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Boundedness and global solvability to a chemotaxis model with nonlinear diffusion

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Abstract

In this paper, we study a chemotaxis model with nonlinear diffusion Δu^m $(m > 1)$. We consider this problem in a bounded domain $\Omega \subset \mathbb{R}^3$ with zero-flux boundary condition, and it is shown that for any large initial datum, for any $m > 1$, the problem admits a global weak solution, which is uniformly bounded.

MSC: 92C17; 35K40; 35K55

Keywords: Chemotaxis system; Nonlinear diffusion; Logistic source; Boundedness

1. Introduction

In this paper, we consider the following chemotaxis model

$$\begin{cases} u_t = \Delta u^m - \nabla \cdot (u \cdot \nabla v) + \mu u(1-u), \text{ in } Q, \\ v_t - \Delta v = -vu, \text{ in } Q, \\ (\nabla u^m - u \cdot \nabla v) \cdot \mathbf{n}\big|_{\partial\Omega} = \dfrac{\partial v}{\partial \mathbf{n}}\Big|_{\partial\Omega} = 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

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where $m > 1$, $Q = \Omega \times \mathbb{R}^+$, $\Omega \subset \mathbb{R}^3$ is a bounded domain, and the boundary $\partial\Omega$ is appropriately smooth, u, v represent the bacterial density, the chemoattractant concentration respectively, $J = u \cdot \nabla v$ is the chemotactic flux, $\mu u(1-u)$ ($\mu > 0$) is the proliferation or death of bacteria according to a generalized logistic law, $-vu$ is the consumption of chemoattractant.

Chemotaxis model was first introduced by Keller and Segel [5] in 1970, since then, a lots of modified chemotaxis models have been widely investigated by many researchers. For the Keller–Segel model of this form

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \cdot \nabla v), \\ \tau v_t - \Delta v + v = u, \end{cases}$$

in which, the chemical not only be consumed, but also be produced by cells. It is well known that the formation of cell aggregates will result in the finite-time blow-up of cell density in spatial dimensions $N \geq 2$, see for example [1,2,11,18]. When a logistic growth term $\mu u(1-u)$ characterizing the death of proliferation of cells is introduced into this model. It is found that the logistic growth term will weaken the aggregation behavior, and prevent blow up, for instance, in two-dimensional space, the solutions will always exist globally [16]; and in three-dimensional space, the solutions will exist globally for large μ [16,21]. For the nonlinear diffusion case, we refer to [8,22].

While if only the consumption of the cells is considered, that is for the Keller–Segel model of this form

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \cdot \nabla v) + \mu u(1-u), \\ v_t - \Delta v = -vu. \end{cases} \tag{1.2}$$

When $\mu = 0$, it is shown that for arbitrarily large initial datum, this problem admits a unique global classical solution in two-dimensional space, admits a global weak solution in three-dimensional space, and more interesting fact is that, the weak solution will become smooth after time T [13]; When $\mu > 0$, in three dimensional case, Zheng, Mu [23] showed that the system (1.2) admits a unique global classical solution if the initial datum of v is small; Lankeit and Wang [7] obtained the global classical solutions for large μ, but without the smallness assumption on the initial data, and for any $\mu > 0$, they also established the existence of global weak solutions. Lankeit also considered a coupled system of (1.2) and the incompressible Navier–Stokes equations [6], in which, the global existence of weak solutions are proved, however, after some time the weak solutions will become smooth and finally converge to a semi-trivial steady state.

However, as indicated by Szymanska, Chaplain et al. [10], the equation modeling the migration of cells should rather be regarded as nonlinear diffusion, in which, the cell mobility is described by a nonlinear function of the cells density, for example, the porous medium diffusion. So, in recent years, many researchers are led to study the model with nonlinear diffusion. For the following system ($m > 1$),

$$\begin{cases} u_t = \Delta u^m - \nabla \cdot (u \cdot \nabla v), \\ v_t - \Delta v = -vu, \end{cases}$$

in two dimensional space, the global solvability of weak solution is established for any $m > 1$ [14]; in three dimensional space, a global bounded weak solution is obtained for $m > \frac{7}{6}$ [19],

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a locally bounded global solution is obtained for $m > \frac{8}{7}$ [15], and still leave a gap for the case $m \in (1, \frac{8}{7}]$.

Inspired by these researches, the purpose of this paper is to show the global solvability of weak solutions to the problem (1.1). The main difficulty lies in the L^∞ estimate of u, we use an iterative technique to improve the regularity constantly, and combining with Moser iteration method, we finally established the L^∞ bound of u. And show that for any $m > 1$, $\mu > 0$ and for any large initial datum, the problem admits a global bounded solution.

We give the assumptions of this paper.

$$\begin{cases} u_0 \in L^\infty(\Omega) \cap W^{1,2}(\Omega),\ v_0 \in C^2(\overline{\Omega}), \\ u_0, v_0 \geq 0, \\ \partial\Omega \in C^{2,\alpha}, \\ \left.\dfrac{\partial v_0}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0. \end{cases} \tag{H}$$

Theorem 1.1. *Assume (H) hold, and $m > 1$. Then for any $\mu > 0$, the problem (1.1) admits a nonnegative weak solution (u, v) with $u \in \mathcal{X}_1$, $v \in \mathcal{X}_2$, where*

$$\mathcal{X}_1 = \{u \in L^\infty(\Omega \times R^+); \nabla u^m \in L^\infty((0,\infty); L^2(\Omega)),$$
$$\left(u^{\frac{m+1}{2}}\right)_t, \nabla u^{\frac{m+1}{2}} \in L^2_{loc}([0,\infty); L^2(\Omega))\},$$

$$\mathcal{X}_2 = \{v \in L^\infty((0,\infty); W^{1,\infty}(\Omega); v_t, \Delta v \in L^p_{loc}([0,\infty); L^p(\Omega))\ \textit{for any}\ \ p > 1\},$$

such that

$$\sup_{t\in(0,\infty)} \left(\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{W^{1,\infty}}\right) \leq M_1, \tag{1.3}$$

$$\sup_{t\in(0,+\infty)} \int_\Omega |\nabla u^m|^2 dx + \sup_{t\in(0,+\infty)} \|u^{\frac{m+1}{2}}\|_{W_2^{1,1}(Q_1(t))} \leq M_2, \tag{1.4}$$

$$\sup_{t\in(0,+\infty)} \|v\|_{W_p^{2,1}(Q_1(t))} \leq M_3, \quad \textit{for any}\ \ p > 1, \tag{1.5}$$

where M_i $(i = 1, 2, 3)$ depend only on μ, u_0, v_0.

2. Preliminaries

We first give some notations, which will be used throughout this paper.

Notations: $\|\cdot\|_{L^p} = \|\cdot\|_{L^p(\Omega)}$, $Q_\tau(t) = \Omega \times (t, t+\tau)$, $Q_T := Q_T(0) = \Omega \times (0, T)$.

Next, we give the definition of weak solutions.

Definition 2.1. (u, v) is called a weak solution of (1.1), if $u \geq 0, v \geq 0$, with $u \in \mathcal{X}$, $v \in W_2^{1,0}(Q_T)$ for any given $T > 0$, and

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$$-\iint\limits_{Q_T} u\varphi_t dxdt - \int\limits_{\Omega} u(x,0)\varphi(x,0)dx + \iint\limits_{Q_T} (\nabla u^m - u\nabla v)\nabla\varphi dxdt$$
$$= \mu \iint\limits_{Q_T} u(1-u)\varphi dxdt,$$
$$-\iint\limits_{Q_T} v\varphi_t dxdt - \int\limits_{\Omega} v(x,0)\varphi(x,0)dx + \iint\limits_{Q_T} \nabla v\nabla\varphi dxdt + \iint\limits_{Q_T} uv\varphi dxdt = 0$$

hold for any $\varphi \in C^\infty(\overline{Q}_T)$ with $\frac{\partial\varphi}{\partial\mathbf{n}}\Big|_{\partial\Omega} = 0$, $\varphi(x,T) = 0$, where $\mathcal{X} = \{u \in L^2(Q_T); \nabla u^m \in L^2(Q_T)\}$.

Before going further, we list some lemmas, which will be used throughout this paper. Firstly, we give the Gagliardo–Nirenberg interpolation inequality as follows.

Lemma 2.1. *For functions $u : \Omega \to \mathbb{R}$ defined on a bounded Lipschitz domain $\Omega \in \mathbb{R}^N$, we have*

$$\|D^j u\|_{L^p} \le C\|D^m u\|_{L^r}^{\alpha}\|u\|_{L^q}^{1-\alpha} + C\|u\|_{L^s},$$

where

$$\frac{1}{p} = \frac{j}{N} + (\frac{1}{r} - \frac{m}{N})\alpha + \frac{1-\alpha}{q}, \quad \frac{j}{m} \le \alpha \le 1,$$

and $s > 0$ is arbitrary.

We list the following lemma [12].

Lemma 2.2. *Let $T > 0$, $\tau \in (0,T)$, $a > 0$, $b > 0$, and suppose that $y : [0,T) \to [0,\infty)$ is absolutely continuous such that*

$$y'(t) + ay(t) \le h(t), \ \textit{for} \ \ t \in [0,T),$$

where $h \ge 0$, $h(t) \in L^1_{loc}([0,T))$ and

$$\int\limits_{t-\tau}^{t} h(s)ds \le b, \ \textit{for all}\ t \in [\tau, T).$$

Then

$$y(t) \le \max\{y(0) + b, \frac{b}{a\tau} + 2b\}\ \textit{for all}\ t \in [0,T).$$

By [3], we also give a similar lemma as Lemma 2.2.

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Lemma 2.3. *Let $T > 0$, $\tau \in (0, T)$, $\alpha > 0$, $\beta > 0$, and suppose that $f : [0, T) \to [0, \infty)$ is absolutely continuous, and satisfies*

$$f'(t) - g(t)f(t) + f^{1+\sigma}(t) \le h(t), t \in \mathbb{R}, \tag{2.1}$$

where $\sigma > 0$ is a constant, $g(t), h(t) \ge 0$ with $g(t), h(t) \in L^1_{loc}([0, T))$ and

$$\int_{t-\tau}^{t} g(s)ds \le \alpha, \quad \int_{t-\tau}^{t} h(s)ds \le \beta, \text{ for all } t \in [\tau, T).$$

Then for any $t > t_0$, we have

$$f(t) \le f(t_0)e^{\int_{t_0}^{t} g(s)ds} + \int_{t_0}^{t} h(\tau)e^{\int_{\tau}^{t} g(s)ds}d\tau, \tag{2.2}$$

and

$$\sup_t f(t) \le \sigma\left(\frac{2A}{1+\sigma}\right)^{\frac{1+\sigma}{\sigma}} + 2B, \tag{2.3}$$

where

$$A = \tau^{-\frac{1}{1+\sigma}}(1+\alpha)^{\frac{1}{1+\sigma}}e^{2\alpha}, \qquad B = \tau^{-\frac{1}{1+\sigma}}\beta^{\frac{1}{1+\sigma}}e^{2\alpha} + 2\beta e^{2\alpha} + f(0)e^{\alpha}.$$

By [4], we have the following Lemma.

Lemma 2.4. *Assume that $u_0 \in W^{2,p}(\Omega)$, and $f \in L^p_{loc}([0, +\infty); L^p(\Omega))$ with*

$$\sup_{t\in(\tau,+\infty)} \int_{t-\tau}^{t} \|f\|^p_{L^p}ds \le A,$$

where $\tau > 0$ is a fixed constant. Then the following problem

$$\begin{cases} u_t - \Delta u + u = f(x,t), \\ \left.\dfrac{\partial u}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0, \\ u(x,0) = u_0(x) \end{cases} \tag{2.4}$$

admits a unique solution u with $u \in L^p_{loc}([0, +\infty); W^{2,p}(\Omega))$, $u_t \in L^p_{loc}([0, +\infty); L^p(\Omega))$ and

$$\sup_{t\in(\tau,+\infty)} \int_{t-\tau}^{t} (\|u\|^p_{W^{2,p}} + \|u_t\|^p_{L^p})ds \le AM\frac{e^{p\tau}}{e^{\frac{p}{2}\tau} - 1} + Me^{\frac{p}{2}\tau}\|u_0\|^p_{W^{2,p}}, \tag{2.5}$$

where M is a constant independent of τ.

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3. Boundedness and global existence of weak solution

We first consider the approximate problems given by

$$\begin{cases} u_{\varepsilon t} = \Delta(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon - \nabla\cdot(u_\varepsilon\cdot\nabla v_\varepsilon) + \mu u_\varepsilon(1-u_\varepsilon), \text{ in } Q, \\ v_{\varepsilon t} - \Delta v_\varepsilon = -v_\varepsilon u_\varepsilon, \text{ in } Q, \\ \left.\dfrac{\partial u_\varepsilon}{\partial \mathbf{n}}\right|_{\partial\Omega} = \left.\dfrac{\partial v_\varepsilon}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0, \\ u_\varepsilon(x,0) = u_{\varepsilon 0}(x), v_\varepsilon(x,0) = v_{\varepsilon 0}(x), \quad x\in\Omega, \end{cases} \tag{3.1}$$

where $u_{\varepsilon 0}, v_{\varepsilon 0}$ satisfy

$$\begin{cases} u_{\varepsilon 0}, v_{\varepsilon 0} \in C^{2+\alpha}(\overline{\Omega}), \left.\dfrac{\partial u_{\varepsilon 0}}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0, \left.\dfrac{\partial v_{\varepsilon 0}}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0, \\ \|u_{\varepsilon 0}\|_{L^\infty} + \|\nabla u_{\varepsilon 0}^m\|_{L^2} + \|v_{\varepsilon 0}\|_{W^{2,\infty}} \le 2(\|u_0\|_{L^\infty} + \|\nabla u_0^m\|_{L^2} + \|v_0\|_{W^{2,\infty}}), \\ u_{\varepsilon 0}\to u_0, v_{\varepsilon 0}\to v_0, \quad \text{uniformly}. \end{cases} \tag{H1}$$

Using a fixed point argument, we can prove the following local existence result, see for example [9,20,14]. We state the local existence result of classical solution to (1.1) as follows.

Lemma 3.1. *Assume that $u_{\varepsilon 0}$, $v_{\varepsilon 0}$ satisfy (H1). Then there exists $T_{\max}\in(0,+\infty]$ such that the problem (3.1) admits a unique classical solution $(u_\varepsilon, v_\varepsilon)\in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T_{\max}))$ with*

$$u_\varepsilon \ge 0, \quad v_\varepsilon \ge 0, \textit{ for all } (x,t)\in\Omega\times(0,T_{\max}), \tag{3.2}$$

such that either $T_{max}=\infty$, or

$$\limsup_{t\nearrow T_{\max}} \left(\|u_\varepsilon(\cdot,t)\|_{L^\infty} + \|v_\varepsilon\|_{W^{1,\infty}}\right) = \infty.$$

In what follows, we show the global existence of solutions. We give the following Proposition.

Proposition 3.1. *Assume that $u_{\varepsilon 0}, v_{\varepsilon 0}$ satisfy (H1). Then for any $\varepsilon>0$, the problem (3.1) admits a unique global classical solution $(u_\varepsilon, v_\varepsilon)$, and*

$$\sup_{t\in(0,\infty)} \left(\|u_\varepsilon(\cdot,t)\|_{L^\infty} + \|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}}\right) \le M_1, \tag{3.3}$$

$$\sup_{t\in(0,+\infty)} \int_\Omega |\nabla u_\varepsilon^m|^2 dx + \sup_{t\in(0,+\infty)} \|u_\varepsilon^{\frac{m+1}{2}}\|_{W_2^{1,1}(Q_1(t))} \le M_2, \tag{3.4}$$

$$\sup_{t\in(0,+\infty)} \|v_\varepsilon\|_{W_p^{2,1}(Q_1(t))} \le M_3, \quad \textit{for any } \ p>1, \tag{3.5}$$

where $M_i\,(i=1,2,3)$ are independent of ε.

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Next, we give some estimates of $(u_\varepsilon, v_\varepsilon)$, we take $\tau = \min\{1, T_{\max}\}$. It is easy to see that $\tau \le 1$. In what follows, all these constants C, C_i, M are independent of τ and $T_{\max}$.

Lemma 3.2. *Assume that $u_{\varepsilon 0}, v_{\varepsilon 0}$ satisfy (H1). Let $(u_\varepsilon, v_\varepsilon)$ be the classical solution of (3.1) in $(0, T_{\max})$. Then*

$$\|v_\varepsilon\|_{L^\infty} \le \|v_{\varepsilon 0}\|_{L^\infty} \le \|v_0\|_{L^\infty}, \tag{3.6}$$

$$\sup_{t\in(0,T_{\max})} \|u_\varepsilon(\cdot,t)\|_{L^1} + \sup_{t\in(\tau,T_{\max})} \int_{t-\tau}^{t} \int_\Omega u_\varepsilon^2 dx \le C_1, \tag{3.7}$$

$$\sup_{t\in(\tau,T_{\max})} \int_{t-\tau}^{t} \|v_\varepsilon\|_{W^{2,2}}^2 ds \le C_2, \tag{3.8}$$

where C_1, C_2 are independent of $T_{\max}$, τ and ε.

Proof. By maximum principle, it is easy to see that (3.6) holds. Integrating the first equation of (3.1) directly, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega u_\varepsilon dx + \mu \int_\Omega u_\varepsilon^2 dx &= \mu \int_\Omega u_\varepsilon dx \\ &\le \frac{\mu}{2}\int_\Omega u_\varepsilon^2 dx + C.\end{aligned}$$

Then by Lemma 2.3, it proves (3.7). Recalling the second equation of (3.1), we see that

$$v_{\varepsilon t} - \Delta v_\varepsilon + v_\varepsilon = v_\varepsilon - v_\varepsilon u_\varepsilon. \tag{3.9}$$

By Lemma 2.4, we obtain (3.8). The proof is complete. □

Lemma 3.3. *Assume that $u_{\varepsilon 0}, v_{\varepsilon 0}$ satisfy (H1). Let $(u_\varepsilon, v_\varepsilon)$ be the classical solution of (3.1) in $(0, T_{\max})$. Then for any $k = 1, 2, 3 \cdots$,*

$$\sup_{t\in(0,T_{\max})} \|u_\varepsilon(\cdot,t)\|_{L^{2m^k-1}}^{2m^k-1} + \sup_{t\in(\tau,T_{\max})} \int_{t-\tau}^{t} \left(\int_\Omega u_\varepsilon^{m+2m^k-4} |\nabla u_\varepsilon|^2 dx + \int_\Omega u_\varepsilon^{2m^k} dx \right) ds \le C_k, \tag{3.10}$$

$$\sup_{t\in(\tau,T_{\max})} \int_{t-\tau}^{t} \left(\|v_\varepsilon\|_{W^{2,2m^k}}^{2m^k} + \|v_{\varepsilon t}\|_{L^{2m^k}}^{2m^k} \right) ds \le \tilde{C}_k, \tag{3.11}$$

where C_k, $\tilde{C}_k$ depends on k, and both of them are independent of τ, $T_{\max}$, and ε.

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Proof. Multiplying the first equation of (3.1) by u_ε^r for any $r>0$, and integrating it over Ω, we obtain

$$\begin{aligned}
&\frac{1}{r+1}\frac{d}{dt}\int_\Omega u_\varepsilon^{r+1}dx + r\int_\Omega u_\varepsilon^{m+r-2}|\nabla u_\varepsilon|^2dx + \mu\int_\Omega u_\varepsilon^{r+2}dx\\
\le&\int_\Omega r u_\varepsilon^r\nabla u_\varepsilon\nabla v_\varepsilon dx + \mu\int_\Omega u_\varepsilon^{r+1}\\
\le&\frac{r}{2}\int_\Omega u_\varepsilon^{m+r-2}|\nabla u_\varepsilon|^2dx + \frac{\mu}{2}\int_\Omega u_\varepsilon^{r+2}dx + \frac{r}{2}\int_\Omega u_\varepsilon^{r+2-m}|\nabla v_\varepsilon|^2dx + 2^{r+1}\mu|\Omega|,
\end{aligned}$$

which means

$$\begin{aligned}
&\frac{1}{r+1}\frac{d}{dt}\int_\Omega u_\varepsilon^{r+1}dx + \frac{r}{2}\int_\Omega u_\varepsilon^{m+r-2}|\nabla u_\varepsilon|^2dx + \frac{\mu}{2}\int_\Omega u_\varepsilon^{r+2}dx\\
\le&\frac{r}{2}\int_\Omega u_\varepsilon^{r+2-m}|\nabla v_\varepsilon|^2dx + 2^{r+1}\mu|\Omega|. \qquad (3.12)
\end{aligned}$$

By Lemma 2.1, for any $k=0,1,2\cdots$, we obtain

$$\begin{aligned}
\int_{t-\tau}^t \|\nabla v_\varepsilon(\cdot,t)\|_{L^{4m^k}}^{4m^k}ds \le& C_1\sup_{t\in(0,T_{\max})}\|v_\varepsilon\|_{L^\infty}^{2m^k(1-\beta)}\int_{t-\tau}^t\|\Delta v_\varepsilon(\cdot,s)\|_{L^{2m^k}}^{2m^k}ds + C_2\|v_\varepsilon\|_{L^\infty}^{4m^k}\\
\le& C_3\left(1+\int_{t-\tau}^t\|\Delta v_\varepsilon(\cdot,s)\|_{L^{2m^k}}^{2m^k}ds\right), \qquad (3.13)
\end{aligned}$$

where C_3 depends on m, k, Ω and $\|v_0\|_{L^\infty}$. Taking $r=2(m-1)$ in (3.12), then we have

$$\begin{aligned}
&\frac{1}{2m-1}\frac{d}{dt}\int_\Omega u_\varepsilon^{2m-1}dx + (m-1)\int_\Omega u_\varepsilon^{3m-4}|\nabla u_\varepsilon|^2dx + \frac{\mu}{2}\int_\Omega u_\varepsilon^{2m}dx\\
\le&(m-1)\int_\Omega u_\varepsilon^m|\nabla v_\varepsilon|^2dx + 2^{2m-1}\mu|\Omega|\\
\le&\frac{\mu}{4}\int_\Omega u_\varepsilon^{2m}dx + C\int_\Omega|\nabla v_\varepsilon|^4dx + 2^{2m-1}\mu|\Omega|. \qquad (3.14)
\end{aligned}$$

Recalling (3.8), taking $k=0$ in (3.13), and using Lemma 2.3, we obtain

$$\sup_{t\in(0,T_{\max})}\|u_\varepsilon(\cdot,t)\|_{L^{2m-1}}^{2m-1} + \sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^t\left(\int_\Omega u_\varepsilon^{3m-4}|\nabla u_\varepsilon|^2dx + \int_\Omega u_\varepsilon^{2m}dx\right)ds \le C.$$

Next, we use recursive method to prove (3.10). Assume that

$$\sup_{t\in(0,T_{\max})}\|u_\varepsilon(\cdot,t)\|_{L^{2m^i-1}}^{2m^i-1}+\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^{t}\left(\int_\Omega u_\varepsilon^{m+2m^i-4}|\nabla u_\varepsilon|^2dx+\int_\Omega u_\varepsilon^{2m^i}dx\right)ds\le C_i. \tag{3.15}$$

Then using Lemma 2.4, we obtain

$$\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^{t}\left(\|v_\varepsilon\|_{W^{2,2m^i}}^{2m^i}+\|v_{\varepsilon t}\|_{L^{2m^i}}^{2m^i}\right)ds\le\tilde{C}_i. \tag{3.16}$$

Taking $r=2m^{i+1}-2$ in (3.12), we see that

$$\begin{aligned}&\frac{1}{2m^i}\frac{d}{dt}\int_\Omega u_\varepsilon^{2m^{i+1}-1}dx+\frac{2m^{i+1}-2}{2}\int_\Omega u_\varepsilon^{m+2m^{i+1}-4}|\nabla u_\varepsilon|^2dx+\frac{\mu}{2}\int_\Omega u_\varepsilon^{2m^{i+1}}dx\\ \le&(m^{i+1}-1)\int_\Omega u_\varepsilon^{2m^{i+1}-m}|\nabla v_\varepsilon|^2dx+2^{2m^{i+1}-1}\mu|\Omega|\\ \le&C\int_\Omega|\nabla v_\varepsilon|^{4m^i}dx+\frac{\mu}{4}\int_\Omega u_\varepsilon^{2m^{i+1}}dx+2^{2m^{i+1}-1}\mu|\Omega|.\end{aligned}$$

Using (3.16), (3.13) and Lemma 2.3, we obtain

$$\sup_{t\in(0,T_{\max})}\|u_\varepsilon(\cdot,t)\|_{L^{2m^{i+1}-1}}^{2m^{i+1}-1}+\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^{t}\left(\int_\Omega u_\varepsilon^{m+2m^{i+1}-4}|\nabla u_\varepsilon|^2dx+\int_\Omega u_\varepsilon^{2m^{i+1}}dx\right)ds\le C_{i+1}.$$

The proof is complete. □

Lemma 3.4. *Assume that $u_{\varepsilon 0}, v_{\varepsilon 0}$ satisfy (H1). Let $(u_\varepsilon, v_\varepsilon)$ be the classical solution of (3.1) in $(0,T_{\max})$. Then*

$$\sup_{t\in(0,T_{\max})}\|u_\varepsilon(\cdot,t)\|_{L^\infty}+\sup_{t\in(0,T_{\max})}\|v\|_{W^{1,\infty}}\le C, \tag{3.17}$$

where C is independent of $T_{\max}$ and ε.

Proof. By Lemma 3.3, we take k appropriately large such that $2m^k\ge 10$. Then, by t-anisotropic embedding theorem, we have

$$\|v_\varepsilon\|_{C^{\frac{3}{2},\frac{3}{4}}}\le C\sup_{t\in(\tau,T_{\max})}\|v_\varepsilon\|_{W_{10}^{2,1}(Q_\tau(t))},$$

where $Q_\tau(t)=\Omega\times(t-\tau,t)$, it means

$$\sup_{t\in(0,T_{\max})} \|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}} \le C.$$

Using this estimate, next, we adapt the classical Moser's iterative technique to prove the L^∞ estimate of u_ε. This method has also been used by some other authors, see for example [17]. The main step is to obtain a estimate similar to (3.19), and after that, the steps are standard.

Multiplying the first equation of (3.1) by u_ε^{r-1} with $r \ge 3m$, and integrating it over Ω, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^r dx + r(r-1)\int_\Omega u_\varepsilon^{m+r-3}|\nabla u_\varepsilon|^2 dx + r\mu\int_\Omega u_\varepsilon^{r+1}dx + \int_\Omega u_\varepsilon^r dx\\
\le& \int_\Omega r(r-1)u_\varepsilon^{r-1}\nabla u_\varepsilon \nabla v_\varepsilon dx + r\mu\int_\Omega u_\varepsilon^r + \int_\Omega u_\varepsilon^r dx\\
\le& \frac{r(r-1)}{2}\int_\Omega \left(u_\varepsilon^{m+r-3}|\nabla u_\varepsilon|^2 + u_\varepsilon^{r+1-m}|\nabla v_\varepsilon|^2\right)dx + (\mu r+1)\int_\Omega u_\varepsilon^r dx\\
\le& \frac{r(r-1)}{2}\int_\Omega u_\varepsilon^{m+r-3}|\nabla u_\varepsilon|^2 dx + Cr^2\int_\Omega u_\varepsilon^{r+1-m}dx + \frac{r\mu}{2}\int_\Omega u_\varepsilon^{r+1}dx,
\end{aligned}$$

which means

$$\frac{d}{dt}\int_\Omega u_\varepsilon^r dx + \int_\Omega |\nabla u_\varepsilon^{\frac{r+m-1}{2}}|^2 dx + \frac{r\mu}{2}\int_\Omega u_\varepsilon^{r+1}dx + \int_\Omega u_\varepsilon^r dx \le Cr^2\int_\Omega u_\varepsilon^{r+1-m}dx. \tag{3.18}$$

Noticing that

$$\begin{aligned}
&Cr^2\|u_\varepsilon\|_{L^{r+1-m}}^{r+1-m} = Cr^2\|u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^{\frac{2(r+1-m)}{r+m-1}}}^{\frac{2(r+1-m)}{r+m-1}}\\
\le& C_1r^2\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^2}^{\frac{2\beta(r+1-m)}{r+m-1}}\|u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^{\frac{r}{m+r-1}}}^{(1-\beta)\frac{2(r+1-m)}{r+m-1}} + C_2r^2\|u_\varepsilon\|_{\frac{r}{2}}^{r+1-m}\\
\le& \frac{1}{2}\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^2}^2 + C_3r^{\frac{6(m-1)+5r}{6(m-1)+r}}\|u_\varepsilon\|_{\frac{r}{2}}^{\frac{r(2m+r-2)}{6m+r-6}} + C_2r^2\|u_\varepsilon\|_{\frac{r}{2}}^{r+1-m}\\
\le& \frac{1}{2}\|\nabla u_\varepsilon^{\frac{r+m-1}{2}}\|_{L^2}^2 + C_3r^5\|u_\varepsilon\|_{\frac{r}{2}}^{\frac{r(2m+r-2)}{6m+r-6}} + C_2r^2\|u_\varepsilon\|_{\frac{r}{2}}^{r+1-m},
\end{aligned}$$

substituting the above inequality into (3.18), we obtain

$$\frac{d}{dt}\int_\Omega u_\varepsilon^r dx + \int_\Omega u_\varepsilon^r dx \le C_3r^5\|u_\varepsilon\|_{\frac{r}{2}}^{\frac{r(2m+r-2)}{6m+r-6}} + C_2r^2\|u_\varepsilon\|_{\frac{r}{2}}^{r+1-m}. \tag{3.19}$$

By (3.10), there exists $k^* > 0$ such that $2m^{k^*} - 1 \ge 3m$ and

$$\sup_{t\in(0,T_{\max})} \|u_\varepsilon(\cdot,t)\|_{L^{2m^{k^*}-1}} \le C_0.$$

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Taking $r_j = 2r_{j-1} = 2^j r_0$, $r_0 = 2m^{k^*} - 1$, $M_j = \max\left\{1, \sup\limits_{t\in(0,T_{\max})} \|u_\varepsilon\|_{L^{r_j}}\right\}$, then we have

$$
\begin{aligned}
M_j \le& (C_2 + C_3)^{\frac{1}{r_j}} r_j^{\frac{5}{r_j}} M_{j-1} = (C_2 + C_3)^{\frac{1}{r_0 2^j}} r_0^{\frac{5}{r_0 2^j}} 2^{\frac{5j}{r_0 2^j}} M_{j-1} \\
\le& (C_2 + C_3)^{\sum_{k=1}^{j} \frac{1}{r_0 2^k}} r_0^{\sum_{k=1}^{j} \frac{5}{r_0 2^k}} 2^{\sum_{k=1}^{j} \frac{5k}{r_0 2^k}} M_0 \le C,
\end{aligned}
$$

where C is independent of j. Letting $j \to \infty$, and (3.17) is obtained. $\square$

Proof of Proposition 3.1. By Lemma 3.1, Lemma 3.4, for any $\varepsilon > 0$, $T_{\max} = +\infty$, that is the problem (3.1) admits a global classical solution. It means that $\tau = 1$. So we replace τ by 1 in Lemma 3.2 and Lemma 3.3.

Multiplying the first equation of (3.1) by u_ε, and integrating it over Ω, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega u_\varepsilon^2 dx + \int_\Omega (u_\varepsilon^2 + \varepsilon)^{\frac{m-1}{2}} |\nabla u_\varepsilon|^2 dx \le& -\int_\Omega u_\varepsilon^2 \Delta v_\varepsilon dx + C \\
\le& C\left(1 + \int_\Omega |\Delta v_\varepsilon|^2 dx\right),
\end{aligned}
$$

which means

$$
\sup_{t\in(1,\infty)} \int_{t-1}^{t} \int_\Omega (u_\varepsilon^2 + \varepsilon)^{\frac{m-1}{2}} |\nabla u_\varepsilon|^2 dx ds \le C, \tag{3.20}
$$

where C is independent of ε. Multiplying the first equation of (3.1) by $\frac{\partial((u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}} u_\varepsilon)}{\partial t}$, and integrating it over Ω gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla((u_\varepsilon^2 + \varepsilon)^{\frac{m-1}{2}} u_\varepsilon)|^2 dx + \int_\Omega (u_\varepsilon^2 + \varepsilon)^{\frac{m-1}{2}} \left|\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx \\
\le& -\int_\Omega \nabla\cdot(u_\varepsilon \cdot \nabla v_\varepsilon) \frac{\partial((u_\varepsilon^2 + \varepsilon)^{\frac{m-1}{2}} u_\varepsilon)}{\partial t} dx + \mu \int_\Omega u_\varepsilon(1 - u_\varepsilon) \frac{\partial((u_\varepsilon^2 + \varepsilon)^{\frac{m-1}{2}} u_\varepsilon)}{\partial t} \\
\le& m^2 \int_\Omega |\nabla\cdot(u_\varepsilon \cdot \nabla v_\varepsilon)|^2 (u_\varepsilon^2 + \varepsilon)^{\frac{m-1}{2}} dx + \frac{1}{2}\int_\Omega (u_\varepsilon^2 + \varepsilon)^{\frac{m-1}{2}} \left|\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx \\
&+ \mu^2 m^2 \int_\Omega u_\varepsilon^2 (1 - u_\varepsilon)^2 (u_\varepsilon^2 + \varepsilon)^{\frac{m-1}{2}} dx \\
\le& C\left(1 + \int_\Omega \left(|\Delta v_\varepsilon|^2 + (u_\varepsilon^2 + \varepsilon)^{\frac{m-1}{2}} |\nabla u_\varepsilon|^2\right) dx\right) + \frac{1}{2}\int_\Omega (u_\varepsilon^2 + \varepsilon)^{\frac{m-1}{2}} \left|\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx,
\end{aligned}
$$

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which means

$$\frac{d}{dt}\int_\Omega |\nabla((u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon)|^2dx+\int_\Omega (u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}\left|\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx$$
$$+\int_\Omega |\nabla((u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon)|^2dx\le C\left(1+\int_\Omega\left(|\Delta v_\varepsilon|^2+(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}|\nabla u_\varepsilon|^2\right)dx\right).$$

By (3.8) and (3.20), and using Lemma 2.2, we further have

$$\sup_{t\in(0,+\infty)}\int_\Omega |\nabla(u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}u_\varepsilon)|^2dx+\sup_{t\in(1,+\infty)}\int_{t-1}^{t}\int_\Omega (u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}}\left|\frac{\partial u_\varepsilon}{\partial t}\right|^2 dx\le C,$$

which implies (3.4). And (3.5) is a direct result of (3.3) by Lemma 2.4. □

Next, we show Theorem 1.1.

Proof of Theorem 1.1. Using Proposition 3.1, for any $T>0$, we have (passing to subsequences if necessary, for simplicity, we still denote them by $u_\varepsilon, v_\varepsilon$)

$$u_\varepsilon \overset{*}{\rightharpoonup} u,\ \text{in } L^\infty(Q_T),$$
$$u_\varepsilon^{\frac{m+1}{2}} \rightharpoonup u^{\frac{m+1}{2}},\ \text{in } W_2^{1,1}(Q_T),$$
$$v_\varepsilon \rightharpoonup v,\ \text{in } W_p^{2,1}(Q_T),\text{ for any } p\in(1,+\infty).$$

By Aubin–Lions theorem, and recalling (3.4), we have

$$u_\varepsilon^{\frac{m+1}{2}} \to u^{\frac{m+1}{2}},\ \text{in } C([0,T];L^2(\Omega)),$$

which means $u_\varepsilon\to u$ a.e. in Q_T, recalling (3.3) and using Lebesgue's dominated convergence theorem,

$$u_\varepsilon\to u,\ \text{in } C([0,T];L^p(\Omega)),\text{ for any } p\in(1,+\infty).$$

By (3.5), and noting that $W_p^{2,1}(Q_T)\hookrightarrow C^{2-\frac{5}{p},1-\frac{5}{2p}}(Q_T)$ for any $p>\frac{5}{2}$, then, in fact, we also have

$$v_\varepsilon\to v,\ \text{uniformly;}$$

And (1.3), (1.4), (1.5) hold. Noticing that $u_\varepsilon\to u$ in $L^p(Q_T)$, $\nabla v_\varepsilon\rightharpoonup\nabla v$ in $L^p(Q_T)$ for any $p>1$, it means that $u_\varepsilon\nabla v_\varepsilon\rightharpoonup u\nabla v$ in $L^q(Q_T)$ for any $q>1$. Recalling that for any $\varphi\in C^\infty(\overline{Q}_T)$ with $\left.\frac{\partial\varphi}{\partial\mathbf{n}}\right|_{\partial\Omega}=0$ and $\varphi(x,T)=0$,

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$$
\begin{aligned}
&-\iint\limits_{Q_T} u_\varepsilon \varphi_t dxdt - \int\limits_\Omega u_{\varepsilon 0}\varphi(x,0)dx - \iint\limits_{Q_T} (u_\varepsilon^2+\varepsilon)^{\frac{m-1}{2}} u_\varepsilon \Delta\varphi dxdt \\
&= \iint\limits_{Q_T} u_\varepsilon \cdot \nabla v_\varepsilon \nabla\varphi dxdt + \iint\limits_{Q_T} \mu u_\varepsilon(1-u_\varepsilon)\varphi dxdt, \\
&\iint\limits_{Q_T} v_\varepsilon \varphi_t dxdt - \int\limits_\Omega v_{\varepsilon 0}\varphi(x,0)dx + \iint\limits_{Q_T} \nabla v_\varepsilon \nabla\varphi dxdt + \iint\limits_{Q_T} u_\varepsilon v_\varepsilon \varphi dxdt = 0,
\end{aligned}
$$

letting $\varepsilon \to 0$, we conclude that

$$
\begin{aligned}
&-\iint\limits_{Q_T} u \varphi_t dxdt - \int\limits_\Omega u_0(x)\varphi(x,0)dx - \iint\limits_{Q_T} u^m \Delta\varphi dxdt \\
&= \iint\limits_{Q_T} u \cdot \nabla v \nabla\varphi dxdt + \iint\limits_{Q_T} \mu u(1-u)\varphi dxdt, \\
&-\iint\limits_{Q_T} v \varphi_t dxdt - \int\limits_\Omega v_0(x)\varphi(x,0)dx + \iint\limits_{Q_T} \nabla v \nabla\varphi dxdt + \iint\limits_{Q_T} uv\varphi dxdt = 0.
\end{aligned}
$$

Noticing that $\nabla u^m \in L^2(Q_T)$, then we also have

$$
\begin{aligned}
&-\iint\limits_{Q_T} u \varphi_t dxdt - \int\limits_\Omega u(x,0)\varphi(x,0)dx + \iint\limits_{Q_T} (\nabla u^m - u\nabla v)\nabla\varphi dxdt \\
&= \mu \iint\limits_{Q_T} u(1-u)\varphi dxdt,
\end{aligned}
$$

which means (u, v) with $u \in \mathcal{X}_1$, $v \in \mathcal{X}_2$ is a global weak solution of (1.1). $\square$

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