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Dynamics and patterns of a diffusive Leslie–Gower prey–predator model with strong Allee effect in prey ☆

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Abstract

This paper is devoted to study the dynamical properties and stationary patterns of a diffusive Leslie–Gower prey–predator model with strong Allee effect in the prey population. We first analyze the non-negative constant equilibrium solutions and their stabilities, and then study the dynamical properties of time-dependent solutions. Moreover, we investigate the stationary patterns induced by diffusions (Turing pattern). Our results show that the impact of the strong Allee effect essentially increases the system spatio-temporal complexity.

Keywords: Diffusive Leslie–Gower model; Strong Allee effect; Dynamical properties; Stationary patterns

1. Introduction

The understanding of mechanisms and patterns of spatial dispersal of interacting species is a central problem in biology and ecology, and biochemical reactions. The interaction between different species will exhibit the diversity and complexity, and generate the complex network of biological species. The spatial dispersal makes the dynamics of the organisms even more complicated. A typical type of interaction is the one between the prey and predator.

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In most works for prey–predator models, the prey is assumed to grow at a logistic pattern. But in recent years it was recognized that the prey species may have a growth rate of Allee effect, as a result of mate limitation, cooperative defense, cooperative feeding, and environmental conditioning [13,30]. The Allee effect named after W.C. Allee [1], has significant contribution to population dynamics. Allee effects are mainly classified into two ways: strong and weak Allee effect [2,6,31,32,34]. The biological invasion dynamics of reaction–diffusion models with Allee effect has been considered in [12,14,29,34], and the spatiotemporal pattern formation of reaction–diffusion prey–predator models with Allee effect has been studied in [7,19,26,27,33].

In [20], we studied the dynamical properties of the following Leslie–Gower prey–predator model with strong Allee effect in prey:

$$\begin{cases} u' = u(1-u)(u/b-1) - \beta uv, & t > 0, \\ v' = \mu v(1-v/u), & t > 0, \end{cases}$$

where $b \in (0, 1)$ represents Allee threshold value, β and μ are positive constants.

Taking into account the inhomogeneous distribution of the prey and predator in different spatial locations within a fixed domain Ω at any given time, and the natural tendency of each species to diffuse to areas of smaller population concentration, we are naturally led to the following initial and boundary value problem of the corresponding reaction diffusion system

$$\begin{cases} u_t - d_1\Delta u = u(1-u)(u/b-1) - \beta uv, & x \in \Omega,\ t > 0, \\ v_t - d_2\Delta v = \mu v(1-v/u), & x > \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t \geq 0, \\ u(x,0) = u_0(x) > 0,\ v(x,0) = v_0(x) \geq, \not\equiv 0, & x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

where the positive constants d_1 and d_2 are the diffusion coefficients corresponding to u and v, respectively, $\Omega \subset \mathbb{R}^N$ is a smooth and bounded domain, ν is the outward unit normal vector over $\partial\Omega$. The homogeneous Neumann boundary condition indicates that this system is self-contained with zero population flux across the boundary. The initial data u_0, v_0 are continuous functions.

The main purpose of this paper is to investigate the dynamical properties and stationary patterns of (1.1). In particular, under some certain conditions, we prove that both prey and predator will extinct if the initial population of predator is larger than that of prey, which is called overexploitation [31,32] and it is a character of many prey–predator systems with strong Allee effect.

Here we mention that the system (1.1) (without initial data) admits a singularity at $(0, 0)$ and one or two possible positive constant equilibria (see the discussion of §2), so it may have bistable structure between $(0, 0)$ and one positive equilibrium.

On the other hand, because of the reaction term

$$\big[u(1-u)(u/b-1) - \beta uv\big](x,t) < 0$$

for $0 < u(x,t) < b$ and $v(x,t) > 0$, the component $u(x,t)$ may tend to zero and thus the term $v(x,t)/u(x,t)$ may be unbounded. Such a bad structure will bring a lot of difficulties to us in the study of (1.1).

The paper is organized as follows. In section 2, we analyze the nonnegative constant equilibrium solutions and their stabilities. In section 3, we prove the global existence and uniqueness, and give some estimates of solutions of the problem (1.1). Section 4 is devoted to study the dynamical properties of solutions of (1.1). Section 5 concerns with the stationary patterns deriving by diffusions.

Before ending this section, we mention that the classical diffusive Leslie–Gower prey–predator model (without Allee effect) is of the form:

$$\begin{cases} u_t = d_1\Delta u + u(1-u) - \beta uv, & x\in\Omega,\ t>0,\\ v_t = d_2\Delta v + \mu v(1-v/u), & x\in\Omega,\ t>0,\\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega,\ t\geq 0,\\ u(x,0)=u_0(x)>0,\ v(x,0)=v_0(x)\geq,\not\equiv 0, & x\in\bar{\Omega}. \end{cases} \tag{1.2}$$

The dynamical properties, Hopf bifurcation and pattern formation of (1.2) and the diffusive Holling–Tanner prey–predator model (without Allee effect) have been studied widely by many authors, please refer to, for instance, [3–5,8–10,15,18,23–25,28,36] and the references cited therein.

2. Nonnegative constant equilibrium solutions and their stabilities

Obviously, $(b,0)$ and $(1,0)$ are nonnegative constant equilibrium solutions of (1.1). On the other hand, the positive constant equilibrium solution of (1.1) has the form $(\tilde{u},\tilde{u})$, where $\tilde{u}$ satisfies

$$\tilde{u}^2 + (\beta b - 1 - b)\tilde{u} + b = 0.$$

The following results concerning with positive equilibrium solutions are obvious:

(i) If $\beta b > (1-\sqrt{b})^2$, then (1.1) has no positive constant equilibrium solution;
(ii) If $\beta b < (1-\sqrt{b})^2$, then (1.1) has two positive constant equilibrium solutions: $\tilde{\mathbf{u}}_1 = (\tilde{u}_1,\tilde{u}_1)$, $\tilde{\mathbf{u}}_2 = (\tilde{u}_2,\tilde{u}_2)$ with

$$\tilde{u}_1 = \frac{1}{2}\left(1+b-\beta b - \sqrt{(1+b-\beta b)^2-4b}\right),$$
$$\tilde{u}_2 = \frac{1}{2}\left(1+b-\beta b + \sqrt{(1+b-\beta b)^2-4b}\right).$$

(iii) If $\beta b = (1-\sqrt{b})^2$, then (1.1) has a unique positive constant equilibrium solution $\tilde{\mathbf{u}}_3 = (\tilde{u}_3,\tilde{u}_3)$ with $\tilde{u}_3 = \sqrt{b}$.

For the simplicity of notations, we denote $\mathbf{u} = (u, v)$ and

$$G(\mathbf{u}) = \begin{pmatrix} f(u,v)\\ g(u,v)\end{pmatrix} = \begin{pmatrix} u(1-u)(u/b-1) - \beta uv\\ \mu v(1-v/u)\end{pmatrix}.$$

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The linearization of $G(\mathbf{u})$ at $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v})$ is

$$G_{\mathbf{u}}(\tilde{\mathbf{u}}) = \begin{pmatrix} 2(1+1/b)\tilde{u} - 3\tilde{u}^2/b - 1 - \beta\tilde{v} & -\beta\tilde{u} \\ \mu\tilde{v}^2/\tilde{u}^2 & \mu - 2\mu\tilde{v}/\tilde{u} \end{pmatrix}. \tag{2.1}$$

Denote

$$A_i = (1+1/b)\tilde{u}_i - 2\tilde{u}_i^2/b,$$

then we get

$$G_{\mathbf{u}}(\tilde{\mathbf{u}}_i) = \begin{pmatrix} A_i & -\beta\tilde{u}_i \\ \mu & -\mu \end{pmatrix}, \tag{2.2}$$

and

$$A_1 > \beta\tilde{u}_1, \ A_2 < \beta\tilde{u}_2, \ A_3 = \beta\tilde{u}_3.$$

In the following we discuss the local stability of the constant equilibrium solutions $(b, 0)$, $(1, 0)$, $\tilde{\mathbf{u}}_1$, $\tilde{\mathbf{u}}_2$ and $\tilde{\mathbf{u}}_3$.

Let $0 = \mu_0 < \mu_1 < \cdots < \mu_i < \cdots$ be the complete set of eigenvalues of the operator $-\Delta$ in Ω with the homogeneous Neumann boundary condition, and $E(\mu_i)$ be the subspace generated by the eigenfunctions corresponding to μ_i. Let m_i be the algebraic multiplicity of μ_i, i.e., $m_i = \dim E(\mu_i)$, and $\{\phi_{ij}\}_{j=1}^{m_i}$ be a basis of $E(\mu_i)$, i.e., $\{\phi_{ij}\}_{j=1}^{m_i}$ constitute a complete set of linearly independent eigenfunctions corresponding to μ_i. Define

$$\mathbf{X}_{ij} = \{c\phi_{ij} : c \in \mathbb{R}^2\}, \quad \mathbf{X}_i = \bigoplus_{j=1}^{m_i} \mathbf{X}_{ij},$$

$$\mathbf{X} = \left\{ (u, v) \in [C^1(\bar{\Omega})]^2 : \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}. \tag{2.3}$$

Then

$$\mathbf{X} = \bigoplus_{i=0}^{\infty} \mathbf{X}_i.$$

The stationary problem of (1.1) is the following elliptic boundary value problem

$$\begin{cases} -d_1\Delta u = u(1-u)(u/b-1) - \beta uv, & x \in \Omega, \\ -d_2\Delta v = \mu v(1 - v/u), & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{2.4}$$

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And the linearization of (2.4) at $\tilde{\mathbf{u}}$ is

$$\begin{cases} -D\Delta\mathbf{u} = G_{\mathbf{u}}(\tilde{\mathbf{u}})\mathbf{u}, & x \in \Omega, \\ \dfrac{\partial \mathbf{u}}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

where

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}.$$

Theorem 2.1. (i) *For all $d_1, d_2, \mu > 0$, the constant equilibrium solutions* $(1, 0)$, $(b, 0)$ *and* $(\tilde{u}_1, \tilde{u}_1)$ *are unstable.*

(ii) *For all $d_1, d_2 > 0$, the constant equilibrium solution $(\tilde{u}_2, \tilde{u}_2)$ is locally asymptotically stable when $\mu > \max\{A_2, \frac{d_2}{d_1}A_2\}$, and is unstable when $\mu < A_2$.*

(iii) *For all $d_1, d_2 > 0$, the constant equilibrium solution $(\tilde{u}_3, \tilde{u}_3)$ is unstable if $\mu < A_3$.*

Remark 2.1. In Theorem 2.1(ii) there is a gap between the stability and instability of $(\tilde{u}_2, \tilde{u}_2)$. For the values of parameters in this gap, the stability depends on d_1, d_2. Indeed, in this gap the Turing instability could occur (stable for ODE but can be unstable for some d_1, d_2). Please refer to [20, Theorem 3.2] and the arguments of §5.3.

Proof of Theorem 2.1. (i) For $\tilde{\mathbf{u}} = (1, 0)$. Taking advantage of (2.1) it follows that

$$G_{\mathbf{u}}(\tilde{\mathbf{u}}) = \begin{pmatrix} 1 - 1/b & -\beta \\ 0 & \mu \end{pmatrix}.$$

As $0 < b < 1$, we have $\mu + 1/b > 1$. It is easy to verify that μ is an eigenvalue of $G_{\mathbf{u}}(\tilde{\mathbf{u}})$ and the corresponding eigenvector is $(\frac{\beta}{1-\mu-1/b}, 1)$. Therefore, $(1, 0)$ is unstable.

For $\tilde{\mathbf{u}} = (b, 0)$, we have

$$G_{\mathbf{u}}(\tilde{\mathbf{u}}) = \begin{pmatrix} 1 - b & -\beta b \\ 0 & \mu \end{pmatrix}.$$

We can verify that μ is an eigenvalue of $G_{\mathbf{u}}(\tilde{\mathbf{u}})$ with eigenvector (ϕ, h): $(\phi, h) = (\frac{\beta b}{1-b-\mu}, 1)$ when $b + \mu \neq 1$, while $(\phi, h) = (1, 0)$ when $b + \mu = 1$. Hence $(b, 0)$ is unstable.

For the cases $\tilde{\mathbf{u}}_i = (u_i, u_i)$, $i = 1, 2, 3$. Denote

$$\mathcal{L} = D\Delta + G_{\mathbf{u}}(\tilde{\mathbf{u}}_i).$$

Then for each $j \in \{0, 1, 2, \ldots\}$, $\mathbf{X}_j$ is invariant under the operator $\mathcal{L}$, and ξ is an eigenvalue of $\mathcal{L}$ on $\mathbf{X}_j$ if and only if ξ is an eigenvalue of the matrix

$$Q_{ij} = -\mu_j D + G_{\mathbf{u}}(\tilde{\mathbf{u}}_i) = \begin{pmatrix} -\mu_j d_1 + A_i & -\beta\tilde{u}_i \\ \mu & -\mu_j d_2 - \mu \end{pmatrix}$$

where $G_{\mathbf{u}}(\tilde{\mathbf{u}}_i)$ is given by (2.2). The direct calculation gives

$$\begin{aligned}\operatorname{Tr} Q_{ij} &= A_i - \mu - (d_1 + d_2)\mu_j,\\ \det Q_{ij} &= d_1 d_2 \mu_j^2 + (\mu d_1 - d_2 A_i)\mu_j + \mu\beta\tilde{u}_i - \mu A_i.\end{aligned}$$

For $i = 1$ and $j = 0$, we have

$$|\xi I - Q_{10}| = \xi^2 - (A_1 - \mu)\xi + \mu\beta\tilde{u}_1 - \mu A_1.$$

Since $\beta\tilde{u}_1 - A_1 < 0$, the two eigenvalues ξ_0^- and ξ_0^+ of Q_{10} satisfy $\xi_0^- < 0 < \xi_0^+$, which implies that $(\tilde{u}_1, \tilde{u}_1)$ is unstable.

Similarly, if $\mu < A_2$, then $(\tilde{u}_2, \tilde{u}_2)$ is unstable.

For $i = 3$, owning to $\mu < A_3$ and $\beta\tilde{u}_3 = A_3$, the two eigenvalues ξ_0^- and ξ_0^+ of Q_{30} satisfy

$$\begin{aligned}\xi_0^- + \xi_0^+ &= \operatorname{Tr} Q_{30} = A_3 - \mu > 0,\\ \xi_0^- \xi_0^+ &= \det Q_{30} = \mu\beta\tilde{u}_3 - \mu A_3 = 0.\end{aligned}$$

Hence $0 = \xi_0^- < \xi_0^+$, which implies that $(\tilde{u}_3, \tilde{u}_3)$ is unstable.

Finally, we show that if $\mu > \max\{A_2, \frac{d_2}{d_1}A_2\}$, then $(\tilde{u}_2, \tilde{u}_2)$ is locally asymptotically stable. In fact, since $\beta\tilde{u}_2 > A_2$, the two eigenvalues ξ_j^- and ξ_j^+ of Q_{2j} satisfy

$$\begin{aligned}\xi_j^- + \xi_j^+ &= \operatorname{Tr} Q_{2j} = A_2 - \mu - (d_2 + d_1)\mu_j < 0,\\ \xi_j^- \xi_j^+ &= \det Q_{2j} = \mu_j(\mu_j d_1 d_2 + \mu d_1 - d_2 A_2) + \mu\beta\tilde{u}_2 - \mu A_2 > 0.\end{aligned}$$

Thus ξ_j^- and ξ_j^+ have negative real parts. Actually

$$\operatorname{Re}\xi_j^- = \operatorname{Re}\left\{\frac{1}{2}\left(\operatorname{Tr} Q_{2j} - \sqrt{(\operatorname{Tr} Q_{2j})^2 - 4\det Q_{2j}}\right)\right\} \le \frac{\operatorname{Tr} Q_{2j}}{2} \le \frac{A_2 - \mu}{2} < 0, \tag{2.5}$$

and when $(\operatorname{Tr} Q_{2j})^2 - 4\det Q_{2j} \le 0$,

$$\operatorname{Re}\xi_j^+ = \operatorname{Re}\left\{\frac{1}{2}\left(\operatorname{Tr} Q_{2j} + \sqrt{(\operatorname{Tr} Q_{2j})^2 - 4\det Q_{2j}}\right)\right\} = \frac{\operatorname{Tr} Q_{2j}}{2} \le \frac{A_2 - \mu}{2} < 0. \tag{2.6}$$

For the case $(\operatorname{Tr} Q_{2j})^2 - 4\det Q_{2j} > 0$, we have

$$\begin{aligned}\operatorname{Re}\xi_j^+ &= \frac{1}{2}\left(\operatorname{Tr} Q_{2j} + \sqrt{(\operatorname{Tr} Q_{2j})^2 - 4\det Q_{2j}}\right)\\ &= \frac{2\det Q_{2j}}{\operatorname{Tr} Q_{2j} - \sqrt{(\operatorname{Tr} Q_{2j})^2 - 4\det Q_{2j}}} < 0,\end{aligned}$$

and $\operatorname{Re}\xi_j^+ \to -\infty$ as $j \to \infty$. Hence, there exists a constant $\sigma > 0$ such that

$$\operatorname{Re}\xi_j^+ < -\sigma, \quad \forall\, j \in \{0, 1, 2, \ldots\}. \tag{2.7}$$

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Inequalities (2.5), (2.6) and (2.7) imply that

$$\operatorname{Re}\xi_j^-,\ \operatorname{Re}\xi_j^+ \le -\min\{\sigma, \frac{-A_2+\mu}{2}\} =: -\varepsilon < 0, \quad \forall\, j \in \{0,1,2,\dots\}.$$

Consequently, the spectrum of $\mathcal{L}$, which consists of eigenvalues, lies in $\{\operatorname{Re}\xi \le -\varepsilon\}$, and local stability of $(\tilde{u}_2, \tilde{u}_2)$ is followed from [11, Theorem 5.1.1]. The proof is complete. □

3. Global existence, uniqueness and estimates of the solutions

In this section, we provide the global existence, uniqueness and estimates of solutions of the problem (1.1).

Theorem 3.1. (i) *The problem* (1.1) *has a unique global solution* $(u(x,t), v(x,t))$ *satisfying* $u(x,t) > 0$, $v(x,t) \ge 0$ *for* $(x,t) \in \bar{\Omega} \times [0,\infty)$, *and*

$$\limsup_{t\to\infty} \max_{\bar{\Omega}} u(x,t) \le 1, \quad \limsup_{t\to\infty} \max_{\bar{\Omega}} v(x,t) \le 1. \tag{3.1}$$

(ii) *There exists a constant* $M > 0$ *such that*

$$\|u(\cdot,t)\|_{C^1(\bar{\Omega})} \le M, \quad \forall\, t \ge 1. \tag{3.2}$$

Proof. (i) It is easy to see that, in the domain $\{u > 0,\ v \ge 0\}$, the problem (1.1) is a mixed quasi-monotone system. Take $\underline{v}(x,t) = 0$ and $(\bar{u}(x,t), \bar{v}(x,t)) = (u^*(t), v^*(t))$, where $(u^*(t), v^*(t))$ is the unique solution of

$$\begin{cases} u' = u(1-u)(u/b-1), & t > 0, \\ v' = \mu v(1-v/u), & t > 0, \\ u(0) = u^*, \quad v(0) = v^*, \end{cases}$$

with $u^* = \max_{x\in\bar{\Omega}} u_0(x) > 0$ and $v^* = \max_{x\in\bar{\Omega}} v_0(x) > 0$. Let $\underline{u}(x,t)$ be the unique positive solution of

$$\begin{cases} u_t = d_1\Delta u + u(1-u)(u/b-1) - \beta u\bar{v}, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t \ge 0, \\ u(x,0) \equiv u_*, & x \in \Omega, \end{cases}$$

where $u_* = \min_{x\in\bar{\Omega}} u_0(x) > 0$. Then $(\bar{u}(x,t), \bar{v}(x,t))$ and $(\underline{u}(x,t), \underline{v}(x,t))$ are the coupled ordered upper and lower solutions of the problem (1.1). Hence (1.1) has a unique global solution $(u(x,t), v(x,t))$ satisfying

$$0 < \underline{u}(x,t) \le u(x,t) \le u^*(t), \quad 0 \le v(x,t) \le v^*(t), \quad \forall\, x \in \bar{\Omega},\ t \ge 0.$$

Moreover, by the strong maximum principle we also have $v(x,t) > 0$ for $x \in \bar{\Omega}$ and $t > 0$.

In the following we prove that (u, v) satisfies (3.1). Clearly, u satisfies

$$\begin{cases} u_t \leq d_1 \Delta u + u(1-u)(u/b-1), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t \geq 0, \\ u(x,0) = u_0(x) > 0, & x \in \Omega. \end{cases} \tag{3.3}$$

It is deduced by the comparison principle that $u(x,t) \leq \varphi(t)$, where $\varphi(t)$ is the unique solution of

$$\varphi' = \varphi(1-\varphi)(\varphi/b-1), \quad t > 0; \qquad \varphi(0) = u^*, \tag{3.4}$$

with $u^* = \max_{x\in\bar{\Omega}} u_0(x) > 0$. Evidently, $\lim\limits_{t\to\infty} \varphi(t) < 1$, this implies $\limsup\limits_{t\to\infty} \max\limits_{\bar{\Omega}} u(x,t) \leq 1$. For any given $\varepsilon > 0$, there is $T > 0$ such that

$$u(x,t) < 1 + \varepsilon, \quad x \in \bar{\Omega},\ t \geq T.$$

It follows from the second equation of (1.1) that

$$v_t < d_2 \Delta v + \mu v\big(1 - v/(1+\varepsilon)\big), \quad x > \Omega,\ t > T. \tag{3.5}$$

Remember the boundary condition $\frac{\partial v}{\partial \nu} = 0$, it is derived by the comparison principle that $v(x,t) \leq \phi(t)$ for $x \in \bar{\Omega}$, $t \geq T$, where $\phi(t)$ is the unique solution of

$$\phi' = \mu\phi\big(1 - \phi/(1+\varepsilon)\big), \quad t > 0; \qquad \phi(T) = \max_{x\in\bar{\Omega}} v(x,T). \tag{3.6}$$

Obviously, $\lim\limits_{t\to\infty} \phi(t) \leq 1 + \varepsilon$, which leads to $\limsup\limits_{t\to\infty} \max\limits_{\bar{\Omega}} v(x,t) \leq 1 + \varepsilon$. The arbitrariness of ε yields $\limsup\limits_{t\to\infty} \max\limits_{\bar{\Omega}} v(x,t) \leq 1$.

(ii) The proof is similar to that of [35, Theorem 2.1], and we just give a brief description. For the integer $k \geq 0$, denote $u^k(x,t) = u(x,t+k)$. Then u^k satisfies

$$\begin{cases} u^k_t - d_1 \Delta u^k = u^k(1-u^k)(u^k/b-1) - \beta u^k v(x,t+k), & x \in \Omega,\ 0 < t \leq 3, \\ \dfrac{\partial u^k}{\partial \nu} = 0, & x \in \partial\Omega,\ t \geq 0, \\ u^k(x,0) = u(x,k) > 0, & x \in \bar{\Omega}, \end{cases}$$

Noticing (3.1), it is easy to see that $\|u^k\|_{L^\infty(\Omega\times(0,3])}$ and $\|v(x,t+k)\|_{L^\infty(\Omega\times(0,3])}$ are bounded in k. Similarly to the arguments in the proof of [35, Theorem 2.1], we can prove (3.2). $\square$

Remark 3.1. Generally, we couldn't get uniform estimates of $v(\cdot,t)$ in $C^1(\bar{\Omega})$ since u may tend to 0 as $t \to \infty$ and v/u may be unbounded in $\bar{\Omega} \times [1,\infty)$.

4. Dynamical properties of the solution

This section is devoted to investigate the dynamical properties of the solution $(u(x,t), v(x,t))$ of (1.1). We first give a general result.

Theorem 4.1. *For all* $d_1, d_2, \beta, \mu > 0$, *if* $u_0(x) \le b$ *and* $(u_0(x), v_0(x)) \not\equiv (b, 0)$, *then* $\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} v(x,t) = 0$ *uniformly on* $\bar{\Omega}$.

Proof. The proof is divided into three cases.

Case 1: $u_0(x) < b$ on $\bar{\Omega}$. By (3.3), (3.4) and $u_0(x) < b$, we obtain that $\lim_{t\to\infty} \varphi(t) = 0$ and $\lim_{t\to\infty} u(x,t) = 0$ uniformly on $\bar{\Omega}$. For any given $\varepsilon > 0$, there is $T > 0$ such that

$$u(x,t) < \varepsilon, \quad \forall x \in \bar{\Omega}, \ t \ge T.$$

Replacing $1+\varepsilon$ with ε in both (3.5) and (3.6), we can obtain similarly

$$\limsup_{t\to\infty} \max_{\bar{\Omega}} v(x,t) \le \lim_{t\to\infty} \phi(t) \le \varepsilon.$$

The arbitrariness of ε gives that $\lim_{t\to\infty} v(x,t) = 0$ uniformly for $x \in \bar{\Omega}$.

Case 2: $u_0(x) \le b$ and $u_0(x) \not\equiv b$. Let $w = b - u$. Then, by (3.3) and (3.4), we have that $0 < u(x,t) \le b$, $0 \le w(x,t) < b$ for $x \in \bar{\Omega}$, $t > 0$, and

$$\begin{cases} w_t - d_1 \Delta w = \dfrac{w}{b}(b-w)(1-b+w) + \beta(b-w)v \ge 0, & x \in \Omega, \ t > 0, \\ \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \ t \ge 0, \\ w(x,0) \ge, \not\equiv 0, & x \in \Omega. \end{cases}$$

It follows from the strong maximum principle that

$$w(x,t) > 0, \quad \forall\, x \in \bar{\Omega}, \ t > 0,$$

i.e., $u(x,t) < b$ for $x \in \bar{\Omega}$, $t > 0$. Similarly to the case 1, we have

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} v(x,t) = 0$$

uniformly for $x \in \bar{\Omega}$.

Case 3: $u_0 \equiv b$ and $v_0(x) \not\equiv 0$ for $x \in \bar{\Omega}$. In view of (3.3) and (3.4), it is easy to see that $u(x,t) \le b$ for $x \in \bar{\Omega}$, $t \ge 0$. If $u(x,t) \equiv b$, we obtain a contradiction since $v(x,t) > 0$ for $x \in \bar{\Omega}$, $t > 0$. Hence there exists $t_0 > 0$ such that $0 < u(x,t_0) \le b$ and $u(x,t_0) \not\equiv b$. Similar to the case 2,

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} v(x,t) = 0$$

uniformly for $x \in \bar{\Omega}$. The proof is complete. □

Let $(u(x,t), v(x,t))$ be the unique solution of (1.1) and

$$s(t) = \min_{x\in\bar{\Omega}} \frac{v(x,t)}{u(x,t)}, \quad t \geq 0. \tag{4.1}$$

For the positive constant λ, we denote

$$\mathcal{R}_\lambda := \{(u,v) : u > 0,\ v \geq \lambda u\}.$$

We call that $\mathcal{R}_\lambda$ is an invariant region of (1.1) if $(u_0(x), v_0(x)) \in \mathcal{R}_\lambda$ for all $x \in \bar{\Omega}$ implies $(u(x,t), v(x,t)) \in \mathcal{R}_\lambda$ for all $x \in \bar{\Omega}$ and $t \geq 0$.

Lemma 4.1. *Suppose that $d_1 = d_2 = d$ and $\mu > 0$. Denote $\lambda_0 = (1-\sqrt{b})^2/(\beta b)$.*

(i) *When $\beta b > (1-\sqrt{b})^2$, the set $\mathcal{R}_\lambda$ is an invariant region of (1.1) for each $\lambda_0 \leq \lambda \leq 1$. Moreover, $s(t)$ is strictly increasing in t provided $\lambda_0 \leq s(t) \leq 1$;*
(ii) *When $\beta b = (1-\sqrt{b})^2$, the set $\mathcal{R}_1$ is an invariant region of (1.1).*

Proof. (i) Let $(u(x,t), v(x,t))$ be the unique solution of (1.1). Set $w = v - \lambda u$ and

$$h(u,w;\lambda) = \frac{1}{d_2} g(u, w+\lambda u) - \frac{\lambda}{d_1} f(u, w+\lambda u). \tag{4.2}$$

Since $d_1 = d_2 = d$, then w satisfies

$$\begin{cases} w_t - d\Delta w = dh(u,w;\lambda), & x \in \Omega,\ t > 0, \\ \dfrac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t \geq 0, \\ w(x,0) = v_0(x) - \lambda u_0(x), & x \in \Omega, \end{cases} \tag{4.3}$$

and

$$\begin{aligned} h(u,0;\lambda) &= \frac{1}{d} g(u,\lambda u) - \frac{1}{d}\lambda f(u,\lambda u) \\ &= \frac{1}{d}\Big(\mu\lambda u(1-\lambda) - \lambda u(1-u)(u/b-1) + \beta\lambda^2 u^2\Big) \\ &= \frac{1}{d}\lambda u[\mu(1-\lambda) + \lambda\beta u - (1-u)(u/b-1)] \\ &= \frac{1}{d}\lambda u[(\beta u - \mu)\lambda + \mu - (1-u)(u/b-1)] \end{aligned}$$

with $u = u(x,t)$. Set

$$k(u,\lambda) = (\beta u - \mu)\lambda + \mu - (1-u)(u/b-1).$$

Notice $\lambda_0 < 1$ when $\beta b > (1-\sqrt{b})^2$. Owing to $u > 0$, we have

$$k(u,\lambda_0) = (1-\lambda_0)\mu + \beta\lambda_0 u - (1-u)(u/b-1) > 0.$$
$$k(u,1) = \beta u - (1-u)(u/b-1) > 0.$$

Therefore, $k(u,\lambda) > 0$ for all $\lambda_0 \le \lambda \le 1$ and $u > 0$, which leads to

$$h(u,0;\lambda) > 0, \quad \forall\, x \in \bar{\Omega},\ \ t \ge 0. \tag{4.4}$$

If $(u_0(x), v_0(x)) \in \mathcal{R}_\lambda$ for all $x \in \bar{\Omega}$, then $w(x,0) \ge 0$ in $\bar{\Omega}$. Noticing (4.4), we can apply the strong maximum principle and Hopf boundary lemma to (4.3) and derive that for $\lambda_0 \le \lambda \le 1$,

$$w(x,t) > 0, \ \text{ i.e., } \ v(x,t) > \lambda u(x,t), \quad \forall\, x \in \bar{\Omega},\ \ t > 0. \tag{4.5}$$

Hence $\mathcal{R}_\lambda$ is an invariant region for each $\lambda_0 \le \lambda \le 1$.

The conclusion (4.5) also suggests that if $\lambda_0 \le s(0) \le 1$ then

$$v(x,t) > s(0)u(x,t), \quad \forall\, x \in \bar{\Omega},\ \ t > 0,$$

i.e., $s(t) > s(0)$ for all $t > 0$. Similarly, it can be shown that if $s(t_1) \le 1$ for some $t_1 > 0$, then $s(t) > s(t_1)$ for all $t > t_1$.

(ii) The condition $\beta b = (1-\sqrt{b})^2$ implies $\lambda_0 = 1$. As $u > 0$, we have

$$h(u,0,\lambda_0) = \frac{1}{d}uk(u,1) = \frac{1}{d}u[\beta u - (1-u)(u/b-1)] = \frac{1}{bd}u(u-\sqrt{b})^2 \ge 0.$$

By the strong maximum principle we have that $\mathcal{R}_{\lambda_0}$ is an invariant region of (1.1). The proof is complete. $\square$

When $\mu > (1-b)^2/(4b)$, we have

$$k(u,0) = \mu - (1-u)(u/b-1) > 0.$$

Combining this formula and $k(u,1) \ge 0$, by the same method as in the proof of Lemma 4.1, we can get the following conclusion.

Corollary 4.1. *Suppose that $d_1 = d_2 = d$, $\beta b \ge (1-\sqrt{b})^2$ and $\mu > (1-b)^2/(4b)$.*

(i) *The set $\mathcal{R}_\lambda$ is an invariant region of (1.1) for each $0 < \lambda \le 1$;*
(ii) *$s(t)$ is strictly increasing in t provided $0 < s(t) < 1$.*

Remark 4.1. Under these assumptions of Corollary 4.1 when we have a more restrictive condition on μ ($\mu > (1-b)^2/(4b)$), it can be shown that the problem (1.1) has neither nonconstant positive time-period solution (see Proposition 4.1) nor nonconstant positive steady states (see Theorem 5.4).

Theorem 4.2. *Assume* $d_1 = d_2 = d$, $\beta b > (1-\sqrt{b})^2$ *and* $\mu > 0$. *If the initial data* (u_0, v_0) *satisfies* $\lambda_0 u_0(x) \le v_0(x)$ *on* $\bar{\Omega}$, *then* $\lim_{t\to\infty}(u(x,t), v(x,t)) = (0,0)$ *uniformly on* $\bar{\Omega}$.

Proof. If $\lambda_0 u_0(x) \le v_0(x)$ on $\bar{\Omega}$, thanks to the conclusion Lemma 4.1(i), it follows that $\lambda_0 u(x,t) < v(x,t)$ for all $x \in \bar{\Omega}$ and $t > 0$, and there exist $\lambda_0 < \lambda_1 < 1$, $t_0 > 0$ such that

$$\lambda_1 u(x,t) \le v(x,t), \quad \forall\, x \in \bar{\Omega},\ t \ge t_0.$$

Consequently, u satisfies

$$\begin{aligned} u_t - d\Delta u &= u(1-u)(u/b-1) - \beta uv \\ &\le u[(1-u)(u/b-1) - \lambda_1 \beta u] \\ &\le ku, \quad \forall\, x \in \bar{\Omega},\ t \ge t_0, \end{aligned}$$

where $k = \max_{y\ge 0}[(1-y)(y/b-1) - \lambda_1\beta y] < 0$. This leads to $\lim_{t\to\infty} u(x,t) = 0$ uniformly on $\bar{\Omega}$. In the same way as in the proof of Theorem 4.1, it can be deduced that $\lim_{t\to\infty} v(x,t) = 0$ uniformly on $\bar{\Omega}$. □

Theorem 4.3. *Assume* $d_1 = d_2 = d$, $\beta b = (1-\sqrt{b})^2$ *and* $\mu > 0$.

(i) *If* $u_0(x) \le \sqrt{b}$ *and* $(u_0(x), v_0(x)) \not\equiv (\sqrt{b}, \sqrt{b})$, *then* $\lim_{t\to\infty}(u(x,t), v(x,t)) = (0,0)$ *uniformly on* $\bar{\Omega}$.
(ii) *If* $u_0(x) \le v_0(x)$ *on* $\bar{\Omega}$, *then* $\lim_{t\to\infty}(u(x,t), v(x,t)) = (0,0)$ *or* $(\sqrt{b}, \sqrt{b})$ *uniformly on* $\bar{\Omega}$.

Proof. (i) By Lemma 4.1(ii), we obtain that $\mathcal{R}_1$ is an invariant region of (1.1) and $v(x,t) \ge u(x,t)$ for $x \in \bar{\Omega}$, $t \ge 0$.

Case 1: $u_0(x) \le v_0(x)$ and $u_0(x) \le, \not\equiv \sqrt{b}$ on $\bar{\Omega}$. Setting $\varphi = \sqrt{b} - u$, we have

$$\begin{cases} \varphi_t - d\Delta\varphi \ge (\sqrt{b}-\varphi)[(\sqrt{b}-\varphi-1)(\dfrac{\sqrt{b}-\varphi}{b} - 1) + \beta(\sqrt{b}-\varphi)] \ge 0, & x \in \Omega,\ t > 0, \\ \dfrac{\partial\varphi}{\partial\nu} = 0, & x \in \partial\Omega,\ t \ge 0, \\ \varphi(x,0) \ge, \not\equiv 0, & x \in \Omega. \end{cases}$$

The strong maximum principle and Hopf boundary lemma yield

$$\varphi(x,t) > 0, \ \text{ i.e., } \ u(x,t) < \sqrt{b}, \quad \forall\, x \in \bar{\Omega},\ t > 0.$$

Set $c = \max_{x\in\bar{\Omega}} u(x,t_0) < \sqrt{b}$ for a fixed $t_0 > 0$. We can prove similarly

$$u(x,t) \le c, \quad \forall\, x \in \bar{\Omega},\ t \ge t_0.$$

Because, in the current situation, $\sqrt{b}$ is the unique root of $(1-y)(y/b-1) - \beta y = 0$, we see that $\gamma := (1-c)(c/b-1) - \beta c < 0$ and $(1-y)(y/b-1) - \beta y < \gamma$ for all $y < c$. Therefore,

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$$(1-u(x,t))(u(x,t)/b-1)-\beta u(x,t)\leq \gamma<0, \quad \forall\, x\in\bar{\Omega},\ t\geq t_0,$$

and

$$u_t-d\Delta u\leq u[(1-u)(u/b-1)-\beta u]\leq \gamma u, \quad \forall\, x\in\Omega,\ t>t_0.$$

Similarly to the proof of Theorem 4.2, it can be deduced that $\lim\limits_{t\to\infty}u(x,t)=\lim\limits_{t\to\infty}v(x,t)=0$ uniformly on $\bar{\Omega}$.

Case 2: $u_0(x)\equiv\sqrt{b}\leq v_0(x)$ and $(u_0(x),v_0(x))\not\equiv(\sqrt{b},\sqrt{b})$ on $\bar{\Omega}$. Similarly to Case 1, we derive that $u(x,t)\leq\sqrt{b}$ for $x\in\bar{\Omega}$, $t\geq 0$. If $u(x,t)\equiv\sqrt{b}$, we get a contradiction since $v(x,t)>0$ for $x\in\bar{\Omega}$, $t>0$. Therefore, there exists a $T>0$ such that $u(x,T)\leq v(x,T)$ and $u(x,T)\leq,\not\equiv\sqrt{b}$ on $\bar{\Omega}$. Using the conclusion of Case 1, we have that $\lim\limits_{t\to\infty}u(x,t)=\lim\limits_{t\to\infty}v(x,t)=0$ uniformly on $\bar{\Omega}$.

(ii) Using Lemma 4.1(ii) again, we have

$$v(x,t)\geq u(x,t), \quad x\in\bar{\Omega},\ t\geq 0. \tag{4.6}$$

Taking advantage of (4.6) and the first equation of (1.1), we get

$$\begin{cases} u_t-d\Delta u\leq -\dfrac{1}{b}u(u-\sqrt{b})^2, & x\in\Omega,\ t>0,\\ \dfrac{\partial u}{\partial\nu}=0, & x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x), & x\in\Omega.\end{cases}$$

It follows that

$$\limsup_{t\to\infty}\max_{x\in\bar{\Omega}}u(x,t)\leq\sqrt{b}. \tag{4.7}$$

According to (4.7), we conclude that either

$$\max_{x\in\bar{\Omega}}u(x,T_0)\leq\sqrt{b} \quad \text{for some } T_0\geq 0, \tag{4.8}$$

or

$$\lim_{t\to\infty}\max_{x\in\bar{\Omega}}u(x,t)=\sqrt{b}, \quad \max_{x\in\bar{\Omega}}u(x,t)>\sqrt{b},\ \ \forall\, t\geq 0. \tag{4.9}$$

When (4.8) holds, making use of (4.6) and the conclusion (i), we have that either $(u(x,t),v(x,t))\equiv(\sqrt{b},\sqrt{b})$ in $\bar{\Omega}\times[T_0,\infty)$ or

$$\lim_{t\to\infty}(u(x,t),v(x,t))=(0,0) \quad \text{uniformly on } \bar{\Omega}.$$

When (4.9) holds, we shall divide the proof into two steps.

Step 1. We first show that $\lim\limits_{t\to\infty} u(x,t) = \sqrt{b}$ uniformly on $\bar{\Omega}$. Denote

$$\bar{u}(t) = \frac{1}{|\Omega|}\int\limits_{\Omega} u(x,t)dx, \quad \varphi_p(t) = \int\limits_{\Omega} u^p(x,t)dx, \quad p \geq 2.$$

Then

$$\bar{u}'(t) \leq -\frac{1}{b|\Omega|}\int\limits_{\Omega} u(u-\sqrt{b})^2 dx \leq 0, \tag{4.10}$$

$$\begin{aligned}
\varphi_p'(t) &= p\int\limits_{\Omega}\left(u^{p-1}[d\Delta u + u(1-u)(\frac{u}{b}-1) - \beta uv]\right)dx \\
&\leq p\int\limits_{\Omega}\left(du^{p-1}\Delta u - \frac{1}{b}u^p(u-\sqrt{b})^2\right)dx \\
&= -p\int\limits_{\Omega}\left(d(p-1)u^{p-2}|\nabla u|^2 + \frac{1}{b}u^p(u-\sqrt{b})^2\right)dx \\
&\leq 0.
\end{aligned} \tag{4.11}$$

Hence $E := \lim\limits_{t\to\infty}\bar{u}(t) \leq \sqrt{b}$ and $\lim\limits_{t\to\infty}\varphi_p(t) \geq 0$, and so the limit $\lim\limits_{t\to\infty}\int\limits_{\Omega} u(u-\sqrt{b})^2 dx$ exists. Based on (4.10), it is easy to see that

$$\lim_{t\to\infty}\int\limits_{\Omega} u(u-\sqrt{b})^2 dx = 0. \tag{4.12}$$

Using (4.7), (4.12), Fatou Lemma and Hölder inequality, we have

$$\begin{aligned}
0 &\leq \int\limits_{\Omega}\liminf_{t\to\infty} u(\sqrt{b}-u)dx \leq \liminf_{t\to\infty}\int\limits_{\Omega} u(\sqrt{b}-u)dx \leq \limsup_{t\to\infty}\int\limits_{\Omega} u(\sqrt{b}-u)dx \\
&\leq \lim_{t\to\infty}\left(\int\limits_{\Omega} udx\right)^{\frac{1}{2}}\left(\int\limits_{\Omega} u(u-\sqrt{b})^2 dx\right)^{\frac{1}{2}} = 0,
\end{aligned}$$

which implies $\lim\limits_{t\to\infty}\int\limits_{\Omega} u(\sqrt{b}-u)dx = 0$. Direct computation yields

$$\begin{aligned}
0 &= \lim_{t\to\infty}\int\limits_{\Omega} u(\sqrt{b}-u)dx = \lim_{t\to\infty}\int\limits_{\Omega}\left(\sqrt{b}u - (u-\bar{u}+\bar{u})^2\right)dx \\
&= \lim_{t\to\infty}\int\limits_{\Omega}\left(\sqrt{b}u - (u-\bar{u})^2 - \bar{u}^2\right)dx.
\end{aligned}$$

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That is,

$$k := \lim_{t\to\infty} \int_\Omega (u-\bar{u})^2 dx = |\Omega| E(\sqrt{b} - E) \geq 0. \tag{4.13}$$

By virtue of the Poincaré inequality, there exists a constant $C > 0$ such that

$$\int_\Omega (u-\bar{u})^2 dx \leq C \int_\Omega |\nabla u|^2 dx, \quad \forall\, t > 0. \tag{4.14}$$

We claim that $E = \lim_{t\to\infty} \bar{u}(t) = \sqrt{b}$. If this is not true, then

$$E < \sqrt{b}, \quad k > 0. \tag{4.15}$$

Take $p = 2$. It follows from (4.11), (4.14) and (4.15) that

$$\begin{aligned} \varphi_2'(t) &\leq -2 \int_\Omega \left(d|\nabla u|^2 + \frac{1}{b} u^2 (u - \sqrt{b})^2 \right) dx \leq -2 \int_\Omega d|\nabla u|^2 dx \\ &\leq -2 \int_\Omega \frac{d}{C} (u-\bar{u})^2 dx \to -\frac{2dk}{C} < 0 \quad \text{as } t \to \infty, \end{aligned}$$

which leads to $\lim_{t\to\infty} \varphi_2(t) = -\infty$. This is a contradiction since $\varphi_2(t) \geq 0$ for all $t \geq 0$. Thus $E = \sqrt{b}$, and upon using (4.13),

$$\lim_{t\to\infty} \int_\Omega (u - \sqrt{b})^2 dx = 0. \tag{4.16}$$

Thanks to (3.2), we see that the set $\{u(\cdot,t) : t \geq 2\}$ is relatively compact in $C(\bar{\Omega})$. Assume that

$$\|u(x,t_k) - u_\infty(x)\|_{C(\bar{\Omega})} \to 0 \quad \text{as } t_k \to \infty$$

for some $u_\infty(x) \in C(\bar{\Omega})$. In view of (4.16) and the uniqueness of limit, it follows that $u_\infty(x) \equiv \sqrt{b}$. Combining this and the relatively compactness of $\{u(\cdot,t) : t \geq 2\}$ in $C(\bar{\Omega})$, we deduce

$$\lim_{t\to\infty} \|u(\cdot,t) - \sqrt{b}\|_{C(\bar{\Omega})} = 0. \tag{4.17}$$

Step 2. Now we prove $\lim_{t\to\infty} \|v(\cdot,t) - \sqrt{b}\|_{C(\bar{\Omega})} = 0$. It follows from (4.6) that

$$\liminf_{t\to\infty} v(x,t) \geq \liminf_{t\to\infty} u(x,t) \equiv \sqrt{b}.$$

Remember (4.17), then similar to the proof of Theorem 3.1(i) we have $\limsup_{t\to\infty} v(x,t) \le \sqrt{b}$. Thus

$$\lim_{t\to\infty} v(x,t) = \sqrt{b}. \tag{4.18}$$

Owing to (4.17) and $u(x,t) > 0$ on $\bar{\Omega} \times [0,\infty)$, we have $\inf_{x\in\bar{\Omega}, t\ge 0} u(x,t) > 0$, and then $v(x,t)/u(x,t)$ is uniformly bounded on $\bar{\Omega} \times [0,\infty)$. In the same way as in the proof of Theorem 3.1(ii), there exists $M_1 = M_1(b,\beta,\mu) > 0$ such that

$$\|v(\cdot,t)\|_{C^1(\bar{\Omega})} \le M_1, \quad \forall\, t \ge 2.$$

Using this estimate and (4.18), we can derive similarly that $\lim_{t\to\infty} \|v(\cdot,t) - \sqrt{b}\|_{C(\bar{\Omega})} = 0$. The proof is complete. □

Proposition 4.1. *Suppose that $d_1 = d_2 = d$ and $\beta b \ge (1-\sqrt{b})^2$. If $\mu > (1-b)^2/(4b)$, then the problem (1.1) has no nonconstant positive time-periodic solution.*

Proof. Suppose on the contrary that $(u(x,t), v(x,t))$ is a nonconstant positive time-periodic solution of (1.1) with period T. Then we have

$$s(t) = s(t+T), \quad t \ge 0, \tag{4.19}$$

where $s(t)$ is given by (4.1). We first prove

$$\limsup_{t\to\infty} s(t) \ge 1. \tag{4.20}$$

If this is not true, then there exist $\varepsilon > 0$ and $T_0 \ge 0$ such that $s(t) < 1-\varepsilon$ for all $t \ge T_0$. By use of Corollary 4.1, it follows that $s(t)$ is strictly increasing and $\lim_{t\to\infty} s(t) \le 1-\varepsilon$. This is a contradiction with (4.19). Thus, (4.20) holds.

For the case $\beta b > (1-\sqrt{b})^2$, we have $\lambda_0 = (1-\sqrt{b})^2/(\beta b) < 1$. Combining this with (4.20), we can find a $T_1 > 0$ such that $\lambda_0 < s(T_1)$, i.e., $\lambda_0 < \frac{v(x,T_1)}{u(x,T_1)}$ for all $x \in \bar{\Omega}$. Then, by Theorem 4.2,

$$\lim_{t\to\infty} (u(x,t), v(x,t)) = (0,0)$$

uniformly on $\bar{\Omega}$. This is a contradiction because $(u(x,t), v(x,t))$ is a positive periodic solution.

For the case $\beta b = (1-\sqrt{b})^2$, thanks to (4.20), we have that either

$$s(T_2) \ge 1 \quad \text{for some } T_2 \ge 0, \tag{4.21}$$

or

$$\lim_{t\to\infty} s(t) = 1, \;\; s(t) < 1, \;\; \forall\, t \ge 0. \tag{4.22}$$

When (4.21) holds, we have $\frac{v(x,T_2)}{u(x,T_2)} \geq s(T_2) \geq 1$, i.e., $v(x,T_2) \geq u(x,T_2)$ on $\bar{\Omega}$. Take advantage of Theorem 4.3, $\lim_{t\to\infty}(u(x,t), v(x,t)) = (0,0)$ or $(\sqrt{b}, \sqrt{b})$ uniformly on $\bar{\Omega}$. This is a contradiction because $(u(x,t), v(x,t))$ is a nonconstant positive periodic solution. When (4.22) holds, by Corollary 4.1, $s(t)$ is strictly increasing for $t > 0$. This contradicts with (4.19). The proof is complete. □

Theorem 4.4. *Assume $d_1 = d_2 = d$, $\beta b < (1-\sqrt{b})^2$ and $\mu > 0$. If $u_0(x) \leq v_0(x)$ and $u_0(x) < \tilde{u}_1$ on $\bar{\Omega}$, then the following hold:*

(i) *$u(x,t) < \tilde{u}_1$ on $\bar{\Omega} \times [0,\infty)$;*
(ii) *The function $\max_{x\in\bar{\Omega}} u(x,t)$ is strictly decreasing in $t \in [0,\infty)$;*
(iii) *$\lim_{t\to\infty}(u(x,t), v(x,t)) = (0,0)$ uniformly on $\bar{\Omega}$.*

Proof. The proof will be divided into three steps.

Step 1. We first show that if $u(x,t) < \tilde{u}_1$ on $\bar{\Omega} \times [0,T]$ for some $0 < T < \infty$, then $\max_{x\in\bar{\Omega}} u(x,t)$ is strictly decreasing for $0 \leq t \leq T$. It is sufficient to prove

$$u(x,t) < \max_{x\in\bar{\Omega}} u_0(x), \quad \forall\, x \in \bar{\Omega},\ 0 < t \leq T.$$

Set $\eta = \max_{x\in\bar{\Omega},\, 0\leq t\leq T} u(x,t)$, then $\eta < \tilde{u}_1$. Because $\tilde{u}_1$ is the first root of

$$\phi(y) := \beta y - (1-y)(y/b - 1) = 0,$$

we have that $\phi(\eta) > 0$ and $\phi(y) > \phi(\eta)$ for all $0 < y < \eta$. Therefore,

$$\begin{aligned} &\beta u(x,t) - [1-u(x,t)](u(x,t)/b - 1) \\ &\quad = \phi(u(x,t)) \geq \phi(\eta) > 0, \quad \forall\, x \in \bar{\Omega},\ 0 \leq t \leq T. \end{aligned} \tag{4.23}$$

Denote $w = v - u$. Then, upon using (4.23), we have

$$h(u,0;1) = \frac{1}{d} u(x,t)\{\beta u(x,t) - [1-u(x,t)](u(x,t)/b - 1)\} > 0, \quad \forall\, x \in \bar{\Omega},\ 0 \leq t \leq T,$$

where $h(u,w;\lambda)$ is given by (4.2). Similarly to the proof of Lemma 4.1(i), it can be deduced that

$$v(x,t) > u(x,t), \quad \forall\, x \in \bar{\Omega},\ 0 < t \leq T.$$

This combined with (4.23) allows us to derive

$$u_t - d\Delta u < u[(1-u)(u/b - 1) - \beta u] \leq -\phi(\eta) u < 0, \quad \forall\, x \in \Omega,\ 0 < t \leq T.$$

It follows from the strong maximum principle that

$$u(x,t) < \max_{x\in\bar{\Omega}} u_0(x) < \tilde{u}_1, \quad \forall\, x \in \bar{\Omega},\ 0 < t \leq T.$$

Step 2. Noticing $u_0(x) < \tilde{u}_1$ on $\bar{\Omega}$, we can define

$$T_1 = \sup\{\tau : u(x,t) < \tilde{u}_1,\ \forall\, x \in \bar{\Omega},\ 0 \le t < \tau\}.$$

It will be shown that $T_1 = \infty$. Suppose on the contrary that $T_1 < \infty$, then there exists $x_0 \in \bar{\Omega}$ such that $u(x_0, T_1) = \tilde{u}_1$ and $u(x,t) < \tilde{u}_1$ for all $x \in \bar{\Omega}$, $0 \le t < T_1$. Obviously, for any $0 < t_2 < T_1$, there holds

$$u(x,t) < \tilde{u}_1, \quad \forall\, x \in \bar{\Omega},\ 0 \le t \le t_2.$$

By the conclusion of Step 1, $\max_{x\in\bar{\Omega}} u(x,t_2) < \max_{x\in\bar{\Omega}} u(x,t_1) < \tilde{u}_1$ for any $0 < t_1 < t_2$. Letting $t_2 \nearrow T_1$ and applying the continuity of $\max_{x\in\bar{\Omega}} u(x,t)$ with respect to t, we obtain that

$$\tilde{u}_1 = \max_{x\in\bar{\Omega}} u(x,T_1) \le \max_{x\in\bar{\Omega}} u(x,t_1) < \tilde{u}_1.$$

This contradiction shows $T_1 = \infty$, and so $u(x,t) < \tilde{u}_1$ on $\bar{\Omega} \times [0,\infty)$. Recalling the result of Step 1, we conclude that $\max_{x\in\bar{\Omega}} u(x,t)$ is strictly decreasing in $t \in [0,\infty)$. The conclusions (i) and (ii) are proved.

Step 3. Now we prove conclusion (iii). From the arguments in Steps 1 and 2 we can see that

$$v(x,t) > u(x,t), \quad \forall\, x \in \bar{\Omega},\ t > 0,$$

and $\max_{x\in\bar{\Omega}} u(x,t)$ is strictly decreasing in $t > 0$. Set $m = \max_{x\in\bar{\Omega}} u_0$ and

$$-k = (1-m)(m/b-1) - \beta m.$$

Then $m < \tilde{u}_1$. Similarly to Step 1, we have that $k > 0$ and

$$u_t - d\Delta u < u[(1-u)(u/b-1) - \beta u] \le -ku, \quad \forall\, x \in \Omega,\ t > 0.$$

Hence, $\lim_{t\to\infty} u(x,t) = 0$ uniformly on $\bar{\Omega}$. By the same way as in the proof of Theorem 4.1, it can be shown that $\lim_{t\to\infty} v(x,t) = 0$ uniformly on $\bar{\Omega}$. The proof is finished. □

Remark 4.2. This theorem shows that both predator and prey will become extinct if the initial density of the former surpasses that of the latter and their initial densities are less than $\tilde{u}_1$. As a result, $(\tilde{u}_1, \tilde{u}_1)$ is unstable.

5. Stationary patterns—nonconstant positive solutions of (2.4)

What is of interest in the prey–predator models is whether the various species can coexist. In the case where the species are homogeneously distributed, this would be indicated by a constant positive solution. In the spatially inhomogeneous case, the existence of nonconstant time-independent positive solutions, also called stationary patterns, is an indication of the dynamical richness of the systems.

In this section we study the existence and non-existence of nonconstant positive solutions of the problem (2.4).

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5.1. Estimates of positive solutions of (2.4)

To discuss the existence and nonexistence of nonconstant positive solutions of the problem (2.4), in this subsection we shall give a priori positive upper and lower bounds for the positive solutions. For this, we first state two known results.

Proposition 5.1 *(Harnack inequality [16]). Let $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a positive solution of $\Delta w(x) + c(x)w(x) = 0$, where $c \in C(\Omega) \cap L^\infty(\Omega)$, satisfying the homogeneous Neumann boundary condition. Then there exists a positive constant $\tilde{C}$ which depends only on M where $\|c\|_\infty \le M$ such that*

$$\max_{\bar{\Omega}} w \le \tilde{C} \min_{\bar{\Omega}} w.$$

Proposition 5.2 *(Maximum principle [17]). Let $g \in C(\bar{\Omega})$, and $b_j \in C(\bar{\Omega})$, $j = 1, 2, \dots, N$.*

(i) *Assume that $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ and satisfies*

$$\begin{cases} \Delta u + \displaystyle\sum_{j=1}^{N} b_j(x)u_{x_j} + g(x) \ge 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} \le 0, & x \in \partial\Omega. \end{cases}$$

If $u(x_0) = \max_{\bar{\Omega}} u$, then $g(x_0) \ge 0$.

(ii) *Assume that $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ and satisfies*

$$\begin{cases} \Delta u + \displaystyle\sum_{j=1}^{N} b_j(x)u_{x_j} + g(x) \le 0, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} \ge 0, & x \in \partial\Omega. \end{cases}$$

If $u(x_0) = \min_{\bar{\Omega}} u$, then $g(x_0) \le 0$.

Making use of Propositions 5.1 and 5.2, we can derive the following results.

Theorem 5.1. *Let (u, v) be a positive solution of (2.4). Then*

$$b \le \max_{\bar{\Omega}} u \le 1, \quad 0 < \min_{\bar{\Omega}} u \le \min_{\bar{\Omega}} v \le \max_{\bar{\Omega}} v \le \max_{\bar{\Omega}} u \le 1.$$

Theorem 5.2. *Let $d^* > 0$ be a fixed constant. Then there exists a constant $c_* = c_*(d^*, b, \beta, \Omega, N) > 0$ such that, for any $d_1 \ge d^*$, every possible positive solution (u, v) of (2.4) satisfies*

$$0 < b \le \max_{\bar{\Omega}} u \le c_* \min_{\bar{\Omega}} u.$$

Corollary 5.1. *Let $d^* > 0$ be a fixed constant. Then there is a constant $\underline{C} = \underline{C}(d^*, b, \beta, \Omega, N) > 0$ such that, for any $d_1 \geq d^*$, every possible positive solution (u, v) of (2.4) satisfies*

$$\min_{x\in\bar{\Omega}} u \geq \underline{C}, \quad \min_{x\in\bar{\Omega}} v \geq \underline{C}.$$

The following result follows by the standard Schauder theory for elliptic equations, its proof will be omitted here.

Theorem 5.3. *Let $d_0 > 0$ be a fixed constant. Then there exists a positive constant $\bar{C} = \bar{C}(d_0, b, \beta, \mu, \Omega, N)$ such that, for all $d_1, d_2 \geq d_0$, every possible positive solution of (2.4) satisfies $(u, v) \in [C^{2+\alpha}(\bar{\Omega})]^2$ and*

$$\|u\|_{2+\alpha} \leq \bar{C}, \quad \|v\|_{2+\alpha} \leq \bar{C}.$$

5.2. Nonexistence of nonconstant positive solutions of (2.4)

In this subsection, we shall give some conditions to guarantee the nonexistence of nonconstant positive solutions of (2.4). We first consider the case $\beta b \geq (1 - \sqrt{b})^2$.

Theorem 5.4. *Assume that $\beta b \geq (1 - \sqrt{b})^2$. If $\mu > \frac{d_2}{d_1}\frac{(1-b)^2}{4b}$, then the problem (2.4) has no nonconstant positive solution.*

Proof. On the contrary we assume that (2.4) has a nonconstant positive solution (u, v). We claim that

$$0 < \lambda_0 := \min_{x\in\bar{\Omega}} \frac{v(x)}{u(x)} < 1. \tag{5.1}$$

If (5.1) is not true, then $v(x) \geq u(x)$ for all $x \in \bar{\Omega}$. It is easy to see that $v \not\equiv u$ since (u, v) is a nonconstant solution of (2.4). Thus, $v \geq, \not\equiv u$, i.e., $v/u \geq, \not\equiv 1$. Integrating the second equation of (2.4) over Ω we have

$$0 = \mu \int_\Omega v(1 - v/u)\mathrm{d}x < 0.$$

This contradiction shows that (5.1) holds.

For $0 < \lambda \leq 1$, we set $w(x) = v(x) - \lambda u(x)$. Then

$$h(u, 0; \lambda) = \lambda u[\frac{1}{d_2}\mu(1 - \lambda) - \frac{1}{d_1}(1 - u)(u/b - 1) + \frac{1}{d_1}\beta\lambda u]$$

where $h(u, w; \lambda)$ is given by (4.2). Define

$$K(u, \lambda) = \frac{1}{d_2}\mu(1 - \lambda) - \frac{1}{d_1}(1 - u)(u/b - 1) + \frac{1}{d_1}\beta\lambda u.$$

Taking advantage of $\mu > \frac{d_2}{d_1}\frac{(1-b)^2}{4b}$ and $\beta b \geq (1-\sqrt{b})^2$, we obtain that, for any $u > 0$,

$$K(u,0) = \frac{1}{d_2}\mu - \frac{1}{d_1}(1-u)(u/b-1) > 0,$$
$$K(u,1) = \frac{1}{d_1}\beta\lambda - \frac{1}{d_1}(1-u)(u/b-1)u \geq 0,$$

which implies $K(u,\lambda) > 0$ for all $0 < \lambda < 1$ and $u > 0$. Hence

$$h(u,0;\lambda) > 0, \quad \forall\, 0 < \lambda < 1,\ x \in \bar{\Omega}. \tag{5.2}$$

By (5.1), there exists $x_0 \in \bar{\Omega}$ such that $0 < \lambda_0 = \frac{v(x_0)}{u(x_0)} < 1$. Certainly, $h(u,0,;\lambda_0) > 0$ by (5.2). Let $w_0(x) = v(x) - \lambda_0 u(x)$. Then $w_0(x)$ satisfies

$$\begin{cases} -\Delta w_0 = h(u,w_0;\lambda_0), & x \in \Omega, \\ \dfrac{\partial w_0}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

According to $\lambda_0 = \frac{v(x_0)}{u(x_0)}$ and (5.1), we have

$$w_0(x_0) = 0, \quad w_0(x) \geq 0 \text{ on } \bar{\Omega}.$$

Since $h(u(x_0),0;\lambda_0) > 0$, by the strong maximum principle and Hopf boundary lemma, it derives that $w_0(x) > 0$ on $\bar{\Omega}$. This contradicts to the fact that $w_0(x_0) = 0$, and so (2.4) has no nonconstant positive solution. $\square$

Now we discuss the case $\beta b < (1-\sqrt{b})^2$.

Lemma 5.1. *Suppose that $\beta b < (1-\sqrt{b})^2$. Let $d_{1j} \in (0,\infty)$, and $(u^{(j)}, v^{(j)})$ be the positive solutions of (2.4) with $d_1 = d_{1j}$. Assume that $d_{1j} \to \hat{d}_1 \in [0,\infty]$, and*

$$(u^{(j)}, v^{(j)}) \to (u^*, v^*)$$

uniformly on $\bar{\Omega}$. If u^*, v^* are positive constants, then $(u^*, v^*) = (\tilde{u}_1, \tilde{u}_1)$ or $(u^*, v^*) = (\tilde{u}_2, \tilde{u}_2)$.

The proof of this lemma is the same as that of [21, Lemma 2], and the details are omitted here.

Theorem 5.5. *Suppose that $\beta b < (1-\sqrt{b})^2$. Then there is a positive constant $\hat{d}_1 \gg 1$ such that, for any $d_1 \geq \hat{d}_1$, the problem (2.4) has no nonconstant positive solution.*

Proof. Step 1. Fix $0 < \alpha < 1$ and define

$$X = \left\{ u \in C^{\alpha}(\bar{\Omega}) : \int_{\Omega} u \mathrm{d}x = 0 \right\},$$

$$Y = \left\{ u \in C^{2+\alpha}(\bar{\Omega}) : \left. \frac{\partial u}{\partial \nu} \right|_{\partial\Omega} = 0 \right\},$$
$$Z = X \bigcap Y.$$

Let $\rho = d_1^{-1}$, and decompose $u = a + z$ with $a \in \mathbb{R}$ and $z \in Z$. Set

$$f_1(\rho, a, z, v) = \frac{1}{|\Omega|} \int_\Omega \left((a+z)(1-a-z)(\frac{a+z}{b} - 1) - \beta(a+z)v \right) \mathrm{d}x,$$
$$f_2(\rho, a, z, v) = \Delta z + \rho(a+z)(1-a-z)(\frac{a+z}{b} - 1) - \rho\beta(a+z)v - \rho f_1,$$
$$f_3(\rho, a, z, v) = d_2 \Delta v + \mu v(1 - v/u),$$
$$F(\rho, a, z, v) = (f_1, f_2, f_3)^T(\rho, a, z, v).$$

Then

$$F : \mathbb{R}^2 \times Z \times Y \to \mathbb{R} \times X \times C^\alpha(\bar{\Omega}),$$

and for any $\rho > 0$, (u, v) is the solution of (2.4) if and only if $F(\rho, a, z, v) = 0$. It is easy to verify that, for all ρ, $F(\rho, \tilde{u}_1, 0, \tilde{u}_1) = 0$.

Let ψ be the Fréchet derivative of F at $(0, \tilde{u}_1, 0, \tilde{u}_1)$ with respect to (a, z, v). A direct computation yields

$$\psi(a, z, v) = \begin{pmatrix} \frac{1}{|\Omega|} \int_\Omega (A_1 a + A_1 z - \beta \tilde{u}_1 v) \mathrm{d}x \\ \Delta z \\ d_2 \Delta v + \mu(a+z) - \mu v \end{pmatrix}, \quad \text{with } A_1 = (1 + 1/b)\tilde{u}_1 - 2\tilde{u}_1^2/b.$$

We will prove that ψ is one-to-one and surjective. It suffices to show that for any given

$$(a_0, z_0, v_0) \in \mathbb{R} \times X \times C^\alpha(\Omega),$$

the equation

$$\psi(\rho, a, z, v) = (a_0, z_0, v_0)$$

has a unique solution $(a, z, v) \in \mathbb{R} \times Z \times Y$, or equivalently, the following system has a unique solution:

$$\frac{1}{|\Omega|} \int_\Omega (A_1 a + A_1 z - \beta \tilde{u}_1 v) \mathrm{d}x = a_0, \tag{5.3}$$

$$\Delta z = z_0, \quad x \in \Omega; \quad \left. \frac{\partial z}{\partial \nu} \right|_{\partial\Omega} = 0; \quad \int_\Omega z \mathrm{d}x = 0, \tag{5.4}$$

$$\left.\begin{array}{ll} -d_2\Delta v+\mu v=\mu(a+z)-v_0, & x\in\Omega, \\ \dfrac{\partial v}{\partial \nu}=0, & x\in\Omega. \end{array}\right\} \tag{5.5}$$

Since $z_0\in X$, the problem (5.4) has a unique solution $z\in Z$. Recall $A_1>\beta\tilde{u}_1$. Take

$$a=\frac{1}{A_1-\beta\tilde{u}_1}\left(a_0-\frac{\beta\tilde{u}_1}{\mu|\Omega|}\int_\Omega v_0\mathrm{d}x\right). \tag{5.6}$$

Then the problem (5.5) has a unique solution v, and v satisfies

$$\mu\int_\Omega v dx=\mu a|\Omega|-\int_\Omega v_0\mathrm{d}x. \tag{5.7}$$

Note that by (5.7) and $\int_\Omega z\mathrm{d}x=0$, it is easy to verify that such a pair (a,v) satisfies (5.3). This shows that the system (5.3)–(5.5) has a solution. On the other hand, any solution (a,v) of (5.3) and (5.5) must satisfy (5.6) and (5.7), i.e., the solution of (5.3)–(5.5) is unique. In conclusion, ψ is a one-to-one and surjective map between two Banach spaces. Therefore, ψ^{-1} exists and is a bounded linear operator.

We note that, by the implicit function theorem, there are two constants $\delta_1>0$ and $\varepsilon_1>0$ such that, for all $0<\rho<\delta_1$, in the neighborhood B_{ε_1} of $(\tilde{u}_1,0,\tilde{u}_1)$, the equation $F(\rho,a,z,v)=0$ admits a unique solution, which must be $(\tilde{u}_1,0,\tilde{u}_1)$. Therefore, when $d_1>1/\delta_1$, in a small neighborhood of $(\tilde{u}_1,\tilde{u}_1)$, the equation (2.4) has only constant solution $(\tilde{u}_1,\tilde{u}_1)$.

Similarly to the above, there exists a constant $\delta_2>0$ such that, when $d_1>1/\delta_2$, in a small neighborhood of $(\tilde{u}_2,\tilde{u}_2)$, the equation (2.4) has only constant solution $(\tilde{u}_2,\tilde{u}_2)$.

Step 2. We will prove the desired conclusion by contradiction. Suppose that $(u^{(j)},v^{(j)})$ are the nonconstant positive solutions of (2.4) with $d_1=d_{1j}$ and $d_{1j}\to\infty$. According to Theorem 5.3, we may assume that $(u^{(j)},v^{(j)})\to(u,v)$ in $[C^2(\bar{\Omega})]^2$. By virtue of Corollary 5.1, $u^{(j)}(x),\ v^{(j)}(x)\geq\underline{C}$, and so $u,\ v\geq\underline{C}$ on $\bar{\Omega}$. It is obvious that $u\equiv u^*$ is a positive constant, and $v>0$ satisfies

$$\begin{cases} -d_2\Delta v=\mu v(1-v/u^*), & x\in\Omega, \\ \dfrac{\partial v}{\partial\nu}=0, & x\in\Omega. \end{cases}$$

Therefore, $v\equiv u^*$. By Lemma 5.1, we have $(u,v)=(\tilde{u}_1,\tilde{v}_1)$ or $(u,v)=(\tilde{u}_2,\tilde{u}_2)$. If $(u,v)=(\tilde{u}_1,\tilde{v}_1)$, then there exists $j_0\gg 1$ such that

$$d_{1j}>\max\{1/\delta_1,1/\delta_2\},\quad \|(u^{(j)},v^{(j)})-(\tilde{u}_1,\tilde{v}_1)\|_{[C(\bar{\Omega})]^2}<\varepsilon_1,\quad \forall\ j\geq j_0.$$

By the conclusion of Step 1, $(u^{(j)},v^{(j)})=(\tilde{u}_1,\tilde{v}_1)$ for each $j\geq j_0$. This contradicts with our assumption. The case $(u,v)=(\tilde{u}_2,\tilde{u}_2)$ is also impossible. $\square$

Finally, we give a nonexistence result without any limitations for the coefficients b, β and μ.

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Theorem 5.6. *There exists a large constant d such that the problem (2.4) has no nonconstant positive solution provided d_1, $d_2 \geq d$.*

Proof. Let (u, v) be a positive solution of (2.4), and denote

$$\bar{u} = \frac{1}{|\Omega|}\int_\Omega u(x)\mathrm{d}x, \quad \bar{v} = \frac{1}{|\Omega|}\int_\Omega v(x)\mathrm{d}x.$$

Then, multiplying the equation of u in (2.4) by $u - \bar{u}$, and integrating the result over Ω, we have

$$\begin{aligned} d_1\int_\Omega |\nabla(u-\bar{u})|^2\mathrm{d}x &= \int_\Omega (u-\bar{u})u(1-u)(u/b-1)\mathrm{d}x - \int_\Omega \beta(u-\bar{u})uv\mathrm{d}x \\ &\equiv I_1 + I_2. \end{aligned}$$

Recalling $\int_\Omega (u-\bar{u})\mathrm{d}x = 0$ and $\bar{u} \leq 1$, it yields

$$\begin{aligned} I_1 &= \frac{1}{b}\int_\Omega (u-\bar{u})\left\{(u-\bar{u})[\alpha - u^2 + (1+b-\bar{u})u] + \alpha\bar{u}\right\}\mathrm{d}x \\ &= \frac{1}{b}\int_\Omega (u-\bar{u})^2[\alpha - u^2 + (1+b-\bar{u})u]\mathrm{d}x \\ &\leq \frac{1}{b}\int_\Omega (u-\bar{u})^2\Big[\frac{1}{4}(1+b-\bar{u})^2 + \bar{u}(1+b-\bar{u}) - b\Big]\mathrm{d}x, \end{aligned}$$

where $\alpha = \bar{u}(1+b-\bar{u}) - b$. By the careful calculation we can get

$$\frac{1}{4}(1+b-\bar{u})^2 + \bar{u}(1+b-\bar{u}) - b \leq \frac{1}{3}(1+b^2-b).$$

Thus

$$I_1 \leq \frac{b^2-b+1}{3b}\int_\Omega (u-\bar{u})^2\mathrm{d}x.$$

Applying Corollary 5.1 with $d^* = 1$, it follows that

$$\begin{aligned} I_2 &= -\beta\int_\Omega [(u-\bar{u})^2 v + \bar{u}(u-\bar{u})v]\mathrm{d}x \\ &\leq -\beta\int_\Omega [\underline{C}(u-\bar{u})^2 + \bar{u}(u-\bar{u})v]\mathrm{d}x \\ &= -\beta\int_\Omega [\underline{C}(u-\bar{u})^2 + \bar{u}(u-\bar{u})(v-\bar{v})]\mathrm{d}x \end{aligned}$$

$$\leq \frac{\beta(1-2\underline{C})}{2}\int_\Omega (u-\bar{u})^2 \mathrm{d}x + \frac{\beta}{2}\int_\Omega (v-\bar{v})^2 \mathrm{d}x.$$

Therefore,

$$d_1 \int_\Omega |\nabla(u-\bar{u})|^2 \mathrm{d}x \leq \left(\frac{\beta(1-2\underline{C})}{2} + \frac{b^2-b+1}{3b}\right)\int_\Omega (u-\bar{u})^2 \mathrm{d}x + \frac{\beta}{2}\int_\Omega (v-\bar{v})^2 \mathrm{d}x.$$

It can be followed from Theorems 5.1 and 5.2 that

$$u(x) \leq c_* v(x), \quad u(x) \leq c_* \bar{v}, \quad \frac{\bar{v}^2}{\bar{u}u(x)} \leq c_*^2, \ \ \forall\, x \in \bar{\Omega}.$$

Using this fact, similarly to the above, we have

$$\begin{aligned}
d_2 \int_\Omega |\nabla(v-\bar{v})|^2 \mathrm{d}x &= \int_\Omega \mu v(v-\bar{v})(1-\frac{v}{u})\mathrm{d}x \\
&= \mu \int_\Omega \left((v-\bar{v})^2(1-\frac{v}{u}) + \bar{v}(v-\bar{v})(1-\frac{v}{u})\right)\mathrm{d}x \\
&= \mu \int_\Omega \left((v-\bar{v})^2(1-\frac{v}{u}) - (v-\bar{v})^2\frac{\bar{v}}{u} - (v-\bar{v})\frac{\bar{v}^2}{u}\right)\mathrm{d}x \\
&= \mu \int_\Omega \left((v-\bar{v})^2(1-\frac{v+\bar{v}}{u}) + (v-\bar{v})\frac{\bar{v}^2}{\bar{u}} - (v-\bar{v})\frac{\bar{v}^2}{u}\right)\mathrm{d}x \\
&\leq \mu \int_\Omega \left((1-\frac{2}{c_*})(v-\bar{v})^2 + (v-\bar{v})\frac{\bar{v}^2}{\bar{u}} - (v-\bar{v})\frac{\bar{v}^2}{u}\right)\mathrm{d}x \\
&= \mu \int_\Omega \left((1-\frac{2}{c_*})(v-\bar{v})^2 + (v-\bar{v})(u-\bar{u})\frac{\bar{v}^2}{\bar{u}u}\right)\mathrm{d}x \\
&\leq \mu\left(1-\frac{2}{c_*}+\frac{c_*^2}{2}\right)\int_\Omega (v-\bar{v})^2 \mathrm{d}x + \frac{1}{2}\mu c_*^2 \int_\Omega (u-\bar{u})^2 \mathrm{d}x.
\end{aligned}$$

Summing up the above estimates we have

$$d_1 \int_\Omega |\nabla(u-\bar{u})|^2 \mathrm{d}x + d_2 \int_\Omega |\nabla(v-\bar{v})|^2 \mathrm{d}x \leq M \int_\Omega \left[(u-\bar{u})^2 + (v-\bar{v})^2\right]\mathrm{d}x$$

for some positive constant $M > 0$. Then, by use of the Poincaré inequality, there exists a constant $C > 0$ such that

$$d_1 \int_\Omega |\nabla(u-\bar{u})|^2 \mathrm{d}x + d_2 \int_\Omega |\nabla(v-\bar{v})|^2 \mathrm{d}x \le C \int_\Omega \left(|\nabla(u-\bar{u})|^2 + |\nabla(v-\bar{v})|^2\right) \mathrm{d}x.$$

It follows that, when $d_1, d_2 \gg 1$, $\nabla(u-\bar{u}) = \nabla(v-\bar{v}) = 0$, i.e., $u \equiv \bar{u}$, $v \equiv \bar{v}$. $\square$

5.3. Existence of nonconstant positive solutions of (2.4)

In this part, we shall discuss the existence of nonconstant positive solutions of (2.4). We assume that $\beta b < (1-\sqrt{b})^2$. Therefore, the problem (2.4) has two positive constant solutions: $\tilde{\mathbf{u}}_1 = (\tilde{u}_1, \tilde{u}_1)$ and $\tilde{\mathbf{u}}_2 = (\tilde{u}_2, \tilde{u}_2)$.

Throughout this subsection, μ_j, $\mathbf{X}_j$, $\mathbf{X}$ and A_i are given in Section 2.

The problem (2.4) can be rewritten as

$$\begin{cases} -D\Delta \mathbf{u} = G(\mathbf{u}), & x \in \Omega, \\ \dfrac{\partial \mathbf{u}}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

where $D = \mathrm{diag}(d_1,\ d_2)$, or equivalently,

$$F(d_1, d_2; \mathbf{u}) = \mathbf{u} - (I-\Delta)^{-1}\{D^{-1}G(\mathbf{u}) + \mathbf{u}\} = 0 \ \text{ on } \mathbf{X}, \tag{5.8}$$

where $(I-\Delta)^{-1}$ is the inverse of $I-\Delta$ with homogeneous Neumann boundary condition. By direct computation, we have

$$D_{\mathbf{u}}F(d_1, d_2; \tilde{\mathbf{u}}_i) = I - (I-\Delta)^{-1}\{D^{-1}G_{\mathbf{u}}(\tilde{\mathbf{u}}_i) + I\} = 0, \quad i = 1, 2.$$

We note that for each $\mathbf{X}_j$, ξ is an eigenvalue of $D_{\mathbf{u}}F(d_1, d_2; \tilde{\mathbf{u}}_i)$ on $\mathbf{X}_j$ if and only if $\xi(1+\mu_j)$ is an eigenvalue of the matrix

$$M_{ij} = \mu_j I - D^{-1}G_{\mathbf{u}}(\tilde{\mathbf{u}}_i) = \begin{pmatrix} \mu_j - A_i/d_1 & \beta\tilde{u}_i/d_1 \\ -\mu/d_2 & \mu/d_2 - \mu_j \end{pmatrix}.$$

The direct computations yield

$$\begin{aligned} \det M_{ij} &= \frac{1}{d_1 d_2}[d_1 d_2 \mu_j^2 + (\mu d_1 - d_2 A_i)\mu_j + \mu(\beta\tilde{u}_i - A_i)], \\ \mathrm{Tr} M_{ij} &= 2\mu_j + \frac{1}{d_2}\mu - \frac{1}{d_1}A_i. \end{aligned}$$

Denote

$$H_i(d_1, d_2; \lambda) := d_1 d_2 \lambda^2 + (\mu d_1 - d_2 A_i)\lambda + \mu(\beta\tilde{u}_i - A_i), \quad i = 1, 2, \tag{5.9}$$

then $H_i(d_1, d_2; \mu_j) = d_1 d_2 \det M_{ij}$.

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For $i = 1$. Since $\beta\tilde{u}_1 < A_1$, we see that $H_1(d_1, d_2, \lambda) = 0$ has two real roots:

$$\lambda_-^1(d_1, d_2) = \frac{d_2 A_1 - \mu d_1 - \sqrt{(d_2 A_1 - \mu d_1)^2 - 4 d_1 d_2 \mu(\beta\tilde{u}_1 - A_1)}}{2 d_1 d_2} < 0,$$
$$\lambda_+^1(d_1, d_2) = \frac{d_2 A_1 - \mu d_1 + \sqrt{(d_2 A_1 - \mu d_1)^2 - 4 d_1 d_2 \mu(\beta\tilde{u}_1 - A_1)}}{2 d_1 d_2} > 0.$$

For $i = 2$. Notice that $\beta\tilde{u}_2 > A_2$. If

$$d_2 A_2 - \mu d_1 > 2\sqrt{d_1 d_2 \mu(\beta\tilde{u}_2 - A_2)}, \tag{5.10}$$

then $H_2(d_1, d_2; \lambda) = 0$ has two positive roots:

$$\lambda_-^2(d_1, d_2) = \frac{d_2 A_2 - \mu d_1 - \sqrt{(d_2 A_2 - \mu d_1)^2 - 4 d_1 d_2 \mu(\beta\tilde{u}_2 - A_2)}}{2 d_1 d_2},$$
$$\lambda_+^2(d_1, d_2) = \frac{d_2 A_2 - \mu d_1 + \sqrt{(d_2 A_2 - \mu d_1)^2 - 4 d_1 d_2 \mu(\beta\tilde{u}_2 - A_2)}}{2 d_1 d_2}.$$

Here we remark that for any fixed $d_1 > 0$, (5.10) must be true for the large d_2.

Set

$$S_p = \{\mu_0, \mu_1, \mu_2, \ldots\},$$
$$B_i(d_1, d_2) = \{\lambda \geq 0 : \lambda_-^{(i)}(d_1, d_2) < \lambda < \lambda_+^{(i)}(d_1, d_2)\}, \quad i = 1, 2,$$

and let $m(\mu_j)$ be the multiplicity of μ_j. It is easy to see that

$$\lim_{d_2 \to \infty} \lambda_-^i(d_1, d_2) = 0, \qquad \lim_{d_2 \to \infty} \lambda_+^i(d_1, d_2) = A_i / d_1, \quad i = 1, 2. \tag{5.11}$$

Now, we state a known lemma (cf. [22, Lemma 5.1]) which gives the formula of index.

Lemma 5.2. *Let $i = 1$ or 2. Suppose that $H_i(d_1, d_2; \mu_j) \neq 0$ for all $\mu_j \in S_p$. Then*

$$\text{index}(F(d_1, d_2; .), \tilde{\mathbf{u}}_i) = (-1)^{\gamma_i},$$

where

$$\gamma_i = \begin{cases} \displaystyle\sum_{\mu_j \in B_i \cap S_p} m(\mu_j) & \text{if } B_i \cap S_p \neq \emptyset, \\ 0 & \text{if } B_i \cap S_p = \emptyset. \end{cases}$$

In particular, if $H_i(d_1, d_2; \lambda) > 0$ for all $\lambda \geq 0$, then $\gamma_i = 0$.

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Theorem 5.7. *Let $d_1 > 0$ be fixed and $\beta b < (1-\sqrt{b})^2$. For the integer $l \geq 1$, we define $\sigma_l = \sum_{i=0}^{l} m(\mu_i)$. Assume that $A_1/d_1 \in (\mu_k, \mu_k + 1)$, $A_2/d_1 \in (\mu_q, \mu_q + 1)$ for some $q \geq 1$, $k \geq 1$. If $\sigma_k + \sigma_q$ is odd, then there is a positive constant d_2^*, such that for any $d_2 \geq d_2^*$, the problem (2.4) has at least one nonconstant positive solution.*

Proof. Firstly, as $\beta\tilde{u}_2 > A_2$, we can find a $\hat{d}_2 \gg 1$ such that (5.10) holds for all $d_2 \geq \hat{d}_2$.

According to $A_1/d_1 \in (\mu_k, \mu_k + 1)$, $A_2/d_1 \in (\mu_q, \mu_q + 1)$ and (5.11), there exists $d_0 > \hat{d}_2$ such that, for all $d_2 \geq d_0$,

$$\begin{cases} \lambda_-^1(d_1, d_2) < 0, \quad \mu_k < \lambda_+^1(d_1, d_2) < \mu_{k+1}, \\ 0 < \lambda_-^2(d_1, d_2) < \mu_1, \quad \mu_q < \lambda_+^2(d_1, d_2) < \mu_{q+1}. \end{cases} \tag{5.12}$$

Taking advantage of Theorem 5.6, there exists $d_1^* > d_0$ such that (2.4) has no nonconstant positive solution for all $d_1, d_2 \geq d_1^*$. Moreover, we can choose d_1^* so large that $A_i/d_1^* < \mu_1$, $i = 1, 2$. Applying (5.11) once again, there exists a constant $d_2^* > d_1^*$ such that

$$\lambda_-^1(d_1^*, d_2^*) < 0 < \lambda_+^1(d_1^*, d_2^*) < \mu_1, \quad 0 < \lambda_-^2(d_1^*, d_2^*) < \lambda_+^2(d_1^*, d_2^*) < \mu_1. \tag{5.13}$$

We will prove that, for any $d_2 \geq d_2^*$, (2.4) has at least one nonconstant positive solution. Suppose, on the contrary that, for some $d_2 \geq d_2^*$, (2.4) has no nonconstant positive solution.

For these fixed d_1^*, d_2^*, d_1 and d_2, we define

$$D(t) = \begin{pmatrix} td_1 + (1-t)d_1^* & 0 \\ 0 & td_2 + (1-t)d_2^* \end{pmatrix}, \quad 0 \leq t \leq 1$$

and consider the problem

$$\begin{cases} -\Delta \mathbf{u} = D^{-1}(t)G(\mathbf{u}), & x \in \Omega, \\ \dfrac{\partial \mathbf{u}}{\partial \nu} = 0, & x \in \Omega. \end{cases} \tag{5.14}$$

Noted that $\mathbf{u}$ is a nonconstant positive solution of (2.4) if and only if it is such a solution of (5.14) for $t = 1$. It is obvious that $\tilde{\mathbf{u}}_1$ and $\tilde{\mathbf{u}}_2$ are constant positive solutions of (5.14). And for any $0 \leq t \leq 1$, $\mathbf{u}$ is a nonconstant solution of (5.14) if and only if it is such a solution of the problem

$$\Phi(\mathbf{u}; t) = \mathbf{u} - (I - \Delta)^{-1}\{D^{-1}(t)G(\mathbf{u}) + \mathbf{u}\} = 0 \quad \text{on } \mathbf{X},$$

where $\mathbf{X}$ is given by (2.3). It is obvious that

$$\Phi(\mathbf{u}; 1) = F(d_1, d_2; \mathbf{u}), \quad \Phi(\mathbf{u}; 0) = F(d_1^*, d_2^*; \mathbf{u}),$$

and

$$D_{\mathbf{u}}F(d_1, d_2; \tilde{\mathbf{u}}_i) = I - (I - \Delta)^{-1}\{D^{-1}G_{\mathbf{u}}(\tilde{\mathbf{u}}_i) + I\} = 0,$$
$$D_{\mathbf{u}}F(d_1^*, d_2^*; \tilde{\mathbf{u}}_i) = I - (I - \Delta)^{-1}\{\tilde{D}^{-1}G_{\mathbf{u}}(\tilde{\mathbf{u}}_i) + I\} = 0,$$

where $F(d_1, d_2; \mathbf{u})$ is defined by (5.8) and $D = \mathrm{diag}(d_1, d_2)$, $\tilde{D} = \mathrm{diag}(d_1^*, d_2^*)$. The above arguments show that both equations $\Phi(\mathbf{u}; 1) = 0$ and $\Phi(\mathbf{u}; 0) = 0$ have no nonconstant positive solution.

It follows from (5.12) and (5.13) that

$$B_1(d_1, d_2) \cap S_p = \{\mu_0, \mu_1, \mu_2 \ldots \mu_k\}, \quad B_1(d_1^*, d_2^*) \cap S_p = \{\mu_0\},$$
$$B_2(d_1, d_2) \cap S_p = \{\mu_1, \mu_2, \ldots, \mu_q\}, \quad B_2(d_1^*, d_2^*) \cap S_p = \emptyset.$$

Therefore

$$\begin{aligned}
&\mathrm{index}(\Phi(\cdot; 1), \tilde{\mathbf{u}}_1) = \mathrm{index}(F(d_1, d_2; \cdot), \tilde{\mathbf{u}}_1) = (-1)^{\sigma_k},\\
&\mathrm{index}(\Phi(\cdot; 1), \tilde{\mathbf{u}}_2) = \mathrm{index}(F(d_1, d_2; \cdot), \tilde{\mathbf{u}}_2) = (-1)^{\sigma_q - 1},\\
&\mathrm{index}(\Phi(\cdot; 0), \tilde{\mathbf{u}}_1) = \mathrm{index}(F(d_1^*, d_2^*; \cdot), \tilde{\mathbf{u}}_1) = -1,\\
&\mathrm{index}(\Phi(\cdot; 0), \tilde{\mathbf{u}}_2) = \mathrm{index}(F(d_1^*, d_2^*; \cdot), \tilde{\mathbf{u}}_2) = 1.
\end{aligned}$$

Since $\sigma_k + \sigma_q - 1$ is even, we have

$$\begin{cases} \mathrm{index}(\Phi(\cdot; 1), \tilde{\mathbf{u}}_1) + \mathrm{index}(\Phi(\cdot; 1), \tilde{\mathbf{u}}_2) = 2 \ \text{ or } -2, \\ \mathrm{index}(\Phi(\cdot; 0), \tilde{\mathbf{u}}_1) + \mathrm{index}(\Phi(\cdot; 0), \tilde{\mathbf{u}}_2) = 0. \end{cases} \tag{5.15}$$

Making use of Theorem 5.1 and Corollary 5.1, there exists a positive constant $\underline{C}$ such that, for all $0 \le t \le 1$, the positive solution (u, v) of (5.14) satisfies $\underline{C} < u(x), v(x) < 2$ on $\bar{\Omega}$. Set

$$\Sigma = \{\mathbf{u} = (u, v) \in \mathbf{X} : \underline{C} < u(x), v(x) < 2 \text{ on } \bar{\Omega}\}.$$

Then $\Phi(u; t) \neq 0$ for all $u \in \partial\Sigma$ and $0 \le t \le 1$. By the homotopy invariance of the Leray–Schauder degree,

$$\deg(\Phi(\cdot; 0), \Sigma, 0)) = \deg(\Phi(\cdot; 1), \Sigma, 0)). \tag{5.16}$$

Since both equations $\Phi(\mathbf{u}; 0) = 0$ and $\Phi(\mathbf{u}; 1) = 0$ have only two positive constant solutions $\tilde{\mathbf{u}}_1$, $\tilde{\mathbf{u}}_2$ in Σ, by (5.15), we have

$$\begin{aligned}
&\deg(\Phi(\cdot; 0), \Sigma, 0) = \mathrm{index}(\Phi(\cdot; 0); \tilde{\mathbf{u}}_1) + \mathrm{index}(\Phi(\cdot; 0); \tilde{\mathbf{u}}_2) = 0,\\
&\deg(\Phi(\cdot; 1), \Sigma, 0) = \mathrm{index}(\Phi(\cdot; 1); \tilde{\mathbf{u}}_1) + \mathrm{index}(\Phi(\cdot; 1); \tilde{\mathbf{u}}_2) = 2 \ \text{ or } -2,
\end{aligned}$$

this contradicts with (5.16), and the proof is complete. □

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