ARTICLE IN PRESS YJDEQ:8479

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Wave-breaking phenomena for the nonlocal Whitham-type equations

Feiyao Ma [a], Yue Liu [a,b], Changzheng Qu [a,*]

[a] *Department of Mathematics, Ningbo University, Ningbo 315211, China*
[b] *Department of Mathematics, University of Texas, Arlington, TX 76019-0408, USA*

Received 8 December 2015

Abstract

In this paper, the formation of singularities for the nonlocal Whitham-type equations is studied. It is shown that if the lowest slope of flows can be controlled by its highest value with the bounded Whitham-type integral kernel initially, then the finite-time blow-up will occur in the form of wave-breaking. This refined wave-breaking result is established by analyzing the monotonicity and continuity properties of a new system of the Riccati-type differential inequalities involving the extremal slopes of flows. Our theory is illustrated via the Whitham equation, Camassa–Holm equation, Degasperis–Procesi equation, and their μ-versions as well as hyperelastic rod equation.

MSC: 35B44; 35G25

Keywords: Wave-breaking; Whitham-type equations; Camassa–Holm equation; μ-Camassa–Holm equation; Degasperis–Procesi equation; Hyperelastic rod equation

* Corresponding author.
E-mail address: quchangzheng@nbu.edu.cn (C. Qu).

http://dx.doi.org/10.1016/j.jde.2016.08.027
0022-0396/

1. Introduction

Consideration here is the issue of wave-breaking phenomena for the general nonlocal Whitham-type equation, namely,

$$\begin{cases} u_t + uu_x + \int_{\mathbb{R}} K(x-\xi)Q(u,u_x)(t,\xi)d\xi = 0, \\ u(0,x) = u_0(x), \end{cases} \qquad t>0, \quad x \in \mathbb{R}, \text{ or } \mathbb{S}. \tag{1.1}$$

Wave-breaking phenomena, that means wave profile remains bounded while its slope becomes infinite in finite time, usually appeared in water waves. It is natural to know what kind of nonlinear model equations can prescribe wave breaking phenomena. This is one of the most intriguing long-standing problems in water-wave theory [35]. To understand this issue, Whitham suggested to consider the model equation (1.1) with the functional $Q(u,u_x) = u_x$ and the singular kernel

$$K(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \left(\frac{\tanh \xi}{\xi}\right)^{\frac{1}{2}} e^{ix\xi} d\xi, \tag{1.2}$$

and conjectured that this model in (1.1) with the kernel (1.2) can describe wave-breaking phenomena [35]. It is noted that some integrable models in water waves, including the Camassa–Holm equation and its μ-version, Degasperis–Procesi equation and its μ-version, can be rewritten as the form of the Whitham-type equation in (1.1). Indeed, these model equations can describe wave breaking phenomena, see the refs. [5,9,12,14,15,18,22,27,29], for instance.

It is known that the Korteweg–de Vries (KdV) equation

$$u_t + u_{xxx} + 6uu_x = 0, \tag{1.3}$$

is the simplest integrable equation and can be used to model the unidirectional propagation of long waves in shallow water. One important feature of the KdV equation (1.3) is that it has the smooth solitons, that is, the solitary waves keep their shape and height after interaction. However, the KdV equation can not describe wave breaking phenomena but permanent waves only [23].

On the other hand, the Camassa–Holm (CH) equation [5,21], which is closely related to the KdV equation [20,21,34], defined in the following form

$$y_t + 2yu_x + uy_x + \gamma u_x = 0, \quad y = u - u_{xx} \tag{1.4}$$

can be rewritten as the form of the Whitham-type equation in (1.1) with the function

$$Q(u,u_x) = u^2 + \frac{1}{2}u_x^2 + \gamma u \tag{1.5}$$

and the kernel K given by

$$K(x) = -\frac{1}{2}e^{-|x|}\,\mathrm{sgn}(x), \tag{1.6}$$

Please cite this article in press as: F. Ma et al., Wave-breaking phenomena for the nonlocal Whitham-type equations, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.08.027

where

$$\operatorname{sgn}(x)=\begin{cases}1, & x>0,\\ -1, & x<0.\end{cases} \tag{1.7}$$

The CH equation was derived as a model for the unidirectional propagation of the shallow water wave over a flat bottom, with $u(t,x)$ representing the water's free surface or velocity in non-dimensional variable [5,14]. It was also found earlier by using the method of recursion operator due to Fuchssteiner and Fokas [21]. Interestingly, it can also be derived by the tri-Hamiltonian duality approach due to Olver and Rosenau [34]. The CH equation admits two remarkable properties, which are not admitted by the KdV equation, one is existence of peakons [1,5], and another one is description of wave breaking phenomena [7–15]. The CH equation has a nonlocal version, that is the so-called μ-CH equation [24]

$$y_t+2yu_x+uy_x+\gamma u_x=0, \quad y=\mu(u)-u_{xx}, \tag{1.8}$$

where $\mu(u)=\int_0^1 u\,dx$.

The μ-CH equation is also integrable and possesses the Lax pair and bi-Hamiltonian structure. It describes the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal with external magnetic field and self-interaction, where the solution $u(t,x)$ of the μ-CH equation is the direct field of a nematic liquid crystal, x is a space variable in a reference frame moving with the linearized wave velocities [24]. Both the periodic CH equation and the μ-CH equation describe the geodesic flow on the diffeomorphism group $\mathcal{D}(\mathbb{S})$ over $\mathbb{S}$.

Another important integrable equation similar to the CH equation is the Degasperis–Procesi (DP) equation

$$y_t+3yu_x+uy_x+\gamma u_x=0, \quad y=u-u_{xx}, \tag{1.9}$$

which belongs to the Whitham-type equation (1.1) with the function

$$Q(u,u_x)=\frac{3}{2}u^2+\gamma u \tag{1.10}$$

and the kernel K given in (1.6). The DP equation can also be derived from the governing equations for water waves [14]. More interestingly, it admits the shock peakons [18]. The μ-version of the DP equation was introduced in [25] by Lenells, Misiołek and Tiğlay, which takes the form (1.9) with $y=\mu(u)-u_{xx}$. Note that both of the μ-CH and μ-DP equations can be rewritten as the Whitham-type equations in (1.1) with the function [25]

$$Q(u,u_x)=\lambda\mu(u)u+\frac{3-\lambda}{2}u_x^2+\gamma u \tag{1.11}$$

and the kernel K defined by [24]

$$K(x)=\begin{cases}0, & x=0,\\ x-\frac{1}{2}, & 0<x<1.\end{cases} \tag{1.12}$$

The choices of $\lambda=2$ and $\lambda=3$ yield the μ-CH and μ-DP equations, respectively.

Please cite this article in press as: F. Ma et al., Wave-breaking phenomena for the nonlocal Whitham-type equations, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.08.027

The wave-breaking of the CH equation, μ-CH equation, DP equation, μ-DP equation and the modified CH equation have been studied extensively, see the references [2–4,6–13,18,19,22,24,25,27–33,35], for instance. In particular, a variety of interesting results on the wave breaking of the CH equation has been obtained within two decades. The early result was shown by Constantin and Escher [9] that if the slope of the initial data is less than $-(1/\sqrt{2})\|u_0\|_{H^1(\mathbb{R})}$, then the wave breaking for the CH equation occurs in finite time. In [31], Mckean proved that the wave breaking happens for solutions of the CH equation if and only if some portion of the positive part of the initial momentum density $y_0 = u_0 - \partial_x^2 u_0$ lies to the left of some portion of its negative part. Recently, Brandolese [2] derived a local-in-space blow-up result. Precisely, it was shown in [2] that if the initial value u_0 satisfies $u_0'(x_0) + |u_0(x_0)| < 0$ at some point $x_0 \in \mathbb{R}$, then the wave-breaking occurs. This result asserts that local perturbation of data around that point does not prevent the singularity formation. For the μ-CH equation, it was shown in [24] that the solutions break down in the case of the zero-mean initial data. While in the case of the nonzero mean initial data, if the initial data satisfies $|\mu_0| \le \frac{1}{4}\|u_0'\|_{L^2(\mathbb{S})}$, then the wave breaks down in finite time. These results are extended in [19] with some weaker initial conditions.

The nonlinear evolution equation

$$u_t - u_{xxt} + 3uu_x - \lambda(2u_x u_{xx} + uu_{xxx}) + \gamma u_x = 0 \tag{1.13}$$

was derived by Dai [17] from a compressible hyperelastic material, where the constant λ depends on the prestress and the material parameters, γ is a constant. When $\lambda = 1$, it reduces to the celebrated CH equations. The wave breaking for the compressible hyperelastic rod equation was also studied extensively, a number of results have been obtained (see [2–4,16], for instance). Note that equation (1.13) can be rewritten as the form of the Whitham-type equation in (1.1) with the function

$$Q(u, u_x) = \frac{3-\lambda}{2}u^2 + \frac{\lambda}{2}u_x^2 + \gamma u \tag{1.14}$$

and the kernel K defined in (1.6).

The aim of the present paper is to investigate the wave-breaking phenomena for various models of the Whitham-type equation in (1.1). Our study is motivated from the work by Constantin and Escher in a study of wave-breaking phenomena to a Whitham-type equation [9]. Wave-breaking for the Whitham-type with $Q = u_x$ and a regular kernel K which is symmetric and monotonically decreasing on $\mathbb{R}_+$ was first studied in [32]. A rigorous argument was then given by Constantin and Escher in [9]. To understand the latter, let $m(t) = \inf_{x\in\mathbb{R}}\{u_x(t,x)\}$, $M(t) = \sup_{x\in\mathbb{R}}\{u_x(t,x)\}$. Then it can be attained at some points $x_1(t)$ and $x_2(t)$, respectively. By differentiating (1.1) with respect to x and evaluating the resulting equation at $x_1(t)$ and $x_2(t)$, the wave breaking result can be obtained from the following Riccati-type differential inequalities, namely,

$$\begin{cases} m'(t) \le -m^2(t) + K(0)\big(M(t) - m(t)\big), \\ M'(t) \le -M^2(t) + K(0)\big(M(t) - m(t)\big), \end{cases} \quad \text{a.e. } in\ t \in [0, T). \tag{1.15}$$

The wave-breaking then occurs at finite time if the initial condition

$$m(0) + M(0) < -2K(0) \tag{1.16}$$

is satisfied. It is of great interest to investigate whether or not the condition on the initial extremes in (1.16) is optimal, or a lower threshold for the breakdown of the solutions can be induced from the information in (1.16). On the other hand, it is observed from the above discussions that many of model equations in the water-wave theory can be rewritten as the Whitham-type equations [32]. To understand features of those models in a case-by-case study naturally led to the study of the general Whitham-type equation (1.1), which is part of work in the present study.

Before starting our findings, let us briefly look at our approach. Inspired from the approach by Constantin and Escher [9], we consider a general system of the Riccati-type differential inequalities for infimum $m(t)$ and supremum $M(t)$ of the slopes u_x to the solutions of the problem (1.1), that is

$$\begin{cases} m'(t) \le -am^2(t) + b(M(t) - m(t)) + c, \\ M'(t) \le -aM^2(t) + b(M(t) - m(t)) + c, \end{cases} \quad \text{a.e. } in\ t \in [0, T), \tag{1.17}$$

where $a > 0$, b, $c \ge 0$ are generic constants. Wave breaks down in finite time T is then understood in the sense that $\liminf_{t\to T^-} m(t) = -\infty$. To prevent the occurrence of the breaking wave, one challenge issue is how to balance between the lower bound of $m(t)$ and the upper bound of $M(t)$. One possible approach is to use the continuity and monotonicity properties, we could find that wave breaks down with $m \to -\infty$ at the finite time before $M \to +\infty$. In the other words, if $m(t)$ is bounded below then $M(t)$ must be bounded above. This led to our main result, Theorem 2.1, in the next section.

And yet there were some questions that remained open to us. For instance, due to the specific structure of the DP equation (in particular, the solution is not uniformly bounded, because of the lower regularity of the conservation laws), we can not deal with the DP equation directly by using the approach on the abstract form (1.15). This issue prompted us to examine the more general system of the Riccati-type differential inequalities

$$\begin{aligned} m'(t) &\le -am^2(t) + b(M(t) - m(t)) + f(t), \\ M'(t) &\le -aM^2(t) + b(M(t) - m(t)) + f(t), \end{aligned} \tag{1.18}$$

where $a > 0$ and $b > 0$ are constants, and $f(t)$ is a positive, continuous and nondecreasing function. Notice that in the case of the DP equation, the function $f(t)$ is a linear function of t. Since we consider that the wave breaks at a finite time T, one could dominate this non-decreasing continuous function $f(t)$ bounded in a larger bounded domain of time $t \in [0, T_0)$, $T_0 > T$ with a careful choice of the initial condition on $m(t)$ and $M(t)$. This thus implies that $f(t)$ can be replaced by its upper constant bound in some sense. A partially nice feature of this observation is that with some necessarily delicate analysis, it is possible to apply the first wave breaking result to study the problem associated with (2.18) and the result will be presented in Theorem 2.2.

It is noted that the results obtained in present paper illustrate the fact that the wave-breaking results of these model equations mentioned above can be improved substantially by applying the optimal conditions of wave-breaking from the general Whitham-type equation in (1.1).

The remainder of the paper is organized as follows. In Section 2, the principal result on the wave-breaking for the functions $m(t)$ and $M(t)$ associated with two different Riccati-type differential evolution systems is described, and proof of the result is then followed in details. As applications of the result, wave-breaking phenomena for the Whitham-type equation, μ-CH equation, DP equation and μ-DP equations as well as the compressible hyperelastic rod equation are further investigated, respectively, in Sections 3, 4, 5 and 6.

Notation. Throughout the paper, given a Banach space X, we denote its norm by $\|\cdot\|_X$. If there is no ambiguity, we omit the domain of function spaces.

2. The Riccati-type differential inequalities

In this, the primary section of the paper, we will investigate wave-breaking conditions on the extremal functions $m(t)$ and $M(t)$, where both $m(t)$ and $M(t)$ satisfy the Riccati-type differential inequalities (1.17) and (1.18). The corresponding wave-breaking results will be used in subsequent sections.

Our main result of the present paper may now be enunciated.

Theorem 2.1. *Let $m(t)$ and $M(t)$ be two continuous and almost everywhere differentiable functions defined in $t \in [0, T)$ with $T \leq \infty$ satisfying*

$$\begin{cases} m'(t) \leq -am^2(t) + b(M(t) - m(t)) + c, \\ M'(t) \leq -aM^2(t) + b(M(t) - m(t)) + c, \end{cases} \quad \text{a.e. in } t \in [0, T), \tag{2.1}$$

where a is a positive constant, b and c are non-negative constants, and $M(t)$ is a nonnegative function of t. Suppose that the initial data $m_0 = m(0)$ and $M_0 = M(0)$ satisfy

$$m_0 < \min\left\{-\frac{1}{a}\left(b + \sqrt{b^2 + ac}\right),\ -\frac{1}{2a}\left(b + \sqrt{b^2 + 4a(bM_0 + c)}\right)\right\}. \tag{2.2}$$

Then $m(t)$ is monotonically decreasing and breaks down in the finite time T_0 with

$$T_0 \leq t^* = \frac{am_0^2 + bm_0 - c}{a(am_0^2 + 2bm_0 - c)\sqrt{a(am_0^2 - b(M_0 - m_0) - c)}}, \tag{2.3}$$

in the sense that

$$\lim_{t \to T_0^-} \inf m(t) = -\infty.$$

In the case of $T_0 = t^*$, the wave-breaking rate can be estimated by

$$m(t) \leq \frac{am_0^2 + bm_0 - c}{am_0^2 + 2bm_0 - c} \frac{1}{t - t^*}.$$

Furthermore, if $m(t)$ is bounded below by some negative constant m_l, i.e. $m(t) \geq m_l$, then $M(t)$ is bounded by

$$M(t) \leq \max\left\{M_0, \frac{b + \sqrt{b^2 + 4a(c - bm_l)}}{2a}\right\}. \tag{2.4}$$

Proof. For convenience, we denote $B = b/a$, $C = c/a$ and

$$\beta(t) := m^2(t) - B\big(M(t) - m(t)\big) - C. \tag{2.5}$$

First we claim that the following two inequalities

$$2m(t) + B < 0, \tag{2.6}$$

and

$$m^2(t) + 2Bm(t) - C > 0 \tag{2.7}$$

hold for $t \in [0, T)$. The proof is approached by a contrary argument. Let

$$t^* = \sup\Big\{t \in [0, T);\ \ 2m(t) + B < 0,\ \ m^2(t) + 2Bm(t) - C > 0\Big\}.$$

It is easy to see from the assumption (2.2) that (2.6) and (2.7) hold at $t = 0$. We now claim that $t^* = T$. Indeed, if not, then there exists t_0 satisfying $t^* \le t_0 < T$ such that

$$2m(t_0) + B = 0 \quad or \quad m^2(t_0) + 2Bm(t_0) - C = 0, \tag{2.8}$$

and both (2.6) and (2.7) hold in $[0, t_0)$. Owing to (2.1), (2.6), (2.7) and (2.8), we deduce that

$$\begin{aligned}
\beta'(t) &= \Big(m^2 - B(M - m)\Big)' \\
&= (2m + B)m' - BM' \\
&\ge a(2m + B)\big(-m^2 + B(M - m) + C\big) - aB\big(-M^2 + B(M - m) + C\big) \\
&= -2am(m^2 + 2Bm - C) + aB(M + m)^2 \\
&> 0, \quad \text{a.e. } in\ [0, t_0).
\end{aligned} \tag{2.9}$$

It follows readily from (2.1) and the assumption (2.2) that

$$m'(t) \le -a\beta(t) \le -a\beta(0) < 0 \quad \text{a.e. } in\ [0, t_0). \tag{2.10}$$

Hence we have

$$(2m + B)' < 0 \quad \text{a.e. } in\ [0, t_0). \tag{2.11}$$

Again from (2.2) $m(0) + B < 0$, we also have

$$m(t) + B \le m(0) + B < 0, \quad t \in [0, t_0).$$

This thus implies the following inequality

$$(m^2(t) + 2Bm(t) - C)' = 2m'(t)(m(t) + B) > 0 \quad \text{a.e. } in\ [0, t_0), \tag{2.12}$$

Please cite this article in press as: F. Ma et al., Wave-breaking phenomena for the nonlocal Whitham-type equations, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.08.027

which is inferred that

$$m^2(t) + 2Bm(t) - C \geq m^2(0)^2 + 2Bm(0) - C, \quad t \in [0, t_0). \tag{2.13}$$

It then follows from (2.2) that

$$2m(0) + B < 0, \quad m^2(0) + 2Bm(0) - C > 0. \tag{2.14}$$

In view of the continuity of the functions m, this together with (2.11) and (2.13) contradicts with (2.8). Consequently, this completes the proof of the claim.

Now on account of (2.2), we have $\beta(0) > 0$. This in turn implies that

$$\beta(t) \geq \beta(0) > 0, \quad t \in [0, T). \tag{2.15}$$

Based on the argument, we are now ready to prove that $\beta(t)$ breaks down in finite time. By the definition of $\beta(t)$, we first have

$$m^2(t) = \beta(t) + B(M - m) + C \geq \beta(t),$$

which implies for sure that

$$-m(t) \geq \beta^{\frac{1}{2}}(t) \quad \text{a.e. } in \ [0, t_0).$$

Using again (2.10) we obtain

$$\begin{aligned}\beta'(t) &\geq -2am(t)(m^2(t) + 2Bm(t) - C) \geq 2a\beta^{\frac{1}{2}}(t)(m^2 + 2Bm - C)\\ &= 2a\frac{m^2 + 2Bm - C}{m^2 - B(M - m) - C}\beta^{\frac{3}{2}}(t).\end{aligned}$$

By (2.12) and (2.15) with the conditions that $B \geq 0$ and $M \geq 0$, we arrive at

$$\frac{m^2 + 2Bm - C}{m^2 - B(M - m) - C} \geq \frac{m^2 + 2Bm - C}{m^2 + Bm - C} > 0.$$

A straightforward computation then shows that

$$\begin{aligned}\left(\frac{m^2 + 2Bm - C}{m^2 + Bm - C}\right)'(t) &= \left(\frac{Bm}{m^2 + Bm - C}\right)'(t)\\ &= -\frac{B(m^2 + C)m'(t)}{(m^2 + Bm - C)^2} > 0 \ \text{ a.e. } in \ [0, T).\end{aligned}$$

It follows that

$$\frac{m^2 + 2Bm - C}{m^2 + Bm - C} > \frac{m_0^2 + 2Bm_0 - C}{m_0^2 + Bm_0 - C}.$$

Please cite this article in press as: F. Ma et al., Wave-breaking phenomena for the nonlocal Whitham-type equations, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.08.027

Hence

$$\beta'(t) \geq 2a\frac{m_0^2+2Bm_0-C}{m_0^2+Bm_0-C}\beta^{\frac{3}{2}}(t) \qquad \text{a.e. } in\ [0,T).$$

Denote $\delta = \frac{a(m_0^2+2Bm_0-C)}{m_0^2+Bm_0-C}$. It thus transpires that $\delta > 0$ and

$$\left(\beta^{-\frac{1}{2}}\right)'(t) = -\frac{1}{2}\beta^{-\frac{3}{2}}\beta'(t) \leq -\delta \quad \text{a.e. } in\ [0,T).$$

Integrating it from 0 to t yields

$$\beta^{-\frac{1}{2}}(t) \leq \beta^{\frac{1}{2}}(0) - \delta t. \tag{2.16}$$

Since $\beta(t) > 0$ for any $t \geq 0$, then $\beta(t)$ and $m(t)$ break down at the finite time

$$T_0 \leq t^* := \frac{1}{\delta\beta(0)^{\frac{1}{2}}} = \frac{m_0^2+Bm_0-C}{a(m_0^2+2Bm_0-C)\sqrt{m_0^2-B(M_0-m_0)-C}}.$$

In the case of $T_0 = t^*$, in view of (2.5) and (2.16), we have

$$m(t) \leq -\beta^{\frac{1}{2}}(t) \leq \frac{1}{\delta(t-t^*)}. \tag{2.17}$$

Consequently, this completes the proof of (2.3). For the case that $m(t)$ is bounded below, it is adduced that

$$\begin{aligned} M'(t) &\leq -aM^2(t) + b(M(t)-m(t)) + c \leq -aM^2(t) + b(M(t)-m_l) + c \\ &= -a\left(M - \frac{b-\sqrt{b^2+4a(c-bm_l)}}{2a}\right)\left(M - \frac{b+\sqrt{b^2+4a(c-bm_l)}}{2a}\right). \end{aligned}$$

Note that $M'(t) < 0$ when $M(t) > \frac{b+\sqrt{b^2+4a(c-bm_l)}}{2a}$ for all $t \in [0,T)$. This implies (2.4) and the proof of the theorem is complete. □

Remark 2.1. The wave-breaking time t^* in Theorem 2.1 could be optimized by more delicate estimate, which can be obtained from replacing $m^2(t) = \beta(t) + B(M(t)-m(t)) + C \geq \beta(t)$ by $m^2(t) \geq \beta(t) + C - Bm_0$. By investigating the proof in Theorem 2.1, we are able to obtain a better estimate of upper bound on wave-breaking time by

$$t^* = \frac{m_0^2+Bm_0-C}{2a\sqrt{C-Bm_0}(m_0^2+2Bm_0-C)} \ln \frac{\sqrt{m_0^2-BM_0}+\sqrt{C-Bm_0}}{\sqrt{m_0^2-BM_0}-\sqrt{C-Bm_0}}.$$

Please cite this article in press as: F. Ma et al., Wave-breaking phenomena for the nonlocal Whitham-type equations, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.08.027

As addressed in the introduction, in the applications for some equations including the DP equation and μ-DP equation, we can also derive the Riccati-type differential inequalities for the extremal values $m(t)$ and $M(t)$ but with a t-dependent function $f(t)$ instead of a constant c. To deal with these issues, we have to reformulate the Riccati-type differential inequalities in Theorem 2.1 then find a balance between wave-breaking time and growing time of the function $f(t)$. The corresponding result is now stated in the following.

Theorem 2.2. *Assume that $f(t)$ is a positive, continuous and nondecreasing function in $[0, T)$. Suppose that the functions $m(t)$ and $M(t) \geq 0$ satisfy*

$$\begin{aligned} m'(t) &\leq -am^2(t) + b(M(t) - m(t)) + f(t), \\ M'(t) &\leq -aM^2(t) + b(M(t) - m(t)) + f(t), \end{aligned} \tag{2.18}$$

with two positive constants a and b. Then there exists a constant m_1 depending on a, b and f such that if $m(0) = m_0 < m_1$, then $m(t)$ is decreasing and breaks down in finite time $T \leq t(m_0)$, where m_1 is the supremum of the set E composed by the real numbers z which guarantee the existence of positive solutions of the following equation for t:

$$t - \frac{\sqrt{a}(az^2 + bz - f(t))}{(az^2 + 2bz - f(t))\sqrt{az^2 - b(M_0 - z) - f(t)}} = 0, \tag{2.19}$$

and also satisfy

$$z \leq \min\left\{-\frac{b + \sqrt{b^2 + af(t(z))}}{a}, -\frac{b + \sqrt{b^2 + 4a(bM_0 + f(t(z)))}}{2a}\right\}, \tag{2.20}$$

where $t(z)$ is the smallest positive solution of (2.19). Furthermore, if $T = t(m_0)$, we have the estimate of wave-breaking rate:

$$m(t) \leq \frac{am_0^2 + bm_0 - f(t(m_0))}{a(am_0^2 + 2bm_0 - f(t(m_0)))} \frac{1}{t - t(m_0)}. \tag{2.21}$$

Proof. First we need to show the set E is not empty. Let

$$F(t, z) := \frac{\sqrt{a}(az^2 + bz - f(t))}{(az^2 + 2bz - f(t))\sqrt{az^2 - b(M_0 - z) - f(t)}}, \tag{2.22}$$

$$g(t, z) := t - F(t, z) \tag{2.23}$$

and

$$z^*(t) := \min\left\{-\frac{b + \sqrt{b^2 + af(t)}}{a}, -\frac{b + \sqrt{b^2 + 4a(bM_0 + f(t))}}{2a}\right\}. \tag{2.24}$$

If $z < z^*(0)$, it then follows from the proof of Theorem 2.1 that $F(0, z) > 0$. Hence, $g(0, z) < 0$. On the other hand, it's easy to see that for any fixed $t_0 > 0$, there exists a constant $z_*(t_0)$ depending only on t_0, a, b and f such that when $z < z_*(t_0)$ we have $F(t_0, z) < \frac{t_0}{2}$. Consequently

$$g(t_0, z) > \frac{t_0}{2} > 0.$$

Therefore, by the mean-value theorem on the continuous function of t, there exists a positive solution t^* of (2.19), $0 < t^* < t_0$ for $z < \min\{z_*(t_0), z^*(0)\}$. And we also deduce that if $z < \min\{z_*(t_0), z^*(t_0)\}$, then z satisfies (2.20), since $z^*(t_0) < z^*(0)$, and then z belongs to E. Hence, E is not empty.

Next, we claim that if z satisfies (2.20), then

$$\frac{\partial F(t,z)}{\partial z} > 0. \tag{2.25}$$

In fact, denote $B = b/a$, $C = f(t)/a$, $A_1 = z^2 + 2Bz - C$, $A_2 = z^2 + B(z - M_0) - C$, and $A_3 = z^2 - Bz - C$. A direct computation then shows that

$$\begin{aligned}\frac{\partial F}{\partial z} &= \frac{A_1A_2(2z+B) - 2A_3A_2(z+B) - A_1A_3(z+\frac{B}{2})}{aA_1^2A_2^{\frac{3}{2}}} \\ &= \frac{A_2B(z^2+C) - A_1A_3(z+\frac{B}{2})}{aA_1^2A_3^{\frac{3}{2}}}.\end{aligned}$$

In view of (2.20), it is found that $A_1, A_2, A_3 > 0$, and $z + B/2 < 0$. This in turn implies that (2.25) is valid.

Finally we show that if $z_1 \in E$, then $(-\infty, z_1) \subset E$. Moreover if $z_0 < z_1$, it must be the case $t(z_0) < t(z_1)$. To this end, it is found from the definition of E that

$$g(t(z_1), z_1) = t(z_1) - F(t(z_1), z_1) = 0.$$

It then follows from (2.25) that

$$\begin{aligned}g(t(z_1), z_0) &= t(z_1) - F(t(z_1), z_0) \\ &= F(t(z_1), z_1) - F(t(z_1), z_0) > 0.\end{aligned}$$

Notice that $g(0, z_0) < 0$. Applying the mean-value theorem again, positive solution of (2.19) exists and $t(z_0) < t(z_1)$. We also have that $z_0 < z_1 \le z^*(t(z_1)) \le z^*(t(z_0))$, hence (2.20) holds and then $z_0 \in E$. This implies $(-\infty, z_1) \subset E$.

It is also found that the set E is bounded above by zero, and the existence of m_1 is guaranteed. Now by the property of E, if $m_0 < m_1$ then $m_0 \in E$ and it is inferred that m_0 must satisfy the inequality

$$m_0 < \min\left\{-\frac{b+\sqrt{b^2+af(t(m_0))}}{a}, -\frac{b+\sqrt{b^2+4a(bM_0+f(t(m_0)))}}{2a}\right\}, \tag{2.26}$$

and $m(t)$, $M(t)$ satisfy (2.18). It then follows from Theorem 2.1 that the function $m(t)$ is decreasing and breaks down in finite time $T \le t(m_0)$. In the case of $T = t(m_0)$, we have the wave-breaking rate estimate (2.21), thereby concluding the proof of Theorem 2.2. □

Please cite this article in press as: F. Ma et al., Wave-breaking phenomena for the nonlocal Whitham-type equations, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.08.027

In particular, one can choose those initial momentum potentials m_0 in the theorem above more precisely to expect an upper bound of the wave-breaking time. This result is stated in the following.

Corollary 2.1. *Assume that $m(t)$ and $M(t)$ satisfy inequality (2.18) almost everywhere for $t \in [0, T)$, where $f(t)$ is positive and nondecreasing in $[0, +\infty)$. For any constant $\epsilon > 0$, if there holds*

$$m_0 < \frac{-(2+\epsilon)b - \sqrt{(2+\epsilon)^2b^2 + 4a(f(t_*) + bM_0)}}{2a}, \tag{2.27}$$

where $t_* = \frac{1}{\epsilon}\sqrt{\frac{2(b+\epsilon)}{(2+\epsilon)b+\sqrt{(2+\epsilon)^2b^2+4abM_0}}}$, then the extremal function $m(t)$ is decreasing and breaks down in finite time $T_0 \le t_*$. Moreover if $m(t)$ breaks down at T_0 with $T_0 = t_*$, then

$$m(t) \le \frac{am_0^2 + bm_0 - f(T_0)}{a(am_0^2 + 2bm_0 - f(T_0))} \cdot \frac{1}{t - T_0}. \tag{2.28}$$

Proof. Since f is nondecreasing, we have

$$\begin{aligned} m'(t) &\le -am^2(t) + b\big(M(t) - m(t)\big) + f(t^*), \\ M'(t) &\le -aM^2(t) + b\big(M(t) - m(t)\big) + f(t^*), \end{aligned} \tag{2.29}$$

for $0 \le t < t^*$, where t^* is an upper bound of the wave-breaking time, which will be determined later.

Let

$$m_* := \frac{-(2+\epsilon)b - \sqrt{(2+\epsilon)^2b^2 + 4a(f(t^*) + bM_0)}}{2a}.$$

Then it is not hard to check that $m_0 < m_*$ implies

$$m_0 < \min\left\{\frac{-b - \sqrt{b^2 + af(t^*)}}{a}, \frac{-b - \sqrt{b^2 + 4a(bM_0 + f(t^*))}}{2a}\right\}.$$

Hence it suffices to show the upper bound of the wave-breaking time in Theorem 2.1 satisfies

$$T_0 := \frac{am_0^2 + bm_0 - f(t^*)}{a(am_0^2 + 2bm_0 - f(t^*))\sqrt{m_0^2 - \frac{b}{a}(M_0 - m_0) - \frac{f(t^*)}{a}}} \le t^*.$$

Therefore, it remains to show that there exists a t^* such that the above inequality holds.

Please cite this article in press as: F. Ma et al., Wave-breaking phenomena for the nonlocal Whitham-type equations, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.08.027

Assume $m_0 < m_*$. Noticing that

$$m_*^2 + \frac{2b+\epsilon}{a} m_* - \frac{b}{a} M_0 - \frac{f(t^*)}{a} = 0, \tag{2.30}$$

it is thereby inferred that

$$\begin{aligned} m_0^2 + \frac{b}{a} m_0 - \frac{b}{a} M_0 - \frac{f(t^*)}{a} &\geq m_*^2 + \frac{b}{a} m_* - \frac{b}{a} M_0 - \frac{f(t^*)}{a} \\ &= -\frac{b+\epsilon}{a} m_* \\ &= \frac{(b+\epsilon)\big((2+\epsilon)b + \sqrt{(2+\epsilon)^2 b^2 + 4a(f(t^*) + bM_0)}\big)}{2a^2} \\ &\geq \frac{(b+\epsilon)\big((1+\frac{\epsilon}{2})b + \sqrt{(1+\frac{\epsilon}{2})^2 b^2 + abM_0}\big)}{a^2}. \end{aligned} \tag{2.31}$$

On the other hand, it is found that

$$\frac{d}{dm_0}\left(\frac{m_0^2 + \frac{b}{a} m_0 - \frac{f(t^*)}{a}}{m_0^2 + \frac{2b}{a} m_0 - \frac{f(t^*)}{a}} \right) \geq 0, \quad \text{as } m_0 < m_*.$$

Hence it is deduced from $m_0 < m_*$ that

$$\frac{m_0^2 + \frac{b}{a} m_0 - \frac{f(t^*)}{a}}{m_0^2 + \frac{2b}{a} m_0 - \frac{f(t^*)}{a}} \leq \frac{(m_*)^2 + \frac{b}{a} m_* - \frac{f(t^*)}{a}}{(m_*)^2 + \frac{2b}{a} m_* - \frac{f(t^*)}{a}} = \frac{\frac{b}{a} M_0 - \frac{b+\epsilon}{a} m_*}{\frac{b}{a} M_0 - \frac{\epsilon}{a} m_*} \leq \frac{b+\epsilon}{\epsilon}, \tag{2.32}$$

which together with (2.31) implies

$$T_0 \leq \frac{1}{\epsilon} \sqrt{\frac{2(b+\epsilon)}{(2+\epsilon)b + \sqrt{(2+\epsilon)^2 b^2 + 4abM_0}}}.$$

The result is thus derived by choosing $t_* = t^* = \frac{1}{\epsilon} \sqrt{\frac{2(b+\epsilon)}{(2+\epsilon)b + \sqrt{(2+\epsilon)^2 b^2 + 4abM_0}}}$ and applying Theorem 2.1. □

3. The Whitham-type equations

We now turn our attention to applications of wave-breaking for the Riccati-type equations in Theorem 2.1. In particular, we focus on the Whitham-type equation in this section, namely,

$$\begin{cases} u_t + uu_x + \int_{\mathbb{R}} K(x-\xi) u_\xi(t,\xi) d\xi = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{3.1}$$

The following wave-breaking result was obtained in an intriguing paper due to Constantin and Escher [9].

Proposition 3.1. *Assume that $K(x) \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ is symmetric and monotonically decreasing on $\mathbb{R}_+$, $K \neq 0$. A sufficiently asymptotic initial profile yields wave breaking. More precisely, if $u_0 \in H^\infty(\mathbb{R})$ satisfies*

$$\inf_{x\in\mathbb{R}}\{u_0'(x)\} + \sup_{x\in\mathbb{R}}\{u_0'(x)\} \le -2K(0) < 0, \tag{3.2}$$

then for the solution of (3.1) with initial data u_0 we observe wave breaking.

It is noted that there are also several models such as the collective motion of cells and traffic flows with Arrhenius look-ahead dynamics where the kernel K is non-symmetric and not monotonic [26]. It is thus quite natural to consider the more generic kernel $K(x)$ in (3.1). Indeed, in the same spirit as Theorem 2.1, we enable to establish the following result with more general kernel K so that the condition on initial data can be improved.

Theorem 3.1. *Let $u \in C^\infty([0,T); H^\infty(\mathbb{R}))$ be a solution of (3.1) before breaking. If $K(x)$ in (3.1) is regular (continuous and integrable over $\mathbb{R}$). Assume that the kernel $K(x)$ has the ordered extreme points (x_i, K_i), $i \in J_{m,n}$, $J_{m,n} := \{-m, -m+1, \cdots, 0, \cdots, n-1, n\}$, $m, n \in \{0\} \cup Z^+ \cup \{+\infty\}$, $K_i = K(x_i)$, and $\sum_{i\in J_{m,n}} |K_i|$ is bounded. Then $m(t) := \inf_{x\in\mathbb{R}} u_x(t,x)$ and $M(t) := \sup_{x\in\mathbb{R}} u_x(t,x)$ satisfy the following inequalities*

$$\begin{aligned} m'(t) &\le -m^2(t) + \sum_{i\in J_{m,n}} |K_i|\big(M(t) - m(t)\big), \\ M'(t) &\le -M^2(t) + \sum_{i\in J_{m,n}} |K_i|\big(M(t) - m(t)\big). \end{aligned} \tag{3.3}$$

If the initial data $m_0 = m(0)$ and $M_0 = M(0)$ satisfy

$$m_0 < \min\left\{-2\tilde{K}, -\frac{1}{2}\left(\tilde{K} + \sqrt{\tilde{K}^2 + 4\tilde{K}M_0}\right)\right\}, \tag{3.4}$$

where $\tilde{K} = \sum_{i\in J_{m,n}} |K_i|$, then $m(t)$ is monotonically decreasing and blows up in finite time T_0 with an estimate as

$$T_0 \le t^* = \frac{m_0 + \tilde{K}}{(m_0 + 2\tilde{K})\sqrt{m_0^2 - \tilde{K}(M_0 - m_0)}}.$$

Proof. Define two functions $m(t)$ and $M(t)$ by

$$\begin{aligned} m(t) &= \inf_{x\in\mathbb{R}} u_x(t,x) = u_x(t,\xi(t)), \\ M(t) &= \sup_{x\in\mathbb{R}} u_x(t,x) = u_x(t,\eta(t)), \end{aligned} \tag{3.5}$$

where $\xi(t)$ and $\eta(t)$ are some points in $\mathbb{R}$. Differentiating equation (3.1) with respect to x and evaluating the resulting equation respectively at $x = \xi(t)$ and $\eta(t)$, we obtain

$$\begin{cases} m'(t) + m^2(t) + \int_{\mathbb{R}} K(z) u_{xx}(t, \xi(t) - z) dz = 0, \\ M'(t) + M^2(t) + \int_{\mathbb{R}} K(z) u_{xx}(t, \eta(t) - z) dz = 0, \end{cases} \qquad \text{a.e. on } [0, T).$$

Using the second mean-value theorem, there exists $\alpha_i \in [x_{i-1}, x_i]$ such that

$$\begin{aligned} &\left| \int_{x_{i-1}}^{x_i} K(z) u_{xx}(t, \xi(t) - z) dz \right| \\ &\quad = |K(x_{i-1}) \int_{x_{i-1}}^{\alpha_i} u_{xx}(t, \xi(t) - z) dz + K(x_i) \int_{\alpha_i}^{x_i} u_{xx}(t, \xi(t) - y) dy| \\ &\quad = |-K_{i-1}[u_x(t, \xi(t) - \alpha_i) - u_x(t, \xi(t) - x_{i-1})] \\ &\qquad - K_i[u_x(t, \xi(t) - x_i) - u_x(t, \xi(t) - \alpha_i)]| \\ &\quad \le (|K_{i-1}| + |K_i|)(M - m). \end{aligned}$$

In the same way, it is then deduced with some $\alpha_i \in [x_{i-1}, x_i]$ and $\beta_i \in [x_i, x_{i+1}]$ that

$$\begin{aligned} &\left| \int_{x_{i-1}}^{x_{i+1}} K(z) u_{xx}(t, \xi(t) - z) dz \right| \\ &= \left| \left(\int_{x_{i-1}}^{x_i} + \int_{x_i}^{x_{i+1}} \right) K(z) u_{xx}(t, \xi(t) - z) dz \right| \\ &= \left| K(x_{i-1}) \int_{x_{i-1}}^{\alpha_i} u_{xx}(t, \xi(t) - z) dz + K(x_i) \int_{\alpha_i}^{x_i} u_{xx}(t, \xi(t) - z) dz \right. \\ &\quad \left. + K(x_i) \int_{x_i}^{\beta_i} u_{xx}(t, \xi(t) - z) dz + K(x_{i+1}) \int_{\beta_i}^{x_{i+1}} u_{xx}(t, \xi(t) - z) dz \right| \\ &= |-K_{i-1}[u_x(t, \xi(t) - \alpha_i) - u_x(t, \xi(t) - x_{i-1})] - K_i[u_x(t, \xi(t) - x_i) - u_x(t, \xi(t) - \alpha_i)] \\ &\quad - K_i[u_x(t, \xi(t) - \beta_i) - u_x(t, \xi(t) - x_i)] - K_{i+1}[u_x(t, \xi(t) - x_{i+1}) - u_x(t, \xi(t) - \beta_i)]| \\ &= |-K_{i-1}[u_x(t, \xi(t) - \alpha_i) - u_x(t, \xi(t) - x_{i-1})] - K_i[u_x(t, \xi(t) - \beta_i) - u_x(t, \xi(t) - \alpha_i)] \\ &\quad - K_{i+1}[u_x(t, \xi(t) - x_{i+1}) - u_x(t, \xi(t) - \beta_i)]| \\ &\le (|K_{i-1}| + |K_i| + |K_{i+1}|)(M - m), \end{aligned}$$

where and hereafter $K_j = K(x_j)$, $j = 1, 2, \cdots$. Hence we deduce that

$$
\begin{aligned}
&\left| \int_{-x_m}^{x_n} K(z)u_{xx}(t, \xi(t) - z)dz \right| \\
&= \left| \left(\int_{x_{-m}}^{x_{-m+1}} + \cdots + \int_{x_{-1}}^{x_0} + \int_{x_0}^{x_1} + \int_{x_1}^{x_2} + \cdots + \int_{x_{n-1}}^{x_n} \right) K(z)u_{xx}(t, \xi(t) - z)dz \right| \\
&\le \tilde{K}(M - m).
\end{aligned}
$$

Since K is integrable, then $\lim_{n\to\infty} |K_n| = 0$ and (3.3) is satisfied. Utilizing Theorem 2.1 and replacing b by $\tilde{K}$ with $a = 1$ and $c = 0$ in (2.1), we get the corresponding result. This completes the proof of Theorem 3.1. □

When $K(x)$ fulfills the same condition as in Proposition 3.1, we can get a more general result, which is a direct conclusion of Theorem 2.1.

Corollary 3.1. *If $K(x)$ in (3.1) is a regular (continuous and integrable over $\mathbb{R}$) and positive symmetric kernel, which decreases monotonically over $\mathbb{R}$. Let $u(t, x)$ be a solution of (3.1) with initial data $u_0(x) \in H^\infty(\mathbb{R})$ satisfying*

$$
m_0 := \inf_{x\in\mathbb{R}} u_x(0, x) < \min \left\{ -2K(0),\ \frac{-K(0) - \sqrt{K(0)^2 + 4K(0)M_0}}{2} \right\}, \tag{3.6}
$$

where $M_0 = \sup_{x\in\mathbb{R}} u_x(0, x)$. Then $m(t) := \inf_{x\in\mathbb{R}} u_x(t, x)$ is monotonically decreasing and blows up in finite time T_0 with an estimate

$$
T_0 \le t^* = \frac{m_0 + K(0)}{(m_0 + 2K(0))\sqrt{m_0^2 - K(0)(M_0 - m_0)}}.
$$

Furthermore, in the case of $T_0 = t^*$, we have the blowup rate estimate

$$
m(t) \le \frac{m_0 + K(0)}{m_0 + 2K(0)} \frac{1}{t - t^*}. \tag{3.7}
$$

Proof. Applying the result in [9], we have

$$
\begin{aligned}
m'(t) &\le -m^2(t) + K(0)(M(t) - m(t)), \\
M'(t) &\le -M^2(t) + K(0)(M(t) - m(t)).
\end{aligned}
$$

The results then can be obtained from Theorem 2.1 by setting $a = 1$, $b = K(0)$ and $c = 0$. □

Please cite this article in press as: F. Ma et al., Wave-breaking phenomena for the nonlocal Whitham-type equations, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.08.027

Remark 3.1. It is observed that the blowup condition in Proposition 3.1 by Constantin and Escher [9] can be improved by that in the above corollary. Indeed, it is easy to derive (3.6) from (3.2). On the other hand, the asymptotical property of conditions between (3.6) and (3.2) as M_0 tends to infinite is different: $|m_0| \gtrsim M_0$ as $M_0 \to +\infty$ from (3.2), while $|m_0| \gtrsim \sqrt{M_0}$ as $M_0 \to +\infty$ from (3.6).

If the kernel K is symmetric, then L^2-norm of solution of the Whitham-type equation (3.1) is conserved [32]. In this case, we have a wave-breaking result in the following.

Theorem 3.2. *Let $u_0 \in H^s(\mathbb{R})$, $s \geq 2$. Assume that the kernel $K \in L^1(\mathbb{R})$ is symmetric and the second derivative of the kernel $K'' \in L^2(\mathbb{R})$. Suppose that there exists $x_0 \in \mathbb{R}$ such that*

$$u_0'(x_0) < -\sqrt{\|K''\|_{L^2}\|u_0\|_{L^2}}.$$

Then the corresponding solution of (3.1) blows up in finite time T_0

$$T_0 \leq -\frac{1}{u_0'(x_0) + \sqrt{-u_0'(x_0)}(\|K''\|_{L^2}\|u_0\|_{L^2})^{\frac{1}{4}}}.$$

Proof. Differentiating (3.1) with respect to x yields

$$u_{tx} + uu_{xx} = -u_x^2 - \int_{\mathbb{R}} K'(x-\xi)u_x(t,\xi)d\xi. \tag{3.8}$$

Let $V(t) = u_x(t, q(t, x_0))$, where $q(t, x)$ is governed by the flow

$$\frac{dq(t,x)}{dt} = u(t,x), q(0,x) = x.$$

It follows from (3.8) that $V(t)$ satisfies

$$\frac{dV}{dt} = -V^2 - \int_{\mathbb{R}} K''(q(t,x_0)-\xi)u(t,\xi)d\xi. \tag{3.9}$$

Applying the Cauchy inequality yields the following estimate

$$\left|\int_{\mathbb{R}} K''(q(t,x_0)-\xi)u(t,\xi)d\xi\right| \leq \|K''\|_{L^2}\|u\|_{L^2} = \|K''\|_{L^2}\|u_0\|_{L^2},$$

which together with (3.9) implies

$$\frac{dV}{dt} = -V^2 + \|K''\|_{L^2}\|u_0\|_{L^2}.$$

If $V(0) < -(\|K''\|_{L^2}\|u_0\|_{L^2})^{\frac{1}{2}}$, then $V(t)$ satisfies

$$\frac{dV}{dt} = -(V^2 - V^2(0)).$$

Hence $V(t) \to -\infty$ as $t \to T_0 < \infty$. Consequently, $\inf_{x\in\mathbb{R}} u_x(t,x) \le V(t) \to -\infty$ as $t \to T_0$. This completes the proof of Theorem 3.2. □

4. The μ-CH equation

In this section, we turn now to consideration of wave-breaking for the μ-CH equation. Consider the following initial-value problem,

$$\begin{cases} y_t + 2yu_x + uy_x + \gamma u_x = 0, & t > 0, \ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \ge 0, \ x \in \mathbb{R}, \end{cases} \tag{4.1}$$

where $y = \mu(u) - u_{xx}$, $\mu(u) = \int_0^1 u(t,x)dx$, and $\gamma \neq 0$ is a constant. The following local-wellposedness result for problem (4.1) can be established by the method as in [24].

Proposition 4.1. *Let $u_0 \in H^s(\mathbb{S})$, $s > 3/2$. Then there exist a maximal life span $T > 0$ and a unique solution $u(t,x)$ to (4.1) such that $u(t,x) \in C([0,T); H^s(\mathbb{S})) \cap C^1([0,T); H^{s-1}(\mathbb{S}))$, which depends continuously on the initial data $u_0(x)$.*

Note that the Cauchy problem (4.1) can be rewritten as the following form

$$\begin{cases} u_t + uu_x + \partial_x K * \left(2\mu(u)u + \frac{1}{2}u_x^2 + \gamma u\right) = 0, & t > 0, \ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \ge 0, \ x \in \mathbb{R}, \end{cases} \tag{4.2}$$

where $K(x)$ is given by (1.12), and $v = K * w$ is determined explicitly by [24,25]

$$\begin{aligned} v(x) = {} & \frac{1}{2}\left(x^2 - x + \frac{13}{6}\right)\mu(w) + \left(x - \frac{1}{2}\right)\int_0^1\int_0^y w(z)dzdy \\ & - \int_0^x\int_0^y w(z)dzdy + \int_0^1\int_0^x\int_0^y w(z)dzdydx. \end{aligned} \tag{4.3}$$

It can be seen that the conservation laws are often useful in the investigation on the wave-breaking of solutions.

Lemma 4.1. *[19] Suppose $u_0 \in H^s$, $s \ge 2$, and let T be the maximum existence time of the solution $u(t,x)$ to the problem (4.1) with initial data $u_0(x)$. Then we have*

Please cite this article in press as: F. Ma et al., Wave-breaking phenomena for the nonlocal Whitham-type equations, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.08.027

$$\mu(u) = \mu(u_0), \quad E(t) = \left(\int_{\mathbb{S}} u_x^2 dx\right)^{\frac{1}{2}} = \left(\int_{\mathbb{S}} u_{0,x}^2 dx\right)^{\frac{1}{2}}.$$

Furthermore, we have the estimate

$$\|u - \mu(u_0)\|_{L^\infty} \le \frac{\sqrt{3}}{6} E(0). \tag{4.4}$$

We now consider $x_1,\ x_2 \in \mathbb{S}$ to be such that

$$u_{0,x}(x_1) = \inf_{x\in\mathbb{S}} u_{0,x}(x), \quad u_{0,x}(x_2) = \sup_{x\in\mathbb{S}} u_{0,x}(x).$$

Let us denote that

$$m(t) = \inf_{x\in\mathbb{S}} u_x(t,x), \quad M(t) = \sup_{x\in\mathbb{S}} u_x(t,x).$$

Then there exist $\xi(t), \eta(t) \in \mathbb{S}$ such that [9]

$$\begin{aligned} m(t) &= u_x(t,\xi(t)) = \inf_{x\in\mathbb{S}} u_x(t,x), \quad t \in (0,T), \\ M(t) &= u_x(t,\eta(t)) = \sup_{x\in\mathbb{S}} u_x(t,x), \quad t \in (0,T), \end{aligned} \tag{4.5}$$

where we can choose $\xi(0) = x_1$, $\eta(0) = x_2$. Differentiating (4.2) with respect to x leads to

$$u_{tx} + uu_{xx} + u_x^2 + \partial_x^2 K * \left(2\mu(u)u + \frac{1}{2}u_x^2 + \gamma u\right) = 0. \tag{4.6}$$

A direct computation then gives

$$u_{tx} + uu_{xx} + \frac{1}{2}u_x^2 - 2\mu(u)u + a + \gamma\partial_x K * u_x = 0, \tag{4.7}$$

where

$$a = \frac{1}{2}\int_{\mathbb{S}} u_x^2 dx + 2\mu^2(u) = \frac{1}{2}E^2(0) + 2\mu^2(u).$$

Furthermore, it is inferred from (4.3) that

$$\begin{aligned} \partial_x K * u_x = {} & \left(x - \frac{1}{2}\right)\int_0^1 (u_x(y) - u_x(0))dy - \int_0^x (u_x(y) - u_x(0))dy \\ & + \int_0^1\int_0^y (u_x(z) - u_x(0))dz dy. \end{aligned} \tag{4.8}$$

In view of the inequality (4.4), plugging (4.8) into (4.6), we arrive at

$$\begin{aligned} u_{tx} + \frac{1}{2}u_x^2 + uu_{xx} \leq & \frac{\sqrt{3}}{3}|\mu(u_0)|E(0) - \frac{1}{2}E^2(0) \\ & -\gamma\left[\left(x-\frac{1}{2}\right)\int_0^1 (u_x(y)-u_x(0))dy - \int_0^x (u_x(y)-u_x(0))dy \right. \\ & \left. + \int_0^1\int_0^y (u_x(z)-u_x(0))dzdy\right]. \end{aligned} \tag{4.9}$$

Setting $x = \xi(t)$ and $x = \eta(t)$ respectively in (4.9), we deduce that

$$\begin{aligned} m'(t) &\leq -\frac{1}{2}m^2(t) + \frac{5}{2}|\gamma|(M(t)-m(t)) + \frac{\sqrt{3}}{3}|\mu(u_0)|E(0) - \frac{1}{2}E^2(0), \\ M'(t) &\leq -\frac{1}{2}M^2(t) + \frac{5}{2}|\gamma|(M(t)-m(t)) + \frac{\sqrt{3}}{3}|\mu(u_0)|E(0) - \frac{1}{2}E^2(0). \end{aligned} \tag{4.10}$$

Using Theorem 2.1, we get the following wave-breaking result for solutions of (4.1).

Theorem 4.1. *Assume that $u \in C([0,T); H^s(\mathbb{S})) \cap C^1([0,T); H^{s-1}(\mathbb{S}))$, $s \geq 2$ is a solution of the Cauchy problem (4.1) with the initial value $u_0 \in H^s(\mathbb{S})$. Suppose that*

$$m_0 < \min\left\{-5|\gamma| - \sqrt{25\gamma^2 + 2C},\ -\frac{5}{2}|\gamma| - \sqrt{\frac{25}{4}\gamma^2 + 5|\gamma|M_0 + 2C}\right\},$$

where $C = \max\{0, \frac{\sqrt{3}}{3}|\mu(u_0)|E(0) - \frac{1}{2}E^2(0)\}$. Then the solution of (4.1) blows up in finite time

$$T_0 \leq t^* = \frac{4(m_0^2 + 5|\gamma|m_0 - 2C)}{(m_0^2 + 10|\gamma|m_0 - 2C)\sqrt{m_0^2 - 5|\gamma|(M_0 - m_0) - 2C}}.$$

Remark 4.1. It was shown in [19] that if the initial data $u_0(x)$ satisfies $|\mu_0| \geq \frac{\pi}{\sqrt{3}}\mu_1$, $\mu_1 = (\int_{\mathbb{S}} u_{0,x}^2 dx)^{\frac{1}{2}}$ and there exists a point x_0 such that $u_0'(x_0) < -\sqrt{\frac{2\sqrt{3}}{3}|\mu_0|\mu_1 - \mu_1^2}$, then the wave breaking occurs. It is obvious that this result is now improved by Theorem 4.1.

5. The DP and μ-DP equations

We now turn our attention to the DP and μ-DP equations. As is known that the solutions for DP and μ-DP equations are uniformly bounded but growing linearly in time t, one can not apply Theorem 2.1 directly but possibly formate it to the case suitable in Theorem 2.2. We first consider the Cauchy problem of the DP equation in the following,

Please cite this article in press as: F. Ma et al., Wave-breaking phenomena for the nonlocal Whitham-type equations, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.08.027

$$\begin{cases} u_t + uu_x + \partial_x K * (\frac{3}{2}u^2 + \gamma u) = 0, & t > 0, \ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{5.1}$$

where $K(x)$ is given in (1.6).

The following local well-posedness result was established in [29].

Lemma 5.1. *[29] Assume $u_0 \in H^s(\mathbb{R})$, $s > 3/2$, there exists a $T = T(u_0) > 0$ and a unique strong solution $u(t,x) \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$ to the problem (5.1) depending continuously on u_0. Moreover, there are the following three conserved densities*

$$E_1(u) = \int_{\mathbb{R}} (u^2 + u_x^2) dx, \quad E_2(u) = \int_{\mathbb{R}} yv dx, \quad E_3(u) = \int_{\mathbb{R}} u^3 dx, \tag{5.2}$$

where $y = (1 - \partial_x^2)u$ and $v = (4 - \partial_x^2)^{-1}u$.

To construct the Riccati-type differential inequalities (2.18), we differentiate DP equation (5.1) with respect to x to get

$$u_{xt} + uu_{xx} + u_x^2 = \frac{3}{2}u^2 - \frac{3}{2}K * u^2 - \gamma \partial_x K * u_x. \tag{5.3}$$

In view of the result in [29], we have the upper bound for solutions of (5.1)

$$\|u\|_{L^\infty} \le 3\|u_0\|_{L^2}^2 t + \|u_0\|_{L^\infty}. \tag{5.4}$$

It follows from (5.3) and (5.4) that $m(t)$ and $M(t)$ defined by (3.5) satisfy

$$m' \le -m^2 + |\gamma|(M - m) + \frac{3}{2}J^2(t),$$
$$M' \le -M^2 + |\gamma|(M - m) + \frac{3}{2}J^2(t),$$

where $J(t) = 3\|u_0\|_{L^2(\mathbb{R})}^2 t + \|u_0\|_{L^\infty(\mathbb{R})}$.

Applying Corollary 2.1 this time, we should have the following result.

Theorem 5.1. *Assume that $u \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$, $s > 3/2$ is a solution of the Cauchy problem (5.1) with the initial value $u_0 \in H^s(\mathbb{R})$. For any constant $\epsilon > 0$, if there holds*

$$m_0 < -(1 + \frac{\epsilon}{2})|\gamma| - \sqrt{(1 + \frac{\epsilon}{2})^2\gamma^2 + \frac{3}{2}J^2(t_*) + |\gamma|M_0}, \tag{5.5}$$

where $t_* = \frac{1}{\epsilon}\sqrt{\frac{2(|\gamma|+\epsilon)}{(2+\epsilon)|\gamma| + \sqrt{(2+\epsilon)^2\gamma^2 + 4|\gamma|M_0}}}$, then the extremal function $m(t)$ is decreasing and breaks down in finite time $T_0 \le t_*$.

Please cite this article in press as: F. Ma et al., Wave-breaking phenomena for the nonlocal Whitham-type equations, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.08.027

Next, we study the case of the μ-DP equation. Consider now the initial-value problem

$$\begin{cases} u_t + uu_x + \partial_x K * \big(3\mu(u)u + \gamma u\big) = 0, & t > 0, \ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(x+1,t) = u(x,t), & t \geq 0, \ x \in \mathbb{R}, \end{cases} \tag{5.6}$$

where $K(x)$ is given in (1.12).

The local well-posedness was already established in [25].

Lemma 5.2. *[25] Assume $u_0 \in H^s(\mathbb{S})$, $s > 3/2$. Then there exists a $T = T(u_0) > 0$ and a unique strong solution $u \in C([0,T); H^s(\mathbb{S})) \cap u \in C^1([0,T); H^{s-1}(\mathbb{S}))$ to the problem (5.6) which depends continuously on u_0. Moreover, there are the following three conserved densities*

$$F_1(u) = \mu(u), \ \ E_2(u) = \int_{\mathbb{S}} u^2 dx, \ \ E_3(u) = \int_{\mathbb{S}} \Big(9\mu(u)(\Lambda^{-1}\partial_x u)^2 + u^3\Big)\, dx. \tag{5.7}$$

It is also noted that [19]

$$|K * u_x| \leq \frac{1}{2}|\mu(u_0)| + 2\mu_2(u_0), \tag{5.8}$$

where $\mu_2(u) = (E_2(u))^{\frac{1}{2}}$.

Along the flow

$$\begin{cases} \frac{dq(t,x)}{dt} = u(t, q(t,x)), & t > 0, \quad x \in \mathbb{S}, \\ q(0,x) = x, \end{cases}$$

there holds for solutions of (5.6)

$$\frac{du}{dt} = -(3\mu_0 + \gamma)K * u_x.$$

It is inferred from the above expression that

$$-h_0 t + u_0(x) \leq u(t, q(t,x)) \leq h_0 t + u_0(x), \tag{5.9}$$

with

$$h_0 = \frac{3}{2}\mu_0^2 + 6|\mu_0|\mu_2 + |\gamma|(\frac{1}{2}|\mu_0| + 2\mu_2).$$

Differentiating (5.6) with respect to x leads to

$$u_{xt} + uu_{xx} + u_x^2 = 3\mu(u)(u - \mu(u)) - \gamma\partial_x K * u_x. \tag{5.10}$$

Owing to (5.8), (5.9) and (5.10), we have the following inequalities:

$$m'(t) \le -m^2(t) + |\gamma|(M(t) - m(t)) + J(t),$$
$$M'(t) \le -M^2(t) + |\gamma|(M(t) - m(t)) + J(t),$$

where $J(t) = 3|\mu_0|(h_0 t + \|u_0\|_{L^\infty} - \mu_0)$.

Making use of Corollary 2.1 again, we arrive at the following result.

Theorem 5.2. *Assume that $u \in C([0,T); H^s(\mathbb{S})) \cap C^1([0,T); H^{s-1}(\mathbb{S}))$, $s > 3/2$, is a solution of the Cauchy problem (5.6) with the initial value $u_0 \in H^s(\mathbb{S})$. For any constant $\epsilon > 0$, if there holds*

$$m_0 < -(1+\frac{\epsilon}{2})|\gamma| - \sqrt{(1+\frac{\epsilon}{2})^2\gamma^2 + J(t_*) + |\gamma| M_0}, \tag{5.11}$$

where $t_* = \frac{1}{\epsilon}\sqrt{\frac{2(|\gamma|+\epsilon)}{(2+\epsilon)|\gamma| + \sqrt{(2+\epsilon)^2\gamma^2 + 4|\gamma|M_0}}}$, then the extremal function $m(t)$ is decreasing and breaks down in finite time $T_0 \le t_*$.

6. The hyperelastic rod equation

In the last section, our attention is paid to the hyperelastic rod equation. Consider the initial-value problem of the hyperelastic rod equation, namely,

$$\begin{cases} u_t + \lambda u u_x + \partial_x K * (\frac{3-\lambda}{2}u^2 + \frac{\lambda}{2}u_x^2 + \gamma u) = 0, & t > 0, \ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{6.1}$$

where $K(x)$ is given by (1.6).

The following results of local well-posedness and wave-breaking criterion were established in [16].

Lemma 6.1. *[16] Assume $u_0 \in H^s(\mathbb{R})$, $s > 3/2$. Then there exist a maximal time $T > 0$ and a unique solution $u \in C([0,T); H^s(\mathbb{R})) \cap u \in C^1([0,T); H^{s-1}(\mathbb{R}))$ to the problem (6.1) depending continuously on $u_0(x)$. Moreover, there are the following two conserved densities*

$$E(u) = \int_{\mathbb{R}} (u^2 + u_x^2)dx, \quad F(u) = \int_{\mathbb{R}} (u^3 + \lambda u u_x^2 + \gamma u^2)dx. \tag{6.2}$$

Lemma 6.2. *[16] Assume $u_0(x) \in H^s(\mathbb{R})$, $s > 3/2$. Let $u(t,x)$ be the corresponding solution of (1.13) with life span T. Then*

$$\sup_{x\in\mathbb{R}, 0\le t<T} |u(t,x)| \le C(\|u_0\|_{H^1}),$$

and T is bounded if and only if

$$\liminf_{t\to T} \inf_{x\in\mathbb{S}} \{\mathrm{sgn}\lambda \, u_x(t,x)\} = -\infty.$$

Again, to apply Theorem 2.1, we need the following lemma.

Lemma 6.3. *[2] Let $0 < \lambda < 4$. Assume $u_0(x) \in H^s(\mathbb{R})$, $s \geq 2$. Then there holds the following inequality*

$$K * \left(\frac{3-\lambda}{2}u^2 + \frac{\lambda}{2}u_x^2\right)(t, \xi(t)) \geq C_1 u^2(t, \xi(t)), \tag{6.3}$$

where K is given in (1.6), and $C_1 = \frac{1}{4}(\sqrt{\lambda(12-3\lambda)} - \lambda)$.

Differentiating (6.1) with respect to x yields

$$u_{xt} + \lambda\left(uu_{xx} + \frac{1}{2}u_x^2\right) + \gamma\partial_x K * u_x + K * \left(\frac{3-\lambda}{2}u^2 + \frac{\lambda}{2}u_x^2\right) - \frac{3-\lambda}{2}u^2 = 0.$$

Using (6.3), we get the following estimate

$$\begin{aligned} & u_{xt} + \lambda\left(uu_{xx} + \frac{1}{2}u_x^2\right) \\ & = -\gamma\partial_x K * u_x - K * \left(\frac{3-\lambda}{2}u^2 + \frac{\lambda}{2}u_x^2\right) + \frac{3-\lambda}{2}u^2 \\ & \leq -\gamma\partial_x K * u_x + \left(\frac{3-\lambda}{2} - C_1\right)u^2 \\ & \leq -\gamma\partial_x K * u_x + C_2\|u_0\|_{H^1}^2, \end{aligned} \tag{6.4}$$

where $C_2 = \frac{1}{2}\max\{\frac{3-\lambda}{2} - C_1, 0\}$. It follows from (6.4) that $m(t)$ and $M(t)$ defined by (3.5) satisfy

$$\begin{aligned} m' &\leq -\frac{\lambda}{2}m^2 + |\gamma|(M-m) + C_2\|u_0\|_{H^1}^2, \\ M' &\leq -\frac{\lambda}{2}M^2 + |\gamma|(M-m) + C_2\|u_0\|_{H^1}^2. \end{aligned}$$

Applying Theorem 2.1, we arrive at the following result.

Theorem 6.1. *Assume that $0 < \lambda < 4$, and $u \in C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}))$, $s \geq 2$ is a solution of the Cauchy problem (6.1) with the initial value $u_0 \in H^s(\mathbb{R})$. Suppose that*

$$m_0 < \min\left\{-\frac{2\left(|\gamma| + \sqrt{\gamma^2 + \frac{\lambda}{2}C_2\|u_0\|_{H^1}^2}\right)}{\lambda}, -\frac{|\gamma| + \sqrt{\gamma^2 + 2\lambda(\lambda M_0 + C_2\|u_0\|_{H^1}^2)}}{\lambda}\right\}.$$

Then the corresponding solution of (6.1) breaks down in finite time T_0 with an estimate as

$$T_0 \leq t^* = \frac{4(\lambda m_0^2 + 2|\gamma|m_0 - 2C_2\|u_0\|_{H^1}^2)}{\lambda(\lambda m_0^2 + 4|\gamma|m_0 - 2C_2\|u_0\|_{H^1}^2)\sqrt{\lambda(\lambda m_0^2 - 2\gamma(M_0 - m_0) - 2C_2\|u_0\|_{H^1}^2)}}.$$

Please cite this article in press as: F. Ma et al., Wave-breaking phenomena for the nonlocal Whitham-type equations, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.08.027

Remark 6.1. The local-in-space criterion for the wave-breaking of solutions to the hyperelastic rod equation (6.1) was given in [2] by a delicate analysis. It was shown in [2] that the wave-breaking occurs if $1 \le \lambda \le 4$ and there exists $x_0 \in \mathbb{R}$ such that the initial data $u_0(x)$ satisfies $u_0'(x_0) < -\beta_\lambda \left| u_0(x_0) + \frac{\gamma}{3-\lambda} \right|$, where β_λ is a nonnegative constant depending on λ. However, it is noted that this local-in-space criterion is not valid for $0 < \lambda < 1$.

Acknowledgments

The work of Ma is partially supported by the National Natural Science Foundation of China #11201250 and Natural Science Foundation of Ningbo #2012A610033. The work of Qu is supported by the National Natural Science Foundation of China #11631007 and #11471174.

References

[1] M.S. Alber, R. Camassa, D.D. Holm, J.E. Marsden, The geometry of peaked solitons and billiard solutions of a class of integrable PDE's, Lett. Math. Phys. 32 (1994) 137–151.
[2] L. Brandolese, Local-in-space criteria for blowup in shallow water and dispersive rod equations, Comm. Math. Phys. 330 (2014) 401–414.
[3] L. Brandolese, M.F. Cortez, On permanent and breaking waves in hyperelastic rods and rings, J. Funct. Anal. 266 (2014) 6954–6987.
[4] L. Brandolese, A Liuoville theorem for the Degasperis–Procesi equation, http://xxx.lanl.gov/abs/arXiv:1405.6675v1, 2014.
[5] R. Camassa, D.D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71 (1993) 1661–1664.
[6] R.C. Chen, Y. Liu, C.Z. Qu, S.H. Zhang, Oscillation induced blow up in the modified Camassa–Holm equation with linear dispersion, Adv. Math. 272 (2015) 225–251.
[7] A. Constantin, Existence of permanent and breaking waves for a shallow water equation: a geometric approach, Ann. Inst. Fourier (Grenoble) 50 (2000) 321–362.
[8] A. Constantin, On the blow-up of solutions of a periodic shallow water equation, J. Nonlinear Sci. 10 (2000) 391–399.
[9] A. Constantin, J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, Acta Math. 181 (1998) 229–243.
[10] A. Constantin, J. Escher, On the structure of a family of quasilinear equations arising in shallow water theory, Math. Ann. 312 (1998) 403–416.
[11] A. Constantin, J. Escher, On the blow-up rate and the blow-up set of breaking waves for a shallow water equation, Math. Z. 233 (2000) 75–91.
[12] A. Constantin, J. Escher, Well-posedness, global existence and blow-up phenomena for a periodic quasilinear hyperbolic equation, Comm. Pure Appl. Math. 51 (1998) 475–504.
[13] A. Constantin, J. Escher, Global existence and blow-up for a shallow water equation, Ann. Sc. Norm. Super. Pisa 26 (1998) 303–328.
[14] A. Constantin, D. Lannes, The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations, Arch. Ration. Mech. Anal. 192 (2009) 165–186.
[15] A. Constantin, H.P. McKean, A shallow water equation on the circle, Comm. Pure Appl. Math. 52 (1999) 949–982.
[16] A. Constantin, W. Strauss, Stability of a class of solitary waves in compressible elastic rods, Phys. Lett. A 270 (2000) 140–148.
[17] H.H. Dai, Model equations for nonlinear dispersive waves in a compressible Mooney–Rivlin rod, Acta Mech. 127 (1998) 193–207.
[18] J. Escher, Y. Liu, Z. Yin, Shock waves and blow-up phenomena for the periodic Degasperis–Procesi equation, Indiana Univ. Math. J. 56 (2007) 87–117.
[19] Y. Fu, Y. Liu, C.Z. Qu, On the blow-up structure for the generalized periodic Camassa–Holm and Degasperis–Procesi equations, J. Funct. Anal. 262 (2012) 3125–3158.
[20] B. Fuchssteiner, Some tricks from the symmetry-toolbox for nonlinear equations: generalizations of the Camassa–Holm equation, Phys. D 95 (1996) 229–243.

[21] B. Fuchssteiner, A.S. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, Phys. D 4 (1981/1982) 47–66.
[22] G.L. Gui, Y. Liu, P. Olver, C.Z. Qu, Wave-breaking and peakons for a modified Camassa–Holm equation, Comm. Math. Phys. 319 (2013) 731–759.
[23] C. Kenig, G. Ponce, L. Vega, Well-posedness and scattering results for the generalized Korteweg–de Vries equation via the contraction principle, Comm. Pure Appl. Math. 46 (1993) 527–620.
[24] B. Khesin, J. Lenells, G. Misiołek, Generalized Hunter–Saxton equation and the geometry of the group of circle diffeomorphisms, Math. Ann. 342 (2008) 617–656.
[25] J. Lenells, G. Misiołek, F. Tiğlay, Integrable evolution equations on spaces of tensor densities and their peakon solutions, Comm. Math. Phys. 299 (2010) 129–161.
[26] Y. Lee, H. Liu, Thresholds for shock formation in traffic flow models with Arrhenius look-ahead dynamics, Discrete Contin. Dyn. Syst. 35 (2015) 323–339.
[27] Y.A. Li, P. Olver, Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation, J. Differential Equations 162 (2000) 27–63.
[28] H.L. Liu, Wave breaking in a class of nonlocal dispersive wave equations, J. Nonlinear Math. Phys. 13 (2006) 441–466.
[29] Y. Liu, Z. Yin, Global existence and blow-up phenomena for the Degasperis–Procesi equation, Comm. Math. Phys. 267 (2006) 801–820.
[30] H. Lundmark, Formation and dynamics of shock waves in the Degasperis–Procesi equation, J. Nonlinear Sci. 17 (2007) 169–198.
[31] H.P. Mckean, Breakdown of a shallow water equation, Asian J. Math. 2 (1998) 867–874.
[32] P. Naumkin, I. Shishmarev, Nonlinear Nonlocal Equations in the Theory of Waves, Transl. Math. Monogr., vol. 133, Amer. Math. Soc., Providence, RI, 1994.
[33] Y. Liu, P. Olver, C.Z. Qu, S.H. Zhang, On the blow-up of solutions to the integrable modified Camassa–Holm equation, Anal. Appl. 12 (2014) 355–368.
[34] P.J. Olver, P. Rosenau, Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support, Phys. Rev. E 53 (1996) 1900–1906.
[35] G.B. Whitham, Linear and Nonlinear Waves, Pure Appl. Math., Wiley–Interscience, New York, London, Sydney, 1974.

Please cite this article in press as: F. Ma et al., Wave-breaking phenomena for the nonlocal Whitham-type equations, J. Differential Equations (2016), http://dx.doi.org/10.1016/j.jde.2016.08.027