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The diffusive logistic equation with a free boundary and sign-changing coefficient ☆

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Abstract

This short paper concerns a diffusive logistic equation with a free boundary and sign-changing coefficient, which is formulated to study the spread of an invasive species, where the free boundary represents the expanding front. A spreading–vanishing dichotomy is derived, namely the species either successfully spreads to the right-half-space as time $t \to \infty$ and survives (persists) in the new environment, or it fails to establish itself and will extinct in the long run. The sharp criteria for spreading and vanishing are also obtained. When spreading happens, we estimate the asymptotic spreading speed of the free boundary.

MSC: 35K57; 35K61; 35R35; 92D25

Keywords: Diffusive logistic equation; Sign-changing coefficient; Free boundary; Spreading–vanishing; Sharp criteria

1. Introduction

Understanding the nature of establishment and spreading of invasive species is a central problem in invasion ecology. A lot of mathematicians have made efforts to develop various invasion models and investigated them from a viewpoint of mathematical ecology, refer to [2–4,7–18, 22,24–29] for example. Most theoretical approaches are based on or start with single-species models. In consideration of the environmental heterogeneity, the following problem

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$$\begin{cases} u_t - d\Delta u = u\big(m(x) - u\big), & t > 0,\ x \in \Omega, \\ B[u] = 0, & t \geq 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega \end{cases}$$

is a typical one to describe the spreading of invasive species and has received an astonishing amount of attention, see, for example [2,21] and the references therein. In this model, $u(t,x)$ represents the population density; constant $d > 0$ denotes the diffusion (dispersal) rate; the function $m(x)$ accounts for the local growth rate (intrinsic growth rate) of the population and is positive on favorable habitats and negative on unfavorable ones; Ω is a bounded domain of $\mathbb{R}^N$; the boundary operator $B[u] = \alpha u + \beta \frac{\partial u}{\partial \nu}$, α and β are non-negative functions and $\alpha + \beta > 0$; ν is the outward unit normal vector of the boundary $\partial\Omega$. The corresponding systems with heterogeneous environment have also been studied extensively, please refer to [3,4,18,21] and the references cited therein.

To realize the spreading mechanism of an invading species (how fast the species spreads into new territory, and what factors influence the successful spread), Du and Lin [11] proposed the following free boundary problem of the diffusive logistic equation

$$\begin{cases} u_t - du_{xx} = u(a - bu), & t > 0,\ 0 < x < h(t), \\ u_x(t,0) = 0, \quad u\big(t, h(t)\big) = 0, & t \geq 0, \\ h'(t) = -\mu u_x\big(t, h(t)\big), & t \geq 0, \\ h(0) = h_0, \quad u(0,x) = u_0(x), & 0 \leq x \leq h_0, \end{cases} \tag{1.1}$$

where $x = h(t)$ is the moving boundary to be determined; a, b, d, h_0 and μ are given positive constants, h_0 denotes the size of initial habitat, μ is the ratio of expanding speed of the free boundary and population gradient at expanding front, it can also be considered as the "moving parameter"; u_0 is a given positive initial function. They have derived various interesting results.

Since then, this kind of problem describing the spread by free boundary has been studied intensively. For example, when the boundary condition $u_x = 0$ at $x = 0$ in (1.1) is replaced by $u = 0$, such free boundary problem was studied by Kaneko and Yamada [17]. Du and Guo [7,8], Du, Guo and Peng [9] and Du and Liang [10] considered the higher space dimensions, heterogeneous environment and time-periodic environment case, where the heterogeneous environment coefficients were required to have positive lower and upper bounds. Peng and Zhao [22] studied the seasonal succession case. When the nonlinear term $u(a - bu)$ is replaced by a general function $f(u)$, this problem has been investigated by Du and Lou [13] and Du, Matsuzawa and Zhou [14]. The diffusive competition system with a free boundary has been studied by Guo and Wu [15], Du and Lin [12] and Wang and Zhao [26]. The diffusive prey–predator model with free boundaries has been studied by Wang and Zhao [24,25,28].

Recently, Zhou and Xiao [29] studied the following diffusive logistic model with a free boundary in heterogeneous environment:

$$\begin{cases} u_t - du_{xx} = u\big(m(x) - u\big), & t > 0,\ 0 < x < h(t), \\ u_x(t,0) = 0, \quad u\big(t, h(t)\big) = 0, & t \geq 0, \\ h'(t) = -\mu u_x\big(t, h(t)\big), & t \geq 0, \\ h(0) = h_0, \quad u(0,x) = u_0(x), & 0 \leq x \leq h_0, \end{cases}$$

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where the initial function $u_0 \in C^2([0, h_0])$, $u_0'(0) = u_0(h_0) = 0$, $u_0'(h_0) < 0$ and $u_0 > 0$ in $(0, h_0)$. In the *strong* heterogeneous environment, i.e.,

(H1) $m \in C^1([0, \infty)) \cap L^\infty([0, \infty))$ and m changes sign in $(0, h_0)$,

Zhou and Xiao took d and μ as variable parameters and derived some sufficient conditions for species spreading (resp. vanishing); while in the *weak* heterogeneous environment, i.e.,

(H2) $m \in C^1([0, \infty))$ and $0 < m_1 \le m(x) \le m_2 < \infty$ for all $x \ge 0$,

they obtained a spreading–vanishing dichotomy and the sharp criteria for spreading and vanishing. When spreading happens, they gave an estimate of the asymptotic spreading speed of the free boundary for $0 < d \le d^*$ with some d^*.

Motivated by the above works, in this paper we consider the following problem

$$\begin{cases} u_t - du_{xx} = u\big(m(x) - u\big), & t > 0,\ 0 < x < h(t), \\ B[u](t, 0) = 0, \quad u\big(t, h(t)\big) = 0, & t \ge 0, \\ h'(t) = -\mu u_x\big(t, h(t)\big), & t \ge 0, \\ h(0) = h_0, \quad u(0, x) = u_0(x), & 0 \le x \le h_0, \end{cases} \tag{1.2}$$

where $B[u] = \alpha u - \beta u_x$, $\alpha, \beta \ge 0$ are constants and $\alpha + \beta = 1$; the initial function $u_0(x)$ satisfies

- $u_0 \in C^2([0, h_0])$, $u_0 > 0$ in $(0, h_0)$, $B[u_0](0) = u_0(h_0) = 0$.

Throughout this paper, we assume that the function $m(x)$ satisfies

(A) $m \in C([0, \infty)) \cap L^\infty([0, \infty))$ and $m(x)$ is positive somewhere in $(0, \infty)$.

Actually, if $m(x) \le 0$ in $(0, \infty)$, the problem (1.2) may not have any biological background.

The objective of this paper is to study the dynamics of (1.2) under weaker assumptions on the heterogeneous environment function $m(x)$. In Section 2, we shall give the global existence, uniqueness, regularity and some estimates of (u, h). Especially, the uniform estimates of $\|u(t, \cdot)\|_{C^1[0,h(t)]}$ for $t \ge 1$ and $\|h'\|_{C^{\nu/2}([n+1,n+3])}$ for $n \ge 0$ are obtained directly regardless of the size of $h_\infty := \lim_{t\to\infty} h(t)$, which is different from the previous works. Section 3 is devoted to the sharp criteria for spreading and vanishing. We shall use the pairs (h_0, μ) and (d, μ), respectively, as varying parameters to describe the sharp criteria. In Section 4, we study the long time behavior of u for spreading case. To this aim, in this section we first discuss the existence and uniqueness of positive solution to the corresponding stationary problem. As a consequence of the results obtained in Sections 3 and 4, a spreading–vanishing dichotomy is obtained. In Section 5 we estimate the asymptotic spreading speed of the free boundary when spreading occurs. The last section is a brief discussion.

We remark that for the higher dimensional and radially symmetric case of (1.2), the methods of this paper are still valid and the corresponding results can be retained. Besides, the present short paper can be regarded as the simplification, improvement and generalization of [29] in some sense.

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2. Global existence, uniqueness and estimates of the solution (u, h)

In this section, we give the existence, uniqueness, regularity and some estimates of the solution.

Theorem 2.1. *Problem (1.2) has a unique global solution (u, h), and for some $\nu \in (0, 1)$,*

$$u \in C^{\frac{1+\nu}{2}, 1+\nu}(D_\infty), \qquad h \in C^{1+\frac{\nu}{2}}(0, \infty), \tag{2.1}$$

where $D_\infty = \{(t, x) : t \in (0, \infty),\ x \in [0, h(t)]\}$. Furthermore, there exist positive constants $M = M(\|m, u_0\|_\infty)$ and $C = C(\mu, \|m, u_0\|_\infty)$, such that

$$0 < u(t, x) \le M, \qquad 0 < h'(t) \le \mu M, \quad \forall t > 0,\ 0 < x < h(t), \tag{2.2}$$

$$\|h'\|_{C^{\nu/2}([n+1,n+3])} \le C, \quad \forall n \ge 0, \qquad \|u(t, \cdot)\|_{C^1([0,h(t)])} \le C, \quad \forall t \ge 1. \tag{2.3}$$

Proof. Noting that the function m is bounded, and applying the methods used in [1,11] with some modifications, we can prove that (1.2) has a unique global solution (u, h), and satisfies (2.1) and the first estimate of (2.2). The details are omitted here. Because of the condition **(A)**, the regularity of (u, h) cannot be promoted.

As $u > 0$ for $0 < x < h(t)$ and $u = 0$ at $x = h(t)$, we see that $u_x(t, h(t)) \le 0$ and so $h'(t) \ge 0$. In view of [20, Lemma 2.6], it can be deduced that $h'(t) > 0$ for $t > 0$. The proof of $h'(t) \le \mu M$ is similar to that in [11]. The estimate (2.2) is obtained.

Now we prove (2.3). Let $y = x/h(t)$ and $w(t, y) = u(t, x)$. A series of detailed calculations yield

$$\begin{cases} w_t - d\zeta(t)w_{yy} - \xi(t, y)w_y = w\big[m\big(h(t)y\big) - w\big], & t > 0,\ 0 < y < 1, \\ \left(\alpha w - \dfrac{\beta}{h(t)} w_y\right)(t, 0) = 0, \quad w(t, 1) = 0, & t \ge 0, \\ w(0, y) = u_0(h_0 y), & 0 \le y \le 1, \end{cases}$$

where $\zeta(t) = h^{-2}(t)$, $\xi(t, y) = yh'(t)/h(t)$. For any integer $n \ge 0$, let $w^n(t, y) = w(t + n, y)$, then we have

$$\begin{cases} w^n_t - d\zeta(t+n)w^n_{yy} - \xi(t+n, y)w^n_y = w^n\big[m\big(h(t+n)y\big) - w^n\big], & t > 0,\ 0 < y < 1, \\ \left(\alpha w^n - \dfrac{\beta}{h(t+n)} w^n_y\right)(t, 0) = 0, \quad w^n(t, 1) = 0, & t \ge 0, \\ w^n(0, y) = u\big(n, h(n)y\big), & 0 \le y \le 1. \end{cases}$$

Noticing (2.2), applying the interior L^p estimate (see [20, Theorems 7.15 and 7.20]) and embedding theorem, we can find a constant $C > 0$ independent of n such that $\|w^n\|_{C^{\frac{1+\nu}{2}, 1+\nu}([1,3]\times[0,1])} \le C$ for all $n \ge 0$. This implies $\|w\|_{C^{\frac{1+\nu}{2}, 1+\nu}(E_n)} \le C$, where $E_n = [n + 1, n + 3] \times [0, 1]$. This fact combined with $h'(t) = -\mu u_x(t, h(t))$, $u_x(t, h(t)) = h^{-1}(t)w_y(t, 1)$ and $0 < h'(t) \le \mu M$, allows us to get the first estimate of (2.3). Since these rectangles E_n overlap and C is independent of n, one has $\|w\|_{C^{0,1}([1,\infty)\times[0,1])} \le C$. Using $u_x = h^{-1}(t)w_y$ again, the second estimate of (2.3) is obtained. □

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It follows from Theorem 2.1 that $h(t)$ is monotonically increasing. Therefore, there exists $h_\infty \in (0, \infty]$ such that $\lim_{t\to\infty} h(t) = h_\infty$.

3. Sharp criteria for spreading and vanishing

We first prove that $h_\infty < \infty$ implies $\lim_{t\to\infty} \max_{0\le x\le h(t)} u(t, x) = 0$. This conclusion will help us to establish the sharp criteria for spreading and vanishing. To this aim, we first give a lemma, its proof is similar to that of [24, Proposition 3.1] and the details will be omitted.

Lemma 3.1. *Let d, μ and B be as above, $c \in \mathbb{R}$. Assume that $s \in C^1([0, \infty))$, $w \in C^{\frac{1+\nu}{2},1+\nu}([0, \infty) \times [0, s(t)])$ and satisfy $s(t) > 0$, $w(t, x) > 0$ for $t \ge 0$ and $0 < x < s(t)$. We further suppose that $\lim_{t\to\infty} s(t) < \infty$, $\lim_{t\to\infty} s'(t) = 0$ and there exists a constant $C > 0$ such that $\|w(t, \cdot)\|_{C^1[0,s(t)]} \le C$ for $t > 1$. If (w, s) satisfies*

$$\begin{cases} w_t - dw_{xx} \ge cw, & t > 0,\ 0 < x < s(t), \\ B[w] = 0, & t \ge 0,\ x = 0, \\ w = 0, \quad s'(t) \ge -\mu w_x, & t \ge 0,\ x = s(t), \end{cases}$$

then $\lim_{t\to\infty} \max_{0\le x\le s(t)} w(t, x) = 0$.

Applying (2.3) and Lemma 3.1, we have the following theorem.

Theorem 3.1. *Let (u, h) be the solution of (1.2). If $h_\infty < \infty$, then $\lim_{t\to\infty} \max_{0\le x\le h(t)} u(t, x) = 0$. This shows that if the species cannot spread successfully, it will be extinct in the long run.*

For any given $\ell > 0$, let $\lambda_1(\ell; d, m)$ be the first eigenvalue of

$$\begin{cases} -d\phi'' - m(x)\phi = \lambda\phi, & 0 < x < \ell, \\ B[\phi](0) = 0, \quad \phi(\ell) = 0. \end{cases} \tag{3.1}$$

Remember the boundary condition $\phi(\ell) = 0$ and $m(x)$ is bounded; the following conclusions are well known (see, for example, [3,21,23]).

Proposition 3.1.

(i) *$\lambda_1(\ell; d, m)$ is continuous in d, m and ℓ;*
(ii) *$\lambda_1(\ell; d, m)$ is strictly increasing in d, strictly decreasing in m and ℓ;*
(iii) $\lim_{d\to\infty} \lambda_1(\ell; d, m) = \lim_{\ell\to 0^+} \lambda_1(\ell; d, m) = \infty$, $\lim_{d\to 0^+} \lambda_1(\ell; d, m) = -\max_{[0,\ell]} m(x)$.

Lemma 3.2. *If $h_\infty < \infty$, then $\lambda_1(h_\infty; d, m) \ge 0$.*

Proof. We assume $\lambda_1(h_\infty; d, m) < 0$ to get a contradiction. By the continuity of $\lambda_1(\ell; d, m)$ in ℓ and $h(t) \to h_\infty$, there exists $\tau \gg 1$ such that $\lambda_1(h(\tau); d, m) < 0$. Let w be the solution of

$$\begin{cases} w_t - dw_{xx} = w\big(m(x) - w\big), & t \ge \tau,\ 0 < x < h(\tau), \\ B[w](t, 0) = w\big(t, h(\tau)\big) = 0, & t \ge \tau, \\ w(\tau, x) = u(\tau, x), & 0 \le x \le h(\tau). \end{cases}$$

Then $u \geq w$ in $[\tau, \infty) \times [0, h(\tau)]$. As $\lambda_1(h(\tau); d, m) < 0$, similarly to the arguments in Section 3.2.2 of [27], we can prove that $\lim_{t\to\infty} w(t,x) = z(x)$ uniformly on $[0, h(\tau)]$, where z is the unique positive solution of

$$\begin{cases} -dz'' = z\big(m(x) - z\big), & 0 < x < h(\tau), \\ B[z](0) = z\big(h(\tau)\big) = 0. \end{cases}$$

Hence, $\liminf_{t\to\infty} u(t,x) \geq z(x) > 0$ in $(0, h(T))$. This contradicts Theorem 3.1. □

The following lemma is an analogue of [11, Lemma 3.5] and the proof will be omitted.

Lemma 3.3 *(Comparison principle). Let $\bar{h} \in C^1([0,\infty))$ and $\bar{h} > 0$ in $[0,\infty)$, $\bar{u} \in C^{0,1}(\overline{O}) \cap C^{1,2}(O)$, with $O = \{(t,x) : t > 0,\ 0 < x < \bar{h}(t)\}$. Assume that $(\bar{u}, \bar{h})$ satisfies*

$$\begin{cases} \bar{u}_t - d\bar{u}_{xx} \geq \bar{u}\big(m(x) - \bar{u}\big), & t > 0,\ 0 < x < \bar{h}(t), \\ B[\bar{u}](t,0) \geq 0, \quad \bar{u}\big(t, \bar{h}(t)\big) = 0, & t \geq 0, \\ \bar{h}'(t) \geq -\mu \bar{u}_x\big(t, \bar{h}(t)\big), & t \geq 0. \end{cases}$$

If $\bar{h}(0) \geq h_0$, $\bar{u}(0,x) \geq 0$ in $[0, \bar{h}(0)]$, and $\bar{u}(0,x) \geq u_0(x)$ in $[0, h_0]$. Then the solution (u,h) of (1.2) satisfies $h(t) \leq \bar{h}(t)$ in $[0,\infty)$, and $u \leq \bar{u}$ in D_∞, where D_∞ was given in Theorem 2.1.

Lemma 3.4. *If $\lambda_1(h_0; d, m) > 0$, then there exists $\mu_0 > 0$, depending on d, h_0, $m(x)$ and $u_0(x)$, such that $h_\infty < \infty$ provided $\mu \leq \mu_0$. By Lemma 3.2, $\lambda_1(h_\infty; d, m) \geq 0$ for $\mu \leq \mu_0$.*

Proof. The idea comes from [11,15,24], but the proof given here is simpler. Let ϕ be the positive eigenfunction corresponding to $\lambda_1 := \lambda_1(h_0; d, m)$. Noting that $\phi'(h_0) < 0$, $\phi(0) > 0$ when $\beta > 0$, and $\phi'(0) > 0$ when $\beta = 0$, it is easy to see that there exists $k > 0$ such that

$$x\phi'(x) \leq k\phi(x), \quad \forall 0 \leq x \leq h_0. \tag{3.2}$$

Let $0 < \delta, \sigma < 1$ and $K > 0$ be constants, which will be determined later. Set

$$s(t) = 1 + 2\delta - \delta e^{-\sigma t}, \qquad v(t,x) = Ke^{-\sigma t}\phi\big(x/s(t)\big), \quad t \geq 0,\ 0 \leq x \leq h_0 s(t).$$

Firstly, for any given $0 < \varepsilon \ll 1$, since $m(x)$ is uniformly continuous in $[0, 3h_0]$, it is easy to see that there exists $0 < \delta_0(\varepsilon) \ll 1$ such that, for all $0 < \delta \leq \delta_0(\varepsilon)$ and $0 < \sigma < 1$,

$$\big|s^{-2}(t)m\big(x/s(t)\big) - m(x)\big| \leq \varepsilon, \quad \forall t > 0,\ 0 \leq x \leq h_0 s(t). \tag{3.3}$$

Denote $y = x/s(t)$. Owing to (3.2), (3.3) and $\lambda_1 > 0$, direct calculations yield

$$\begin{aligned} v_t - dv_{xx} - v\big(m(x) - v\big) &= v\left(-\sigma + \frac{m(y)}{s^2(t)} - m(x) - \frac{y\phi'(y)}{\phi(y)}\frac{\sigma\delta}{s(t)}e^{-\sigma t} + \frac{\lambda_1}{s^2(t)}\right) + v^2 \\ &\geq v(-\sigma - \varepsilon - k\sigma + \lambda_1/4) > 0, \quad \forall t > 0,\ 0 < x < h_0 s(t) \end{aligned} \tag{3.4}$$

provided $0 < \sigma, \varepsilon \ll 1$. Evidently, $v(t, h_0 s(t)) = K\mathrm{e}^{-\sigma t}\phi(h_0) = 0$. If either $\alpha = 0$ or $\beta = 0$, then $B[v](t, 0) = 0$. If $\alpha, \beta > 0$, then $\alpha\phi(0) = \beta\phi'(0)$ and $\phi'(0) > 0$. Therefore, $B[v](t, 0) = \beta K\mathrm{e}^{-\sigma t}\phi'(0)[1 - 1/s(t)] > 0$ due to $s(t) > 1$. In a word,

$$B[v](t,0) \ge 0, \qquad v\big(t, h_0 s(t)\big) = 0, \quad \forall t \ge 0. \tag{3.5}$$

Fix $0 < \sigma, \varepsilon \ll 1$ and $0 < \delta \le \delta_0(\varepsilon)$. Thanks to the regularities of $u_0(x)$ and $\phi(x)$, we can choose a $K \gg 1$ such that

$$u_0(x) \le K\phi\big(x/(1+\delta)\big) = v(0,x), \quad \forall 0 \le x \le h_0. \tag{3.6}$$

Thanks to $h_0 s'(t) = h_0\sigma\delta\mathrm{e}^{-\sigma t}$ and $v_x(t, h_0 s(t)) = \frac{1}{s(t)} K\mathrm{e}^{-\sigma t}\phi'(h_0)$, there exists $\mu_0 > 0$ such that

$$h_0 s'(t) \ge -\mu v_x\big(t, h_0 s(t)\big), \quad \forall 0 < \mu \le \mu_0,\ t \ge 0. \tag{3.7}$$

Remember (3.4)–(3.7). An application of Lemma 3.3 to (u, h) and $(v, h_0 s)$ yields that $h(t) \le h_0 s(t)$ for all $t \ge 0$. Hence $h_\infty \le h_0 s(\infty) = h_0(1 + 2\delta)$ for all $0 < \mu \le \mu_0$. □

Using η instead of K, from the proof of Lemma 3.4 we see that the following lemma holds.

Lemma 3.5. *If $\lambda_1(h_0; d, m) > 0$, then there exist $\delta, \eta > 0$, such that $h_\infty < \infty$ provided $u_0(x) \le \eta\phi(x/(1+\delta))$ in $[0, h_0]$.*

Lemma 3.6. *Let $C > 0$ be a constant. For any given constants $\bar h_0, H > 0$, and any function $\bar u_0 \in C^2([0, \bar h_0])$ satisfying $B[\bar u_0](0) = \bar u_0(\bar h_0) = 0$ and $\bar u_0 > 0$ in $(0, \bar h_0)$, there exists $\mu^0 > 0$, depending on d, C, $\bar u_0(x)$ and $\bar h_0$, such that when $\mu \ge \mu^0$ and $(\bar u, \bar h)$ satisfies*

$$\begin{cases} \bar u_t - d\bar u_{xx} \ge -C\bar u, & t > 0,\ 0 < x < \bar h(t), \\ B[\bar u](t,0) = 0 = \bar u\big(t, \bar h(t)\big), & t \ge 0, \\ \bar h'(t) = -\mu \bar u_x\big(t, \bar h(t)\big), & t \ge 0, \\ \bar h(0) = \bar h_0, \quad \bar u(0,x) = \bar u_0(x), & 0 \le x \le \bar h_0, \end{cases}$$

we must have $\lim_{t\to\infty} \bar h(t) > H$.

This lemma can be proved by the similar method to that of [26, Lemma 3.2] and we omit the details here.

To establish the sharp criteria, we define two sets. For any given d, let $\Sigma_d = \{\ell > 0 : \lambda_1(\ell; d, m) = 0\}$. By the monotonicity of $\lambda_1(\ell; d, m)$ in ℓ, the set Σ_d contains at most one element. For any given ℓ, we define $\Sigma_\ell = \{d > 0 : \lambda_1(\ell; d, m) = 0\}$. Similarly, it contains at most one element.

Remark 3.1. For the fixed $d > 0$, due to $\lim_{\ell\to 0^+} \lambda_1(\ell; d, m) = \infty$ and $\lim_{\ell\to\infty} \lambda_1(\ell; d, m) := \lambda_1^\infty(d, m)$ exists, we have that $\Sigma_d \ne \emptyset$ is equivalent to $\lambda_1^\infty(d, m) < 0$. As a consequence, if m satisfies one of the following assumptions:

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(A1) there exist a constant $\rho > 0$ and two sequences $\{x_n\}$, $\{y_n\}$ satisfying $y_n > x_n > 0$ and $y_n - x_n \to \infty$ as $n \to \infty$, such that $m(x) \geq \rho$ in $[x_n, y_n]$;
(A2) there exist three constants $\rho > 0$, $k > 1$, $-2 < \gamma \leq 0$ and a sequence $\{x_n\}$ satisfying $x_n \to \infty$ as $n \to \infty$, such that $m(x) \geq \rho x^\gamma$ in $[x_n, kx_n]$,

then $\lambda_1^\infty(d, m) < 0$, and so $\Sigma_d \neq \emptyset$ for all $d > 0$.

In fact, when the condition **(A1)** holds, we use the following expression of $\lambda_1(\ell; d, m)$:

$$\lambda_1(\ell; d, m) = \inf_{\phi \in H^1((0,\ell))} \frac{E[\phi](0) + d\int_0^\ell (\phi'(x))^2 \mathrm{d}x - \int_0^\ell m(x)\phi^2(x)\mathrm{d}x}{\int_0^\ell \phi^2(x)\mathrm{d}x},$$

where, $E[\phi](0) = 0$ if $\alpha\beta = 0$, $E[\phi](0) = \frac{d\alpha}{\beta}\phi^2(0)$ if $\alpha, \beta > 0$. Take a function ϕ_n with $\phi_n(x) = 0$ in $[0, x_n]$, $\phi_n(x) = x - x_n$ in $[x_n, x_n + 1]$, $\phi_n(x) = 1$ in $[x_n + 1, y_n - 1]$ and $\phi_n(x) = y_n - x$ in $[y_n - 1, y_n]$. Then $\phi_n \in H^1((0, y_n))$, $\phi_n(0) = 0$, and

$$\int_0^{y_n} \big(\phi_n'(x)\big)^2 \mathrm{d}x = 2, \qquad \int_0^{y_n} m(x)\phi_n^2(x)\mathrm{d}x > \rho(y_n - x_n - 2), \qquad \int_0^{y_n} \phi_n^2(x)\mathrm{d}x < y_n - x_n.$$

Hence, for any fixed $d > 0$, we have

$$\lambda_1^\infty(d, m) < \lambda_1(y_n; d, m) \leq \frac{2d - \rho(y_n - x_n - 2)}{y_n - x_n} \to -\rho < 0 \quad \text{as } n \to \infty.$$

When the condition **(A2)** holds, we use the idea of [5, Lemma 3.1] to derive our conclusion. Let $\lambda_1(n)$ be the principal eigenvalue of

$$-d\psi'' = \lambda\psi, \quad x_n < x < kx_n; \qquad \psi(x_n) = \psi(kx_n) = 0,$$

and $\psi(x)$ be the corresponding positive eigenfunction. Through a simple rescaling $\psi(x) = \Psi(x/x_n) := \Psi(y)$, we see that $\Psi(y)$ satisfies

$$-d\Psi''(y) = x_n^2\lambda_1(n)\Psi(y), \quad 1 < y < k; \qquad \Psi(1) = \Psi(k) = 0.$$

Since $\Psi > 0$, we have $\lambda_1^* = x_n^2\lambda_1(n)$, where λ_1^* is the principal eigenvalue of

$$-d\phi'' = \lambda\phi, \quad 1 < x < k; \qquad \phi(1) = \phi(k) = 0.$$

Make the zero extension of ψ to $[0, x_n)$, then $\psi(0) = 0$ and

$$\begin{aligned}\int_0^{kx_n} \big[d\big(\psi'\big)^2 - m(x)\psi^2\big]\mathrm{d}x &= \int_{x_n}^{kx_n} \big[d\big(\psi'\big)^2 - m(x)\psi^2\big]\mathrm{d}x \\ &= \int_{x_n}^{kx_n} \big[\lambda_1(n)\psi^2 - m(x)\psi^2\big]\mathrm{d}x\end{aligned}$$

$$\leq \int_{x_n}^{kx_n} \left(x_n^{-2}\lambda_1^* - \rho k^\gamma x_n^\gamma\right)\psi^2 \mathrm{d}x$$

$$= x_n^{-2} \int_{x_n}^{kx_n} \left(\lambda_1^* - \rho k^\gamma x_n^{2+\gamma}\right)\psi^2 \mathrm{d}x < 0 \quad \text{as } n \gg 1$$

due to $x_n \to \infty$ and $2 + \gamma > 0$. This implies $\lambda_1(kx_n; d, m) < 0$ for $n \gg 1$, and then $\lambda_1^\infty(d, m) < 0$.

The conditions **(A1)** and **(A2)** seem to be "weaker" because $m(x)$ may be "very negative" in the sense that both $|\{x : m(x) > 0\}| \ll |\{x : m(x) < 0\}|$ and $\int_0^\infty m(x)\mathrm{d}x = -\infty$ are allowed, where $|A|$ means the measure of set A.

Remark 3.2. For each fixed $\ell > 0$, as $\lim_{d\to\infty} \lambda_1(\ell; d, m) = \infty$, $\lim_{d\to 0^+} \lambda_1(\ell; d, m) = -\max_{[0,\ell]} m(x)$, we see that $\Sigma_\ell \neq \emptyset$ is equivalent to $\max_{[0,\ell]} m(x) > 0$. By the condition **(A)**, we have $\max_{[0,\ell]} m(x) > 0$ for each suitably large ℓ. So, $\Sigma_\ell \neq \emptyset$ for such ℓ.

We first fix d, and consider h_0 and μ as varying parameters to depict the sharp criteria for spreading and vanishing. In the following Corollary 3.1 and Theorem 3.2, we assume that $\Sigma_d \neq \emptyset$ and let $h^* = h^*(d) \in \Sigma_d$. Recalling the estimate (2.2), as a consequence of Lemmas 3.2, 3.4 and 3.6, we have

Corollary 3.1.

(i) *If $h_\infty < \infty$, then $h_\infty \leq h^*$. Hence, $h_0 \geq h^*$ implies $h_\infty = \infty$ for all $\mu > 0$;*
(ii) *When $h_0 < h^*$, there exist $\mu_0, \mu^0 > 0$, which depend on d, $m(x)$, $u_0(x)$ and h_0, such that $h_\infty \leq h^*$ for $\mu \leq \mu_0$, $h_\infty = \infty$ for $\mu \geq \mu^0$.*

Finally, we give the sharp criteria for spreading and vanishing.

Theorem 3.2.

(i) *If $h_0 \geq h^* = h^*(d)$, then $h_\infty = \infty$ for all $\mu > 0$;*
(ii) *If $h_0 < h^*$, then there exists $\mu^* > 0$, depending on d, $m(x)$, $u_0(x)$ and h_0, such that $h_\infty = \infty$ for $\mu > \mu^*$, while $h_\infty \leq h^*$ for $\mu \leq \mu^*$.*

Proof. Noticing Corollary 3.1, by use of Lemma 3.3 and the continuity method, we can prove Theorem 3.2. Please refer to the proof of [11, Theorem 3.9] for details. □

When h_0 is fixed, d and μ are regarded as the varying parameters, we have the following sharp criteria for spreading and vanishing.

Theorem 3.3. *Assume that* $\max_{[0,h_0]} m(x) > 0$, *and let $d^* = d^*(h_0) \in \Sigma_{h_0}$ (see Remark 3.2).*

(i) *If $d \leq d^*$, then $h_\infty = \infty$ for all $\mu > 0$;*
(ii) *If $d > d^*$ and $\Sigma_d \neq \emptyset$, then there exists $\mu^* > 0$, depending on d, $m(x)$, $u_0(x)$ and h_0, such that $h_\infty = \infty$ when $\mu > \mu^*$, $h_\infty < \infty$ when $\mu \leq \mu^*$.*

Remark 3.3. If one of **(A1)** and **(A2)** holds, then $\Sigma_d \neq \emptyset$ for any $d > 0$ (see Remark 3.1).

Proof of Theorem 3.3. (i) When $d < d^*$, we have $\lambda_1(h_0; d, m) < \lambda_1(h_0; d^*, m) = 0$. So, $\Sigma_d \neq \emptyset$ and $h_0 > h^*(d)$. When $d = d^*$, we have $\lambda_1(h_0; d, m) = 0$ and $h_0 = h^*(d)$. By Theorem 3.2(i), $h_\infty = \infty$ for all $\mu > 0$.

(ii) For the fixed $d > d^*$, we have $\lambda_1(h_0; d, m) > \lambda_1(h_0; d^*, m) = 0$. By Lemma 3.4, there exists $\mu_0 > 0$ such that $h_\infty < \infty$ for $\mu \le \mu_0$. On the other hand, as $\Sigma_d \neq \emptyset$, there exists $H \gg 1$ such that $\lambda_1(H; d, m) < 0$. In view of Lemma 3.6, there exists $\mu^0 > 0$ such that $h_\infty > H$ provided $\mu \ge \mu^0$, which implies $\lambda_1(h_\infty; d, m) < \lambda_1(H; d, m) < 0$. Hence, $h_\infty = \infty$ for $\mu \ge \mu^0$ by Lemma 3.2. The remaining proof is the same as that of [11, Theorem 3.9]. $\square$

When $\alpha = 0$ and the condition **(H2)** holds, Theorem 3.3 has been given by [29, Theorem 5.2].

4. Long time behavior of u for the spreading case: $h_\infty = \infty$

For the vanishing case: $h_\infty < \infty$, we have known $\lim_{t\to\infty} \max_{0\le x\le h(t)} u(t, x) = 0$ (cf. Theorem 3.1). In this section we study the long time behavior of u for the spreading case: $h_\infty = \infty$. To this aim, we first study the existence and uniqueness of positive solution to the stationary problem:

$$\begin{cases} -du'' = u\big(m(x) - u\big), & 0 < x < \infty, \\ B[u](0) = 0. \end{cases} \tag{4.1}$$

The following lemma is a special case of [19, Proposition 2.2].

Lemma 4.1 *(Comparison principle). Let $\ell > 0$, $u_1, u_2 \in C^1([0, \ell))$ be positive functions in $(0, \ell)$ and satisfy in the sense of distribution that*

$$-du_1'' - m(x)u_1 + u_1^2 \ge 0 \ge -du_2'' - m(x)u_2 + u_2^2$$

and

$$B[u_1](0) \ge 0 \ge B[u_2](0), \qquad \limsup_{x\to\ell}\big(u_2^2 - u_1^2\big) \le 0.$$

Then $u_1 \ge u_2$ in $(0, \ell)$.

Theorem 4.1. *Assume that there exist constants $-2 < \gamma \le 0$ and $m_1, m_2 > 0$, such that*

$$m_1 = \liminf_{x\to\infty} \frac{m(x)}{x^\gamma}, \qquad m_2 = \limsup_{x\to\infty} \frac{m(x)}{x^\gamma}. \tag{4.2}$$

Then (4.1) has a unique positive solution $\hat{u}$ and

$$m_1 \le \liminf_{x\to\infty} \frac{\hat{u}(x)}{x^\gamma}, \qquad \limsup_{x\to\infty} \frac{\hat{u}(x)}{x^\gamma} \le m_2. \tag{4.3}$$

Proof. The existence of positive solution to (4.1) can be proved as that of [6, Lemma 7.16]. In fact, for any large $\ell > 0$, in the same way as that of [19], we can prove that the problem

$$\begin{cases} -du'' = u\big(m(x) - u\big), & 0 < x < \ell, \\ B[u](0) = 0, \quad u(\ell) = \infty \end{cases}$$

has a unique positive solution u_ℓ (when $\beta = 0$, this conclusion is exactly [6, Theorem 6.15]). Following the proof of [6, Lemma 7.16] step by step (using Lemma 4.1 instead of Lemma 5.6 there), we can prove that (4.1) has at least one positive solution.

The uniqueness of positive solution to (4.1) and the conclusion (4.3) can be proved by the similar way to that of [6, Theorem 7.12] with suitable modifications. We omit the details here. Actually, proofs of the uniqueness and (4.3) only rely on the properties of m and u at infinity, and have nothing to do with the condition of u at $x = 0$. □

It is easy to see that if the condition (4.2) holds, then the assumption **(A2)** must be true. Therefore, $\Sigma_d \neq \emptyset$ by Remark 3.1.

Lemma 4.2. *Assume that (4.2) holds. Let $h^* = h^*(d)$ satisfy $\lambda_1(h^*; d, m) = 0$. For $\ell > h^*$, which implies $\lambda_1 := \lambda_1(\ell; d, m) < 0$, let $u_\ell(x)$ be the unique positive solution of*

$$\begin{cases} -du'' = u\big(m(x) - u\big), & 0 < x < \ell, \\ B[u](0) = 0, \quad u(\ell) = 0. \end{cases} \tag{4.4}$$

Then $\lim_{\ell\to\infty} u_\ell(x) = \hat{u}(x)$ uniformly in $[0, L]$ for any $L > 0$.

Proof. Let ϕ be the positive eigenfunction of (3.1) corresponding to λ_1. Since $\lambda_1 < 0$, it is easy to verify that $\varepsilon\phi$ and $\sup_{x\geq 0} m(x)$ are the ordered lower and upper solutions to (4.4) provided $0 < \varepsilon \ll 1$. So, the problem (4.4) has at least one positive solution. The uniqueness of positive solution to (4.4) is followed by Lemma 4.1.

By Lemma 4.1, $u_\ell \leq \hat{u}$ in $[0, \ell]$, and u_ℓ is increasing in ℓ. Utilizing the regularity theory and compactness argument, it follows that there exists a positive function u, such that $u_\ell \to u$ in $C^2_{\rm loc}([0, \infty))$ as $\ell \to \infty$, and u solves (4.1). By the uniqueness, $u = \hat{u}$. □

Finally, we give the main result of this section.

Theorem 4.2. *Let (4.2) hold. If $h_\infty = \infty$, then $\lim_{t\to\infty} u(t, x) = \hat{u}(x)$ in $C_{\rm loc}([0, \infty))$.*

Proof. Choose $K > 1$ such that $K\hat{u} \geq u_0$ in $[0, h_0]$. Then $\varphi := K\hat{u}$ satisfies $\varphi_t - d\varphi_{xx} > \varphi(m(x) - \varphi)$ for $x > 0$. Let w be the solution of

$$\begin{cases} w_t - dw_{xx} = w\big(m(x) - w\big), & t > 0,\ 0 < x < \infty, \\ B[w](t, 0) = 0, & t > 0, \\ w(0, x) = K\hat{u}(x), & x \geq 0. \end{cases}$$

Then $u \leq w$, and w is monotone decreasing in t. Because $\hat{u}$ is the unique positive solution of (4.1), by the standard method we can prove that $\lim_{t\to\infty} w(t, x) = \hat{u}(x)$ uniformly in $[0, L]$ for any $L > 0$. As $h_\infty = \infty$, it follows that $\limsup_{t\to\infty} u(t, x) \leq \hat{u}(x)$ uniformly in $[0, L]$.

Let $h^* = h^*(d)$ be such that $\lambda_1(h^*; d, m) = 0$. When $\ell > h^*$, we have $\lambda_1 := \lambda_1(\ell; d, m) < 0$. As $h_\infty = \infty$, there exists $T \gg 1$ such that $h(t) > \ell$ for all $t \ge T$. Let ϕ be the positive eigenfunction of (3.1) corresponding to λ_1. Choose $0 < \sigma \ll 1$ such that $u(T, x) \ge \sigma\phi(x)$ in $[0, \ell]$ and $\sigma\phi$ is a lower solution of (4.4). Let u^ℓ be the unique solution of

$$\begin{cases} u_t - du_{xx} = u\big(m(x) - u\big), & t \ge T,\ 0 < x < \ell, \\ B[u](t,0) = 0, \quad u(t,\ell) = 0, & t \ge T, \\ u(T, x) = \sigma\phi(x), & x \in [0, \ell]. \end{cases}$$

Then $u \ge u^\ell$ in $[T, \infty) \times [0, \ell]$, and u^ℓ is increasing in t. So, $\lim_{t\to\infty} u^\ell(t, x) = u_\ell(x)$ uniformly in $[0, \ell]$ since u_ℓ is the unique positive solution of (4.4). Hence, $\liminf_{t\to\infty} u(t, x) \ge u_\ell(x)$ uniformly in $[0, \ell]$. By Lemma 4.2, $\liminf_{t\to\infty} u(t, x) \ge \hat{u}(x)$ uniformly in $[0, L]$ for any $L > 0$. □

Here we remark that, when $\alpha = 0$, Theorem 4.2 has been obtained by [29] under one of the following assumptions:

(i) the condition **(H2)** holds (see [29, Lemma 5.2]);
(ii) the function $m \in C^1([0, \infty))$, is positive somewhere in $(0, h_0)$ and satisfies (4.2) with $\gamma = 0$. The diffusion rate d satisfies $0 < d \le d^*$ for some $d^* > 0$ (see [29, Lemma 6.2]).

Obviously, **(H2)** implies (4.2) with $\gamma = 0$.

Combining Theorems 3.1, 3.2, 3.3 and 4.2, we have the following two theorems concerning spreading–vanishing dichotomy and sharp criteria for spreading and vanishing.

Theorem 4.3. *Let (4.2) hold, $d > 0$ be fixed and $h^* = h^*(d)$ satisfy $\lambda_1(h^*; d, m) = 0$. Then either*

(i) *Spreading: $h_\infty = \infty$ and $\lim_{t\to\infty} u(t, x) = \hat{u}(x)$ uniformly in $[0, L]$ for any $L > 0$, where $\hat{u}(x)$ is the unique positive solution of (4.1); or*
(ii) *Vanishing: $h_\infty \le h^*$ and $\lim_{t\to\infty} \max_{0\le x\le h(t)} u(t, x) = 0$.*

Moreover,

(iii) *If $h_0 \ge h^*$, then $h_\infty = \infty$ for all $\mu > 0$;*
(iv) *If $h_0 < h^*$, then there exists $\mu^* > 0$, depending on d, $m(x)$, $u_0(x)$ and h_0, such that $h_\infty = \infty$ for $\mu > \mu^*$, while $h_\infty \le h^*$ for $\mu \le \mu^*$.*

Theorem 4.4. *Assume that (4.2) holds, $h_0 > 0$ is fixed and $\max_{[0,h_0]} m(x) > 0$. Let $d^* = d^*(h_0) \in \Sigma_{h_0}$. Then either*

(i) *Spreading: $h_\infty = \infty$ and $\lim_{t\to\infty} u(t, x) = \hat{u}(x)$ uniformly in $[0, L]$ for any $L > 0$; or*
(ii) *Vanishing: $h_\infty < \infty$ and $\lim_{t\to\infty} \max_{0\le x\le h(t)} u(t, x) = 0$.*

Moreover,

(iii) *If $d \le d^*$, then $h_\infty = \infty$ for all $\mu > 0$;*
(iv) *If $d > d^*$, then there exists $\mu^* > 0$, depending on d, $m(x)$, $u_0(x)$ and h_0, such that $h_\infty = \infty$ for $\mu > \mu^*$, while $h_\infty < \infty$ for $\mu \le \mu^*$.*

5. Asymptotic spreading speed

In this section, we shall estimate the asymptotic spreading speed of the free boundary $h(t)$ when spreading occurs. Throughout this section, we assume that (4.2) holds with $\gamma = 0$, which implies $\Sigma_d \neq \emptyset$ for all $d > 0$.

Let us first state a known result, which plays an important role in later discussion.

Proposition 5.1. (See [11].) *Let d and c be given positive constants. Then for any $0 \le k < 2\sqrt{cd}$, the problem*

$$\begin{cases} -dw'' + kw' = w(c - w), & 0 < x < \infty, \\ w(0) = 0, \quad w(\infty) = c \end{cases}$$

has a unique positive solution $w_k(x)$. Moreover, for each $\mu > 0$, there exists a unique $k_0 = k_0(\mu, c) \in (0, 2\sqrt{cd}\,)$ such that $\mu w'_{k_0}(0) = k_0$.

Theorem 5.1. *When $h_\infty = \infty$, we have (no other restrictions on d, h_0, m and u_0)*

$$k_0(\mu, m_1) \le \liminf_{t\to\infty} \frac{h(t)}{t}, \qquad \limsup_{t\to\infty} \frac{h(t)}{t} \le k_0(\mu, m_2). \tag{5.1}$$

Proof. The proof is similar to those of [11, Theorem 4.2], [7, Theorem 3.6] and [29, Theorem 6.1]. Here we give the sketch for completeness and readers' convenience.

For any given $0 < \varepsilon \ll 1$, by (4.2) and (4.3) with $\gamma = 0$, there exists $\ell = \ell(\varepsilon) \gg 1$ such that

$$m_1 - \varepsilon < m(x) < m_2 + \varepsilon, \qquad m_1 - \varepsilon < \hat{u}(x) < m_2 + \varepsilon, \quad \forall x \ge \ell.$$

Taking advantage of $h_\infty = \infty$ and Theorem 4.2, we can see that there exists $T = T(\ell) \gg 1$ such that

$$h(T) > 2\ell, \qquad m_1 - 2\varepsilon < u(t + T, \ell) < m_2 + 2\varepsilon, \quad \forall t > 0.$$

Following the proof of [7, Theorem 3.6] or [29, Theorem 6.1] step by step, we can get (5.1). The details are omitted here. $\square$

When $\alpha = 0$, Theorem 5.1 has been given in [29] for the case that $0 < d \le d^*$ with some $d^* > 0$.

6. Conclusion

From the above discussions we have seen that $\lambda_1^\infty(d, m) := \lim_{\ell\to\infty} \lambda_1(\ell; d, m) < 0$ is an essential condition. This number is only characterized by d and m, and is independent of the moving parameter μ and initial value $u_0(x)$. It seems that $\lambda_1^\infty(d, m)$ is determined by d and $\int_0^\infty m(x)\mathrm{d}x$.

The main conclusions of this paper can be briefly summarized as follows:

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(I) If one of the following holds:

(i) d is suitably small (h_0 and $m(x)$ are fixed, and $m(x)$ is positive somewhere in $(0, h_0)$),
(ii) $m(x)$ is suitably large in the sense of "distribution" (h_0 and d are fixed),
(iii) h_0 is suitably large (d and $m(x)$ are fixed, and $m(x)$ satisfies either **(A1)** or **(A2)**),

then the species will successfully spread and survive in the new environment (maintain a positive density distribution), regardless of initial population size and value of the moving parameter.

(II) When the above situations do not occur, we can control the moving parameter μ and find a critical value μ^* such that the species will spread successfully when $\mu > \mu^*$, the species fails to establish itself and will be extinct in the long run when $\mu \le \mu^*$. A better way to reduce the moving parameter is to control the surrounding environment.

These theoretical results may be helpful in the prediction and prevention of biological invasions.

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