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Large time behavior for the fast diffusion equation with critical absorption

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Abstract

We study the large time behavior of nonnegative solutions to the Cauchy problem for a fast diffusion equation with critical zero order absorption

$$\partial_t u - \Delta u^m + u^q = 0 \qquad \text{in } (0,\infty) \times \mathbb{R}^N ,$$

with $m_c := (N-2)_+/N < m < 1$ and $q = m + 2/N$. Given an initial condition u_0 decaying arbitrarily fast at infinity, we show that the asymptotic behavior of the corresponding solution u is given by a Barenblatt profile with a logarithmic scaling, thereby extending a previous result requiring a specific algebraic lower bound on u_0. A by-product of our analysis is the derivation of sharp gradient estimates and a universal lower bound, which have their own interest and hold true for general exponents $q > 1$.

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Keywords: Large time behavior; Fast diffusion; Critical absorption; Gradient estimates; Lower bound

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1. Introduction and main results

In this paper, we deal with the large time behavior of a fast diffusion equation with absorption, in a special case when the exponent of the absorption term is critical. More precisely, we consider the following Cauchy problem

$$\partial_t u - \Delta u^m + u^q = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^N, \tag{1.1}$$

with initial condition

$$u(0,x) = u_0(x), \qquad x\in\mathbb{R}^N, \tag{1.2}$$

where $N \geq 1$,

$$u_0 \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N), \quad u_0 \geq 0,\ u_0 \not\equiv 0, \tag{1.3}$$

and the parameters m and q satisfy

$$m_c := \frac{(N-2)_+}{N} < m < 1, \quad q = q_* := m + \frac{2}{N}. \tag{1.4}$$

Degenerate and singular parabolic equations with absorption such as (1.1) have been the subject of intensive research during the last decades. In (1.1), the main feature is the competition between the diffusion Δu^m and the absorption $-u^q$ which turns out to depend heavily on the exponents $m > 0$ and $q > 0$. More precisely, a critical exponent $q_* = m + 2/N$ has been uncovered which separates different dynamics and the large time behavior for non-critical exponents $q \neq q_*$ is now well understood. Indeed, for the semilinear case $m = 1$ and the slow diffusion case $m > 1$, it has been shown that, when $q > q_*$, the effect of the absorption is negligible, and the large time behavior is given by the diffusion alone, leading to either Gaussian or Barenblatt profiles [9,14,15,18,19,21].

A more interesting case turns out to be the intermediate range of the absorption exponent $q \in (m, q_*)$, where the competition of the two effects is balanced. For $m \geq 1$, the study of this range has led to the discovery of some special self-similar solutions called *very singular solutions* which play an important role in the description of the large time behavior, see [3,6,9,15,19–21, 26] for instance. This was an important improvement, as the existence of very singular solutions has been later established for many other different equations.

The study of the fast diffusion case $0 < m < 1$ was performed later, but restricted to the range of exponents $m_c < m < 1$, as the singular phenomenon of finite time extinction occurs when $m \in (0, m_c)$. When $m \in (m_c, 1)$, the asymptotic behavior has been also identified for any $q \neq q_*$, $q > 1$, and again very singular solutions play an important role [25,27,28]. Later, also the extinction case when q ranges in $(0, 1)$ has been studied [7,8], although there are still many open problems in these ranges, as most of the results are valid only in dimension $N = 1$.

In this paper we focus on the critical absorption exponent $q = q_*$ which is the limiting case above which the effect of the absorption term is negligible in the large time dynamics. That the diffusion is almost governing the asymptotic behavior is revealed by the fact that the asymptotic profile is given by the diffusion, but the scaling is modified as a result of the influence of the absorption term and additional logarithmic factors come into play. More precisely, the solutions converge to a Gaussian or Barenblatt type profile, subject to corrections in x and u of type powers of $\log t$. The semilinear case $m = 1$ and $q = q_*$ is investigated in [4,9,14,15] in any space dimension. For slow diffusion $m > 1$, still considering the case $q = q_*$, the asymptotic behavior of nonnegative solutions to (1.1)–(1.2) is investigated in [11] (and previously [10] in dimension $N = 1$), where a new dynamical systems approach, well-known nowadays as the *S-theorem*, is introduced to deal with small asymptotic non-autonomous perturbations of autonomous equations. This approach became then common when dealing with critical exponents, and a survey of it can be found in the book [12]. The precise identification of the large time limit is however only achieved for compactly supported solutions in [11] and this restriction is successfully removed in [30], extending the result to the wider class of solutions emanating from initial data decaying more rapidly than $|x|^{-N}$ as $|x| \to \infty$. While the final step of the proof performed in [30] relies on the stability technique developed in [11] the main novelty is the construction of a non-compactly supported supersolution with the expected temporal decay and a spatial decay complying with that of the initial condition.

Main results. However, in spite of the general interest in literature, the problem of studying the asymptotic behavior for the fast diffusion case $m_c < m < 1$ with critical exponent $q = q_*$ and establishing an analogous result as the one by Galaktionov and Vázquez [11] still remains open for a wide class of non-negative initial data u_0, including in particular compactly supported ones, at least for $N \ge 2$, see [8, Section 11.1] for a sketch of proof when $N = 1$. The main difficulty to be overcome seems to be the following: due to the infinite speed of propagation, a property which contrasts markedly with the range $m > 1$, and to the nonlinearity of the diffusion which is the main difference with the semilinear case $m = 1$, a suitable control of the tail as $|x| \to \infty$ of $u(t,x)$ is needed for positive times. Of particular importance is the derivation of a sharp lower bound which allows one to exclude the convergence to zero in the scaling variables. This difficulty is by-passed in [31] by establishing the required sharp lower bound as soon as the initial condition $u_0(x)$ behaves as $C|x|^{-l}$ as $|x| \to \infty$ for some $C > 0$ and $l < 2/(1-m)$. This is done by constructing a subsolution having the right temporal behavior [31]. However the above decay assumption clearly excludes a broad class of "classical" initial data, including compactly supported ones, and our aim in this paper is to get rid of such a decay assumption. An intermediate step is to figure out how does the solution u to the Cauchy problem (1.1)–(1.2) behave as $|x| \to \infty$ for positive times $t > 0$ if it starts from a, say, compactly supported initial condition u_0.

We actually provide an answer to this question, in the form of a sharp lower bound for solutions to (1.1)–(1.2), which is valid **for any** $q > 1$:

Theorem 1.1. *Consider an initial condition u_0 satisfying (1.3), $m \in (m_c, 1)$, $q > 1$, and let u be the corresponding solution to (1.1)–(1.2). Then*

$$u(t,x) \ge \ell_u(t)\, (1+|x|)^{-2/(1-m)}\,, \qquad (t,x) \in (0,\infty) \times \mathbb{R}^N\,, \tag{1.5}$$

with

$$\ell_u(t) := \left[u^{(m-1)/2}(t,0) + \sqrt{B_0} \left(1 + \sqrt{(3-m-2q)_+} \|u_0\|_\infty^{(q-1)/2} \sqrt{t} \right) t^{-1/2} \right]^{2/(m-1)}\,,$$

and

$$B_0 := \frac{1-m}{2m(mN-N+2)} > 0\,. \tag{1.6}$$

Note that ℓ_u depends on u and converges to zero as $t \to 0$ and $t \to \infty$ since $m < 1$, the latter being a consequence of the decay to zero of $\|u(t)\|_\infty$ as $t \to \infty$, see (2.3) below. Moreover, note also that the space dependence of this lower bound is sharp. Indeed, it says: whatever the initial condition is, during the later evolution, the solution to the Cauchy problem has a spatial decay at infinity slower than the decay of a Barenblatt self-similar profile, a property which is inherited from the fast diffusion equation [16, Theorem 2.4]. In particular, let us point out a curious *jump of the tails*: if u_0 is compactly supported (no tail at all), or decays as $|x| \to \infty$ with a tail of the form $|x|^{-l}$, $l > 2/(1-m)$, then its tail jumps immediately to a slower decaying one for positive times. This peculiar property does not seem to have been noticed in [31] where it is rather shown that (1.5) holds true provided u_0 does not decay too fast as $|x| \to \infty$, namely $u_0(x) \sim C|x|^{-l}$ as $|x| \to \infty$ for some $C > 0$ and $l < 2/(1-m)$.

The proof of Theorem 1.1 is based on some *sharp gradient estimates* for well-chosen negative powers of u which have their own interest and are given in Theorem 2.2 below.

This universal lower bound allows for a comparison from below of general solutions with suitable constructed subsolutions. This is the main technical tool that enables us to establish the asymptotic behavior of solutions for a very general class of initial data. More precisely, our main result is:

Theorem 1.2. *Consider an initial condition u_0 satisfying (1.3), $m \in (m_c, 1)$, $q = q_* = m + 2/N$, and assume further that u_0 satisfies*

$$u_0(x) \le K|x|^{-k}\,, \quad x \in \mathbb{R}^N\,, \quad \textit{with}\ k := N + \frac{mN(mN-N+2)}{2[2-m+mN(1-m)]}\,, \tag{1.7}$$

for some $K > 0$. Let u be the solution to the Cauchy problem (1.1)–(1.2). Then

$$\lim_{t\to\infty} (t\log t)^{1/(q-1)} \left| u(t,x) - \frac{1}{(t\log t)^{1/(q-1)}}\sigma_{A_*}\left(\frac{x}{t^{1/N(q-1)}(\log t)^{(1-m)/2(q-1)}}\right)\right| = 0, \tag{1.8}$$

uniformly in $\mathbb{R}^N$, where

$$\sigma_A(y) = \left(A + B_0|y|^2\right)^{1/(m-1)}, \quad B_0 = \frac{1-m}{2m(mN-N+2)}, \quad A > 0,$$

and A_* is uniquely determined and given by

$$A_* := \left(\frac{2\displaystyle\int_0^\infty (1+r)^{q_*/(m-1)} r^{(N-2)/2}\,dr}{N\displaystyle\int_0^\infty (1+r)^{1/(m-1)} r^{(N-2)/2}\,dr}\right)^{N(1-m)/(2(Nm+2-N))}.$$

The first step towards the proof of Theorem 1.2 is to establish that the temporal decay of the solution u to (1.1)–(1.2) is the expected one, see (1.8). To this end the usual approach is to construct suitable subsolutions and supersolutions with the expected temporal decay to which u can be compared. This is in particular the approach used in [31] and the construction of the supersolution performed therein requires the upper bound (1.7) on the initial condition u_0 while that of the subsolution requires the already mentioned lower bound $u_0(x) \geq C|x|^{-l}$ as $|x| \to \infty$ for some $k \leq l < 2/(1-m)$. To get rid of this last assumption we take another route and exploit the following property described in Theorem 1.1: whatever the initial condition the corresponding solution u to (1.1)–(1.2) is bounded from below by $C(t)|x|^{-2/(1-m)}$ as $|x| \to \infty$ for all positive times t. It is worth emphasizing that this feature of the equation not only dispenses us from assuming an algebraic lower bound but also provides a lower bound matching that of the Barenblatt profile and allows us to construct subsolutions which are simpler than the ones from [31]. Concerning the supersolution we use the one constructed in [31] which requires the upper bound (1.7) and point out that extending Theorem 1.2 to any $k > N$ seems to be an open problem. The final step of the proof relies on the stability technique developed in [11] but we set it up in a slightly different way from [31].

Remarks. (i) We point out that the profile σ_A is the well-known Barenblatt profile from the theory of the standard fast diffusion equation

$$\partial_t \varphi = \Delta \varphi^m \quad \text{in } (0,\infty) \times \mathbb{R}^N \tag{1.9}$$

in the supercritical range $m \in (m_c, 1)$, see [32] for more information.

(ii) As already mentioned, Shi & Wang prove Theorem 1.2 in [31] under more restrictive conditions on the initial data u_0. More precisely, they assume the initial condition to satisfy:

$$\lim_{|x|\to\infty} |x|^l u_0(x) = C > 0 \quad \text{for } k \leq l < \frac{2}{1-m} ,$$

where k defined in (1.7) satisfies $k \in (N, 2/(1-m))$, since $(N-2)_+/N < m < 1$. This condition works well in view of comparison from below with rescaled Barenblatt-type profiles, but it has the drawback of not allowing some natural choices of initial data to be considered: in particular, initial data u_0 with compact support, or fast decay at infinity, or even with the same decay at infinity as the Barenblatt profiles (that is, with $l = 2/(1-m)$ in the condition above) fail to enter the framework of [31]. Our analysis removes the previous condition and allows us to consider all these ranges of initial data. However, we will use (and recall when necessary) some of the technical steps and results in [31], especially those concerning the use of the general stability technique to show the convergence part of the proof of Theorem 1.2.

(iii) Let us finally mention that less results seem to be available for the p-Laplacian equation with critical absorption

$$\partial_t u - \Delta_p u + u^{(p(N+1)-N)/N} = 0 \quad \text{in} \quad (0,\infty) \times \mathbb{R}^N ,$$

the parameter p ranging in $(2N/(N+1), \infty)$. As far as we know, a result similar to Theorem 1.2 is shown in [11] for compactly supported solutions when $p > 2$ while the case $p \in (2N/(N+1), 2)$ is considered in [5] without identifying the limit. When the absorption exponent is not critical the large time behavior of non-negative solutions to the p-Laplacian equation with absorption is studied in [5,22,24,33].

Organization of the paper. In *Section 2*, we prove some sharp gradient estimates for (1.1), which are valid for any $q > 1$; this result is new and interesting by itself, and it is stated in Theorem 2.2. We next prove Theorem 1.1, which turns out to be a rather simple consequence of Theorem 2.2. In *Section 3*, we construct suitable subsolutions that can be used for comparison from below, in view of the previous lower bound. This is the most involved part of the work, from the technical point of view, since the approach of [31] does not seem to work. Let us emphasize here that our construction relies on the fact that the solution u to (1.1)–(1.2) enjoys suitable decay properties after waiting for some time, as a consequence of Theorem 1.1. Finally, we prove Theorem 1.2 in *Section 4*, as a consequence of the previous analysis and of techniques from [11,12,31].

2. Gradient estimates and lower bound

In this section we consider $m \in (m_c, 1)$, $q > 1$, and an initial condition u_0 satisfying (1.3). By [27, Theorem 2.1] the Cauchy problem (1.1)–(1.2) has a unique non-negative solution $u \in BC((0,\infty) \times \mathbb{R}^N)$ and classical arguments entail that $u \in C([0,\infty); L^1(\mathbb{R}^N))$. In addition u enjoys the same positivity property as the solutions to the fast diffusion equation (1.9).

Lemma 2.1. *Consider $q > 1$ and an initial condition u_0 satisfying (1.3). Then the corresponding solution u to (1.1)–(1.2) satisfies $u(t,x) > 0$ for all $(t,x) \in (0,\infty) \times \mathbb{R}^N$.*

Proof. Let σ be the solution to the fast diffusion equation

$$\partial_t \sigma - \Delta \sigma^m = 0\,, \quad (t,x) \in (0,\infty) \times \mathbb{R}^N\,,$$
$$\sigma(0) = u_0\,, \quad x \in \mathbb{R}^N\,.$$

We set $a := \|u_0\|_\infty^{q-1} > 0$ and

$$\lambda(t) := e^{-at}\,, \qquad s(t) := \frac{e^{(1-m)at} - 1}{(1-m)a}\,, \qquad \ell(t,x) := \lambda(t)\sigma(s(t),x)$$

for $(t,x) \in (0,\infty) \times \mathbb{R}^N$. Introducing the parabolic operator

$$\mathcal{L}z := \partial_t z - \Delta z^m + a\, z\,,$$

we infer from (1.1), the non-negativity of u, and the comparison principle that

$$\mathcal{L}u = \left(\|u_0\|_\infty^{q-1} - u^{q-1}\right) u \ge 0 \quad \text{in} \quad (0,\infty) \times \mathbb{R}^N\,.$$

Next, for $(t,x) \in (0,\infty) \times \mathbb{R}^N$,

$$\mathcal{L}\ell(t,x) = (\lambda' + a\lambda)(t)\sigma(s(t),x) + \left(\lambda s' - \lambda^m\right)(t)\partial_t\sigma(s(t),x) = 0\,.$$

Since $u(0,x) = u_0(x) = \sigma(0,x) = \ell(0,x)$ for $x \in \mathbb{R}^N$, the comparison principle entails that $u \ge \ell$ in $(0,\infty)\times\mathbb{R}^N$. Owing to [1, Théorème 3], the function σ is positive in $(0,\infty)\times\mathbb{R}^N$ and so are ℓ and u. $\square$

An immediate consequence of Lemma 2.1 and classical parabolic regularity is that $u \in C^\infty((0,\infty)\times\mathbb{R}^N)$.

We next turn to estimates on the gradient of solutions to (1.1)–(1.2).

Theorem 2.2. *Consider an initial condition u_0 satisfying (1.3) and let u be the corresponding solution to (1.1)–(1.2). Then*

$$\left|\nabla u^{(m-1)/2}(t,x)\right| \le \sqrt{(3-m-2q)_+ B_0}\ \|u_0\|_\infty^{(q-1)/2} + \sqrt{\frac{B_0}{t}} \tag{2.1}$$

for $(t,x)\in(0,\infty)\times\mathbb{R}^N$, the constant B_0 being defined in (1.6). In addition,

$$0 < u(t,x) \le \left(\|u_0\|_\infty^{1-q} + (q-1)t\right)^{-1/(q-1)}, \quad (t,x)\in(0,\infty)\times\mathbb{R}^N . \tag{2.2}$$

Remark 2.3. According to [23, Theorem A] and [25, Theorem 1] there are self-similar solutions U to (1.1) for every $q \in (1, m+(2/N))$ which read

$$U(t,x) = t^{-1/(q-1)}\phi\left(|x|t^{-(q-m)/2(q-1)}\right), \quad (t,x)\in(0,\infty)\times\mathbb{R}^N .$$

The analysis performed in [23, Proposition 2.3] reveals that $\left(\phi^{(m-1)/2}\right)'(r)$ has a positive limit λ as $r\to\infty$ so that

$$\left|\nabla U^{(m-1)/2}(t,x)\right| = t^{-1/2}\left|\left(\phi^{(m-1)/2}\right)'\left(|x|t^{-(q-m)/2(q-1)}\right)\right| \sim \lambda t^{-1/2} \ \text{ as }\ |x|\to\infty ,$$

thereby indicating that (2.1) is optimal (at least for $q \ge (3-m)/2$).

Proof. The proof of Theorem 2.2 relies on a modified Bernstein technique and the nonlinear diffusion is handled as in [2], see also [34] for positive solutions.

Step 1. We first assume that $u_0 \in W^{1,\infty}(\mathbb{R}^N)$ and there is $\varepsilon > 0$ such that $u_0 \ge \varepsilon$ in $\mathbb{R}^N$. The comparison principle then provides the following lower and upper bounds

$$0 < \left(\varepsilon^{1-q} + (q-1)t\right)^{-1/(q-1)} \le u(t,x) \le \left(\|u_0\|_\infty^{1-q} + (q-1)t\right)^{-1/(q-1)} \le \|u_0\|_\infty \tag{2.3}$$

for $(t,x)\in(0,\infty)\times\mathbb{R}^N$.

Let $\varphi \in C^2(0,\infty)$ be a positive and monotone function and set $u := \varphi(v)$ and $w := |\nabla v|^2$. We infer from (1.1) that

$$\partial_t v = \frac{(\varphi^m)'}{\varphi'}(v)\ \Delta v + \frac{(\varphi^m)''}{\varphi'}(v)\ w - \frac{\varphi^q}{\varphi'}(v), \quad (t,x)\in(0,\infty)\times\mathbb{R}^N . \tag{2.4}$$

We next recall that

$$\Delta w = 2 \sum_{i,j=1}^{N} \left(\partial_i \partial_j v\right)^2 + 2\nabla v \cdot \nabla \Delta v\ . \tag{2.5}$$

It then follows from (2.4) and (2.5) that

$$\begin{aligned}\partial_t w = {} & 2\ \nabla v \cdot \left[\frac{(\varphi^m)'}{\varphi'}(v)\ \nabla \Delta v + \left(\frac{(\varphi^m)'}{\varphi'} \right)'(v)\ \Delta v\ \nabla v + \frac{(\varphi^m)''}{\varphi'}(v)\ \nabla w \right] \\ & + 2 \left(\frac{(\varphi^m)''}{\varphi'} \right)'(v)\ w^2 - 2 \left(\frac{\varphi^q}{\varphi'} \right)'(v)\ w\end{aligned}$$

or equivalently

$$\begin{aligned}\partial_t w = {} & \frac{(\varphi^m)'}{\varphi'}(v) \left[\Delta w - 2 \sum_{i,j=1}^{N} \left(\partial_i \partial_j v\right)^2 \right] + 2 \left(\frac{(\varphi^m)'}{\varphi'} \right)'(v)\ w\ \Delta v \\ & + 2 \frac{(\varphi^m)''}{\varphi'}(v)\ \nabla v \cdot \nabla w + 2 \left(\frac{(\varphi^m)''}{\varphi'} \right)'(v)\ w^2 - 2 \left(\frac{\varphi^q}{\varphi'} \right)'(v)\ w\ .\end{aligned}$$

It also reads

$$\begin{aligned}& \partial_t w - \frac{(\varphi^m)'}{\varphi'}(v) \Delta w - \left[2 \frac{(\varphi^m)''}{\varphi'} + \left(\frac{(\varphi^m)'}{\varphi'} \right)' \right](v)\ \nabla v \cdot \nabla w \\ & \qquad + \mathcal{S} - 2 \left(\frac{(\varphi^m)''}{\varphi'} \right)'(v)\ w^2 + 2 \left(\frac{\varphi^q}{\varphi'} \right)'(v)\ w = 0\ ,\end{aligned} \tag{2.6}$$

where

$$\mathcal{S} := 2 \frac{(\varphi^m)'}{\varphi'}(v) \sum_{i,j=1}^{N} \left(\partial_i \partial_j v\right)^2 + 2 \left(\frac{(\varphi^m)'}{\varphi'} \right)'(v) \left[\frac{1}{2} \nabla v \cdot \nabla w - w \Delta v \right] .$$

We now use Bénilan's trick [2] to obtain

$$\begin{aligned}\mathcal{S} = {} & 2m\varphi^{m-1}(v) \sum_{i,j=1}^{N} \left(\partial_i \partial_j v\right)^2 + 2m(m-1) \left(\varphi^{m-2} \varphi' \right)(v) \left[\sum_{i,j=1}^{N} \partial_i v\ \partial_j v\ \partial_i \partial_j v - w \sum_{i=1}^{N} \partial_i^2 v \right] \\ = {} & 2m\varphi^{m-1}(v) \sum_{i=1}^{N} \left[\left(\partial_i^2 v \right)^2 + (m-1) \frac{\varphi'}{\varphi}(v) \left((\partial_i v)^2 - w \right) \partial_i^2 v \right] \\ & + 2m\varphi^{m-1}(v) \sum_{i \neq j} \left[\left(\partial_i \partial_j v \right)^2 + (m-1) \frac{\varphi'}{\varphi}(v)\ \partial_i v\ \partial_j v\ \partial_i \partial_j v \right] .\end{aligned}$$

We further estimate $\mathcal{S}$ as follows

$$\begin{aligned}
\mathcal{S} = & \, 2m\varphi^{m-1}(v)\sum_{i=1}^{N}\left[\partial_i^2 v + \frac{m-1}{2}\frac{\varphi'}{\varphi}(v)\left((\partial_i v)^2 - w\right)\right]^2 \\
& - 2m\varphi^{m-1}(v)\sum_{i=1}^{N}\frac{(m-1)^2}{4}\left(\frac{\varphi'}{\varphi}\right)^2(v)\ \left((\partial_i v)^2 - w\right)^2 \\
& + 2m\varphi^{m-1}(v)\sum_{i\neq j}\left[\partial_i\partial_j v + \frac{m-1}{2}\frac{\varphi'}{\varphi}(v)\ \partial_i v\ \partial_j v\right]^2 \\
& - 2m\varphi^{m-1}(v)\sum_{i\neq j}\frac{(m-1)^2}{4}\left(\frac{\varphi'}{\varphi}\right)^2(v)\ (\partial_i v)^2\ \left(\partial_j v\right)^2 \\
\geq & -\frac{m(m-1)^2}{2}\left(\varphi^{m-3}(\varphi')^2\right)(v)(N-1)\ w^2\ .
\end{aligned}$$

Consequently, inserting the previous lower bound in (2.6), we find

$$\mathcal{H}w \leq 0\ , \quad (t,x) \in (0,\infty)\times\mathbb{R}^N\ , \tag{2.7}$$

the parabolic operator $\mathcal{H}$ being defined by

$$\mathcal{H}z := \partial_t z - m\varphi^{m-1}(v)\Delta z - \left[2\frac{(\varphi^m)''}{\varphi'} + \left(\frac{(\varphi^m)'}{\varphi'}\right)'\right](v)\ \nabla v\cdot\nabla z + \mathcal{R}_1(v)\ z^2 + \mathcal{R}_2(v)\ z,$$

with

$$\mathcal{R}_1 := -2\left(\frac{(\varphi^m)''}{\varphi'}\right)' - \frac{m(m-1)^2(N-1)}{2}\varphi^{m-3}(\varphi')^2\ , \tag{2.8}$$

$$\mathcal{R}_2 := 2\left(\frac{\varphi^q}{\varphi'}\right)'\ . \tag{2.9}$$

We now choose $\varphi(r) = r^{2/(m-1)}$, $r > 0$. Then

$$\left(\frac{(\varphi^m)''}{\varphi'}\right)(r) = \frac{m(m+1)}{m-1}\ r\ , \quad \left(\varphi^{m-3}(\varphi')^2\right)(r) = \frac{4}{(m-1)^2}\ ,$$

so that

$$\mathcal{R}_1(v) = \frac{2m}{1-m}\ (mN+2-N)\ , \quad \mathcal{R}_2(v) = (2q+m-3)\ v^{2(q-1)/(m-1)}\ .$$

Observe that $mN+2-N > 0$ due to $m > (N-2)_+/N$ so that $\mathcal{R}_1(v) > 0$.

We next divide the analysis into two cases depending on the sign of $2q+m-3$.

(a) If $q \ge (3-m)/2$, it follows that $\mathcal{R}_2(v) \ge 0$. Recalling that the constant B_0 is defined in (1.6), the function

$$W_1(t) := \frac{B_0}{t} , \quad t > 0 ,$$

clearly satisfies

$$\mathcal{H}W_1 \ge 0 \text{ in } (0,\infty) \times \mathbb{R}^N \text{ with } W_1(0) = \infty .$$

We infer from (2.7) and the comparison principle that

$$\left|\nabla u^{(m-1)/2}(t,x)\right| \le \sqrt{\frac{B_0}{t}} , \quad (t,x) \in (0,\infty) \times \mathbb{R}^N ,$$

recalling that $u^{(m-1)/2}$ is well-defined since $u > 0$ by (2.3). We have thus proved (2.1) in that case.

(b) In the complementary case $q \in (1,(3-m)/2)$, set

$$A := (3-m-2q)\|u_0\|_\infty^{q-1} B_0 > 0 \text{ and } W_2(t) := A + \frac{B_0}{t} , \quad t > 0 ,$$

the constant B_0 being defined in (1.6). We infer from (2.3) and the definition of v that

$$\begin{aligned}
\mathcal{H}W_2 &= -\frac{B_0}{t^2} + \frac{1}{B_0}\left(A + \frac{B_0}{t}\right)^2 - (3-m-2q)\left(A + \frac{B_0}{t}\right) u(t,x)^{q-1} \\
&\ge \frac{A^2}{B_0} + \frac{2A}{t} - (3-m-2q)\left(A + \frac{B_0}{t}\right) \|u_0\|_\infty^{q-1} \\
&\ge \frac{A}{B_0}\left(A - (3-m-2q)\|u_0\|_\infty^{q-1} B_0\right) + \frac{2}{t}\left(A - \frac{(3-m-2q)B_0}{2}\|u_0\|_\infty^{q-1}\right) \\
&\ge 0 .
\end{aligned}$$

Thus

$$\mathcal{H}W_2 \ge 0 \text{ in } (0,\infty) \times \mathbb{R}^N \text{ with } W_2(0) = \infty .$$

The comparison principle and (2.7) imply that

$$w(t,x) \le W_2(t) , \quad (t,x) \in (0,\infty) \times \mathbb{R}^N .$$

Combining this estimate with the subadditivity of the square root gives (2.1).

Step 2. We now consider u_0 satisfying (1.3) and denote the corresponding solution to (1.1)–(1.2) by u. For $\varepsilon > 0$, classical approximation arguments allow us to construct a family of functions $(u_{0,\varepsilon})_\varepsilon$ such that $\varepsilon < u_{0,\varepsilon} < \|u_0\|_\infty + 2\varepsilon$, $u_{0,\varepsilon} \in W^{1,\infty}(\mathbb{R}^N)$, and $(u_{0,\varepsilon})_\varepsilon$ converges a.e.

in $\mathbb{R}^N$ towards u_0 as $\varepsilon \to 0$. Denoting the corresponding solution to (1.1)–(1.2) with initial condition $u_{0,\varepsilon}$ by u_ε, it follows from Step 1 that u_ε satisfies (2.1). Classical stability results and [28, Theorem 2.1] guarantee that $(u_\varepsilon)_\varepsilon$ converges towards u uniformly on compacts subsets of $(0,\infty)\times\mathbb{R}^N$ as $\varepsilon \to 0$. Since $u > 0$ in $(0,\infty)\times\mathbb{R}^N$ by Lemma 2.1, the validity of the estimate (2.1) for u is a consequence of the estimate (2.1) for u_ε and the upper bound on $\|u_{0,\varepsilon}\|_\infty$.

Finally, the bounds (2.2) readily follow from Lemma 2.1 and (2.3). □

Thanks to the just established gradient estimate, we can improve the positivity statement of Lemma 2.1 and prove Theorem 1.1, which is now a simple consequence of Lemma 2.1.

Proof of Theorem 1.1. We infer from the positivity of u (see Lemma 2.1) and (2.1) that, for $(t,x) \in (0,\infty)\times\mathbb{R}^N$,

$$\begin{aligned}
u^{(m-1)/2}(t,x) &\le u^{(m-1)/2}(t,0) + \left\| \nabla u^{(m-1)/2}(t) \right\|_\infty |x| \\
&\le u^{(m-1)/2}(t,0) + \sqrt{B_0}\left(\sqrt{(3-m-2q)_+}\|u_0\|_\infty^{(q-1)/2} + t^{-1/2} \right) |x| \\
&\le \ell_u(t)^{(m-1)/2}(1+|x|) \ .
\end{aligned}$$

We thus obtain the estimate (1.5) in Theorem 1.1, since $m<1$. □

We end up this section by reporting a further consequence of Theorem 2.2, which is a somewhat less precise version of Theorem 1.1 but will be needed in the sequel.

Proposition 2.4. *Consider $q>1$ and an initial condition u_0 satisfying (1.3) and let u be the corresponding solution to (1.1)–(1.2). Given $\varepsilon \in (0,1)$, there are $\tau_\varepsilon \ge 1/\varepsilon$ and $\kappa_\varepsilon \ge 1/\varepsilon$ depending on N, m, q, u_0, and ε such that*

$$u^{m-1}(\tau_\varepsilon, x) \le \kappa_\varepsilon + \varepsilon |x|^2 \ , \quad x \in \mathbb{R}^N \ . \tag{2.10}$$

Proof. Let $(t,x) \in (0,\infty)\times\mathbb{R}^N$. We infer from (2.1), (2.2), and the positivity of u established in Lemma 2.1 that

$$\begin{aligned}
u^{(m-1)/2}(t,x) &\le u^{(m-1)/2}(t,0) + \|\nabla u^{(m-1)/2}(t)\|_\infty |x| \\
&\le u^{(m-1)/2}(t,0) + C_1 \left[\left\| u\left(\frac{t}{2}\right)\right\|_\infty^{(q-1)/2} + \sqrt{\frac{2}{t}} \right] |x| \\
&\le u^{(m-1)/2}(t,0) + C_1 \left[\sqrt{\frac{2}{(q-1)t}} + \sqrt{\frac{2}{t}} \right] |x|,
\end{aligned}$$

for some $C_1>0$ depending only on N, m, and q, hence

$$\begin{aligned}
u^{m-1}(t,x) &\le 2u^{m-1}(t,0) + 4C_1^2 \left[\frac{2}{(q-1)t} + \frac{2}{t} \right] |x|^2 \\
&\le 2u^{m-1}(t,0) + \frac{8qC_1^2}{(q-1)t}\, |x|^2 \ .
\end{aligned}$$

It follows from the previous estimate that there is $t_\varepsilon > 1/\varepsilon$ depending only on N, m, q, and ε such that

$$u^{m-1}(t,x) \le 2u^{m-1}(t,0) + \varepsilon|x|^2 \,, \quad (t,x) \in (t_\varepsilon, \infty) \times \mathbb{R}^N \,.$$

Using once more (2.2) together with $m < 1$ gives the existence of $\tau_\varepsilon > t_\varepsilon$ such that $\kappa_\varepsilon := 2u^{m-1}(\tau_\varepsilon, 0) > 1/\varepsilon$ and completes the proof. □

3. Subsolutions and supersolutions

We restrict our analysis to the critical case $q = q_*$ from now on. Consider an initial condition u_0 satisfying (1.3) and let u be the corresponding solution to the Cauchy problem (1.1)–(1.2). Fix $T > 0$. We perform the change to self-similar variables

$$\begin{cases} v(s,y) := \big[(T+t)\log(T+t)\big]^{1/(q-1)} u(t,x), \\ y := \dfrac{x}{(T+t)^{1/N(q-1)}(\log(T+t))^{(1-m)/2(q-1)}}, \quad s := \log(T+t), \end{cases} \tag{3.1}$$

and notice that (1.1) implies that v solves

$$\partial_s v - \mathcal{L}v = 0 \quad \text{in} \quad (\log T, \infty) \times \mathbb{R}^N \,, \tag{3.2}$$

with

$$v(\log T, y) = (T\log T)^{1/(q-1)} u_0\left(yT^{1/N(q-1)}(\log T)^{(1-m)/2(q-1)}\right) \,, \qquad y \in \mathbb{R}^N \,,$$

where $\mathcal{L}$ is the following nonlinear differential operator:

$$\begin{aligned} \mathcal{L}z := {} & \Delta z^m + \frac{1}{N(q-1)}(Nz + y\cdot\nabla z) \\ & + \frac{1}{(q-1)s}\left(z + \frac{1-m}{2} y\cdot\nabla z\right) - \frac{z^q}{s}, \end{aligned} \tag{3.3}$$

with $q = q_* = m + 2/N$.

The aim of this section is to construct subsolutions and supersolutions to (3.2) having the correct time scale and a form similar to the expected asymptotic profile.

Construction of subsolutions. We recall that the Barenblatt profiles are defined by

$$\sigma_A(y) = \left(A + B_0|y|^2\right)^{1/(m-1)}, \quad B_0 = \frac{1-m}{2m(Nm-N+2)}, \tag{3.4}$$

where $A > 0$ is a free parameter (to be chosen later according to our aims) and $B_0 > 0$, since $m_c < m < 1$. With the above notations, we have the following result:

Lemma 3.1. *There is $A_{sub} > 0$ depending only on N and m such that:*

(i) *If $m \in [(N-1)/N, 1)$, then*

$$w_A(s, y) := \sigma_A(y) , \quad (s, y) \in (0, \infty) \times \mathbb{R}^N ,$$

is a subsolution to (3.2) in $(0, \infty) \times \mathbb{R}^N$ for any $A \geq A_{sub}$.

(ii) *If $m_c < m < (N-1)/N$, the function*

$$w_A(s, y) := \sigma_A(y) \left(1 - \frac{\gamma}{s}\right), \quad (s, y) \in (0, \infty) \times \mathbb{R}^N , \quad \gamma := \frac{1}{2(1-m)} > 0, \tag{3.5}$$

is a subsolution to (3.2) in $(s_0, \infty) \times \mathbb{R}^N$ for $A \geq A_{sub}$ and

$$s_0 := \max \left\{ \frac{4q}{1-m}, \frac{2^{m+2} q}{q-1} \right\} .$$

Proof. (i) It is easy to check that

$$\begin{aligned} \Delta\sigma_A^m(y) &= \frac{4B_0 m}{(m-1)^2} \frac{B_0|y|^2}{A + B_0|y|^2} \sigma_A(y) + \frac{2NB_0 m}{m-1} \sigma_A(y) \\ &= \left[\frac{4B_0 m}{(m-1)^2} + \frac{2NB_0 m}{m-1} \right] \sigma_A(y) - \frac{4B_0 m}{(m-1)^2} \frac{A}{A + B_0|y|^2} \sigma_A(y) , \end{aligned}$$

and

$$\frac{1}{N(q-1)} \left(N\sigma_A(y) + y \cdot \nabla\sigma_A(y) \right) = -\frac{\sigma_A(y)}{1-m} + \frac{2\sigma_A(y)}{(1-m)(mN - N + 2)} \frac{A}{A + B_0|y|^2} ,$$

and moreover

$$\sigma_A(y) + \frac{1-m}{2} y \cdot \nabla\sigma_A(y) = \frac{A}{A + B_0|y|^2} \sigma_A(y) .$$

Consequently, by direct calculation, we find that

$$\begin{aligned} &\frac{1}{\sigma_A(y)} \left(\partial_s\sigma_A - \mathcal{L}\sigma_A \right)(y) \\ &= \frac{1}{1-m} - \frac{2B_0 m(mN - N + 2)}{(1-m)^2} \\ &+ \frac{2}{mN - N + 2} \left[-\frac{1}{1-m} + \frac{2B_0 m(mN - N + 2)}{(1-m)^2} \right] \frac{A}{A + B_0|y|^2} \\ &+ \frac{1}{(q-1)s} \frac{1}{A + B_0|y|^2} \left[(q-1) \left(A + B_0|y|^2 \right)^{(q-1)/(m-1)+1} - A \right] \\ &= \frac{1}{(q-1)s} \frac{1}{A + B_0|y|^2} \left[(q-1) \left(A + B_0|y|^2 \right)^{(q-1)/(m-1)+1} - A \right], \end{aligned}$$

after noticing that (3.4) ensures

$$\frac{1}{1-m} - \frac{2B_0 m(mN-N+2)}{(1-m)^2} = 0 \,.$$

Since $(N-1)/N \le m < 1$, we remark that

$$\frac{q-1}{m-1} + 1 = \frac{2}{m-1}\left(m - \frac{N-1}{N}\right) \le 0,$$

hence

$$\begin{aligned}\frac{1}{\sigma_A(y)}\left(\partial_s\sigma_A - \mathcal{L}\sigma_A\right)(y) &\le \frac{1}{(q-1)(A+B_0|y|^2)s}\left[(q-1)A^{(q+m-2)/(m-1)} - A\right] \\ &= \frac{A^{(q+m-2)/(m-1)}}{(q-1)(A+B_0|y|^2)s}\left[(q-1) - A^{(q-1)/(1-m)}\right] \le 0,\end{aligned}$$

for A sufficiently large, which ends the proof of (i).

(ii) Let w_A be defined in (3.5) and set $\xi = B_0|y|^2$. According to [31, Proof of Lemma 3.2], we have, in our notation, that

$$\begin{aligned}\left(\partial_s w_A - \mathcal{L}w_A\right)(s,y) = {}& \frac{1}{N(q-1)}\left[NA + \frac{N(1-m)-2}{1-m}\xi\right]\frac{\sigma_A(y)}{A+\xi}\left[\left(1-\frac{\gamma}{s}\right)^m - \left(1-\frac{\gamma}{s}\right)\right] \\ &+ \left(1-\frac{\gamma}{s}\right)^q \frac{\sigma_A(y)^q}{s} - \frac{\sigma_A(y)}{(q-1)s}\left(\frac{A}{A+\xi}\right)\left(1-\frac{\gamma}{s}\right) + \frac{\gamma\sigma_A(y)}{s^2},\end{aligned}$$

hence, after some easy rearranging,

$$\begin{aligned}\frac{s}{\sigma_A(y)}\left(\partial_s w_A - \mathcal{L}w_A\right)(s,y) = {}& \frac{s}{q-1}\left(\frac{A}{A+\xi}\right)\left(1-\frac{\gamma}{s}\right)^m\left[1-\left(1-\frac{\gamma}{s}\right)^{1-m}\right] \\ &- \frac{s}{1-m}\left(\frac{\xi}{A+\xi}\right)\left(1-\frac{\gamma}{s}\right)^m\left[1-\left(1-\frac{\gamma}{s}\right)^{1-m}\right] \\ &+ \left(1-\frac{\gamma}{s}\right)^q\sigma_A(y)^{q-1} + \frac{\gamma}{s}\left[1+\frac{A}{(q-1)(A+\xi)}\right] \\ &- \frac{A}{(q-1)(A+\xi)}.\end{aligned} \tag{3.6}$$

We next note that

$$1-\left(1-\frac{\gamma}{s}\right)^{1-m} = (1-m)\int_{-\gamma/s}^{0}(1+r)^{-m}\,dr,$$

hence

$$(1-m)\frac{\gamma}{s} \le 1-\left(1-\frac{\gamma}{s}\right)^{1-m} \le (1-m)\left(1-\frac{\gamma}{s}\right)^{-m}\frac{\gamma}{s}.$$

Using the previous inequalities to estimate the first two terms of (3.6) and the choice of γ, we get

$$\begin{aligned}\frac{s}{\sigma_A(y)}\left(\partial_s w_A - \mathcal{L}w_A\right)(s,y) &\le \frac{(1-m)\gamma}{q-1}\left(\frac{A}{A+\xi}\right) - \gamma\left(1-\frac{\gamma}{s}\right)^m \frac{\xi}{A+\xi} \\ &\quad + \sigma_A(y)^{q-1} + \frac{\gamma q}{(q-1)s} - \frac{1}{q-1}\left(\frac{A}{A+\xi}\right) \\ &= (A+\xi)^{(q-1)/(m-1)} + \frac{\gamma q}{(q-1)s} - \frac{1}{2(q-1)}\left(\frac{A}{A+\xi}\right) \\ &\quad - \gamma\left(1-\frac{\gamma}{s}\right)^m \frac{\xi}{A+\xi}. \end{aligned} \tag{3.7}$$

Since $m \in (m_c, (N-1)/N)$, we notice that

$$0 < \frac{q-1}{m-1} + 1 = \frac{2-m-q}{1-m} < 1\,. \tag{3.8}$$

Let $R > 0$ be chosen later. We split the analysis into two regions according to the relative position of ξ and R.

Case 1. If $\xi \in [0, R]$, then we infer from (3.7) that

$$\begin{aligned}\frac{s}{\sigma_A(y)}\left(\partial_s w_A - \mathcal{L}w_A\right)(s,y) &\le \frac{1}{A+\xi}\left[(A+\xi)^{(2-m-q)/(1-m)} - \frac{A}{2(q-1)}\right] + \frac{\gamma q}{(q-1)s} \\ &\le \frac{1}{A+\xi}\left[(A+R)^{(2-m-q)/(1-m)} - \frac{A}{2(q-1)}\right] + \frac{\gamma q}{(q-1)s}\,.\end{aligned}$$

Taking into account (3.8), we realize that, if A is large enough, we can choose R such that

$$(A+R)^{(2-m-q)/(1-m)} \le \frac{A}{4(q-1)}\,. \tag{3.9}$$

With such a choice of R, we deduce

$$\begin{aligned}\frac{s}{\sigma_A(y)}\left(\partial_s w_A - \mathcal{L}w_A\right)(s,y) &\le -\frac{A}{4(q-1)(A+\xi)} + \frac{\gamma q}{(q-1)s} \\ &\le \frac{\gamma q}{(q-1)s} - \frac{A}{4(q-1)(A+R)} \le 0,\end{aligned}$$

provided

$$s \ge 4q\gamma\frac{A+R}{A}\,. \tag{3.10}$$

Case 2. If $\xi \geq R$ and $s \geq 2\gamma$, then $(1-\gamma/s)^m \geq 2^{-m}$ and we infer from (3.7) and (3.8) that

$$\begin{aligned}
&\frac{s}{\sigma_A(y)} \left(\partial_s w_A - \mathcal{L} w_A\right)(s,y) \\
&\quad \leq (A+\xi)^{(q-1)/(m-1)} + \frac{\gamma q}{(q-1)s} - \gamma \left(1-\frac{\gamma}{s}\right)^m \frac{\xi}{A+\xi} \\
&\quad \leq \frac{(A+\xi)^{(2-m-q)/(1-m)}}{A+\xi} + \frac{\gamma q}{(q-1)s} - \frac{\gamma}{2^m}\frac{\xi}{A+\xi} \\
&\quad \leq \frac{1}{A+\xi}\left[(A+\xi)^{(2-m-q)/(1-m)} - \frac{\gamma}{2^{m+1}}\xi\right] \\
&\quad + \frac{\gamma q}{(q-1)s} - \frac{\gamma\xi}{2^{m+1}(A+\xi)} \\
&\quad \leq \frac{1}{A+\xi}\left[A^{(2-m-q)/(1-m)} + \xi^{(2-m-q)/(1-m)} - \frac{\gamma}{2^{m+1}}\xi\right] \\
&\quad + \frac{\gamma q}{(q-1)s} - \frac{\gamma R}{2^{m+1}(A+R)} \\
&\quad \leq \frac{1}{A+\xi}\left[A^{(2-m-q)/(1-m)} + \left(R^{(q-1)/(m-1)} - \frac{\gamma}{2^{m+1}}\right)\xi\right] \\
&\quad + \gamma\left[\frac{q}{(q-1)s} - \frac{R}{2^{m+1}(A+R)}\right].
\end{aligned} \tag{3.11}$$

Choosing now $R > 0$ and s such that

$$R^{(q-1)/(m-1)} \leq \frac{\gamma}{2^{m+2}} \quad \text{and} \quad \frac{2^{m+1}q(A+R)}{(q-1)R} \leq s\,, \tag{3.12}$$

we derive from (3.11) that

$$\begin{aligned}
\frac{s}{\sigma_A(y)} \left(\partial_s w_A - \mathcal{L} w_A\right)(s,y) &\leq \frac{1}{A+\xi}\left[A^{(2-m-q)/(1-m)} - \frac{\gamma}{2^{m+2}}\xi\right] \\
&\leq \frac{1}{A+\xi}\left[A^{(2-m-q)/(1-m)} - \frac{\gamma}{2^{m+2}}R\right] \leq 0,
\end{aligned}$$

if

$$A^{(2-m-q)/(1-m)} \leq 2^{-(m+2)}\gamma R\,. \tag{3.13}$$

Gathering the two cases, we have thus shown that $(\partial_s w_A - \mathcal{L} w_A)(s,y) \leq 0$ for $y \in \mathbb{R}^N$ provided the conditions (3.9), (3.10), (3.12), (3.13), and $s \geq 2\gamma$ are satisfied simultaneously by R, A, and s. We now let $R = A$, so that these conditions become

$$(2A)^{(2-m-q)/(1-m)} \leq \frac{A}{4(q-1)}, \qquad A^{(q-1)/(m-1)} \leq \frac{\gamma}{2^{m+2}}$$

or equivalently

$$A^{(q-1)/(m-1)} \leq \min\left\{\frac{2^{(m+q-2)/(1-m)}}{4(q-1)}, \frac{\gamma}{2^{m+2}}\right\}, \tag{3.14}$$

and

$$s \geq s_0 := \max\left\{8\gamma q, 2\gamma, \frac{2^{m+2}q}{q-1}\right\}.$$

Since $(q-1)/(m-1) < 0$, we notice that (3.14) is satisfied provided A is sufficiently large. We have thereby shown that w_A is a subsolution to (3.2) in $(s_0, \infty) \times \mathbb{R}^N$ for A large enough. □

Comparison with subsolutions. We show now that the subsolutions constructed above are indeed useful to investigate the large time asymptotics of (1.1)–(1.2). Let u be the solution to the Cauchy problem (1.1)–(1.2) with initial condition u_0 satisfying (1.3) and exponents (m, q) given by (1.4). Then the rescaled function v obtained from u via the transformation (3.1) enjoys the following property:

Proposition 3.2. *Let u_0 be an initial condition satisfying (1.3) and denote the corresponding solution to (1.1)–(1.2) by u. Let v be its rescaled version defined by (3.1) and consider $T \geq e^{s_0}$. There are $A_T \geq A_{sub}$, $s_T > 0$, and $\gamma_T > 0$ depending only on N, m, u_0, and T such that*

$$v(s, y) \geq \left(1 - \frac{\gamma_T}{s}\right)\left(1 - \gamma_T e^{-s}\right)^{1/(1-m)} w_{A_T}(s, y), \quad (s, y) \in (s_T, \infty) \times \mathbb{R}^N, \tag{3.15}$$

where w_{A_T} is defined in Lemma 3.1.

Proof. For $t \geq 1$ we define

$$a_t := (t \log t)^{1/(q-1)}, \quad b_t := t^{1/N(q-1)} (\log t)^{(1-m)/2(q-1)},$$

and

$$c_t := \begin{cases} 1 & \text{if } m \in \left[\dfrac{N-1}{N}, 1\right), \\ 1 - \dfrac{1}{2(1-m)\log t} & \text{if } m \in \left(\dfrac{N-2}{N}, \dfrac{N-1}{N}\right). \end{cases}$$

Fix $\varepsilon_T \in (0, B_0 c_T^{m-1}/T)$ such that $c_T^{1-m} a_T^{m-1} \geq \varepsilon_T A_{sub}$. According to Proposition 2.4 there are $\tau_T \geq 1/\varepsilon_T$ and $\kappa_T \geq 1/\varepsilon_T$ such that

$$u^{m-1}(\tau_T, x) \leq \kappa_T + \varepsilon_T |x|^2, \quad x \in \mathbb{R}^N. \tag{3.16}$$

Define now the function V by

$$V(\log(T+t), y) := a_{T+t} u(t + \tau_T, y b_{T+t}), \quad (t, y) \in [0, \infty) \times \mathbb{R}^N.$$

Note that V is defined by (3.1) with $u(\cdot+\tau_T)$ instead of u and thus satisfies

$$\partial_s V - \mathcal{L}V = 0 \quad \text{in} \quad (\log T, \infty) \times \mathbb{R}^N \tag{3.17}$$

with $V(\log T, y) = a_T u(\tau_T, y b_T)$ for $y \in \mathbb{R}^N$. Moreover, thanks to (3.16),

$$V^{m-1}(\log T, y) = a_T^{m-1} u^{m-1}(\tau_T, y b_T) \le a_T^{m-1}\kappa_T + \varepsilon_T a_T^{m-1} b_T^2 |y|^2 .$$

Since $a_T^{m-1} b_T^2 = T$ and $\varepsilon_T c_T^{1-m} T \le B_0$, we obtain

$$V^{m-1}(\log T, y) \le c_T^{m-1}\left(c_T^{1-m} a_T^{m-1}\kappa_T + \varepsilon_T c_T^{1-m} T|y|^2\right) \le c_T^{m-1}\left(c_T^{1-m} a_T^{m-1}\kappa_T + B_0|y|^2\right) .$$

Recalling that $w_{A_T}(\log T, y) = c_T \sigma_{A_T}(y)$ and $m<1$, we end up with

$$V(\log T, y) \ge w_{A_T}(\log T, y) , \quad y \in \mathbb{R}^N , \quad \text{with} \;\; A_T := c_T^{1-m} a_T^{m-1}\kappa_T .$$

Now the properties of κ_T and ε_T ensure that $A_T \ge c_T^{1-m} a_T^{m-1}/\varepsilon_T \ge A_{sub}$, so that w_{A_T} is a subsolution to (3.2) in $(\log T, \infty) \times \mathbb{R}^N$ by Lemma 3.1. Taking into account (3.17), the comparison principle entails that

$$V(s, y) \ge w_{A_T}(s, y) , \quad (s, y) \in [\log T, \infty) \times \mathbb{R}^N .$$

Equivalently

$$u(t+\tau_T, x) \ge \frac{c_{T+t}}{a_{T+t}}\left(A_T + B_0 \frac{|x|^2}{b_{T+t}^2}\right)^{1/(m-1)} , \quad (t, x) \in [0, \infty) \times \mathbb{R}^N .$$

Recalling that $a_{T+t}^{m-1} b_{T+t}^2 = T+t$ and $m<1$ we realize that

$$\begin{aligned}
u(t+\tau_T, x) &\ge c_{T+t}\frac{b_{T+t}^{2/(1-m)}}{a_{T+t}}\left(A_T b_{T+t}^2 + B_0|x|^2\right)^{1/(m-1)} \\
&\ge c_{T+t}(T+t)^{1/(1-m)}\left(A_T b_{T+\tau_T+t}^2 + B_0|x|^2\right)^{1/(m-1)} \\
&\ge c_{T+t}\left(\frac{T+t}{T+\tau_T+t}\right)^{1/(1-m)} \frac{b_{T+\tau_T+t}^{2/(1-m)}}{a_{T+\tau_T+t}}\left(A_T b_{T+\tau_T+t}^2 + B_0|x|^2\right)^{1/(m-1)} \\
&\ge c_{T+t}\left(\frac{T+t}{T+\tau_T+t}\right)^{1/(1-m)} \frac{1}{a_{T+\tau_T+t}} w_{A_T}\left(\log(T+\tau_T+t), \frac{x}{b_{T+\tau_T+t}}\right) .
\end{aligned}$$

Since

$$c_{T+t} \ge 1 - \frac{1}{2(1-m)\log(T+t)} \ge 1 - \frac{\log(T+\tau_T)}{2(1-m)\log(T)}\,\frac{1}{\log(T+\tau_T+t)} , \quad t \ge 0 ,$$

and

$$\frac{T+t}{T+\tau_T+t} = 1 - \frac{\tau_T}{T+\tau_T+t} , \quad t \ge 0 ,$$

we end up with

$$u(t,x) \ge \left(1 - \frac{\gamma_T}{\log(T+t)}\right) \left(1 - \frac{\gamma_T}{T+t}\right)^{1/(1-m)} \frac{1}{a_{T+t}} w_{A_T} \left(\log(T+t), \frac{x}{b_{T+t}}\right)$$

for $(t,x) \in (\tau_T, \infty) \times \mathbb{R}^N$, where

$$\gamma_T := \max\left\{\tau_T, \frac{2\log(T+\tau_T)}{(1-m)\log T}\right\} .$$

The inequality (3.15) then readily follows after setting $s_T := \log(T+\tau_T)$ and using (3.1). □

Construction of supersolutions. A class of supersolutions to (3.2) is identified in [31]. Using our notation, we recall in the next result the outcome of the construction performed in [31, Lemma 3.2].

Lemma 3.3. *Define*

$$z_A(s,y) := \sigma_A(y) \left[1 + \frac{1}{s}(A + B_0|y|^2)^\delta\right] , \quad (s,y) \in (0,\infty) \times \mathbb{R}^N , \tag{3.18}$$

where $A > 0$ and $\delta := 1/(1-m) - (k/2)$, the parameter k being defined in (1.7) and σ_A and B_0 in (3.4). There are $s_1 > 0$ and $A_{sup} \in (0,1)$ depending only on N and m such that z_A is a supersolution to (3.2) in $(s_1,\infty) \times \mathbb{R}^N$ for $A \in (0, A_{sup})$.

The statement given in [31, Lemma 3.2] is somewhat less precise with respect to the dependence of s_1, but a careful inspection of the proof allows one to check that it does not depend on $A \in (0,1)$.

Proposition 3.4. *Let u_0 be an initial condition satisfying (1.3) and (1.7) and denote the corresponding solution to (1.1)–(1.2) by u. Let v be its rescaled version defined by (3.1). There exists $T(K) > e^{s_1}$ depending only on N, m, and K, with K given in (1.7), such that, given $T \ge T(K)$, there is $A_T' \in (0, A_{sup})$ depending only on N, m, u_0, and T such that*

$$v(s,y) \le z_{A_T'}(s,y) , \qquad (s,y) \in (\log T, \infty) \times \mathbb{R}^N . \tag{3.19}$$

Proof. Let $T(K) \ge e^{s_1}$ be such that

$$(2B_0)^{k/2} K \le T^{(k-N)/N(q-1)} (\log T)^{(k(1-m)-2q)/2(q-1)} \quad \text{for all} \quad T \ge T(K) , \tag{3.20}$$

the existence of $T(K)$ being guaranteed by the inequality $k > N$. Consider $T \ge T(K)$ and let $A > 0$ be specified later. On the one hand, if $y \in \mathbb{R}^N$ satisfies $|y|^2 \ge A/B_0$, we deduce from (1.7), (3.18), and (3.20) that

$$z_A(\log T, y) \geq \frac{\left(A + B_0|y|^2\right)^{-k/2}}{\log T} \geq \frac{(2B_0)^{-k/2}}{\log T}|y|^{-k}$$
$$\geq K a_T \left(|y| b_T\right)^{-k} \geq a_T u_0(y b_T) = v(\log T, y) ,$$

where

$$a_T = (T \log T)^{1/(q-1)} , \quad b_T = T^{1/N(q-1)} (\log T)^{(1-m)/2(q-1)} .$$

On the other hand, if $y \in \mathbb{R}^N$ satisfies $|y|^2 < A/B_0$, then

$$v(\log T, y) \leq a_T \|u_0\|_\infty \quad \text{and} \quad z_A(\log T, y) \geq \left(A + B_0|y|^2\right)^{1/(m-1)} \geq (2A)^{1/(m-1)} .$$

Therefore, if $A \leq (a_T \|u_0\|_\infty)^{m-1}/2$, then

$$v(\log T, y) \leq z_A(\log T, y) , \quad y \in B_{\sqrt{A/B_0}}(0) .$$

We have thus shown that, if $T \geq T(K)$ and $A \leq (a_T \|u_0\|_\infty)^{m-1}/2$, then

$$v(\log T, y) \leq z_A(\log T, y) , \quad y \in \mathbb{R}^N .$$

Pick now $A'_T \in (0, A_{sup}) \cap \left(0, (a_T \|u_0\|_\infty)^{m-1}/2\right]$. The above analysis guarantees that $v(\log T) \leq z_{A'_T}(\log T)$ in $\mathbb{R}^N$, while v and $z_{A'_T}$ are a solution and a supersolution to (3.2), respectively, by (3.2) and Lemma 3.3. Applying the comparison principle completes the proof of Proposition 3.4. □

4. Convergence

The convergence (1.8) is now a consequence of the previous analysis and the stability technique developed in [11,12], the latter having already been used in [31] for (1.1)–(1.2). We briefly recall it for the sake of completeness in the Appendix and sketch its application in our framework below.

We fix an initial condition u_0 satisfying (1.3) and (1.7) and $T \geq 1 + e^{s_0} + T(K)$, the parameters s_0 and $T(K)$ being defined in Lemma 3.1 and Proposition 3.4, respectively. We denote the corresponding solution to (1.1)–(1.2) by u and define its rescaled version v by (3.1). We set $A_1 := A_T \geq A_{sub}$ and $A_2 := A'_T \in (0, A_{sup})$ where A_T and A'_T are defined in Proposition 3.2 and Proposition 3.4, respectively, and consider the complete metric space

$$X := \left\{ \vartheta_0 \in L^1(\mathbb{R}^N) \ : \ w_{A_1}(\log T, y) \leq \vartheta_0(y) \leq z_{A_2}(\log T, y) , \ \ y \in \mathbb{R}^N \right\} \tag{4.1}$$

endowed with the distance induced by the L^1-norm. Recall that w_{A_1} and z_{A_2} are defined in Lemma 3.1 and (3.18), respectively.

Let $\vartheta_0 \in X$ and consider the solution ϑ to

$$\partial_s \vartheta - \mathcal{L}\vartheta = 0 \ \text{ in } \ (\log T, \infty) \times \mathbb{R}^N , \quad \vartheta(\log T) = \vartheta_0 \ \text{ in } \ \mathbb{R}^N , \tag{4.2}$$

where $\mathcal{L}$ is defined in (3.3). Observe that ϑ is actually given by

$$\vartheta(s,y) = \left(se^s\right)^{1/(q-1)} u_\vartheta \left(e^s - T, s^{(1-m)/2(q-1)} e^{s/N(q-1)} y\right) \tag{4.3}$$

for $(s,y) \in (\log T, \infty) \times \mathbb{R}^N$, where u_ϑ denotes the unique solution to (1.1) with initial condition

$$u_\vartheta(0,x) = \frac{1}{(T\log T)^{1/(q-1)}} \vartheta_0 \left(\frac{x}{T^{1/N(q-1)}(\log T)^{(1-m)/2(q-1)}}\right), \quad x \in \mathbb{R}^N,$$

which exists as $\vartheta_0 \in X \subset L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. This formula guarantees in particular the existence and uniqueness of ϑ. Furthermore, ϑ enjoys several useful properties which we collect now. First, since $T \ge \max\{e^{s_0}, e^{s_1}\}$ with s_1 defined in Lemma 3.3, we infer from Lemma 3.1, Lemma 3.3, and the comparison principle that

$$w_{A_1}(\log T, y) \le w_{A_1}(s,y) \le \vartheta(s,y) \le z_{A_2}(s,y) \le z_{A_2}(\log T, y) \tag{4.4}$$

for $(s,y) \in (\log T, \infty) \times \mathbb{R}^N$. Consequently,

$$\vartheta(s) \in X, \qquad s \ge \log T. \tag{4.5}$$

It next follows from (4.3) and Theorem 2.2 that

$$\begin{aligned}
\left|\nabla\vartheta^{(m-1)/2}(s,y)\right| &= e^{s/2} \left|\nabla u_\vartheta^{(m-1)/2}\left(e^s - T, s^{(1-m)/2(q-1)} e^{s/N(q-1)} y\right)\right| \\
&\le e^{s/2}\left[\sqrt{(3-m-2q)_+ B_0} \left\|u_\vartheta\left(\frac{e^s}{2} - T\right)\right\|_\infty^{(q-1)/2} + \sqrt{2B_0 e^{-s}}\right] \\
&\le e^{s/2} \sqrt{\frac{(3-m-2q)_+ B_0}{(s-\log 2)e^{s-\log 2}}} \, \|\vartheta(s-\log 2)\|_\infty^{(q-1)/2} + \sqrt{2B_0}.
\end{aligned}$$

We then use (4.4) and the boundedness of z_{A_2} to conclude that

$$\left|\nabla\vartheta^{(m-1)/2}(s,y)\right| \le C(T), \qquad (s,y) \in (\log T, \infty) \times \mathbb{R}^N, \tag{4.6}$$

for some positive constant $C(T)$ depending only on N, m, u_0, and T. Since

$$\nabla\vartheta = \frac{2}{m-1}\vartheta^{(3-m)/2}\nabla\vartheta^{(m-1)/2} \quad \text{and} \quad \nabla\vartheta^m = \frac{2m}{m-1}\vartheta^{(m+1)/2}\nabla\vartheta^{(m-1)/2},$$

the following bounds are a straightforward consequence of (4.4), (4.6), and the boundedness of z_{A_2}:

$$|\nabla\vartheta(s,y)| + \left|\nabla\vartheta^m(s,y)\right| \le C(T), \qquad (s,y) \in (\log T, \infty) \times \mathbb{R}^N. \tag{4.7}$$

We then infer from [13,17,29] and (4.7) that, given $R > 0$, there are $\zeta > 0$ and $C(R,\zeta) > 0$ depending only on N, m, and T such that, for $s_2 > s_1 \geq \log T$ satisfying $|s_2 - s_1| \leq \zeta$, there holds:

$$|\vartheta(s_2, y) - \vartheta(s_1, y)| \leq C(R,\zeta)\sqrt{|s_2 - s_1|}\ , \quad y \in B_R(0)\ . \tag{4.8}$$

Combining the time continuity of u_ϑ in $L^1(\mathbb{R}^N)$ with (4.4) and (4.8) gives

$$\vartheta \in C([\log T, \infty); L^1(\mathbb{R}^N))\ . \tag{4.9}$$

Collecting the information obtained so far on the solutions ϑ to (4.2) associated to initial data in X we realize that we are in a position to check the validity of the three assumptions **(H1)–(H3)** required to apply the stability theory from [12] which are recalled in the Appendix. In our setting the non-autonomous operator $\mathcal{L}$ is defined in (3.3) with the metric space X introduced in (4.1), its autonomous counterpart being

$$Lz := \Delta z^m + \frac{1}{N(q-1)}(Nz + y \cdot \nabla z)\ . \tag{4.10}$$

The evolution equation

$$\partial_s \Phi - L\Phi = 0 \quad \text{in} \quad (\log T, \infty) \times \mathbb{R}^N\ , \tag{4.11}$$

is related to the fast diffusion equation (1.9) by a (self-similar) change of variables. The bounds (4.4), (4.6), (4.7), and (4.8) ensure that both **(H1)** and **(H2)** are satisfied, after noticing that

$$|L\vartheta(s,y) - \mathcal{L}\vartheta(s,y)| \leq \frac{C}{s}\ , \qquad (s,y) \in (\log T, \infty) \times \mathbb{R}^N\ .$$

As for **(H3)**, it involves only to the fast diffusion equation (1.9) and its self-similar form (4.11) and we refer to [12,32] for its proof.

We may thus apply Theorem A.1 below to deduce that the ω-limit set of any solution ϑ to (4.2) starting from an initial condition in X is a subset of

$$\Omega := \left\{ \sigma_A : w_{A_1}(\log T, y) \leq \sigma_A(y) \leq z_{A_2}(\log T, y),\ \ y \in \mathbb{R}^N \right\}\ .$$

Since σ_A is strictly decreasing with respect to A, we obtain that there are $0 < A_3 < A_4$ such that $\sigma_A \in \Omega$ if and only if $A \in [A_3, A_4]$. The remainder of the proof proceeds along the same lines as in [31, Section 4] to which we refer. We nevertheless mention that the S-theorem provides only the convergence in $L^1(\mathbb{R}^N)$ (which is the topology of X) and a further step is needed to achieve the uniform convergence, see [11, Section 5] and [31, Section 4].

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Appendix A. The stability theorem

We briefly recall here for the reader's convenience the S-theorem introduced by Galaktionov and Vázquez in [11,12] and used in Section 4 to complete the proof of Theorem 1.2. As a general framework, consider a non-autonomous evolution equation

$$\partial_s \vartheta = \mathcal{L}\vartheta \ , \tag{A.1}$$

that can be seen as a *small perturbation* of an autonomous evolution equation with good asymptotic properties

$$\partial_s \Phi = L\Phi \ , \tag{A.2}$$

in the sense described by the three assumptions below. There is a complete metric space (X, d) which is positively invariant for both (A.1) and (A.2) and:

(H1) The orbit $\{\vartheta(t)\}_{t>0}$ of a solution $\vartheta \in C([0,\infty); X)$ to (A.1) is relatively compact in X. Moreover, if we let

$$\vartheta^\tau(t) := \vartheta(t+\tau) \ , \quad t \geq 0 \ , \quad \tau > 0,$$

then $\{\vartheta^\tau\}_{\tau>0}$ is relatively compact in $L^\infty_{\mathrm{loc}}([0,\infty); X)$.

(H2) Given a solution $\vartheta \in C([0,\infty); X)$ to (A.1), assume that there is a sequence of positive times $(t_k)_{k\geq 1}$, $t_k \to \infty$ such that $\vartheta(\cdot + t_k) \to \tilde{\vartheta}$ in $L^\infty_{\mathrm{loc}}([0,\infty); X)$ as $k \to \infty$. Then $\tilde{\vartheta}$ is a solution to (A.2).

(H3) Define the ω-limit set Ω of (A.2) in X as the set of $f \in X$ such that there are a solution $\Phi \in C([0,\infty); X)$ to (A.2) and a sequence of positive times $(t_k)_{k\geq 1}$ such that $t_k \to \infty$ and $\Phi(t_k) \longrightarrow f$ in X. Then Ω is non-empty, compact and uniformly stable, that is: for any $\varepsilon > 0$, there exists $\delta > 0$ such that if Φ is any solution to (A.2) with $d(\Phi(0), \Omega) \leq \delta$, then $d(\Phi(t), \Omega) \leq \varepsilon$ for any $t > 0$, where d is the distance in the complete metric space X.

The S-theorem then reads:

Theorem A.1. *If (H1)–(H3) above are satisfied, then the ω-limit set of any solution $\vartheta \in C([0,\infty); X)$ to (A.1) is contained in Ω.*

For a detailed proof we refer the reader to [11,12].

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