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Wong–Zakai approximations and center manifolds of stochastic differential equations ☆

Jun Shen [a], Kening Lu [a,b,*]

[a] School of Mathematics, Sichuan University, Chengdu, Sichuan 610064, PR China
[b] Department of Mathematics, Brigham Young University, Provo, UT 84602, USA

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Abstract

In this paper, we study the Wong–Zakai approximations given by a stationary process via the Wiener shift and their associated dynamics of the stochastic differential equation driven by a l-dimensional Brownian motion. We prove that the solutions of Wong–Zakai approximations converge in the mean square to the solutions of the Stratonovich stochastic differential equation. We also show that for a simple multiplicative noise, the center-manifold of the Wong–Zakai approximations converges to the center-manifold of the Stratonovich stochastic differential equation.
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1. Introduction

In this paper, we study the Wong–Zakai approximations given by a stationary process via the Wiener shift and their associated dynamics of the following stochastic differential equation in $\mathbb{R}^n$:

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* Corresponding author.
E-mail addresses: junshen85@163.com (J. Shen), klu@math.byu.edu (K. Lu).

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$$du = f(u)\,dt + \sigma(u) \circ dW, \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times l}$ are nonlinear functions, $W(t,\omega)$ is a l-dimensional Brownian motion, and $\circ\, dW(t,\omega)$ denotes the Stratonovich differential.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the classical Wiener probability space, where

$$\Omega = C_0(\mathbb{R}, \mathbb{R}^l) := \{\omega \in C(\mathbb{R}, \mathbb{R}^l) : \ \omega(0) = 0\}$$

with the open compact topology, $\mathcal{F}$ is its Borel σ-algebra, and $\mathbb{P}$ is the Wiener measure. The Brownian motion has the form $W(t,\omega) = \omega(t)$. Consider the Wiener shift θ_t defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\theta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t).$$

It is known that the probability measure $\mathbb{P}$ is an ergodic invariant measure for θ_t. $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ forms a metric dynamical system, see Arnold [2].

For each $\delta > 0$, let $\mathcal{G}_\delta : \Omega \to \mathbb{R}^l$ denote the random variable

$$\mathcal{G}_\delta(\omega) = \frac{1}{\delta}\omega(\delta).$$

Then we have

$$\mathcal{G}_\delta(\theta_t\omega) = \frac{1}{\delta}(\omega(t+\delta) - \omega(t)). \tag{1.2}$$

From the properties of Brownian motions, it follows that $\mathcal{G}_\delta(\theta_t\omega)$ is a stationary stochastic process with a normal distribution and is unbounded in t for almost all ω. $\mathcal{G}_\delta(\theta_t\omega)$ may be viewed as an approximation of white noise in the sense

$$\lim_{\delta\to 0^+} \sup_{t\in[0,T]} \left| \int_0^t \mathcal{G}_\delta(\theta_s\omega)ds - \omega(t) \right| = 0, a.s.$$

for each $T > 0$, which will be proved in Section 3.

We consider the following Wong–Zakai approximation of equation (1.1) driven by a multiplicative noise of $\mathcal{G}_\delta(\theta_t\omega)$:

$$\dot{u}_\delta = f(u_\delta) + \sigma(u_\delta)\mathcal{G}_\delta(\theta_t\omega). \tag{1.3}$$

Note that the above equation is a random differential equation driven by the stationary stochastic process $\mathcal{G}_\delta(\theta_t\omega)$. As a consequence, its solutions generate a random dynamical system. Thus one can study its sample-wise (or pathwise) dynamical properties.

In current paper, we first study the limit behavior of solutions of equation (1.3) as $\delta \to 0^+$ and show that u_δ converges in the mean square to a solution of equation (1.1). Then, to illustrate that the dynamics of this Wong–Zakai approximations converge to ones of the stochastic equation (1.3), we show that for a simple multiplicative noise, the center-manifold of (1.3) converges to the center-manifold of stochastic equation (1.1).

Our first result is about the mean square convergence of solutions of Wong–Zakai approximations.

Theorem A. *Let $u(t,\omega,x)$ and $u_\delta(t,\omega,x)$ be solutions of equations (1.1) and (1.3) with the initial data x at $t=0$, respectively. Assume that $f^i \in C_b^1(\mathbb{R}^n)$ and $\sigma^{ij} \in C_b^2(\mathbb{R}^n)$ for all $i = 1,\cdots,n$ and $j = 1,\cdots,l$. Then, for every $T>0$ we have*

$$\lim_{\delta\to 0^+} E\left[\sup_{t\in[0,T]} |u_\delta(t,\omega,x) - u(t,\omega,x)|^2\right] = 0,$$

where $C_b^k(\mathbb{R}^n)$ is the usual space of C^k smooth functions from $\mathbb{R}^n$ to $\mathbb{R}$ with bounded derivatives up to order $k \in \mathbb{N}$.

Our second result is on the convergence of approximations of center manifolds. We consider the stochastic differential equation of form

$$du = (Au + f(u))\,dt + u \circ dW \tag{1.4}$$

and its Wong–Zakai approximation

$$\dot{u}_\delta = Au_\delta + f(u_\delta) + u_\delta \mathcal{G}_\delta(\theta_t\omega). \tag{1.5}$$

Here A is a $n\times n$ matrix and f is a high order term. Note that in this case, $u=0$ is a stationary solution.

Theorem B. *Assume that A is a partially hyperbolic $n\times n$ matrix and f is globally Lipschitz continuous with $f(0)=0$. Then, there exists $\epsilon_0>0$ such that if the Lipschitz constant of f $Lip(f)<\epsilon_0$, then both equation (1.4) and equation (1.5) have global center manifolds, and the center manifold of equation (1.5) converges pathwise to that of equation (1.4).*

The study of approximations of stochastic differential equations by using pathwise deterministic differential equations dates back to Wong and Zakai [41,42] in which they studied a scalar stochastic differential equation

$$du = f(u,t)\,dt + \sigma(u,t)\circ dW \tag{1.6}$$

and its approximation

$$du_n = f(u_n,t)\,dt + \sigma(u_n,t)\,dW_n, \tag{1.7}$$

where f and σ are scalar functions, W is a 1-dimensional Brownian motion and W_n is its approximation. In [41], they proved that under the conditions that $\frac{\partial}{\partial x}\sigma(x,t)$ is continuous in (x,t) and $f(x,t)$, $\sigma(x,t)$, $\frac{\partial}{\partial x}\sigma^2(x,t)$ are continuous in t and Lipschitz in x, if $W_n(t,\omega)$ is a continuous piecewise linear approximation of the Brownian motion $W(t,\omega)$, then solution u_n of equation (1.7) converges in the mean square to solution u of equation (1.6). In another paper [42], Wong and Zakai studied the piecewise smooth approximations $W_n(t,\omega)$ of the 1-dimensional Brownian motion $W(t,\omega)$ and proved that solution $u_n(t,\omega)$ of equation (1.7) converges almost

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surely to solution $u(t,\omega)$ of equation (1.6) under additional assumptions on $\sigma(x,t)$, which are $\sigma(x,t) > \beta > 0$ (or $\sigma(x,t) < -\beta < 0$) and $|\frac{\partial}{\partial t}\sigma| \leq K\sigma^2$. Generally, Wong–Zakai's results for piecewise smooth approximations of a Brownian motion with dimension $d \geq 2$ may not hold. In [23], McShane gave a counter example to show that Wong–Zakai's result does not hold for a 2-dimensional Brownian motion approximated by smooth functions.

Stroock and Varadhan [36] studied Wong–Zakai's piecewise linear (polygonal) approximations for high-dimensional Brownian motions and proved that the stochastic process determined by the corresponding approximated integral equation converges in law and used it to determine the support of diffusion processes.

In [37,38], Sussmann extended Wong–Zakai's results to high dimensional stochastic differential equations driven by 1-dimensional noise and studied C^1 smooth approximations of 1-dimensional Brownian motion. Assuming that the drift term is locally Lipschitz and satisfies a linear growth condition, the diffusion term is C^1 and its partial derivatives are locally Lipschitz and are uniformly bounded, Sussmann proved that the solutions of the approximated equation converge almost surely to the ones of the stochastic differential equation uniformly on compact time intervals. He gave an example to show that when the noise is high dimensional, his result does not hold. He also gave a counter example to show the assumption that the partial derivatives of σ are uniformly bounded cannot be replaced.

Ikeda, Nakao and Yamato [15] studied the Wong–Zakai approximations of high-dimensional Brownian motion by incorporating the shift operator $\theta_t\omega(\cdot) = \omega(t+\cdot)$ into approximations. They considered the piecewise smooth approximations $W_\delta(t,\omega)$ of a d-dimensional Brownian motion $W(t,\omega)$ which satisfy a set of conditions including at each partition point $k\delta$, $W_\delta(k\delta,\omega) = W(k\delta,\omega)$, and at its shift point $t+k\delta$, $W_\delta(t+k\delta,\omega) = W(t,\theta_{k\delta}\omega)$. They proved that if $\sigma \in C_b^2(\mathbb{R}^d, \mathbb{R}^{d\times d})$, then there exists a sequence $\delta_n \to 0$ such that the solutions u_{δ_n} of

$$du_{\delta_n} = \sigma(u_{\delta_n})\, dW_{\delta_n}$$

converge to solutions of

$$du = \sigma(u)\, dW + S * \sigma(u)\, dt,$$

for each $t \in [0,T]$, where S is a $d \times d$ matrix depending on the choice of approximation W_δ.

Later, Ikeda and Watanabe [14] continued to study the Wong–Zakai approximations of a high-dimensional Brownian motion by incorporating the Wiener shift operator $\theta_t\omega(\cdot) = \omega(t+\cdot) - \omega(t)$ into the piecewise smooth approximations. A major change is that at each partition point $k\delta$, $W_\delta(t+k\delta,\omega) = W_\delta(t,\theta_{k\delta}\omega) + W(k\delta,\omega)$. They considered the stochastic differential equation

$$du = f(u)\, dt + \sigma(u) \circ dW, \tag{1.8}$$

and proved that if $f \in C_b^1(\mathbb{R}^d)$ and $\sigma \in C_b^2(\mathbb{R}^d, \mathbb{R}^{d\times l})$, then solution u_δ of

$$du_\delta = f(u_\delta)\, dt + \sigma(u_\delta)\, dW_\delta$$

converges to solution u of

$$du = f(u)\, dt + \sigma(u)\, dW + S * \sigma(u)\, dt,$$

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in the mean square uniformly in $[0, T]$ as $\delta \to 0$. Here again S is a $l \times l$ matrix depending on the choice of approximation W_δ. We note that this limit equation is not the original Stratonovich stochastic differential equation (1.8).

Recently, Kelly and Melbourne [18] studied a class of smooth approximations given by

$$W_n(t) = \frac{1}{\sqrt{n}} \int_0^{nt} v \circ \phi_s \, ds$$

where ϕ_t is a C^2 uniformly hyperbolic flow on a compact manifold and v is a smooth observable function. They proved that $W_n(t)$ and its second level $\mathbb{W}_n(t)$ converge weakly to a Brownian motion $W(t)$ and its second level, respectively. They considered the stochastic differential equation

$$du = f(u)\, dt + \sigma(u) \circ dW. \tag{1.9}$$

Using the rough path theory, they showed that solutions of

$$du_n = f(u_n)\, dt + \sigma(u_n)\, dW_n$$

converge weakly to solutions of the following stochastic differential equation

$$du = f(u)\, dt + \sigma(u) * dW, \tag{1.10}$$

where $\sigma(u) * dW$ depends on the observable function v and the hyperbolic dynamical system ϕ_t, and the limit equation (1.10) is not the original Stratonovich equation (1.9).

The study of the Wong–Zakai approximations has also been extended to stochastic differential equations driven by martingales and semimartingales, see for example, Nakao–Yamato [27], Konecny [16], Protter [31], Nakao [26], and Kurtz–Protter [19,20]. Recently, there are works on the Wong–Zakai approximations of solutions to reflecting stochastic differential equations, see, Pettersson [29], Evans–Stroock [10], Aida–Sasaki [1], Zhang [43], Slomiński [35], Ren–Wu [32] and their references therein.

In current paper, we use the stationary process $\mathcal{G}(\theta_t\omega) = (\omega(t+\delta) - \omega(t))/\delta$ to approximate the white noise of high dimension in the sense

$$\lim_{\delta \to 0^+} \sup_{t \in [0,T]} \left| \int_0^t \mathcal{G}_\delta(\theta_s\omega) ds - W(t,\omega) \right| = 0, a.s.$$

for each $T > 0$. The approximation of $W(t,\omega)$ is

$$W_\delta(t,\omega) = \int_0^t \mathcal{G}_\delta(\theta_s\omega) ds.$$

An advantage of such approximations is that the corresponding approximated equation (1.3) generates a random dynamical system and the solutions of the approximated equation (1.3) converge to the solutions of the original Stratonovich stochastic differential equation (1.1).

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We note that the approximation $W_\delta(t,\omega)$ we study here is different from the ones studied by Ikeda, Nakao and Yamato in [15] and Ikeda and Watanabe in [14] since at each partition point $k\delta$, it does not satisfy neither $W_\delta(k\delta,\omega)=W(k\delta,\omega)$ in [15] nor $W_\delta(t+k\delta,\omega)=W_\delta(t,\theta_{k\delta}\omega)+W(k\delta,\omega)$ in [14].

Instead of white noise, for each $\delta>0$, our random driving force $\mathcal{G}_\delta(\theta_t\omega)$ is a stationary Gaussian process and for almost all sample path $\omega\in\Omega$, it is an unbounded function in $t\in\mathbb{R}$. Furthermore, unlike Brownian motion $W(t,\omega)$, the approximation $W_\delta(t,\omega)=\int_0^t\mathcal{G}_\delta(\theta_s\omega)ds$ has a short term memory of range δ. But this property is sufficient for us to establish our main result.

In [21], Lu and Wang studied equation (1.3) in 2-dimension driven by a 1-dimensional noise. Assuming the equation with only drift term has a homoclinic orbit to a saddle fixed point, they proved that if the diffusion term σ is not completely tangent to the homoclinic orbit, then for almost all sample pathes of the Brownian motion, the forced equation (1.3) admits a topological horseshoe of infinitely many branches, thus is chaotic. They also applied the result to the randomly forced Duffing equation and the pendulum equation. Later, their work was extended to the system with a heterclinic loop in Shen–Lu–Zhang [34].

The theory of invariant manifolds is a fundamental tool for describing and understanding nonlinear dynamical systems. They are widely used to investigate the qualitative behavior of the flows, bifurcation characteristics and linearization, etc. The study of invariant manifolds dates back to Hadamard [11], Lyapunov [22] and Perron [28]. Since then, there is an extensive literature on invariant manifolds, included the stable, unstable, center, center-stable and center-unstable manifolds for finite or infinite deterministic dynamical systems, see for example, on center-manifolds, Pliss [30], Kelley [17], Hale [12], Henry [13], Carr [4], Vanderbauwhede–Van Gils [39], Chow–Lu [6,7], Bates–Jones [3]. The works on invariant manifolds for random dynamical systems can be founded, for example, in Wanner [40], Arnold [2], Mohammed–Scheutzow [24], Schmalfuss [33], Duan–Lu–Schmalfuss [8,9], and Mohammed–Zhang–Zhao [25]. In this paper, we show that when the noise is linear multiplicative, both equations (1.1) and (1.3) have center manifolds, and the center manifold of the approximated equation (1.3) converges to the center manifold of the original Stratonovich stochastic differential equation (1.1).

We organize this paper as follows. In section 2, we study equations (1.1) and (1.3) and establish the convergence of Wong–Zakai approximations. In section 3, we prove the existence of center manifolds of equation (1.5) and equation (1.4) and that the center manifolds of equation (1.5) converge pathwisely to ones of equation (1.4) as $\delta\to 0^+$.

2. Wong–Zakai approximations of SDEs

In this section, we study the Wong–Zakai approximations of stochastic differential equation

$$du=f(u)dt+\sigma(u)\circ dW,\ u_0=x,\ x\in\mathbb{R}^n \tag{2.1}$$

of the form

$$\dot{u}_\delta=f(u_\delta)+\sigma(u_\delta)\,\mathcal{G}(\theta_t\omega),\ u_\delta(0)=x,\ x\in\mathbb{R}^n, \tag{2.2}$$

where $\mathcal{G}(\theta_t\omega)$ is given by (1.2).

Throughout of this section, we assume that f and σ are Lipschitz continuous functions, i.e., there is a constant $L>0$ such that for all $u,v\in\mathbb{R}^n$

$$|f(u) - f(v)| + |\sigma(u) - \sigma(v)| \le L|u - v|. \tag{2.3}$$

For the classical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$, from the law of logarithms, it follows that there exists a θ_t-invariant subset $\bar{\Omega}$ of Ω of full measure with sublinear growth:

$$\lim_{s\to\pm\infty} \frac{|\omega(s)|}{|s|} = 0.$$

Let

$$C_\omega = \sup_{s\in\mathbb{Q}} \frac{|\omega(s)|}{|s|+1},$$

where $\mathbb{Q}$ is the set of rational numbers. Since for each s, $\omega(s) : \Omega \to \mathbb{R}^l$ is measurable and supremum is finite, $C_\omega : \bar{\Omega} \to \mathbb{R}^+$ is a measurable function and

$$|\omega(s)| \le C_\omega(|s| + 1)$$

for all $s \in \mathbb{R}$. Recall that $\theta_t\omega(s) = \omega(s + t) - \omega(t)$, it then follows that

$$C_{\theta_t\omega} \le 2C_\omega(|t| + 1).$$

Thus

$$|\mathcal{G}_\delta(\theta_t\omega)| \le K_\delta C_\omega(|t| + 1), \tag{2.4}$$

where $K_\delta := \frac{2}{\delta}(\delta + 1)$. This estimate plays a key role in the proof of well-posedness of equation (2.2).

We now replace $\mathcal{F}$ by

$$\bar{\mathcal{F}} = \{\bar{\Omega} \cap A,\ A \in \mathcal{F}\}.$$

The probability measure on $\bar{\Omega}$ is the restriction of the Wiener measure to this new σ-algebra, which is also denoted by $\mathbb{P}$. We will restrict our study in this probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \mathbb{P})$. For simplicity, we denote this space again by $(\Omega, \mathcal{F}, \mathbb{P})$. Without loss of generality, we may assume that $\mathcal{F}$ is complete. Let

$$\mathcal{F}_t := \vee_{s\le t}\mathcal{F}_s^t \ \forall t \in \mathbb{R}$$

with

$$\mathcal{F}_s^t := \sigma(\omega(p) - \omega(q) : s \le p \le q \le t) \vee \mathcal{N} \ \forall\ s \le t,$$

where $\sigma(\omega(p) - \omega(q) : s \le p \le q \le t)$ is the smallest σ-algebra generated by the random variable $\omega(p) - \omega(q)$ for $s \le p \le q \le t$ and $\mathcal{N}$ is a null set of $\mathcal{F}$. By Arnold [2, p. 91], we have

$$\mathcal{F}_s^{t+} := \cap_{u>t}\mathcal{F}_s^u = \mathcal{F}_s^t, \quad \mathcal{F}_{s-}^t := \cap_{u<s}\mathcal{F}_u^t = \mathcal{F}_s^t$$

and $\theta_p^{-1}\mathcal{F}_s^t = \mathcal{F}_{s+p}^{t+p}$ for $s \le t$. Hence $(\Omega, \mathcal{F}, (\theta_t)_{t\in\mathbb{R}}, (\mathcal{F}_s^t)_{s\le t})$ is a filtered dynamical system.

We first note that condition (2.3) guarantees the existence and uniqueness of solutions of stochastic differential equation (2.1). It together with (2.4) also give the existence and uniqueness of solution of equation (2.2) and their properties, which we summarize in the following proposition.

Proposition 2.1. *Assume (2.3) holds. Then, for each $\delta > 0$ we have the following*

(i) *equation (2.2) has a unique solution $u_\delta(t, \omega, x)$ defined for all $0 \le t < +\infty$;*
(ii) *$u_\delta(t, \omega, x)$ is Lipschitz continuous in x;*
(iii) *$u_\delta(t, \cdot, x)$ is $\mathcal{F}_0^{t+\delta}$ measurable;*
(iv) *$u_\delta(\cdot, \cdot, \cdot)$ is $\mathcal{B}([0, +\infty)) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^n)$ measurable;*
(v) *$u_\delta(t, \omega, x)$ generates a random dynamical system.*

We point out that unlike the solutions of equation (2.1), the solution $u_\delta(t, \omega, x)$ is not adapted to the filtration $\mathcal{F}_t$. Since the proof of this proposition follows from the standard arguments, we omit it.

Let

$$G_\delta(t, \omega) := (G_\delta(t, \omega_1), G_\delta(t, \omega_2), \ldots, G_\delta(t, \omega_l)),$$

where for each $1 \le i \le l$,

$$G_\delta(t, \omega_i) := \int_0^t \mathcal{G}_\delta(\theta_s \omega_i) ds.$$

Then random differential equation (2.2) can be written as

$$\dot{u}_\delta = f(u_\delta) + \sigma(u_\delta)\dot{G}_\delta(t, \omega), \ \ u_\delta(0) = x, \ \ x \in \mathbb{R}^n. \tag{2.5}$$

For any $T > 0$, in what follows, we shall show that the solutions of equation (2.5) converge in mean square to the solutions of equation (2.1) uniformly on $[0, T]$ as $\delta \to 0^+$.

For simplicity, in this section we let K denote a generic constant whose value may change from line to line, but does not depend on δ.

The following lemma is a summary of basic properties of the approximations of a Brownian motion.

Lemma 2.1. *For every $1 \le i \le l$, we have*

(1) $G_\delta(t + k\delta, \omega_i) = G_\delta(t, \theta_{k\delta}\omega_i) + G_\delta(k\delta, \omega_i)$ $\forall k \in \mathbb{N}$;
(2) $G_\delta(0, \omega_i) = 0$;
(3) $E\left(\int_{k\delta}^{(k+1)\delta} |\dot{G}_\delta(s, \omega_i)| ds\right)^6 \le K\delta^3$ $\forall k \in \mathbb{N}$;
(4) *for any $i_1, \ldots, i_m \in \{1, \ldots, l\}$, if $p_1, p_2, \cdots, p_m \ge 1$ and $p_1 + p_2 + \cdots + p_m \le 6$,*

$$E\left[\left(\int_{k\delta}^{(k+1)\delta} |\dot{G}_\delta(s, \omega_{i_1})| ds\right)^{p_1} \left(\int_{k\delta}^{(k+1)\delta} |\dot{G}_\delta(s, \omega_{i_2})| ds\right)^{p_2} \cdots \left(\int_{k\delta}^{(k+1)\delta} |\dot{G}_\delta(s, \omega_{i_m})| ds\right)^{p_m}\right]$$

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$$\leq K\delta^{\frac{1}{2}(p_1+p_2+\cdots+p_m)} \ \forall k \in \mathbb{N};$$

(5) *for any* $i_1, \ldots, i_m \in \{1, \ldots, l\}$ *and* $n_1, n_2, \ldots, n_m \in \mathbb{N}^+$*, if* $p_1, p_2, \cdots, p_m \geq 1$ *and* $p_1 + p_2 + \cdots + p_m \leq 6$*,*

$$E\left[\left(\int_0^{n_1\delta} |\dot{G}_\delta(s, \omega_{i_1})|ds\right)^{p_1} \left(\int_0^{n_2\delta} |\dot{G}_\delta(s, \omega_{i_2})|ds\right)^{p_2} \cdots \left(\int_0^{n_m\delta} |\dot{G}_\delta(s, \omega_{i_m})|ds\right)^{p_m}\right]$$
$$\leq K n_1^{p_1} n_2^{p_2} \cdots n_m^{p_m} \delta^{\frac{1}{2}(p_1+p_2+\cdots+p_m)}.$$

Proof. It is obvious that Property (1) and (2) hold. Property (4) follows from the Hölder inequality and Property (3). To show Property (3), we first use Property (1) and the invariance of the probability measure $\mathbb{P}$ under θ_t to have

$$E\left(\int_{k\delta}^{(k+1)\delta} |\dot{G}_\delta(s, \omega_i)|ds\right)^6 = E\left(\int_0^{\delta} |\dot{G}_\delta(s, \omega_i)|ds\right)^6.$$

Then, by the Hölder inequality, Fubini's theorem, and the Brownian scaling property, we obtain

$$E\left(\int_0^{\delta} |\dot{G}_\delta(s, \omega_i)|ds\right)^6 \leq \delta^{-1}\int_0^{\delta} E|\omega_i(s+\delta) - \omega_i(s)|^6 ds \leq \delta^{-1}\int_0^{\delta} E|\omega_i(\delta)|^6 ds = K\delta^3,$$

where $K = E|\omega_i(1)|^6$. Similarly, for each $j = 1, \ldots, m$, we have

$$E\left(\int_0^{n_j\delta} |\dot{G}_\delta(s, \omega_{i_j})|ds\right)^6 \leq K n_j^6 \delta^3,$$

which together with the Hölder inequality yield that Property (5) holds. This completes the proof of the lemma. □

From Lemma 2.1(1), we see that our approximated noise $G_\delta(t, \omega)$ is different from Ikeda and Watanabe's condition $G_\delta(t + k\delta, \omega) = G_\delta(t, \theta_{k\delta}\omega) + W(k\delta, \omega)$.

The next lemma is on the moments of the approximations.

Lemma 2.2. *For each* $1 \leq i \leq l$*,* $EG_\delta(\sigma, \omega_i)^2 = -\frac{1}{3}\delta + \sigma$ *for all* $\sigma \geq \delta$*,* $EG_\delta(\sigma, \omega_i)^4 \leq K(\sigma + \delta)^2$*, and* $E[G_\delta(\sigma, \omega_i)G_\delta(\delta, \omega_i)] \equiv \frac{5\delta}{6}$ *for all* $\sigma \geq 2\delta$*.*

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Proof. We first compute

$$
\begin{aligned}
& EG_\delta(\sigma,\omega_i)^2 \\
&= \frac{1}{\delta^2} \iint\limits_{[0,\sigma]\times[0,\sigma]} \Big\{ E[\omega_i(\delta+s)\omega_i(\delta+r)] - E[\omega_i(\delta+s)\omega_i(r)] \\
&\qquad\qquad - E[\omega_i(s)\omega_i(\delta+r)] + E[\omega_i(s)\omega_i(r)] \Big\} drds \\
&= \frac{1}{\delta^2} \iint\limits_{[0,\sigma]\times[0,\sigma]} (\delta+s)\wedge(\delta+r)drds - \frac{1}{\delta^2} \iint\limits_{[0,\sigma]\times[0,\sigma]} (\delta+s)\wedge r drds \\
&\quad - \frac{1}{\delta^2} \iint\limits_{[0,\sigma]\times[0,\sigma]} s\wedge(\delta+r)drds + \frac{1}{\delta^2} \iint\limits_{[0,\sigma]\times[0,\sigma]} s\wedge r drds \\
&= \frac{\sigma^2(\sigma+3\delta)}{3\delta^2} - \frac{2\sigma^3+\delta^3+3\sigma^2\delta-3\sigma\delta^2}{6\delta^2} - \frac{-3\sigma\delta^2+2\sigma^3+\delta^3+3\delta\sigma^2}{6\delta^2} + \frac{\sigma^3}{3\delta^2} \\
&= -\frac{1}{3}\delta + \sigma.
\end{aligned}
$$

To get the estimate of $G_\delta(\sigma,\omega_i)$, we rewrite $G_\delta(\sigma,\omega_i)$ as

$$
G_\delta(\sigma,\omega_i) = -\int_0^\delta \frac{\omega_i(s)}{\delta} ds + \int_\sigma^{\sigma+\delta} \frac{\omega_i(s)}{\delta} ds.
$$

By the Hölder inequality, Fubini's theorem and the Brownian scaling property, we obtain

$$
\begin{aligned}
EG_\delta(\sigma,\omega_i)^4 &\le KE\left(\int_0^\delta \frac{\omega_i(s)}{\delta} ds\right)^4 + KE\left(\int_\sigma^{\sigma+\delta} \frac{\omega_i(s)}{\delta} ds\right)^4 \\
&\le K\delta^3 \int_0^\delta E\left(\frac{\omega_i(s)}{\delta}\right)^4 ds + K\delta^3 \int_\sigma^{\sigma+\delta} E\left(\frac{\omega_i(s)}{\delta}\right)^4 ds \\
&\le K\delta^2 + K(\sigma+\delta)^2 \le K(\sigma+\delta)^2.
\end{aligned}
$$

Since a Brownian motion has independent increments, for $\sigma > 2\delta$ we have that

$$
\int_{2\delta}^\sigma \mathcal{G}(\theta_s\omega)\, ds \quad \text{and} \quad \int_0^\delta \mathcal{G}(\theta_s\omega)\, ds
$$

are independent. Thus,

$$
E[G_\delta(\sigma,\omega_i)G_\delta(\delta,\omega_i)] = E[G_\delta(2\delta,\omega_i)G_\delta(\delta,\omega_i)].
$$

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A similar calculation to $EG_\delta(\sigma, \omega_i)^2$ yields $E[G_\delta(2\delta, \omega_i)G_\delta(\delta, \omega_i)] = \frac{5\delta}{6}$. The proof of Lemma 2.2 is complete. $\square$

We write equation (2.5) and equation (2.1) in terms of coordinates and their integral forms. We have for equation (2.5)

$$u_\delta^i(t,\omega) - x^i = \int_0^t f^i(u_\delta(s,\omega))ds + \sum_{j=1}^{l} \int_0^t \sigma^{ij}(u_\delta(s,\omega))\dot{G}_\delta(s,\omega_j)ds, \tag{2.6}$$

and for equation (2.1) using Ito integral

$$\begin{aligned} u^i(t,\omega) - x^i = &\int_0^t f^i(u(s,\omega))ds + \sum_{j=1}^{l} \int_0^t \sigma^{ij}(u(s,\omega))dw_j \\ &+ \frac{1}{2}\sum_{j=1}^{l}\sum_{\alpha=1}^{n} \int_0^t (\sigma^{\alpha j}\partial_\alpha \sigma^{ij})(u(s,\omega))ds \quad \forall i = 1, \cdots, n, \end{aligned} \tag{2.7}$$

where $\partial_\alpha \sigma^{ij} = \frac{\partial \sigma^{ij}}{\partial x^\alpha}$. Now we state our main result as follows.

Theorem 2.1. *Assume that $f^i \in C_b^1(\mathbb{R}^n)$ and $\sigma^{ij} \in C_b^2(\mathbb{R}^n)$ for all $i = 1, \cdots, n$ and $j = 1, \cdots, l$. For every $T > 0$ we have*

$$\lim_{\delta \to 0^+} E\left[\sup_{t \in [0,T]} |u_\delta(t,\omega) - u(t,\omega)|^2\right] = 0.$$

The proof of this theorem consists of four main parts on estimates of difference of solutions of equations (2.6) and (2.7) over various time intervals. We formulate them in the following four lemmas.

As in [14], we choose an integer function $n : (0, 1] \to \mathbb{N}$ that satisfies $n(\delta)^4\delta \downarrow 0$ and $n(\delta) \uparrow +\infty$ as $\delta \downarrow 0$. Let $\tilde{\delta} := n(\delta)\delta$. For any $s \in \mathbb{R}^+$, if $k\tilde{\delta} \le s < (k+1)\tilde{\delta}$, we define $\lceil s \rceil(\tilde{\delta}) := (k+1)\tilde{\delta}$ and $\lfloor s \rfloor(\tilde{\delta}) := k\tilde{\delta}$, respectively. Set $m(s) := \lfloor s \rfloor(\tilde{\delta})/\tilde{\delta}$. We divide the interval $[0, +\infty)$ into equal subintervals of length $\tilde{\delta}$ by using partition points:

$$0 = \bar{t}_0 < \bar{t}_1 < \bar{t}_2 < \cdots < \bar{t}_k < \cdots,$$

where $\bar{t}_k = k\tilde{\delta}$.

For each $i = 1, \ldots, n$, using (2.6) and (2.7), we write the difference $u_\delta^i(t,\omega) - u^i(t,\omega)$ as a sum of four terms, i.e.,

$$u_\delta^i(t,\omega) - u^i(t,\omega) = \Lambda(t) + \Pi(0,\tilde{\delta}) + \Pi(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) + \Pi(\lfloor t \rfloor(\tilde{\delta}), t),$$

where

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$$\Lambda(t) := \int_0^t f^i(u_\delta(s,\omega))ds - \int_0^t f^i(u(s,\omega))ds,$$

$$\Pi(t_1,t_2) := \Pi_1(t_1,t_2) + \Pi_2(t_1,t_2) + \Pi_3(t_1,t_2), \tag{2.8}$$

$$\Pi_1(t_1,t_2) := \sum_{j=1}^{l} \int_{t_1}^{t_2} \sigma^{ij}(u_\delta(s,\omega))\dot{G}_\delta(s,\omega_j)ds,$$

$$\Pi_2(t_1,t_2) := -\sum_{j=1}^{l} \int_{t_1}^{t_2} \sigma^{ij}(u(s,\omega))dw_j,$$

$$\Pi_3(t_1,t_2) := -\frac{1}{2}\sum_{j=1}^{l}\sum_{\alpha=1}^{n} \int_{t_1}^{t_2} (\sigma^{\alpha j}\partial_\alpha \sigma^{ij})(u(s,\omega))ds$$

for $0 \le t_1 < t_2$.

First, for $\Lambda(t)$, it is clear that we have for $s_1 \in [0,T]$

Lemma 2.3. *There is a constant $K > 0$ such that for each $s_1 \in [0,T]$*

$$E\left[\sup_{t\in[0,s_1]} |\Lambda(t)|^2\right] \le K \int_0^{s_1} E|u_\delta(s,\omega) - u(s,\omega)|^2 ds.$$

Next, we estimate $\Pi(\lfloor t\rfloor(\tilde{\delta}), t)$ and have the following lemma.

Lemma 2.4. *As $\delta \to 0^+$, we have*

$$E\left[\sup_{t\in[0,T]} |\Pi(\lfloor t\rfloor(\tilde{\delta}), t)|^2\right] = o(1).$$

Proof. Recall from (2.8) that

$$\Pi(\lfloor t\rfloor(\tilde{\delta}), t) = \Pi_1(\lfloor t\rfloor(\tilde{\delta}), t) + \Pi_2(\lfloor t\rfloor(\tilde{\delta}), t) + \Pi_3(\lfloor t\rfloor(\tilde{\delta}), t).$$

For $\Pi_1(\lfloor t\rfloor(\tilde{\delta}), t)$, using the Hölder inequality, we have

$$E\left[\sup_{t\in[0,T]} |\Pi_1(\lfloor t\rfloor(\tilde{\delta}), t)|^2\right]$$
$$\le K\sum_{j=1}^{l} E \sup_{t\in[0,T]} \left(\int_{\lfloor t\rfloor(\tilde{\delta})}^{t} |\dot{G}_\delta(s,\omega_j)|ds\right)^2$$

$$\leq K \sum_{j=1}^{l} \left(E \left(\sup_{t\in[0,T]} \int_{\lfloor t\rfloor(\tilde{\delta})}^{t} |\dot{G}_\delta(s,\omega_j)|ds \right)^4 \right)^{\frac{1}{2}} \tag{2.9}$$

$$= K \sum_{j=1}^{l} \left[E \left(\max_{0\leq k\leq m(T)} \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} |\dot{G}_\delta(s,\omega_j)|ds \right)^4 \right]^{\frac{1}{2}}$$

$$\leq K \sum_{j=1}^{l} \left[\sum_{k=0}^{m(T)} E \left(\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} |\dot{G}_\delta(s,\omega_j)|ds \right)^4 \right]^{\frac{1}{2}}.$$

By Lemma 2.1 (1) and (5) and the invariance of probability measure $\mathbb{P}$, we have that

$$E\Big(\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} |\dot{G}_\delta(s,\omega_j)|ds \Big)^4$$

$$= E\Big(\int_0^{\tilde{\delta}} |\dot{G}_\delta(s+k\tilde{\delta},\omega_j)|ds \Big)^4$$

$$= E\Big(\int_0^{\tilde{\delta}} |\dot{G}_\delta(s,\theta_{k\tilde{\delta}}\omega_j)|ds \Big)^4$$

$$= E\Big(\int_0^{\tilde{\delta}} |\dot{G}_\delta(s,\omega_j)|ds \Big)^4 \leq K n(\delta)^4 \delta^2.$$

Thus, from (2.9), we have

$$E\left[\sup_{t\in[0,T]} |\Pi_1(\lfloor t\rfloor(\tilde{\delta}),t)|^2 \right]$$

$$\leq K[m(T)n(\delta)^4\delta^2]^{\frac{1}{2}} \tag{2.10}$$

$$\leq K \left[\frac{n(\delta)^4\delta^2}{n(\delta)\delta} \right]^{\frac{1}{2}} = K(n^3(\delta)\delta)^{\frac{1}{2}} \to 0 \text{ as } \delta \to 0^+.$$

For $\Pi_2(\lfloor t\rfloor(\tilde{\delta}),t)$, we split it into two parts.

$$\Pi_2(\lfloor t\rfloor(\tilde{\delta}),t) = -\sum_{j=1}^{l} \Pi_{21}(\lfloor t\rfloor(\tilde{\delta}),t) - \sum_{j=1}^{l} \Pi_{22}(\lfloor t\rfloor(\tilde{\delta}),t),$$

where

$$\Pi_{21}(\lfloor t \rfloor(\tilde{\delta}), t) := \sigma^{ij}(u(\lfloor t \rfloor(\tilde{\delta}), \omega))[\omega_j(t) - \omega_j(\lfloor t \rfloor(\tilde{\delta}))],$$

$$\Pi_{22}(\lfloor t \rfloor(\tilde{\delta}), t) := \int_{\lfloor t \rfloor(\tilde{\delta})}^{t} \left[\sigma^{ij}(u(s,\omega)) - \sigma^{ij}(u(\lfloor s \rfloor(\tilde{\delta}), \omega))\right] dw_j(s).$$

For $\Pi_{21}(\lfloor t \rfloor(\tilde{\delta}), t)$, we obtain

$$\begin{aligned}
&E\left[\sup_{t\in[0,T]} |\Pi_{21}(\lfloor t \rfloor(\tilde{\delta}), t)|^2\right] \\
&\le K E[\sup_{t\in[0,T]} |\omega_j(t) - \omega_j(\lfloor t \rfloor(\tilde{\delta}))|^2] \\
&\le K \left[E \max_{0\le k\le m(T)} \sup_{0\le t\le\tilde{\delta}} |\omega_j(t + k\tilde{\delta}) - \omega_j(k\tilde{\delta})|^4\right]^{\frac{1}{2}} \\
&\le K \left[\sum_{k=0}^{m(T)} E|\omega_j((k+1)\tilde{\delta}) - \omega_j(k\tilde{\delta})|^4\right]^{\frac{1}{2}} \\
&\le K \left[m(T)\tilde{\delta}^2\right]^{\frac{1}{2}} \le K\tilde{\delta}^{\frac{1}{2}} \to 0 \text{ as } \delta \to 0^+,
\end{aligned}$$

where the third inequality follows from martingale inequality and the fourth one follows from $\omega_j((k+1)\tilde{\delta}) - \omega_j(k\tilde{\delta})$ has the same distribution as $\omega_j(\tilde{\delta})$ and the Brownian scaling property.

For $\Pi_{22}(\lfloor t \rfloor(\tilde{\delta}), t)$ we get that

$$\begin{aligned}
&E\left[\sup_{t\in[0,T]} |\Pi_{22}(\lfloor t \rfloor(\tilde{\delta}), t)|^2\right] \\
&\le E\left\{\sup_{0\le k\le m(T)} \sup_{t\in[0,\tilde{\delta}]} \left[\int_{k\tilde{\delta}}^{k\tilde{\delta}+t} [\sigma^{ij}(u(s,\omega)) - \sigma^{ij}(u(\lfloor s \rfloor(\tilde{\delta}), \omega))] dw_j(s)\right]^2\right\} \\
&\le \sum_{k=0}^{m(T)} E\left\{\sup_{t\in[0,\tilde{\delta}]} \left[\int_{k\tilde{\delta}}^{k\tilde{\delta}+t} [\sigma^{ij}(u(s,\omega)) - \sigma^{ij}(u(\lfloor s \rfloor(\tilde{\delta}), \omega))] dw_j(s)\right]^2\right\} \\
&\le K \sum_{k=0}^{m(T)} E\left\{\left[\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} [\sigma^{ij}(u(s,\omega)) - \sigma^{ij}(u(\lfloor s \rfloor(\tilde{\delta}), \omega))] dw_j(s)\right]^2\right\}.
\end{aligned}$$

Here the martingale inequality is used to get the last estimate. Then, using the Itô isometry, we obtain

$$\sum_{k=0}^{m(T)} E\left\{\left[\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} [\sigma^{ij}(u(s,\omega)) - \sigma^{ij}(u(\lfloor s\rfloor(\tilde{\delta}),\omega))]dw_j(s)\right]^2\right\}$$
$$= \sum_{k=0}^{m(T)} E\left\{\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} \left[\sigma^{ij}(u(s,\omega)) - \sigma^{ij}(u(\lfloor s\rfloor(\tilde{\delta}),\omega))\right]^2 ds\right\}$$
$$= \sum_{k=0}^{m(T)} \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} E\left[\sigma^{ij}(u(s,\omega)) - \sigma^{ij}(u(\lfloor s\rfloor(\tilde{\delta}),\omega))\right]^2 ds.$$

Since $f \in C_b^1$ and $\sigma \in C_b^2$, using (2.7) and the Itô isometry, we have that for $t > s$

$$E|u^i(t,\omega) - u^i(s,\omega)|^2 \le KE\left(\int_s^t f^i(u(r,\omega))dr\right)^2 + KE\left(\sum_{j=1}^{l}\int_s^t \sigma^{ij}(u(r,\omega))dw_j(r)\right)^2$$
$$+K\left(\frac{1}{2}\sum_{j=1}^{l}\sum_{\alpha=1}^{n}\int_s^t (\sigma^{\alpha j}\partial_\alpha\sigma^{ij})(u(r,\omega))dr\right)^2$$
$$\le K((t-s)^2 + (t-s)) \quad \forall i = 1,\cdots,n.$$

Hence,

$$\sum_{k=0}^{m(T)} \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} E\left[\sigma^{ij}(u(s,\omega)) - \sigma^{ij}(u(\lfloor s\rfloor(\tilde{\delta}),\omega))\right]^2 ds$$
$$\le Km(T)\tilde{\delta}^2(\tilde{\delta}+1) \le K\tilde{\delta}(\tilde{\delta}+1) \to 0 \text{ as } \delta \to 0^+.$$

Therefore, we have

$$E\left[\sup_{t\in[0,T]} |\Pi_{22}(\lfloor t\rfloor(\tilde{\delta}),t)|^2\right] = o(1) \quad \text{as} \quad \delta \to 0^+. \tag{2.11}$$

Since σ has bounded derivatives, we have

$$E\left[\sup_{t\in[0,T]} |\Pi_3(\lfloor t\rfloor(\tilde{\delta}),t)|^2\right] \le K\tilde{\delta}^2. \tag{2.12}$$

Combining (2.8), (2.10) and (2.11)–(2.12), we have

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$$E\left[\sup_{t\in[0,T]}|\Pi(\lfloor t\rfloor(\tilde{\delta}),t)|^2\right]=o(1)\text{ as }\delta\to 0^+.$$

This completes the proof of the lemma. □

Since $|\Pi(0,\tilde{\delta})|^2\le\sup_{t\in[0,T]}|\Pi(\lfloor t\rfloor(\tilde{\delta}),t)|^2$, we have

Lemma 2.5. *The following holds*

$$E\left[|\Pi(0,\tilde{\delta})|^2\right]=o(1)\ \textit{as}\ \delta\to 0^+.$$

Finally, we estimate $\Pi(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))$. Recall from (2.8) that

$$\Pi(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))=\Pi_1(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))+\Pi_2(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))+\Pi_3(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta})),$$

where

$$\Pi_1(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta})):=\sum_{j=1}^{l}\int_{\tilde{\delta}}^{\lfloor t\rfloor(\tilde{\delta})}\sigma^{ij}(u_\delta(s,\omega))\dot{G}_\delta(s,\omega_j)ds,$$

$$\Pi_2(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta})):=-\sum_{j=1}^{l}\int_{\tilde{\delta}}^{\lfloor t\rfloor(\tilde{\delta})}\sigma^{ij}(u(s,\omega))dw_j,$$

$$\Pi_3(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta})):=-\frac{1}{2}\sum_{j=1}^{l}\sum_{\alpha=1}^{n}\int_{\tilde{\delta}}^{\lfloor t\rfloor(\tilde{\delta})}(\sigma^{\alpha j}\partial_\alpha\sigma^{ij})(u(s,\omega))ds.$$

We first rewrite $\Pi_1(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))$ and use integration by parts to have

$$\begin{aligned}\Pi_1(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))=&-\sum_{j=1}^{l}\sum_{k=1}^{m(t)-1}\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}\sigma^{ij}(u_\delta(s,\omega))d\Big(G_\delta((k+1)\tilde{\delta},\omega_j)-G_\delta(s,\omega_j)\Big)\\ =&\sum_{j=1}^{l}\Pi_{11}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))+\sum_{j=1}^{l}\sum_{\alpha=1}^{n}\Pi_{12}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta})),\end{aligned}$$

where

$$\Pi_{11}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta})):=\sum_{k=1}^{m(t)-1}\sigma^{ij}(u_\delta(k\tilde{\delta},\omega))\Big(G_\delta((k+1)\tilde{\delta},\omega_j)-G_\delta(k\tilde{\delta},\omega_j)\Big),$$

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$$\Pi_{12}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) := \sum_{k=1}^{m(t)-1} \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} \partial_\alpha \sigma^{ij}(u_\delta(s,\omega))\Big(f^\alpha(u_\delta(s,\omega)) + \sum_{\beta=1}^{l} \sigma_{\alpha\beta}(u_\delta(s,\omega))\dot{G}_\delta(s,\omega_\beta)\Big)\Big(G_\delta((k+1)\tilde{\delta},\omega_j) - G_\delta(s,\omega_j)\Big) ds.$$

Note that

$$\int_{\tilde{\delta}}^{\lfloor t \rfloor(\tilde{\delta})} \sigma^{ij}(u_\delta(\lfloor s \rfloor(\tilde{\delta}) - \delta, \omega))\, dw_j(s) = \sum_{k=1}^{m(t)-1} \sigma^{ij}(u_\delta(k\tilde{\delta} - \delta, \omega))\Big(\omega_j((k+1)\tilde{\delta}) - \omega_j(k\tilde{\delta})\Big).$$

We then write $\Pi_{11}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta}))$ as a sum of three parts.

$$\Pi_{11}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) = \int_{\tilde{\delta}}^{\lfloor t \rfloor(\tilde{\delta})} \sigma^{ij}(u_\delta(\lfloor s \rfloor(\tilde{\delta}) - \delta, \omega))\, dw_j(s) + \Pi_{111}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) + \Pi'_{11}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})),$$

where

$$\begin{aligned}
&\Pi_{111}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) \\
&:= \sum_{k=1}^{m(t)-1} \Big(\sigma^{ij}(u_\delta(k\tilde{\delta},\omega)) - \sigma^{ij}(u_\delta(k\tilde{\delta} - \delta, \omega))\Big)\Big(G_\delta((k+1)\tilde{\delta},\omega_j) - G_\delta(k\tilde{\delta},\omega_j)\Big), \\
&\Pi'_{11}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) \\
&:= \sum_{k=1}^{m(t)-1} \sigma^{ij}(u_\delta(k\tilde{\delta} - \delta, \omega))\Big(G_\delta((k+1)\tilde{\delta},\omega_j) - G_\delta(k\tilde{\delta},\omega_j) - \big(\omega_j((k+1)\tilde{\delta}) - \omega_j(k\tilde{\delta})\big)\Big).
\end{aligned}$$

Since $G_\delta(t,\omega_j)$ has no independent increments, we split $\Pi'_{11}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta}))$ into

$$\Pi'_{11}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) = \Pi_{112}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) + \Pi_{113}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})),$$

where

$$\begin{aligned}
&\Pi_{112}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) := \sum_{k=1}^{m(t)-1} \sigma^{ij}(u_\delta(k\tilde{\delta} - \delta, \omega)) A^j_{k,\tilde{\delta},\delta}, \\
&\Pi_{113}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) := \sum_{k=1}^{m(t)-1} \sigma^{ij}(u_\delta(k\tilde{\delta} - \delta, \omega))\Big(G_\delta((k+1)\tilde{\delta},\omega_j) - G_\delta((k+1)\tilde{\delta} - \delta,\omega_j)\Big), \\
&A^j_{k,\tilde{\delta},\delta} := G_\delta((k+1)\tilde{\delta} - \delta,\omega_j) - G_\delta(k\tilde{\delta},\omega_j) - \big(\omega_j((k+1)\tilde{\delta}) - \omega_j(k\tilde{\delta})\big).
\end{aligned}$$

Then $\Pi_{11}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta}))$ can be rewritten as

$$\Pi_{11}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) = \int_{\tilde{\delta}}^{\lfloor t \rfloor(\tilde{\delta})} \sigma^{ij}(u_\delta(\lfloor s \rfloor(\tilde{\delta}) - \delta, \omega))dw_j(s) + \Pi_{111}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) + \Pi_{112}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) + \Pi_{113}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})).$$

Hence, we can write $\Pi(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta}))$ as

$$\Pi(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) = \sum_{j=1}^{l} \Big(\Upsilon_1^j(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) + \Pi_{111}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) + \Pi_{112}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) + \Pi_{113}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) + \sum_{\alpha=1}^{n} \Upsilon_2^{j,\alpha}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) \Big), \tag{2.13}$$

where

$$\Upsilon_1^j(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) := \int_{\tilde{\delta}}^{\lfloor t \rfloor(\tilde{\delta})} \Big(\sigma^{ij}(u_\delta(\lfloor s \rfloor(\tilde{\delta}) - \delta, \omega)) - \sigma^{ij}(u(s, \omega)) \Big) \, dw_j(s),$$

$$\Upsilon_2^{j,\alpha}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) := \Pi_{12}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) - \frac{1}{2} \int_{\tilde{\delta}}^{\lfloor t \rfloor(\tilde{\delta})} (\sigma^{\alpha j} \partial_\alpha \sigma^{ij})(u(s, \omega))ds.$$

Next, we estimate each term on the right hand of equation (2.13), respectively. We first summarize them as follows.

Lemma 2.6. *We have the following estimates as $\delta \to 0^+$*

$$E\Big[\sup_{t \in [0, s_1]} |\Upsilon_1^j(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta}))|^2 \Big] \leq K \int_0^{s_1} E\big| u_\delta(s, \omega) - u(s, \omega) \big|^2 ds + o(1), \tag{2.14}$$

$$E\Big[\sup_{t \in [0, T]} |\Pi_{111}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta}))|^2 \Big] = o(1), \tag{2.15}$$

$$E\Big[\sup_{t \in [0, T]} |\Pi_{112}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta}))|^2 \Big] = o(1), \tag{2.16}$$

$$E\Big[\sup_{t \in [0, T]} |\Pi_{113}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta}))|^2 \Big] = o(1), \tag{2.17}$$

$$E\Big[\sup_{t \in [0, s_1]} |\Upsilon_2^{j,\alpha}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta}))|^2 \Big] \leq K \int_0^{s_1} E\big| u_\delta(s, \omega) - u(s, \omega) \big|^2 ds + o(1). \tag{2.18}$$

Proof. We prove the lemma in the order of list above.

(I) Estimate of $\Upsilon_1^j(\tilde\delta, \lfloor t\rfloor(\tilde\delta))$

By using the martingale inequality and the Itô isometry, we obtain

$$E\left[\sup_{t\in[0,s_1]} |\Upsilon_1^j(\tilde\delta, \lfloor t\rfloor(\tilde\delta))|^2\right] \le K \int_{\tilde\delta}^{\lfloor s_1\rfloor(\tilde\delta)} E\left|u_\delta(\lfloor s\rfloor(\tilde\delta)-\delta,\omega)-u(s,\omega)\right|^2 ds$$

$$\le K\int_0^{s_1} E|u_\delta(s,\omega)-u(s,\omega)|^2 ds$$

$$+K\int_{\tilde\delta}^{\lfloor s_1\rfloor(\tilde\delta)} E|u_\delta(\lfloor s\rfloor(\tilde\delta)-\delta,\omega)-u_\delta(s,\omega)|^2 ds.$$

From (2.6), we have

$$|u_\delta^i(s,\omega)-u_\delta^i(\lfloor s\rfloor(\tilde\delta)-\delta,\omega)| \le K\left[\tilde\delta+\delta+\sum_{d=1}^{l}\int_{\lfloor s\rfloor(\tilde\delta)-\delta}^{\lceil s\rceil(\tilde\delta)} |\dot G_\delta(r,\omega_d)|dr\right].$$

Then, changing variable r to $r+\lfloor s\rfloor(\tilde\delta)-\delta$ in the integral, and using Lemma 2.1 (1) and (5) and the θ_t-invariance of $\mathbb{P}$, we have

$$E\left[\sup_{t\in[0,s_1]} |\Upsilon_1^j(\tilde\delta, \lfloor t\rfloor(\tilde\delta))|^2\right] \le K\int_0^{s_1} E\left|u_\delta(s,\omega)-u(s,\omega)\right|^2 ds$$

$$+K\int_{\tilde\delta}^{T}\left[(\tilde\delta+\delta)^2+\sum_{d=1}^{l} E\Big(\int_{\lfloor s\rfloor(\tilde\delta)-\delta}^{\lceil s\rceil(\tilde\delta)} |\dot G_\delta(r,\omega_d)|dr\Big)^2\right]ds$$

$$\le K\int_0^{s_1} E\left|u_\delta(s,\omega)-u(s,\omega)\right|^2 ds + K[(\tilde\delta+\delta)^2+(n(\delta)+1)^2\delta]$$

$$= K\int_0^{s_1} E\left|u_\delta(s,\omega)-u(s,\omega)\right|^2 ds + o(1).$$

This completes the proof of property (2.14).

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(II) Estimate of $\Pi_{111}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta}))$

Recall that

$$\Pi_{111}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) := \sum_{k=1}^{m(t)-1} \Big(\sigma^{ij}(u_\delta(k\tilde{\delta}, \omega)) - \sigma^{ij}(u_\delta(k\tilde{\delta}-\delta, \omega))\Big)\Big(G_\delta((k+1)\tilde{\delta}, \omega_j) - G_\delta(k\tilde{\delta}, \omega_j)\Big).$$

Using the Cauchy inequality, we have

$$\begin{aligned} E\left[\sup_{t\in[0,T]} |\Pi_{111}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta}))|^2\right] \le\ & E\left\{\sum_{k=1}^{m(T)-1} \Big(\sigma^{ij}(u_\delta(k\tilde{\delta}, \omega)) - \sigma^{ij}(u_\delta(k\tilde{\delta}-\delta, \omega))\Big)^2 \right. \\ & \left. \times \sum_{k=1}^{m(T)-1} \Big(G_\delta((k+1)\tilde{\delta}, \omega_j) - G_\delta(k\tilde{\delta}, \omega_j)\Big)^2\right\} \\ \le\ & K\left\{m(T)\sum_{k=1}^{m(T)-1}\sum_{i=1}^{n} E\Big|u_\delta^i(k\tilde{\delta}, \omega) - u_\delta^i(k\tilde{\delta}-\delta, \omega)\Big|^4 \right. \\ & \left. \times m(T)\sum_{k=1}^{m(T)-1} E\Big(G_\delta((k+1)\tilde{\delta}, \omega_j) - G_\delta(k\tilde{\delta}, \omega_j)\Big)^4\right\}^{\frac{1}{2}}. \end{aligned}$$

From (2.6), we get

$$|u_\delta^i(k\tilde{\delta}, \omega) - u_\delta^i(k\tilde{\delta}-\delta, \omega)| \le K\left[\delta + \sum_{d=1}^{l}\int_{k\tilde{\delta}-\delta}^{k\tilde{\delta}} |\dot{G}_\delta(s, \omega_d)|ds\right].$$

Thus, by using Lemma 2.1(1) and the θ_t-invariance of $\mathbb{P}$, we have that

$$\begin{aligned} E\left[\sup_{t\in[0,T]} |\Pi_{111}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta}))|^2\right] \le K\left\{m(T)\left[\sum_{k=1}^{m(T)-1}\left(\delta^4 + \sum_{d=1}^{l} E\Big(\int_{k\tilde{\delta}-\delta}^{k\tilde{\delta}} |\dot{G}_\delta(s, \omega_d)|ds\Big)^4\right)\right]\right. \\ \left. \times\, m(T)^2 E G_\delta(\tilde{\delta}, \omega_j)^4\right\}^{\frac{1}{2}}. \end{aligned}$$

Then, changing variable s to $s + k\tilde{\delta} - \delta$ and using Lemma 2.1 (1) and (5) and the θ_t-invariance of $\mathbb{P}$, we have that

$$\begin{aligned} E\left[\sup_{t\in[0,T]} |\Pi_{111}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta}))|^2\right] \le\ & K[m(T)^2(\delta^4 + \delta^2) \times m(T)^2 E G_\delta(\tilde{\delta}, \omega_j)^4]^{\frac{1}{2}} \\ \le\ & K[m(T)^2(\delta^4 + \delta^2) m(T)^2 \tilde{\delta}^2]^{\frac{1}{2}} \le K n(\delta)^{-1} \to 0 \text{ as } \delta \to 0^+, \end{aligned}$$

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where in the second inequality Lemma 2.2 is used, and the fact $m(T)\tilde{\delta} \le T$ and $\tilde{\delta} = n(\delta)\delta$ are also used. This completes the proof of property (2.15).

(III) Estimate of $\Pi_{112}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta}))$

Note that

$$\Pi_{112}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) = \sum_{k=1}^{m(t)-1} \sigma^{ij}(u_\delta(k\tilde{\delta} - \delta, \omega)) A^j_{k,\tilde{\delta},\delta},$$

where $A^j_{k,\tilde{\delta},\delta} = G_\delta((k+1)\tilde{\delta} - \delta, \omega_j) - G_\delta(k\tilde{\delta}, \omega_j) - \big(\omega_j((k+1)\tilde{\delta}) - \omega_j(k\tilde{\delta})\big)$.

In order to estimate $\Pi_{112}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta}))$, we set

$$M_n(\omega) := \sum_{k=1}^{n} \sigma^{ij}(u_\delta(k\tilde{\delta} - \delta, \omega)) A^j_{k,\tilde{\delta},\delta}.$$

Put $\mathcal{J}_n := \mathcal{F}_0^{(n+1)\tilde{\delta}}$. Since $\sigma^{ij}(u_\delta(k\tilde{\delta} - \delta, \omega))$ is $\mathcal{F}_0^{k\tilde{\delta}}$-measurable by Proposition 2.1 (iii) and $A^j_{k,\tilde{\delta},\delta}$ is $\mathcal{F}_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}$-measurable, $M_n(\omega)$ is a $\mathcal{J}_n$-martingale. In fact, we have

$$\begin{aligned} E[M_{n+1}(\omega)|\mathcal{J}_n] &= M_n(\omega) + E[\sigma^{ij}(u_\delta((n+1)\tilde{\delta} - \delta, \omega)) A^j_{n+1,\tilde{\delta},\delta} | \mathcal{J}_n] \\ &= M_n(\omega) + \sigma^{ij}(u_\delta((n+1)\tilde{\delta} - \delta, \omega)) E[A^j_{n+1,\tilde{\delta},\delta} | \mathcal{J}_n] \\ &= M_n(\omega) + \sigma^{ij}(u_\delta((n+1)\tilde{\delta} - \delta, \omega)) E[A^j_{n+1,\tilde{\delta},\delta}] = M_n(\omega). \end{aligned}$$

Hence, by the martingale inequality we obtain

$$\begin{aligned} E\left[\sup_{t\in[0,T]} |\Pi_{112}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta}))|^2\right] &\le K E\left[|M_{m(T)-1}(\omega)|^2\right] \\ &= K \left\{ \sum_{k=1}^{m(T)-1} E\left[\left(\sigma^{ij}(u_\delta(k\tilde{\delta} - \delta, \omega))\right)^2 \left(A^j_{k,\tilde{\delta},\delta}\right)^2 \right] \right\} \\ &\le K \left\{ \sum_{k=1}^{m(T)-1} E|A^j_{k,\tilde{\delta},\delta}|^2 \right\}, \end{aligned} \tag{2.19}$$

where the equality follows from the fact for each $1 \le k_1 < k_2 \le m(T) - 1$,

$$\begin{aligned} &E[\sigma^{ij}(u_\delta(k_1\tilde{\delta} - \delta, \omega)) A^j_{k_1,\tilde{\delta},\delta} \times \sigma^{ij}(u_\delta(k_2\tilde{\delta} - \delta, \omega)) A^j_{k_2,\tilde{\delta},\delta}] \\ &= E[\sigma^{ij}(u_\delta(k_1\tilde{\delta} - \delta, \omega)) A^j_{k_1,\tilde{\delta},\delta} \times \sigma^{ij}(u_\delta(k_2\tilde{\delta} - \delta, \omega))] E[A^j_{k_2,\tilde{\delta},\delta}] = 0. \end{aligned}$$

For each $1 \le j \le l$, by an elementary calculation, we have that

$$A^j_{k,\tilde{\delta},\delta} = \int_{k\tilde{\delta}}^{k\tilde{\delta}+\delta} \frac{\omega_j(k\tilde{\delta}) - \omega_j(s)}{\delta} ds + \int_{(k+1)\tilde{\delta}-\delta}^{(k+1)\tilde{\delta}} \frac{\omega_j(s) - \omega_j((k+1)\tilde{\delta})}{\delta} ds.$$

Hence,

$$E|A^j_{k,\tilde{\delta},\delta}|^2 \leq K\delta,$$

which together with (2.19) implies that

$$E\left[\sup_{t\in[0,T]} |\Pi_{112}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta}))|^2\right] \leq Km(T)\delta \leq Kn(\delta)^{-1} \to 0 \text{ as } \delta \to 0^+.$$

This completes the proof of property (2.16).

(IV) Estimate of $\Pi_{113}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta}))$

Since

$$\Pi_{113}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta})) = \sum_{k=1}^{m(t)-1} \sigma^{ij}(u_\delta(k\tilde{\delta} - \delta, \omega))\Big(G_\delta((k+1)\tilde{\delta}, \omega_j) - G_\delta((k+1)\tilde{\delta} - \delta, \omega_j)\Big),$$

we consider

$$N_n(\omega) := \sum_{k=1}^{n} \sigma^{ij}(u_\delta(k\tilde{\delta} - \delta, \omega))[G_\delta((k+1)\tilde{\delta}, \omega_j) - G_\delta((k+1)\tilde{\delta} - \delta, \omega_j)]$$

and set $\mathcal{K}_n := \mathcal{F}_0^{(n+1)\tilde{\delta}+\delta}$. Since $\sigma^{ij}(u_\delta(k\tilde{\delta} - \delta, \omega))$ is $\mathcal{F}_0^{k\tilde{\delta}}$-measurable by Proposition 2.1 (iii) and $G_\delta((k+1)\tilde{\delta}, \omega_j) - G_\delta((k+1)\tilde{\delta} - \delta, \omega_j)$ is $\mathcal{F}^{(k+1)+\tilde{\delta}}_{(k+1)\tilde{\delta}-\delta}$-measurable, we have

$$\begin{aligned} &E[N_{n+1}(\omega)|\mathcal{K}_n] \\ &\quad = N_n(\omega) + E\{\sigma^{ij}(u_\delta((n+1)\tilde{\delta} - \delta, \omega))[G_\delta((n+2)\tilde{\delta}, \omega_j) - G_\delta((n+2)\tilde{\delta} - \delta, \omega_j)]|\mathcal{K}_n\} \\ &\quad = N_n(\omega) + \sigma^{ij}(u_\delta((n+1)\tilde{\delta} - \delta, \omega))E[G_\delta((n+2)\tilde{\delta}, \omega_j) - G_\delta((n+2)\tilde{\delta} - \delta, \omega_j)|\mathcal{K}_n] \\ &\quad = N_n(\omega) + \sigma^{ij}(u_\delta((n+1)\tilde{\delta} - \delta, \omega))E[G_\delta((n+2)\tilde{\delta}, \omega_j) - G_\delta((n+2)\tilde{\delta} - \delta, \omega_j)] \\ &\quad = N_n(\omega). \end{aligned}$$

Here we use the fact $n(\delta) > 2$ for small δ. Thus, $N_n(\omega)$ is a $\mathcal{K}_n$ martingale. By the martingale inequality, Lemma 2.1 (1) and (5), and the θ_t-invariance of $\mathbb{P}$, we have that

$$\begin{aligned} E\left[\sup_{t\in[0,T]} |\Pi_{113}(\tilde{\delta}, \lfloor t \rfloor(\tilde{\delta}))|^2\right] &\leq KE\left[|N_{m(T)-1}(\omega)|^2\right] \\ &= K\sum_{k=1}^{m(T)-1} E\Big\{[\sigma^{ij}(u_\delta(k\tilde{\delta} - \delta, \omega))]^2 \end{aligned}$$

$$\times[G_\delta((k+1)\tilde{\delta},\omega_j)-G_\delta((k+1)\tilde{\delta}-\delta,\omega_j)]^2\Big\}$$

$$\leq K\sum_{k=1}^{m(T)-1}E\left(\int_{(k+1)\tilde{\delta}-\delta}^{(k+1)\tilde{\delta}}|\dot{G}_\delta(s,\omega_j)|ds\right)^2$$

$$\leq Km(T)\delta\leq Kn(\delta)^{-1}\to 0 \text{ as } \delta\to 0^+.$$

Here the last equality follows from the fact for each $1\leq k_1<k_2\leq m(T)-1$,

$$E\Big\{\sigma^{ij}(u_\delta(k_1\tilde{\delta}-\delta,\omega))[G_\delta((k_1+1)\tilde{\delta},\omega_j)-G_\delta((k_1+1)\tilde{\delta}-\delta,\omega_j)]$$

$$\times\sigma^{ij}(u_\delta(k_2\tilde{\delta}-\delta,\omega))[G_\delta((k_2+1)\tilde{\delta},\omega_j)-G_\delta((k_2+1)\tilde{\delta}-\delta,\omega_j)]\Big\}$$

$$=E\left\{\sigma^{ij}(u_\delta(k_1\tilde{\delta}-\delta,\omega))[G_\delta((k_1+1)\tilde{\delta},\omega_j)-G_\delta((k_1+1)\tilde{\delta}-\delta,\omega_j)]\times\sigma^{ij}(u_\delta(k_2\tilde{\delta}-\delta,\omega))\right\}$$

$$\times E[G_\delta((k_2+1)\tilde{\delta},\omega_j)-G_\delta((k_2+1)\tilde{\delta}-\delta,\omega_j)]=0.$$

This completes the proof of property (2.17).

(V) Estimate of $\Upsilon_2^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))$

We first recall that

$$\Upsilon_2^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))=\Pi_{12}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))-\frac{1}{2}\int_{\tilde{\delta}}^{\lfloor t\rfloor(\tilde{\delta})}(\sigma^{\alpha j}\partial_\alpha\sigma^{ij})(u(s,\omega))ds,$$

where

$$\Pi_{12}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta})):=\sum_{k=1}^{m(t)-1}\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}\partial_\alpha\sigma^{ij}(u_\delta(s,\omega))\Big(f^\alpha(u_\delta(s,\omega))$$

$$+\sum_{\beta=1}^{l}\sigma_{\alpha\beta}(u_\delta(s,\omega))\dot{G}_\delta(s,\omega_\beta)\Big)\Big(G_\delta((k+1)\tilde{\delta},\omega_j)-G_\delta(s,\omega_j)\Big)ds.$$

To estimate $\Upsilon_2^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))$, we rewrite it as

$$\Upsilon_2^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))=\sum_{\beta=1}^{l}\Upsilon_{21}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))+\Upsilon_{22}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))+\Upsilon_{23}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))+\Upsilon_{24}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta})),$$

where

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$$\Upsilon_{21}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta})) := \sum_{k=1}^{m(t)-1}\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}\Big((\sigma^{\alpha\beta}\partial_\alpha\sigma^{ij})(u_\delta(s,\omega)) - (\sigma^{\alpha\beta}\partial_\alpha\sigma^{ij})(u_\delta(k\tilde{\delta},\omega))\Big)\dot{G}_\delta(s,\omega_\beta)$$
$$\times\Big(G_\delta((k+1)\tilde{\delta},\omega_j) - G_\delta(s,\omega_j)\Big)ds,$$

$$\Upsilon_{22}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta})) := \sum_{k=1}^{m(t)-1}\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}(f^\alpha\partial_\alpha\sigma^{ij})(u_\delta(s,\omega))\Big(G_\delta((k+1)\tilde{\delta},\omega_j) - G_\delta(s,\omega_j)\Big)ds,$$

$$\Upsilon_{23}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta})) := \sum_{\beta=1}^{l}\sum_{k=1}^{m(t)-1}\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}(\sigma^{\alpha\beta}\partial_\alpha\sigma^{ij})(u_\delta(k\tilde{\delta},\omega))\Big\{\dot{G}_\delta(s,\omega_\beta)$$
$$\times\Big(G_\delta((k+1)\tilde{\delta},\omega_j) - G_\delta(s,\omega_j)\Big) - \frac{1}{2}\delta_{\beta j}\Big\}ds,$$

$$\Upsilon_{24}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta})) := \frac{1}{2}\sum_{k=1}^{m(t)-1}\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}\Big((\sigma^{\alpha j}\partial_\alpha\sigma^{ij})(u_\delta(k\tilde{\delta},\omega)) - (\sigma^{\alpha j}\partial_\alpha\sigma^{ij})(u(s,\omega))\Big)ds,$$

where $\delta_{\beta j}$ is Kronecker delta. As for $\Upsilon_{21}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))$, we have

$$E\left[\sup_{t\in[0,T]}|\Upsilon_{21}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))|^2\right] \le K\sum_{i=1}^{l}E\Bigg\{\sum_{k=1}^{m(T)-1}\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}|u_\delta^i(s,\omega) - u_\delta^i(k\tilde{\delta},\omega)||\dot{G}_\delta(s,\omega_\beta)|ds$$
$$\times\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}|\dot{G}_\delta(s,\omega_j)|ds\Bigg\}^2. \tag{2.20}$$

Recall (2.6). We have

$$|u_\delta^i(s,\omega) - u_\delta^i(k\tilde{\delta},\omega)| \le K\left(\tilde{\delta} + \sum_{d=1}^{l}\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}|\dot{G}_\delta(r,\omega_d)|dr\right).$$

Then by (2.20) and Lemma 2.1(5) we get that

$$E\left[\sup_{t\in[0,T]}|\Upsilon_{21}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))|^2\right] \le KE\Bigg\{\sum_{k=1}^{m(T)-1}\left(\tilde{\delta} + \sum_{d=1}^{l}\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}|\dot{G}_\delta(r,\omega_d)|dr\right)$$
$$\times\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}|\dot{G}_\delta(s,\omega_\beta)|ds\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}|\dot{G}_\delta(s,\omega_j)|ds\Bigg\}^2$$

$$\le Km(T)E\left\{\sum_{k=1}^{m(T)-1}\left[\tilde{\delta}^2+\sum_{d=1}^{l}\left(\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}|\dot{G}_\delta(r,\omega_d)|dr\right)^2\right]\right.$$
$$\left.\times\left(\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}|\dot{G}_\delta(s,\omega_\beta)|ds\right)^2\left(\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}|\dot{G}_\delta(s,\omega_j)|ds\right)^2\right\}$$
$$\le Km(T)^2[\tilde{\delta}^2 n(\delta)^4\delta^2+n(\delta)^6\delta^3]\le Kn(\delta)^4\delta\to 0 \text{ as } \delta\to 0^+.$$

For $\Upsilon_{22}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))$, by Lemma 2.1(4) we find

$$E\left[\sup_{t\in[0,T]}|\Upsilon_{22}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))|^2\right]\le K\tilde{\delta}^2E\left(\sum_{k=1}^{m(T)-1}\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}|\dot{G}_\delta(s,\omega_j)|ds\right)^2$$
$$\le K\tilde{\delta}^2m(T)\sum_{k=1}^{m(T)-1}E\left(\int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}}|\dot{G}_\delta(s,\omega_j)|ds\right)^2$$
$$\le K\tilde{\delta}^2m(T)^2n(\delta)^2\delta\le Kn(\delta)^2\delta\to 0 \text{ as } \delta\to 0^+.$$

For $\Upsilon_{24}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))$ we have that

$$E\left[\sup_{t\in[0,s_1]}|\Upsilon_{24}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))|^2\right]\le K\int_0^{s_1}E|(\sigma^{\alpha j}\partial_\alpha\sigma^{ij})(u_\delta(\lfloor s\rfloor(\tilde{\delta}),\omega))-(\sigma^{\alpha j}\partial_\alpha\sigma^{ij})(u(s,\omega))|^2ds$$
$$\le K\int_0^{s_1}E|u(s,\omega)-u_\delta(\lfloor s\rfloor(\tilde{\delta}),\omega)|^2ds$$
$$\le K\int_0^{s_1}E|u(s,\omega)-u_\delta(s,\omega)|^2ds$$
$$+K\int_0^{s_1}E|u_\delta(s,\omega)-u_\delta(\lfloor s\rfloor(\tilde{\delta}),\omega)|^2ds.$$

Using (2.6), Lemma 2.1 (1) and (5), and the θ_t-invariance of $\mathbb{P}$, we have for each $1\le i\le n$

$$E|u_\delta^i(s,\omega)-u_\delta^i(\lfloor s\rfloor(\tilde{\delta}),\omega)|^2\le KE\left(\tilde{\delta}+\sum_{d=1}^{l}\int_{\lfloor s\rfloor(\tilde{\delta})}^{\lceil s\rceil(\tilde{\delta})}|\dot{G}_\delta(s,\omega_d)|ds\right)^2\le K(\tilde{\delta}^2+n(\delta)^2\delta).$$

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Hence, we have

$$E\left[\sup_{t\in[0,s_1]}|\Upsilon_{24}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))|^2\right]\le K\int_0^{s_1}E|u(s,\omega)-u_\delta(s,\omega)|^2ds$$
$$+K(\tilde{\delta}^2+n(\delta)^2\delta).$$

Finally we will prove $E\left[\sup_{t\in[0,T]}|\Upsilon_{23}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))|^2\right]=o(1)$ as $\delta\to 0^+$. We write

$$\Upsilon_{23}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta})):=\sum_{\beta=1}^{l}\Upsilon_{231}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))+\sum_{\beta=1}^{l}\Upsilon_{232}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))+\Upsilon_{233}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta})),$$

where

$$\Upsilon_{231}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta})):=\sum_{k=1}^{m(t)-1}\int_{k\tilde{\delta}}^{k\tilde{\delta}+\delta}(\sigma^{\alpha\beta}\partial_\alpha\sigma^{ij})(u_\delta(k\tilde{\delta},\omega))\Big\{\dot{G}_\delta(s,\omega_\beta)$$
$$\times\Big(G_\delta((k+1)\tilde{\delta},\omega_j)-G_\delta(s,\omega_j)\Big)-\frac{1}{2}\delta_{\beta j}\Big\}ds,$$

$$\Upsilon_{232}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta})):=\sum_{k=1}^{m(t)-1}\int_{k\tilde{\delta}+\delta}^{(k+1)\tilde{\delta}}(\sigma^{\alpha\beta}\partial_\alpha\sigma^{ij})(u_\delta(k\tilde{\delta},\omega))\Big\{\dot{G}_\delta(s,\omega_\beta)$$
$$\times\Big(G_\delta((k+1)\tilde{\delta},\omega_j)-G_\delta(s,\omega_j)\Big)+\Big(\frac{1}{6}\delta(\tilde{\delta}-\delta)^{-1}-\frac{1}{2}\Big)\delta_{\beta j}\Big\}ds,$$

$$\Upsilon_{233}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta})):=-\frac{1}{6}\delta\times\sum_{k=1}^{m(t)-1}(\sigma^{\alpha j}\partial_\alpha\sigma^{ij})(u_\delta(k\tilde{\delta},\omega)).$$

For $\Upsilon_{231}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))$, we find

$$E\left[\sup_{t\in[0,T]}|\Upsilon_{231}^{j,\alpha}(\tilde{\delta},\lfloor t\rfloor(\tilde{\delta}))|^2\right]$$
$$\le KE\left[\sum_{k=1}^{m(T)-1}\int_{k\tilde{\delta}}^{k\tilde{\delta}+\delta}|\dot{G}_\delta(s,\omega_\beta)|\times|G_\delta((k+1)\tilde{\delta},\omega_j)-G_\delta(s,\omega_j)|ds\right]^2+Km(T)^2\delta^2$$
$$\le Km(T)^2E\left[\int_0^{\delta}|\dot{G}_\delta(s,\omega_\beta)||G_\delta(\tilde{\delta},\omega_j)-G_\delta(s,\omega_j)|ds\right]^2+Km(T)^2\delta^2.$$

We continue to estimate the last term above.

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$$
\begin{aligned}
&Km(T)^2E\Big[\int_0^\delta |\dot G_\delta(s,\omega_\beta)||G_\delta(\tilde\delta,\omega_j)-G_\delta(s,\omega_j)|ds\Big]^2+Km(T)^2\delta^2\\
&\le Km(T)^2\Big\{E\Big[\int_0^\delta |\dot G_\delta(s,\omega_\beta)||G_\delta(\delta,\omega_j)-G_\delta(s,\omega_j)|ds\Big]^2\\
&\quad+E\Big[\int_0^\delta |\dot G_\delta(s,\omega_\beta)||G_\delta(\tilde\delta,\omega_j)-G_\delta(\delta,\omega_j)|ds\Big]^2\Big\}+Km(T)^2\delta^2\\
&\le Km(T)^2\Big\{E\Big[\Big(\int_0^\delta |\dot G_\delta(s,\omega_\beta)|ds\Big)^2\Big(\int_0^\delta |\dot G_\delta(s,\omega_j)|ds\Big)^2\Big]\\
&\quad+E\Big[\Big(\int_0^\delta |\dot G_\delta(s,\omega_\beta)|ds\Big)^2\Big(|G_\delta(\tilde\delta,\omega_j)|^2+|G_\delta(\delta,\omega_j)|^2\Big)\Big]\Big\}+Km(T)^2\delta^2\\
&\le Km(T)^2\Big(2\delta^2+\delta\tilde\delta\Big)+Km(T)^2\delta^2\to 0 \text{ as } \delta\to 0^+.
\end{aligned}
$$

Here in the second inequality we use the θ_t-invariance of $\mathbb{P}$ and in the last one we use Lemma 2.1 (4), the Hölder inequality, and Lemma 2.2.

For $\Upsilon_{232}^{j,\alpha}(\tilde\delta,\lfloor t\rfloor(\tilde\delta))$, similarly to $\Pi_{112}(\tilde\delta,\lfloor t\rfloor(\tilde\delta))$, for each $\beta=1,\cdots,l$ we consider

$$
\begin{aligned}
R_n:=\sum_{k=1}^n\int_{k\tilde\delta+\delta}^{(k+1)\tilde\delta}(\sigma^{\alpha\beta}\partial_\alpha\sigma^{ij})(u_\delta(k\tilde\delta,\omega))\Big\{&\dot G_\delta(s,\omega_\beta)[G_\delta((k+1)\tilde\delta,\omega_j)-G_\delta(s,\omega_j)]\\
&+[\frac{1}{6}\delta(\tilde\delta-\delta)^{-1}-\frac{1}{2}]\delta_{\beta j}\Big\}ds.
\end{aligned}
$$

We note that $(\sigma^{\alpha\beta}\partial_\alpha\sigma^{ij})(u_\delta(k\tilde\delta,\omega))$ is $\mathcal{F}_0^{k\tilde\delta+\delta}$-measurable by Proposition 2.1 (iii) and

$$
\dot G_\delta(s,\omega_\beta)[G_\delta((k+1)\tilde\delta,\omega_j)-G_\delta(s,\omega_j)]
$$

is $\mathcal{F}_{k\tilde\delta+\delta}^{(k+1)\tilde\delta+\delta}$-measurable for any $k\tilde\delta+\delta\le s\le(k+1)\tilde\delta$. Then we have

$$
\begin{aligned}
E(R_{n+1}|\mathcal{K}_n)=R_n+E\Big\{&\int_{(n+1)\tilde\delta+\delta}^{(n+2)\tilde\delta}(\sigma^{\alpha\beta}\partial_\alpha\sigma^{ij})(u_\delta((n+1)\tilde\delta,\omega))\Big[\dot G_\delta(s,\omega_\beta)\\
&\times\Big(G_\delta((n+2)\tilde\delta,\omega_j)-G_\delta(s,\omega_j)\Big)+\Big(\frac{1}{6}\delta(\tilde\delta-\delta)^{-1}-\frac{1}{2}\Big)\delta_{\beta j}\Big]ds\Big|\mathcal{K}_n\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= R_n + (\sigma^{\alpha\beta}\partial_\alpha\sigma^{ij})(u_\delta((n+1)\tilde{\delta},\omega))E\Bigg\{\int\limits_{(n+1)\tilde{\delta}+\delta}^{(n+2)\tilde{\delta}}\Big[\dot{G}_\delta(s,\omega_\beta) \\
&\quad\times\Big(G_\delta((n+2)\tilde{\delta},\omega_j)-G_\delta(s,\omega_j)\Big)+\Big(\frac{1}{6}\delta(\tilde{\delta}-\delta)^{-1}-\frac{1}{2}\Big)\delta_{\beta j}\Big]ds\Big|\mathcal{K}_n\Bigg\}.
\end{aligned}
$$

By the property of conditional expectation, we further get

$$
\begin{aligned}
E(R_{n+1}|\mathcal{K}_n) = \; & R_n + (\sigma^{\alpha\beta}\partial_\alpha\sigma^{ij})(u_\delta((n+1)\tilde{\delta},\omega))\Bigg\{E\int\limits_{(n+1)\tilde{\delta}+\delta}^{(n+2)\tilde{\delta}}\Big[\dot{G}_\delta(s,\omega_\beta) \\
&\times[G_\delta((n+2)\tilde{\delta},\omega_j)-G_\delta(s,\omega_j)]\Big]ds-\frac{1}{2}(\tilde{\delta}-\delta)\,\delta_{\beta j}+\frac{1}{6}\delta\,\delta_{\beta j}\Bigg\}.
\end{aligned}
$$

Clearly, as $\beta \neq j$, $E(R_{n+1}|\mathcal{K}_n) = R_n$. As $\beta = j$, set

$$
\Theta := \int\limits_{(n+1)\tilde{\delta}+\delta}^{(n+2)\tilde{\delta}} \dot{G}_\delta(s,\omega_j)\Big(G_\delta((n+2)\tilde{\delta},\omega_j)-G_\delta(s,\omega_j)\Big)ds.
$$

Changing variable s to $s+(n+1)\tilde{\delta}$ in the integration and using Lemma 2.1(1) we have

$$
\Theta = \int\limits_{\delta}^{\tilde{\delta}} \dot{G}_\delta(s,\theta_{(n+1)\tilde{\delta}}\omega_j)\Big(G_\delta(\tilde{\delta},\theta_{(n+1)\tilde{\delta}}\omega_j)-G_\delta(s,\theta_{(n+1)\tilde{\delta}}\omega_j)\Big)ds.
$$

Integrating by parts, we further obtain

$$
\begin{aligned}
\Theta = &-\Big(G_\delta(\tilde{\delta},\theta_{(n+1)\tilde{\delta}}\omega_j)-G_\delta(\delta,\theta_{(n+1)\tilde{\delta}}\omega_j)\Big)G_\delta(\delta,\theta_{(n+1)\tilde{\delta}}\omega_j) \\
&+\frac{1}{2}\Big(G_\delta(\tilde{\delta},\theta_{(n+1)\tilde{\delta}}\omega_j)^2-G_\delta(\delta,\theta_{(n+1)\tilde{\delta}}\omega_j)^2\Big).
\end{aligned}
$$

Thus, using the θ_t-invariance of $\mathbb{P}$ and Lemma 2.2, we have

$$
\begin{aligned}
E\Theta = \; & -E\{[G_\delta(\tilde{\delta},\omega_j)-G_\delta(\delta,\omega_j)]G_\delta(\delta,\omega_j)\}+\frac{1}{2}[EG_\delta(\tilde{\delta},\omega_j)^2-EG_\delta(\delta,\omega_j)^2] \\
&= -\frac{1}{6}\delta+\frac{1}{2}(\tilde{\delta}-\delta).
\end{aligned}
$$

Hence, we have that $E(R_{n+1}|\mathcal{K}_n) = R_n$ for $\beta = j$. So, R_n is a $\mathcal{K}_n$-martingale. Consequently, for each $\beta = 1, \cdots, l$, by the martingale inequality we have

$$
\begin{aligned}
&E\left[\sup_{t\in[0,T]} |\Upsilon^{j,\alpha}_{232}(\tilde{\delta}, \lfloor t\rfloor(\tilde{\delta}))|^2\right] \\
&\le K E\left[|R_{m(T)-1}(\omega)|^2\right] \\
&= K\sum_{k=1}^{m(T)-1} E\Bigg\{(\sigma^{\alpha\beta}\partial_\alpha\sigma^{ij})^2(u_\delta(k\tilde{\delta},\omega))\Bigg\{\int_{k\tilde{\delta}+\delta}^{(k+1)\tilde{\delta}}\Bigg[\dot{G}_\delta(s,\omega_\beta) \\
&\quad\times\Big(G_\delta((k+1)\tilde{\delta},\omega_j) - G_\delta(s,\omega_j)\Big) + \Big(\frac{1}{6}\delta(\tilde{\delta}-\delta)^{-1} - \frac{1}{2}\Big)\delta_{\beta j}\Bigg]ds\Bigg\}^2\Bigg\} \\
&\le K(m(T)-1)E\Bigg\{\int_0^{\tilde{\delta}-\delta}\Bigg[\dot{G}_\delta(s,\omega_\beta)\times\Big(G_\delta(\tilde{\delta}-\delta,\omega_j) - G_\delta(s,\omega_j)\Big)\Bigg]ds \\
&\quad+\Big(\frac{1}{6}\delta - \frac{1}{2}(\tilde{\delta}-\delta)\Big)\delta_{\beta j}\Bigg\}^2.
\end{aligned}
$$

Here the first equality follows from the fact that for each $1 \le k_1 < k_2 \le m(T) - 1$,

$$
\begin{aligned}
&E\Bigg[(\sigma^{\alpha\beta}\partial_\alpha\sigma^{ij})(u_\delta(k_1\tilde{\delta},\omega))\Gamma^{\beta j}_\delta(k_1) \times (\sigma^{\alpha\beta}\partial_\alpha\sigma^{ij})(u_\delta(k_2\tilde{\delta},\omega))\Gamma^{\beta j}_\delta(k_2)\Bigg] \\
&= E\Bigg[(\sigma^{\alpha\beta}\partial_\alpha\sigma^{ij})(u_\delta(k_1\tilde{\delta},\omega))\Gamma^{\beta j}_\delta(k_1) \times (\sigma^{\alpha\beta}\partial_\alpha\sigma^{ij})(u_\delta(k_2\tilde{\delta},\omega))\Bigg] \times E[\Gamma^{\beta j}_\delta(k_2)] = 0
\end{aligned}
$$

where

$$
\Gamma^{\beta j}_\delta(k) := \int_{k\tilde{\delta}+\delta}^{(k+1)\tilde{\delta}} \Bigg\{\dot{G}_\delta(s,\omega_\beta)\Big(G_\delta((k+1)\tilde{\delta},\omega_j) - G_\delta(s,\omega_j)\Big) + [\frac{1}{6}\delta(\tilde{\delta}-\delta)^{-1} - \frac{1}{2}]\delta_{\beta j}\Bigg\}ds,
$$

for $1 \le k \le m(T) - 1$. Hence,

$$
\begin{aligned}
E\left[\sup_{t\in[0,T]} |\Upsilon^{j,\alpha}_{232}(\tilde{\delta}, \lfloor t\rfloor(\tilde{\delta}))|^2\right] \le\ & Km(T)E\left[\left(\int_0^{\tilde{\delta}-\delta}|\dot{G}_\delta(s,\omega_\beta)|ds\right)^2\left(\int_0^{\tilde{\delta}-\delta}|\dot{G}_\delta(s,\omega_j)|ds\right)^2\right] \\
& + Km(T)\delta^2 + Km(T)\tilde{\delta}^2 \\
\le\ & Km(T)n(\delta)^4\delta^2 + Km(T)\delta^2 + Km(T)\tilde{\delta}^2 \\
\le\ & K(n(\delta)^3\delta + n(\delta)^{-1}\delta + \tilde{\delta}) \to 0 \text{ as } \delta \to 0^+.
\end{aligned}
$$

For $\Upsilon^{j,\alpha}_{233}(\tilde{\delta}, \lfloor t\rfloor(\tilde{\delta}))$, we obtain

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$$E\left[\sup_{t\in[0,T]} |\Upsilon_{233}^{j,\alpha}(\tilde{\delta}, \lfloor t\rfloor(\tilde{\delta}))|^2\right] \le K\delta^2 m(T)^2 \le Kn(\delta)^{-2} \to 0 \text{ as } \delta \to 0^+.$$

Summarizing above gives (2.18). □

Combining estimates (2.14)–(2.18), we have

Lemma 2.7. *As* $\delta \to 0^+$,

$$E\left[\sup_{t\in[0,s_1]} |\Pi(\tilde{\delta}, \lfloor t\rfloor(\tilde{\delta}))|^2\right] \le K\int_0^{s_1} E|u_\delta(s,\omega) - u(s,\omega)|^2 ds + o(1).$$

Proof of Theorem 2.1. Using Lemma 2.3, Lemma 2.4, Lemma 2.5, and Lemma 2.7, we have that as $\delta \to 0^+$,

$$E\left[\sup_{t\in[0,s_1]} |u_\delta(t,\omega) - u(t,\omega)|^2\right] \le K\int_0^{s_1} E|u_\delta(s,\omega) - u(s,\omega)|^2 ds + o(1).$$

Then, Theorem 2.1 follows from Gronwall's inequality. □

As a consequence of this theorem, the distribution of $u_\delta(\cdot,\omega)$ converges to the distribution of $u(\cdot,\omega)$ in C_T as $\delta \to 0^+$.

3. Wong–Zakai approximations and center manifolds

In this section, we consider the approximations of center manifolds of stochastic differential equation

$$du = (Au + f(u))dt + u \circ dW, \tag{3.1}$$

where A is $n\times n$ matrix with zero real parts of eigenvalues, f is globally Lipschitz continuous with $f(0) = 0$, and $\mathcal{G}_\delta(\theta_t\omega)$ and $W(t,\omega)$ are defined in section 1 for $l = 1$. We prove that the center manifold of the Wong–Zakai approximation

$$\dot{u}_\delta = Au_\delta + f(u_\delta) + u_\delta \mathcal{G}_\delta(\theta_t\omega) \tag{3.2}$$

converges to the center manifold of equation (3.1).

Before we prove our main result in this section, we first introduce several basic lemmas. The first lemma shows that $\int_0^t \mathcal{G}_\delta(\theta_s\omega)ds$ is an approximation of $W(t,\omega)$.

Lemma 3.1. *For each* $T_1, T_2 \in \mathbb{R}$ *and* $T_1 < T_2$, *and each continuous path* $\omega(t)$, *we have*

$$\lim_{\delta\to 0^+} \sup_{t\in[T_1,T_2]} \left|\int_{T_1}^t \mathcal{G}_\delta(\theta_s\omega)ds - \omega(t) + \omega(T_1)\right| = 0. \tag{3.3}$$

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Proof. First, for each $T_1, T_2 \in \mathbb{R}$ and $T_1 < T_2$, we note that

$$\int_{T_1}^{t} \mathcal{G}_\delta(\theta_s\omega)ds = \left(-\int_{T_1}^{T_1+\delta} + \int_{t}^{t+\delta} \right) \frac{\omega(s)}{\delta} ds, \qquad \forall t \in [T_1, T_2].$$

Then, we have

$$\left| \int_{T_1}^{t} \mathcal{G}_\delta(\theta_s\omega)ds - \omega(t) + \omega(T_1) \right| \leq \left| \int_{T_1}^{T_1+\delta} \frac{\omega(T_1) - \omega(s)}{\delta} ds \right| + \left| \int_{t}^{t+\delta} \frac{\omega(s) - \omega(t)}{\delta} ds \right|.$$

Thus, by using that ω is uniformly continuous on $[T_1, T_2 + \delta]$, we complete the proof of Lemma 3.1. □

Next, we consider a linear stochastic differential equation:

$$dz = -zdt + dW. \tag{3.4}$$

A solution of this equation is called an Ornstein–Uhlenbeck process. It follows from [8, Lemma 2.1] that

Lemma 3.2.

(1) *There exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant set $\Omega_1 \in \mathcal{B}(C_0(\mathbb{R}, \mathbb{R}))$ of full measure with sublinear growth:*

$$\lim_{t\to\pm\infty} \frac{|\omega(t)|}{|t|} = 0 \ \forall\, \omega \in \Omega_1.$$

(2) *For $\omega \in \Omega_1$ the random variable*

$$z(\omega) = -\int_{-\infty}^{0} e^r \omega(r)dr$$

exists and generates a unique stationary solution of (3.4) given by

$$\Omega_1 \times \mathbb{R} \ni (\omega, t) \to z(\theta_t\omega) = -\int_{-\infty}^{0} e^r \theta_t\omega(r)dr = -\int_{-\infty}^{0} e^r \omega(r+t)dr + \omega(t).$$

The mapping $t \to z(\theta_t\omega)$ is continuous.

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(3) *In particular, on* Ω_1 *we have*

$$\lim_{t\to\pm\infty}\frac{|z(\theta_t\omega)|}{|t|}=0,$$

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t z(\theta_r\omega)dr=0.$$

Note that from the arguments of [8, Lemma 2.1], one can have $\Omega_1\subset\bar{\Omega}$, where $\bar{\Omega}$ is given in section 2.

For each $\delta>0$, replace the white noise in equation (3.4) with the Gaussian one $\mathcal{G}_\delta(\theta_t\omega)$. We get that the following random differential equation

$$\dot{z}_\delta=-z_\delta+\mathcal{G}_\delta(\theta_t\omega). \tag{3.5}$$

Let $z(t,\omega,x)$ and $z_\delta(t,\omega,x)$ denote the solutions of equations (3.4) and (3.5) through the point x at time $t=0$, respectively.

The following lemma shows that $z_\delta(t,\omega,x)$ converges to $z(t,\omega,x)$.

Lemma 3.3. *Let* $z(t,\omega,x)$ *and* $z_\delta(t,\omega,x)$ *be given as above. Then for any* $T_1<T_2$*, we have*

$$\lim_{\delta\to0^+}\|z_\delta(\cdot,\omega,x)-z(\cdot,\omega,x)\|_{C([T_1,T_2])}=0.$$

Here $C([T_1,T_2])$ is the usual space of continuous functions defined on $[T_1,T_2]$.

Proof. We first chose a positive constant T such that $[T_1,T_2]\subset[-T,T]$. Note that $z(t,\omega,x)$ and $z_\delta(t,\omega,x)$ satisfy the following, respectively

$$z(t,\omega,x)=x-\int_0^t z(s,\omega,x)ds+\omega(t),$$

$$z_\delta(t,\omega,x)=x-\int_0^t z_\delta(s,\omega,x)ds+\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr,$$

which yield

$$|z_\delta(t,\omega,x)-z(t,\omega,x)|\le|\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr-\omega(t)|+\left|\int_0^t|z_\delta(s,\omega,x)-z(s,\omega,x)|ds\right|.$$

By Gronwall's inequality we get

$$|z_\delta(t,\omega,x) - z(t,\omega,x)| \le \sup_{t\in[0,T]} |\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr - \omega(t)|e^T \quad \forall t \in [0,T],$$

$$|z_\delta(t,\omega,x) - z(t,\omega,x)| \le \sup_{t\in[-T,0]} |\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr - \omega(t)|e^T \quad \forall t \in [-T,0].$$

Then, using Lemma 3.1, we have completed the proof of Lemma 3.3. □

Comparing with equation (3.4), equation (3.5) does not involve Itô differential. It can be interpreted as a deterministic equation with random parameters. The next lemma gives a stationary solution of equation (3.5) and shows that the stationary solution of equation (3.5) is pathwise convergence to that of equation (3.4).

Lemma 3.4. *Let $z(\theta_t\omega)$, Ω_1, T_1 and T_2 be given as above. Then the following results hold:*

(1) *For each $\delta > 0$, on $\omega \in \Omega_1$ the random variable*

$$z_\delta(\omega) = \int_{-\infty}^0 e^r \mathcal{G}_\delta(\theta_r\omega)dr$$

exists and generates a stationary solution of (3.5) given by

$$\Omega_1 \times \mathbb{R} \ni (\omega,t) \to z_\delta(\theta_t\omega) = \int_{-\infty}^0 e^r \mathcal{G}_\delta(\theta_{r+t}\omega)dr.$$

The mapping $t \to z_\delta(\theta_t\omega)$ is continuous.

(2) *In particular, for each $\delta > 0$, on Ω_1 we have*

$$\lim_{t\to\pm\infty} \frac{|z_\delta(\theta_t\omega)|}{|t|} = 0, \qquad \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z_\delta(\theta_r\omega)\,dr = 0,$$

uniformly with respect to $\delta \in (0,1]$.

(3) *In addition, for $\omega \in \Omega_1$,*

$$\lim_{\delta\to 0^+} \|z_\delta(\theta.\omega) - z(\theta.\omega)\|_{C([T_1,T_2])} = 0.$$

Proof. We first let $\omega \in \Omega_1$ be fixed. To show (1) holds, we recall (2.4), i.e.,

$$|\mathcal{G}_\delta(\theta_t\omega)| \le K_\delta C_\omega(|t|+1).$$

Thus,

$$z_\delta(\omega) = \int_{-\infty}^{0} e^r \mathcal{G}_\delta(\theta_r\omega)dr$$

is well-defined. We note that $z_\delta(\theta_t\omega)$ can be written as

$$z_\delta(\theta_t\omega) = \int_{-\infty}^{t} e^{r-t} \mathcal{G}_\delta(\theta_r\omega)dr.$$

Then, $z_\delta(\theta_t\omega)$ satisfies equation (3.5) and is a stationary process.

(2) We write $z_\delta(\theta_t\omega)$ as

$$z_\delta(\theta_t\omega) = \frac{e^{-\delta}-1}{\delta} \int_{-\infty}^{t} e^{r-t}\omega(r)dr + \int_{t}^{t+\delta} e^{r-t-\delta}\frac{\omega(r)}{\delta}dr. \tag{3.6}$$

By Lemma 3.2 (3), we have

$$\lim_{t\to\pm\infty} \frac{1}{t} \int_{-\infty}^{t} e^{r-t}\omega(r)dr = 0. \tag{3.7}$$

Note that

$$\int_{t}^{t+\delta} e^{r-t-\delta}\frac{\omega(r)}{\delta}dr = \int_{0}^{\delta} e^{r-\delta}\frac{\omega(r+t)}{\delta}dr = e^{\delta^*-\delta}\omega(\delta^*+t),$$

where δ^* is in between 0 and δ, which together with Lemma 3.2(1) implies that

$$\lim_{t\to\pm\infty} \frac{1}{t} \left| \int_{t}^{t+\delta} e^{r-t-\delta}\frac{\omega(r)}{\delta}dr \right| = 0. \tag{3.8}$$

Combing (3.7) and (3.8), for each $\delta > 0$ we have

$$\lim_{t\to\pm\infty} \frac{|z_\delta(\theta_t\omega)|}{t} = 0$$

uniformly with respect to $\delta \in (0, 1]$.

By using (3.6) and the integration by parts, we have

$$\int_{0}^{t} z_\delta(\theta_r\omega)dr = \frac{e^{-\delta}-1}{\delta}\left(-\int_{-\infty}^{t} e^{r-t}\omega(r)dr + \int_{-\infty}^{0} e^{r}\omega(r)dr + \int_{0}^{t}\omega(r)dr \right) + I_3, \tag{3.9}$$

where

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$$I_3 = -e^{-t}\int_t^{t+\delta} e^{r-\delta}\frac{\omega(r)}{\delta}dr + \int_0^{\delta} e^{r-\delta}\frac{\omega(r)}{\delta}dr + \int_{\delta}^{t+\delta}\frac{\omega(r)}{\delta}dr - e^{-\delta}\int_0^t \frac{\omega(r)}{\delta}dr.$$

We rewrite I_3 as

$$I_3 = \int_t^{t+\delta}(1-e^{r-t-\delta})\frac{\omega(r)}{\delta}dr + \int_0^{\delta}(e^{r-\delta}-1)\frac{\omega(r)}{\delta}dr + \frac{1-e^{-\delta}}{\delta}\int_0^t \omega(r)dr. \tag{3.10}$$

For each $1 \le j \le l$, observe that

$$\begin{aligned}\int_t^{t+\delta}(1-e^{r-t-\delta})\frac{\omega(r)}{\delta}dr + \int_0^{\delta}(e^{r-\delta}-1)\frac{\omega(r)}{\delta}dr &= \int_0^{\delta}(1-e^{r-\delta})\frac{\omega(r+t)-\omega(r)}{\delta}dr \\ &= (1-e^{\delta^{**}-\delta})[\omega(\delta^{**}+t)-\omega(\delta^{**})],\end{aligned} \tag{3.11}$$

where δ^{**} is in between 0 and δ. Combing (3.9), (3.10) and (3.11), we have

$$\begin{aligned}\int_0^t z_\delta(\theta_r\omega)\,dr = \frac{e^{-\delta}-1}{\delta}&\left(-\int_{-\infty}^t e^{r-t}\omega(r)dr + \int_{-\infty}^0 e^r\omega(r)dr\right) \\ &+\left(1-e^{\delta^{**}-\delta}\right)\left(\omega(\delta^{**}+t)-\omega(\delta^{**})\right).\end{aligned}$$

Then, using (3.7) and Lemma 3.2 (1), we have

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t z_\delta(\theta_r\omega)dr = 0$$

uniformly with respect to $\delta \in (0, 1]$.

(3) By the integral transformations we rewrite $z_\delta(\omega)$ as

$$z_\delta(\omega) = \frac{1}{\delta}\left[\int_{-\infty}^{\delta} e^{r-\delta}\omega(r)dr - \int_{-\infty}^0 e^r\omega(r)dr\right].$$

Applying L'Hospital's rule, we get that

$$\lim_{\delta\to 0^+} z_\delta(\omega) = z(\omega). \tag{3.12}$$

In the end, we observe that $z(\theta_t\omega)$ and $z_\delta(\theta_t\omega)$ satisfy respectively

$$z(\theta_t\omega) = z(\omega) - \int_0^t z(\theta_s\omega)ds + \omega(t),$$

$$z_\delta(\theta_t\omega) = z_\delta(\omega) - \int_0^t z_\delta(\theta_s\omega)ds + \int_0^t \mathcal{G}_\delta(\theta_r\omega)dr,$$

which along with Gronwall's inequality implies that

$$|z_\delta(\theta_t\omega) - z(\theta_t\omega)| \leq \left[|z_\delta(\omega) - z(\omega)| + \sup_{t\in[0,T]} |\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr - \omega(t)| \right] e^T \text{ for all } t \in [0, T],$$

$$|z_\delta(\theta_t\omega) - z(\theta_t\omega)| \leq \left[|z_\delta(\omega) - z(\omega)| + \sup_{t\in[-T,0]} |\int_0^t \mathcal{G}_\delta(\theta_r\omega)dr - \omega(t)| \right] e^T \text{ for all } t \in [-T, 0],$$

where T is a positive constant such that $[T_1, T_2] \subset [-T, T]$. It follows from (3.12) and (3.3) that

$$\lim_{\delta\to 0^+} \|z_\delta(\theta.\omega) - z(\theta.\omega)\|_{C([T_1,T_2])} = 0.$$

This completes the proof of this Lemma. □

For the remainder of this section, we restrict θ_t on this invariant set Ω_1 of full measure instead of Ω and work with the corresponding probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P})$, but we still denote it by $(\Omega, \mathcal{F}, \mathbb{P})$.

We now are ready to study the center manifolds of equation (3.1) and Wong–Zakai approximated equation (3.2). We first show that the solution of (3.1) defines a random dynamical system. To see this, we consider the random differential equation

$$\frac{dv}{dt} = Av + z(\theta_t\omega)v + F(\theta_t\omega, v), \tag{3.13}$$

where $F(\omega, v) = e^{-z(\omega)} f(e^{z(\omega)}v)$. Note that for each $\omega \in \Omega$, the function F has the same global Lipschitz constant L as f. In contrast to the original stochastic differential equation, no stochastic integral appears here. By the usual theorem of existence and uniqueness of solutions, this equation has a unique solution for every $\omega \in \Omega$. No exceptional sets appear. Hence the solution mapping

$$(t, \omega, x) \mapsto v(t, \omega, x)$$

generates a random dynamical system, i.e., v is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^n)$ measurable and forms a cocycle:

$$v(0, \omega, x) = x, \quad \text{for all } \omega \in \Omega,$$

$$v(t + s, \omega, x) = v(t, \theta_s\omega, \cdot) \circ v(s, \omega, x), \quad \text{for all } t, s \in \mathbb{R}, \quad \omega \in \Omega.$$

Accordingly, for Wong–Zakai approximated equation (3.2), we consider

$$\frac{dv_\delta}{dt} = Av_\delta + z_\delta(\theta_t\omega)v_\delta + F_\delta(\theta_t\omega, v_\delta), \tag{3.14}$$

where $F_\delta(\omega, v) = e^{-z_\delta(\omega)} f(e^{z_\delta(\omega)}v)$. The same conclusions as for equation (3.13) hold for equation (3.14), especially, the solution map

$$(t, \omega, x) \mapsto v_\delta(t, \omega, x)$$

generates a random dynamical system.

For each $x \in \mathbb{R}^n$ and $\omega \in \Omega$ we introduce the following random transformations

$$T_\delta(\omega, x) := e^{-z_\delta(\omega)}x \text{ and } T(\omega, x) := e^{-z(\omega)}x.$$

Clearly, for fixed $\omega \in \Omega$ their inverse transformations are

$$T_\delta^{-1}(\omega, x) = e^{z_\delta(\omega)}x \text{ and } T^{-1}(\omega, x) = e^{z(\omega)}x,$$

respectively. For the sake of convenience, we denote $z_0(\omega) = z(\omega)$, $T_0(\omega, x) = T(\omega, x)$, $F_0 = F$ and $v_0(t, \omega, x) = v(t, \omega, x)$.

Proposition 3.1. *Suppose that v_δ is the random dynamical system generated by equation (3.13) (resp. (3.14)). Then*

$$(t, \omega, x) \mapsto T_\delta^{-1}(\theta_t\omega, \cdot) \circ v_\delta(t, \omega, T_\delta(\omega, x)) =: \hat{v}_\delta(t, \omega, x) \tag{3.15}$$

is a random dynamical system. For any $x \in \mathbb{R}^n$ this process is a solution of equation (3.1) (resp. (3.2)) and forms a random dynamical system.

Proof. For $\delta = 0$, applying the Ito formula to $T_0(\theta_t\omega, \hat{v}_0(t, \omega, T_0^{-1}(\omega, x)))$ gives a solution of equation (3.13). The converse is also true, since $T_0^{-1}(\theta_t\omega, x)$ and $v_0(t, \omega, x)$ are defined for *each* $\omega \in \Omega$ and T_0^{-1} is the inverse of T, and thus

$$(t, \omega, x) \mapsto T_0^{-1}(\theta_t\omega, v_0(t, \omega, T_0(\omega, x)))$$

gives a solution of equation (3.1) for each $\omega \in \Omega$. It is easy to check that (3.15) defines a random dynamical system. Since v_0 is measurable with respect to $\mathcal{F}$ so is this $\hat{v}_0$. For $\delta > 0$, since $z_\delta(\theta_t\omega)$ is differentiable, the proof of this lemma is straightforward. □

We write the spectrum $\sigma(A)$ of matrix A as

$$\sigma(A) = \sigma_u \cup \sigma_c \cup \sigma_s,$$

where $\sigma_u := \{\lambda \in \sigma(A) \mid \mathrm{Re}\lambda > 0\}$, $\sigma_c := \{\lambda \in \sigma(A) \mid \mathrm{Re}\lambda = 0\}$ and $\sigma_s := \{\lambda \in \sigma(A) \mid \mathrm{Re}\lambda < 0\}$. By the assumption, $\sigma_c \neq \emptyset$. Let E^u, E^c and E^s denote the generalized eigenspaces corresponding to σ_u, σ_c and σ_s, respectively. Then

$$R^n = E^u \oplus E^c \oplus E^s$$

with corresponding projections $P^u : \mathbb{R}^n \to E^u$, $P^c : \mathbb{R}^n \to E^c$ and $P^s : \mathbb{R}^n \to E^s$. Let

$$0 < \beta < \min\{|\mathrm{Re}\lambda| \;\big|\; \lambda \in \sigma_u \cup \sigma_s\}.$$

It is well-known that for each $0 < \alpha < \beta$, there is a $K \geq 1$ such that

$$\begin{aligned} &|e^{At} P^c| \leq K e^{\alpha|t|}, t \in \mathbb{R}, \\ &|e^{At} P^u| \leq K e^{\beta t}, t \leq 0, \\ &|e^{At} P^s| \leq K e^{-\beta t}, t \geq 0. \end{aligned} \tag{3.16}$$

For the remainder of this section, we will fix such α. For each $\gamma \in (\alpha, \beta)$, we define the Banach spaces

$$C_{\gamma,\delta} := \{\varphi \in C(\mathbb{R}, \mathbb{R}^n) | \sup_{t\in\mathbb{R}} e^{-\gamma|t| - \int_0^t z_\delta(\theta_r\omega)dr} |\varphi(t)| < +\infty\}$$

with the norm

$$|\varphi|_{\gamma,\delta} = \sup_{t\in\mathbb{R}} e^{-\gamma|t| - \int_0^t z_\delta(\theta_r\omega)dr} |\varphi(t)|,$$

where $C_{\gamma,0}$ and $|\cdot|_{\gamma,0}$ are also denoted by C_γ and $|\cdot|_\gamma$. Then for all $\delta \geq 0$ we define

$$M_\delta^c(\omega) := \{x_0 \in \mathbb{R}^n \;\big|\; v_\delta(\cdot, \omega, x_0) \in C_{\gamma,\delta}\}.$$

We will prove that $M_\delta^c(\omega)$ is given by the graph of a Lipschitz function for all small $\delta \geq 0$. To see this, we need the following lemma.

Lemma 3.5. *For each $\gamma \in (\alpha, \beta)$, $x_0 \in M_\delta^c(\omega)$ if and only if there exists a function $v_\delta(\cdot) \in C_{\gamma,\delta}$ with the initial value $v_\delta(0) = x_0$ and satisfies*

$$\begin{aligned} v_\delta(t) = e^{At + \int_0^t z_\delta(\theta_r\omega)dr} \xi &+ \int_0^t e^{A(t-s) + \int_s^t z_\delta(\theta_r\omega)dr} P^c F_\delta(\theta_s\omega, v_\delta(s)) ds \\ &+ \int_{+\infty}^t e^{A(t-s) + \int_s^t z_\delta(\theta_r\omega)dr} P^u F_\delta(\theta_s\omega, v_\delta(s)) ds \\ &+ \int_{-\infty}^t e^{A(t-s) + \int_s^t z_\delta(\theta_r\omega)dr} P^s F_\delta(\theta_s\omega, v_\delta(s)) ds, \end{aligned} \tag{3.17}$$

where $\xi = P^c x_0$.

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Proof. Let $x_0 \in M^c_\delta(\omega)$. By the variation of constants formula, for $\tau, t \in \mathbb{R}$ we have that

$$v_\delta(t,\omega,x_0) = e^{A(t-\tau)+\int_\tau^t z_\delta(\theta_r\omega)dr} v_\delta(\tau,\omega,x_0) + \int_\tau^t e^{A(t-s)+\int_s^t z_\delta(\theta_r\omega)dr} F_\delta(\theta_s\omega, v_\delta(s,\omega,x_0))ds.$$

Taking $\tau = 0$, we find

$$P^c v_\delta(t,\omega,x_0) = e^{At+\int_0^t z_\delta(\theta_r\omega)dr} P^c x_0 + \int_0^t e^{A(t-s)+\int_s^t z_\delta(\theta_r\omega)dr} P^c F_\delta(\theta_s\omega, v_\delta(s,\omega,x_0))ds. \tag{3.18}$$

Note that

$$\begin{aligned} P^u v_\delta(t,\omega,x_0) =& e^{A(t-\tau)+\int_\tau^t z_\delta(\theta_r\omega)dr} P^u v_\delta(\tau,\omega,x_0) \\ &+ \int_\tau^t e^{A(t-s)+\int_s^t z_\delta(\theta_r\omega)dr} P^u F_\delta(\theta_s\omega, v_\delta(s,\omega,x_0))ds. \end{aligned} \tag{3.19}$$

By (3.16), for $\tau > \max\{t, 0\}$ we observe that

$$\begin{aligned} &|e^{A(t-\tau)+\int_\tau^t z_\delta(\theta_r\omega)dr} P^u v_\delta(\tau,\omega,x_0)| \\ &\quad \le K e^{\beta t+\int_0^t z_\delta(\theta_r\omega)dr} e^{(\gamma-\beta)\tau} |v_\delta(\cdot,\omega,x_0)|_{\gamma,\delta} \to 0 \text{ as } \tau \to +\infty. \end{aligned}$$

Taking the limit $\tau \to +\infty$ in (3.19), we obtain

$$P^u v_\delta(t,\omega,x_0) = \int_{+\infty}^t e^{A(t-s)+\int_s^t z_\delta(\theta_r\omega)dr} P^u F_\delta(\theta_s\omega, v_\delta(s,\omega,x_0))ds. \tag{3.20}$$

Similarly, with (3.16) we have

$$P^s v_\delta(t,\omega,x_0) = \int_{-\infty}^t e^{A(t-s)+\int_s^t z_\delta(\theta_r\omega)dr} P^s F_\delta(\theta_s\omega, v_\delta(s,\omega,x_0))ds. \tag{3.21}$$

Combining (3.18), (3.20) and (3.21), we get (3.17). The converse follows from a straight-forward computation. This completes the proof. □

The following theorem gives the existence of center manifolds for equations (3.14) and (3.13).

Theorem 3.1. *If*

$$KL\left(\frac{1}{\gamma-\alpha}+\frac{2}{\beta-\gamma}\right)<1, \tag{3.22}$$

then there exists a Lipschiz center manifold for the random differential equation (3.14) as $\delta>0$ (resp. (3.13) as $\delta=0$) which is given by

$$M_\delta^c(\omega)=\{\xi+h_\delta^c(\xi)|\xi\in E^c\},$$

where $h_\delta^c: E^c\to E^u\oplus E^s$ is a Lipschitz continuous mapping and satisfies $h_\delta^c(0)=0$.

Proof. We first prove that equation (3.17) has a unique solution $v^\delta=v^\delta(\cdot,\omega,\xi)$ in $C_{\gamma,\delta}$ which is Lipschitz continuous with $\xi\in E^c$. To see this, we replace $v_\delta(s)$ with $v(s)$ on the right hand side of equation (3.17) and denote it by $\mathcal{J}_\delta^c(v,\omega,\xi)$. Multiplying both sides of (3.17) by $e^{-\gamma|t|-\int_0^t z_\delta(\theta_r\omega)dr}$, it then follows from (3.16) that

$$\begin{aligned}
e^{-\gamma|t|-\int_0^t z_\delta(\theta_r\omega)dr}|\mathcal{J}_\delta^c(v,\omega,\xi)| &\le K|\xi|+KL\Big(\Big|\int_0^t e^{\alpha|t-s|+\gamma|s|-\gamma|t|}ds\Big|+\int_t^{+\infty} e^{\beta(t-s)+\gamma|s|-\gamma|t|}ds\\
&\quad+\int_{-\infty}^t e^{-\beta(t-s)+\gamma|s|-\gamma|t|}ds\Big)|v|_{\gamma,\delta}\\
&\le K|\xi|+KL\left(\frac{1}{\gamma-\alpha}+\frac{2}{\beta-\gamma}\right)|v|_{\gamma,\delta},
\end{aligned}$$

where L is Lipschitz constant of f. This implies that the operator $\mathcal{J}_\delta^c(\cdot,\omega,\xi)$ maps $C_{\gamma,\delta}$ into $C_{\gamma,\delta}$. For each $v,\bar v\in C_{\gamma,\delta}$, we have that

$$\begin{aligned}
|\mathcal{J}_\delta^c(v,\omega,\xi)-\mathcal{J}_\delta^c(\bar v,\omega,\xi)| &\le \Big|\int_0^t |e^{A(t-s)+\int_s^t z_\delta(\theta_r\omega)dr}P^c||F_\delta(\theta_s\omega,v(s))-F_\delta(\theta_s\omega,\bar v(s))|ds\Big|\\
&\quad+\int_t^{+\infty} |e^{A(t-s)+\int_s^t z_\delta(\theta_r\omega)dr}P^u||F_\delta(\theta_s\omega,v(s))-F_\delta(\theta_s\omega,\bar v(s))|ds\\
&\quad+\int_{-\infty}^t |e^{A(t-s)+\int_s^t z_\delta(\theta_r\omega)dr}P^s||F_\delta(\theta_s\omega,v(s))-F_\delta(\theta_s\omega,\bar v(s))|ds,
\end{aligned}$$

which in view of (3.16) yields

$$|\mathcal{J}_\delta^c(v,\omega,\xi)-\mathcal{J}_\delta^c(\bar v,\omega,\xi)|_{\gamma,\delta}\le KL\left(\frac{1}{\gamma-\alpha}+\frac{2}{\beta-\gamma}\right)|v-\bar v|_{\gamma,\delta}.$$

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By (3.22), we have that $\mathcal{J}_\delta^c(\cdot,\omega,\xi)$ is a uniform contraction with respect to the parameter (ω,ξ). Using the contraction mapping principle, $\mathcal{J}_\delta^c(\cdot,\omega,\xi)$ has a unique fixed point $v^\delta(\cdot,\omega,\xi)$ for each $\xi\in E^c$. Clearly, $v^\delta(\cdot,\omega,0)=0$ since $f(0)=0$. Similarly, for all $\xi,\xi_0\in E^c$ we get that

$$
\begin{aligned}
&|v^\delta(t,\omega,\xi)-v^\delta(t,\omega,\xi_0)|e^{-\gamma|t|-\int_0^t z_\delta(\theta_r\omega)dr}\\
&\le K|\xi-\xi_0|+KL\left(\frac{1}{\gamma-\alpha}+\frac{2}{\beta-\gamma}\right)|v^\delta(\cdot,\omega,\xi)-v^\delta(\cdot,\omega,\xi_0)|_{\gamma,\delta}.
\end{aligned}
$$

Hence we have

$$
|v^\delta(\cdot,\omega,\xi)-v^\delta(\cdot,\omega,\xi_0)|_{\gamma,\delta}\le\frac{K}{1-KL\left(\frac{1}{\gamma-\alpha}+\frac{2}{\beta-\gamma}\right)}|\xi-\xi_0|.
$$

Moreover, since $v^\delta(\cdot,\omega,\xi)$ can be an ω-wise limit of the iteration of contraction mapping $\mathcal{J}_\delta^c$ starting at 0 and mapping a $\mathcal{F}$-measurable function to a measurable function, $v^\delta(\cdot,\omega,\xi)$ is $\mathcal{F}$-measurable. On the other hand, since $v^\delta(\cdot,\omega,\xi)$ is Lipschitz continuous in ξ, by Castaing and Valadier [5, Lemma III.14], $v^\delta(\cdot,\omega,\xi)$ is measurable with respect to (ω,ξ). Put $h_\delta^c(\omega,\xi)=P^uv^\delta(0,\omega,\xi)+P^sv^\delta(0,\omega,\xi)$. Then

$$
\begin{aligned}
h_\delta^c(\omega,\xi)=&\int_{+\infty}^{0}e^{-As+\int_s^0 z_\delta(\theta_r\omega)dr}P^uF_\delta(\theta_s\omega,v^\delta(s,\omega,\xi))ds\\
&+\int_{-\infty}^{0}e^{-As+\int_s^0 z_\delta(\theta_r\omega)dr}P^sF_\delta(\theta_s\omega,v^\delta(s,\omega,\xi))ds
\end{aligned}
$$

and $h_\delta^c(\omega,0)=0$. Taking (3.16), for any $\xi,\xi_0\in E^c$ we have

$$
|h_\delta^c(\omega,\xi)-h_\delta^c(\omega,\xi_0)|\le\frac{2K^2L}{(\beta-\gamma)\left[1-KL\left(\frac{1}{\gamma-\alpha}+\frac{2}{\beta-\gamma}\right)\right]}|\xi-\xi_0|. \tag{3.23}
$$

Using Theorem III.14 in Castaing and Valadier [5] again, we clearly see that h_δ^c is measurable with respect to (ω,ξ). From Lemma 3.5 and the definition of $h_\delta^c(\omega,\xi)$, we find

$$
M_\delta^c=\{\xi+h_\delta^c(\omega,\xi)|\xi\in E^c\}.
$$

Next we prove $M_\delta^c(\omega)$ is a random set, i.e., for any $x\in\mathbb{R}^n$

$$
\omega\mapsto\inf_{y\in\mathbb{R}^n}|x-\left(P^cy+h_\delta^c(\omega,P^cy)\right)| \tag{3.24}
$$

is measurable. In fact, the right hand side of (3.24) is equal to

$$
\omega\mapsto\inf_{y\in\mathbb{Q}^n}|x-\left(P^cy+h_\delta^c(\omega,P^cy)\right)|
$$

which follows immediately by the continuity of $h^c_\delta(\omega, \cdot)$. The measurability of any expression under the infimum of (3.24) follows from that $\omega \to h^c_\delta(\omega, P^c y)$ is measurable for any $y \in \mathbb{R}^n$.

In the end, we claim that $M^c_\delta(\omega)$ is invariant, i.e., for all $s \in \mathbb{R}$

$$v_\delta(s, \omega, M^c_\delta(\omega)) = M^c_\delta(\theta_s\omega). \tag{3.25}$$

We first note that for each fixed $s \in \mathbb{R}$ and $x_0 \in M^c_\delta(\omega)$, $v_\delta(t + s, \omega, x_0)$ is a solution of

$$\dot{v}_\delta = Av_\delta + z(\theta_t(\theta_s\omega))v_\delta + F_\delta(\theta_t(\theta_s\omega), v_\delta),\ v_\delta(0) = v_\delta(s, \omega, x_0).$$

Thus $v_\delta(t, \theta_s\omega, v_\delta(s, \omega, x_0)) = v_\delta(t + s, \omega, x_0)$. Since

$$v_\delta(\cdot, \omega, x_0) \in C_{\gamma,\delta}, \quad v_\delta(\cdot, \theta_s\omega, v_\delta(s, \omega, x_0)) \in C_{\gamma,\delta}.$$

Therefore, $v_\delta(s, \omega, x_0) \in M^c_\delta(\theta_s\omega)$, which implies that

$$v_\delta(s, \omega, M^c_\delta(\omega)) \subset M^c_\delta(\theta_s\omega). \tag{3.26}$$

Since $v_\delta(s, \omega) := v_\delta(s, \omega, \cdot)$ is a cocycle of homeomorphisms of $\mathbb{R}^n$, by (3.26) we have

$$M^c_\delta(\omega) \subset v_\delta(s, \omega)^{-1} M^c_\delta(\theta_s\omega) = v_\delta(-s, \theta_s\omega) M^c_\delta(\theta_s\omega) \subset M^c_\delta(\omega).$$

Thus (3.25) holds. Then we complete the proof of Theorem 3.1. □

Theorem 3.2. *$\tilde{M}^c_\delta(\omega) = T^{-1}_\delta(\omega, M^c_\delta(\omega))$ is a Lipschiz center manifold of equation (3.2) as $\delta > 0$ (resp. (3.1) as $\delta = 0$).*

Proof. By using Proposition 3.1 and (3.25), we have for $\delta \geq 0$

$$\begin{aligned} u_\delta(t, \omega, \tilde{M}^c_\delta(\omega)) &= T^{-1}_\delta(\theta_t\omega, v_\delta(t, \omega, T_\delta(\omega, \tilde{M}^c_\delta(\omega)))) = T^{-1}_\delta(\theta_t\omega, v_\delta(t, \omega, M^c_\delta(\omega))) \\ &= T^{-1}_\delta(\theta_t\omega, M^c_\delta(\theta_t\omega)) = \tilde{M}^c_\delta(\theta_t\omega), \end{aligned}$$

which implies that $\tilde{M}^c_\delta(\omega)$ is an invariant set. Moreover

$$\begin{aligned} \tilde{M}^c_\delta(\omega) =& T^{-1}_\delta(\omega, M^c_\delta(\omega)) = \{T^{-1}_\delta(\omega, \xi + h^c_\delta(\omega, \xi)) | \xi \in E^c\} = \{e^{z_\delta(\omega)}(\xi + h^c_\delta(\omega, \xi)) | \xi \in E^c\} \\ =& \{\xi + e^{z_\delta(\omega)} h^c_\delta(\omega, e^{-z_\delta(\omega)}\xi)) | \xi \in E^c\}. \end{aligned}$$

This implies $\tilde{M}^c_\delta(\omega)$ is a Lipschitz center manifold. The proof is complete. □

Finally, we give our main result in this section.

Theorem 3.3. *Set $h^c := h^c_0$. Under the assumption of Theorem 3.1, for any $(\omega, \xi) \in \Omega \times E^c$*

$$\lim_{\delta \to 0^+} e^{z_\delta(\omega)} h^c_\delta(\omega, e^{-z_\delta(\omega)}\xi) = e^{z(\omega)} h^c(\omega, e^{-z(\omega)}\xi),$$

i.e., the Lipschitz center manifold of equation (3.2) converges pathwise to that of equation (3.1).

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Proof. We first prove that

$$\lim_{\delta\to 0^+} |v^\delta(\cdot,\omega,\xi) - v(\cdot,\omega,\xi)|_\gamma = 0, \tag{3.27}$$

where $v(\cdot,\omega,\xi) := v^0(\cdot,\omega,\xi)$.

We choose ζ such that $\alpha < \gamma - 4\zeta < \gamma < \beta$ and

$$KL\left(\frac{1}{\gamma - j\zeta - \alpha} + \frac{2}{\beta - \gamma + j\zeta}\right) < 1 \quad \text{for all } j = 1, \ldots, 4.$$

Following the proof of Theorem 3.1, the above conditions imply that $\mathcal{J}^\delta$ is a contraction from $C_{(\gamma - j\zeta),\delta}$ to itself with the contraction constant

$$KL\left(\frac{1}{\gamma - j\zeta - \alpha} + \frac{2}{\beta - \gamma + j\zeta}\right)$$

for $j = 1, 2, 3, 4$. This yields that $v^\delta \in C_{\gamma - j\zeta,\delta}$ and $v \in C_{\gamma - j\zeta}$. We note that the following continuous embedding:

$$C_{\gamma-4\zeta} \subset C_{\gamma-3\zeta,\delta} \subset C_{\gamma-2\zeta} \subset C_{\gamma-\zeta,\delta} \subset C_\gamma,$$
$$C_{\gamma-4\zeta,\delta} \subset C_{\gamma-3\zeta} \subset C_{\gamma-2\zeta,\delta} \subset C_{\gamma-\zeta} \subset C_\gamma.$$

Using Lemmas 3.2 and 3.4, there exists $T_1 > 0$ such that

$$\left|\int_0^t \Big(z_\delta(\theta_r\omega) - z(\theta_r\omega)\Big)\, dr\right| < \zeta|t| \quad \text{and} \quad |z_\delta(\theta_t\omega) - z(\theta_t\omega)| < \zeta|t| \tag{3.28}$$

for any $|t| \ge T_1$, $\delta \in (0, 1]$. Applying Lemma 3.4(3), there exists $\delta_0 \in (0, 1]$ such that

$$\|z_\delta(\theta.\omega) - z(\theta.\omega)\|_{C([-T_1,T_1])} < \zeta \quad \text{for } 0 < \delta < \delta_0, \tag{3.29}$$

which yields that for $|t| \le T_1$ and $0 < \delta < \delta_0$, we have

$$\left|\int_0^t \Big(z_\delta(\theta_r\omega) - z(\theta_r\omega)\Big)\, dr\right| \le \|z_\delta(\theta.\omega) - z(\theta.\omega)\|_{C([-T_1,T_1])}|t| < \zeta|t|. \tag{3.30}$$

From (3.28)–(3.30), we obtain

$$\left|\int_0^t z_\delta(\theta_r\omega) - z(\theta_r\omega) dr\right| < \zeta|t|,\ |z_\delta(\theta_t\omega) - z(\theta_t\omega)| < \zeta(|t| + 1) \tag{3.31}$$

for all $0 < \delta < \delta_0$ and $t \in \mathbb{R}$.

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Let $u^\delta(t,\omega,\xi) = v^\delta(t,\omega,\xi) - v(t,\omega,\xi)$. For simplicity, we also denote $u^\delta(t) = u^\delta(t,\omega,\xi)$ and $v(t) = v(t,\omega,\xi)$. Similarly to Lemma 3.5, we see for all $t \in \mathbb{R}$, $u^\delta(t)$ satisfies the following integral equation

$$
\begin{aligned}
u^\delta(t) = & \int_0^t e^{A(t-s)+\int_s^t z_\delta(\theta_r\omega)dr} P^c \\
& \times \Big(F_\delta(\theta_s\omega, u^\delta(s)+v(s)) - F(\theta_s\omega, v(s)) + \big(z_\delta(\theta_s\omega) - z(\theta_s\omega)\big)v(s)\Big)dr \\
& + \int_{+\infty}^t e^{A(t-s)+\int_s^t z_\delta(\theta_r\omega)dr} P^u \\
& \times \Big(F_\delta(\theta_s\omega, u^\delta(s)+v(s)) - F(\theta_s\omega, v(s)) + \big(z_\delta(\theta_s\omega) - z(\theta_s\omega)\big)v(s)\Big)dr \\
& + \int_{-\infty}^t e^{A(t-s)+\int_s^t z_\delta(\theta_r\omega)dr} P^s \\
& \times \Big(F_\delta(\theta_s\omega, u^\delta(s)+v(s)) - F(\theta_s\omega, v(s)) + \big(z_\delta(\theta_s\omega) - z(\theta_s\omega)\big)v(s)\Big)dr.
\end{aligned}
$$

We denote above the first, the second, and the third integrals by $I_{\delta,1}$, $I_{\delta,2}$, and $I_{\delta,3}$, respectively. To estimate these integrals, we first estimate

$$
\begin{aligned}
& |F_\delta(\theta_s\omega, u^\delta(s)+v(s)) - F(\theta_s\omega, v(s))| \\
& \le Le^{(\gamma-\zeta)|s|+\int_0^s z_\delta(\theta_r\omega)dr}\Big(|u^\delta(\cdot)|_{\gamma-\zeta,\delta} + 2e^{-2\zeta|s|}|e^{z_\delta(\theta_s\omega)-z(\theta_s\omega)} - 1||v(\cdot)|_{\gamma-3\zeta,\delta}\Big).
\end{aligned}
\tag{3.32}
$$

We first estimate $I_{\delta,1}$. Using (3.32), we have

$$
e^{-(\gamma-\zeta)|t|-\int_0^t z_\delta(\theta_r\omega)dr}|I_{\delta,1}| \le KLe^{-(\gamma-\zeta)|t|}\left|\int_0^t e^{\alpha|t-s|+(\gamma-\zeta)|s|}H_\delta(s)ds\right|,
$$

where

$$
\begin{aligned}
H_\delta(s) := & |u^\delta(\cdot)|_{\gamma-\zeta,\delta} + 2e^{-2\zeta|s|}|e^{z_\delta(\theta_s\omega)-z(\theta_s\omega)} - 1||v(\cdot)|_{\gamma-3\zeta,\delta} \\
& + e^{-2\zeta|s|}|z_\delta(\theta_s\omega) - z(\theta_s\omega)||v(\cdot)|_{\gamma-3\zeta,\delta}.
\end{aligned}
$$

We claim that

$$
|I_{\delta,1}|_{\gamma-\zeta,\delta} \le \frac{KL}{\gamma-\zeta-\alpha}|u^\delta(\cdot)|_{\gamma-\zeta,\delta} + o(1), \quad \text{as } \delta \to 0^+. \tag{3.33}
$$

Since $H_\delta(s)$ has three terms, we estimate the corresponding three integrals. The first one is

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$$\sup_{t\in\mathbb{R}}\left\{e^{-(\gamma-\zeta)|t|}\left|\int_0^t KLe^{\alpha|t-s|+(\gamma-\zeta)|s|}|u^\delta(\cdot)|_{\gamma-\zeta,\delta}ds\right|\right\}\le\frac{KL}{\gamma-\zeta-\alpha}|u^\delta(\cdot)|_{\gamma-\zeta,\delta}. \quad (3.34)$$

Next, we show

$$\sup_{t\in\mathbb{R}}\left\{e^{-(\gamma-\zeta)|t|}\left|\int_0^t 2KLe^{\alpha|t-s|+(\gamma-3\zeta)|s|}|e^{z_\delta(\theta_s\omega)-z(\theta_s\omega)}-1||v(\cdot)|_{\gamma-3\zeta,\delta}ds\right|\right\}=o(1). \quad (3.35)$$

For any $\varepsilon>0$, let $T_2>T_1$ be large enough so that

$$\frac{4KLe^\zeta|v(\cdot)|_{\gamma-4\zeta}}{\gamma-2\zeta-\alpha}e^{-\zeta T_2}<\varepsilon.$$

For $0<\delta<\delta_0$ and $|t|\ge T_2$, by (3.31), we have

$$\begin{aligned}
&2KLe^{-(\gamma-\zeta)|t|}\left|\int_0^t e^{\alpha|t-s|+(\gamma-3\zeta)|s|}|e^{z_\delta(\theta_s\omega)-z(\theta_s\omega)}-1||v(\cdot)|_{\gamma-3\zeta,\delta}ds\right|\\
&\le 2KL|v(\cdot)|_{\gamma-3\zeta,\delta}e^{-(\gamma-\zeta)|t|}\left|\int_0^t e^{\alpha|t-s|+(\gamma-3\zeta)|s|}|e^{(1+|s|)\zeta}-1|ds\right|\\
&\le 4KLe^\zeta|v(\cdot)|_{\gamma-3\zeta,\delta}e^{-(\gamma-\zeta)|t|}\left|\int_0^t e^{\alpha|t-s|+(\gamma-2\zeta)|s|}ds\right|\\
&\le\frac{4KLe^\zeta|v(\cdot)|_{\gamma-3\zeta,\delta}}{\gamma-2\zeta-\alpha}e^{-\zeta T_2}\\
&\le\frac{4KLe^\zeta|v(\cdot)|_{\gamma-4\zeta}}{\gamma-2\zeta-\alpha}e^{-\zeta T_2}<\varepsilon.
\end{aligned}$$

For $|t|\le T_2$,

$$\begin{aligned}
&2KLe^{-(\gamma-\zeta)|t|}\left|\int_0^t e^{\alpha|t-s|+(\gamma-3\zeta)|s|}|e^{z_\delta(\theta_s\omega)-z(\theta_s\omega)}-1||v(\cdot)|_{\gamma-3\zeta,\delta}ds\right|\\
&\le 2KL(e^{\|z_\delta(\theta.\omega)-z(\theta.\omega)\|_{C([-T_2,T_2])}}-1)|v(\cdot)|_{\gamma-3\zeta,\delta}\times\frac{1}{\gamma-\zeta-\alpha}\\
&\le 2KL(e^{\|z_\delta(\theta.\omega)-z(\theta.\omega)\|_{C([-T_2,T_2])}}-1)|v(\cdot)|_{\gamma-2\zeta}\times\frac{1}{\gamma-\zeta-\alpha}<\varepsilon
\end{aligned}$$

provided that δ is sufficiently small since $\|z_\delta(\theta.\omega)-z(\theta.\omega)\|_{C([-T_2,T_2])}\to 0$ as $\delta\to 0^+$ by Lemma 3.4(3). Hence (3.35) holds.

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Finally, we show that

$$\sup_{t\in\mathbb{R}}\left\{e^{-(\gamma-\zeta)|t|}\left|\int_0^t KLe^{\alpha|t-s|+(\gamma-3\zeta)|s|}|z_\delta(\theta_s\omega)-z(\theta_s\omega)||v(\cdot)|_{\gamma-3\zeta,\delta}ds\right|\right\}=o(1). \tag{3.36}$$

Note that by (3.31), $|z_\delta(\theta_s\omega)-z(\theta_s\omega)|\leq(1+|s|)\zeta$. Let

$$K^*=\sup_{t\in\mathbb{R}}e^{-\zeta|t|}(1+|t|).$$

Choose $T_3>T_2$ large enough such that

$$\frac{KL\zeta K^*|v(\cdot)|_{\gamma-4\zeta}}{\gamma-2\zeta-\alpha}e^{-\zeta T_3}<\varepsilon.$$

Then, for $|t|>T_3$, we have

$$\begin{aligned}&e^{-(\gamma-\zeta)|t|}\left|\int_0^t KLe^{\alpha|t-s|+(\gamma-3\zeta)|s|}|z_\delta(\theta_s\omega)-z(\theta_s\omega)||v(\cdot)|_{\gamma-3\zeta,\delta}ds\right|\\&\leq\frac{KL\zeta K^*|v(\cdot)|_{\gamma-3\zeta,\delta}}{\gamma-2\zeta-\alpha}e^{-\zeta T_3}\\&\leq\frac{KL\zeta K^*|v(\cdot)|_{\gamma-4\zeta}}{\gamma-2\zeta-\alpha}e^{-\zeta T_3}<\varepsilon.\end{aligned}$$

For $|t|\leq T_3$, we have

$$\begin{aligned}&e^{-(\gamma-\zeta)|t|}\left|\int_0^t KLe^{\alpha|t-s|+(\gamma-3\zeta)|s|}|z_\delta(\theta_s\omega)-z(\theta_s\omega)||v(\cdot)|_{\gamma-3\zeta,\delta}ds\right|\\&\leq\frac{KL|v(\cdot)|_{\gamma-3\zeta,\delta}\|z_\delta(\theta.\omega)-z(\theta.\omega)\|_{C([-T_3,T_3])}}{\gamma-\zeta-\alpha}\\&\leq\frac{KL|v(\cdot)|_{\gamma-4\zeta}\|z_\delta(\theta.\omega)-z(\theta.\omega)\|_{C([-T_3,T_3])}}{\gamma-\zeta-\alpha}<\varepsilon\end{aligned}$$

when δ is sufficiently small. Hence (3.36) holds. Combing (3.34), (3.35), and (3.36) gives (3.33). In the same fashing as estimating $I_{\delta,1}$, we have

$$|I_{\delta,2}|_{\gamma-\zeta,\delta}\leq\frac{KL}{\beta-\gamma+\zeta}|u^\delta(\cdot)|_{\gamma-\zeta,\delta}+o(1),\quad\text{as }\delta\to0^+ \tag{3.37}$$

and

$$|I_{\delta,3}|_{\gamma-\zeta,\delta}\leq\frac{KL}{\beta-\gamma+\zeta}|u^\delta(\cdot)|_{\gamma-\zeta,\delta}+o(1),\quad\text{as }\delta\to0^+. \tag{3.38}$$

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Therefore, by (3.33), (3.37) and (3.38) we have

$$|u^\delta(\cdot)|_{\gamma-\zeta,\delta} \le KL\Big(\frac{1}{\gamma-\zeta-\alpha}+\frac{2}{\beta-\gamma+\zeta}\Big)|u^\delta(\cdot)|_{\gamma-\zeta,\delta}+o(1).$$

Since

$$KL\Big(\frac{1}{\gamma-\zeta+\alpha}+\frac{2}{\beta-\gamma+\zeta}\Big)<1,$$

we have

$$|u^\delta(\cdot)|_\gamma \le |u^\delta(\cdot)|_{\gamma-\zeta,\delta}=o(1), \quad \text{as } \delta\to 0^+.$$

Namely,

$$\lim_{\delta\to 0^+}|v^\delta(\cdot,\omega,\xi)-v(\cdot,\omega,\xi)|_\gamma=0.$$

It follows from (3.27) that

$$\lim_{\delta\to 0^+} h^c_\delta(\omega,\xi)=h^c(\omega,\xi)\ \forall(\omega,\xi)\in\Omega\times E^c. \tag{3.39}$$

Employing (3.23), we get that

$$\begin{aligned}
&|h^c_\delta(\omega,e^{-z_\delta(\omega)}\xi)-h^c(\omega,e^{-z(\omega)}\xi)|\\
&\le |h^c_\delta(\omega,e^{-z_\delta(\omega)}\xi)-h^c_\delta(\omega,e^{-z(\omega)}\xi)|+|h^c_\delta(\omega,e^{-z(\omega)}\xi)-h^c(\omega,e^{-z(\omega)}\xi)|\\
&\le \frac{2K^2L}{(\beta-\gamma)\Big[1-KL\Big(\frac{1}{\gamma-\alpha}+\frac{2}{\beta-\gamma}\Big)\Big]}\times|(e^{-z_\delta(\omega)}-e^{-z(\omega)})\xi|\\
&+|h^c_\delta(\omega,e^{-z(\omega)}\xi)-h^c(\omega,e^{-z(\omega)}\xi)|.
\end{aligned}$$

Together with Lemma 3.4(3) and (3.39), we have

$$\lim_{\delta\to 0^+} h^c_\delta(\omega,e^{-z_\delta(\omega)}\xi)=h^c(\omega,e^{-z(\omega)}\xi)\ \forall(\omega,\xi)\in\Omega\times E^c.$$

Using Lemmas 3.4(3) again, Theorem 3.3 is established. □

Remark. (1) The center manifolds are C^1 smooth. (2) The results in this section hold for stable and unstable manifolds. (3) If f is a C^1 function with $f(0)=0$ and $Df(0)=0$, one can use the standard procedure to modify f by using a smooth cut-off function such that the modified function is globally Lipschitz continuous with a desired small Lipschitz constant. Thus, applying the results obtained here, one can get the convergence of local center-manifolds of Wong–Zakai approximations.

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