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ScienceDirect

J. Differential Equations 260 (2016) 445–477

Journal of Differential Equations

www.elsevier.com/locate/jde

Diffraction of a shock into an expansion wavefront for the transonic self-similar nonlinear wave system in two space dimensions

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Received 10 January 2015

Available online 12 September 2015

Abstract

We consider a configuration where a planar shock reflects and diffracts as it hits a semi-infinite rigid screen. The diffracted reflected shock meets the diffracted expansion wave, created by the incident shock that does not hit the screen, and changes continuously from a shock into an expansion. The governing equation changes its type and becomes degenerate as the wave changes continuously from a shock to an expansion. Furthermore the governing equation has multiple free boundaries (transonic shocks) and an additional degenerate sonic boundary (the expansion wave). We develop an analysis to understand the solution structure near which the shock strength approaches zero and the shock turns continuously into an expansion wavefront, and show the existence of the global solution to this configuration for the nonlinear wave system. Moreover we provide an asymptotic analysis to estimate the position of the change of the wave, and present intriguing numerical results.

MSC: primary 76L05, 35L65; secondary 65M06, 35M33

Keywords: Transonic shock; Shock diffraction; Riemann problem; Conservation laws; Nonlinear wave system

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[1] Research supported in part by the National Science Foundation under Grants DMS-1212142 and DMS-1351898.
[2] Research supported by the National Science Foundation under Grant DMS-1109202.

http://dx.doi.org/10.1016/j.jde.2015.08.051

1. Introduction

This paper addresses one of the longstanding open problems in fluid mechanics and multidimensional conservation laws: shock diffraction by a screen. For the weak diffracted waves, there are perturbation analyses by Lighthill [18] for a finite strength shock along a flat wall with an angle near π, by Ting and Ludloff [27] for a finite strength shock along a thin airfoil with an angle close to zero, and by Keller and Blank [10] for a weak shock at any angle by a corner of any angle. Hunter and Keller [7] worked on the weak shock diffraction by a rigid wedge and presented the asymptotic analysis of nonlinear geometrical acoustics by using Whitham's nonlinearization technique [28]. Recently Hunter and Tesdall [8] provided an interesting numerical result on the asymptotically reduced problem, known as the unsteady transonic small disturbance equation, to understand the local structure of self-similar solutions near a point at which the diffracted reflected shock meets the diffracted expansion wave, and showed that the shock diffracts nonlinearly into the expansion region. They [8] noted that the nonlinearity is important as the standard transonic scaling and matching with the global linearized solution is not sufficient to understand this region where the shock changes to an expansion wave. Those asymptotic analyses and numerical results provide some understanding of solutions to the shock diffraction problems.

Despite the physical importance of the problem, however, there are very few rigorous results available. The purpose of this paper is to establish the global solution to the shock diffraction problem by a screen. More specifically, we consider a planar shock of constant strength hitting a semi-infinite, rigid screen at normal incidence. The incident shock, the part that does not hit the screen, diffracts around the screen, and creates an expansion wave behind it. The reflected shock also diffracts past the screen, and also creates a diffracted expansion wave behind the reflected shock. The diffracted reflected shock meets the diffracted expansion wave, and the wave changes continuously from a shock into an expansion. See Fig. 1. We study the solution structure of this transonic shock diffraction problem for the nonlinear wave system in two space dimensions.

The nonlinear wave system, which can be considered as wave motions of shallow water and multidimensional p-systems, is a reduced system from the compressible Euler system for isentropic, irrotational flow in two space dimensions [2,3]. The nonlinear wave system can be also considered as a part of an operator splitting scheme in numerics, where the compressible Euler system can be split into the nonlinear wave system (the pressure system) and the pressure-less system (the gradient flow). In fact [30] noted that the Euler system can be split into the pressure-gradient system and the pressure-less system, and the pressure-gradient system has been studied in [25,29] and the references therein. Note that the pressure-gradient system is a special case of the nonlinear wave system. Note also that the pressure-less system is well understood by [24]. Hence if one understands the solution structure of the nonlinear wave system, one can construct the solution for the Euler system successively by using the splitting method. Furthermore, there are many similarities on the structures of both the nonlinear wave system and the Euler system, see [2,22]. As such, it is crucial to understand the nonlinear wave system in order to study the Euler system.

There are recent progresses on the transonic self-similar nonlinear wave system for different configurations including global solutions to Mach stems for interacting shocks [3], local solutions to regular shock reflections [9], a global solution to a shock diffraction [12,14], weak solutions to regular shock reflections by a wedge [23], a global solution to a shock diffraction by a convex corner [4], and numerical solutions to the triple point paradox [26].

Typically the position of the transonic shock is unknown a priori and hence it gives rise to a free boundary problem. In this configuration, we have two free boundary problems corresponding to transonic diffracted shocks; one is created by the incident shock and the other is by the reflected shock. These two free boundary problems are connected by a sonic boundary which becomes a degenerate Dirichlet boundary condition, and the position of the sonic boundary is in part unknown as the change of the wave is not known a priori. This is a new type of free boundary problems and little progress has been made to rigorously understand such problems. It is also of interest to understand how the diffracted wave changes from the compression to the expansion wave. This paper provides a correct framework to establish the existence result and an analysis to understand how the type of the diffracted wave changes.

Our analysis is advanced from the benchmark work [12,13,15], where the mathematical frameworks for self-similar transonic shock problems were provided; in particular, we utilize the iteration methods developed in [12,13,15]. We note that, however, those earlier studies are focused on the configurations that consist of one transonic shock (free boundary) and one sonic (degenerate) boundary and the estimates developed there reply on the wave structures that the density of the flow attains its minimum value at the degenerate boundary and they have a specific geometry of the boundary. Our new configuration gives rise to more complicated boundary value problems with two free boundaries and two sonic boundaries, one of which becomes a free boundary additionally. The wave structure in this configuration is much more involved (one can consider it as an interaction of two flows; one created by the reflected diffracted shock and the other by the diffracted expansion wave) and more refined estimates are required to understand the problem.

In order to analyze such interaction of the waves, we provide a new local estimate in the region where the wave changes. More precisely we establish the estimate for the case when the governing equation becomes degenerate and at the same time the corresponding directional derivative condition also becomes degenerate. And we provide improved estimates, where conditions given in [12,13,15] are relaxed, and show that both transonic shock boundaries (the diffracted reflected shock and the diffracted shock) and the sonic boundary (the diffracted expansion wave created by the reflected shock) are in C^{1,α_Σ} where $0 < \alpha_\Sigma < 1$ depends only on the incident shock strength, and the solution is continuous up to the sonic boundary (the diffracted expansion wave) including the point at which the wave changes from the shock to the expansion.

The main contributions of this paper are the following. We first formulate the corresponding boundary value problem for the self-similar nonlinear wave system. We next establish local estimates near the region on which the shock changes to an expansion wave, and provide the existence result for the global solution to this configuration. We also present the nonlinear asymptotic analysis to the nonlinear wave system in the same spirit of [7], and provide approximate solutions to locate the position at which the diffracted shock meets the diffracted expansion wave. Numerical results by using CLAWPACK for certain pressures are presented as well.

We believe our results will add to the understanding of transonic shocks and lead to further developments of systematic theories for multi-dimensional conservation laws. Interested readers can refer the survey paper [5] for the comprehensive references and the recent progresses in the transonic problems.

2. Formulation and main result

From the compressible Euler system for isentropic flow in two space dimensions, ignoring the nonlinear velocity terms (assuming low velocities) and assuming irrotational flows, we can deduce a simpler system, the nonlinear wave system [2,3]

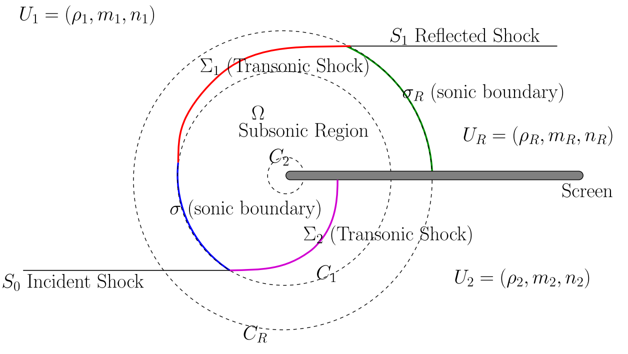

Fig. 1. The transonic shock diffraction.

$$
\begin{aligned}
\rho_t + (\rho u)_x + (\rho v)_y &= 0, \\
(\rho u)_t + p_x &= 0, \\
(\rho v)_t + p_y &= 0.
\end{aligned} \tag{1}
$$

Here $\rho(t, x, y)$ is the density, $u(t, x, y)$ and $v(t, x, y)$ are the x and y components of the velocity, respectively, and $p(\rho)$ is the pressure satisfying a polytropic gas law

$$
\frac{dp}{d\rho} = c^2(\rho) = k\gamma\rho^{\gamma-1},
$$

with constants k (we let $k = 1/\gamma$ for simplicity) and $1 < \gamma < \infty$, and a local sound speed $c^2(\rho)$. This system can be considered as wave motions of shallow water ($\gamma = 2$) and multidimensional p-systems [2,3].

We let the momentum $(\rho u, \rho v) = (m, n)$ and use U to denote (ρ, m, n):

$$
U = (\rho, m, n) = (\rho, \rho u, \rho v).
$$

We consider a configuration in which a planar incident shock S_0 moves downward and half of it hits the screen located on the positive x-axis, creates a reflected shock S_1, and the other half of the incident shock moves downward continuously. The reflected shock S_1 diffracts, creating a transonic shock which we denote S. The diffracted wave changes continuously from a compression to an expansion wave and creates a sonic curve, denoted by σ. The incident shock S_0 also diffracts and becomes a transonic shock, and we denote this diffracted shock S_d. In order to avoid any complications due to the edge of the screen, we consider the boundary associated with the screen (the surface of the screen) is smooth, that is, the screen has a small thickness with a rounded edge, see Fig. 1. We also ignore the viscous separation of a vortex sheet from the edge of the screen.

Let two positive constants satisfying $\rho_1 > \rho_2 > 0$ be given. For $x < 0$, the incident shock S_0 separates U_1 and U_2 where

$$U(x, y, 0) = \begin{cases} U_1 = (\rho_1, 0, n_1) & \text{if } y > 0, \\ U_2 = (\rho_2, 0, 0) & \text{if } y < 0, \end{cases} \tag{2}$$

with $n_1 < 0$ so that S_0 moves downward.

In the self-similar coordinates $\xi = x/t$ and $\eta = y/t$, the system (1) can be written in a conserved form

$$(F - \xi U)_\xi + (G - \eta U)_\eta = -2U, \tag{3}$$

where $F = (m, p, 0)$ and $G = (n, 0, p)$, or equivalently,

$$\begin{aligned} -\xi\rho_\xi - \eta\rho_\eta + m_\xi + n_\eta &= 0, \\ -\xi m_\xi - \eta m_\eta + c^2(\rho)\rho_\xi &= 0, \\ -\xi n_\xi - \eta n_\eta + c^2(\rho)\rho_\eta &= 0. \end{aligned} \tag{4}$$

This system can be written as a second order equation of ρ:

$$((c^2 - \xi^2)\rho_\xi - \xi\eta\rho_\eta)_\xi + ((c^2 - \eta^2)\rho_\eta - \xi\eta\rho_\xi)_\eta + \xi\rho_\xi + \eta\rho_\eta = 0. \tag{5}$$

The system is sonic when $c^2(\rho) = \xi^2 + \eta^2$, supersonic when $c^2(\rho) < \xi^2 + \eta^2$, and subsonic when $c^2(\rho) > \xi^2 + \eta^2$.

Across the horizontal incident shock S_0, the Rankine–Hugoniot jump conditions in (3) become

$$[G - \eta_0 U] = 0, \tag{6}$$

or equivalently

$$[n] = \eta_0[\rho], \quad [p] = \eta_0[n], \quad 0 = \eta_0[m],$$

where η_0 is the shock speed. Then since $[n]^2 = [p][\rho]$, $\eta_0^2 = [p]/[\rho]$, and $n_1 < 0$, we obtain

$$n_1 = -\sqrt{(p(\rho_1) - p(\rho_2))(\rho_1 - \rho_2)}, \tag{7}$$

$$\eta_0 = -\sqrt{\frac{p(\rho_1) - p(\rho_2)}{(\rho_1 - \rho_2)}}. \tag{8}$$

The reflected shock S_1 is a plane, and a new state $U_R = (\rho_R, m_R, n_R)$, where $\rho_R > \rho_1$, is created below the reflected shock S_1 (in-between S_1 and the screen). Note that we consider a configuration in which the screen does not absorb the incident shock and it simply reflects the incident shock, that is, the shock strength across S_1 is preserved to be the same as the shock strength across S_0. The Rankine–Hugoniot jump conditions, in particular $[n]^2 = [p][\rho]$ and $(n_R - n_1)^2 = (n_1 - n_2)^2$, across S_1 imply

$$(p(\rho_R) - p(\rho_1))(\rho_R - \rho_1) = (p(\rho_1) - p(\rho_2))(\rho_1 - \rho_2), \quad \text{and} \quad \rho_R > \rho_1. \tag{9}$$

Hence with ρ_R satisfying the above stated Rankine–Hugoniot condition (9) for given ρ_1 and ρ_2, we have

$$n_R = 0, \tag{10}$$

$$\eta_R = \sqrt{\frac{p(\rho_R) - p(\rho_1)}{(\rho_R - \rho_1)}}. \tag{11}$$

We denote sonic circles C_i, where $i = 1, 2, R$, which correspond to $C_i = \{c^2(\rho_i) = r^2\}$, and write Ω to be the subsonic region of this configuration. We write $\sigma \subset C_1$ to be the sonic boundary on which $\rho = \rho_1$. Note that this sonic boundary σ is also unknown a priori, and thus it becomes a free boundary.

We also denote two free boundaries due to the transonic shocks; Σ_1 to be the diffracted regular shock curve originated from the reflected shock S_1, and Σ_2 to be the diffracted shock curve from the incident shock S_0. The sonic boundary (the expansion wave) behind the reflected shock S_1 is denoted by σ_R where $\sigma_R \subset C_R$. Finally we denote Σ_S to be the boundary of the screen. Hence the subsonic region is bounded by two sonic boundaries σ_R and σ where σ is a free boundary, two transonic shock curves Σ_i, $i = 1, 2$ as free boundaries, and the screen Σ_S.

It is convenient to work in polar coordinates $\xi = r\cos\theta$ and $\eta = r\sin\theta$, as the sonic line becomes the circle for our system. Then the governing equation (5) in the subsonic region Ω is written as

$$\left((c^2 - r^2)\rho_r\right)_r + \frac{c^2}{r}\rho_r + \left(\frac{c^2}{r^2}\rho_\theta\right)_\theta = 0, \tag{12}$$

or in a non-divergence form

$$(c^2 - r^2)\rho_{rr} + \frac{c^2}{r^2}\rho_{\theta\theta} + (c^2)'\left(\rho_r^2 + \frac{1}{r^2}\rho_\theta^2\right) + \frac{c^2}{r}\rho_r - 2r\rho_r = 0. \tag{13}$$

Next we derive the shock evolution equations along Σ_j. To this end, we first rewrite the system in conservation form $\widetilde{F}_r + \widetilde{G}_\theta = \widetilde{H}$, where

$$\widetilde{F} = \begin{pmatrix} -r\rho + \cos\theta m + \sin\theta n \\ p(\rho)\cos\theta - rm \\ p(\rho)\sin\theta - rn \end{pmatrix}, \quad \text{and} \quad \widetilde{G} = \frac{1}{r}\begin{pmatrix} -\sin\theta m + \cos\theta n \\ -p(\rho)\sin\theta \\ p(\rho)\cos\theta \end{pmatrix}.$$

Then the Rankine–Hugoniot conditions in polar coordinates along the shocks become $[\widetilde{F}] = dr/d\theta[\widetilde{G}]$:

$$-r[\rho] + \cos\theta[m] + \sin\theta[n] = \frac{dr}{d\theta}\left(-\frac{\sin\theta}{r}[m] + \frac{\cos\theta}{r}[n]\right), \tag{14}$$

$$[p]\cos\theta - r[m] = -\frac{dr}{d\theta}[p]\frac{\sin\theta}{r}, \tag{15}$$

$$[p]\sin\theta - r[n] = \frac{dr}{d\theta}[p]\frac{\cos\theta}{r}. \tag{16}$$

Solving for $[m]$ in (15) and for $[n]$ in (16), we have

$$[m] = \frac{dr}{d\theta}[p]\frac{\sin\theta}{r^2} + [p]\frac{\cos\theta}{r} = \frac{[p]}{r^2}(\frac{r'}{r}\eta + \xi), \tag{17}$$

$$[n] = -\frac{dr}{d\theta}[p]\frac{\cos\theta}{r^2} + [p]\frac{\sin\theta}{r} = \frac{[p]}{r^2}(-\frac{r'}{r}\xi + \eta). \tag{18}$$

From (14) eliminate $[m]$ and $[n]$ by using (17) and (18) to obtain the following shock evolution equations on Σ_j, $j = 1, 2$,

$$\frac{dr}{d\theta} = -r\sqrt{\frac{r^2[\rho] - [p]}{[p]}} = -r\sqrt{\frac{r^2 - \bar{c}_j}{\bar{c}_j}} = -s(r, \rho, \rho_j), \tag{19}$$

where

$$\bar{c}_j = \frac{[p]}{[\rho]} = \frac{p(\rho) - p(\rho_j)}{\rho - \rho_j}.$$

Note that we have chosen the negative sign in (19) so that $dr/d\theta \leq 0$ because the other one is not physical in our configuration.

While the Rankine–Hugoniot conditions (19) can be used as a shock evolution equation, the problem then becomes underdetermined since the position of the transonic shock is unknown a priori. In order to obtain the well-posed problem, it is necessary to impose an appropriate boundary condition on Σ_j, $j = 1, 2$. We use the following nonlinear oblique derivative boundary condition:

$$M^j\rho = \beta^j \cdot \nabla\rho = 0 \quad \text{on} \quad \Sigma_j,$$

where $\nabla\rho = (\rho_r, \rho_\theta)$ and $\beta^j = (\beta_1^j, \beta_2^j)$ are

$$\beta_1^j = r'\left(-c^2(r^2 - \bar{c}_j) + 3\bar{c}_j(c^2 - r^2)\right), \qquad \beta_2^j = -3c^2(r^2 - \bar{c}_j) + \bar{c}_j(c^2 - r^2).$$

Here the obliqueness, denoted by μ, becomes

$$\mu^j = \beta^j \cdot (-1, r') = -2r'\left(\bar{c}_j(c^2 - r^2) + c^2(r^2 - \bar{c}_j)\right).$$

The oblique boundary condition is derived by taking the tangential derivative along the shock (assuming the shock position is differentiable) and using the first order system $\widetilde{F}_r + \widetilde{G}_\theta = \widetilde{H}$, see [12,15] for details on the derivation.

It is noteworthy to point out that [1] is the first to derive the oblique derivative boundary condition on the transonic shock for a different model (the steady small disturbance equation) as it is done in this paper. Since their work, by now, it became a convention to have the oblique boundary condition on the transonic shock. The rational to use the oblique boundary condition along the transonic shock is following: In order to establish the existence of the subsonic solution, a standard of procedure is to apply an appropriate fixed point type argument which requires a certain compactness. The correct boundary condition would then lead to the regularity estimates and thus to obtain the compactness. The oblique derivative boundary condition has been a conventional boundary condition for the transonic shock problems for the nonlinear wave system to obtain the necessary compactness.

Note that [23] derived an oblique derivative boundary condition for the nonlinear wave system in a different way; from the second order density equation in a weak form with a test function in a linear form with two independent parameters. They deduced two Rankine–Hugoniot conditions; one is exactly the same as the shock evolution equation as in this paper, and the other becomes an oblique boundary condition after simplifying higher-order derivative terms by using tangential derivatives of the shock evolution equation along the transonic shock. Their oblique boundary condition is very similar to ours, in particular, both have the same mathematical difficulties (the degeneracies appeared in both conditions are exactly the same). Hence whichever the boundary value problem one imposes, the degeneracies must be handled and thus the techniques developed in this paper will be applicable for both boundary problems.

Finally, since the transonic shock hits the screen Σ_S perpendicularly due to the no-flow condition on the screen, we impose $\partial\rho/\partial n = 0$ on Σ_S.

The governing boundary value problem for ρ in the subsonic region Ω becomes;

$$Q\rho = \left((c^2 - r^2)\rho_r\right)_r + \frac{c^2}{r}\rho_r + \left(\frac{c^2}{r^2}\rho_\theta\right)_\theta = 0, \quad c^2(\rho) > r^2, \quad \text{in } \Omega, \tag{20}$$

$$\rho = \rho_R \quad \text{on } \sigma_R, \tag{21}$$

$$M^1\rho = \beta^1 \cdot \nabla\rho = 0, \quad \frac{dr}{d\theta} = -s(r, \rho, \rho_1) = -r\sqrt{\frac{r^2 - \bar{c}_1}{\bar{c}_1}} \quad \text{on } \Sigma_1, \tag{22}$$

$$c^2(\rho) = r^2 \quad \text{on } \sigma, \tag{23}$$

$$M^2\rho = \beta^2 \cdot \nabla\rho = 0, \quad \frac{dr}{d\theta} = -s(r, \rho, \rho_2) = -r\sqrt{\frac{r^2 - \bar{c}_2}{\bar{c}_2}} \quad \text{on } \Sigma_2, \tag{24}$$

$$\frac{\partial\rho}{\partial n} = 0 \quad \text{on } \Sigma_S. \tag{25}$$

The hyperbolic state U_1 remains to be a constant, and consequently along the diffracted reflected shock S on Σ_1, the density ρ must hold $\rho > \rho_1$. Similarly the other diffracted shock S_d must stay in between two sonic circles C_1 and C_2 in order to be physical. Hence we have the following shock entropy conditions:

E1. $C_1 < S < C_R$,
E2. $C_2 < S_d < C_1$.

The diffracted reflected shock S loses its strength and becomes sonic, that is, $\rho = \rho_1$ at the point where the shock becomes sonic. In order words, the shock becomes sonic when it hits the sonic circle C_1, and thus the sonic curve $\sigma \subset C_1$. Hence ρ decreases from ρ_R to ρ_1 as it changes from the shock to the expansion wave. However, it is not known a priori where the shock becomes sonic, that is, the position, denoted by $\Xi_1 = \overline{\Sigma}_1 \cap \sigma$, is unknown a priori. The remaining sonic boundary σ is located on the sonic circle C_1.

We denote corner points by

$$\begin{aligned}
\Xi_R &= (c_R, \theta_R) = \sigma_R \cap \overline{\Sigma}_1, \\
\Xi_1 &= (c_1, \theta_1) = \sigma \cap \overline{\Sigma}_1, \\
\Xi_2 &= (c_1, \theta_2) = \sigma \cap \overline{\Sigma}_2, \\
\Xi_s &= (c_s, \theta_s) = \overline{\Sigma}_2 \cap \Sigma_S,
\end{aligned}$$

where $\theta_R = \arctan(\eta_R/\xi_R)$, η_R satisfying (11) and $\xi_R = \sqrt{c^2(\rho_R) - \eta_R^2}$, and $\theta_2 = \arctan(\eta_0/\xi_0)$, η_0 satisfying (8) and $\xi_0 = -\sqrt{c^2(\rho_1) - \eta_0^2}$. We point out that θ_1 is unknown a priori. Also, we note that $r'(\theta_s) = 0$ and thus the obliqueness becomes degenerate ($\mu = 0$) at $\theta = \theta_s$. We denote a set $V = \{\Xi_R, \Xi_1, \Xi_2, \Xi_s\}$ to be the collection of the corner points.

Due to the corner points, the solution may not be smooth near these points and consequently we consider the weighted Hölder space with the weighted norms.

Recall the following standard norms [6]

$$\begin{aligned}
|w|_{0:\Omega} &= \sup_{\Omega} |w|, \\
|w|_{a:\Omega} &= \sum_{|\beta|<k} |D^\beta w|_{0:\Omega} + \sum_{|\beta|=k} \sup_{x \neq y \in \Omega} \frac{|D^\beta w(x) - D^\beta w(y)|}{|x-y|^\alpha},
\end{aligned}$$

where $a = k + \alpha$ with a nonnegative integer k and $0 < \alpha < 1$. We then define weighted norms

$$|w|_{a:\Omega}^{(b)} = \sup_{\delta>0} \delta^{a+b} |w|_{a:\Omega_\delta} \quad \text{where} \quad \Omega_\delta = \{x \in \Omega : dist(x, V) > \delta\}.$$

We now state the main theorem.

Theorem 2.1. *Assume that Riemann data given in (2) satisfy (7) and (8). Then the free boundary problem consisting of (20)–(25) has a classical solution* $\rho \in C^{2,\alpha}(\Omega \cup \Sigma_S) \cap C^{1,\alpha}(\Omega \cup \Sigma_1 \cup \Sigma_2) \cap C^{\alpha_\Sigma}(\Omega \cup \{\Xi_R, \Xi_2, \Xi_s\}) \cap C^{0,1}(\Omega \cup \sigma_R \setminus \{\Xi_R\}) \cap C^0(\overline{\Omega})$ *satisfying* $\rho_2 < \rho < \rho_R$, *where* ρ_R *with* $\rho_R > \rho_1$ *is the solution of (9). Moreover, the free boundary* $r(\theta)$ *satisfies* **E1**, **E2**, *and (19), strictly decreases in* θ*-direction for* $\theta \in [\theta_R, \theta_1) \cup (\theta_2, \theta_s]$, *and is in* $C^{1,\alpha_\Sigma}([\theta_R, \theta_s] \setminus \{\theta_2\}) \cap C^{0,1}([\theta_R, \theta_s])$, *where* $0 < \alpha, \alpha_\Sigma < 1$ *are determined by the Riemann data of the problem.*

The proof comprises the following procedures of iteration methods. We first impose several cut-offs (f will be used for the ellipticity and g, h will be used for the obliqueness and well-posedness of the shock evolution equation respectively), and elliptic and oblique regularizations (ε, δ will be used as the corresponding parameters). We then construct a sequence of solutions by using the fixed point methods and regularity results. We next establish uniform barriers near the point at which the wave changes from a shock into an expansion, and remove the cut-off functions g and h. The oblique regularization parameter, δ, is then removed. We next show that the solutions are strictly elliptic inside of the subsonic region, that is the cut-off function f is removed. We then find the limiting solution ($\varepsilon \to 0$) by using the local compactness argument. We finally construct upper barrier functions to show the continuity of the limiting solution up to the degenerate boundary.

3. Regularized problems

Since the governing equation becomes sonic on σ and σ_R, we first consider regularized problems. That is, for given $0 < \varepsilon < 1$ we write

$$Q^\varepsilon \rho = Q\rho + \varepsilon \Delta \rho = 0, \quad \text{in} \quad \Omega,$$

and for $\rho \in C^2$ we can write the governing equation in the non-divergent form:

$$\begin{aligned} Q^\varepsilon \rho &= a_{ii}^\varepsilon D_{ii}\rho + a(\rho_r^2 + \frac{\rho_\theta^2}{r^2}) + b\rho_r \\ &= (c^2 - r^2 + \varepsilon)\rho_{rr} + (\frac{c^2}{r^2} + \varepsilon)\rho_{\theta\theta} + (c^2)'(\rho_r^2 + \frac{\rho_\theta^2}{r^2}) + \frac{1}{r}(c^2 - 2r^2)\rho_r = 0. \end{aligned}$$

We use the standard fixed point iteration method, see for example [3,12], to establish the solutions of the regularized problems.

We define a set $\mathcal{R} \subset C^{1,\alpha_0}([\theta_R, \theta_2) \cup (\theta_2, \theta_s]) \cap C^{\alpha_0}([\theta_R, \theta_s])$ with $0 < \alpha_0 < 1$, where $r(\theta) \in \mathcal{R}$ satisfies

R1. $r(\theta_R) = c_R$, and $r'(\theta_R) = -s(c_R, \rho_R, \rho_1) = -c_R\sqrt{\frac{c_R^2 - \bar{c}_1(\rho_R)}{\bar{c}_1(\rho_R)}}$,
R2. $c_1 \le r(\theta) \le c_R$ for $\theta_R \le \theta \le \theta_2$,
R3. $-s(c_R, \rho_R, \rho_1) \le r'(\theta) \le 0$ for $\theta_R \le \theta \le \theta_2$,
R4. $r(\theta_2) = c_1$, and $r'(\theta_2) = -s(c_1, \rho_1, \rho_2) = -c_1\sqrt{\frac{c_1^2 - \bar{c}_2(\rho_1)}{\bar{c}_2(\rho_1)}}$,
R5. $r'(\theta_s) = 0$,
R6. $c_2 \le r(\theta) \le c_1$ for $\theta_2 \le \theta \le \theta_s$,
R7. $-s(c_1, \rho_1, \rho_2) \le r'(\theta) \le 0$ for $\theta_2 \le \theta \le \theta_s$.

Since the problem is nonlinear, the ellipticity, obliqueness and well-posedness of the shock evolution equation are not known a priori. To get around these difficulties, we introduce cut-off functions in the equations $Q\rho = 0$, $M^j\rho = 0$ and $r'(\theta) = -s(r, \rho, \rho_j)$, $j = 1, 2$. We define cut-off functions for ρ in Ω, and $\bar{c}$ and $r^2 - \bar{c}$ on Σ_j respectively:

$$\begin{aligned} f(t) &= \max\{t, (c^2)^{-1}(r^2)\} \quad \text{in} \quad \Omega, \\ g_j(t) &= \min\{t, r^2 - \min\{\tau_0, \tau_* d^q\}\} \quad \text{on} \quad \Sigma_j, \\ h_j(t) &= \max\{t, \min\{\tau_0, \tau_* d^q\}\} \quad \text{on} \quad \Sigma_j, \end{aligned}$$

where $d = d(X) = |X - \Xi_k|$, $k = m, s$ and $\Xi_m = (c_1, \theta_m)$, and θ_m is defined in (26). The positive constants τ_0, τ_*, q will be determined later. As discussed before, it is not known a priori whether the governing equation becomes elliptic in the subsonic region, and the directional derivative boundary conditions becomes oblique on the boundary. The standard of procedure is that first impose appropriate cut-off functions as necessary and next construct upper/lower barriers via Maximum principle arguments to remove the cut-off functions. For this configuration, the problem is nonlinear, which includes several degeneracies (the governing equation becomes sonic on the part of the boundary, and the directional derivative becomes degenerate at the exact same point where the governing equation becomes sonic), and has mixed boundary conditions. Although the cut-off functions applied in the problem may appear to be cumbersome, however, they are crucial to establish ellipticity and obliqueness, and most importantly to develop the estimates near degenerate boundaries. The roles of cut-off functions are following: The cut-off functions f is to ensure the ellipticity within the subsonic region. The cut-off functions g_j, $j = 1, 2$ ensure the uniform obliqueness on the transonic shock boundaries. The cut-off functions h_j, $j = 1, 2$ ensures the well-posedness on the shock evolution equation, and consequently the shock position is at least Lipschitz.

We then consider the following modified equations:

$$
\begin{aligned}
Q^+\rho &= (c^2(f(\rho)) - r^2)\rho_{rr} + \frac{c^2(f(\rho))}{r^2}\rho_{\theta\theta} + (c^2)'(f(\rho))\left(\rho_r^2 + \frac{1}{r^2}\rho_\theta^2\right)\\
&\quad + \frac{c^2(f(\rho))}{r}\rho_r - 2r\rho_r\\
&= a_{ii}^+ D_{ii}\rho + a^+(\rho_r^2 + \frac{1}{r^2}\rho_\theta^2) + b^+\rho_r = 0,\\
M^{j,+}\rho &= \beta_i^{j,+} D_i\rho = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_1^{j,+} &= r'\left(-c^2(f(\rho))(r^2 - g_j(\bar{c})) + 3g_j(\bar{c})(c^2(f(\rho)) - r^2)\right),\\
\beta_2^{j,+} &= -3c^2(f(\rho))(r^2 - g_j(\bar{c})) + g_j(\bar{c})(c^2(f(\rho)) - r^2).
\end{aligned}
$$

Hence the obliqueness becomes

$$
\mu^+ = \beta^+ \cdot (-1, r') = -2r'\left(g_j(\bar{c})(c^2(f(\rho)) - r^2) + c^2(f(\rho))(r^2 - g_j(\bar{c}))\right) \geq 0.
$$

We next consider, for $j = 1, 2$,

$$
\frac{dr}{d\theta} = -s^+(r, \rho, \rho_j) = -r\sqrt{\frac{h_j(r^2 - \bar{c}_j)}{\bar{c}_j}} \quad \text{on} \quad \Sigma_j.
$$

3.1. Regularized solutions

For a given $r \in \mathcal{R}$, we let

$$
\theta_m = \min_\theta\{r(\theta) = c_1\}. \tag{26}
$$

Note that there exists a positive constant δ_R such that $\theta_R + \delta_R \leq \theta_m \leq \theta_2$ since $r'(\theta_R) < 0$. We first consider the following *fixed* boundary problem:

$$
\begin{aligned}
Q^{+,\varepsilon}\rho = a_{ii}^{+,\varepsilon}(\Xi,\rho)D_{ii}\rho + \tilde{b}(\Xi,\rho,D\rho) = 0 \quad &\text{in} \quad \Omega,\\
\rho = \rho_R \quad \text{on} \quad \sigma_R = \{(r=c_R,\theta): 0\le\theta\le\theta_R\} \subset C_R,&\\
M^{1,+,\delta}\rho = \beta_i^{1,+}(\Xi,\rho)D_i\rho + \delta\frac{\partial\rho}{\partial\nu} = 0 \quad \text{on} \quad &\Sigma_1 = \{(r(\theta),\theta): \theta_R<\theta<\theta_m\},\\
\rho = \rho_1 \quad \text{on} \quad \sigma = \{(r,\theta): \theta_m\le\theta\le\theta_2\}\subset C_1,&\\
M^{2,+,\delta}\rho = \beta_i^{2,+}(\Xi,\rho)D_i\rho + \delta\frac{\partial\rho}{\partial\nu} = 0 \quad \text{on} \quad &\Sigma_2 = \{(r(\theta),\theta): \theta_2<\theta<\theta_s-\delta\},\\
\rho = (\bar{c})^{-1}(r^2(\theta)-\tau_* d^q) \ge \rho_2 \quad \text{on} \quad &\Sigma_\delta = \{(r(\theta),\theta): \theta_s-\delta\le\theta\le\theta_s\},\\
\partial\rho/\partial n = 0 \quad \text{on} \quad &\Sigma_s = 0,
\end{aligned} \tag{27}
$$

where ν is the inward normal on Σ_j, and δ is a sufficiently small positive parameter. For the regularization, we consider Σ_δ, a small neighborhood of Ξ_s on Σ_2, and impose a Dirichlet boundary condition on Σ_δ. Thus as $\delta \to 0$ the limiting solution satisfies the point valued Dirichlet boundary condition at Ξ_s. We note that the limiting solution is obtained after we establish the uniform obliqueness, and the obtained estimates are independent of δ and ε.

The existence result of the fixed boundary value problem (27) is given in the following theorem.

Theorem 3.1. *For a given $r \in \mathcal{R}$, there exists a solution $\rho^{\varepsilon,\delta} \in C^{\gamma_{\varepsilon,\delta}}(\overline{\Omega}) \cap C^{1,\alpha_{\varepsilon,\delta}}(\overline{\Omega}\setminus V)$, where V is the set of all the corner points, which satisfies (27) and*

$$
\rho_2 < \rho^{\varepsilon,\delta} < \rho_R \text{ in } \overline{\Omega}\setminus(\sigma_R\cup\Sigma_\delta), \quad |\rho^{\varepsilon,\delta}|^{(-\gamma_{\varepsilon,\delta})}_{1+\alpha_{\varepsilon,\delta}} < K \text{ in } \overline{\Omega}, \tag{28}
$$

for a positive constant $K = K(\varepsilon,\delta)$ and $0<\alpha_{\varepsilon,\delta},\gamma_{\varepsilon,\delta}<1$.

Proof. Since the problem is homogeneous, and ρ_2 and ρ_R are constants, the standard maximum principle applies to have the uniform bound, $\rho_2 < \rho^{\varepsilon,\delta} < \rho_R$ in $\overline{\Omega}\setminus(\sigma_R\cup\Sigma_\delta)$ in (28). In addition, the existence of the solution for the fixed boundary value problem (27) can be followed by using the similar argument as in [12, Lemma 2.1].

For the second inequality in (28), with the uniform bound $\rho_2 < \rho^{\varepsilon,\delta} < \rho_R$, apply the interior Hölder estimates in [6, Lemma 15.4], the local estimates near the Dirichlet boundaries [6, Corollary 9.29] where the quadratic gradient terms can be handled as in [6, Lemma 15.4], the local estimates near the oblique boundaries in the remark after [17, Theorem 2.3], the corner estimates obtained in [16, Lemmas 4.1 and 4.2], and piece these estimates as in [6, Theorem 8.29] to obtain the Hölder estimates in $\overline{\Omega}$ with $0<\gamma_{\varepsilon,\delta}<1$. Next, we treat the governing equation $Q^{+,\varepsilon}\rho = 0$ to be linear so that we can apply the Schauder estimates in [16, Lemma 3.1] to obtain $|\rho^{\varepsilon,\delta}|^{(-\gamma_{\varepsilon,\delta})}_{1+\alpha_{\varepsilon,\delta}} < K$ in $\overline{\Omega}$, where the quadratic gradient terms can be handled as in [12, Lemma 2.1]. This completes the proof. □

Now with the solution ρ satisfying (27) and (28), we define

$$
\tilde{\theta}_m = \min_{\theta}\{\bar{c}_1(\rho(c_1,\theta)) = c_1^2\}, \tag{29}
$$

and define a map J on $\mathcal{R}$ such that $Jr = \tilde{r}$ and $\tilde{r}$ satisfies

$$\tilde{r}'(\theta) = -s^+(r, \rho, \rho_1) = -r\sqrt{\frac{h(r^2 - \bar{c}_1(\rho(r,\theta)))}{\bar{c}_1(\rho(r,\theta))}}, \ \theta \in (\theta_R, \tilde{\theta}_m), \quad \tilde{r}(\theta_R) = c_R,$$

$$\tilde{r}(\theta) = c_1, \quad \theta \in [\tilde{\theta}_m, \theta_2],$$

$$\tilde{r}'(\theta) = -s^+(r, \rho, \rho_2) = -r\sqrt{\frac{h(r^2 - \bar{c}_2(\rho(r,\theta)))}{\bar{c}_2(\rho(r,\theta))}}, \ \theta \in (\theta_2, \theta_s), \quad \tilde{r}(\theta_2) = c_1. \tag{30}$$

By showing J is compact in $\mathcal{R}$ and by the Schauder fixed point theorem, we next establish the existence of regularized *free* boundary problems:

Theorem 3.2. *There exist solutions $\rho^{\varepsilon,\delta} \in C^{1,\alpha}(\overline{\Omega} \setminus V) \cap C^{\alpha_0}(\overline{\Omega})$ and $r^{\varepsilon,\delta} \in C^{1,\alpha_0}([\theta_R, \theta_s] \setminus \{\theta_2\}) \cap C^{\alpha_0}([\theta_R, \theta_s])$ satisfying (27) and $r' = -s^+(r, \rho, \rho_j)$ on Σ_j, $j = 1, 2$, where $0 < \alpha, \alpha_0 < 1$.*

Proof. By ρ being a solution to the fixed boundary problem (27) and satisfying (28), J maps $\mathcal{R}$ into itself. In particular, at $\tilde{\Xi}_m = (c_1, \tilde{\theta}_m)$, the solution ρ satisfies $\bar{c}(\rho(\tilde{\Xi}_m)) = r^2(\tilde{\theta}_m) = c_1^2$, and thus $\tilde{r}'(\tilde{\theta}_m) = 0$. By the definition of the map J, clearly $\tilde{r}' = 0$ when $\theta \in [\tilde{\theta}_m, \theta_2]$. In addition since we showed that $|\rho|^{(-\gamma_{\varepsilon,\delta})}_{1+\alpha_{\varepsilon,\delta}} < K$, and since $\tilde{r}$ satisfies (30), we now have $|\tilde{r}|_{1+\gamma_{\varepsilon,\delta}:[\theta_R,\theta_s]\setminus\{\theta_2\}} \leq K_1$, and thus $|\tilde{r}|^{(-\gamma_{\varepsilon,\delta})}_{1+\gamma_{\varepsilon,\delta}} \leq K_1$ on Σ (where the weight is given at θ_2). This shows that the map J is compact in $\mathcal{R}$, when α_0 is chosen sufficiently small, and is continuous.

Hence we apply the Schauder fixed point theorem to obtain a fixed point $\tilde{r} = r$ in $\mathcal{R} \subset C^{1,\alpha_0}([\theta_R, \theta_s] \setminus \{\theta_2\} \cap C^{\alpha_0}([\theta_R, \theta_s])$ where $r = r^{\varepsilon,\delta}$. Using the fact that the fixed boundary problem (27) has a solution for the corresponding $\sigma = \sigma^{\varepsilon,\delta}$, we establish the existence of a solution $(\rho^{\varepsilon,\delta}, r^{\varepsilon,\delta})$ where $\rho^{\varepsilon,\delta} \in C^{1,\alpha}(\overline{\Omega} \setminus V) \cap C^{\alpha_0}(\overline{\Omega})$ and $r^{\varepsilon,\delta} \in C^{1,\alpha_0}([\theta_R, \theta_s] \setminus \{\theta_2\}) \cap C^{\alpha_0}([\theta_R, \theta_s])$ of the free boundary problem $Q^{+,\varepsilon}\rho = 0$ in Ω, $M^{j,+,\delta}\rho = 0$ and $r' = -s^+(r, \rho, \rho_j)$ on Σ_j, $j = 1, 2$, $\rho \mid_{\sigma_R} = \rho_R$, $\rho \mid_{\sigma} = \rho_1$, and $\partial\rho/\partial n \mid_{\Sigma_s} = 0$, for sufficiently small $\alpha = \alpha(\varepsilon, \delta)$ and $\alpha_0 = \alpha_0(\varepsilon, \delta)$. The regularity argument, such as Theorem 6.2 in [6], ensures that the solution $\rho^{\varepsilon,\delta} \in C^{1,\alpha}(\overline{\Omega} \setminus V) \cap C^{\alpha_0}(\overline{\Omega})$ is in fact in $C^{2,\alpha}(\Omega^{\varepsilon,\delta})$. This completes the proof. □

4. The limiting solution

The main difficulty of constructing the limiting solution is lack of uniform estimates near the point at which the shock becomes sonic; both the governing equation and directional derivative boundary condition become degenerate. Furthermore, the position of the degenerate point is not known a priori. As discussed before, the degeneracy that we have in this paper appears to be much more involved than the one in the earlier studies. More precisely the governing equation degenerates and at the same time both components of the directional derivatives β_1, β_2 become zero simultaneously whereas the position of the change of the wave is unknown. Hence we must establish new estimates near the degenerate point.

We first show the uniform obliqueness and well-posedness of the shock evolution equations in Lemma 4.1. That is, we can remove the cut-off functions g_j and h_j, $j = 1, 2$. With the uniform obliqueness, we next push $\delta \to 0$ so that the sequence of solutions is now ρ^{ε}. With the uniform obliqueness, we then establish the uniform ellipticity near the degenerate point in Lemma 4.6. This will allow us to remove the cut-off function f and push $\varepsilon \to 0$ to obtain the limiting solution.

4.1. The uniform obliqueness and well-posedness of the shock evolution equations

In this section, we show $r^2 - \bar{c}_j(\rho^{\varepsilon,\delta}) > 0$ on Σ_j, $j = 1, 2$, which implies that the shock evolution equations are well-posed and the directional derivative is oblique. Consequently we can remove the related cut-off functions g_j and h_j, $j = 1, 2$.

We first establish the following key lemma, Lemma 4.1, to construct the lower bound for $r^2 - \bar{c}$ near the sonic point Ξ_m. The earlier work in [12,13,15] depends on the conditions that the density attains its minimum at the degenerate point and the boundary has a perpendicular angle at the degenerate point. Our result does not rely on those restrictions, is allowed to have degeneracies arising in the governing equation and on the both components of the directional derivative vector, and is independent of the degenerate point.

Lemma 4.1. *There exist positive constants τ_0, R_0, τ_*, q independent of ε, δ, depending only on c_1, c_R and γ, and satisfying $\tau_* R_0^q = \tau_0$, such that*

$$r^2 - \bar{c} \geq \tau_* d^q = \tau_* |X - \Xi_m|^q, \tag{31}$$

where $X \in \Sigma_1(R_0) = \{X \in \Sigma_1 : |X - \Xi_m| < R_0\}$.

Proof. The solutions $\rho^{\varepsilon,\delta}$ are depending on the regularization parameters. However, for the notational simplicity, throughout the proof, we write $\rho \equiv \rho^{\varepsilon,\delta}$. Also we write $\beta^1 = \beta$, and $\Sigma_1 = \Sigma$.

We write $\Omega_1 = \{\rho > \rho_1\} \subset \Omega$. We consider local polar coordinates $d = |X - \Xi_m|$ where $X \in \overline{\Omega}_1$ and $\varphi = \tan^{-1}((\eta - \eta_m)/(\xi - \xi_m))$ where $\varphi_0 \leq \varphi \leq \varphi_1$ so that $\varphi = \varphi_1$ on $\Sigma \cap B_{R'}(\Xi_m)$, and $\varphi = \varphi_0$ on $\partial\Omega_1 \cap B_{R'}(\Xi_m)$. We consider $D = \{|r^2 - \bar{c}| \leq \tilde{\tau}\} \subset B_{R'}(\Xi_m) \cap \overline{\Omega}_1$ for some constants $R' > 0$ and $\tilde{\tau} > 0$.

We define

$$w = A[(\varphi - \varphi_0)^s - (\varphi_1 - \varphi_0)^s] + B(\varphi_1 - \varphi) - C(\varphi_1 - \varphi)^t,$$

where A is a positive constant to be determined, with $B = sA(\varphi_1 - \varphi_0)^{s-1}$, $C = 2(s-1)A(\varphi_1 - \varphi_0)^{s-t}$, $s < \alpha$, and with appropriate $1 < s < 2 < t$ so that $w'' > 0$ when $\varphi_0 \leq \varphi \leq \varphi_1$.

We next define

$$u = \bar{c} - r^2 + \chi + w, \quad \chi = k(\varphi)d^\alpha, \quad \text{and} \quad v = u\zeta(R_0^h - d^h), \tag{32}$$

where $h \geq 2\alpha > 0$ are constants to be determined, and k with $k(\varphi_1) = \tau_* > 0$ and a cut-off function ζ with $\zeta(0) = 0$, $\zeta > 0$ otherwise, are functions to be determined. We denote $\mathcal{O} = \{X \in D : v > 0\}$. We now fix A in w so that $v \leq 0$ when $\varphi = \varphi_0$, that is the set $\mathcal{O}$ does not intersect on $\varphi = \varphi_0$.

Assume that $\mathcal{O} \cap \Sigma \neq \emptyset$ (otherwise we are done). Then there exists $X_c \in \mathcal{O} \cap \Sigma$ so that X_c is either a local maximum point or a saddle point of v. We show by contradictions that both cases cannot occur and thus deduce $\mathcal{O} \cap \Sigma = \emptyset$.

Suppose X_c is a local maximum point. Then by using $M^{+,\delta}\rho = 0$ and $\bar{c}$ is a function of ρ, we evaluate and denote

$$\begin{aligned} M^{+,\delta}u &= \beta_i^+ D_i u + \delta \partial u/\partial \nu \\ &= -2r\beta_1^+ + \beta_i^+ D_i \chi + \delta(2r + \partial\chi/\partial\nu) \equiv G. \end{aligned}$$

Note that since $w'(\varphi_1) = 0$, the term $M^{+,\delta}w$ is negligible. For $d \le R_0$ small and $\alpha > 1$, we have

$$2r + \partial\chi/\partial\nu = 2r + d^{\alpha-1}(-k'(\varphi_1) - r' r\alpha k(\varphi_1)) \ge 0.$$

Thus, on $\Sigma \cap \mathcal{O}$, we obtain

$$\begin{aligned} G &\ge -2r\beta_1^+ + d^{\alpha-1}(k'(\varphi_1)\beta_1^+ - r\alpha k(\varphi_1)\beta_2^+) \\ &= \bar{c}(c^2 - r^2)[-6rr' + d^{\alpha-1}(3k'(\varphi_1)r' - r\alpha k(\varphi_1))] \\ &\quad + c^2(r^2 - \bar{c})[2rr' + d^{\alpha-1}(-k'(\varphi_1)r' + 3r\alpha k(\varphi_1))] \\ &\ge g_0 d^\alpha \tau_*^{3/2} \\ &> 0, \end{aligned}$$

by choosing $k(\varphi_1) = \tau_*$ satisfying

$$6\frac{r}{\sqrt{\bar{c}}} > \sqrt{\tau_*} > \frac{2}{3}\frac{r}{\sqrt{\bar{c}}}, \tag{33}$$

for sufficiently small $d \le R_0$ with $\alpha = q = 2$. Here $g_0 > 0$ is a constant independent of ρ and r.

Hence at X_c noting that $X_c \ne X_m$ so $\zeta(X_c) = \zeta_c > 0$, for $d \le R_0$ sufficiently small if necessary and $h > \alpha + 1$, we obtain

$$\begin{aligned} 0 \ge M^{+,\delta}v(X_c) &= \zeta M^{+,\delta}u + uM^{+,\delta}\zeta \\ &> \zeta_c g_0 \tau_*^{3/2} d^\alpha + u\zeta' h d^{h-1}\beta_2^+ \\ &> 0, \end{aligned}$$

which leads to a contradiction.

Next, if X_c is a saddle point then there exists an interior maximum point X_i so that $v(X_i) \ge v(X_c)$ where $X_i \in \Omega$ and $X_i \ne X_c$.

By multiplying $\bar{c}'$ over $Q^{+,\varepsilon}\rho = 0$ we have

$$\begin{aligned} 0 &= \bar{c}' Q^{+,\varepsilon}\rho \\ &= a_{ii}^\varepsilon (D_{ii}\bar{c} - \frac{\bar{c}''}{(\bar{c}')^2}|D_i\bar{c}|^2) + \frac{a}{\bar{c}'}(\bar{c}_r^2 + \frac{1}{r^2}\bar{c}_\theta^2) + b\bar{c}_r, \end{aligned}$$

where $a_{ii}^\varepsilon = a_{ii}^+ + \varepsilon$, and thus we can write

$$0 = a_{ii}^\varepsilon D_{ii}\bar{c} + a_1\bar{c}_r^2 + a_2\bar{c}_\theta^2 + b\bar{c}_r,$$

where $a_1 = -a_{11}^\varepsilon \frac{\bar{c}''}{(\bar{c}')^2} + \frac{a}{\bar{c}'}$ and $a_2 = -a_{22}^\varepsilon \frac{\bar{c}''}{(\bar{c}')^2} + \frac{a}{r^2\bar{c}'}$. We then obtain and denote

$$\begin{aligned} 0 &= a_{ii}^\varepsilon D_{ii}\bar{c} + a_i|D\bar{c}|^2 + b\bar{c}_r \\ &= a_{ii}^\varepsilon D_{ii}(u + r^2 - \chi - w) + a_i|D_i(u + r^2 - \chi - w)|^2 + b(u + r^2 - \chi - w)_r \\ &\equiv Lu + F_1 + F_2, \end{aligned}$$

where

$$
\begin{aligned}
Lu &= a^{\varepsilon}_{ii} D_{ii} u + a_i |D_i u|^2 + b_i D_i u \\
&= a^{\varepsilon}_{ii} D_{ii} u + a_i |D_i u|^2 + (2a_1(2r - \chi_r - w_r) + b) u_r - 2a_2(\chi_\theta + w_\theta) u_\theta, \\
F_1 &= -a^{\varepsilon}_{ii} D_{ii} \chi + a_i |D_i \chi|^2 - (4ra_1 + b)\chi_r + 2a^{\varepsilon}_{11} + 4r^2 a_1 + 2rb, \\
F_2 &= -a^{\varepsilon}_{ii} D_{ii} w + a_i |D_i w|^2 - (4ra_1 + b) w_r + 2a_i D_i \chi D_i w.
\end{aligned}
$$

We will seek χ in $\mathcal{O}$ satisfying

$$
-F_1 = a^{\varepsilon}_{ii} D_{ii} \chi - a_i |D_i \chi|^2 + (4ra_1 + b)\chi_r - 2a^{\varepsilon}_{11} - 4r^2 a_1 - 2rb > 0.
$$

Notice that

$$
(4ra_1 + b)\chi_r - 4r^2 a_1 - 2rb = (4ra_1 + b)(\chi_r - r) - rb,
$$

and

$$
4ra_1 + b = -4r \frac{\bar{c}''}{(\bar{c}')^2}(c^2 - r^2 + \varepsilon) + 4r\frac{a}{\bar{c}'} + \frac{1}{r}(c^2 - r^2) - r.
$$

Hence we can write

$$
\begin{aligned}
&(4ra_1 + b)\chi_r - 4r^2 a_1 - 2rb \\
&= \left(-4r \frac{\bar{c}''}{(\bar{c}')^2}(c^2 - r^2 + \varepsilon) + 4r\frac{a}{\bar{c}'} + \frac{1}{r}(c^2 - r^2) - r\right)(\chi_r - r) - (c^2 - r^2) + r^2 \\
&= -4r \frac{\bar{c}''}{(\bar{c}')^2}(c^2 - r^2 + \varepsilon)(\chi_r - r) + (c^2 - r^2)\left(\frac{1}{r}(\chi_r - r) - 1\right) \\
&\quad + r^2 + \left(4r\frac{a}{\bar{c}'} - r\right)(\chi_r - r).
\end{aligned}
$$

When $\rho > \rho_1$ and $d(X) = |X - X_m| \le R$ small where $\rho(X_m) = \rho_1$, we have $a/\bar{c}' \le 1 + \varepsilon_0$ where $0 \le \varepsilon_0 \le 1/4$ so that

$$
\left(4r\frac{a}{\bar{c}'} - r\right)(\chi_r - r) = ((4\varepsilon_0 - 1)r + 4r)(\chi_r - r) \ge 4r(\chi_r - r),
$$

where $\chi_r = k' d^{\alpha-1} < r$ since d is small and $\alpha > 1$. Thus we have

$$
\begin{aligned}
-F_1 \ge\ & (c^2 - r^2 + \varepsilon)(d^{\alpha-2}k'' - 2) + \frac{c^2}{r^2}\alpha(\alpha - 1)d^{\alpha-2}kr^2 - a_0 d^{2\alpha-2}(k^2 + (k')^2) \\
& - 4r \frac{\bar{c}''}{(\bar{c}')^2}(c^2 - r^2 + \varepsilon)(k'd^{\alpha-1} - r) + (c^2 - r^2)\left(\frac{1}{r}(k'd^{\alpha-1} - r) - 1\right) \\
& + r^2 + 4r(k'd^{\alpha-1} - r)
\end{aligned}
$$

$$
\begin{aligned}
&= (c^2 - r^2 + \varepsilon)\left(d^{\alpha-2}k'' - 2 - 4r\frac{\bar{c}''}{(\bar{c}')^2}(k'd^{\alpha-1} - r)\right) \\
&\quad + \frac{c^2}{r^2}\alpha(\alpha-1)d^{\alpha-2}kr^2 - a_0 d^{2\alpha-2}(k^2 + (k')^2) \\
&\quad + (c^2 - r^2)\frac{1}{r}(k'd^{\alpha-1} - 2r) + 4rk'd^{\alpha-1} - 3r^2.
\end{aligned}
$$

Notice that $\alpha = 2$ and thus by choosing $k \geq 3/2$, which also satisfying (33), we see that

$$
\frac{c^2}{r^2}\alpha(\alpha-1)d^{\alpha-2}kr^2 - 3r^2 = 2kc^2 - 3r^2 \geq 3(c^2 - r^2).
$$

In addition, we can decrease $d \leq R$ further if necessary to have

$$
4rk'd^{\alpha-1} - a_0 d^{2\alpha-2}(k^2 + (k')^2) \geq rk'd > 0.
$$

Hence by choosing $k'' > 1$ large if necessary, we have

$$
\begin{aligned}
-F_1 &\geq (c^2 - r^2 + \varepsilon)\left(d^{\alpha-2}k'' - 2 - 4r\frac{\bar{c}''}{(\bar{c}')^2}(k'd^{\alpha-1} - r)\right) \\
&\quad + (c^2 - r^2)(\frac{k'}{r}d^{\alpha-1} - 2) + 3(c^2 - r^2) + rk'd \\
&\geq (c^2 - r^2 + \varepsilon)\frac{k''}{2} + (c^2 - r^2)(\frac{k'}{r}d^{\alpha-1} + 1) + rk'd \\
&\geq (c^2 - r^2)\frac{k''}{2} + rk'd \\
&\geq rk'd,
\end{aligned}
$$

for the interior point in $\mathcal{O}$.

For F_2, since we have chosen $w'' > 0$ (for given φ_1, we can choose φ_0 so that $\varphi_1 - \varphi_0$ sufficiently small if necessary so that $w'' > 0$ is the dominant positive term) whereas the lower order terms are of order $O((\varphi - \varphi_0)^{s-1})$, we have $-F_2 \geq 0$ for $d \leq R_0$ sufficiently small.

Now evaluate

$$
\begin{aligned}
D_i v &= \zeta D_i u + u D_i \zeta \\
D_{ii} v &= \zeta D_{ii} u + 2D_i u D_i \zeta + u D_{ii}\zeta = \zeta D_{ii} u + \frac{2}{\zeta}(D_i v - u D_i \zeta) D_i \zeta + u D_{ii}\zeta,
\end{aligned}
$$

and write

$$
0 = \zeta(Lu + F_1 + F_2) = L_1 v + F,
$$

where

$$L_1 v = a^\varepsilon_{ii} D_{ii} v - \frac{2}{\zeta} a^\varepsilon_{ii} D_i\zeta D_i v + \frac{1}{\zeta} a_i |D_i v|^2 - \frac{2}{\zeta} u a_i D_i v D_i \zeta + b_i D_i v,$$

$$F = \zeta(F_1 + F_2) - a^\varepsilon_{ii} u D_{ii}\zeta + 2\frac{a^\varepsilon_{ii}}{\zeta} u |D_i\zeta|^2 + \frac{a_i}{\zeta} u^2 |D_i\zeta|^2 - b_i u D_i\zeta.$$

Evaluate

$$\begin{aligned}
&a^\varepsilon_{ii} D_{ii}\zeta - 2\frac{a^\varepsilon_{ii}}{\zeta}|D_i\zeta|^2 - \frac{a_i}{\zeta} u |D_i\zeta|^2 + b_i D_i\zeta \\
&\quad = a^\varepsilon_{ii}(\zeta'' h^2 d^{2h-2}|D_i d|^2 - h(h-1)\zeta' d^{h-2}|D_i d|^2 - \zeta' h d^{h-1} D_{ii} d) \\
&\qquad - (2\frac{a^\varepsilon_{ii}}{\zeta} + \frac{a_i}{\zeta} u)(\zeta')^2 h^2 d^{2h-2}|D_i d|^2 - \zeta' h d^{h-1} b_i D_i d \\
&\quad = O(d^{h-2}).
\end{aligned}$$

Thus by choosing $h \geq 4$ and $d \leq R_0$ sufficiently small if necessary, (notice that $\zeta \geq \zeta_0 > 0$ where $v(X_i) \geq v(X_c)$ and $X_i \notin \{d = R'\}$,) and at the interior maximum point, we have

$$0 \geq L_1 v = -F > 0,$$

which is a contradiction. This completes the proof. □

Remark 4.2. We note that the proof of Lemma 4.1 does not depend on the strict ellipticity and the position Ξ_m. Lemma 4.1 can be generalized to the case when the degenerate corner point (where the governing equation and both β_1 and β_2 become zero simultaneously) satisfies an exterior cone condition, and the estimate depends only on the angle of the exterior cone, and nonlinear structures of the governing equation and the boundary conditions.

With a simple modification of the proof of Lemma 4.1, we establish the following lemma.

Lemma 4.3. *There exist positive constants R_0, τ_*, q, independent of ε and δ, depending only on c_1, c_2 and γ, such that*

$$r^2 - \bar{c} \geq \tau_* d^q = \tau_* |X - \Xi_s|^q, \quad \textit{for} \quad \Sigma_2(R_0) = \{X \in \Sigma_2 : |X - \Xi_s| < R_0\}. \tag{34}$$

Proof. Replace $\beta^2 = \beta$ and $\Sigma_2 = \Sigma$. Consider local polar coordinates $d = |X - \Xi_s|$ where $X \in \Omega' = \{\rho < \rho_1\} \subset \Omega$, and $\varphi = \tan^{-1}((\eta - \eta_s)/(\xi - \xi_s))$ where $\varphi_0 \leq \varphi \leq \varphi_1$ so that $\varphi = \varphi_1$ on $\Sigma \cap B'_R(\Xi_s)$, and $\varphi = \varphi_0$ on $\Sigma_S \cap B_{R'}(\Xi_s)$. Then the result follows immediately by repeating the same argument as we did in the proof of Lemma 4.1. □

Note that the same result can be found in [15, Lemma 3.7], however as noted before, Lemma 4.3 does not depend on the condition that ρ attains its minimum at Ξ_s, and the angle of the corner point can be other than $\pi/2$.

We next establish the following lemma, which is an improved version of [12, Lemma 3.2].

Lemma 4.4. *There exists a positive constant $\tau_0 = \tau_* R_0^q$ depending only on c_1, γ, and c_R for Σ_1, and c_2 for Σ_2, such that*

$$r^2 - \bar{c} \geq \tau_0 \quad on \quad \Sigma_j \setminus \Sigma_j(R_0), \quad j = 1, 2. \tag{35}$$

Proof. As before we write $\rho = \rho^{\varepsilon,\delta}$ and assume that there exists a non-empty set $\Sigma' = \{\Xi \in \Sigma_j \setminus \Sigma_j(R_0) : \bar{c} - r^2 > -\tau_0\}$ where $\tau_0 = \tau_* R_0^q$, $j = 1, 2$. Then there exists a point $X_c = (r_c, \theta_c) \in \Sigma'$ such that $\max_{\overline{\Sigma'}} \bar{c}(\rho) - r^2 + \tau_0 = (\bar{c}(\rho) - r^2)(X_c) + \tau_0 = m > 0$ for some constant m. This X_c can be either a local maximum point or a saddle point in $\Omega \cup \Sigma'$.

Define $d = r_c - r + r'(\theta_c)(\theta - \theta_x)$ and $B = \{d > 0\} \cap \{\bar{c} - r^2 + \tau_0 > m\} \subset \Omega$. Next for $d_0 > 0$, define

$$u = \bar{c} - r^2 + \chi, \quad \chi = \chi(d), \quad \text{and} \quad v = u\zeta(d_0^h - d^h),$$

where $h \geq 4$, and χ is the function to be determined.

If X_c is a local maximum point then by using $M^{j,+,\delta}\rho = 0$, and choosing $\chi' \geq c_R$, we have

$$\begin{aligned} M^{j,+,\delta}u &= -2r\beta_1^{j,+} + \chi'\beta_i^{j,+} D_i d + \delta(2r + \chi'\partial\chi/\partial\nu) \\ &\geq (-2r')\Big(c^2(r^2 - g_j)(-r + \chi') + g_j(c^2 - r^2)(3r + \chi')\Big) \\ &> (-2r')c^2(r^2 - g_j)(-r + \chi') > 0. \end{aligned}$$

Hence the contradiction follows by repeating the same argument as we did in the proof of Lemma 4.1.

Next, if X_c is a saddle point then there exists an interior maximum point X_i so that $v(X_i) \geq v(X_c)$ where $X_i \in \Omega$ and $X_i = X_c$. Then from

$$\begin{aligned} 0 &= a_{ii}^{\varepsilon} D_{ii}\bar{c} + a_i|D\bar{c}|^2 + b\bar{c}_r \\ &= a_{ii}^{\varepsilon} D_{ii}(u + r^2 - \chi) + a_i|D_i(u + r^2 - \chi)|^2 + b(u + r^2 - \chi)_r \\ &\equiv Lu + F, \end{aligned}$$

where

$$\begin{aligned} Lu &= a_{ii}^{\varepsilon} D_{ii}u + a_i|D_iu|^2 + b_iD_iu \\ &= a_{ii}^{\varepsilon} D_{ii}u + a_i|D_iu|^2 + (2a_1(2r - \chi_r) + b)u_r - 2a_2\chi_\theta u_\theta, \\ F &= -a_{ii}^{\varepsilon} D_{ii}\chi + a_i|D_i\chi|^2 - (4ra_1 + b)\chi_r + 2a_{11}^{\varepsilon} + 4r^2a_1 + 2rb, \end{aligned}$$

evaluate

$$\begin{aligned} -F = \chi''(a_{11}^{\varepsilon} + (r'(\theta_c))^2 a_{22}^{\varepsilon}) - (\chi')^2(a_1 + (r'(\theta_c))^2 a_2) \\ - (4ra_1 + b)\chi' - 2a_{11}^{\varepsilon} - 4r^2a_1 - 2rb. \end{aligned}$$

Notice that $r'(\theta_c) = -r_c\sqrt{h_j(r_c^2 - \bar{c}_j)/\bar{c}_j}$ and thus $(r'(\theta_c))^2 = \tau_0 > 0$. Thus by choosing $\chi'' \geq K(\chi')^2 > 0$ with $\chi(0) = 0$ for $K > 1$ sufficiently large if necessary; for example

$$\chi(d) = \frac{1}{K}\left(\ln\frac{1}{c_R} - \ln\left(\frac{1}{c_R} - Kd\right)\right) \quad \text{where} \quad d \leq d_0 < \frac{1}{c_R K};$$

we have $F > 0$ in B. Therefore by choosing $h \geq 4$ and $d \leq d_0$ sufficiently small, we deduce the contradiction and this completes the proof. □

Thus from Lemmas 4.1, 4.3 and 4.4, the cut-off functions g_j on the directional derivative conditions on $M^{j,+,\delta}\rho = 0$, and h_j on the evolution equations $r' = -s(r, \rho, \rho_j)$, for $j = 1, 2$, are removed.

We now push $\delta \to 0$ (by using diagonalization arguments if necessary) so that the sequence is depending only on ε. The limiting solution for $\delta \to 0$ is constructed in a standard way. Since the elliptic regularity ε is still at present, one can use the Hölder gradient estimates (for instance, see [3,11]) and obtain a convergent subsequence. Then use Lemmas 4.1, 4.3 and 4.4 to establish the limit. Hence we now have ρ^ε, and $M^{j,+,\delta}\rho = M^{j,+}\rho = 0$ and $r' = -s(r, \rho, \rho_j)$ for $j = 1, 2$. Note that the cut-off function f in β^j is still present and so we write $M^{j,+}\rho = 0$ with understanding that g_j and h_j are removed.

4.2. *The uniform ellipticity*

We next establish the uniform ellipticity locally away from the boundary of the domain, which then implies the domain is subsonic, in the following lemma. Thus the cut-off function f throughout the governing equation is then removed, and $Q^{+,\varepsilon}\rho = Q^\varepsilon\rho = 0$ and $M^{j,+}\rho = M^j\rho = 0$, $j = 1, 2$.

Lemma 4.5. *There exist positive constants R and δ_0 depending only on ρ_R, ρ_1, ρ_2, γ (independent of ε), such that*

$$c^2(\rho^\varepsilon) - r^2 \geq \delta_0\left(1 - \frac{|X - X_0|^2}{R^2}\right), \tag{36}$$

where $B_R = \{|X - X_0|^2 < R^2\} \subset \Omega$ and $\partial B_R \cap \partial\Omega = \{X_a\}$ where X_a is an arbitrary point on $\partial\Omega$.

Proof. For the notational simplicity, throughout the proof, we write $\rho = \rho^\varepsilon$.

Let $B_R = \{|X - X_0|^2 < R^2\} \subset \Omega$ and $\partial B_R \cap \partial\Omega = \{X_a\}$, where $X_0 \in \Omega$ and $X_a \in \partial\Omega$ are arbitrary points. Write $\varphi = \delta(1 - \frac{|X-X_0|^2}{R^2})$. Suppose that $c^2(\rho) - r^2 < \varphi$ for all sufficiently small $\delta > 0$, since otherwise we are done.

By the maximum principle we have $\rho > \rho_2$ in $\overline{\Omega}$. Thus for $B_R \subset \{r^2 < c_2^2\} \cap \Omega$, the inequality (36) where δ_0 and R depends only on ρ_2 follows immediately by observing

$$c^2(\rho) - r^2 > c_2^2 - r^2 > 0.$$

We now consider $\Omega_a = \Omega \setminus \{r^2 < c_2^2\}$. By multiplying $dc^2/d\rho = (\gamma - 1)\rho^{\gamma-2}$ throughout the equation $Q^\varepsilon\rho = 0$ and letting $c^2(\rho) = u$, we have

$$Lu = (a_{ii}^+ + \varepsilon)D_{ii}u + a_i|D_i u|^2 + bu_r = 0,$$

where

$$a_1 = -\frac{\gamma-2}{(\gamma-1)\rho^{\gamma-1}}(a_{11}^+ + \varepsilon) + \frac{a}{(\gamma-1)\rho^{\gamma-2}},$$

$$\text{and} \quad a_2 = -\frac{\gamma-2}{(\gamma-1)\rho^{\gamma-1}}(a_{22}^+ + \varepsilon) + \frac{a}{r^2(\gamma-1)\rho^{\gamma-2}}.$$

Note $\frac{a}{(\gamma-1)\rho^{\gamma-2}} \geq 1$ due the cut-off function f. Define $v = u - r^2 - \varphi$. Using $Lu = 0$ we have

$$\begin{aligned}
0 &= Lu \\
&= (a_{ii}^+ + \varepsilon)D_{ii}(v + r^2 + \varphi) + a_i|D_i(v + r^2 + \varphi)|^2 + b(v + r^2 + \varphi)_r \\
&= L_1 v + L_2\varphi,
\end{aligned}$$

where

$$\begin{aligned}
L_1 v &= (a_{ii}^+ + \varepsilon)D_{ii}v + a_i|D_i v|^2 + \{2a_1(2r + \varphi_r) + b\}v_r + a_2\varphi_\theta v_\theta, \\
L_2\varphi &= (a_{ii}^+ + \varepsilon)D_{ii}\varphi + a_i|D_i\varphi|^2 + (4ra_1 + b)\varphi_r + \{2(a_{11} + \varepsilon) + 4r^2 a_1 + 2rb\}.
\end{aligned}$$

We evaluate

$$\begin{aligned}
&2(a_{11}^+ + \varepsilon) + 4r^2 a_1 + 2rb \\
&\quad \geq 2(c^2 - r^2 + \varepsilon)(1 - 2r^2\frac{\gamma-2}{(\gamma-1)\rho^{\gamma-1}}) + 4r^2 + 2(c^2 - 2r^2) \\
&\quad = \frac{2(c^2 - r^2 + \varepsilon)}{(\gamma-1)\rho^{\gamma-1}}[(\gamma-1)(c^2 - r^2) + r^2 - (\gamma-2)r^2] + 2(c^2 - r^2) + 2r^2.
\end{aligned}$$

The last inequality is bounded below by $2c_2^2 > 0$ when $1 < \gamma \leq 3$. For $\gamma > 3$, we choose $\varepsilon_0 = \varepsilon_0(\gamma, c_2)$ small so that for any $\varepsilon, \delta \leq \varepsilon_0$, the last inequality becomes

$$\begin{aligned}
&\frac{2(c^2 - r^2 + \varepsilon)}{(\gamma-1)\rho^{\gamma-1}}[(\gamma-1)(c^2 - r^2) + r^2 - (\gamma-2)r^2] + 2(c^2 - r^2) + 2r^2 \\
&\quad \geq 2r^2\left(\frac{c^2 - r^2 + \varepsilon_0}{c_2^2}(3-\gamma) + 1\right) \\
&\quad \geq 2r^2\left(\frac{\delta + \varepsilon_0}{c_2^2}(3-\gamma) + 1\right) \geq r^2,
\end{aligned}$$

when $c^2 - r^2 < \varphi \leq \delta$ for small $\delta \leq \varepsilon_0$, so that the lower bound becomes c_2^2. The rest terms with φ in the equation $L_2\varphi$ is of order δ and thus by choosing $0 < \delta \leq \delta_0 = \varepsilon_0$ sufficiently small we have

$$L_2\varphi > c_2^2 + O(\delta) > 0.$$

Thus we have $L_1 v = -L_2\varphi < 0$ in $B_R \cap \{c^2 - r^2 < \varphi\}$. By the standard maximum principle, v must attain its minimum on $\partial B_R \cap \{c^2 - r^2 \leq \varphi\}$ or $B_R \cap \{c^2 - r^2 = \varphi\}$, that is, $v = c^2 - r^2 - \varphi \geq 0$ on $B_R \cap \{c^2 - r^2 < \varphi\}$, which is a contradiction. This completes the proof. □

We next construct uniform barriers locally away from $\sigma \cup \sigma_R$.

Lemma 4.6. *There exists a function $\varphi = \delta_0 \zeta^\tau$ where $\delta_0 = \delta_0(\rho_1, \rho_R, \gamma)$ and $\tau \geq 2$ are positive constants and $0 \leq \zeta \leq 1$ independent of ε such that $c^2(\rho^\varepsilon) - r^2 \geq \varphi$ in $\overline{\Omega} \setminus \{\sigma \cup \sigma_R\}$.*

Moreover, there exists a positive constant λ_0 independent of ε such that ρ^ε satisfies

$$c^2(\rho^\varepsilon) - r^2 \geq \lambda_0 |X - \Xi_m|^2, \quad X \in B_R(\Xi_m) \cap \overline{\Omega}. \tag{37}$$

Proof. The first part of the statement immediately follows from the results of [12, Lemma 3.6] and [15, Lemma 3.10]. Hence we show (37) in this proof. For the notational simplicity, throughout the proof, we write $\rho = \rho^\varepsilon$, $\beta^1 = \beta$ and $\Sigma_1 = \Sigma$.

Notice that using $c^2 - r^2 \geq \varphi$ in $\overline{\Omega} \setminus \{\Xi_m\}$, for $R \leq R_0$, we find $0 < \delta_1 \leq \delta_0$ small so that $c^2 - r^2 \geq \delta_1$ on $\partial B_R(\Xi_m) \cap \overline{\Omega}$.

We write $\varphi = \lambda d^2$ where $d = |X - \Xi_m|$ and $\lambda > 0$ is to be determined. On $\{d = R\} \cap \overline{\Omega}$ we use the local ellipticity established in Lemma 4.5 and the ellipticity away from Ξ_m on Σ, to obtain

$$c^2 - r^2 \geq \lambda_0 > 0. \tag{38}$$

Let $v = c^2(\rho) - r^2 - \varphi$ and $\lambda R^2 \leq \lambda_0$. From Lemma 4.5, we find an interior ball $B_R \subset \Omega$ such that $\partial B_R \cap \overline{\Omega} = \Xi_m$ and $c^2(\rho) - r^2 \geq \delta_0(1 - |X - X_0|^2/R^2)$ in B_R. Hence we choose $\lambda \leq \delta_0$ so that $v \geq 0$ on the edge of the cone with a vertex at Ξ_m where the edge is located interior to Ω. We now consider a cone with a vertex at Ξ_m and an angle $\varphi_0 \leq \varphi \leq \varphi_1$ where $\varphi = \varphi_0$ is interior to $B_R \subset \Omega$ so that $v \geq 0$ is ensured and $\varphi = \varphi_1$ is on Σ. Assume that there exists a nonempty set $\mathcal{O} = \{v > 0\}$ in the cone we have defined.

By the same calculation as before in Lemma 4.5, we have $L_1 v = -L_2\varphi < 0$ with a sufficiently small λ on the interior set $\{c^2 - r^2 < \varphi\} \subset \Omega$. We evaluate

$$Mv = (c^2)'\beta_i D_i \rho - 2r\beta_1 - \beta_i D_i \varphi = -2r\beta_1 - \beta_i D_i \varphi,$$

and observe that

$$\begin{aligned} \beta_1 &= r\sqrt{\frac{r^2 - \bar{c}}{\bar{c}}}(c^2(r^2 - \bar{c}) - 3\bar{c}(c^2 - r^2)) \\ &> r\sqrt{\frac{\tau_*}{\bar{c}}} d(c^2 \tau_* d^2 - 3\bar{c}\lambda d^2) > \frac{r}{2}\bar{c}^{1/2}\tau_*^{3/2} d^3 \end{aligned}$$

by choosing

$$\lambda < \frac{\tau_*}{6}. \tag{39}$$

Also since β_i s are in terms of $c^2 - r^2$ and $r^2 - \bar{c}$, where notice that $r^2 - \bar{c} \le r^2 - c_1^2 \le d^2$, we have

$$\begin{aligned}|\beta_i D_i \varphi| &= 2\lambda d|\beta_i D_i d| \\ &\le 2\lambda d \left(c^2(r^2 - \bar{c})|r' d_r + 3d_\theta| + \bar{c}(c^2 - r^2)|3r' d_r + d_\theta|\right) \\ &\le C_0 \lambda d^3,\end{aligned}$$

where C_0 is a positive constant.

Hence by choosing λ further sufficiently small if necessary, on the set $\mathcal{O} = \{c^2 - r^2 < \varphi\} \cap \Sigma$, we have

$$Mv < -r^2 \bar{c}^{1/2} \tau_*^{3/2} d^3 + C_0 \lambda d^3 < 0.$$

Thus by the maximum principle applied on the set $\mathcal{O}$ to $L_1 v < 0$ and $Mv < 0$, we have $v \ge \min_{\partial\mathcal{O}} v = 0$, which is a contradiction. □

4.3. The limiting solution

Since we now have strict ellipticity locally away from the sonic boundary, by using local compactness and diagonalization arguments, see for example [3,12], we next establish the limiting solution.

Lemma 4.7. *There exist a limit $\rho \in C^{2,\alpha}(\Omega)$ for some $0 < \alpha < 1$ and a limit $r(\theta) \in C^{1,\alpha_\Sigma}([\theta_R, \theta_s] \setminus \{\theta_2\}) \cap C^{0,1}([\theta_R, \theta_s])$ for some $0 < \alpha_\Sigma < 1$ such that ρ and r satisfy the governing equation $Q\rho = 0$ in Ω, $M^j \rho = 0$ and $r' = -s(r, \rho, \rho_j)$ on Σ_j, $j = 1, 2$.*

We have now established the limit r from r^ε, and thus we let Ξ_1 be the corresponding limit of Ξ_m. We next construct a uniform barrier function to show the continuity of the limiting solution at Ξ_1. A new barrier function is necessary in our problem because the proof in the earlier work [15, Lemma 3.11] depends on the fact that the solution ρ attains its minimum at the degenerate point whereas our configuration is not the case.

Lemma 4.8. *There exists a function $\psi = K d^t$ independent of ε where*

$$d = r - c_1 + q_0\sqrt{(\theta - \theta_1)^2 + q_1} - q_0\sqrt{q_1}, \quad 0 < q_0, q_1 < 1, \quad 0 < t < 1/2, \quad \textit{and} \quad K \ge 1,$$

with $\Xi_1 = (c_1, \theta_1)$ such that $c^2(\rho^\varepsilon) - r^2 \le \psi$ in $\{d < d_0\} \cap \overline{\Omega}$ where $d_0 > 0$ is independent of ε.

Proof. For the notational simplicity, throughout the proof, we let $\rho^\varepsilon = \rho$, $\beta^1 = \beta$ and $\Sigma_1 = \Sigma$. Since $r'(\theta_1) = 0$ and $r \in C^{1+\alpha_\Sigma}$ we can find $0 < q_0 < 1$ so that $r(\theta) \ge c_1 - q_0|\theta - \theta_1|$ for $|\theta - \theta_1| < 1$ small. Hence $d = r - c_1 + q_0\sqrt{(\theta - \theta_1)^2 + q_1} - q_0\sqrt{q_1} > 0$ whenever $\theta \ne \theta_1$ and $d = 0$ at Ξ_1.

As we did in the proof of Lemma 4.6, multiply $dc^2/d\rho = (\gamma - 1)\rho^{\gamma - 2}$ throughout the equation $Q^\varepsilon \rho = 0$ and denote $c^2(\rho) = u$ to have;

$$Lu = (a_{ii} + \varepsilon)D_{ii}u + a_i|D_i u| + bu_r = 0.$$

Define $v = c^2 - r^2 - \psi$ and assume that there exists a non-empty set $\mathcal{O} = \{X \in \Omega_1 : v > 0\}$ (if $\mathcal{O}$ is an empty set then we are done). From $Lu = 0$, we have

$$\begin{aligned}
0 &= Lu \\
&= a^{\varepsilon}_{ii} D_{ii}(v + r^2 + \psi) + a_i|D_i(v + r^2 + \psi)|^2 + b(v + r^2 + \psi)_r \\
&= L_1 v + L_2 \psi
\end{aligned}$$

where

$$\begin{aligned}
L_2\psi &= (a_{11} + \varepsilon)t(t-1)Kd^{t-2} + a_1 t^2 K^2 d^{2t-2} - (4ra_1 + b)tKd^{t-1} \\
&\quad + (a_{22} + \varepsilon)[t(t-1)Kd^{t-2}d_\theta^2 + tKd^{t-1}d_{\theta\theta}] + a_2 t^2 K^2 d^{2t-2} d_\theta^2 \\
&\quad + 2(a_{11} + \varepsilon) + 4r^2 a_1 + 2rb \\
&< t(2t-1)K^2 d^{2t-2} - (4ra_1 + b)tKd^{t-1} + 2(a_{11} + \varepsilon) + 4r^2 a_1 + 2rb \\
&< 0,
\end{aligned}$$

with $0 < t < 1/2$ and K sufficiently large if necessary.

Next evaluate

$$Mv = -2r\beta_1 - \beta_i D_i \psi = -2r\beta_1 - Ktd^{t-1}\beta_i D_i d.$$

Notice that $D_i d = (1, d_\theta)$ while the inward normal on $\Sigma = (r(\theta), \theta)$ is $\nu = (-1, r'(\theta))$, and thus by the choice of q_i, $i = 0, 1$ small, we have $\beta_i D_i d \leq 0$. Moreover, on the set $\mathcal{O}$ we have $c^2 - r^2 > \psi = Kd^t$ and $c^2(\rho) > \bar{c} > c_1^2$ while $\rho > \rho_1$, and thus

$$\begin{aligned}
-2r\beta_1 &= 2r^2\sqrt{\frac{r^2 - \bar{c}}{\bar{c}}}(3\bar{c}(c^2 - r^2) - c^2(r^2 - \bar{c})) \\
&> 2r^2\sqrt{\frac{r^2 - \bar{c}}{\bar{c}}}(3\bar{c}Kd^t + c^2(c_1^2 - r^2)) \\
&\geq 2r^2\sqrt{\frac{r^2 - \bar{c}}{\bar{c}}}(3\bar{c}Kd^t - c^2 d^2) \\
&> 0,
\end{aligned}$$

for sufficiently small $d \leq R'$ and sufficiently large K if necessary.

By the choice of $K > 1$ sufficiently large, we have $v \leq 0$ on $\Sigma(R_0)$ and $\Omega \cap \partial\Omega(R')$. Hence we apply the maximum principle type argument applied in the set $\mathcal{O}$ to obtain $v \leq 0$. Thus we establish the result. □

Remark 4.9. The barrier function in Lemma 4.8 is constructed locally and it can be used also the neighboring points near Ξ_1 on σ. Away from Ξ_1, since $\sigma \subset C_1$, we can construct the barrier function $\psi = K[(c_1 - r)^t + (\theta - \theta_0)^2]$ locally for $\Xi_0 = (c_1, \theta_0) \in \sigma$. That is, for each $\Xi_0 \in \sigma$, since we can find a neighborhood B of Ξ_0 so that $c_1 - r > 0$ in B, and repeat the similar argument as in Lemma 4.8 and $c^2 - r^2 \le \psi$ on $\overline{B} \cap \sigma$ to show ψ is the local upper barrier at $\Xi_0 \in \sigma$.

Using these barrier functions ψ, we obtain that the limiting solution is continuous on σ.

Remark 4.10. On σ_R, since the solutions $\rho < \rho_R$ in Ω, while $c^2(\rho) - r^2 > r^2 - r_R^2$ in Ω, we have

$$\lim_{r \to r_R} \frac{|c^2(\rho) - r^2|}{r_R - r} \le 2r_R. \tag{40}$$

That is $\rho \in C^{0,1}(\Omega \cup \sigma_R)$.

4.4. *Proof of the main theorem*

Finally we now establish the main theorem.

Proof of Theorem 2.1. Lemmas 4.4 and 4.5, and Remarks 4.9 and 4.10 show that existence of a solution pair (ρ, r) where $\rho \in C^{2,\alpha}(\Omega \cup \Sigma_S) \cap C^{1,\alpha}(\Omega \cup \Sigma_1 \cup \Sigma_2) \cap C^{\gamma}(\Omega \cup \{\Xi_R, \Xi_2, \Xi_s\}) \cap C^{0,1}(\Omega \cup \sigma_R \setminus \{\Xi_R\}) \cap C^0(\overline{\Omega})$ satisfying $\rho_2 < \rho < \rho_R$, and $r \in C^{1,\gamma}([\theta_R, \theta_s] \setminus \{\theta_2\}) \cap C([\theta_R, \theta_s])$, where $0 < \alpha, \alpha_\Sigma < 1$, satisfying (20)–(25). This completes the proof. □

5. Asymptotic analysis

In this section, we discuss the asymptotic analysis for weak shocks of the nonlinear wave system in the same spirit of Hunter and Keller [7]. The approximate solution to the linear acoustics near the diffracted wavefront is given by Keller and Blank [10], and from [7] it can be written as

$$\rho = \rho_i + \delta\rho_i \frac{k(\theta)}{r^{1/2}}(c_i t - r)^{1/2} + O[\delta(c_i t - r)^{3/2}] + O(\delta^2), \tag{41}$$

where δ is the shock strength and

$$\begin{aligned} k(\theta) &= \frac{1}{2^{1/2}\pi \cos\theta}\left(2^{1/2}\cos\frac{\theta}{2}\left(1 + \frac{\rho_R - \rho_1}{\rho_1 - \rho_2}\right) + \left(1 - \frac{\rho_R - \rho_1}{\rho_1 - \rho_2}\right)\right) \\ &= \frac{2\cos(\theta/2)}{\pi\cos\theta} + O(\rho_1 - \rho_2). \end{aligned} \tag{42}$$

As in [7], we write the corresponding nonlinear approximate solution of the diffracted wave for the nonlinear wave system,

$$\begin{pmatrix} \rho \\ u \\ v \end{pmatrix} = \begin{pmatrix} \rho_i \\ u_i \\ v_i \end{pmatrix} + \frac{\varepsilon k(\theta)}{c_i r^{1/2}}\left(\frac{\zeta}{\varepsilon}\right)^{1/2}\begin{pmatrix} \rho_i \phi_t \\ -(c_i\phi_x + u_i\phi_t) \\ -(c_i\phi_y + v_i\phi_t) \end{pmatrix}, \tag{43}$$

where

$$\varepsilon = \delta^2, \quad \phi = c_i t - r,$$

and the nonlinear effect ζ is

$$\zeta = c_i t - r + (\gamma - 1)(\varepsilon r \zeta)^{1/2} k(\theta). \tag{44}$$

When $k(\theta) < 0$, the diffracted wave is a rarefaction and ζ has the unique nonnegative solution

$$\zeta^{1/2} = \left(c_i t - \left(1 - \varepsilon \frac{(\gamma - 1)^2}{4} k^2 \right) r \right)^{1/2} + \frac{\gamma - 1}{2} k(\varepsilon r)^{1/2}. \tag{45}$$

Hence for $r < c_i t$, the rarefaction wave becomes

$$\begin{aligned} \begin{pmatrix} \rho \\ u \\ v \end{pmatrix} &= \begin{pmatrix} \rho_i \\ u_i \\ v_i \end{pmatrix} \\ &\quad + \varepsilon^{1/2} \frac{k(\theta)}{r^{1/2}} \left[\left(c_i t - \left(1 - \varepsilon \frac{(\gamma - 1)^2}{4} k^2 \right) r \right)^{1/2} + \frac{\gamma - 1}{2} k(\varepsilon r)^{1/2} \right] \\ &\quad \times \begin{pmatrix} \rho_i \\ c_i \frac{x}{r} - u_i \\ c_i \frac{y}{r} - v_i \end{pmatrix}. \end{aligned} \tag{46}$$

The jump in the gradient across the wave front, ρ_r, can be written as

$$[\rho_r] = \rho_i \frac{1}{(\gamma - 1) r}. \tag{47}$$

When $k(\theta) > 0$, the diffracted wave is compressive and the wave front becomes a shock. By checking an envelope equation on which the characteristic lines $\zeta = \mathit{constant} \leq 0$ overlap, the shock location is given by

$$c_i t = \left(1 - \frac{\varepsilon (\gamma - 1)^2}{4} k^2(\theta) \right) r, \tag{48}$$

and the corresponding wave is

$$\begin{pmatrix} \rho \\ u \\ v \end{pmatrix} = \begin{pmatrix} \rho_i \\ u_i \\ v_i \end{pmatrix} + \varepsilon \frac{\gamma - 1}{2} k^2(\theta) \begin{pmatrix} \rho_i \\ c_i \frac{x}{r} - u_i \\ c_i \frac{y}{r} - v_i \end{pmatrix}. \tag{49}$$

The density jump across the shock can be written as

$$[\rho] = \varepsilon \rho_i \frac{\gamma - 1}{2} k^2(\theta) + O(\varepsilon^2). \tag{50}$$

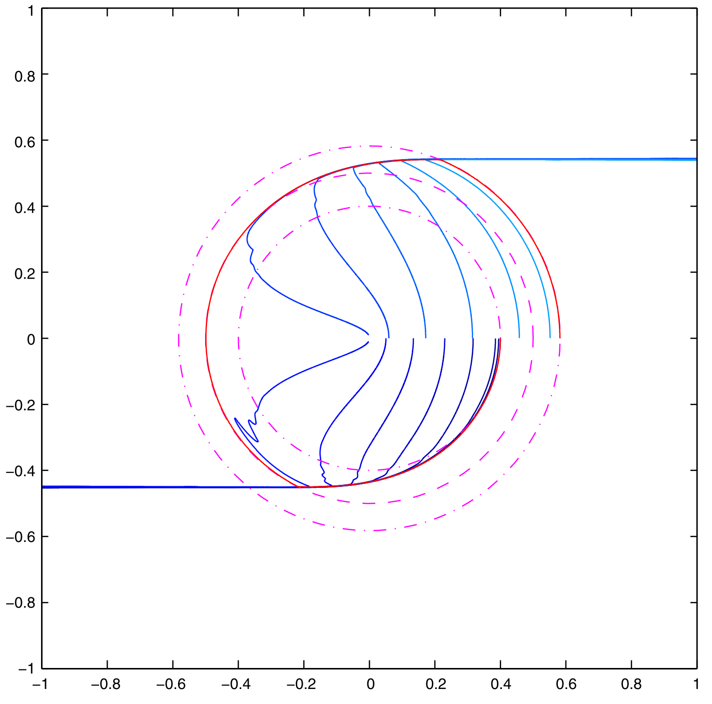

Fig. 2. Density ρ contour plot where $\gamma = 3$, $\rho_1 = 0.5$, $\rho_2 = 0.4$, and the corresponding $\rho_R = 0.5832$. Magenta dashed circles are sonic circles C_1,C_2, and C_R. Red solid curve is the sonic line. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The diffracted wavefront is a shock when $k(\theta) \geq 0$ and then it becomes a rarefaction when $k(\theta) \leq 0$. Hence, from the equation for $k(\theta)$ in (42), we obtain an approximation of θ_1, denoted by θ_a, where the diffracted shock meets the diffracted expansion wave:

$$\theta_a = 2\cos^{-1}\left(2^{-1/2}\frac{\rho_R - 2\rho_1 + \rho_2}{\rho_R - \rho_2}\right). \tag{51}$$

We now show that $\theta_a > \pi$ and $\theta_a \to \pi^+$ as $\rho_1 - \rho_2 \to 0^+$.

Lemma 5.1. *The approximation θ_a from (51) of θ_1 holds*

$$\theta_a > \pi, \quad \theta_a \to \pi^+, \tag{52}$$

as $\rho_1 - \rho_2 \to 0^+$.

Proof. Recall (9) whence

$$(\rho_R - \rho_1)^2 \frac{\rho_R^\gamma - \rho_1^\gamma}{\rho_R - \rho_1} = (\rho_1 - \rho_2)^2 \frac{\rho_1^\gamma - \rho_2^\gamma}{\rho_1 - \rho_2}. \tag{53}$$

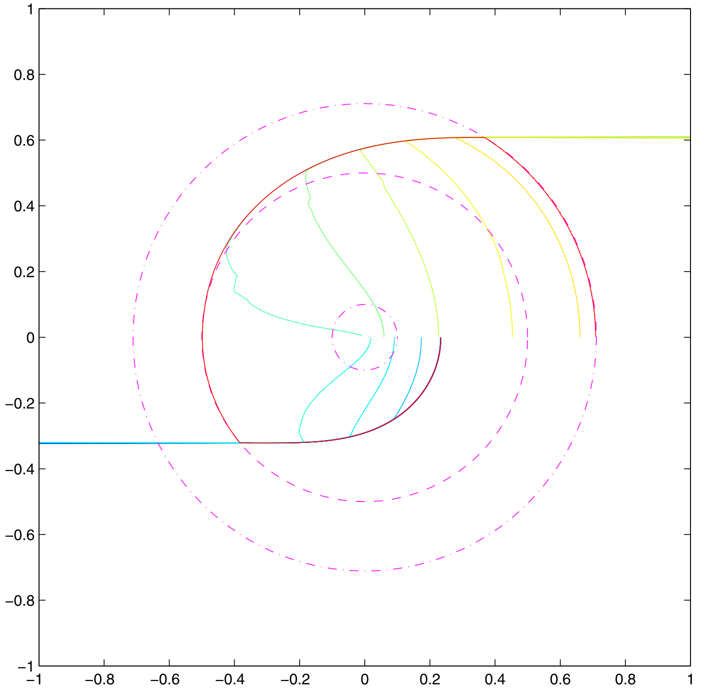

Fig. 3. Density ρ contour plot where $\gamma = 3$, $\rho_1 = 0.5$, $\rho_2 = 0.1$, and the corresponding $\rho_R = 0.7112$. Magenta dashed circles are sonic circles C_1,C_2, and C_R. Red solid curve is the sonic line. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Since $y = x^\gamma$, $\gamma > 1$ is convex and $\rho_R > \rho_1 > \rho_2$, we have $\frac{\rho_R^\gamma - \rho_1^\gamma}{\rho_R - \rho_1} > \frac{\rho_1^\gamma - \rho_2^\gamma}{\rho_1 - \rho_2}$, which in turn implies

$$0 < \rho_R - \rho_1 < \rho_1 - \rho_2 \quad \Longrightarrow \quad 0 < \frac{\rho_R - \rho_1}{\rho_1 - \rho_2} < 1.$$

Then

$$\frac{\rho_R - 2\rho_1 + \rho_2}{\rho_R - \rho_2} = \frac{\frac{\rho_R - \rho_1}{\rho_1 - \rho_2} - 1}{\frac{\rho_R - \rho_1}{\rho_1 - \rho_2} + 1} < 0$$

which shows $\theta_a > \pi$. If $\rho_2 \to \rho_1$, then $\rho_R \to \rho_1$ and thus from (53), we see that $\frac{\rho_R - \rho_1}{\rho_1 - \rho_2} \to 1$. Therefore, we deduce that $\theta_a \to \pi^+$ when $\rho_1 - \rho_2 \to 0^+$. $\square$

This computation based on the above weakly nonlinear approximation alludes that the point, at which the diffracted reflected weak shock wave changes into an expansion wave, is located just below the negative x-axis.

Remark 5.2. On the other hand, θ_R and θ_2 representing the other two corner points are explicitly given and so their limits when $\rho_1 - \rho_2 \to 0^+$ are. To compute the limits, we recall

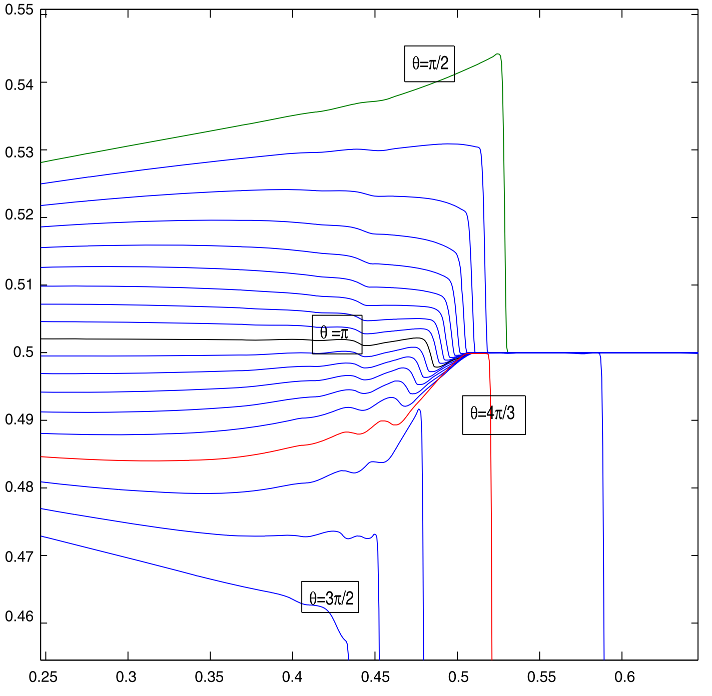

Fig. 4. Density ρ plot in the radial coordinate for given θ from $\theta = \pi/2$ to $3\pi/2$. $\gamma = 3$, $\rho_1 = 0.5$, $\rho_2 = 0.4$.

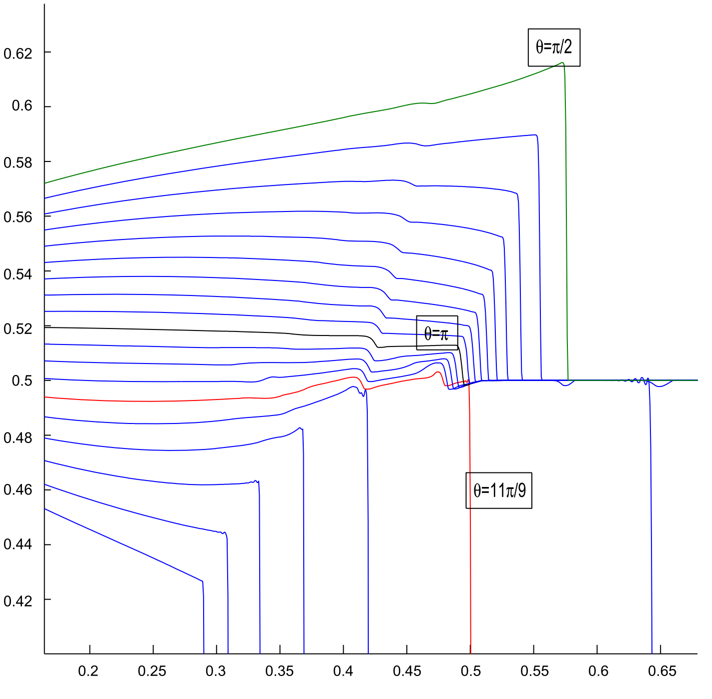

Fig. 5. Density ρ plot in the radial coordinate for given θ from $\theta = \pi/2$ to $3\pi/2$.where $\gamma = 3$, $\rho_1 = 0.5$, $\rho_2 = 0.1$.

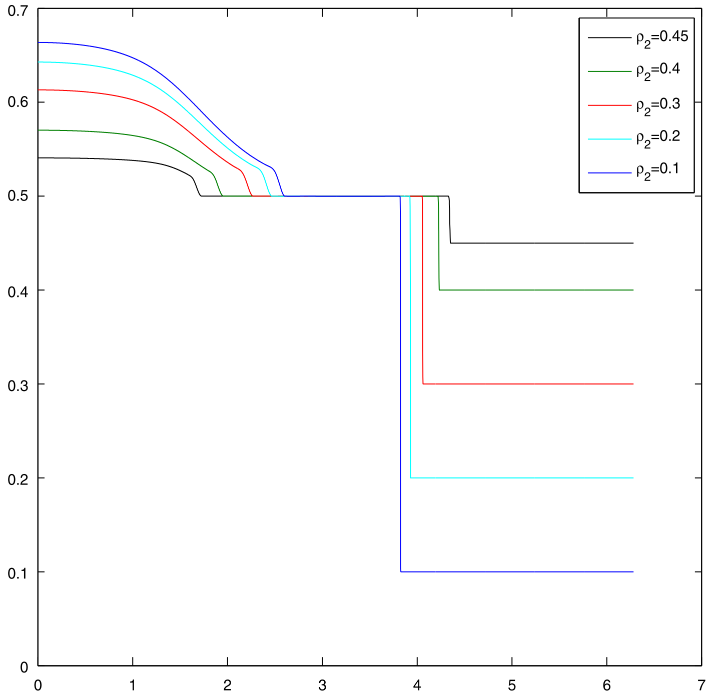

Fig. 6. Density ρ plot with respect to θ for $r = 0.5$ (this corresponds to the sonic circle C_1) and $\rho_2 = 0.1, 0.2, 0.3, 0.4, 0.45$. $\gamma = 3$, $\rho_1 = 0.5$.

$$\theta_R = \arctan\left(\frac{\eta_R}{\xi_R}\right), \quad \eta_R = \sqrt{\frac{p(\rho_R) - p(\rho_1)}{(\rho_R - \rho_1)}}, \quad \xi_R = \sqrt{c^2(\rho_R) - \eta_R^2}.$$

Write

$$\frac{\eta_R}{\xi_R} = \sqrt{\frac{\frac{\rho_R^\gamma - \rho_1^\gamma}{\rho_R - \rho_1}}{\gamma \rho_R^{\gamma-1} - \frac{\rho_R^\gamma - \rho_1^\gamma}{\rho_R - \rho_1}}}.$$

Since $\frac{\rho_R^\gamma - \rho_1^\gamma}{\rho_R - \rho_1} \to \gamma \rho_1^{\gamma-1}$ as $\rho_R \to \rho_1$, we see that $\eta_R/\xi_R \to \infty$, that is $\theta_R \to \pi/2^-$. Similarly, one can compute the limit of $\theta_2 = \arctan(\eta_0/\xi_0)$ as $3\pi/2^-$.

6. Numerical results

In this section we present our intriguing numerical results of this configuration. The numerical results are obtained by using CLAWPACK [20]. We implement Roe average methods [21] and finite volume methods on quadrilateral grids. More precisely we implement Roe average methods in a uniform grid in polar coordinates as our computational domain along with a coordinate mapping and an appropriate scaling of the flux differences. The scaling is done by using the area ratio “capacity” of the computational cell which is determined by the size of the corresponding physical cell [19].

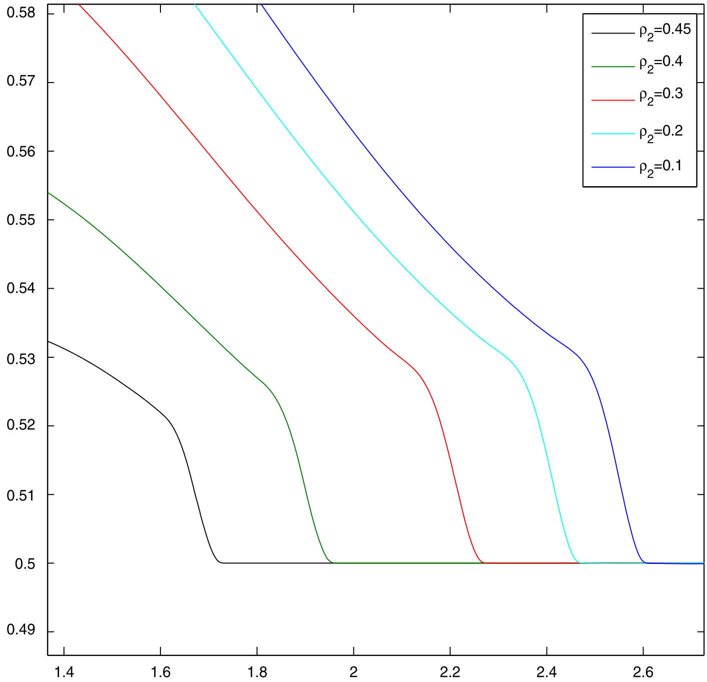

Fig. 7. Enlarged figure of density ρ plot with respect to θ for $r = 0.5$ (this corresponds to the sonic circle C_1) and $\rho_2 = 0.1, 0.2, 0.3, 0.4, 0.45$. $\gamma = 3$, $\rho_1 = 0.5$.

The numerical results presented in this paper are with the computational domain $10^{-2} \leq r \leq 1.5$ and $0 \leq \theta \leq 2\pi$ where (r, θ) are polar coordinates, and mesh sizes $\Delta r = 1.49/2400 \approx 6.2083 \times 10^{-4}$ and $\Delta\theta = 2\pi/3600 \approx 1.7453 \times 10^{-3}$, and with the final time $T = 1$.

We have tested different γ values where γ ranges from 2 to 4. However in all cases the numerics are similar and we only present the results with $\gamma = 3$ in this paper. In all numerics we fix $\rho_1 = 0.5$ and vary ρ_2 where $0 < \rho_2 < \rho_1$. Figs. 2, 3, 4 and 5 are results with different values of ρ_2. We discuss them in the following sections.

Figs. 2 and 4 are the results with $\rho_2 = 0.4$. In this case, the incident shock strength is relatively small with the corresponding $\rho_R = 0.5832$. Fig. 2 depicts the density plot for given θ in the radial coordinate, where θ is ranging from $\pi/2$ to $3\pi/2$ incrementing by 10 degrees for each plot. In this case since $c^2(\rho) = \rho^2$, the shock becomes sonic when $\rho = 0.5$. However the numerical result appears to be the case that the wave continues to be compressed even inside of the sonic circle C_1, and may stay as an infinitesimally small shock inside of the subsonic region. On the other hand, for the nonlinear wave system, the shock must become sonic upon it meets to the sonic circle C_1.

Similar phenomenons appear for larger incident shock strengths. Figs. 3 and 5 are the results with $\rho_2 = 0.1$. Fig. 5 is the density plot in the radial coordinates for given θ ranging from $\theta = \pi/2$ to $3\pi/2$. Again it shows that the diffracted shock is apparent inside of the sonic circle C_1.

At this point, we do not have a sharp estimate on the location at which the shock becomes sonic, $\Xi_1 = \partial\Sigma_1 \cap \sigma$. The asymptotic analysis shown in Lemma 5.1 in the previous section suggests that the position Ξ_1 is near $\theta = \pi$ for a small incident shock. More precisely, we have shown in Lemma 5.1 that $\theta_a \to \pi^+$, as $\rho_1 - \rho_2 \to 0$, where θ_a is an approximation to θ_1. From equation (51), we can evaluate θ_a explicitly: $\theta_a = 3.2714$ (187.4358 in degrees) when $\theta_2 = 0.4$ and $\theta_a = 3.532$ (205.2332 in degrees) when $\theta_2 = 0.1$. However Fig. 2 (when $\rho_2 = 0.4$) indicates that the sonic curve meets the sonic circle C_1 already near $3\pi/4$. On the other hand Fig. 3 (when $\rho_2 = 0.1$) shows that the sonic curve meets the sonic circle C_1 near π. As mentioned before, the numerics suggest that the wave continues to be compressed even inside of the sonic circle C_1, and thus our numerical results are insufficient at this point to find a correlation between the incident shock strengths and the location Ξ_1, as similar to that of [8].

Fig. 6 is the density plot with respect to $0 \le \theta \le 2\pi$ for $r = 0.5$ which corresponds to the sonic circle C_1 and ρ_2 varying from 0.1 to 0.45, for $\gamma = 3$ and $\rho_1 = 0.5$. We also provide Fig. 7 for the enlargement of Fig. 6 when ρ is near 0.5. These figures suggest that the angle θ values at which the shock becomes sonic maybe away from π even close to $\pi/2$ for the small incident shock strength (see the case when $\rho_2 = 0.45$).

The statement in this section is based on an observation from our numerical results. Further refined numerical schemes may provide better results and we leave this question to interested readers.

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