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Lower bounds of eigenvalues for a class of bi-subelliptic operators [☆]

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Abstract

Let Ω be a bounded open domain in $\mathbb{R}^n$ with smooth boundary and $X = (X_1, X_2, \cdots, X_m)$ be a system of real smooth vector fields defined on Ω with the boundary $\partial\Omega$ which is non-characteristic for X. If X satisfies the Hörmander's condition, then the vector fields are finitely degenerate and the sum of square operators $\triangle_X = \sum_{i=1}^m X_i^2$ is a subelliptic operator. Let λ_k be the k-th eigenvalue for the bi-subelliptic operator $\triangle_X^2$ on Ω. In this paper, we introduce the generalized Métivier's condition and study the lower bounds of Dirichlet eigenvalues for the operator $\triangle_X^2$ on some finitely degenerate systems of vector fields X which satisfy the Hörmander's condition or the generalized Métivier's condition. By using the subelliptic estimates, we shall give a explicit lower bound estimates of λ_k which is polynomial increasing in k with the order relating to the Hörmander index or the generalized Métivier index.

Keywords: Eigenvalues; Finitely degenerate elliptic operators; Hörmander's condition; Métivier's condition; Subelliptic estimate; Bi-subelliptic operator

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1. Introduction and main results

For $n \geq 2$, the systems of real smooth vector fields $X = (X_1, X_2, \cdots, X_m)$ are defined on an open domain W in $\mathbb{R}^n$. Let $J = (j_1, \cdots, j_l)$ with $1 \leq j_i \leq m$, $X^J = X_{j_1} X_{j_2} X_{j_3} \cdots X_{j_{l-1}} X_{j_l}$, $|J| = l$; and $X^J = id$ if $|J| = 0$. Then we introduce following function space (cf. [21,24,27]):

$$H_X^2(W) = \{u \in L^2(W) \mid X^J u \in L^2(W)\ ,\ |J| \leq 2\},$$

which is a Hilbert space with norm $\|u\|^2_{H_X^2(W)} = \sum\limits_{|J|\leq 2} \|X^J u\|^2_{L^2(W)}$.

We say that $X = (X_1, X_2, \cdots, X_m)$ satisfies the Hörmander's condition on W if there exists a positive integer Q, such that for any $|J| = k \leq Q$, X together with all k-th repeated commutators

$$X_J = [X_{j_1}, [X_{j_2}, [X_{j_3}, \cdots, [X_{j_{k-1}}, X_{j_k}] \cdots]]]$$

span the tangent space at each point of W. Here Q is called the Hörmander index of X on W, which is defined as the smallest positive integer for the Hörmander's condition above being satisfied (cf. [1,10]).

Let $\Omega \subset W$ be a bounded open subset with smooth boundary $\partial\Omega$ which is non-characteristic for X. If X satisfies the Hörmander's condition on Ω with $1 \leq Q < +\infty$, then we say that X is a finitely degenerate system of vector fields on Ω and the finitely degenerate elliptic operator $\triangle_X = \sum_{i=1}^m X_i^2$ is a subelliptic operator. Let $H^2_{X,0}(\Omega)$ be a subspace defined as a closure of $C_0^\infty(\Omega)$ in $H_X^2(W)$. Then $H^2_{X,0}(\Omega)$ is also a Hilbert space.

In this paper, we consider the following eigenvalue problems of bi-subelliptic operators in $H^2_{X,0}(\Omega)$,

$$\begin{cases} \triangle_X^2 u = \lambda u, & \text{in } \Omega, \\ u = 0,\ Xu = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Xu denotes the gradient $(X_1 u, \cdots, X_m u)$ related to the finitely degenerate system of vector fields X.

Before we state our results, we would like to remark on the estimates of eigenvalues for the classical biharmonic operator $\triangle^2$, which is generally called the eigenvalue problem for the clamped plate.

If $X = (\partial_{x_1}, \cdots, \partial_{x_n})$, then $\triangle_X = \triangle$ is standard Laplacian. In this case, there are a lot of results on the estimates of the eigenvalues, e.g. [5–7,15,17–19,23,26]. Also, in this case the operator $\triangle_X^2 = \triangle^2$ is the standard bi-harmonic operator with Hörmander index $Q = 1$. In this aspect, Levine and Protter [16] proved that the eigenvalues $\{\lambda_k\}_{k\geq 1}$ of the clamped plate problem

$$\begin{cases} \triangle^2 u = \lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

satisfy

$$\sum_{i=1}^k \lambda_k \geq \frac{16\pi^4 n}{n+4} \left(\frac{\omega_{n-1}|\Omega|_n}{n}\right)^{-\frac{4}{n}} k^{1+\frac{4}{n}}, \tag{1.3}$$

where $\frac{\partial u}{\partial \nu}$ denotes the derivative of u with respect to the outer unit normal vector ν, and $\{\lambda_k\}_{k\geq 1}$ are eigenvalues for $\triangle^2$, $|\Omega|_n$ is the n-dimensional Lebesgue measure of Ω and ω_{n-1} denotes the area of the unit ball in $\mathbb{R}^n$.

If X is a finitely degenerate system of vector fields on Ω with its Hörmander index $1 < Q < +\infty$, then $\triangle_X = \sum_{i=1}^m X_i^2$ is a finitely degenerate elliptic operator. For this case, many results are obtained in [2,3,8,9,11–14,20] (some situations for infinitely degenerate cases, one can see [2,4]).

In the first part of this paper, we shall study the general finitely degenerate systems of vector fields $X = (X_1, X_2, \cdots, X_m)$ which satisfy the Hörmander's condition and get the lower bound estimates of the eigenvalues for the problem (1.1).

In the finitely degenerate case, we know that there is a sequence of discrete eigenvalues $\{\lambda_k\}_{k\geq 1}$ for the problem (1.1) satisfying $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \cdots$ and $\lambda_k \to +\infty$ as $k \to +\infty$. By using the extension of Li and Yau's approach in [17], we can prove the following lower bounds of the eigenvalues for the bi-subelliptic operator $\triangle_X^2$.

Theorem 1.1. *If $X = (X_1, X_2, \cdots, X_m)$ satisfies the Hörmander's condition with its Hörmander index $1 \leq Q < +\infty$, λ_i is the i-th eigenvalue of the problem (1.1), then for all $k \geq 1$*

$$\sum_{i=1}^{k} \lambda_i \geq C k^{1+\frac{4}{nQ}} - \frac{\widetilde{C}(Q)}{C(Q)} k \tag{1.4}$$

with

$$C = \frac{n^{1+\frac{4}{nQ}} Q (2\pi)^{\frac{4}{Q}}}{C(Q)(nQ+4)(|\Omega|_n \omega_{n-1})^{\frac{4}{nQ}}},$$

where $C(Q)$, $\widetilde{C}(Q)$ are the constants in the subelliptic estimates (2.1) below, ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, $|\Omega|_n$ is the volume of Ω.

Remark 1.1. (1) Since $k\lambda_k \geq \sum_{i=1}^k \lambda_i$, then Theorem 1.1 shows that the eigenvalues λ_k satisfy

$$\lambda_k \geq C k^{\frac{4}{nQ}} - \frac{\widetilde{C}(Q)}{C(Q)}, \quad for\ all\ k \geq 1. \tag{1.5}$$

(2) If $\triangle_X^2 = \triangle^2$ is the standard biharmonic operator, then $Q = 1$, $C(Q) = 1$, $\widetilde{C}(Q) = 0$, $C = \frac{16\pi^4 n}{n+4} \left(\frac{\omega_{n-1}|\Omega|_n}{n} \right)^{-4/n}$. Thus the result of Theorem 1.1 is the same as the estimate of Levine and Protter [16].

Furthermore, if X satisfies the Hörmander's condition on Ω with the Hörmander index Q, then for each $1 \leq j \leq Q$ and $x \in \Omega$, we denote by $V_j(x)$ the subspace of the tangent space $T_x(\Omega)$ which is spanned by the vector fields X_J with $|J| \leq j$ (by convention the X_i themselves will be regarded as commutators of length one). If the dimension of $V_j(x)$ is constant v_j in a neighborhood of each $x \in \Omega$, then we say the system of the vector fields X satisfies the so called Métivier's condition on Ω and the Métivier index is defined as

$$v = \sum_{j=1}^{Q} j(v_j - v_{j-1}), \text{ here } v_0 = 0, \tag{1.6}$$

where v is also called the Hausdorff dimension of Ω related to the subelliptic metric induced by the vector fields X.

In 1976, Métivier [20] proved a asymptotic result for the problem (1.1) under the Métivier's condition

$$\lambda_k \approx k^{\frac{2}{v}}, \text{ as } k \to +\infty, \tag{1.7}$$

where v is the Métivier index above.

In case of $Q > 1$, we know that $n + Q - 1 \le v < nQ$. Thus the increasing order $\frac{4}{nQ}$ of k for λ_k in Theorem 1.1 will be smaller than $\frac{4}{v}$. That means the lower bounds of the eigenvalues for general cases in Theorem 1.1 will be not optimal. Next, we can obtain the optimal increasing order for the lower bounds of the eigenvalues for some kinds of Grushin type degenerate vector fields, which will be related to the following generalized Métivier's index (cf. [2]).

Let us introduce the following generalized Métivier index. By using the same notation we denote here $v_j(x) = \dim V_j(x)$, $v(x) = \sum_{j=1}^{Q} j(v_j(x) - v_{j-1}(x))$, with $v_0(x) = 0$. Then we define

$$\widetilde{v} = \max_{x \in \Omega} v(x) \tag{1.8}$$

as the generalized Métivier index. Thus a degenerate vector fields X always have the generalized Métivier index $\widetilde{v}$ on Ω even if the Métivier's condition will be not satisfied for X. Observe $\tilde{v} = v$ if the Métivier's condition is satisfied.

In the second part of the paper, we shall study the bi-subelliptic operators $\triangle_X^2$ on two kinds of Grushin type vector fields which satisfy the generalized Métivier's condition and get the optimal lower bound estimates for the eigenvalues.

Theorem 1.2. *Let $X = (\partial_{x_1}, \cdots, \partial_{x_{n-1}}, x_1^p \partial_{x_n})$, $n \ge 2$ and $p \in \mathbb{Z}_+$, $\Omega \cap \{x_1 = 0\} \ne \emptyset$. Then X is a Grushin type system of vector fields on Ω with one degenerate direction and the Hörmander index $Q = p + 1$. Also the generalized Métivier index $\widetilde{v} = n + Q - 1$. Suppose λ_i is the i-th eigenvalue of the problem (1.1), then for $k \ge 1$,*

$$\sum_{i=1}^{k} \lambda_i \ge \widetilde{C}(Q) k^{1+\frac{4}{\tilde{v}}} - \frac{C_2(Q)}{C_1(Q)} k, \tag{1.9}$$

where

$$\widetilde{C}(Q) = \frac{A_Q}{C_1(Q) n^2 (n + Q + 3)} \left(\frac{(2\pi)^n}{Q \omega_{n-1} |\Omega|_n} \right)^{\frac{4}{n+Q-1}} (n + Q - 1)^{\frac{n+Q+3}{n+Q-1}},$$

and

$$A_Q = \begin{cases} min\{1, n^{\frac{3-Q}{2}}\}, Q \geq 2, \\ n, \qquad\qquad\quad Q = 1; \end{cases}$$

$C_1(Q), C_2(Q)$ are the constants in Proposition 2.3 below, ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, $|\Omega|_n$ is the volume of Ω.

Remark 1.2. (1) Since $k\lambda_k \geq \sum_{i=1}^k \lambda_i$, then Theorem 1.2 shows that the eigenvalues λ_k satisfy

$$\lambda_k \geq \widetilde{C}(Q)k^{\frac{4}{\tilde{\nu}}} - \frac{C_2(Q)}{C_1(Q)}, \ for\ all\ k \geq 1. \tag{1.10}$$

(2) If $\triangle_X^2 = \triangle^2$ is the standard biharmonic operator, then $Q = 1$, $C_1(Q) = 1$, $C_2(Q) = 0$, $\widetilde{C}(Q) = \frac{16\pi^4 n}{n+4}\left(\frac{\omega_{n-1}|\Omega|_n}{n}\right)^{-4/n}$. Thus the result of Theorem 1.2 is the same as the estimate of Levine and Protter [16].

Theorem 1.3. *Let $X = (\partial_{x_1}, \cdots, x_i^p \partial_{x_{n-1}}, x_j^q \partial_{x_n})$, $n \geq 3$, p, $q \in \mathbb{Z}_+$, i, $j \in \{1, 2, \cdots, n-2\}$, X be a Grushin type system of vector fields on Ω with two degenerate directions. If $\Omega \cap \{x_i = 0\} \neq \emptyset$ and $\Omega \cap \{x_j = 0\} \neq \emptyset$, then X satisfies the Hörmander's condition on Ω with $Q = \max\{p, q\} + 1$ and its generalized Métivier index $\tilde{\nu} = n + p + q$. Suppose λ_i is the i-th eigenvalue of the problem (1.1), then for $k \geq 1$,*

$$\sum_{i=1}^k \lambda_i \geq \widetilde{C}(p,q)k^{1+\frac{4}{\tilde{\nu}}} - \frac{C_2(p,q)}{C_1(p,q)}k \tag{1.11}$$

with

$$\widetilde{C}(p,q) = \frac{2^n}{5C_1(p,q)n^{\frac{n+p+q+6}{2}}}\left(\frac{n+p+q}{(p+1)(q+1)\omega_{n-1}}\right)^{1+\frac{4}{n+p+q}}\left(\frac{(2\pi)^n}{|\Omega|_n}\right)^{\frac{4}{n+p+q}},$$

where $C_1(p,q)$ and $C_2(p,q)$ are the corresponding subelliptic estimate constants in Proposition 2.4.

Remark 1.3. (1) Since $k\lambda_k \geq \sum_{i=1}^k \lambda_i$, then Theorem 1.3 shows that the eigenvalues λ_k satisfy

$$\lambda_k \geq \widetilde{C}(p,q)k^{\frac{4}{\tilde{\nu}}} - \frac{C_2(p,q)}{C_1(p,q)}, \ for\ all\ k \geq 1. \tag{1.12}$$

(2) The result of Theorem 1.3 can be extended to the case in which the vector fields X have more than two degenerate directions.

Our paper is organized as follows. In the section 2, we introduce some preliminaries about subelliptic estimate and maximally hypoellipticity. In the section 3, we prove Theorem 1.1. In the section 4, we prove Theorem 1.2. Finally, we prove Theorem 1.3 in the section 5.

2. Preliminaries

Firstly, we introduce the subelliptic estimates as follows.

Proposition 2.1. *The system of vector fields $X = (X_1, \cdots, X_m)$ satisfies the Hörmander's condition on Ω with its Hörmander index $Q \geq 1$, if and only if the following subelliptic estimate*

$$\left\| |\nabla|^{\frac{2}{Q}} u \right\|^2_{L^2(\Omega)} \leq C(Q)\|\triangle_X u\|^2_{L^2(\Omega)} + \widetilde{C}(Q)\|u\|^2_{L^2(\Omega)} \tag{2.1}$$

holds for all $u \in C_0^\infty(\Omega)$, where $\nabla = (\partial_{x_1}, ..., \partial_{x_m})$, $|\nabla|^{\frac{2}{Q}}$ is a pseudo-differential operator with the symbol $|\xi|^{\frac{2}{Q}}$, the constants $C(Q) > 0$, $\widetilde{C}(Q) \geq 0$ depending on Q.

Proof. Refer to [11] and [24] to show that the subelliptic operator $\triangle_X = \sum\limits_{i=1}^{m} X_i^2$ satisfies

$$\|u\|_{(2\epsilon)} \leq C_1\|\triangle_X u\|_{L^2(\Omega)} + C_2\|u\|_{L^2(\Omega)}$$

with $\epsilon = \frac{1}{Q}$, where $\|u\|_{(2\epsilon)}$ is the Sobolev norm of order 2ϵ. On the other hand, we have

$$\begin{aligned} \|u\|_{(\frac{2}{Q})} &= \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^{\frac{2}{Q}} |\widehat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &\geq \left(\int_{\mathbb{R}^n} |\xi|^{\frac{4}{Q}} |\widehat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &= \left\| |\nabla|^{\frac{2}{Q}} u \right\|_{L^2(\mathbb{R}^n)} = \left\| |\nabla|^{\frac{2}{Q}} u \right\|_{L^2(\Omega)}. \end{aligned}$$

Using the Cauchy–Schwarz inequality we get the estimate

$$\left\| |\nabla|^{\frac{2}{Q}} u \right\|^2_{L^2(\Omega)} \leq C(Q)\|\triangle_X u\|^2_{L^2(\Omega)} + \widetilde{C}(Q)\|u\|^2_{L^2(\Omega)} \qquad \square$$

Proposition 2.2. *(cf. [22,24,25]) If the system of vector fields $X = (X_1, \cdots, X_m)$ satisfies the Hörmander's condition at any point of Ω, then the operator $\triangle_X = \sum_{i=1}^m X_i^2$ is maximally hypoelliptic, i.e., there exists a constant $C > 0$ such that*

$$\sum_{|\alpha| \leq 2} \|X^\alpha u\|^2_{L^2(\Omega)} \leq C(\|\triangle_X u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}), \forall u \in C_0^\infty(\Omega),$$

where $\alpha = (\alpha_1, \cdots, \alpha_m)$ is a multi-index with $|\alpha| = \alpha_1 + \cdots + \alpha_m$ and $X^\alpha = X_1^{\alpha_1} \cdots X_m^{\alpha_m}$.

Proposition 2.3. *If $X = (X_1, X_2, \cdots, X_{n-1}, X_n) = (\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_{n-1}}, x_1^p \partial_{x_n})$ and its Hörmander index $Q = p + 1$, then we have the following subelliptic estimate*

$$\sum_{i=1}^{n-1} \|\partial_{x_i}^2 u\|_{L^2(\Omega)}^2 + \left\| |\partial_{x_n}|^{\frac{2}{Q}} u \right\|_{L^2(\Omega)}^2 \le C_1(Q) \|\triangle_X u\|_{L^2(\Omega)}^2 + C_2(Q) \|u\|_{L^2(\Omega)}^2, \tag{2.2}$$

for all $u \in C_0^\infty(\Omega)$, where $|\partial_{x_n}|^{\frac{2}{Q}}$ is a pseudo-differential operator with the symbol $|\xi_n|^{\frac{2}{Q}}$, $C_1(Q) > 0,\ C_2(Q) \ge 0$ are constants depending on Q.

Proof. From the Plancherel's formula, we have

$$\begin{aligned} \left\| |\partial_{x_n}|^{\frac{2}{Q}} u \right\|_{L^2(\Omega)}^2 &= \left\| |\xi_n|^{\frac{2}{Q}} \widehat{u} \right\|_{L^2(\mathbb{R}^n)}^2 \\ &\le \left\| |\xi|^{\frac{2}{Q}} \widehat{u} \right\|_{L^2(\mathbb{R}^n)}^2 = \left\| |\nabla|^{\frac{2}{Q}} u \right\|_{L^2(\mathbb{R}^n)}^2 \\ &= \left\| |\nabla|^{\frac{2}{Q}} u \right\|_{L^2(\Omega)}^2 . \end{aligned} \tag{2.3}$$

Also, from the maximally hypoelliptic estimate of Proposition 2.2, we can deduce that

$$\sum_{i=1}^{n-1} \|\partial_{x_i}^2 u\|_{L^2(\Omega)}^2 \le \sum_{|\alpha| \le 2} \|X^\alpha u\|_{L^2(\Omega)}^2 \le C(\|\triangle_X u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2). \tag{2.4}$$

Combining (2.1), (2.3) and (2.4) we can deduce that

$$\sum_{i=1}^{n-1} \|\partial_{x_i}^2 u\|_{L^2(\Omega)}^2 + \left\| |\partial_{x_n}|^{\frac{2}{Q}} u \right\|_{L^2(\Omega)}^2 \le C_1(Q) \|\triangle_X u\|_{L^2(\Omega)}^2 + C_2(Q) \|u\|_{L^2(\Omega)}^2. \qquad \square$$

Proposition 2.4. *If $X = (X_1, X_2, \cdots, X_{n-1}, X_n) = (\partial_{x_1}, \partial_{x_2}, \cdots, x_i^p \partial_{x_{n-1}}, x_j^q \partial_{x_n})$, $n \ge 3$, $p,\ q \in \mathbb{Z}_+$, $i,\ j \in \{1, 2, \cdots, n-2\}$, then we have the following subelliptic estimate*

$$\begin{aligned} &\sum_{i=1}^{n-2} \|\partial_{x_i}^2 u\|_{L^2(\Omega)}^2 + \left\| |\partial_{x_{n-1}}|^{\frac{2}{p+1}} u \right\|_{L^2(\Omega)}^2 + \left\| |\partial_{x_n}|^{\frac{2}{q+1}} u \right\|_{L^2(\Omega)}^2 \\ &\quad \le C_1(p,q) \|\triangle_X u\|_{L^2(\Omega)}^2 + C_2(p,q) \|u\|_{L^2(\Omega)}^2, \end{aligned} \tag{2.5}$$

for all $u \in C_0^\infty(\Omega)$, where $|\partial_{x_{n-1}}|^{\frac{2}{p+1}}$ is a pseudo-differential operator with the symbol $|\xi_{n-1}|^{\frac{2}{p+1}}$ and $|\partial_{x_n}|^{\frac{2}{q+1}}$ is a pseudo-differential operator with the symbol $|\xi_n|^{\frac{2}{q+1}}$, $C_1(p,q) = C_1(p+1) + C_1(q+1) > 0,\ C_2(p,q) = C_2(p+1) + C_2(q+1) \ge 0$. $C_1(p+1),\ C_1(q+1),\ C_2(p+1),\ C_2(q+1) \ge 0$ are the corresponding constants in Proposition 2.3.

Proof. We consider the system of vector fields $\tilde{X} = (\partial_{x_1}, \cdots, \partial_{x_{n-2}}, x_i^p \partial_{x_{n-1}})$ defined on the projection $\Omega_{x'}$ of Ω on the direction $x' = (x_1, \cdots, x_{n-1})$. Similarly to Proposition 2.3, we have

$$\sum_{i=1}^{n-2} \|\partial_{x_i}^2 u\|_{L^2(\Omega_{x'})}^2 + \left\| |\partial_{x_{n-1}}|^{\frac{2}{p+1}} u \right\|_{L^2(\Omega_{x'})}^2 \le C_1(p+1)\|\triangle_{\tilde{X}} u\|_{L^2(\Omega_{x'})}^2 + C_2(p+1)\|u\|_{L^2(\Omega_{x'})}^2.$$

Then we have

$$\sum_{i=1}^{n-2} \|\partial_{x_i}^2 u\|_{L^2(\Omega)}^2 + \left\| |\partial_{x_{n-1}}|^{\frac{2}{p+1}} u \right\|_{L^2(\Omega)}^2 \le C_1(p+1)\|\triangle_X u\|_{L^2(\Omega)}^2 + C_2(p+1)\|u\|_{L^2(\Omega)}^2. \tag{2.6}$$

Similarly, we can deduce that

$$\sum_{i=1}^{n-2} \|\partial_{x_i}^2 u\|_{L^2(\Omega)}^2 + \left\| |\partial_{x_n}|^{\frac{2}{q+1}} u \right\|_{L^2(\Omega)}^2 \le C_1(q+1)\|\triangle_X u\|_{L^2(\Omega)}^2 + C_2(q+1)\|u\|_{L^2(\Omega)}^2. \tag{2.7}$$

Finally, we get the subelliptic estimate from (2.6) and (2.7)

$$\begin{aligned}
&\sum_{i=1}^{n-2} \|\partial_{x_i}^2 u\|_{L^2(\Omega)}^2 + \left\| |\partial_{x_{n-1}}|^{\frac{2}{p+1}} u \right\|_{L^2(\Omega)}^2 + \left\| |\partial_{x_n}|^{\frac{2}{q+1}} u \right\|_{L^2(\Omega)}^2 \\
&\le \big(C_1(p+1) + C_1(q+1)\big)\|\triangle_X u\|_{L^2(\Omega)}^2 + \big(C_2(p+1) + C_2(q+1)\big)\|u\|_{L^2(\Omega)}^2. \qquad \square
\end{aligned}$$

3. Proof of Theorem 1.1

Lemma 3.1. *For the system of vector fields $X = (X_1, \cdots, X_m)$, suppose $\{\phi_j\}_{j=1}^k$ is the set of orthonormal eigenfunctions corresponding to the eigenvalues $\{\lambda_j\}_{j=1}^k$. Define*

$$\Phi(x, y) = \sum_{j=1}^{k} \phi_j(x)\phi_j(y).$$

Then for the partial Fourier transformation of $\Phi(x, y)$ with respect to the x-variable,

$$\widehat{\Phi}(z, y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \Phi(x, y) e^{-ix\cdot z} dx,$$

we have

$$\int_{\Omega}\int_{\mathbb{R}^n} \left|\widehat{\Phi}(z, y)\right|^2 dz dy = k, \int_{\Omega} \left|\widehat{\Phi}(z, y)\right|^2 dy \le (2\pi)^{-n} |\Omega|_n.$$

Proof. Since

$$\int_{\mathbb{R}^n} \Phi^2(x,y)dx = \int_{\mathbb{R}^n} \left|\widehat{\Phi}(z,y)\right|^2 dz.$$

By the orthonormality of $\{\phi_j\}_{j=1}^k$, it follows that

$$\int_\Omega \int_{\mathbb{R}^n} \left|\widehat{\Phi}(z,y)\right|^2 dzdy = \int_\Omega \int_{\mathbb{R}^n} |\Phi(x,y)|^2 dxdy = \int_\Omega \int_\Omega |\Phi(x,y)|^2 dxdy = k.$$

On the other hand,

$$\int_{\mathbb{R}^n} \left|\widehat{\Phi}(z,y)\right|^2 dy = \int_\Omega (2\pi)^{-n} \left|\int_{\mathbb{R}^n} \Phi(x,y)e^{-ix\cdot z}dx\right|^2 dy = \int_\Omega (2\pi)^{-n} \left|\int_\Omega \Phi(x,y)e^{-ix\cdot z}dx\right|^2 dy.$$

Using the Fourier expansion for the function $e^{-ix\cdot z}$, i.e.

$$e^{-ix\cdot z} = \sum_{j=1}^{\infty} a_j(z)\phi_j(x), \quad \text{with} \quad a_j(z) = \int_\Omega e^{-ix\cdot z}\phi_j(x)dx,$$

we obtain that

$$\sum_{j=1}^{\infty} \left|a_j(z)\right|^2 = \int_\Omega |e^{-ix\cdot z}|^2 dx = |\Omega|_n.$$

Thus

$$\left|\int_\Omega \Phi(x,y)e^{-ix\cdot z}dx\right| \le \left|\int_\Omega \sum_{j=1}^{k}\sum_{l=1}^{\infty} a_l(z)\phi_l(x)\phi_j(x)\phi_j(y)dx\right| = \left|\sum_{j=1}^{k} a_j(z)\phi_j(y)\right|.$$

Using the estimates above, we have

$$\int_\Omega \left|\widehat{\Phi}(z,y)\right|^2 dy \le (2\pi)^{-n}\int_\Omega \left|\sum_{j=1}^{k} a_j(z)\phi_j(y)\right|^2 dy = (2\pi)^{-n}\sum_{j=1}^{k}\left|a_j(z)\right|^2 \le (2\pi)^{-n}|\Omega|_n. \qquad \square$$

Lemma 3.2. *Let f be a real-valued function defined on $\mathbb{R}^n$ with $0 \le f \le M_1$, and for $Q \in \mathbb{N}^+$,*

$$\int_{\mathbb{R}^n} |z|^{\frac{4}{Q}} f(z)dz \le M_2.$$

Then

$$\int_{\mathbb{R}^n} f(z)dz \le \left(\frac{nQ+4}{Q}\right)^{\frac{nQ}{nQ+4}} \frac{(M_1\omega_{n-1})^{\frac{4}{nQ+4}}}{n} M_2^{\frac{nQ}{nQ+4}},$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$.

Proof. First, we choose R such that

$$\int_{\mathbb{R}^n} |z|^{\frac{4}{Q}} g(z)dz = M_2,$$

where

$$g(z) = \begin{cases} M_1, & |z| \le R, \\ 0, & |z| > R. \end{cases}$$

Then $\left(|z|^{\frac{4}{Q}} - R^{\frac{4}{Q}}\right)(f(z) - g(z)) \ge 0$. Hence we have

$$R^{\frac{4}{Q}} \int_{\mathbb{R}^n} (f(z) - g(z))dz \le \int_{\mathbb{R}^n} |z|^{\frac{4}{Q}} (f(z) - g(z))dz \le 0.$$

That means

$$\int_{\mathbb{R}^n} f(z)dz \le \int_{\mathbb{R}^n} g(z)dz. \tag{3.1}$$

Now we have

$$M_2 = \int_{\mathbb{R}^n} |z|^{\frac{4}{Q}} g(z)dz = M_1 \int_{B_R} |z|^{\frac{4}{Q}} dz = M_1 \int_0^R r^{n-1+\frac{4}{Q}} \omega_{n-1} dr = \frac{M_1 Q \omega_{n-1} R^{n+\frac{4}{Q}}}{nQ+4}, \tag{3.2}$$

where $B_R = \{z \in \mathbb{R}^n, |z| \le R\}$, ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$.

On the other hand, we know that

$$\int_{\mathbb{R}^n} g(z)dz = \frac{M_1 \omega_{n-1} R^n}{n} \tag{3.3}$$

Finally, (3.1), (3.2), (3.3) imply that

$$\int_{\mathbb{R}^n} f(z)dz \le \int_{\mathbb{R}^n} g(z)dz = \left(\frac{nQ+4}{Q}\right)^{\frac{nQ}{nQ+4}} \frac{(M_1\omega_{n-1})^{\frac{4}{nQ+4}}}{n} M_2^{\frac{nQ}{nQ+4}}. \qquad \square$$

Proof of Theorem 1.1. Let $\{\lambda_k\}_{k\ge1}$ be a sequence of the eigenvalues for the problem (1.1), $\{\phi_k(x)\}_{m\ge1}$ be the corresponding eigenfunctions, then $\{\phi_k(x)\}_{k\ge1}$ constitute an orthonormal basis of the Sobolev space $H^2_{X,0}(\Omega)$. Let $\Phi(x,y)=\sum_{j=1}^k \phi_j(x)\phi_j(y)$. By using Plancherel's formula, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\Omega}|z|^{\frac{4}{Q}}|\widehat{\Phi}(z,y)|^2dydz&=\int_{\mathbb{R}^n}\int_{\Omega}\left||\nabla|^{\frac{2}{Q}}\Phi(x,y)\right|^2dydx\\&=\int_{\Omega}\int_{\Omega}\left||\nabla|^{\frac{2}{Q}}\Phi(x,y)\right|^2dydx.\end{aligned}\tag{3.4}$$

Also, from Proposition 2.1,

$$\left\||\nabla|^{\frac{2}{Q}}u\right\|^2_{L^2(\Omega)}\le C(Q)\|\triangle_X u\|^2_{L^2(\Omega)}+\widetilde{C}(Q)\|u\|^2_{L^2(\Omega)},\tag{3.5}$$

and combining (3.4) with the subelliptic estimate (3.5), we get the following inequality

$$\int_{\mathbb{R}^n}\int_{\Omega}|z|^{\frac{4}{Q}}\left|\widehat{\Phi}(z,y)\right|^2dydz\le C(Q)\int_{\Omega}\int_{\Omega}|\triangle_X\Phi(x,y)|^2dxdy+\widetilde{C}(Q)\int_{\Omega}\int_{\Omega}\Phi^2(x,y)dxdy.\tag{3.6}$$

Next, by using integration-by-parts, we have

$$\begin{aligned}\sum_{i=1}^k\lambda_i&=\sum_{i=1}^k\int_{\Omega}\lambda_i\phi_i(x)\cdot\phi_i(x)dx=\sum_{i=1}^k\int_{\Omega}\triangle_X^2\phi_i(x)\cdot\phi_i(x)dx\\&=\sum_{i=1}^k\int_{\Omega}X(\triangle_X\phi_i(x))\cdot X\phi_i(x)dx=\sum_{i=1}^k\int_{\Omega}\triangle_X\phi_i(x)\cdot\triangle_X\phi_i(x)dx\\&=\int_{\Omega}\int_{\Omega}\sum_{i=1}^k|\triangle_X\phi_i(x)\phi_i(y)|^2dxdy=\int_{\Omega}\int_{\Omega}|\triangle_X\Phi(x,y)|^2dxdy.\end{aligned}\tag{3.7}$$

Hence we obtain the following result from Lemma 3.1, (3.6) and (3.7)

$$\int_{\mathbb{R}^n}\int_{\Omega}|z|^{\frac{4}{Q}}\left|\widehat{\Phi}(z,y)\right|^2dydz\le C(Q)\sum_{i=1}^k\lambda_i+\widetilde{C}(Q)k.$$

Now we choose

$$f(z)=\int_{\Omega}\left|\widehat{\Phi}(z,y)\right|^2dy,\quad M_1=(2\pi)^{-n}|\Omega|_n,\quad M_2=C(Q)\sum_{i=1}^k\lambda_i+\widetilde{C}(Q)k.$$

From Lemma 3.2, for any $k\ge 1$,

$$k \le \left(\frac{nQ+4}{Q}\right)^{\frac{nQ}{nQ+4}} \frac{((2\pi)^{-n}|\Omega|_n\omega_{n-1})^{\frac{4}{nQ+4}}}{n} \left(C(Q)\sum_{i=1}^{k}\lambda_i + \widetilde{C}(Q)k\right)^{\frac{nQ}{nQ+4}}.$$

This means, for any $k \ge 1$,

$$\sum_{i=1}^{k}\lambda_i \ge Ck^{1+\frac{4}{nQ}} - \frac{\widetilde{C}(Q)}{C(Q)}k$$

with

$$C = \frac{n^{1+\frac{4}{nQ}}Q(2\pi)^{\frac{4}{Q}}}{C(Q)(nQ+4)(|\Omega|_n\omega_{n-1})^{\frac{4}{nQ}}}.$$

The proof of Theorem 1.1 is complete. □

4. Proof of Theorem 1.2

Lemma 4.1. *Let f be a real-valued function defined on $\mathbb{R}^n$ with $0 \le f \le M_1$, and for $Q \in \mathbb{N}^+$,*

$$\int_{\mathbb{R}^n} \left(\sum_{i=1}^{n-1} z_i^2 + |z_n|^{\frac{2}{Q}}\right)^2 f(z)dz \le M_2.$$

Then

$$\int_{\mathbb{R}^n} f(z)dz \le \int_{\mathbb{R}^n} g(z)dz \le \frac{(QM_1\omega_{n-1})^{\frac{4}{n+Q+3}}}{n+Q-1}\left(\frac{n(n+Q+3)}{A_Q}\right)^{\frac{n+Q-1}{n+Q+3}} M_2^{\frac{n+Q-1}{n+Q+3}}$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, and

$$A_Q = \begin{cases} min\{1, n^{\frac{3-Q}{2}}\}, & Q \ge 2, \\ n, & Q = 1. \end{cases}$$

Proof. First, we choose R such that

$$\int_{\mathbb{R}^n} \left(\sum_{i=1}^{n-1} z_i^2 + |z_n|^{\frac{2}{Q}}\right)^2 g(z)dz = M_2,$$

where

$$g(z) = \begin{cases} M_1, & \sum_{i=1}^{n-1} z_i^2 + |z_n|^{\frac{2}{Q}} \le R^2, \\ 0, & \sum_{i=1}^{n-1} z_i^2 + |z_n|^{\frac{2}{Q}} > R^2. \end{cases}$$

Then $\left(\sum\limits_{i=1}^{n-1} z_i^2 + |z_n|^{\frac{2}{Q}} - R^2\right)(f(z) - g(z)) \geq 0$. Hence we have

$$R^2 \int\limits_{\mathbb{R}^n} (f(z) - g(z))dz \leq \int\limits_{\mathbb{R}^n} \left(\sum_{i=1}^{n-1} z_i^2 + |z_n|^{\frac{2}{Q}}\right)(f(z) - g(z))dz \leq 0,$$

then

$$\int\limits_{\mathbb{R}^n} f(z)dz \leq \int\limits_{\mathbb{R}^n} g(z)dz. \tag{4.1}$$

Now we have

$$M_2 = \int\limits_{\mathbb{R}^n} \left(\sum_{i=1}^{n-1} z_i^2 + |z_n|^{\frac{2}{Q}}\right)^2 g(z)dz = M_1 \int\limits_{\widetilde{B}_R} \left(\sum_{i=1}^{n-1} z_i^2 + |z_n|^{\frac{2}{Q}}\right)^2 dz,$$

where

$$\widetilde{B}_R = \left\{z \in \mathbb{R}^n, \quad \sum_{i=1}^{n-1} z_i^2 + |z_n|^{\frac{2}{Q}} \leq R^2\right\}, \quad B_R = \{z \in \mathbb{R}^n, \quad |z| \leq R\}.$$

By using the change of variables,

$$z_i = z_i' \quad (i = 1, 2, \cdots, n-1), \quad z_n = |z_n'|^Q,$$

the determinant of Jacobian

$$\det\Big(\frac{\partial(z_1, \cdots, z_n)}{\partial(z_1', \cdots, z_n')}\Big) = Q|z_n'|^{Q-1}.$$

Hence

$$\begin{aligned} M_2 &= M_1 \int\limits_{\widetilde{B}_R} \left(\sum_{i=1}^{n-1} z_i^2 + |z_n|^{\frac{2}{Q}}\right)^2 dz = M_1 Q \int\limits_{B_R} |z|^4 |z_n|^{Q-1} dz \\ &= \frac{M_1 Q}{n} \int\limits_{B_R} |z|^4 \sum_{i=1}^{n} |z_i|^{Q-1} dz. \end{aligned}$$

On the other hand,

$$\sum_{i=1}^{n} |z_i|^{Q-1} = |z|^{Q-1} \sum_{i=1}^{n} \left(\frac{|z_i|}{|z|}\right)^{Q-1} \geq A_Q |z|^{Q-1},$$

where

$$A_Q = \begin{cases} min\{1, n^{\frac{3-Q}{2}}\}, Q \geq 2, \\ n, \qquad\qquad\qquad Q = 1. \end{cases}$$

Then we have

$$M_2 \geq \frac{M_1 Q A_Q}{n} \int_{B_R} |z|^{Q+3} dz = \frac{M_1 Q A_Q \omega_{n-1}}{n(n+Q+3)} R^{n+Q+3}. \tag{4.2}$$

From the definition of $g(z)$, we know that

$$\begin{aligned} \int_{\mathbb{R}^n} g(z)dz &= M_1 \int_{\widetilde{B}_R} dz = M_1 Q \int_{B_R} |z_n|^{Q-1} dz \\ &\leq M_1 Q \int_{B_R} |z|^{Q-1} dz = \frac{M_1 Q \omega_{n-1}}{n+Q-1} R^{n+Q-1}. \end{aligned} \tag{4.3}$$

Combining (4.1), (4.2) and (4.3), we obtain

$$\int_{\mathbb{R}^n} f(z)dz \leq \int_{\mathbb{R}^n} g(z)dz \leq \frac{(Q M_1 \omega_{n-1})^{\frac{4}{n+Q+3}}}{n+Q-1} \left(\frac{n(n+Q+3)}{A_Q}\right)^{\frac{n+Q-1}{n+Q+3}} M_2^{\frac{n+Q-1}{n+Q+3}}.$$

Lemma 4.1 is proved. □

Proof of Theorem 1.2. Let $\{\lambda_k\}_{k\geq 1}$ be a sequence of the eigenvalues for the problem (1.1), $\{\phi_k(x)\}_{m\geq 1}$ be the corresponding eigenfunctions, then $\{\phi_k(x)\}_{m\geq 1}$ constitute an orthonormal basis of the Sobolev space $H^2_{X,0}(\Omega)$.

Let $\Phi(x, y) = \sum_{j=1}^{k} \phi_j(x)\phi_j(y)$, we use Cauchy–Schwarz inequality to get

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_{\Omega} \left(\sum_{i=1}^{n-1} z_i^2 + |z_n|^{\frac{2}{Q}}\right)^2 \left|\widehat{\Phi}(z, y)\right|^2 dydz \\ \leq n &\int_{\mathbb{R}^n} \int_{\Omega} \left(\sum_{i=1}^{n-1} z_i^4 + |z_n|^{\frac{4}{Q}}\right) \left|\widehat{\Phi}(z, y)\right|^2 dydz. \end{aligned} \tag{4.4}$$

Similarly to the result of (3.7), we can deduce that

$$\sum_{i=1}^{k} \lambda_i = \int_{\Omega} \int_{\Omega} |\triangle_X \Phi(x, y)|^2 dxdy. \tag{4.5}$$

Then by using Plancherel's formula and Proposition 2.3, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\Omega}\left(\sum_{i=1}^{n-1} z_i^2 + |z_n|^{\frac{2}{Q}}\right)^2 \left|\widehat{\Phi}(z,y)\right|^2 dydz \\
&\le n\int_{\mathbb{R}^n}\int_{\Omega}\left(\sum_{i=1}^{n-1} z_i^4 + |z_n|^{\frac{4}{Q}}\right) \left|\widehat{\Phi}(z,y)\right|^2 dydz \\
&= n\int_{\mathbb{R}^n}\int_{\Omega}\left(\sum_{i=1}^{n-1} |\partial_{x_i}^2 \Phi(x,y)|^2 + \left||\partial_{x_n}|^{\frac{2}{Q}}\Phi(x,y)\right|^2\right) dydx \\
&= n\int_{\Omega}\int_{\Omega}\left(\sum_{i=1}^{n-1} |\partial_{x_i}^2 \Phi(x,y)|^2 + \left||\partial_{x_n}|^{\frac{2}{Q}}\Phi(x,y)\right|^2\right) dydx \\
&\le n\left[C_1(Q)\int_{\Omega}\int_{\Omega} |\triangle_X \Phi(x,y)|^2 dxdy + C_2(Q)\int_{\Omega}\int_{\Omega}|\Phi(x,y)|^2 dxdy\right].
\end{aligned} \tag{4.6}
$$

Thus from (4.5) and Lemma 3.1 above, we can deduce that

$$
\int_{\mathbb{R}^n}\int_{\Omega}\left(\sum_{i=1}^{n-1} z_i^2 + |z_n|^{\frac{2}{Q}}\right)^2 \left|\widehat{\Phi}(z,y)\right|^2 dydz \le n\left(C_1(Q)\sum_{i=1}^{k}\lambda_i + C_2(Q)k\right).
$$

Next, we choose

$$
f(z) = \int_{\Omega}\left|\widehat{\Phi}(z,y)\right|^2 dy, \quad M_1 = (2\pi)^{-n}|\Omega|_n, \quad M_2 = n\left(C_1(Q)\sum_{i=1}^{k}\lambda_i + C_2(Q)k\right).
$$

Then from the result of Lemma 4.1, we know that for any $k \ge 1$,

$$
k \le \frac{Q\omega_{n-1}(2\pi)^{-n}|\Omega|_n}{n+Q-1}\left(\frac{n(n+Q+3)}{(2\pi)^{-n}|\Omega|_n Q A_Q \omega_{n-1}}\right)^{\frac{n+Q-1}{n+Q+3}}\left(n\left(C_1(Q)\sum_{i=1}^{k}\lambda_i + C_2(Q)k\right)\right)^{\frac{n+Q-1}{n+Q+3}}.
$$

This means, for any $k \ge 1$,

$$
\sum_{i=1}^{k}\lambda_i \ge \widetilde{C}(Q)k^{1+\frac{4}{\tilde{v}}} - \frac{C_2(Q)}{C_1(Q)}k
$$

with

$$
\widetilde{C}(Q) = \frac{A_Q}{C_1(Q)n^2(n+Q+3)}\left(\frac{(2\pi)^n}{Q\omega_{n-1}|\Omega|_n}\right)^{\frac{4}{n+Q-1}}(n+Q-1)^{\frac{n+Q+3}{n+Q-1}},
$$

and

$$A_Q = \begin{cases} min\{1, n^{\frac{3-Q}{2}}\}, Q \geq 2, \\ n, \qquad\qquad\quad Q = 1; \end{cases}$$

The proof of Theorem 1.2 is complete. □

5. Proof of Theorem 1.3

Lemma 5.1. *Let f be a real-valued function defined on $\mathbb{R}^n$ with $0 \leq f \leq M_1$, and for $p, q \in \mathbb{N}^+$,*

$$\int_{\mathbb{R}^n} \left(\sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}} \right)^2 f(z)dz \leq M_2.$$

Then

$$\int_{\mathbb{R}^n} f(z)dz \leq \frac{(p+1)(q+1)\omega_{n-1}}{n+p+q} M_1^{\frac{4}{n+p+q+4}} \left(\frac{5n^{\frac{n+p+q+4}{2}}}{2^{n+2}} \right)^{\frac{n+p+q}{n+p+q+4}} M_2^{\frac{n+p+q}{n+p+q+4}},$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$.

Proof. First, we choose R such that

$$\int_{\mathbb{R}^n} \left(\sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}} \right)^2 g(z)dz = M_2,$$

where

$$g(z) = \begin{cases} M_1, & \sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}} \leq R^2, \\ 0, & \sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}} > R^2. \end{cases}$$

Then $\left(\sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}} - R^2 \right) (f(z) - g(z)) \geq 0$. Hence we have

$$R^2 \int_{\mathbb{R}^n} (f(z) - g(z))dz \leq \int_{\mathbb{R}^n} \left(\sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}} \right) (f(z) - g(z))dz \leq 0,$$

that means

$$\int_{\mathbb{R}^n} f(z)dz \leq \int_{\mathbb{R}^n} g(z)dz. \tag{5.1}$$

Now we have

$$M_2 = \int\limits_{\mathbb{R}^n} \left(\sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}} \right)^2 g(z)dz = M_1 \int\limits_{\widetilde{B}_R} \left(\sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}} \right)^2 dz,$$

where

$$\widetilde{B}_R = \left\{ z \in \mathbb{R}^n, \quad \sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}} \leq R^2 \right\}.$$

Now we change the variables as follows,

$$z_i = z_i' \quad (i = 1, 2, \cdots, n-2), \quad z_{n-1} = |z_{n-1}'|^{p+1}, \quad z_n = |z_n'|^{q+1},$$

then we have the following determinant of Jacobian,

$$\det\Big(\frac{\partial(z_1, \cdots, z_n)}{\partial(z_1', \cdots, z_n')}\Big) = (p+1)(q+1)|z_{n-1}'|^p |z_n'|^q.$$

Hence

$$\begin{aligned} M_2 &= M_1 \int\limits_{\widetilde{B}_R} \left(\sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}} \right)^2 dz \\ &= M_1(p+1)(q+1) \int\limits_{B_R} |z|^4 |z_{n-1}|^p |z_n|^q dz \\ &\geq M_1(p+1)(q+1) \int\limits_{A_R} |z|^4 |z_{n-1}|^p |z_n|^q dz \end{aligned}$$

where

$$B_R = \{z \in \mathbb{R}^n, |z| \leq R\}, \ A_R = \left\{ z \in \mathbb{R}^n, |z_i| \leq \frac{R}{\sqrt{n}}, i = 1, \cdots, n \right\}.$$

On the other hand,

$$\int\limits_{A_R} |z|^4 |z_{n-1}|^p |z_n|^q dz \geq \int\limits_{A_R} |z_1|^4 |z_{n-1}|^p |z_n|^q dz = \frac{2^n}{5(p+1)(q+1)} n^{-\frac{n+p+q+4}{2}} R^{n+p+q+4}.$$

Then we have

$$M_2 \geq \frac{2^n M_1}{5} n^{-\frac{n+p+q+4}{2}} R^{n+p+q+4}. \tag{5.2}$$

From the definition of $g(z)$, we know that

$$\begin{aligned}\int_{\mathbb{R}^n} g(z)dz &= M_1 \int_{\widetilde{B}_R} dz = M_1(p+1)(q+1)\int_{B_R} |z_{n-1}|^p |z_n|^q dz \\ &\le \int_{B_R} |z|^{p+q} dz = \frac{M_1(p+1)(q+1)\omega_{n-1}}{n+p+q} R^{n+p+q}.\end{aligned} \tag{5.3}$$

From (5.1), (5.2) and (5.3), we obtain

$$\int_{\mathbb{R}^n} f(z)dz \le \int_{\mathbb{R}^n} g(z)dz \le \frac{(p+1)(q+1)\omega_{n-1}}{n+p+q} M_1^{\frac{4}{n+p+q+4}} \left(\frac{5n^{\frac{n+p+q+4}{2}}}{2^n}\right)^{\frac{n+p+q}{n+p+q+4}} M_2^{\frac{n+p+q}{n+p+q+4}}.$$

Lemma 5.1 is proved. □

Proof of Theorem 1.3. Let $\{\lambda_k\}_{k\ge1}$ be a sequence of the eigenvalues for the problem (1.1), $\{\phi_k(x)\}_{k\ge1}$ be the corresponding eigenfunctions, then $\{\phi_k(x)\}_{k\ge1}$ constitute an orthonormal basis of the Sobolev space $H^2_{X,0}(\Omega)$.

Let $\Phi(x,y) = \sum_{j=1}^k \phi_j(x)\phi_j(y)$. Thus, by using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\Omega} \left(\sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}}\right)^2 \left|\widehat{\Phi}(z,y)\right|^2 dydz \\ \le n &\int_{\mathbb{R}^n}\int_{\Omega} \left(\sum_{i=1}^{n-2} z_i^4 + |z_{n-1}|^{\frac{4}{p+1}} + |z_n|^{\frac{4}{q+1}}\right) \left|\widehat{\Phi}(z,y)\right|^2 dydz.\end{aligned} \tag{5.4}$$

Similarly to the result of (3.7), we obtain that

$$\sum_{i=1}^k \lambda_i = \int_{\Omega}\int_{\Omega} |\triangle_X \Phi(x,y)|^2 dxdy. \tag{5.5}$$

Then by using Plancherel's formula and Proposition 2.4, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\Omega} \left(\sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}}\right)^2 \left|\widehat{\Phi}(z,y)\right|^2 dydz \\ \le n &\int_{\mathbb{R}^n}\int_{\Omega} \left(\sum_{i=1}^{n-2} |\partial^2_{x_i}\Phi(x,y)|^2 + \left||\partial_{x_{n-1}}|^{\frac{2}{p+1}}\Phi(x,y)\right|^2 + \left||\partial_{x_n}|^{\frac{2}{q+1}}\Phi(x,y)\right|^2\right) dydx \\ = n &\int_{\Omega}\int_{\Omega} \left(\sum_{i=1}^{n-2} |\partial^2_{x_i}\Phi(x,y)|^2 + \left||\partial_{x_{n-1}}|^{\frac{2}{p+1}}\Phi(x,y)\right|^2 + \left||\partial_{x_n}|^{\frac{2}{q+1}}\Phi(x,y)\right|^2\right) dydx\end{aligned}$$

$$\le n\left[C_1(p,q)\int_\Omega\int_\Omega |\triangle_X\Phi(x,y)|^2dxdy + C_2(p,q)\int_\Omega\int_\Omega |\Phi(x,y)|^2dxdy\right].$$

Thus from (5.5) and Lemma 3.1 above, we can deduce that

$$\int_{\mathbb{R}^n}\int_\Omega \left(\sum_{i=1}^{n-2} z_i^2 + |z_{n-1}|^{\frac{2}{p+1}} + |z_n|^{\frac{2}{q+1}}\right)^2 \left|\widehat{\Phi}(z,y)\right|^2 dydz \le n\left(C_1(p,q)\sum_{i=1}^k \lambda_i + C_2(p,q)k\right).$$

Finally, we choose

$$f(z) = \int_\Omega |\widehat{\Phi}(z,y)|^2dy, \quad M_1 = (2\pi)^{-n}|\Omega|_n, \quad M_2 = n\left(C_1(p,q)\sum_{i=1}^k \lambda_i + C_2(p,q)k\right).$$

Then from the Lemma 5.1, we have for any $k \ge 1$,

$$\begin{aligned} k \le & \frac{(p+1)(q+1)\omega_{n-1}}{n+p+q}((2\pi)^{-n}|\Omega|_n)^{\frac{4}{n+p+q+4}}\left(\frac{5n^{\frac{n+p+q+4}{2}}}{2^n}\right)^{\frac{n+p+q}{n+p+q+4}} \\ & \times \left(n\left(C_1(p,q)\sum_{i=1}^k \lambda_i + C_2(p,q)k\right)\right)^{\frac{n+p+q}{n+p+q+4}}. \end{aligned}$$

This means, for any $k \ge 1$,

$$\sum_{i=1}^k \lambda_i \ge \widetilde{C}(p,q)k^{1+\frac{4}{\tilde{v}}} - \frac{C_2(p,q)}{C_1(p,q)}k$$

with

$$\widetilde{C}(p,q) = \frac{2^n}{5C_1(p,q)n^{\frac{n+p+q+6}{2}}}\left(\frac{n+p+q}{(p+1)(q+1)\omega_{n-1}}\right)^{1+\frac{4}{n+p+q}}\left(\frac{(2\pi)^n}{|\Omega|_n}\right)^{\frac{4}{n+p+q}},$$

and the constant $C_1(p,q) = C_1(p+1) + C_1(q+1) > 0$, $C_2(p,q) = C_2(p+1) + C_2(q+1) \ge 0$. $C_1(p+1), C_1(q+1) > 0$ and $C_2(p+1), C_2(q+1) \ge 0$ are the corresponding subelliptic estimate constants in Proposition 2.3. The proof of Theorem 1.3 is complete. □

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