\lambda}\big(\mathcal{E}(\rho)-\mathcal{E}(\rho_\infty)\big)}. \tag{2.7}$$

We can derive similar results for the ε problems.

Proposition 2.4. *Let $s \in (0,1]$, $\lambda > 0$, $0 < \varepsilon < \lambda/2\pi$, $\rho \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C^\alpha(\mathbb{R})$ nonnegative where $\alpha > 1-2s$ and with $\int \rho = 1$, and ρ_∞^ε the minimum point of $\mathcal{E}_\varepsilon$ on $\mathcal{P}_2(\mathbb{R})$. Then*

$$\mathcal{E}_\varepsilon(\rho) - \mathcal{E}_\varepsilon\big(\rho_\infty^\varepsilon\big) \le \sqrt{\mathcal{I}_\varepsilon(\rho)}\, W_2\big(\rho, \rho_\infty^\varepsilon\big) - \frac{\lambda}{2} W_2^2\big(\rho, \rho_\infty^\varepsilon\big).$$

Proof. The proof is basically the same, but since we have a new term inside the respective diffusion, we shall include it for completeness.

As in the previous theorem, let $K\rho(x) = \partial_x(-\partial_{xx})^{-s}\rho(x)$ and θ be such that $\theta\#\rho = \rho_\infty^\varepsilon$. Then, we decompose the inequality as

$$\sqrt{\mathcal{I}_\varepsilon(\rho)}\, W_2\big(\rho, \rho_\infty^\varepsilon\big) - \frac{\lambda}{2} W_2^2\big(\rho, \rho_\infty^\varepsilon\big) - \mathcal{E}_\varepsilon(\rho) + \mathcal{E}_\varepsilon\big(\rho_\infty^\varepsilon\big) = T_1 + T_2 + T_3$$

where

$$\begin{aligned}
T_1 :=& \left(\int \big|K\rho(x) + \lambda x + \varepsilon\partial_x \log\rho(x)\big|^2 d\rho(x)\right)^{1/2} \left(\int \big|x - \theta(x)\big|^2 d\rho(x)\right)^{1/2} \\
&- \int \big(K\rho(x) + \lambda x + \varepsilon\partial_x \log\rho(x)\big)\big(x - \theta(x)\big)\, d\rho(x), \\
T_2 :=& -\int \big(\varepsilon\partial_x \log\rho(x) + \lambda x\big)\big(\theta(x) - x\big)\, d\rho - \int \left(\frac{\lambda}{2}x^2 + \varepsilon\log\rho\right) d\rho \\
&+ \int \left(\frac{\lambda}{2}x^2 + \varepsilon\log\rho_\infty^\varepsilon\right) d\rho_\infty^\varepsilon - \frac{\lambda}{2}\int \big|x - \theta(x)\big|^2 d\rho(x), \\
T_3 :=& \frac{c_{1,s}}{2}\int \frac{d\rho(x)\, d\rho(y)}{|\theta(x) - \theta(y)|^{1-2s}} - \frac{c_{1,s}}{2}\int \frac{d\rho(x)\, d\rho(y)}{|x-y|^{1-2s}} \\
&- \int K\rho(x)\big(\theta(x) - x\big)\, d\rho(x).
\end{aligned}$$

By the same arguments, we conclude that $T_1, T_3 \ge 0$. Now, for T_2, let us define the following functional

$$H(f|g) := \int f(x)\log\left(\frac{f(x)}{g(x)}\right) dx$$

for all nonnegative $f, g \in L^1(\mathbb{R})$ with $g > 0$. Then we can rewrite T_2 in the following way

$$\begin{aligned}
T_2 =& \varepsilon\bigg(-\int \partial_x \log\left(\frac{\rho(x)}{e^{-\pi x^2}}\right)\big(\theta(x) - x\big)\, d\rho(x) - H\big(\rho|e^{-\pi x^2}\big) \\
&+ H\big(\rho_\infty^\varepsilon|e^{-\pi x^2}\big) + \pi\int \big|\theta(x) - x\big|^2 d\rho\bigg) \\
&+ \left(1 - \frac{2\pi}{\lambda}\varepsilon\right)\int \left\{-\lambda x\big(\theta(x) - x\big) - \frac{\lambda}{2}x^2 + \frac{\lambda}{2}\theta(x)^2 + \frac{\lambda}{2}\big(\theta(x) - x\big)^2\right\} d\rho(x).
\end{aligned}$$

Note that the second line is equal to $(\lambda - 2\pi\varepsilon)\int |\theta(x) - x|^2\, dx$, which is nonnegative for $\varepsilon < \lambda/2\pi$. For the first line, we can use the proof of the HWI inequality made in [25]. Actually, Otto and Villani showed that whenever $\rho, \rho_\infty^\varepsilon \in C_c^\infty(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ and $V \in C^2(\mathbb{R})$ is such that $\int e^{-V}\, dx = 1$ and $V'' \geq \lambda$ for some constant $\lambda \in \mathbb{R}$, then

$$H\big(\rho_\infty^\varepsilon \big| e^{-V}\big) - H\big(\rho \big| e^{-V}\big) - \int \partial_x \log \frac{\rho(x)}{e^{-V(x)}}\big(\theta(x) - x\big)\rho(x)\, dx$$
$$- \frac{\lambda}{2}\int \big|\theta(x) - x\big|^2 \rho(x)\, dx \geq 0,$$

and for the density argument given in the proof of Theorem 9.17 of [21], we have that this inequality holds for all $\rho, \rho_\infty^\varepsilon \in L^1(\mathbb{R}) \cap \mathcal{P}_2(\mathbb{R})$. So, applying this for $V(x) = \pi x^2$ we have that $\lambda = 2\pi$ and we conclude that $T_2 \geq 0$. □

Remark 2.5. By the same arguments given for (2.6) and s(2.7), we conclude that the following Log-Sobolev and Talagrand inequalities hold for $\mathcal{E}_\varepsilon$, as long as ρ satisfies the assumptions of Proposition 2.4:

$$\mathcal{E}_\varepsilon(\rho) - \mathcal{E}_\varepsilon\big(\rho_\infty^\varepsilon\big) \leq \frac{1}{2\lambda}\mathcal{I}_\varepsilon(\rho),$$
$$W_2\big(\rho, \rho_\infty^\varepsilon\big) \leq \sqrt{\frac{2}{\lambda}\big(\mathcal{E}_\varepsilon(\rho) - \mathcal{E}_\varepsilon\big(\rho_\infty^\varepsilon\big)\big)}. \tag{2.8}$$

Remark 2.6. These results also work for a general confinement potential $V : \mathbb{R} \to \mathbb{R}$ instead of the quadratic one $\frac{\lambda}{2}x^2$, as long as $V - \frac{\lambda}{2}x^2$ is convex.

Finally, let us prove the following lemma that shall be used in the last section for the convergence in entropy of the solutions of the approximate problems. The proof uses similar arguments given in Theorem 1.4 of [33]. Let us just remind that a sequence $\{\rho_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(\mathbb{R})$ is said to converge in the weak-$*$ sense to $\rho \in \mathcal{P}(\mathbb{R})$, $\rho_n \overset{*}{\rightharpoonup} \rho$ if

$$\lim_{n\to\infty} \int_{\mathbb{R}} \varphi(x)\, d\rho_n(x) = \int_{\mathbb{R}} \varphi(x)\, d\rho(x), \quad \text{for all } \varphi \in C_0(\mathbb{R})$$

where $C_0(\mathbb{R})$ is the space of continuous functions on $\mathbb{R}$ that goes to zero at infinity. It is clear that convergence in W_2 implies weak convergence and weak convergence implies weak-$*$ convergence.

Lemma 2.7. *The entropy $\mathcal{E}_\varepsilon$ is weak-$*$ lower semi-continuous for all $\varepsilon \geq 0$ on $\mathcal{P}_{ac}(\mathbb{R})$.*

Proof. We know from [24] that the functional

$$\rho \mapsto \int \rho \log \rho$$

is weak-$*$ lower semi-continuous on $\mathcal{P}_{ac}(\mathbb{R})$, so we just need to show the result for $\mathcal{E}$. For this, let us write it in the following way:

$$\mathcal{E}(\rho) = \int_{\mathbb{R}^2} F(x,y)\, d\rho(x)\, d\rho(y),$$

where

$$F(x,y) = \begin{cases} \frac{\lambda}{4}(x^2+y^4) + \frac{c_{1,s}}{2}\frac{1}{|x-y|^{1-2s}}, & \text{if } x \neq y, \\ +\infty, & \text{if } x = y. \end{cases}$$

Since F is nonnegative and smooth outside the diagonal $x = y$, we can find a sequence $\{F_k\}_{k\in\mathbb{N}} \subset C_0(\mathbb{R}^2)$ such that $F_k(x,y) \nearrow F(x,y)$ for all $(x,y) \in \mathbb{R}^2$. Therefore, by the monotone convergence theorem and the fact that $\rho_n \times \rho_n \overset{*}{\rightharpoonup} \rho \times \rho$ if $\rho_n \overset{*}{\rightharpoonup} \rho$, we have that

$$\begin{aligned} \mathcal{E}(\rho) &= \int F(x,y)\, d\rho(x)\, d\rho(y) = \lim_{k\to\infty} \int F_k(x,y)\, d\rho(x)\, d\rho(y) \\ &= \lim_{k\to\infty}\lim_{n\to\infty} \int F_k(x,y)\, d\rho_n(x)\, d\rho_n(y) \le \lim_{n\to\infty} \int F(x,y)\, d\rho_n(x)\, d\rho_n(y) \\ &= \liminf_{n\to\infty} \mathcal{E}(\rho_n). \qquad \square \end{aligned}$$

3. Exponential convergence

In this section we shall prove that the energy of the solution decays exponentially fast for the regularized equation with mollified initial data, and then passing the limit on these regularizing parameters.

Theorem 3.1. *Let $\rho_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that*

$$0 \le \rho_0(x) \le Ae^{-a|x|},$$

for some constants $a, A > 0$. Then, for each $0 < s < 1/2$, the solution $\rho(t,\cdot)$ of (1.1) with initial data ρ_0 satisfies

$$\mathcal{E}\big(\rho(t)\big) - \mathcal{E}(\rho_\infty) \le e^{-2\lambda t}\big(\mathcal{E}(\rho_0) - \mathcal{E}(\rho_\infty)\big).$$

Proof. In order to use the results of Section 2, firstly we shall assume that

$$\rho_0 \in C^\infty(\mathbb{R}) \quad \text{and} \quad \int_{\mathbb{R}} \rho_0(x)\, dx = 1. \tag{3.1}$$

Let $\rho_\infty, \rho_\infty^\varepsilon \in \mathcal{P}(\mathbb{R})$ be the minimizers for $\mathcal{E}$ and $\mathcal{E}_\varepsilon$ respectively. By the assumption on ρ_0 we know from the proofs of Theorems 4.1 and 4.2 in [1] that the solutions ρ and ρ^ε to

$$\begin{cases} \partial_t \rho = \partial_x\big(\rho \partial_x(-\partial_{xx})^{-s}\rho + \lambda x \rho\big), & \text{in } \mathbb{R}\times(0,\infty), \\ \rho(0) = \rho_0, & \text{in } \mathbb{R}, \end{cases} \tag{3.2}$$

and

$$\begin{cases} \partial_t \rho^\varepsilon = \partial_x \big(\rho^\varepsilon \partial_x (-\partial_{xx})^{-s} \rho^\varepsilon + \lambda x \rho^\varepsilon\big) + \varepsilon \partial_{xx} \rho^\varepsilon, & \text{in } \mathbb{R} \times (0,\infty), \\ \rho^\varepsilon(0) = \rho_0, & \text{in } \mathbb{R}, \end{cases} \tag{3.3}$$

satisfy $\rho \in C([0,\infty); L^1(\mathbb{R}))$ and $\rho^\varepsilon \in C^1((0,\infty)\times\mathbb{R})$ for all $\varepsilon > 0$ sufficiently small. Because of the regularization in (3.3), for fixed time $t > 0$, $\rho^\varepsilon(t,.)$ is in fact in $C^2(\mathbb{R})$. Moreover, there exist $C(t), a(t) > 0$, such that

$$0 \le \rho(t,x), \rho^\varepsilon(t,x) \le C(t) e^{-a(t)|x|}. \tag{3.4}$$

Since $\rho^\varepsilon(t)$ is smooth, we can apply the Log-Sobolev inequality (2.8) for $\mathcal{E}_\varepsilon$ and obtain that for all $t \ge 0$,

$$\mathcal{E}_\varepsilon\big(\rho^\varepsilon(t)\big) - \mathcal{E}_\varepsilon\big(\rho^\varepsilon_\infty\big) \le \frac{1}{2\lambda}\mathcal{I}_\varepsilon\big(\rho^\varepsilon(t)\big).$$

Making use of the fact that

$$\frac{d}{dt}\mathcal{E}_\varepsilon\big(\rho^\varepsilon(t)\big) = -\mathcal{I}_\varepsilon\big(\rho^\varepsilon(t)\big),$$

we conclude that

$$\mathcal{E}_\varepsilon\big(\rho^\varepsilon(t)\big) - \mathcal{E}_\varepsilon\big(\rho^\varepsilon_\infty\big) \le e^{-2\lambda t}\big(\mathcal{E}_\varepsilon(\rho_0) - \mathcal{E}_\varepsilon\big(\rho^\varepsilon_\infty\big)\big). \tag{3.5}$$

To take the limits as $\varepsilon \to 0^+$, let us analyze each term on both sides of (3.5) separately:

i) The easiest one is the limit $\mathcal{E}_\varepsilon(\rho_0)$, since $\lim_{\varepsilon\to 0^+}\mathcal{E}_\varepsilon(\rho_0) = \mathcal{E}(\rho_0)$ holds as long as $\mathcal{E}_\varepsilon(\rho_0) < \infty$ for some $\varepsilon > 0$, which is true by the assumptions on ρ_0.

ii) For the term $\mathcal{E}_\varepsilon(\rho^\varepsilon_\infty)$, let us first define the following auxiliary functional on $\mathcal{P}_{2,ac}(\mathbb{R})$:

$$\mathcal{H}(\rho) := \mathcal{H}\big(\rho \big| e^{-\pi x^2}\big) = \pi \int x^2 \rho + \int \rho \log \rho.$$

Since $\int e^{-\pi x^2}\,dx = 1$, we can write

$$\begin{aligned}\mathcal{H}(\rho) &= \int \frac{\rho}{e^{-\pi x^2}} \log\left(\frac{\rho}{e^{-\pi x^2}}\right) e^{-\pi x^2}\,dx \\ &= \int \left[\frac{\rho}{e^{-\pi x^2}} \log\left(\frac{\rho}{e^{-\pi x^2}}\right) - \frac{\rho}{e^{-\pi x^2}} + 1\right] e^{-\pi x^2}\,dx,\end{aligned}$$

which is nonnegative by Jensen's inequality.
Let us prove that $\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon(\rho^\varepsilon_\infty) \le \mathcal{E}(\rho_\infty)$. Using the fact that ρ^ε_∞ is the minimum for $\mathcal{E}_\varepsilon$, we obtain the following inequality

$$\mathcal{E}_\varepsilon\big(\rho^\varepsilon_\infty\big) \le \mathcal{E}_\varepsilon(\rho_\infty) = \mathcal{E}(\rho_\infty) + \varepsilon \int \rho_\infty \log \rho_\infty. \tag{3.6}$$

By the characterization of the minimum ρ_∞ in [2,26], we know that $\rho_\infty \in \mathcal{P}^2_{ac}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and hence the second term on the right-hand side of (3.6) is finite. Thus, we can take the limit $\varepsilon \to 0$ and obtain that $\limsup_{\varepsilon\to 0^+} \mathcal{E}_\varepsilon(\rho^\varepsilon_\infty) \le \mathcal{E}(\rho_\infty)$.
For the opposite inequality $\liminf_{\varepsilon\to 0^+} \mathcal{E}_\varepsilon(\rho^\varepsilon_\infty) \ge \mathcal{E}(\rho_\infty)$, we can use the fact that ρ_∞ is the minimum for $\mathcal{E}$ and write

$$\mathcal{E}(\rho_\infty) \le \mathcal{E}\big(\rho^\varepsilon_\infty\big) = \mathcal{E}_\varepsilon\big(\rho^\varepsilon_\infty\big) - \varepsilon\mathcal{H}\big(\rho^\varepsilon_\infty\big) + \varepsilon\pi \int x^2 \rho^\varepsilon_\infty \tag{3.7}$$

$$\le \mathcal{E}_\varepsilon\big(\rho^\varepsilon_\infty\big) + \varepsilon\pi \int x^2 \rho^\varepsilon_\infty. \tag{3.8}$$

So, it is sufficient to prove that the second moments of ρ^ε_∞ are uniformly bounded for $\varepsilon > 0$ sufficiently small. For this, note that

$$\begin{aligned} 0 &\le \frac{\lambda}{4}\int x^2\rho^\varepsilon_\infty \le \frac{(1-\varepsilon\pi)\lambda}{2}\int x^2\rho^\varepsilon_\infty \\ &\le \frac{(1-\varepsilon\pi)\lambda}{2}\int x^2\rho^\varepsilon_\infty + \frac{c_{1,s}}{2}\int \frac{d\rho^\varepsilon_\infty(x)\,d\rho^\varepsilon_\infty(y)}{|x-y|^{1-2s}} + \frac{\varepsilon\lambda}{2}\mathcal{H}\big(\rho^\varepsilon_\infty\big) \\ &= \mathcal{E}_\varepsilon\big(\rho^\varepsilon_\infty\big) \le \mathcal{E}_\varepsilon(\rho_\infty) \le \mathcal{E}(\rho_\infty) + \left|\int \rho_\infty \log \rho_\infty\right| \end{aligned}$$

for all $0 < \varepsilon < 1/2\pi$. Therefore, by (3.7) and (3.8)

$$\mathcal{E}(\rho_\infty) \le \liminf_{\varepsilon\to 0^+} \mathcal{E}_\varepsilon\big(\rho^\varepsilon_\infty\big) + \lim_{\varepsilon\to 0^+} \varepsilon\pi \int x^2\rho^\varepsilon_\infty = \liminf_{\varepsilon\to 0^+} \mathcal{E}_\varepsilon\big(\rho^\varepsilon_\infty\big).$$

Hence, as ε goes to zero from above, we have that the minimum of $\mathcal{E}_\varepsilon(\rho)$ indeed converge to the minimum of $\mathcal{E}(\rho)$, i.e., $\mathcal{E}(\rho_\infty) = \lim_{\varepsilon\to 0^+} \mathcal{E}_\varepsilon(\rho^\varepsilon_\infty)$.

iii) Finally, let us prove that $\mathcal{E}(\rho(t)) \le \liminf_{\varepsilon\to 0^+} \mathcal{E}_\varepsilon(\rho^\varepsilon(t))$, as a consequence of the convergence of $\rho^\varepsilon(t)$ to $\rho(t)$ in $\mathcal{P}_{2,ac}(\mathbb{R})$ and the lower semi-continuity of the energy $\mathcal{E}_\varepsilon$. For this we can use the bound (3.4) to obtain

$$\lim_{R\to\infty} \sup_{\varepsilon>0} \int\limits_{|x|>R} \rho^\varepsilon(t,x)\,dx \le \lim_{R\to\infty} C(t) \int\limits_{|x|>R} e^{-a(t)|x|}\,dx = 0,$$

which means that $\rho^\varepsilon(t)$ is a tight family of probability measures and by Prokhorov Theorem, there exist a sequence $\varepsilon_n \to 0^+$ such that $\rho^{\varepsilon_n}(t) \rightharpoonup \rho(t)$, i.e.,

$$\int\limits_{\mathbb{R}} \varphi(x)\rho^{\varepsilon_n}(t,x)\,dx \to \int\limits_{\mathbb{R}} \varphi(x)\rho(t,x)\,dx, \quad \forall \varphi \in C_b(\mathbb{R}). \tag{3.9}$$

Moreover, due to uniform exponential bound, we also have that

$$\lim_{R\to\infty} \sup_{\varepsilon_n\to 0} \int\limits_{|x|\ge R} x^2\rho^{\varepsilon_n}(t,x)\,dx \le \lim_{R\to\infty} C(t) \int\limits_{|x|>R} x^2 e^{-a(t)|x|}\,dx = 0. \tag{3.10}$$

Therefore, by the equivalent conditions of weak convergence we have that (3.9) and (3.10) imply that $\rho^{\varepsilon_n}(t)$ converges to $\rho(t)$ in $(\mathcal{P}_2(\mathbb{R}), W_2)$. Now, for the following inequality

$$\begin{aligned}\mathcal{E}\big(\rho^{\varepsilon_n}(t)\big) &= \mathcal{E}_{\varepsilon_n}\big(\rho^{\varepsilon_n}(t)\big) - \varepsilon_n \mathcal{H}\big(\rho^{\varepsilon_n}(t)\big) + \pi\varepsilon_n \int x^2 \rho^{\varepsilon_n}(t,x) \\ &\le \mathcal{E}_{\varepsilon_n}\big(\rho^{\varepsilon_n}\big) + \pi\varepsilon_n \int x^2 \rho^{\varepsilon_n}(t,x),\end{aligned}$$

and by the fact that $\mathcal{E}$ is lower semi-continuous in $(\mathcal{P}_2(\mathbb{R}), W_2)$ and the second moments of $\rho^{\varepsilon_n}(t)$ are uniformly bounded w.r.t. n, we obtain

$$\mathcal{E}\big(\rho(t)\big) \le \liminf_{n\to\infty} \mathcal{E}\big(\rho^{\varepsilon_n}(t)\big) \le \liminf_{n\to\infty} \mathcal{E}_{\varepsilon_n}\big(\rho^{\varepsilon_n}(t)\big).$$

Putting all the limits as ε goes to zero together, we can conclude the exponential convergence of $\mathcal{E}(\rho(t)) - \mathcal{E}(\rho_\infty)$, that is,

$$\begin{aligned}\mathcal{E}\big(\rho(t)\big) - \mathcal{E}(\rho_\infty) &\le \liminf_{n\to\infty} \mathcal{E}_{\varepsilon_n}\big(\rho^{\varepsilon_n}(t)\big) - \lim_{n\to\infty} \mathcal{E}_{\varepsilon_n}\big(\rho_\infty^{\varepsilon_n}\big) \\ &= \liminf_{n\to\infty} \big(\mathcal{E}_{\varepsilon_n}\big(\rho^{\varepsilon_n}(t)\big) - \mathcal{E}_{\varepsilon_n}\big(\rho_\infty^{\varepsilon_n}\big)\big) \\ &\le e^{-2\lambda t} \liminf_{n\to\infty} \big(\mathcal{E}_{\varepsilon_n}(\rho_0) - \mathcal{E}_{\varepsilon_n}\big(\rho_\infty^{\varepsilon}\big)\big) \\ &= e^{-2\lambda t}\big(\mathcal{E}(\rho_0) - \mathcal{E}(\rho_\infty)\big).\end{aligned}$$

If the regularity assumption in (3.1) is not true, we can proceed the above argument with the mollified initial data $\rho_{0,\delta} = \eta_\delta * \rho_0$, which has the same bound and mass as ρ_0. Since we still have the same exponential bounds for the respective solutions $\rho_\delta(t)$, we can argue as above and conclude that $\mathcal{E}(\rho(t)) \le \liminf_{\delta\to 0} \mathcal{E}(\rho_\delta(t))$ holds for all $t > 0$. For $t = 0$ we can use the exponential bound of the initial data and the Dominated Convergence Theorem to conclude that $\lim_{\delta\to 0} \mathcal{E}(\rho_{\delta,0}) = \mathcal{E}(\rho_0)$. □

As a direct consequence of the Talagrand inequality in (2.7), we also obtain the exponential decay in Wasserstein distance.

Corollary 3.2. *Assume that ρ_0 satisfies $0 \le \rho_0(x) \le Ae^{-a|x|}$ for all $x \in \mathbb{R}$ and some $a, A > 0$. Then, for each $0 < s < 1/2$, the solution of (1.1) with initial data ρ_0 satisfies*

$$W_2\big(\rho(t), \rho_\infty\big) \le e^{-\lambda t}\sqrt{\frac{2}{\lambda}\big(\mathcal{E}(\rho_0) - \mathcal{E}(\rho_\infty)\big)}.$$

For the Fokker–Planck equation or the classic Porous Medium Equations, exponential convergence of the relative entropy $\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty)$ implies convergence of ρ to the steady states ρ_∞ in some classical L^p norms, using for this the classical Csiszár–Kullback–Pinsker inequality as in [28,29]. Here we can show that the convergence in the relative entropy implies the convergence of the norm $\|(-\partial_{xx})^{-\frac{s}{2}}(\rho - \rho_\infty)\|_2$.

Lemma 3.3. *Let ρ_∞ be the unique minimizer of $\mathcal{E}$, then for any $\rho \in \mathcal{P}_2(\mathbb{R})$,*

$$\frac{1}{2}\left\|(-\partial_{xx})^{-\frac{s}{2}}(\rho-\rho_\infty)\right\|_2^2 \le \mathcal{E}(\rho)-\mathcal{E}(\rho_\infty).$$

Proof. The characterization (2.1a) and (2.1b) of the global minimizer ρ_∞ and the nonnegativity of $\rho-\rho_\infty$ outside of the support of ρ_∞ imply that

$$0 = C_* \int_{\mathbb{R}} (\rho-\rho_\infty) \le \int_{\mathbb{R}} \left((-\Delta)^{-s}\rho_\infty(x) + \lambda\frac{|x^2|}{2}\right)(\rho-\rho_\infty).$$

Therefore, we deduce

$$\begin{aligned}\mathcal{E}(\rho)-\mathcal{E}(\rho_\infty) &= \frac{1}{2}\int_{\mathbb{R}} \rho(-\partial_{xx})^{-s}\rho - \frac{1}{2}\int_{\mathbb{R}} \rho(-\partial_{xx})^{-s}\rho_\infty + \frac{\lambda}{2}\int_{\mathbb{R}} |x|^2(\rho-\rho_\infty)\\ &\ge \frac{1}{2}\int_{\mathbb{R}} \rho(-\partial_{xx})^{-s}\rho - \frac{1}{2}\int_{\mathbb{R}} \rho(-\partial_{xx})^{-s}\rho_\infty - \int_{\mathbb{R}} (\rho-\rho_\infty)(-\partial_{xx})^{-s}\rho_\infty\\ &= \frac{1}{2}\int_{\mathbb{R}} (\rho-\rho_\infty)(-\partial_{xx})^{-s}(\rho-\rho_\infty) = \frac{1}{2}\left\|(-\partial_{xx})^{-\frac{s}{2}}(\rho-\rho_\infty)\right\|_2^2. \qquad \square\end{aligned}$$

Since $\|(-\partial_{xx})^{-\frac{s}{2}}(\rho-\rho_\infty)\|_2$ is the $H^{-s/2}$-norm of $\rho-\rho_\infty$, it is unlikely to produce a bound on any stronger L^p norm for the difference $\rho-\rho_\infty$. One way to show the exponential convergence of $\rho(t)$ to ρ_∞ is to assume a uniform bound on a higher order norm of $\rho-\rho_\infty$. For example, if $\|(-\partial_{xx})^{\frac{s}{2}}(\rho-\rho_\infty)\|_2$ is uniformly bounded, then we have (easy to establish in Fourier space)

$$\|\rho-\rho_\infty\|_2^2 \le \left\|(-\partial_{xx})^{\frac{s}{2}}(\rho-\rho_\infty)\right\|_2 \left\|(-\partial_{xx})^{-\frac{s}{2}}(\rho-\rho_\infty)\right\|_2$$

and $\|\rho-\rho_\infty\|_2$ converges to zero also exponentially fast, but with a smaller rate.

Let us prove that in fact the exponential convergence also holds in L^2 without any additional hypothesis. For this, since $(-\partial_{xx})^{-\frac{s}{2}}u$ usually has more regularity than u, we need to look for an interpolation inequality containing some sort of fractional differentiation, which in our case, it seems natural to be a Hölder semi-norm, i.e., for every $\alpha \in (0,1]$ and $v \in C^\alpha(\mathbb{R})$ we denote the α-Hölder semi-norm of v by

$$[v]_\alpha := \sup_{x\neq y} \frac{|v(x)-v(y)|}{|x-y|^\alpha}$$

Therefore, to obtain the desired decay in L^2 we shall use the following new interpolation inequality, that we will prove for any dimension $d \ge 1$.

Theorem 3.4. *Let $0<\alpha\le 1$ and $0<s<d/2$ and $0<r<\alpha/2$. There exists a constant $C = C(d,s,\alpha)$ such that*

$$\|u\|_2 \le C\left\|(-\Delta)^{-\frac{s}{2}}u\right\|_2^{\sigma_1}[u]_\alpha^{\sigma_2}\|u\|_1^{\sigma_3} \tag{3.11}$$

for all $u \in L^1(\mathbb{R}^d) \cap C^\alpha(\mathbb{R}^d)$ *with*

$$\sigma_1 = \frac{r}{s+r}, \qquad \sigma_2 = \frac{s(d+2r)}{2(d+\alpha)(s+r)}, \qquad \sigma_3 = \frac{s(d+2\alpha-2r)}{2(d+\alpha)(s+r)}.$$

Proof. We first use Fourier variables, Plancherel's formula, and the Hölder's inequality to interpolate between $\dot{H}^r(\mathbb{R}^d)$ and $(-\Delta)^{-\frac{s}{2}}u \in L^2(\mathbb{R}^d)$ obtaining

$$\begin{aligned} \|u\|_2^2 = \int_{\mathbb{R}^d} \left|\widehat{u}(\xi)\right|^2 d\xi &\le \left(\int_{\mathbb{R}^d} \left|\widehat{u}(\xi)\right|^2 |\xi|^{-2s}\, d\xi \right)^{\sigma_1} \left(\int_{\mathbb{R}^d} \left|\widehat{u}(\xi)\right|^2 |\xi|^{2r}\, d\xi \right)^{1-\sigma_1} \\ &= \left\|(-\Delta)^{-\frac{s}{2}}u\right\|_2^{2\sigma_1} \left(\int_{\mathbb{R}^d} \left|\widehat{u}(\xi)\right|^2 |\xi|^{2r}\, d\xi \right)^{1-\sigma_1} \end{aligned} \tag{3.12}$$

where $\sigma_1 = r/(s+r)$, for all $0 < s < 1/2$ and $r > 0$.

Our aim now is to bound $\dot{H}^r(\mathbb{R}^d)$ by $[u]_\alpha$ and $\|u\|_1$. We write the singular integral representation of this norm (Proposition 3.4 of [34]) and we split it as

$$\begin{aligned} \|u\|_{\dot{H}^r}^2 &= \int_{\mathbb{R}^d} \left|\widehat{u}(\xi)\right|^2 |\xi|^{2r}\, d\xi = C_{d,r} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(u(x)-u(y))^2}{|x-y|^{d+2r}}\, dx\, dy \\ &= C_{d,r} \iint_{|x-y|\le R} \frac{(u(x)-u(y))^2}{|x-y|^{d+2r}}\, dx\, dy + C_{d,r} \iint_{|x-y|> R} \frac{(u(x)-u(y))^2}{|x-y|^{d+2r}}\, dx\, dy \\ &:= I_1 + I_2. \end{aligned}$$

To estimate I_1, we make use of $|u(x)-u(y)| \le [u]_\alpha |x-y|^\alpha$ to get, by the change of variables $(z,w) = (x-y, x+y)$, that

$$\begin{aligned} I_1 &= C_{d,r} \iint_{|x-y|\le R} \frac{(u(x)-u(y))^2}{|x-y|^{d+2r}}\, dx\, dy \le C_{d,r}[u]_\alpha \iint_{|x-y|\le R} \frac{|u(x)-u(y)|}{|x-y|^{d+2r-\alpha}}\, dx\, dy \\ &\le C[u]_\alpha \|u\|_1 \int_{|z|\le R} |z|^{\alpha-2r-d}\, dz \le C[u]_\alpha \|u\|_1 R^{\alpha-2r}, \end{aligned}$$

where the last step is allowed since $2r < \alpha$. On the other hand, we can similarly estimate the far field term as

$$I_2 = C_{d,r} \iint_{|x-y|\ge R} \frac{(u(x)-u(y))^2}{|x-y|^{d+2r}}\, dx\, dy \le 4C_{d,r} \int_{\mathbb{R}^d} \left|u(x)\right|^2 dx \int_{|z|\ge R} \frac{dz}{|z|^{d+2r}} \le C\|u\|_2^2 R^{-2r}.$$

Joining the two integrals and optimizing in R, we infer

$$\|u\|_{\dot{H}^r}^2 \le C\|u\|_2^{2(\alpha-2r)/\alpha} \|u\|_1^{2r/\alpha} [u]_\alpha^{2r/\alpha}. \tag{3.13}$$

We finally use the classical interpolation results between $L^p(\mathbb{R}^d)$ and $C^\alpha(\mathbb{R}^d)$ spaces due to L. Nirenberg in [35], see also [36] for a full statement. This interpolation inequality ensures the existence of a constant depending on α and d such that

$$\|u\|_2^2 \le C\|u\|_1^{(d+2\alpha)/(\alpha+d)}[u]_\alpha^{d/(\alpha+d)}.$$

Putting it together with (3.13), it yields

$$\|u\|_{\dot{H}^r}^2 \le C\|u\|_1^{(d+2\alpha-2r)/(d+\alpha)}[u]_\alpha^{(d+2r)/(d+\alpha)}.$$

Finally, we plug this into (3.12) to conclude (3.11). $\square$

Therefore, from Theorem 3.1 and Theorem 3.4, we derive the following decay towards the stationary state under the L^2 norm.

Corollary 3.5. *Assume that ρ_0 satisfies $0 \le \rho_0(x) \le Ae^{-a|x|}$ for all $x \in \mathbb{R}$ and some $a, A \ge 0$. Then, for each $0 < s < 1/2$, the solution of (1.1) with initial data ρ_0 satisfies*

$$\|\rho(t) - \rho_\infty\|_2 \le C\big(1 + [\rho_\infty]_\alpha\big)^{\sigma_2}\big(\mathcal{E}(\rho_0) - \mathcal{E}(\rho_\infty)\big)^{\frac{\sigma_1}{2}} e^{-\lambda\sigma_1 t}.$$

Proof. Given ρ_0 under the conditions above, we know from Theorem 5.1 of [37] that there exists an $\alpha \in (0, 1)$ such that the solution ρ of (1.1) satisfies $\rho(t) \in C^\alpha(\mathbb{R})$ for all $t > 0$ with a uniform bound in time. Since ρ_∞ is $(1 - s)$-Hölder continuous, we can use inequality (3.11) for $u = \rho(t) - \rho_\infty$ and $0 < r < 2\min(\alpha, 1 - s)$ to conclude. $\square$

Let us point out that the decay of the entropy in Theorem 3.1 implies a uniform in time control of the second moment of the solutions trivially at least for $0 < s < 1/2$. Otherwise, one has to work a bit due to the sign of the constant in the fractional operator. In any case, a uniform in time control of the second moments together with the L^2-decay rates implies L^1-decay rates of the form

$$\begin{aligned}
\|\rho(t) - \rho_\infty\|_1 &\le \int\limits_{|x|<R} \big|\rho(t,x) - \rho_\infty(x)\big|\,dx + \int\limits_{|x|\ge R} \big|\rho(t,x) - \rho_\infty(x)\big|\,dx \\
&\le C\left(R^{d/2}\|\rho(t) - \rho_\infty\|_2 + R^{-2}\int\limits_{\mathbb{R}^d} |x|^2\big(\rho(t,x) + \rho_\infty(x)\big)\,dx\right) \\
&\le C\big(\mathcal{E}(\rho_0) + \mathcal{E}(\rho_\infty)\big)^{d/(d+4)}\|\rho(t) - \rho_\infty\|_2^{4/(d+4)}, \qquad (3.14)
\end{aligned}$$

by choosing $R \sim ((\mathcal{E}(\rho_0) + \mathcal{E}(\rho_\infty))/\|\rho(t) - \rho_\infty\|_2)^{2/(d+4)}$; see a similar calculation in [38, Lemma 2.24] for instance. In one dimension, using Corollary 3.5, we obtain the decay rate $e^{-4\lambda\sigma_1 t/5}$ for $\|\rho(t) - \rho_\infty\|_1$.

We finally remark that the decay in L^p-norms obtained via Corollary 3.5 and (3.14) are translated through the change of variables (1.4)–(1.5) into algebraic decay rates toward self-similar solutions of the original fractional porous medium equation (1.2).

4. Fractional diffusion in higher dimensions and open problems

In this section, we first show some formal computations using the Bakry–Émery strategy [27] to identify the main technical problem with passing from dimension $d = 1$ to $d > 1$ in the results about the exponentially fast decay of the relative entropy $\mathcal{E}(\rho|\rho_\infty) := \mathcal{E}(\rho) - \mathcal{E}(\rho_\infty)$. We shall do this by taking the second order time derivative of $\mathcal{E}(\rho|\rho_\infty)$ along the evolution equation (1.1).

We first rewrite Eq. (1.1) as

$$\rho_t = \nabla \cdot (\rho \nabla \xi) \quad \text{with } \xi := (-\Delta)^{-s}\rho + \lambda|x|^2/2. \tag{4.1}$$

Assuming that ρ (and thus ξ) is smooth enough, taking the time derivative of the entropy dissipation rate $\mathcal{I}(\rho)$ along the evolution equation, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{I}(\rho) &= \int \rho_t |\nabla\xi|^2 + 2\int \rho\nabla\xi \cdot \nabla\xi_t \\ &= \int \nabla\cdot(\rho\nabla\xi)|\nabla\xi|^2 + 2\int \rho\nabla\xi\cdot\nabla\big[(-\Delta)^{-s}\big(\nabla\cdot(\rho\nabla\xi)\big)\big].\end{aligned}$$

Using the fact $D^2\xi = D^2(-\Delta)^{-s}\rho + \lambda I$ for the Hessian matrix of ξ, the first term on the right-hand side above can be written as

$$\int \nabla\cdot(\rho\nabla\xi)|\nabla\xi|^2 = -2\int \rho\big\langle D^2\xi\cdot\nabla\xi, \nabla\xi\big\rangle = -2\lambda\mathcal{I}(\rho) - 2\int \rho\big\langle D^2(-\Delta)^{-s}\rho\cdot\nabla\xi, \nabla\xi\big\rangle.$$

Therefore, $d\mathcal{I}(\rho)/dt = -2\lambda\mathcal{I}(\rho) - 2\mathcal{R}(\rho)$ with

$$\mathcal{R}(\rho) = \int \rho\big\langle D^2(-\Delta)^{-s}\rho\cdot\nabla\xi, \nabla\xi\big\rangle - \int \rho\nabla\xi\cdot\nabla\big[(-\Delta)^{-s}\big(\nabla\cdot(\rho\nabla\xi)\big)\big]. \tag{4.2}$$

The entropy–entropy dissipation method can be summarized as follows: if $\mathcal{R}(\rho) \geq 0$ for the solution ρ, then from the conditions $d\mathcal{E}(\rho)/dt = -\mathcal{I}(\rho)$ and $d\mathcal{I}(\rho)/dt \leq -2\lambda\mathcal{I}(\rho)$, we can conclude that $\mathcal{I}(\rho)(t) \leq \mathcal{I}(\rho)(0)e^{-2\lambda t}$ and $\mathcal{E}(\rho)(t) - \mathcal{E}(\rho_\infty) \leq (\mathcal{E}(\rho)(0) - \mathcal{E}(\rho_\infty))e^{-2\lambda t}$, or the exponential convergence of both $\mathcal{I}(\rho)(t)$ and $\mathcal{E}(\rho)(t) - \mathcal{E}(\rho_\infty)$ towards zero.

When $s = 0$, Eq. (1.1) reduces to the standard porous medium equation with quadratic nonlinearity. In this special case, the nonnegativity of $\mathcal{R}(\rho)$ was established in [19] using several integration by parts, leading to (with $\xi = \rho + \lambda|x|^2/2$)

$$\mathcal{R}(\rho) = \frac{1}{2}\int \rho^2\big[(\Delta\xi)^2 + \|D^2\xi\|_F^2\big] \geq 0.$$

Here $\|A\|_F = \sqrt{\mathrm{tr}(A^T A)}$ is the Frobenius norm of the matrix A. Consequently, by deducing various decay on the norms of $\rho(t,\cdot) - \rho_\infty(\cdot)$, the solution ρ converges to its steady state exponentially fast.

However, in the case $s \in (0, 1)$ considered here, it is not immediately clear whether $\mathcal{R}(\rho)$ given in (4.2) above is nonnegative or not. To simplify $\mathcal{R}(\rho)$, we need more explicit expressions of $D^2(-\Delta)^{-s}\rho$ and $\nabla[(-\Delta)^{-s}(\nabla\cdot(\rho\nabla\xi))]$, or the second order derivatives of the Riesz potential of ρ and $\rho\nabla\xi$ respectively. Since these derivatives cannot be applied to the corresponding kernel $\mathcal{K}(x) = c_{d,s}|x|^{2s-d}$ directly, we have to invoke the following technical lemma. For the sake of completeness, we shall also include a proof of it.

Lemma 4.1. *If ρ is a smooth function on $\mathbb{R}^d$, then the components of the Hessian matrix of the Riesz potential $(-\Delta)^{-s}\rho$ are given by*

$$D_{ij}(-\Delta)^{-s}\rho(x) = \partial_{ij}(-\Delta)^{-s}\rho(x) = -c_{d,s}^{+}\int K_{ij}(x-y)\big(\rho(x)-\rho(y)\big)\,dy, \tag{4.3}$$

where $K_{ij}(x) = |x|^{2s-2-d}((d+2-2s)x_i x_j/|x|^2 - \delta_{ij})$ and $c_{d,s}^{+} = (d-2s)c_{d,s}$.

Proof. Since the Riesz potential $(-\Delta)^{-s}$ is a singular integral, these second order derivatives cannot be applied to the kernel (1.3) directly, but can be derived from several equivalent approaches. We shall interpret $D_{ij}(-\Delta)^{-s}\rho$ as distributional derivatives, and obtain the expressions using the definition in a similar way as representing the velocity gradient using vorticity in fluid mechanics [39].

For any function $\phi \in \mathcal{S}(\mathbb{R}^d)$ (the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$), the distributional derivative $D_{ij}(-\Delta)^{-s}\rho$ is defined as

$$\big\langle D_{ij}(-\Delta)^{-s}\rho, \phi\big\rangle := \big\langle(-\Delta)^{-s}\rho, D_{ij}\phi\big\rangle = c_{d,s}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\frac{\rho(y)}{|x-y|^{d-2s}}\frac{\partial^2\phi(x)}{\partial x_i\partial x_j}\,dy\,dx.$$

Next, we use integration by parts to shift the derivatives from the test function ϕ to the singular integral $(-\Delta)^{-s}\rho$, by writing the above expression as a limit outside a ball. More precisely,

$$\begin{aligned}\big\langle(-\Delta)^{-s}\rho, D_{ij}\phi\big\rangle &= \lim_{\epsilon\to 0^+} c_{d,s}\int\limits_{\mathbb{R}^d}\rho(y)\Bigg[\int\limits_{B(y,\epsilon)^c}\frac{1}{|x-y|^{d-2s}}\frac{\partial^2\phi(x)}{\partial x_i\partial x_j}\,dx\Bigg]dy \\ &= \lim_{\epsilon\to 0^+}(d-2s)c_{d,s}\int\limits_{\mathbb{R}^d}\rho(y)\Bigg[\int\limits_{B(y,\epsilon)^c}\frac{x_i-y_i}{|x-y|^{d+2-2s}}\frac{\partial\phi(x)}{\partial x_j}\,dx\Bigg]dy,\end{aligned}$$

where $B(y,\epsilon)^c$ is the complement of the ball $B(y,\epsilon) = \{x\in\mathbb{R}^d \mid |x-y|<\epsilon\}$ and the integration on the boundary $\partial B(y,\epsilon)$ vanishes in the limit. Integrating by parts again, we obtain (the unit outer normal at $x\in B(y,\epsilon)^c$ is $-(x-y)/|x-y|$)

$$\lim_{\epsilon\to 0^+} c_{d,s}^{+}\int\limits_{\mathbb{R}^d}\rho(y)\Bigg[\int\limits_{B(y,\epsilon)^c}K_{ij}(x-y)\phi(x)\,dx - \int\limits_{\partial B(y,\epsilon)}\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^{d+3-2s}}\phi(x)\,dS_x\Bigg]dy, \tag{4.4}$$

where $c_{d,s}^{+} = (d-2s)c_{d,s}$ and

$$K_{ij}(x) = \frac{1}{d-2s}\frac{\partial^2}{\partial x_i\partial x_j}|x|^{2s-d} = \frac{1}{|x-y|^{d+2-2s}}\bigg((d+2-2s)\frac{x_i x_j}{|x|^2} - \delta_{ij}\bigg).$$

Since for any $x\in\partial B(y,\epsilon)$, $\phi(x) = \phi(y) + (x-y)\cdot\nabla\phi(y) + O(|x-y|^2)$, we can replace $\phi(x)$ by $\phi(y)$ in the boundary integral in (4.4), i.e.,

$$\lim_{\epsilon\to 0^+} \int\limits_{\partial B(y,\epsilon)} \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^{d+3-2s}}\phi(x)\,dS_x = \phi(y) \lim_{\epsilon\to 0^+} \int\limits_{\partial B(y,\epsilon)} \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^{d+3-2s}}\,dS_x.$$

It is easy to see that for $j \neq i$,

$$\int\limits_{\partial B(y,\epsilon)} \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^{d+3-2s}}\,dS_x = \int\limits_{B(y,\epsilon)^c} K_{ij}(x-y)\,dx = 0,$$

and for $j = i$,

$$\int\limits_{\partial B(y,\epsilon)} \frac{(x_i-y_i)(x_i-y_i)}{|x-y|^{d+3-2s}}\,dS_x = \int\limits_{B(y,\epsilon)^c} K_{ii}(x-y)\,dx = \frac{|\mathbb{S}^{d-1}|}{d}\epsilon^{2s-2},$$

where $|\mathbb{S}^{d-1}|$ is the area of the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d \mid |x| = 1\}$.

Therefore, the distributional derivative $\langle D_{ij}(-\Delta)^{-s}\rho, \phi\rangle$ written as the limit (4.4) can be simplified as

$$\begin{aligned}
&\big\langle D_{ij}(-\Delta)^{-s}\rho, \phi\big\rangle \\
&= \lim_{\epsilon\to 0^+} c_{d,s}^+ \int\limits_{\mathbb{R}^d} \rho(y)\Bigg[\int\limits_{B(y,\epsilon)^c} K_{ij}(x-y)\phi(x)\,dy - \phi(y)\int\limits_{B(y,\epsilon)^c} K_{ij}(x-y)\,dy\Bigg]\,dy \\
&= \lim_{\epsilon\to 0^+} c_{d,s}^+ \iint\limits_{|x-y|>\epsilon} K_{ij}(x-y)\big(\rho(y)\phi(x) - \rho(y)\phi(y)\big)\,dy\,dx \\
&= -\lim_{\epsilon\to 0^+} c_{d,s}^+ \int\limits_{\mathbb{R}^d} \phi(x)\Bigg[\int\limits_{B(x,\epsilon)} K_{ij}(x-y)\big(\phi(x)-\phi(y)\big)\,dy\Bigg]\,dx.
\end{aligned}$$

This implies the following singular integral represent of the Hessian matrix of $(-\Delta)^{-s}\rho$:

$$D_{ij}(-\Delta)^{-s}\rho(x) = -c_{d,s}^+ \int\limits_{\mathbb{R}^d} K_{ij}(x-y)\big(\rho(x)-\rho(y)\big)\,dy.$$

In particular, we can write the fractional Laplacian $(-\Delta)^{1-s}\rho$ as

$$\begin{aligned}
(-\Delta)^{1-s}\rho(x) &= -\sum_{i=1}^{d} D_{ii}(-\Delta)^{-s}\rho(x) = c_{d,s}^+ \int\limits_{\mathbb{R}^d} K_{ij}(x-y)\big(\rho(x)-\rho(y)\big)\,dy \\
&= c_{d,s}^+ \int\limits_{\mathbb{R}^d} \frac{\rho(x)-\rho(y)}{|x-y|^{d+2-2s}}\,dy,
\end{aligned}$$

recovering its standard singular integral representation [3,4]. $\square$

Using the singular integral representation (4.3), we obtain

$$
\begin{aligned}
\mathcal{R}(\rho) &= \sum_{i,j} \int_{\mathbb{R}^d} \big\{ \rho(x)\partial_i\xi(x)\partial_j\xi(x) D_{ij}(-\Delta)^{-s}\rho(x) - \rho(x)\partial_i\xi(x) D_{ij}(-\Delta)^{-s}[\rho\partial_j\xi](x) \big\}\, dx \\
&= -c_{d,s}^{+} \sum_{i,j} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(x)\partial_i\xi(x) K_{ij}(x-y) \\
&\quad \times \big\{ \partial_j\xi(x)\big(\rho(x)-\rho(y)\big) - \rho(x)\partial_j\xi(x) + \rho(y)\partial_j\xi(y) \big\}\, dy\, dx \\
&= c_{d,s}^{+} \sum_{i,j} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(x)\rho(y)\partial_i\xi(x) K_{ij}(x-y) \big\{ \partial_j\xi(x) - \partial_j\xi(y) \big\}\, dy\, dx \\
&= \frac{c_{d,s}^{+}}{2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(x)\rho(y) \big\langle \nabla\xi(x) - \nabla\xi(y), \mathbf{K}(x-y)\big(\nabla\xi(x)-\nabla\xi(y)\big)\big\rangle\, dy\, dx,
\end{aligned} \tag{4.5}
$$

where $\mathbf{K}(x)$ is a matrix with entries $K_{ij}(x)$ and the integrand is symmetrized in the last step.

Similar expressions like

$$
\mathcal{R}(\rho) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(x)\rho(y) \big\langle \big(\nabla\xi(x) - \nabla\xi(y)\big), D^2\mathcal{K}(x-y)\nabla\xi(x) - \nabla\xi(y) \big\rangle\, dy\, dx
$$

already appear in the context of nonlocal equations for granular flow or biological swarms [21], when the interaction kernel $\mathcal{K}$ in the equation $\rho_t = \nabla \cdot (\rho\nabla\mathcal{K} * \rho)$ is smooth. Therefore, we recover the following proposition.

Proposition 4.2. *Let ρ_t be a smooth solution of $\rho_t = \nabla \cdot (\rho\nabla\mathcal{K} * \rho)$ where $\mathcal{K}$ is a C^2 smooth kernel such that the Hessian $\mathbf{K} = D^2\mathcal{K}$ is a nonnegative and locally integrable matrix function. Then the second time derivative of the energy $\mathcal{E}[\rho] = \frac{1}{2}\int \rho\mathcal{K} * \rho$ is nonnegative.*

In one dimension, $\mathbf{K}(x) = (2-2s)|x|^{2s-3}$ is a positive function almost everywhere and $\mathcal{R}(\rho) \geq 0$ for any nonnegative density ρ, leading to the desired exponential convergence. However, in higher dimensions, the matrix $\mathbf{K}(x)$ can be written as

$$
\mathbf{K}(x) = |x|^{2s-2-d}\big((d+2-2s)x \otimes x/|x|^2 - I\big),
$$

which has one positive eigenvalue $\lambda_1 = (d+1-2s)|x|^{2s-d-2}$ and $d-1$ negative eigenvalues $\lambda_i = -|x|^{2s-d-2}$, $i = 2, \dots, d$. Therefore, it is not known from (4.5) whether $\mathcal{R}(\rho)$ is positive or not. We can conclude that both the relative entropy $\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty)$ and the entropy dissipation rate converge to zero exponentially fast in dimension $d = 1$, but in the cases of $d > 1$, it is unknown whether there is always exponentially fast convergence.

The above approach for the exponential decay in one dimension can be proved rigorously, by establishing the results for mollified solutions to the regularized equation (with linear diffusion for example). One of the main difficulties in our case lies in the definition and continuity of the entropy dissipation $\mathcal{I}(\rho)$. The set of functions for which $\mathcal{I}$ is finite is difficult to handle. Therefore, passing to the limit the exponential decay of the entropy dissipation using density argument is a complicated task in our case.

Another possible way to obtain the exponential decay for higher dimensions could be the following: we know from Section 2 that the exponential decay for the entropy follows from the generalized Log-Sobolev inequality

$$\mathcal{E}(\rho) - \mathcal{E}(\rho_\infty) \leq \frac{1}{2\lambda}\mathcal{I}(\rho). \tag{4.6}$$

Therefore, in order to obtain (4.6) for other dimensions than one, one idea is to follow a similar approach as Del Pino and Bolbeault [20] by expanding both sides of (4.6) and obtaining the equivalent inequality

$$\lambda\Bigg[\int_{\mathbb{R}^d} \rho(x)(-\Delta)^{-s}\rho(x)\,dx - 2\int_{\mathbb{R}^d} \rho(x)x\cdot\nabla(-\Delta)^{-s}\rho(x)\,dx\Bigg]$$
$$\leq 2\lambda\mathcal{E}(\rho_\infty) + \int_{\mathbb{R}^d} \rho(x)\big|\nabla(-\Delta)^{-s}\rho(x)\big|^2\,dx.$$

The second term on the left-hand side can be simplified using the definition of $(-\Delta)^{-s}\rho$ as the Riesz integral

$$(-\Delta)^{-s}\rho(x) = c_{d,s}\int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-2s}}\rho(y)\,dy,$$

and consequently

$$\begin{aligned}
&-2\int_{\mathbb{R}^d} \rho(x)x\cdot\nabla(-\Delta)^{-s}\rho(x)\,dx \\
&\quad= 2(d-2s)c_{d,s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(x)\rho(y)x\cdot(x-y)|x-y|^{2s-d-2}\,dy\,dx \\
&\quad= (d-2s)c_{d,s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(x)\rho(y)|x-y|^{2s-d}\,dy\,dx \\
&\quad= (d-2s)\int_{\mathbb{R}^d} \rho(x)(-\Delta)^{-s}\rho(x)\,dx.
\end{aligned}$$

Therefore, the inequality (4.6) becomes

$$\lambda(d+1-2s)\int_{\mathbb{R}^d} \rho(x)(-\Delta)^{-s}\rho(x)\,dx \leq 2\lambda\mathcal{E}(\rho_\infty) + \int_{\mathbb{R}^d} \rho(x)\big|\nabla(-\Delta)^{-s}\rho(x)\big|^2\,dx. \tag{4.7}$$

To get a self-consistent inequality, we have to write $\mathcal{E}(\rho_\infty)$ in terms of some functionals of ρ, which is established through the total conserved mass, $M = \int\rho = \int\rho_\infty$. Using the explicit expression for $\rho_\infty(x) = K_{d,s}(R^2 - |x|^2)_+^{1-s}$, the identity (2.2) implies that

$$(-\Delta)^{-s}\rho_\infty(x) = \frac{\lambda}{2}\frac{d}{d-2s}R^2 - \frac{\lambda}{2}|x|^2, \quad \text{for } |x| \le R.$$

Therefore, we conclude that

$$\begin{aligned}
\mathcal{E}(\rho_\infty) &= \frac{1}{2}\int\limits_{\mathbb{R}^d} \rho_\infty(x)\big((-\Delta)^{-s}\rho_\infty(x) + \lambda|x|^2\big)\,dx \\
&= \frac{\lambda K_{d,s}}{4}\int\limits_{\mathbb{R}^d} \big(R^2-|x|^2\big)_+^{1-s}\left(\frac{d}{d-2s}R^2+|x|^2\right)dx \\
&= \frac{\lambda K_{d,s}}{4}\frac{d\pi^{d/2}(d+2-2s)\Gamma(2-s)}{(d-2s)\Gamma(d/2+3-s)}R^{d+4-2s} = \tilde{K}_{d,s}\Bigg(\int\limits_{\mathbb{R}^d}\rho(x)\,dx\Bigg)^{\frac{d+4-2s}{d+2-2s}},
\end{aligned}$$

where (1.6) is used in the last step, together with the constant

$$\tilde{K}_{d,s} = \frac{d(d+2-2s)^{(d+4-2s)/(d+2-2s)}\lambda^{(d-2s)/(d+2-2s)}}{(d-2s)(d+4-2s)2^{(d+2-s)/(d+2-2s)}\pi^{d/(d-2-2s)}}.$$

Therefore, (4.6) is reduced to an inequality bounding the integral $\int \rho(-\Delta)^{-s}\rho\,dx$ by $\int\rho\,dx$ and $\int\rho|\nabla(-\Delta)^{-s}\rho|^2\,dx$, that is,

$$\lambda(d+1-2s)\int\limits_{\mathbb{R}^d}\rho(-\Delta)^{-s}\rho\,dx \le 2\lambda\tilde{K}_{d,s}\Bigg(\int\limits_{\mathbb{R}^d}\rho dx\Bigg)^{\frac{d+4-2s}{d+2-2s}} + \int\limits_{\mathbb{R}^d}\rho\big|\nabla(-\Delta)^{-s}\rho\big|^2\,dx,$$

where the equality holds for the steady state ρ_∞. The last inequality and (4.6) are implied by the equivalent inequality

$$\int\limits_{\mathbb{R}^d}\rho(-\Delta)^{-s}\rho\,dx \le C\Bigg(\int\limits_{\mathbb{R}^d}\rho\,dx\Bigg)^{2-3\theta}\Bigg(\int\limits_{\mathbb{R}^d}\rho\big|\nabla(-\Delta)^{-s}\rho\big|^2\,dx\Bigg)^{\theta} \tag{4.8}$$

in the product form, where $\theta = \frac{d-2s}{2d+2-4s}$ is determined by the homogeneity and C is given by any function $\rho(x) = A(R^2-|x-x_0|^2)_+^{1-s}$ (which is independent of A, R and x_0).

However, unlike the case of porous medium equation [20], we still do not know how to prove (4.8) to establish the Log-Sobolev inequality (4.6). The main difficulty lies in the integral $\int_{\mathbb{R}^d}\rho|\nabla(-\Delta)^{-s}\rho|^2$, where basic questions like monotonicity under symmetric decreasing rearrangement are not clear. Because of the equivalence between (4.6) and (4.8), we showed in Section 2 that (4.8) holds in one dimension and it is a consequence of the HWI inequalities, but it remains an open problem to prove or disprove (4.8) in higher dimensions.

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