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Linear elliptic equations in composite media with anisotropic fibres

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Abstract

Linear elliptic equations in composite media with anisotropic fibres are concerned. The media consist of a periodic set of anisotropic fibres with low conductivity, included in a connected matrix with high conductivity. Inside the anisotropic fibres, the conductivity in the longitudinal direction is relatively high compared with that in the transverse directions. The coefficients of the elliptic equations depend on the conductivity. This work is to derive the Hölder and the gradient L^p estimates (uniformly in the period size of the set of anisotropic fibres as well as in the conductivity ratio of the fibres in the transverse directions to the connected matrix) for the solutions of the elliptic equations. Furthermore, it is shown that, inside the fibres, the solutions have higher regularity along the fibres than in the transverse directions.

MSC: 35J05; 35J15; 35J25; 35J70

Keywords: Morrey space; Campanato space; Anisotropic fibres; Longitudinal direction; Transverse direction

1. Introduction

This work presents uniform estimate for the solutions of linear elliptic equations in composite media with anisotropic fibres. The media contain a periodic set of anisotropic fibres with low conductivity, included in a connected matrix with high conductivity. Inside the anisotropic fibres, the conductivity in the longitudinal direction is relatively high compared with that in the trans-

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verse directions. The diffusion coefficients of the elliptic equations depend on the conductivity of the media. Due to the conductivity, the equations are non-uniform elliptic equations.

Ω denotes a bounded domain in $\mathbb{R}^n$ for $n \geq 2$, $\partial\Omega$ is the boundary of Ω, $\epsilon \in (0, 1]$, $\Omega(2\epsilon) \equiv \{y \in \Omega | dist(y, \partial\Omega) > 2\epsilon\}$, $Y \equiv [0, 1)^n$ consists of a smooth sub-domain Y_m completely surrounded by another connected sub-domain Y_f $(\equiv Y \setminus \overline{Y_m})$, $\epsilon(Y_m + j) \equiv \{y | y = (\epsilon(z_1 + j_1), \cdots, \epsilon(z_n + j_n)), (z_1, \cdots, z_n) \in Y_m, j = (j_1, \cdots, j_n)\}$, $\Omega^\epsilon_m \equiv \{y | y \in \epsilon(Y_m + j) \subset \Omega(2\epsilon)$ for some $j \in \mathbb{Z}^n\}$ is a disconnected subset of Ω, Ω^ϵ_f $(\equiv \Omega \setminus \overline{\Omega^\epsilon_m})$ represents a connected sub-region of Ω. Let $\mathcal{D} \equiv \Omega \times (0, L)$ denote the composite media, $\mathcal{D}^\epsilon_m \equiv \Omega^\epsilon_m \times (0, L)$ the periodic set of anisotropic fibres, and $\mathcal{D}^\epsilon_f \equiv \Omega^\epsilon_f \times (0, L)$ the connected matrix. Set $x \equiv (x', x_{n+1}) \in \mathbb{R}^{n+1}$, $x' \in \mathbb{R}^n$ and define a $(n + 1) \times (n + 1)$ matrix function $\mathbf{E}^{\omega^\tau}_{\omega^2,\epsilon}$ as

$$\mathbf{E}^{\omega^\tau}_{\omega^2,\epsilon}(x', x_{n+1}) \equiv \begin{cases} \mathbf{K}(\frac{x'}{\epsilon})I & \text{for } (x', x_{n+1}) \in \mathcal{D}^\epsilon_f, \\ \mathbf{K}(\frac{x'}{\epsilon})\, diag(\omega^2, \cdots, \omega^2, \omega^\tau) & \text{for } (x', x_{n+1}) \in \mathcal{D}^\epsilon_m, \end{cases}$$

where $\omega \in (0, 1]$, $\tau \in [0, 2]$, $\mathbf{K}$ is a positive periodic function in $\mathbb{R}^n$ with period Y, I is the $(n + 1) \times (n + 1)$ identity matrix, and $diag(\omega^2, \cdots, \omega^2, \omega^\tau)$ is a $(n + 1) \times (n + 1)$ diagonal matrix with ω^2 in the first n entries and ω^τ in the $(n + 1)$-entry. The elliptic equations are

$$\begin{cases} -\nabla \cdot (\mathbf{E}^{\omega^\tau}_{\omega^2,\epsilon} \nabla U) = F & \text{in } \mathcal{D}, \\ U = 0 & \text{on } \partial\mathcal{D}, \end{cases} \tag{1.1}$$

where $\omega, \epsilon \in (0, 1]$, $\tau \in [0, 2]$, and F is a given function.

The problem has applications in oil recovering industry, photonic crystal fibers, the stress in composite materials, and so on (see [3,8,9,15,16] and references therein). For $\tau = 0$ and $\omega^2 = \epsilon^2$, (1.1) was used to describe the Darcy's velocity for two-phase flows in fractured reservoirs [9]. If F is bounded in $\mathcal{D}$, a solution of (1.1) in Hilbert space $H^1(\mathcal{D})$ exists uniquely for each $\omega(= \epsilon)$. In addition that F is small in $\mathcal{D}^\epsilon_m$, the L^2 norm of the gradient of the solution of (1.1) in the connected matrix $\mathcal{D}^\epsilon_f$ is bounded uniformly in ω, ϵ. However, this is not for the solutions in the anisotropic fibres $\mathcal{D}^\epsilon_m$. Furthermore, the homogenized limit of (1.1) as $\epsilon \to 0$ exhibits a non-locality due to the conductivity of the media [7]. For $\tau = 2$ and $\omega^2 = 0$ (that is, a perforated domain case), the homogenization problem of (1.1) was considered in [8]. For $\tau = 2$ and $\omega^2 = \epsilon$, the homogenized limit and convergence of the solution of (1.1) were proved in [3]. For $\tau = 2$ and $\omega^2 = \epsilon^2$, the homogenization of two-phase immiscible flows was studied in [16]. The homogenization of a diffusion equation corresponding to (1.1) with $\tau = 2, \omega = \epsilon^\alpha, \alpha > 0$ was obtained in [18].

Concerning the regularity of the solution, the Hölder estimate of the solution of (1.1) for each fixed ω, ϵ, τ can be obtained by the De Giorgi–Nash–Moser Theorem [11]. The article [6] study equation (1.1) for $\tau = 2 = n, 0 < \epsilon \leq 1 \leq \omega^2 < \infty$ case. Under the assumptions $\omega^2|\mathcal{D}^\epsilon_m| \sim 1$ and $|\mathcal{D}^\epsilon_m| \ll \epsilon$, uniform $W^{1,6}$ bound and uniform $C^{1,\alpha}$ convergence estimate of the solutions were obtained in an interior region of $\mathcal{D}^\epsilon_f$. There are some other works related to our problem. For example, uniform estimates under $C^{0,\alpha}$, $W^{1,p}$, $W^{2,p}$, and $W^{1,\infty}$ norms for uniform elliptic equations with periodic oscillatory coefficients were considered in [5,17]. For non-uniform elliptic equations with smooth periodic coefficients, existence of $C^{2,\alpha}$ solution could be found in [12]. Uniform gradient L^p estimate for non-uniform elliptic equations with discontinuous coefficients was shown in [20].

Problem (1.1) is a non-uniform elliptic equation. Uniform bound of the solution of (1.1) is related to the shape of the anisotropic fibres and the conductivity of the composite media. By separation of variables, the $(n+1)$-dimensional problem (1.1) can be reduced to n-dimensional Helmholtz-type equations with simpler conductivity ratios in periodic domains. A three-step compactness argument [5] is employed to study the uniform regularity of the reduced equations. Finally we obtain the uniform Hölder and uniform gradient L^p estimates in ω, ϵ, τ for the solution of problem (1.1) by applying the Hölder inequality to the uniform estimates of the solutions of the reduced equations. Inside the fibres $\mathcal{D}^\epsilon_m$, it is shown that the solution has higher regularity along the fibres than in the transverse directions. However, the estimates here may not be optimal here. Different from [6], our uniform estimates in ω, ϵ, τ for the solution of (1.1) hold in the whole composite media. Also constraints on the parameters ω, ϵ and the size of $\mathcal{D}^\epsilon_m$ (that is, $\omega^2|\mathcal{D}^\epsilon_m| \sim 1$ and $|\mathcal{D}^\epsilon_m| \ll \epsilon$) are not required here.

The rest of this work is organized as follows: Notation and main results are stated in section 2. The main results are proved in section 3. To prove the main results, we reduce the $(n+1)$-dimensional problem (1.1) to n-dimensional Helmholtz-type equations. Uniform estimates for the solution of (1.1) are based on the uniform estimates for these reduced problems (see Lemma 3.1 and Lemma 3.3). In section 4, some a priori estimates for interface problems are derived. In section 5, uniform Hölder estimate for reduced Helmholtz-type transmission equations (that is, Lemma 3.1) is shown. Section 6 gives the proof of uniform gradient L^p estimate for the reduced problems (that is, Lemma 3.3).

2. Notation and main result

$C^{k,\alpha}$, L^p, $W^{k,p}$, $L^{k,p}$, and $\mathcal{L}^{k,p}$ are used for the Hölder space, Lebesgue space, the Sobolev space, Morrey space, and Campanato space respectively [19]. $C^\infty_{per}(\mathbb{R}^n)$ is the space of infinitely differentiable Y-periodic functions in $\mathbb{R}^n$, $H^1_{per}(\mathbb{R}^n)$ (resp. $C^{1,0}_{per}(\mathbb{R}^n)$) is the closure of $C^\infty_{per}(\mathbb{R}^n)$ under H^1 (resp. $C^{1,0}$) norm. $\mathcal{O}^\nu_m \equiv \bigcup_{j\in\mathbb{Z}^n} \nu(Y_m + j)$ and $\mathcal{O}^\nu_f \equiv \mathbb{R}^n \setminus \overline{\mathcal{O}^\nu_m}$ for $\nu > 0$. $(\lambda-2)_+ \equiv \max\{\lambda-2, 0\}$. Let $\|\varphi_1,\cdots,\varphi_m\|_{\mathbf{B}_1} \equiv \|\varphi_1\|_{\mathbf{B}_1} + \cdots + \|\varphi_m\|_{\mathbf{B}_1}$, $\|\varphi\|_{\mathbf{B}_1\cup\mathbf{B}_2} \equiv \|\varphi\|_{\mathbf{B}_1} + \|\varphi\|_{\mathbf{B}_2}$, $y = (y_1, \hat{y}) \in \mathbb{R}^n$, $B_r(z) \equiv \{y \in \mathbb{R}^n |\ \|y - z\| < r\}$, $B^+_r(z) \equiv \{y \in B_r(z) |\ y_1 > 0\}$, $B^-_r(z) \equiv \{y \in B_r(z) |\ y_1 < 0\}$. For any set S, $\overline{S}$ is the closure of S, ∂S is the boundary of S, $|S|$ is the volume of S, $\mathcal{X}_S$ is the characteristic function on S, and $\frac{1}{r}S = S/r \equiv \{y | ry \in S\}$. $[\varphi]_{C^{0,\alpha}_{x'}(\overline{\mathcal{D}^\epsilon_f})}$ means $\sup_{x',y'\in\Omega^\epsilon_f;x_{n+1}\in[0,L]} \frac{|\varphi(x',x_{n+1})-\varphi(y',x_{n+1})|}{\|x'-y'\|^\alpha}$. Similar definition for $[\varphi]_{C^{0,\alpha}_{x'}(\overline{\mathcal{D}^\epsilon_m})}$. If $\mathbb{A}$ is a positive function, we define, for $y = (y_1, \hat{y}) \in \mathbb{R}^n$,

$$(\varphi)_{z,r,\mathbb{A}} \equiv \int_{B_r(z)} \mathbb{A}\varphi(\xi)d\xi \Big/ \int_{B_r(z)} \mathbb{A}(\xi)d\xi, \tag{2.1}$$

$$(\varphi)^\pm_{z,r,\mathbb{A}}(y) \equiv \begin{cases} \displaystyle\int_{B^+_r(z)} \mathbb{A}\varphi(\xi)d\xi \Big/ \int_{B^+_r(z)} \mathbb{A}(\xi)d\xi & \text{if } y_1 > 0, \\ \displaystyle\int_{B^-_r(z)} \mathbb{A}\varphi(\xi)d\xi \Big/ \int_{B^-_r(z)} \mathbb{A}(\xi)d\xi & \text{if } y_1 < 0. \end{cases} \tag{2.2}$$

If $\vec{\mathbf{n}}_y$ is an outward normal vector on ∂Y_m, we define, for any function φ in Y and $y \in \partial Y_m$,

$$\varphi_{,\pm}(y) \equiv \lim_{t\to 0^+} \varphi(y \pm t\vec{\mathbf{n}}_y), \qquad \lfloor \varphi \rfloor(y) = \varphi_{,+}(y) - \varphi_{,-}(y). \tag{2.3}$$

Define, for $\omega \in [0, 1]$ and $\nu, r \in (0, \infty)$,

$$\begin{cases} \mathbf{K}_{\omega,\nu}(x', x_{n+1}) \equiv \mathbf{K}(\frac{x'}{\nu})(\mathcal{X}_{\Omega_f^\nu}(x') + \omega\mathcal{X}_{\Omega_m^\nu}(x')), \\ \mathbb{K}_{\omega,\nu,r}(x', x_{n+1}) \equiv \mathbf{K}(\frac{rx'}{\nu})(\mathcal{X}_{\Omega_f^\nu/r}(x') + \omega\mathcal{X}_{\Omega_m^\nu/r}(x')), \\ \mathbb{E}_{\omega,\nu}(x', x_{n+1}) \equiv \mathbf{K}(\frac{x'}{\nu})(\mathcal{X}_{\mathcal{O}_f^\nu}(x') + \omega\mathcal{X}_{\mathcal{O}_m^\nu}(x')). \end{cases} \tag{2.4}$$

Π_ϵ denotes the extension operator in Theorem 2.1 [1] and $\Pi_\epsilon \Phi|_{\Omega_f^\epsilon}$ is the extension function of $\Phi|_{\Omega_f^\epsilon}$ in Ω.

Our main results are

Theorem 2.1. *Suppose*

A1. Ω *is a bounded smooth domain in* $\mathbb{R}^n$ *for* $n \geq 2$,
A2. Y_m *is a smooth simply-connected sub-domain of* Y,
A3. $0 < \mathbf{K} \in C^{1,0}_{per}(\mathbb{R}^n)$, $\omega, \epsilon \in (0, 1]$,
A4. $\tau \in [1, 2]$,
A5. $\sigma \in [2-\tau, 1]$, $\delta > 0$, $\mu \equiv \frac{\delta}{n+\delta}$, $3 - \mu - \frac{n}{2} > 0$, $4 < \mu t$, $\frac{1}{t} + \frac{1}{s} = 1$,

any solution of (1.1) satisfies

$$\begin{aligned} &[U]_{C^{0,\mu/4}_{x'}(\overline{\mathcal{D}^\epsilon_f})} + \omega^\sigma [U]_{C^{0,\mu/4}_{x'}(\overline{\mathcal{D}^\epsilon_m})} + \|\mathbf{K}_{\omega^{\tau/p},\epsilon}\partial_{x_{n+1}} U\|_{L^p(\mathcal{D})} \\ &\quad \leq c\left(\sum_{k=1}^\infty \left(\|\mathbf{K}_{\omega^{\sigma-2},\epsilon}\mathbb{F}_k\|_{L^{n+\delta}(\Omega)} + \mathcal{H}_k\right)^s\right)^{1/s}, \end{aligned} \tag{2.5}$$

where c is a constant independent of ω, ϵ. *See (2.4) for* $\mathbf{K}_{\omega,\nu}$. *Here* $\mathbb{F}_k : \Omega \to \mathbb{R}$ *for* $k \in \mathbb{N}$ *are the Fourier sine coefficients of F defined as*

$$F(x', x_{n+1}) = \sum_{k=1}^\infty \mathbb{F}_k(x') \sin\left(\frac{k\pi}{L} x_{n+1}\right) \tag{2.6}$$

and

$$p \equiv \begin{cases} 2 & \text{if } n = 2 \\ \frac{2n}{n-2} & \text{if } n \geq 3 \end{cases}, \qquad \mathcal{H}_k \equiv \begin{cases} 0 & \text{if } n = 2 \\ \|\mathbf{K}_{\omega^{\frac{-\tau(n+2)}{2n}},\epsilon}\mathbb{F}_k\|_{L^{\frac{2n}{n-2}}(\Omega)} & \text{if } n \geq 3 \end{cases}.$$

Because of the requirement $3 - \mu - \frac{n}{2} > 0$, the possible dimensions that Theorem 2.1 holds are 2, 3, 4, 5. Theorem 2.1 implies that the Hölder norm of the solution of (1.1) in $\mathcal{D}^\epsilon_f$ is bounded independent of ω, ϵ, τ if the right hand side of (2.5) is bounded. In general, this is not for the solution in the anisotropic fibres $\mathcal{D}^\epsilon_m$. In the special case $\tau = 2$ and $\sigma = 0$, the Hölder norm of the solution of (1.1) in the horizontal directions is bounded uniformly in ω, ϵ, τ in the whole domain $\mathcal{D}$ if the right hand side of (2.5) is bounded.

Theorem 2.2. *Suppose A1–A3, $\tau \in [0, 2]$, and*

A6. $2 < q < p < \infty$, $\frac{2(1-\lambda)q}{2-q\lambda} = p$, $\lambda \in (0,1)$, $\lambda t > 1$, $\frac{1}{t} + \frac{1}{s} = 1$,

any solution of (1.1) satisfies

$$\begin{aligned} &\|\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)+2},\epsilon} \partial_{x'} U\|_{L^q(\mathcal{D})} + \|\mathbf{K}_{\omega^{\frac{\tau}{p}},\epsilon} \partial_{x_{n+1}} U\|_{L^p(\mathcal{D})} \\ &\quad \le c \left(\sum_{k=1}^{\infty} \|\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)},\epsilon} \mathbb{F}_k\|^s_{L^p(\Omega)} \right)^{1/s}, \end{aligned} \tag{2.7}$$

where c is a constant independent of ω, ϵ, τ. See (2.4) for $\mathbf{K}_{\omega,\nu}$ and (2.6) for $\mathbb{F}_k$.

Different from Theorem 2.1, Theorem 2.2 holds for $p \in (2, \infty)$, $n \ge 2$, and $\tau \in [0, 2]$. Similar to Theorem 2.1, Theorem 2.2 implies that the gradient L^p estimate of the solution of (1.1) in $\mathcal{D}^\epsilon_f$ is bounded independently of ω, ϵ, τ if the right hand side of (2.7) is bounded. But this is not the case for the solution in the anisotropic fibres $\mathcal{D}^\epsilon_m$. Also note that the solution in the anisotropic fibres has higher regularity along the fibres than in the transverse directions.

By Sobolev imbedding Theorem [11] and Theorem 2.2, we see

Corollary 2.1. *Besides the assumptions of Theorem 2.2, if*

A7. $n < q$, $\mu \equiv 1 - \frac{n}{q}$,

any solution of (1.1) satisfies

$$[U]_{C^{0,\mu}(\overline{\mathcal{D}^\epsilon_f})} + \omega^{\tau(\frac{1}{p}-1)+2} [U]_{C^{0,\mu}(\overline{\mathcal{D}^\epsilon_m})} \le c \left(\sum_{k=1}^{\infty} \|\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)},\epsilon} \mathbb{F}_k\|^s_{L^p(\Omega)} \right)^{1/s},$$

where c is a constant independent of ω, ϵ, τ. See (2.4) for $\mathbf{K}_{\omega,\nu}$ and (2.6) for $\mathbb{F}_k$.

Corollary 2.1 holds for $p \in (2, \infty)$, $n \ge 2$ and $\tau \in [0, 2]$ but it does not guarantee that the Hölder norm of the solution of (1.1) in $\mathcal{D}^\epsilon_m$ is bounded uniformly in ω, ϵ, τ even when $\tau = 2$ case. By Theorem 2.2 and a limiting argument, we also obtain the following result for problems in perforated domains:

Corollary 2.2. *Suppose A1–A3 and A6, any solution of*

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{0,\epsilon} \nabla U) = F & \text{in } \mathcal{D}^\epsilon_f \\ \mathbf{K}_{0,\epsilon} \nabla U \cdot \vec{\mathbf{n}}_\epsilon = 0 & \text{on } \partial \mathcal{D}^\epsilon_f \setminus \partial \mathcal{D} \\ U = 0 & \text{on } \partial \mathcal{D}^\epsilon_f \cap \partial \mathcal{D} \end{cases}$$

satisfies

$$\|\partial_{x'} U\|_{L^q(\mathcal{D}_f^\epsilon)} + \|\partial_{x_{n+1}} U\|_{L^p(\mathcal{D}_f^\epsilon)} \le c\Big(\sum_{k=1}^{\infty} \|\mathbb{F}_k\|^s_{L^p(\Omega_f^\epsilon)}\Big)^{1/s},$$

where c is a constant independent of ϵ. $\vec{\mathbf{n}}_\epsilon$ is the unit vector normal to $\partial\mathcal{D}_f^\epsilon$, see (2.4) for $\mathbf{K}_{\omega,\nu}$, and see (2.6) for $\mathbb{F}_k$.

Next we give a uniform gradient L^p estimate for the elliptic solution of (1.1). The estimate holds in the whole domain and in all dimensions $n \ge 2$.

Theorem 2.3. *Suppose A1–A3, $\tau = 2$, and*

A8. $2 < q < p < \infty$, $\alpha, \lambda \in (0,1)$, $\begin{cases} \alpha p < 2 & \text{if } n = 2 \\ \frac{2n}{n-2} = \frac{2(1-\alpha)p}{2-\alpha p} & \text{if } n \ge 3 \end{cases}$, $\frac{2(1-\lambda)q}{2-q\lambda} = p$, $(\alpha+\lambda-1)t > 1$, $\frac{1}{t} + \frac{1}{s} = 1$,

any solution of (1.1) satisfies

$$\|\partial_{x'} U\|_{L^q(\mathcal{D})} + \|\partial_{x_{n+1}} U\|_{L^p(\mathcal{D})} \le c\Big(\sum_{k=1}^{\infty} \|\mathbf{K}_{\omega^{-2},\epsilon}\mathbb{F}_k\|^s_{L^p(\Omega)}\Big)^{1/s}, \tag{2.8}$$

where c is a constant independent of ω, ϵ. See (2.4) for $\mathbf{K}_{\omega,\nu}$ and (2.6) for $\mathbb{F}_k$.

In Theorem 2.3, the number q in (2.8) can not be large. When dimension n becomes large, q becomes small and is closer to 2.

3. Proof of the main result

First we give some auxiliary lemmas.

Lemma 3.1. *Assume A1–A4, $\sigma \in [2-\tau, 1]$, $\delta, \Lambda \in (0,\infty)$, $3 - \mu - \frac{n}{2} > 0$. Any solution of*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon}\nabla\Phi) + \Lambda^2 \mathbf{K}_{\omega^\tau,\epsilon}\Phi = G & \text{in } \Omega \\ \Phi = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

satisfies

$$\max\{1, \Lambda^{\mu/4}\}\big([\Phi]_{C^{0,\mu/4}(\overline{\Omega_f^\epsilon})} + \omega^\sigma [\Phi]_{C^{0,\mu/4}(\overline{\Omega_m^\epsilon})}\big) \le c\big(\|\mathbf{K}_{\omega^{\sigma-2},\epsilon} G\|_{L^{n+\delta}(\Omega)} + \mathcal{H}\big),$$

where c is independent of $\omega, \epsilon, \Lambda$. See (2.4) for $\mathbf{K}_{\omega,\nu}$. Here $\mu \equiv \frac{\delta}{n+\delta}$ and

$$\mathcal{H} \equiv \begin{cases} 0 & \text{if } n = 2, \\ \|\mathbf{K}_{\omega^{\frac{-\tau(n+2)}{2n}},\epsilon} G\|_{L^{\frac{2n}{n-2}}(\Omega)} & \text{if } n \ge 3. \end{cases}$$

Proof of Lemma 3.1 is given in section 5.

Lemma 3.2. *Assume A1–A3, $\tau \in [0,2]$, $\sigma \in [1-\frac{\tau}{2},\infty)$, $\Lambda \in (0,\infty)$, $G \in L^p(\Omega)$, and $p \in [2,\infty)$. Any solution of (3.1) satisfies*

$$\begin{cases} \left\| \mathbf{K}_{\omega,\epsilon} \Lambda \nabla\Phi |\Lambda^2\Phi|^{\frac{p}{2}-1} \right\|^{2/p}_{L^2(\Omega)} + \|\mathbf{K}_{\omega^{\frac{\tau}{p}},\epsilon} \Lambda^2\Phi\|_{L^p(\Omega)} \le c\|\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)},\epsilon} G\|_{L^p(\Omega)}, \\ \|\Lambda \mathbf{K}_{\omega^\sigma,\epsilon}\nabla\Phi,\, \Lambda^2\mathbf{K}_{\omega^{\frac{\tau}{2}+\sigma-1},\epsilon}\Phi\|_{L^2(\Omega)} \le c\|\mathbf{K}_{\omega^{-\frac{\tau}{2}},\epsilon}G,\, \mathbf{K}_{\omega^{\sigma-\frac{\tau}{2}-1},\epsilon}G\|_{L^2(\Omega)}, \end{cases} \tag{3.2}$$

where c is independent of $\omega, \epsilon, \tau, \sigma, \Lambda, p$.

Remark 3.1. Assume A1–A3, $\tau = 2$, $\sigma = 0$, and $\Lambda \in (0,\infty)$. Poincaré inequality and $(3.2)_2$ imply

$$\|\Lambda\nabla\Phi,\, \Lambda^2\Phi,\, \Lambda\Phi\|_{L^2(\Omega)} \le c\|\mathbf{K}_{\omega^{-2},\epsilon}G\|_{L^2(\Omega)}. \tag{3.3}$$

(3.3) implies, by Sobolev imbedding Theorem [11],

$$\begin{cases} \|\Lambda\Phi\|_{L^2(\Omega)} \le c\min\{1,\Lambda^{-1}\}\|\mathbf{K}_{\omega^{-2},\epsilon}G\|_{L^2(\Omega)}, & \\ \|\Phi\|_{L^r(\Omega)} \le c\Lambda^{-1}\|\mathbf{K}_{\omega^{-2},\epsilon}G\|_{L^2(\Omega)} & \text{for any } r>2 \text{ if } n=2, \\ \|\Phi\|_{L^{\frac{2n}{n-2}}(\Omega)} \le c\Lambda^{-1}\|\mathbf{K}_{\omega^{-2},\epsilon}G\|_{L^2(\Omega)} & \text{if } n\ge 3. \end{cases} \tag{3.4}$$

Let $\alpha \in (0,1)$ and $p > 2$ such that $\begin{cases} \alpha p < 2 & \text{if } n=2 \\ \frac{2n}{n-2} = \frac{2(1-\alpha)p}{2-\alpha p} & \text{if } n \ge 3 \end{cases}$. Then, by (3.4),

$$\begin{aligned} \int_\Omega |\Lambda^\alpha\Phi|^p = \int_\Omega |\Lambda\Phi|^{\alpha p}\Phi^{(1-\alpha)p} &\le \|\Lambda\Phi\|^{\alpha p}_{L^2(\Omega)}\|\Phi\|^{(1-\alpha)p}_{L^{\frac{2(1-\alpha)p}{2-\alpha p}}(\Omega)} \\ &\le c\min\{\Lambda^{-p},\Lambda^{(\alpha-1)p}\}\|\mathbf{K}_{\omega^{-2},\epsilon}G\|^p_{L^2(\Omega)}. \end{aligned}$$

So we obtain $\|\Lambda^2\Phi\|_{L^p(\Omega)} \le c\min\{\Lambda,\Lambda^{1-\alpha}\}\|\mathbf{K}_{\omega^{-2},\epsilon}G\|_{L^2(\Omega)}$.

Proof. $(3.2)_1$ is proved by multiplying (3.1) by $|\Lambda^2\Phi|^{p-2}\Lambda^2\Phi$ as well as employing integration by parts and Hölder inequality. Next we consider the following equation:

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon}\nabla\Phi) + \Lambda^2\mathbf{K}_{\omega^\tau,\epsilon}\Phi = G & \text{in } \Omega^\epsilon_m, \\ \Phi = \Pi_\epsilon\Phi|_{\Omega^\epsilon_f} & \text{on } \partial\Omega^\epsilon_m. \end{cases}$$

See section 2 for the extension function $\Pi_\epsilon\Phi|_{\Omega^\epsilon_f}$. Let $\varphi \equiv \Phi - \Pi_\epsilon\Phi|_{\Omega^\epsilon_f}$ in Ω^ϵ_m, then

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon}\nabla\varphi + \mathbf{K}_{\omega^2,\epsilon}\nabla\Pi_\epsilon\Phi|_{\Omega^\epsilon_f}) + \Lambda^2\mathbf{K}_{\omega^\tau,\epsilon}\,\varphi & \\ \qquad = G - \Lambda^2\mathbf{K}_{\omega^\tau,\epsilon}\Pi_\epsilon\Phi|_{\Omega^\epsilon_f} & \text{in } \Omega^\epsilon_m, \\ \varphi = 0 & \text{on } \partial\Omega^\epsilon_m. \end{cases}$$

Multiply above by $\Lambda^2\varphi$ to get, by energy method and extension theorem [1],

$$\|\Lambda\nabla\Phi, \omega^{\frac{\tau}{2}-1}\Lambda^2\Phi\|_{L^2(\Omega_m^\epsilon)} \le c(\omega^{-\frac{\tau}{2}-1}\|G\|_{L^2(\Omega_m^\epsilon)} + \|\Lambda\nabla\Phi, \omega^{\frac{\tau}{2}-1}\Lambda^2\Phi\|_{L^2(\Omega_f^\epsilon)}).$$

Together with $(3.2)_1$, we get $(3.2)_2$. $\square$

Lemma 3.3. *Under A1–A3, $\sigma \in [0, 2]$, $p \in (1, \infty)$, $Q \in L^p(\Omega)$, and $G \in W^{-1,p}(\Omega)$, a $W^{1,p}(\Omega)$ solution of*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon}\nabla\Phi + Q) = G & \text{in } \Omega \\ \Phi = 0 & \text{on } \partial\Omega \end{cases}$$

exists uniquely and satisfies

$$\begin{cases} \|\mathbf{K}_{\omega^\sigma/\epsilon,\epsilon}\Phi, \mathbf{K}_{\omega^\sigma,\epsilon}\nabla\Phi\|_{L^p(\Omega)} \le c(\|\mathbf{K}_{\omega^{\sigma-2},\epsilon}Q\|_{L^p(\Omega)} & \\ \qquad +\|G\|_{W^{-1,p}(\Omega)} + \omega^{\sigma-2}\|G\|_{W^{-1,p}(\Omega_m^\epsilon)}) & \text{if } \frac{\omega^\sigma}{\epsilon} \le 1, \\ \|\Phi, \mathbf{K}_{\omega^\sigma,\epsilon}\nabla\Phi\|_{L^p(\Omega)} \le c(\|\mathbf{K}_{\omega^{\sigma-2},\epsilon}Q\|_{L^p(\Omega)} & \\ \qquad +\|G\|_{W^{-1,p}(\Omega)} + \omega^{\sigma-2}\|G\|_{W^{-1,p}(\Omega_m^\epsilon)}) & \text{if } \frac{\omega^\sigma}{\epsilon} \ge 1, \end{cases}$$

where c is a constant independent of ω, ϵ, σ. See (2.4) for $\mathbf{K}_{\omega,\nu}$.

Proof of Lemma 3.3 is given in section 6.

Lemma 3.4. *Assume A1–A3, $\tau \in [0, 2]$, $\Lambda \in (0, \infty)$, $2 < q < p < \infty$, $\frac{2(1-\lambda)q}{2-q\lambda} = p$, $\lambda \in (0, 1)$, and $G \in L^p(\Omega)$. Any solution of (3.1) satisfies*

$$\|\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)+2},\epsilon}\Lambda^\lambda\nabla\Phi\|_{L^q(\Omega)} \le c\|\mathbf{K}_{\omega^{-\frac{\tau}{2}},\epsilon}G, \mathbf{K}_{\omega^{\tau(\frac{1}{p}-\frac{3}{2})+1},\epsilon}G\|^{\lambda}_{L^2(\Omega)}\|\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)},\epsilon}G\|^{1-\lambda}_{L^p(\Omega)},$$

where c is independent of $\omega, \epsilon, \tau, \Lambda$.

Proof. Let c denote a constant independent of $\omega, \epsilon, \tau, \Lambda$. By (3.2) of Lemma 3.2,

$$\begin{cases} \|\mathbf{K}_{\omega^{\frac{\tau}{p}},\epsilon}\Lambda^2\Phi\|_{L^p(\Omega)} \le c\|\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)},\epsilon}G\|_{L^p(\Omega)}, \\ \|\Lambda\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)+2},\epsilon}\nabla\Phi\|_{L^2(\Omega)} \le c\|\mathbf{K}_{\omega^{-\frac{\tau}{2}},\epsilon}G, \ \mathbf{K}_{\omega^{\tau(\frac{1}{p}-\frac{3}{2})+1},\epsilon}G\|_{L^2(\Omega)}. \end{cases} \tag{3.5}$$

We write (3.1) as

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon}\nabla\Phi) = G - \Lambda^2\mathbf{K}_{\omega^\tau,\epsilon}\Phi & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 3.3 and $(3.5)_1$,

$$\|\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)+2},\epsilon}\nabla\Phi\|_{L^p(\Omega)} \le c\|\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)},\epsilon}G\|_{L^p(\Omega)}. \tag{3.6}$$

Since $\lambda + \frac{p(2-q\lambda)}{2q} = 1$, we see, by $(3.5)_2$ and (3.6),

$$\int_\Omega |\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)+2},\epsilon}\Lambda^\lambda\nabla\Phi|^q dx = \int_\Omega |\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)+2},\epsilon}\Lambda\nabla\Phi|^{q\lambda}|\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)+2},\epsilon}\nabla\Phi|^{(1-\lambda)q}dx$$
$$\leq c\|\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)+2},\epsilon}\Lambda\nabla\Phi\|^{q\lambda}_{L^2(\Omega)}\|\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)+2},\epsilon}\nabla\Phi\|^{q(1-\lambda)}_{L^p(\Omega)}$$
$$\leq c\|\mathbf{K}_{\omega^{-\frac{\tau}{2}},\epsilon}G,\ \mathbf{K}_{\omega^{\tau(\frac{1}{p}-\frac{3}{2})+1},\epsilon}G\|^{q\lambda}_{L^2(\Omega)}\|\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)},\epsilon}G\|^{q(1-\lambda)}_{L^p(\Omega)}.$$

The lemma follows from the above inequality. □

Lemma 3.5. *Assume A1–A3, $\tau = 2$, $\Lambda \in (0,\infty)$, $2 < q < p < \infty$, $\frac{2(1-\lambda)q}{2-q\lambda} = p$, $\alpha, \lambda \in (0,1)$, and $\begin{cases} \alpha p < 2 & \text{if } n = 2 \\ \frac{2n}{n-2} = \frac{2(1-\alpha)p}{2-\alpha p} & \text{if } n \geq 3 \end{cases}$. Any solution of (3.1) satisfies*

$$\|\Lambda^\lambda\nabla\Phi\|_{L^q(\Omega)} \leq c(\|\mathbf{K}_{\omega^{-2},\epsilon}G\|_{L^p(\Omega)} + \min\{\Lambda, \Lambda^{1-\alpha}\}\|\mathbf{K}_{\omega^{-2},\epsilon}G\|_{L^2(\Omega)}),$$

where c is independent of $\omega, \epsilon, \Lambda$.

Proof. Let c denote a constant independent of $\omega, \epsilon, \Lambda$. By $(3.2)_2$ of Lemma 3.2,

$$\|\Lambda\nabla\Phi\|_{L^2(\Omega)} \leq c\|\mathbf{K}_{\omega^{-2},\epsilon}G\|_{L^2(\Omega)}. \tag{3.7}$$

By Remark 3.1,

$$\|\Lambda^2\Phi\|_{L^p(\Omega)} \leq c\min\{\Lambda, \Lambda^{1-\alpha}\}\|\mathbf{K}_{\omega^{-2},\epsilon}G\|_{L^2(\Omega)}. \tag{3.8}$$

We write (3.1) as

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon}\nabla\Phi) = G - \Lambda^2\mathbf{K}_{\omega^2,\epsilon}\Phi & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 3.3 and (3.8),

$$\|\nabla\Phi\|_{L^p(\Omega)} \leq c(\|\mathbf{K}_{\omega^{-2},\epsilon}G\|_{L^p(\Omega)} + \min\{\Lambda, \Lambda^{1-\alpha}\}\|\mathbf{K}_{\omega^{-2},\epsilon}G\|_{L^2(\Omega)}). \tag{3.9}$$

Since $2 < q < p$ and $\lambda + \frac{p(2-q\lambda)}{2q} = 1$, we obtain, by (3.7) and (3.9),

$$\int_\Omega |\Lambda^\lambda\nabla\Phi|^q dx = \int_\Omega |\Lambda\nabla\Phi|^{q\lambda}|\nabla\Phi|^{(1-\lambda)q}dx \leq c\|\Lambda\nabla\Phi\|^{q\lambda}_{L^2(\Omega)}\|\nabla\Phi\|^{q(1-\lambda)}_{L^p(\Omega)}$$
$$\leq c\|\mathbf{K}_{\omega^{-2},\epsilon}G\|^{q\lambda}_{L^2(\Omega)}(\|\mathbf{K}_{\omega^{-2},\epsilon}G\|_{L^p(\Omega)} + \min\{\Lambda, \Lambda^{1-\alpha}\}\|\mathbf{K}_{\omega^{-2},\epsilon}G\|_{L^2(\Omega)})^{q(1-\lambda)}.$$

Which implies the lemma. □

To study the uniform estimate of equation (1.1), we reduce (1.1) to n-dimensional problems by representing functions in terms of sine functions. Write

$$\begin{cases} U(x', x_{n+1}) = \sum_{k=1}^{\infty} \mathbb{U}_k(x') \sin\big(\frac{k\pi}{L} x_{n+1}\big), \\ F(x', x_{n+1}) = \sum_{k=1}^{\infty} \mathbb{F}_k(x') \sin\big(\frac{k\pi}{L} x_{n+1}\big). \end{cases}$$

Then equation (1.1) implies, for any $k \in \mathbb{N}$,

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \mathbb{U}_k) + \frac{k^2\pi^2}{L^2} \mathbf{K}_{\omega^\tau,\epsilon} \mathbb{U}_k = \mathbb{F}_k & \text{in } \Omega, \\ \mathbb{U}_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

Proof of Theorem 2.1. Under the assumptions of Theorem 2.1, the solution of (3.10) satisfies, by Lemma 3.1 and Lemma 3.2,

$$\begin{aligned} &k^{\mu/4}\big([\mathbb{U}_k]_{C^{0,\mu/4}(\overline{\Omega^\epsilon_f})} + \omega^\sigma [\mathbb{U}_k]_{C^{0,\mu/4}(\overline{\Omega^\epsilon_m})}\big) + k^2 \|\mathbf{K}_{\omega^{\tau/p},\epsilon} \mathbb{U}_k\|_{L^p(\Omega)} \\ &\quad \le c(\|\mathbf{K}_{\omega^{\sigma-2},\epsilon} \mathbb{F}_k\|_{L^{n+\delta}(\Omega)} + \mathcal{H}_k), \end{aligned}$$

where c is independent of $\omega, \epsilon, \tau, \sigma, k$. See Theorem 2.1 for $\sigma, \mu, \delta, p, \mathcal{H}_k, t, s$. Since $4 < \mu t$ and $\frac{1}{t} + \frac{1}{s} = 1$,

$$\begin{aligned} &[U]_{C^{0,\mu/4}_{x'}(\overline{\mathcal{D}^\epsilon_f})} + \omega^\sigma [U]_{C^{0,\mu/4}_{x'}(\overline{\mathcal{D}^\epsilon_m})} + \|\mathbf{K}_{\omega^{\tau/p},\epsilon} \partial_{x_{n+1}} U\|_{L^p(\mathcal{D})} \\ &\quad \le c \sum_{k=1}^{\infty} ([\mathbb{U}_k]_{C^{0,\mu/4}(\overline{\Omega^\epsilon_f})} + \omega^\sigma [\mathbb{U}_k]_{C^{0,\mu/4}(\overline{\Omega^\epsilon_m})} + k \|\mathbf{K}_{\omega^{\tau/p},\epsilon} \mathbb{U}_k\|_{L^p(\Omega)}) \\ &\quad \le c \sum_{k=1}^{\infty} k^{-\mu/4}\big(\|\mathbf{K}_{\omega^{\sigma-2},\epsilon} \mathbb{F}_k\|_{L^{n+\delta}(\Omega)} + \mathcal{H}_k\big) \\ &\quad \le c \Big(\sum_{k=1}^{\infty} k^{-\mu t/4}\Big)^{1/t} \Big(\sum_{k=1}^{\infty} \big(\|\mathbf{K}_{\omega^{\sigma-2},\epsilon} \mathbb{F}_k\|_{L^{n+\delta}(\Omega)} + \mathcal{H}_k\big)^s\Big)^{1/s}. \end{aligned}$$

Which implies Theorem 2.1. □

Proof of Theorem 2.2. Under the assumptions of Theorem 2.2, the solution of (3.10) satisfies, by Lemmas 3.2, 3.4,

$$k^\lambda \|\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)+2},\epsilon} \nabla \mathbb{U}_k\|_{L^q(\Omega)} + k^2 \|\mathbf{K}_{\omega^{\frac{\tau}{p}},\epsilon} \mathbb{U}_k\|_{L^p(\Omega)} \le c \|\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)},\epsilon} \mathbb{F}_k\|_{L^p(\Omega)},$$

where $k \in \mathbb{N}$ and c is independent of $\omega, \epsilon, \tau, k$. See Theorem 2.2 for λ, p, q, t, s. If $1 < \lambda t$ and $\frac{1}{t} + \frac{1}{s} = 1$,

$$
\begin{aligned}
&\|\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)+2},\epsilon}\partial_{x'}U\|_{L^q(\mathcal{D})}+\|\mathbf{K}_{\omega^{\frac{\tau}{p}},\epsilon}\partial_{x_{n+1}}U\|_{L^p(\mathcal{D})}\\
&\quad\le\sum_{k=1}^{\infty}\|\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)+2},\epsilon}\nabla\mathbb{U}_k\|_{L^q(\Omega)}+k\|\mathbf{K}_{\omega^{\frac{\tau}{p}},\epsilon}\mathbb{U}_k\|_{L^p(\Omega)}\\
&\quad\le c\sum_{k=1}^{\infty}k^{-\lambda}\|\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)},\epsilon}\mathbb{F}_k\|_{L^p(\Omega)}\\
&\quad\le c\Big(\sum_{k=1}^{\infty}k^{-\lambda t}\Big)^{1/t}\Big(\sum_{k=1}^{\infty}\|\mathbf{K}_{\omega^{\tau(\frac{1}{p}-1)},\epsilon}\mathbb{F}_k\|^s_{L^p(\Omega)}\Big)^{1/s}.
\end{aligned}
\tag{3.11}
$$

Theorem 2.2 follows from (3.11). □

Proof of Theorem 2.3. Under the assumptions of Theorem 2.3, the solution of (3.10) satisfies, by Remark 3.1 and Lemma 3.5,

$$
k^{\lambda}\|\nabla\mathbb{U}_k\|_{L^q(\Omega)}+k^2\|\mathbb{U}_k\|_{L^p(\Omega)}\le c(\|\mathbf{K}_{\omega^{-2},\epsilon}\mathbb{F}_k\|_{L^p(\Omega)}+k^{1-\alpha}\|\mathbf{K}_{\omega^{-2},\epsilon}\mathbb{F}_k\|_{L^2(\Omega)}),
$$

where c is independent of ω,ϵ,τ,k. See Theorem 2.3 for λ,p,q,t,s,α. Since we know $(\alpha+\lambda-1)t>1$ and $\frac{1}{t}+\frac{1}{s}=1$,

$$
\begin{aligned}
&\|\partial_{x'}U\|_{L^q(\mathcal{D})}+\|\partial_{x_{n+1}}U\|_{L^p(\mathcal{D})}\\
&\quad\le\sum_{k=1}^{\infty}\|\nabla\mathbb{U}_k\|_{L^q(\Omega)}+k\|\mathbb{U}_k\|_{L^p(\Omega)}\le c\sum_{k=1}^{\infty}k^{-(\alpha+\lambda-1)}\|\mathbf{K}_{\omega^{-2},\epsilon}\mathbb{F}_k\|_{L^p(\Omega)}\\
&\quad\le c\Big(\sum_{k=1}^{\infty}k^{-(\alpha+\lambda-1)t}\Big)^{1/t}\Big(\sum_{k=1}^{\infty}\|\mathbf{K}_{\omega^{-2},\epsilon}\mathbb{F}_k\|^s_{L^p(\Omega)}\Big)^{1/s}.
\end{aligned}
\tag{3.12}
$$

Theorem 2.3 follows from (3.12). □

4. A priori estimates

In this section, we study the regularity of the solutions of some interface problems. Two main results are obtained. The first one is a Hölder estimate for a Helmholtz-type interface problem (i.e., Lemma 4.7) and the second one is a gradient Hölder estimate for a Poisson interface problem (i.e., Remark 4.2). Let $r>0$, $\mathcal{I}_r(z)\equiv\{y=(y_1,\hat{y})\in B_r(z)|y_1=0\}$, and

$$
\mathbb{T}_{\omega}(y)\equiv\begin{cases}A_1 & \text{if } y_1\ge 0,\\ \omega A_2 & \text{if } y_1<0,\end{cases}
\tag{4.1}
$$

where $\omega\in(0,1]$ and A_1,A_2 are positive definite constant matrices.

Lemma 4.1. *Let $\omega,r\in(0,1]$, $\tau\in[0,2]$, $\sigma\in[1-\frac{\tau}{2},2]$, $\Lambda\in[0,\infty)$. The solution of*

$$
\begin{cases}-\nabla\cdot(\mathbb{T}_{\omega^2}\nabla\phi+\mathbb{Q})+\Lambda^2\mathbb{T}_{\omega^{\tau}}\phi=\mathbb{G} & \text{in } B_r(0)\\ \phi=0 & \text{on } \partial B_r(0)\end{cases}
$$

satisfies

$$\|\mathbb{T}_{\omega^\sigma}\nabla\phi,\, \Lambda\mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}}\phi\|_{L^2(B_r(0))} \le c\left(\|\mathbb{T}_{\omega^{\sigma-2}}\mathbb{Q}\|_{L^2(B_r(0))} + \mathcal{G}\right), \tag{4.2}$$

where c is a constant independent of $\omega, \tau, \sigma, r, \Lambda$. Here $\mathcal{G}$ is one of the following expressions:

$$\mathcal{G} \equiv \begin{cases} \|\mathbb{T}_{\omega^{\sigma-2}}\mathbb{G}\|_{H^{-1}(B_r^+(0))\cup H^{-1}(B_r^-(0))}, \\ r\|\mathbb{T}_{\omega^{\sigma-2}}\mathbb{G}\|_{L^2(B_r(0))}, \\ \|\Lambda^{-1}\mathbb{T}_{\omega^{\sigma-\frac{\tau}{2}-1}}\mathbb{G}\|_{L^2(B_r(0))} & \text{if } \Lambda > 0. \end{cases} \tag{4.3}$$

Proof. Let c denote a constant independent of $\omega, \tau, \sigma, r, \Lambda$. First we consider the case $\mathbb{Q} \in H_0^1(B_r^+(0)) \cup H_0^1(B_r^-(0))$. We find a $\eta \in H_0^1(B_r^+(0)) \cup H_0^1(B_r^-(0))$ such that

$$-\nabla\cdot(\mathbb{T}_{\omega^2}\nabla\eta + \mathbb{Q}) + \Lambda^2\mathbb{T}_{\omega^\tau}\eta = \mathbb{G} \qquad \text{in } B_r^+(0)\cup B_r^-(0).$$

By energy method,

$$\|\mathbb{T}_{\omega^\sigma}\nabla\eta,\ \Lambda\mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}}\eta\|_{L^2(\mathbf{S})} \le c\left(\|\mathbb{T}_{\omega^{\sigma-2}}\mathbb{Q}\|_{L^2(B_r(0))} + \mathcal{G}\right), \tag{4.4}$$

where $\mathcal{G}$ is defined in (4.3) and $\mathbf{S} = B_r^+(0)$ or $B_r^-(0)$. Suppose $\psi = \phi - \eta$, then

$$\begin{cases} -\nabla\cdot(\mathbb{T}_{\omega^2}\nabla\psi) + \Lambda^2\mathbb{T}_{\omega^\tau}\psi = 0 & \text{in } B_r^+(0)\cup B_r^-(0), \\ \psi = 0 & \text{on } \partial B_r(0), \\ \lfloor\psi\rfloor = 0 & \text{on } B_r(0)\cap\mathcal{I}_1(0), \\ \lfloor\mathbb{T}_{\omega^2}\nabla\psi\cdot\vec{e}_1\rfloor = -\lfloor\mathbb{T}_{\omega^2}\nabla\eta\cdot\vec{e}_1\rfloor \equiv \zeta & \text{on } B_r(0)\cap\mathcal{I}_1(0), \end{cases} \tag{4.5}$$

where $\vec{e}_1$ is the unit normal vector on $\mathcal{I}_1(0)$. See (2.3) for $\lfloor\psi\rfloor$. Let $\widetilde{\psi}$ denote the even extension function of $\psi|_{B_r^+(0)}$ with respect to $y_1 = 0$ in $B_r(0)$. Multiply $(4.5)_1$ by $\psi - \widetilde{\psi}$ and use integration by parts to get

$$\begin{aligned} \|\mathbb{T}_{\omega^\sigma}\nabla\psi, \Lambda\mathbb{T}_{\omega^{\sigma+\tau/2-1}}\psi\|_{L^2(B_r(0))} &\le c\|\nabla\psi, \Lambda\psi\|_{L^2(B_r^+(0))} \\ &\le c\|\zeta\|_{H^{-1/2}(B_r(0)\cap\mathcal{I}_1(0))}. \end{aligned} \tag{4.6}$$

In (4.6), we use trace theorem and Poincaré inequality [11]. (4.4) and (4.6) imply (4.2). For general $\mathbb{Q} \in L^2(B_r(0))$ case, (4.2) is proved by a limiting argument. □

Lemma 4.2. *Let $\omega \in (0,1]$, $\tau \in [0,2]$, $\sigma \in [1-\frac{\tau}{2}, 2]$, $0 < r \le \frac{1}{3}$, $\Lambda \in (0,\infty)$, $z \in \mathcal{I}_{1/3}(0)$, $i \in \mathbb{N}$. There is a constant c independent of $\omega, \tau, \sigma, r, \Lambda, z$ such that any solution of*

$$-\nabla\cdot(\mathbb{T}_{\omega^2}\nabla\Phi) + \Lambda^2\mathbb{T}_{\omega^\tau}\Phi = 0 \quad \text{in } B_1(0) \tag{4.7}$$

satisfies

$$\begin{cases} \|\mathbb{T}_{\omega^\sigma}\nabla\Phi,\ \Lambda\mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}}\Phi\|_{L^2(B_r(z))} \le \frac{c}{r}\|\mathbb{T}_{\omega^\sigma}\Phi\|_{L^2(B_{2r}(z))}, \\ \|\Lambda\mathbb{T}_{\omega^\sigma}\nabla\Phi,\ \Lambda^2\mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}}\Phi\|_{L^2(B_r(z))} \le \frac{c}{r^2}\|\mathbb{T}_{\omega^\sigma}\Phi\|_{L^2(B_{2r}(z))}, \\ \|\mathbb{T}_{\omega^{\sigma+1-\frac{\tau}{2}}}\partial_1^2\Phi,\ \mathbb{T}_{\omega^\sigma}\partial_k\nabla\Phi,\ \Lambda\mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}}\partial_k\Phi\|_{L^2(B_r(z))} \le \frac{c}{r^2}\|\mathbb{T}_{\omega^\sigma}\Phi\|_{L^2(B_{2r}(z))}, \\ \|\nabla^{i+1}\Phi\|_{L^2(B_r^+(z))} \le \frac{c}{r^{i+1}}\|\mathbb{T}_{\omega^2}\Phi\|_{L^2(B_{2r}(z))}, \end{cases} \tag{4.8}$$

where $k \in \{2, \cdots, n\}$ and ∂_s means the partial derivative in the s-th direction.

Proof. Let c denote a constant independent of $\omega, \tau, \sigma, r, \Lambda, z$. Let $\eta \in C_0^\infty(B_{2r}(z))$ be a bell-shaped function satisfying $\eta \in [0, 1]$ and $\eta = 1$ in $B_r(z)$. Multiply (4.7) by η and employ $(4.3)_1$ of Lemma 4.1 to get $(4.8)_{1,2}$.

Differentiate (4.7) with respect to the variable y_k for $k \in \{2, \cdots, n\}$ and employ $(4.8)_1$ to get

$$\|\mathbb{T}_{\omega^\sigma}\partial_k\nabla\Phi,\ \Lambda\mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}}\partial_k\Phi\|_{L^2(B_r(z))} \le \frac{c}{r}\|\mathbb{T}_{\omega^\sigma}\partial_k\Phi\|_{L^2(B_{2r}(z))}. \tag{4.9}$$

By (4.7), $(4.8)_2$, and (4.9),

$$\|\mathbb{T}_{\omega^{\sigma-\frac{\tau}{2}+1}}\partial_1^2\Phi\|_{L^2(B_r(z))} \le \frac{c}{r^2}\|\mathbb{T}_{\omega^\sigma}\Phi\|_{L^2(B_{2r}(z))}.$$

So we get $(4.8)_3$. $(4.8)_4$ is proved by induction and a similar argument as $(4.8)_3$. $\square$

Lemma 4.3. *Let $\omega \in (0, 1]$, $\tau \in [0, 2]$, $0 < r < \frac{1}{3}$, $\Lambda \in (0, \infty)$, $z \in \mathcal{I}_{2/3}(0)$, $i \in \mathbb{N} \cup \{0\}$, $\Phi_{\mathcal{I}} \equiv \Phi|_{\mathcal{I}_1(0)}$. Any solution Φ of (4.7) satisfies*

$$\rho \int_{\mathcal{I}_\rho(z)} |\partial_k^i\Phi_{\mathcal{I}}|^2 d\hat{y} \le c\Big|\frac{\rho}{r}\Big|^n \int_{B_r(z)} |\mathbb{T}_{\omega^2}\partial_k^i\Phi|^2 dy \qquad \text{for } 0 < \rho \le \tfrac{r}{2}, \tag{4.10}$$

where c is independent of $\omega, \tau, \Lambda, r, z, \rho$. Here $k \in \{2, \cdots, n\}$ and $y = (y_1, \hat{y})$.

Proof. We fix $0 < \rho \le \frac{r}{2}$ and m is the smallest integer satisfying $2m > n - 1$. By $(4.8)_4$ of Lemma 4.2,

$$\begin{aligned} \rho \int_{\mathcal{I}_\rho(z)} |\Phi_{\mathcal{I}}|^2 d\hat{y} &\le c(n)\rho^n \|\Phi_{\mathcal{I}}\|^2_{L^\infty(\mathcal{I}_{\frac{r}{2}}(z))} \le c(n,r)\rho^n \|\Phi_{\mathcal{I}}\|^2_{H^m(\mathcal{I}_{\frac{r}{2}}(z))} \\ &\le c(n,r)\rho^n \sum_{\beta \le m} \|\partial_k^\beta\Phi\|^2_{L^2(\mathcal{I}_{\frac{r}{2}}(z))} \le c(n,r)\rho^n \sum_{\beta\le m} \|\partial_k^\beta\Phi\|^2_{H^1(B^+_{r/2}(z))} \\ &\le c(n,r,m)\rho^n \|\mathbb{T}_{\omega^2}\Phi\|^2_{L^2(B_r(z))}, \end{aligned}$$

where ∂_k^β means the β-th tangential derivative. By a similarity transformation, we find $c(n, r, m) \le \frac{c}{r^n}$. So we prove (4.10) for $i = 0$ case. For $i > 0$ case, (4.10) is proved in a similar way. $\square$

Lemma 4.4. *Let* $\omega \in (0,1]$, $\tau \in [0,2]$, $\sigma \in \mathbb{R}$, $0 < \rho \le r \le \frac{1}{6}$, $\Lambda \in [0,\infty)$, $z \in \mathcal{I}_{2/3}(0)$, $\Phi_{\mathcal{I}} \equiv \Phi|_{\mathcal{I}_1(0)}$. *Any solution* Φ *of (4.7) satisfies*

$$\begin{aligned}\int_{\mathbf{S}_\rho} \Big(|\mathbb{T}_{\omega^\sigma} \nabla\Phi|^2 + |\Lambda \mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}} \Phi|^2 \Big)\, dy \\ \le c\Big|\frac{\rho}{r}\Big|^n \int_{\mathbf{S}_r} \Big(|\mathbb{T}_{\omega^\sigma} \nabla\Phi|^2 + |\Lambda \mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}} \Phi|^2 \Big)\, dy \\ + c \int_{\mathbf{S}_r} \Big(|\mathbb{T}_{\omega^\sigma} \nabla\Phi_{\mathcal{I}}|^2 + |\Lambda \mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}} \Phi_{\mathcal{I}}|^2 \Big)\, dy,\end{aligned} \tag{4.11}$$

where c is independent of $\omega, \tau, \sigma, \rho, r, \Lambda, z$. *Here* $(\mathbf{S}_\rho, \mathbf{S}_r) \equiv (B^+_\rho(z), B^+_r(z))$ *or* $(B^-_\rho(z), B^-_r(z))$.

Proof. Suppose c is independent of $\omega, \tau, \sigma, \rho, r, \Lambda, z$. Let $y = (y_1, \hat{y}) \in \mathbb{R}^n$ and set $\psi(y_1, \hat{y}) \equiv \Phi(y_1, \hat{y}) - \Phi_{\mathcal{I}}(0, \hat{y})$. For any $\zeta \in H^1_0(B_1(0))$,

$$\int_{B_1(0)} \Big(\mathbb{T}_{\omega^2} \nabla\psi \nabla\zeta + \Lambda^2 \mathbb{T}_{\omega^\tau} \psi\zeta \Big)\, dy = - \int_{B_1(0)} \Big(\mathbb{T}_{\omega^2} \nabla\Phi_{\mathcal{I}} \nabla\zeta + \Lambda^2 \mathbb{T}_{\omega^\tau} \Phi_{\mathcal{I}}\zeta \Big)\, dy. \tag{4.12}$$

We find a $\phi \in H^1(B^+_r(z))$ such that

$$\begin{cases} -\nabla\cdot(\mathbb{T}_{\omega^2}\nabla\phi) + \Lambda^2 \mathbb{T}_{\omega^\tau}\phi = 0 & \text{in } B^+_r(z), \\ \phi = \psi & \text{on } \partial B^+_r(z). \end{cases} \tag{4.13}$$

By a similar argument as Theorem 6.2.4 [19] and Poincaré inequality, the solution of (4.13) satisfies

$$\int_{B^+_\rho(z)} (|\nabla\phi|^2 + |\Lambda\phi|^2) dy \le c\Big|\frac{\rho}{r}\Big|^n \int_{B^+_r(z)} (|\nabla\phi|^2 + |\Lambda\phi|^2) dy \qquad \text{for } \rho \le r. \tag{4.14}$$

Take $\zeta = (\psi - \phi)\mathcal{X}_{B^+_r(z)}$ in (4.12) as well as multiply $(4.13)_1$ by ζ and employ integration by parts to see

$$\begin{cases} \displaystyle\int_{B^+_r(z)} (|\nabla\zeta|^2 + |\Lambda\zeta|^2) dy \le c \int_{B^+_r(z)} (|\nabla\Phi_{\mathcal{I}}|^2 + |\Lambda\Phi_{\mathcal{I}}|^2) dy, \\ \displaystyle\int_{B^+_r(z)} (|\nabla\phi|^2 + |\Lambda\phi|^2) dy \le c \int_{B^+_r(z)} (|\nabla\psi|^2 + |\Lambda\psi|^2) dy. \end{cases} \tag{4.15}$$

Equations (4.14)–(4.15) imply, for $0 < \rho \le r$,

$$\int_{B^+_\rho(z)} (|\nabla\psi|^2 + |\Lambda\psi|^2)dy \le \int_{B^+_\rho(z)} (|\nabla\phi|^2 + |\Lambda\phi|^2 + |\nabla\zeta|^2 + |\Lambda\zeta|^2)dy$$
$$\le c\Big|\frac{\rho}{r}\Big|^n \int_{B^+_r(z)} (|\nabla\psi|^2 + |\Lambda\psi|^2)dy + c\int_{B^+_r(z)} (|\nabla\Phi_{\mathcal{I}}|^2 + |\Lambda\Phi_{\mathcal{I}}|^2)dy.$$

So we obtain the estimate (4.11) in the upper domain. The estimate (4.11) in the lower domain is proved in a similar way as above. □

The following is a Campanato-type estimate for the solution Φ of (4.7).

Lemma 4.5. *Let $\omega \in (0, 1]$, $\tau \in [0, 2]$, $\sigma \in [1 - \frac{\tau}{2}, 2]$, $0 < \rho < r < \frac{1}{3}$, $\Lambda \in (0, \infty)$, and $z \in \mathcal{I}_{2/3}(0)$. Any solution Φ of (4.7) satisfies, for $0 < \mathbf{e} \ll 1$,*

$$\int_{B_\rho(z)} (|\mathbb{T}_{\omega^\sigma}\nabla\Phi|^2 + |\Lambda\mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}}\Phi|^2)dy$$
$$\le c\Big|\frac{\rho}{r}\Big|^{n-\mathbf{e}} \int_{B_r(z)} (|\mathbb{T}_{\omega^\sigma}\nabla\Phi|^2 + |\Lambda\mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}}\Phi|^2)dy, \tag{4.16}$$

where c is independent of $\omega, \tau, \sigma, \rho, r, \Lambda, z, \mathbf{e}$.

Proof. We first consider $0 < \rho \le t \le \frac{r}{2}$ case. By Lemma 4.3,

$$\int_{B_t(z)} (|\partial_k\Phi_{\mathcal{I}}|^2 + |\Lambda\Phi_{\mathcal{I}}|^2)dy \le c_0\Big|\frac{t}{r}\Big|^n \int_{B_r(z)} (|\mathbb{T}_{\omega^2}\partial_k\Phi|^2 + |\Lambda\mathbb{T}_{\omega^2}\Phi|^2)dy, \tag{4.17}$$

where $k = \{2, \cdots, n\}$ and c_0 is independent of $\omega, \tau, \Lambda, t, r, z$. We introduce the notation

$$\mathbf{M}_1 \equiv c_0 r^{-n+\mathbf{e}} \int_{B_r(z)} (|\mathbb{T}_{\omega^2}\partial_k\Phi|^2 + |\Lambda\mathbb{T}_{\omega^2}\Phi|^2)dy \qquad \text{for } 0 < \mathbf{e} \ll 1.$$

(4.17), $\sigma \in [1 - \frac{\tau}{2}, 2]$, and Lemma 4.4 imply that

$$\mathcal{V}_1(\rho, z) \equiv \int_{B_\rho(z)} (|\mathbb{T}_{\omega^\sigma}\nabla\Phi|^2 + |\Lambda\mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}}\Phi|^2)dy$$
$$\le c_1\Big|\frac{\rho}{t}\Big|^n \int_{B_t(z)} (|\mathbb{T}_{\omega^\sigma}\nabla\Phi|^2 + |\Lambda\mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}}\Phi|^2)dy$$
$$+ c_1c_0\frac{t^{n-\mathbf{e}}t^{\mathbf{e}}}{r^n} \int_{B_r(z)} (|\mathbb{T}_{\omega^2}\partial_k\Phi|^2 + |\Lambda\mathbb{T}_{\omega^2}\Phi|^2)dy$$
$$\le c_1\Big|\frac{\rho}{t}\Big|^n \mathcal{V}_1(t, z) + c_1\mathbf{M}_1 t^{n-\mathbf{e}},$$

where c_1 is independent of $\omega, \tau, \sigma, \rho, t, r, \Lambda, z, \mathbf{e}$. By Lemma 0.1 [4], we have

$$\mathcal{V}_1(\rho, z) \leq c_2 \Big(\Big| \frac{\rho}{t} \Big|^{n-\mathbf{e}} \mathcal{V}_1(t, z) + \mathbf{M}_1\, \rho^{n-\mathbf{e}} \Big) \qquad \text{for } 0 < \rho \leq t \leq \tfrac{r}{2}. \tag{4.18}$$

As $t \nearrow \frac{r}{2}$, we have, by (4.18),

$$\mathcal{V}_1(\rho, z) \leq c_3 \Big| \frac{\rho}{r} \Big|^{n-\mathbf{e}} \mathcal{V}_1(r, z) \qquad \text{for } 0 < \rho \leq \tfrac{r}{2},\ 0 < \mathbf{e} \ll 1. \tag{4.19}$$

An inequality of the form (4.19) for $\rho \in (\frac{r}{2}, r)$ is obvious. So we obtain (4.16). □

Let $\omega > 0$, $y = (y_1, \hat{y})$, and

$$\mathcal{P}_\omega(y) \equiv \begin{cases} \mathbb{A} & \text{if } y_1 \geq 0, \\ \omega \mathbb{A} & \text{if } y_1 < 0, \end{cases} \tag{4.20}$$

where $\mathbb{A}$ is a positive continuous function in $\overline{B_2(0)}$. We consider the problem

$$-\nabla \cdot (\mathcal{P}_{\omega^2} \nabla \Phi + \mathbb{Q}) + \Lambda^2 \mathcal{P}_{\omega^\tau} \Phi = \mathbb{G} \quad \text{in } B_2(0). \tag{4.21}$$

Lemma 4.6. *Let $\omega \in (0, 1]$, $\tau \in [0, 2]$, $\sigma \in [1 - \frac{\tau}{2}, 2]$, $\lambda \in (0, n)$, $\Lambda \in (0, \infty)$, $\ell \in (0, 1)$, $0 < \mathbb{A} \in C(\overline{B_2(0)})$. Any solution Φ of (4.21) satisfies*

$$\begin{aligned} &\|\mathcal{P}_{\omega^\sigma} \nabla \Phi, \Lambda \mathcal{P}_{\omega^{\sigma+\frac{\tau}{2}-1}} \Phi\|_{L^{2,\lambda}(B_{1-\ell}(0))} \leq c\big(\|\mathcal{P}_{\omega^\sigma} \Phi\|_{L^2(B_2(0))} \\ &\qquad + \|\mathcal{P}_{\omega^{\sigma-2}} \mathbb{Q}\|_{L^{2,\lambda}(B_2(0))} + \|\mathcal{P}_{\omega^{\sigma-2}} \mathbb{G}\|_{L^{2,(\lambda-2)_+}(B_2(0))}\big), \end{aligned} \tag{4.22}$$

$$\begin{aligned} &\|\Lambda \mathcal{P}_{\omega^\sigma} \nabla \Phi, \Lambda^2 \mathcal{P}_{\omega^{\sigma+\frac{\tau}{2}-1}} \Phi\|_{L^{2,\lambda}(B_{1-\ell}(0))} \leq c\big(\|\mathcal{P}_{\omega^\sigma} \Phi\|_{L^2(B_2(0))} \\ &\qquad + \|\Lambda \mathcal{P}_{\omega^{\sigma-2}} \mathbb{Q}, \mathcal{P}_{\omega^{\sigma-\frac{\tau}{2}-1}} \mathbb{Q}, \mathcal{P}_{\omega^{\sigma-\frac{\tau}{2}-1}} \mathbb{G}\|_{L^{2,\lambda}(B_2(0))} \\ &\qquad + \|\mathcal{P}_{\omega^{\sigma-2}} \mathbb{Q}\|_{L^2(B_2(0))} + \|\mathcal{P}_{\omega^{\sigma-2}} \mathbb{G}\|_{H^{-1}(B_2^+(0)) \cup H^{-1}(B_2^-(0))}\big), \end{aligned} \tag{4.23}$$

where c is a constant independent of $\omega, \tau, \sigma, \Lambda$. See §2 for $(\lambda - 2)_+$.

In case of $\lambda \in (n-2, n)$, (4.22)–(4.23) imply

$$[\Phi]_{C^{0,\mu}(\overline{B^+_{1-\ell}(0)})} + \omega^\sigma [\Phi]_{C^{0,\mu}(\overline{B^-_{1-\ell}(0)})} \leq \text{the right hand side of (4.22)}, \tag{4.24}$$

$$\Lambda[\Phi]_{C^{0,\mu}(\overline{B^+_{1-\ell}(0)})} + \omega^\sigma \Lambda[\Phi]_{C^{0,\mu}(\overline{B^-_{1-\ell}(0)})} \leq \text{the right hand side of (4.23)}, \tag{4.25}$$

where $\mu \equiv \frac{\lambda - n + 2}{2}$ and c is a constant independent of $\omega, \tau, \sigma, \Lambda$.

Proof. Step 1: Assume $0 < \rho < r < \ell$ and $z \in \mathcal{I}_{1-\ell}(0)$, and set

$$\mathbb{T}_\omega(y) \equiv \begin{cases} \mathbb{A}(z) & \text{if } y \in B^+_\ell(z), \\ \omega \mathbb{A}(z) & \text{if } y \in B^-_\ell(z). \end{cases} \tag{4.26}$$

So $\mathbb{T}_\omega$ is a piecewise constant function. Let $\psi \in H^1(B_r(z))$ be the weak solution of

$$\begin{cases} -\nabla\cdot(\mathbb{T}_{\omega^2}\nabla\psi) + \Lambda^2\mathbb{T}_{\omega^\tau}\psi = 0 & \text{in } B_r(z), \\ \psi = \Phi & \text{on } \partial B_r(z). \end{cases} \tag{4.27}$$

By (4.3)$_3$ of Lemma 4.1, the solution of (4.27) satisfies

$$\begin{aligned} &\int\limits_{B_r(z)} (|\mathbb{T}_{\omega^\sigma}\nabla\psi|^2 + |\Lambda\mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}}\psi|^2)dy \\ &\qquad \le c \int\limits_{B_r(z)} (|\mathbb{T}_{\omega^\sigma}\nabla\Phi|^2 + |\Lambda\mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}}\Phi|^2)dy, \end{aligned} \tag{4.28}$$

where c is independent of $\omega, \tau, \sigma, \Lambda, r, z$. Let $\zeta \equiv \Phi - \psi$. Then (4.21) and (4.27) imply

$$\begin{cases} -\nabla\cdot(\mathbb{T}_{\omega^2}\nabla\zeta + (\mathcal{P}_{\omega^2} - \mathbb{T}_{\omega^2})\nabla\Phi + \mathbb{Q}) + \Lambda^2\mathbb{T}_{\omega^\tau}\zeta & \\ \qquad = \mathbb{G} + \Lambda^2(\mathbb{T}_{\omega^\tau} - \mathcal{P}_{\omega^\tau})\Phi & \text{in } B_r(z), \\ \zeta = 0 & \text{in } \partial B_r(z). \end{cases}$$

By (4.3)$_{2,3}$ of Lemma 4.1,

$$\begin{aligned} &\int\limits_{B_r(z)} (|\mathbb{T}_{\omega^\sigma}\nabla\zeta|^2 + |\Lambda\mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}}\zeta|^2)dy \\ &\quad \le c\Big(\|\Delta^\dagger\mathcal{P}\|^2_{L^\infty(B_r(z))} \int\limits_{B_r(z)} (|\mathbb{T}_{\omega^\sigma}\nabla\Phi|^2 + |\Lambda\mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}}\Phi|^2)dy + \mathbf{M}_2\, r^\lambda \Big), \end{aligned} \tag{4.29}$$

where $\Delta^\dagger\mathcal{P} \equiv (\mathbb{T}_1 - \mathcal{P}_1)\mathbb{T}_1^{-1}$, $\mathbf{M}_2 \equiv \|\mathbb{T}_{\omega^{\sigma-2}}\mathbb{Q}\|^2_{L^{2,\lambda}(B_1(0))} + \|\mathbb{T}_{\omega^{\sigma-2}}\mathbb{G}\|^2_{L^{2,(\lambda-2)_+}(B_1(0))}$, and c is independent of $\omega, \tau, \sigma, \Lambda, r, z$. By Lemma 4.5, the solution ψ of (4.27) satisfies, for $0 < \mathbf{e} \ll 1$ and $0 < \rho < r < \ell$,

$$\begin{aligned} &\int\limits_{B_\rho(z)} (|\mathbb{T}_{\omega^\sigma}\nabla\psi|^2 + |\Lambda\mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}}\psi|^2)dy \\ &\qquad \le c\Big|\frac{\rho}{r}\Big|^{n-\mathbf{e}} \int\limits_{B_r(z)} (|\mathbb{T}_{\omega^\sigma}\nabla\psi|^2 + |\Lambda\mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}}\psi|^2)dy, \end{aligned} \tag{4.30}$$

where c is independent of $\omega, \tau, \sigma, \Lambda, \rho, r, z, \mathbf{e}$. Since $\mathbb{A} \in C(\overline{B_2(0)})$, (4.28)–(4.30) imply, for $0 < \mathbf{e} \ll 1$, $0 < \rho < r < \ell$, and r small,

$$\begin{aligned} \mathcal{V}_2(\rho, z) &\equiv \int\limits_{B_\rho(z)} (|\mathcal{P}_{\omega^\sigma}\nabla\Phi|^2 + |\Lambda\mathcal{P}_{\omega^{\sigma+\frac{\tau}{2}-1}}\Phi|^2)dy \\ &\le c_0\Big(\Big|\frac{\rho}{r}\Big|^{n-\mathbf{e}} + \|\Delta^\dagger\mathcal{P}\|^2_{L^\infty(B_r(z))}\Big)\mathcal{V}_2(r, z) + c\mathbf{M}_2\, r^\lambda, \end{aligned}$$

where c_0 is independent of $\omega, \tau, \sigma, \Lambda, \rho, r, z, \mathbf{e}$. Let us fix $\mathbf{e} < n - \lambda$. By Lemma 0.1 [4], there exists $\theta(c_0, n-\mathbf{e}, \lambda)$ such that if r is small so that $\|\Delta^\dagger \mathcal{P}\|_{L^\infty(B_r(z))} \le \theta$, then

$$\mathcal{V}_2(\rho, z) \le c\left(\left|\frac{\rho}{r}\right|^\lambda \mathcal{V}_2(r, z) + \mathbf{M}_2\, \rho^\lambda\right), \tag{4.31}$$

for $0 < \mathbf{e} \ll 1$ and $0 < \rho < r < \ell$, where c is independent of $\omega, \tau, \sigma, \Lambda, \rho, r, z$.

Since $\mathbb{A} \in C(\overline{B_2(0)})$, there is a $r^* < \ell/2$ such that we have the inequality (4.31) for all $z \in \mathcal{I}_{1-\ell}(0)$ and $0 < \rho < r^* < \ell/2$. That is,

$$\begin{aligned} &\sup_{\substack{\rho < r^* < \ell/2 \\ z \in \mathcal{I}_{1-\ell}(0)}} \frac{1}{\rho^\lambda} \int_{B_\rho(z)} (|\mathcal{P}_{\omega^\sigma} \nabla \Phi|^2 + |\Lambda \mathcal{P}_{\omega^{\sigma+\frac{\tau}{2}-1}} \Phi|^2) dy \\ &\qquad \le c\left(\|\mathcal{P}_{\omega^\sigma} \nabla \Phi, \Lambda \mathcal{P}_{\omega^{\sigma+\frac{\tau}{2}-1}} \Phi\|^2_{L^2(B_1(0))} + \mathbf{M}_2\right). \end{aligned} \tag{4.32}$$

Now let r^* be fixed.

Step 2: Suppose $z \in \mathcal{I}_{1-\ell}(0) \times [-\frac{r^*}{2}, \frac{r^*}{2}]$, $0 < \rho < \frac{r^*}{2}$, $B_\rho(z) \cap \mathcal{I}_{1-\ell}(0) \neq \emptyset$. If $y^* \in B_\rho(z) \cap \mathcal{I}_{1-\ell}(0)$, we obtain, by (4.32),

$$\begin{aligned} &\frac{1}{\rho^\lambda} \int_{B_\rho(z)} |\mathcal{P}_{\omega^\sigma} \nabla \Phi|^2 + |\Lambda \mathcal{P}_{\omega^{\sigma+\frac{\tau}{2}-1}} \Phi|^2 \le \frac{c}{|2\rho|^\lambda} \int_{B_{2\rho}(y^*)} |\mathcal{P}_{\omega^\sigma} \nabla \Phi|^2 + |\Lambda \mathcal{P}_{\omega^{\sigma+\frac{\tau}{2}-1}} \Phi|^2 \\ &\quad \le c\left(\|\mathcal{P}_{\omega^\sigma} \nabla \Phi, \Lambda \mathcal{P}_{\omega^{\sigma+\frac{\tau}{2}-1}} \Phi\|^2_{L^2(B_1(0))} + \mathbf{M}_2\right), \end{aligned}$$

where c is independent of $\omega, \tau, \sigma, \Lambda, \rho, z$.

Suppose $z \in \mathcal{I}_{1-\ell}(0) \times [-\frac{r^*}{2}, \frac{r^*}{2}]$, $0 < \rho < \frac{r^*}{2}$, $B_\rho(z) \cap \mathcal{I}_{1-\ell}(0) = \emptyset$ or suppose $z \in \mathcal{I}_{1-\ell}(0) \times ([-r^*, r^*] \setminus [-\frac{r^*}{2}, \frac{r^*}{2}])$, $0 < \rho < \frac{r^*}{2}$, (4.32) holds by following the argument in Step 1.

Step 3: By the estimates for $\mathcal{V}_2(\rho, z)$ in Steps 1,2, we obtain the inequality (4.31) for all $z \in \overline{B_{1-\ell}(0)}$ and $0 < \rho < \frac{r^*}{2} < \ell$. Consequence,

$$\begin{aligned} &\sup_{\substack{\rho < r^*/2 \\ z \in \overline{B_{1-\ell}(0)}}} \frac{1}{\rho^\lambda} \int_{B_\rho(z)} (|\mathcal{P}_{\omega^\sigma} \nabla \Phi|^2 + |\Lambda \mathcal{P}_{\omega^{\sigma+\frac{\tau}{2}-1}} \Phi|^2) dy \\ &\qquad \le c\left(\|\mathcal{P}_{\omega^\sigma} \nabla \Phi, \Lambda \mathcal{P}_{\omega^{\sigma+\frac{\tau}{2}-1}} \Phi\|^2_{L^2(B_1(0))} + \mathbf{M}_2\right). \end{aligned}$$

Also note, by Lemma 4.1,

$$\|\mathcal{P}_{\omega^\sigma} \nabla \Phi, \Lambda \mathcal{P}_{\omega^{\sigma+\frac{\tau}{2}-1}} \Phi\|_{L^2(B_1(0))} \le c\|\mathcal{P}_{\omega^\sigma} \Phi, \mathcal{P}_{\omega^{\sigma-2}} \mathbb{Q}, \mathcal{P}_{\omega^{\sigma-2}} \mathbb{G}\|_{L^2(B_2(0))}.$$

Which implies the inequality (4.22).

Step 4: To prove (4.23), we follow the above procedure in Step 1–Step 3. In the proof, (4.29) in Step 1 is replaced by, based on $(4.3)_3$ of Lemma 4.1,

$$\int_{B_r(z)} (|\Lambda \mathbb{T}_{\omega^\sigma} \nabla\zeta|^2 + |\Lambda^2 \mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}} \zeta|^2) dy$$
$$\le c\Big(\|\Delta^\dagger \mathcal{P}\|^2_{L^\infty(B_r(z))} \int_{B_r(z)} (|\Lambda \mathbb{T}_{\omega^\sigma} \nabla\Phi|^2 + |\Lambda^2 \mathbb{T}_{\omega^{\sigma+\frac{\tau}{2}-1}} \Phi|^2) dy + \mathbf{M}_3\, r^\lambda \Big),$$

where $\mathbf{M}_3 \equiv \|\Lambda \mathbb{T}_{\omega^{\sigma-2}} \mathbb{Q}, \mathbb{T}_{\omega^{\sigma-\frac{\tau}{2}-1}} \mathbb{G}\|^2_{L^{2,\lambda}(B_1(0))}$ and c is independent of $\omega, \tau, \sigma, \Lambda, r, z$. Other part of the proof is similar to the arguments above.

Step 5: If $\lambda \in (n-2, n)$, the estimates (4.24)–(4.25) follow from (4.22)–(4.23) and Morrey's Theorem [19]. □

Next we give a Hölder estimate.

Lemma 4.7. *Assume A1–A3, $\tau \in [0,2]$, $\sigma \in [1-\frac{\tau}{2}, 2]$, $\lambda \in (0,n)$, $\Lambda \in (0,\infty)$. Any solution of*

$$-\nabla\cdot(\mathbb{E}_{\omega^2,1}\nabla\Phi + \mathbb{Q}) + \Lambda^2 \mathbb{E}_{\omega^\tau,1}\Phi = \mathbb{G} \quad \text{in } Y \tag{4.33}$$

satisfies

$$\begin{aligned} &\|\mathbb{E}_{\omega^\sigma,1}\nabla\Phi, \Lambda \mathbb{E}_{\omega^{\sigma+\frac{\tau}{2}-1},1}\Phi\|_{L^{2,\lambda}(\mathbf{S})} \le c\big(\|\mathbb{E}_{\omega^\sigma,1}\Phi\|_{L^2(Y)} \\ &\qquad + \|\mathbb{E}_{\omega^{\sigma-2},1}\mathbb{Q}\|_{L^{2,\lambda}(Y)} + \|\mathbb{E}_{\omega^{\sigma-2},1}\mathbb{G}\|_{L^{2,(\lambda-2)_+}(Y)}\big), \end{aligned} \tag{4.34}$$

$$\begin{aligned} &\|\Lambda \mathbb{E}_{\omega^\sigma,1}\nabla\Phi, \Lambda^2 \mathbb{E}_{\omega^{\sigma+\frac{\tau}{2}-1},1}\Phi\|_{L^{2,\lambda}(\mathbf{S})} \le c\big(\|\mathbb{E}_{\omega^\sigma,1}\Phi\|_{L^2(Y)} \\ &\qquad + \|\Lambda \mathbb{E}_{\omega^{\sigma-2},1}\mathbb{Q}, \mathbb{E}_{\omega^{\sigma-\frac{\tau}{2}-1},1}\mathbb{Q}, \mathbb{E}_{\omega^{\sigma-\frac{\tau}{2}-1},1}\mathbb{G}, \|_{L^{2,\lambda}(Y)} \\ &\qquad + \|\mathbb{E}_{\omega^{\sigma-2},\epsilon}\mathbb{Q}\|_{L^2(Y)} + \|\mathbb{E}_{\omega^{\sigma-2},\epsilon}\mathbb{G}\|_{H^{-1}(Y_f)\cup H^{-1}(Y_m)}\big), \end{aligned} \tag{4.35}$$

where c is independent of $\omega, \tau, \sigma, \Lambda$. Here $Y_m \subset\subset \mathbf{S} \subset\subset Y$.

In case of $\lambda \in (n-2, n)$, (4.34)–(4.35) imply

$$[\Phi]_{C^{0,\mu}(\overline{\mathbf{S}\backslash Y_m})} + \omega^\sigma [\Phi]_{C^{0,\mu}(\overline{Y_m})} \le \text{the right hand side of (4.34)}, \tag{4.36}$$

$$\Lambda[\Phi]_{C^{0,\mu}(\overline{\mathbf{S}\backslash Y_m})} + \omega^\sigma \Lambda[\Phi]_{C^{0,\mu}(\overline{Y_m})} \le \text{the right hand side of (4.35)}, \tag{4.37}$$

where $\mu \equiv \frac{\lambda-n+2}{2}$ and c is independent of $\omega, \tau, \sigma, \Lambda$. See (2.4) for $\mathbb{E}_{\omega,1}$.

Proof. For any $z \in \partial Y_m$, there exists a small neighborhood $\mathcal{N}(z)$ of z and a $C^{1,0}$-mapping $\mathcal{E}$ so that $\mathcal{N}(z)$ can be mapped into a ball $B_1(0)$ by A2. Moreover,

$$\mathcal{E}(\mathcal{N}(z)\cap Y_f) = B_1^+(0), \quad \mathcal{E}(\mathcal{N}(z)\cap Y_m) = B_1^-(0), \quad \mathcal{E}(\mathcal{N}(z)\cap \partial Y_m) = \mathcal{I}_1(0).$$

Under the mapping and the assumptions of Lemma 4.7, equation (4.33) is transformed into equation (4.21) as well as the assumptions of Lemma 4.6 are satisfied. So the estimates (4.34)–(4.37) are obtained by Lemma 4.6 and partition of unity. □

Remark 4.1. By modifying the arguments of Lemmas 4.6, 4.7, it is not difficult to see

Let $\omega \in (0, 1]$, $\sigma \in [0, 2]$, and $\lambda \in (0, n)$. Any solution of

$$-\nabla \cdot (\mathcal{P}_{\omega^2} \nabla \Phi + \mathbb{Q}) = \mathbb{G} \quad \text{in } B_2(0)$$

satisfies

$$\begin{aligned}\|\mathcal{P}_{\omega^\sigma} \nabla \Phi\|_{L^{2,\lambda}(B_1(0))} &\le c\big(\|\mathcal{P}_{\omega^\sigma} \Phi\|_{L^2(B_2(0))} \\ &\quad + \|\mathcal{P}_{\omega^{\sigma-2}} \mathbb{Q}\|_{L^{2,\lambda}(B_2(0))} + \|\mathcal{P}_{\omega^{\sigma-2}} \mathbb{G}\|_{L^{2,(\lambda-2)_+}(B_2(0))}\big),\end{aligned}$$

where c is a constant independent of ω, σ. See (4.20) for $\mathcal{P}_\omega$.

Suppose A1–A3, $\sigma \in [0, 2]$, $\lambda \in (0, n)$. Any solution of

$$-\nabla \cdot (\mathbb{E}_{\omega^2,1} \nabla \Phi + \mathbb{Q}) = \mathbb{G} \quad \text{in } Y$$

satisfies

$$\begin{aligned}\|\mathbb{E}_{\omega^\sigma,1} \nabla \Phi\|_{L^{2,\lambda}(\mathbf{S})} &\le c\big(\|\mathbb{E}_{\omega^\sigma,1} \Phi\|_{L^2(Y)} \\ &\quad + \|\mathbb{E}_{\omega^{\sigma-2},1} \mathbb{Q}\|_{L^{2,\lambda}(Y)} + \|\mathbb{E}_{\omega^{\sigma-2},1} \mathbb{G}\|_{L^{2,(\lambda-2)_+}(Y)}\big),\end{aligned} \tag{4.38}$$

where c is independent of ω, σ. Here $Y_m \subset\subset \mathbf{S} \subset\subset Y$. In case of $\lambda \in (n-2, n)$,

$$[\Phi]_{C^{0,\mu}(\overline{\mathbf{S} \setminus Y_m})} + \omega^\sigma [\Phi]_{C^{0,\mu}(\overline{Y_m})} \le \text{the right hand side of (4.38)},$$

where $\mu \equiv \frac{\lambda-n+2}{2}$ and c is independent of ω, σ. See (2.4) for $\mathbb{E}_{\omega,1}$.

Now we begin to study the gradient Hölder estimate.

Lemma 4.8. *Let $\omega \in (0, 1]$, $\sigma \in [0, 2]$, $0 < \rho < r \le \frac{1}{3}$, $z \in \mathcal{I}_{1/3}(0)$. There is a constant c independent of $\omega, \sigma, \rho, r, z$ such that any solution of*

$$-\nabla \cdot (\mathbb{T}_{\omega^2} \nabla \Phi) = 0 \quad \text{in } B_1(0) \tag{4.39}$$

satisfies

$$\int_{B_\rho(z)} \mathbb{T}_{\omega^{2\sigma}} |\Phi - (\Phi)_{z,\rho,\mathbb{T}_{\omega^{2\sigma}}}|^2 dy \le c\Big|\frac{\rho}{r}\Big|^{n+2} \int_{B_r(z)} \mathbb{T}_{\omega^{2\sigma}} |\Phi - (\Phi)_{z,r,\mathbb{T}_{\omega^{2\sigma}}}|^2 dy.$$

See (2.1) for $(\Phi)_{z,\rho,\mathbb{T}_{\omega^{2\sigma}}}$ and (4.1) for $\mathbb{T}_\omega$.

Proof. Let c denote a constant independent of $\omega, \sigma, \rho, r, z$. We note

$$\begin{aligned}\int_{B_\rho(z)} \mathbb{T}_{\omega^{2\sigma}} |\Phi - (\Phi)_{z,\rho,\mathbb{T}_{\omega^{2\sigma}}}|^2 dy &\le \int_{B_\rho(z)} \mathbb{T}_{\omega^{2\sigma}} |\Phi - \Phi(z)|^2 dy \\ &\le c\rho^{n+2} \|\mathbb{T}_{\omega^\sigma} \nabla \Phi\|^2_{L^\infty(B_\rho(z))}.\end{aligned}$$

By a similar argument as (4.8), any solution of (4.39) satisfies

$$\|\mathbb{T}_{\omega^\sigma}\nabla\Phi\|^2_{L^\infty(B_\rho(z))} \le c \int\limits_{B_1(z)} \mathbb{T}_{\omega^{2\sigma}}|\Phi - (\Phi)_{z,1,\mathbb{T}_{\omega^{2\sigma}}}|^2 dy.$$

Together with a scaling argument, we prove the lemma. $\square$

Define $\mathbb{Q}^\pm(y) \equiv \begin{cases} \mathbb{Q}^+ & \text{if } y_1 \ge 0 \\ \mathbb{Q}^- & \text{if } y_1 < 0 \end{cases}$, where $\mathbb{Q}^+, \mathbb{Q}^-$ are constant vectors.

Corollary 4.1. *Let $\omega \in (0,1]$, $\sigma \in [0,2]$, $0 < \rho < r \le \frac{1}{3}$, $z \in \mathcal{I}_{\frac{1}{3}}(0)$. Any solution of*

$$-\nabla \cdot (\mathbb{T}_{\omega^2}\nabla\Phi + \mathbb{Q}^\pm) = 0 \quad in\ B_1(0)$$

satisfies

$$\int\limits_{B_\rho(z)} \mathbb{T}_{\omega^{2\sigma}}|\partial_k\Phi - (\partial_k\Phi)_{z,\rho,\mathbb{T}_{\omega^{2\sigma}}}|^2 dy \le c\Big|\frac{\rho}{r}\Big|^{n+2} \int\limits_{B_r(z)} \mathbb{T}_{\omega^{2\sigma}}|\partial_k\Phi - (\partial_k\Phi)_{z,r,\mathbb{T}_{\omega^{2\sigma}}}|^2 dy, \tag{4.40}$$

where ∂_s means the partial derivative in the s-th direction and c is a constant independent of $\omega, \sigma, \rho, r, z$. Here $k \in \{2, \cdots, n\}$ and see (4.1) for $\mathbb{T}_\omega$.

Proof. $\partial_k\Phi$ is a solution of (4.39), so (4.40) follows from Lemma 4.8. $\square$

Next is a gradient Hölder estimate around the interfaces.

Lemma 4.9. *Let $\omega \in (0,1]$, $\sigma \in [0,2]$, $0 < \mathbb{A} \in C^{1,0}(\overline{B_1(0)})$, $\mu \in (0,1)$, $\ell \in (0,\frac{1}{3})$, $\mathbb{Q} \in \mathcal{L}^{2,n+2\mu}(B_1^+(0)) \cup \mathcal{L}^{2,n+2\mu}(B_1^-(0))$, and $\mathbb{G} \in L^{2,n+2\mu-2}(B_1(0))$. There is a $r^* \in (0,\ell/2)$ such that any solution of*

$$-\nabla \cdot (\mathcal{P}_{\omega^2}\nabla\Phi + \mathbb{Q}) = \mathbb{G} \quad in\ B_1(0) \tag{4.41}$$

satisfies

$$\begin{aligned}\sup_{\substack{\rho \le r^*/2 < \ell/4 \\ z \in B_{1-2\ell}(0)}} \frac{1}{\rho^{n+2\mu}} \int\limits_{B_\rho(z)} \mathcal{P}_{\omega^{2\sigma}}|\nabla\Phi - (\nabla\Phi)^\pm_{z,\rho,\mathcal{P}_{\omega^{2\sigma}}}|^2 \le c\mathbf{M}_4 \\ \equiv c\Bigg(\|\mathcal{P}_{\omega^\sigma}\nabla\Phi, \mathcal{P}_{\omega^{\sigma-2}}\mathbb{G}\|^2_{L^{2,n+2\mu-2}(B_1(0))} \\ + \sup_{\substack{r \le r^* < \ell/2 \\ z \in B_{1-\ell}(0)}} \frac{1}{r^{n+2\mu}} \int\limits_{B_r(z)} |\mathcal{P}_{\omega^{\sigma-2}}(\mathbb{Q} - (\mathbb{Q})^\pm_{z,r,I})|^2\Bigg),\end{aligned}$$

where c is independent of $\omega, \sigma, \ell, \rho$. Here I is the identity function. See (4.20) for $\mathcal{P}_\omega$ and (2.2) for $(\mathbb{Q})^{\pm}_{z,r,I}$ and $(\nabla\Phi)^{\pm}_{z,\rho,\mathcal{P}_{\omega^{2\sigma}}}$.

Proof. Let $r^* \in (0, \frac{\ell}{2})$ be a number to be determined.
Step 1. For any fixed $z \in \mathcal{I}_{1-\ell}(0)$ and $0 < t < r^*$, we define $\mathbb{T}_\omega$ as (4.26) and obtain $\psi(t,\cdot)$ by solving

$$\begin{cases} -\nabla \cdot (\mathbb{T}_{\omega^2} \nabla \psi(t,\cdot) + (\mathbb{Q})^{\pm}_{z,t,I}) = 0 & \text{in } B_t(z), \\ \psi(t,\cdot) = \Phi & \text{on } \partial B_t(z), \end{cases} \tag{4.42}$$

where Φ is a solution of (4.41). See the beginning of this section for $\mathcal{I}_{1-\ell}(0)$. If $\phi(t,\cdot) \equiv \psi(t,\cdot) - \Phi$, (4.41) and (4.42) imply

$$\begin{cases} -\nabla \cdot \Big(\mathbb{T}_{\omega^2} \nabla \phi(t,\cdot) + \big(\mathbb{T}_{\omega^2} - \mathcal{P}_{\omega^2}\big) \nabla \Phi + (\mathbb{Q})^{\pm}_{z,t,I} - \mathbb{Q}\Big) = -\mathbb{G} & \text{in } B_t(z), \\ \phi(t,\cdot) = 0 & \text{on } \partial B_t(z). \end{cases}$$

By $(4.3)_2$ of Lemma 4.1 and $\mathbb{A} \in C^{1,0}(\overline{B_1(0)})$,

$$\begin{aligned} \int_{B_t(z)} |\mathbb{T}_{\omega^\sigma} \nabla \phi(t,\cdot)|^2 &\le c \int_{B_t(z)} |t\mathcal{P}_{\omega^\sigma} \nabla \Phi|^2 + |\mathcal{P}_{\omega^{\sigma-2}}(\mathbb{Q} - (\mathbb{Q})^{\pm}_{z,t,I})|^2 + |t\mathcal{P}_{\omega^{\sigma-2}}\mathbb{G}|^2 \\ &\le ct^{n+2\mu} \|\mathcal{P}_{\omega^\sigma} \nabla \Phi, \mathcal{P}_{\omega^{\sigma-2}}\mathbb{G}\|^2_{L^{2,n+2\mu-2}(B_t(z))} + c \int_{B_t(z)} |\mathcal{P}_{\omega^{\sigma-2}}(\mathbb{Q} - (\mathbb{Q})^{\pm}_{z,t,I})|^2 \\ &\le ct^{n+2\mu} \mathbf{M}_4, \end{aligned} \tag{4.43}$$

where c is independent of ω, σ, z, t.
Step 2. If $z \in \mathcal{I}_{1-\ell}(0)$ and $0 < \rho < r < r^*$, any solution of (4.42) with $t = r$ satisfies, by Corollary 4.1,

$$\begin{aligned} &\int_{B_\rho(z)} \mathbb{T}_{\omega^{2\sigma}} |\partial_k \psi(r,\cdot) - (\partial_k \psi(r,\cdot))_{z,\rho,\mathbb{T}_{\omega^{2\sigma}}}|^2 dy \\ &\qquad \le c\Big|\frac{\rho}{r}\Big|^{n+2} \int_{B_r(z)} \mathbb{T}_{\omega^{2\sigma}} |\partial_k \psi(r,\cdot) - (\partial_k \psi(r,\cdot))_{z,r,\mathbb{T}_{\omega^{2\sigma}}}|^2 dy, \end{aligned}$$

where c is independent of $\omega, \sigma, \rho, r, z$. Here $k \in \{2, \cdots, n\}$. By the result in Step 1 and $\mathbb{A} \in C^{1,0}(\overline{B_1(0)})$, if r^* is small, then

$$\begin{aligned} &\int_{B_\rho(z)} \mathcal{P}_{\omega^{2\sigma}} |\partial_k \Phi - (\partial_k \Phi)_{z,\rho,\mathcal{P}_{\omega^{2\sigma}}}|^2 dy \\ &\qquad \le c\Big|\frac{\rho}{r}\Big|^{n+2} \int_{B_r(z)} \mathcal{P}_{\omega^{2\sigma}} |\partial_k \Phi - (\partial_k \Phi)_{z,r,\mathcal{P}_{\omega^{2\sigma}}}|^2 dy + cr^{n+2\mu} \mathbf{M}_4, \end{aligned}$$

where c is independent of $\omega, \sigma, \rho, r, z$. For $0 < \rho < r < r^*$, by Lemma 0.1 [4],

$$\rho^{-(n+2\mu)} \int_{B_\rho(z)} \mathcal{P}_{\omega^{2\sigma}} |\partial_k \Phi - (\partial_k \Phi)_{z,\rho,\mathcal{P}_{\omega^{2\sigma}}}|^2 dy \\ \leq c r^{-(n+2\mu)} \int_{B_r(z)} \mathcal{P}_{\omega^{2\sigma}} |\partial_k \Phi - (\partial_k \Phi)_{z,r,\mathcal{P}_{\omega^{2\sigma}}}|^2 dy + c\mathbf{M}_4, \tag{4.44}$$

where c is independent of $\omega, \sigma, \rho, r, z$.

Step 3. For $z \in \mathcal{I}_{1-\ell}(0)$ and $0 < \rho < r^*$, any solution $\psi(\rho, \cdot)$ of (4.42) with $t = \rho$ satisfies, by mean value theorem,

$$\left\| \mathbb{T}_{\omega^\sigma} (\partial_1 \psi(\rho, \cdot) - (\partial_1 \psi(\rho, \cdot))^{\pm}_{z,\rho/2,\mathbb{T}_{\omega^{2\sigma}}}) \right\|_{L^2(B_{\rho/2}(z))} \\ \leq c\rho^{\frac{n+2}{2}} \left\| \mathbb{T}_{\omega^\sigma} \nabla^2 \psi(\rho, \cdot) \right\|_{L^\infty(B^+_{\rho/2}(z)) \cup L^\infty(B^-_{\rho/2}(z))} \\ \leq c\rho^{\frac{n+2}{2}} \| \mathbb{T}_{\omega^\sigma} \nabla \partial_k \psi(\rho, \cdot) \|_{L^\infty(B^+_{\rho/2}(z)) \cup L^\infty(B^-_{\rho/2}(z))}, \tag{4.45}$$

where $k \in \{2, \cdots, n\}$. Since $\partial_k \psi(\rho, \cdot) - (\partial_k \Phi)_{z,\rho,\mathbb{T}_{\omega^{2\sigma}}}$ for $k \in \{2, \cdots, n\}$ is a solution of (4.39), a similar argument as (4.8) gives

$$\rho^{\frac{n+2}{2}} \| \mathbb{T}_{\omega^\sigma} \nabla \partial_k \psi(\rho, \cdot) \|_{L^\infty(B^+_{\rho/2}(z)) \cup L^\infty(B^-_{\rho/2}(z))} \\ \leq c \left\| \mathbb{T}_{\omega^\sigma} \left(\partial_k \psi(\rho, \cdot) - (\partial_k \Phi)_{z,\rho,\mathbb{T}_{\omega^{2\sigma}}} \right) \right\|_{L^2(B_\rho(z))} \\ \leq c \left(\| \mathbb{T}_{\omega^\sigma} \partial_k \phi(\rho, \cdot) \|_{L^2(B_\rho(z))} + \| \mathbb{T}_{\omega^\sigma} \left(\partial_k \Phi - (\partial_k \Phi)_{z,\rho,\mathbb{T}_{\omega^{2\sigma}}} \right) \|_{L^2(B_\rho(z))} \right), \tag{4.46}$$

where $\phi(\rho, \cdot) \equiv \psi(\rho, \cdot) - \Phi$. So, by $\mathbb{A} \in C^{1,0}(\overline{B_1(0)})$, (4.43) with $t = \rho$, and (4.45)–(4.46),

$$\int_{B_{\rho/2}(z)} \mathbb{T}_{\omega^{2\sigma}} |\partial_1 \Phi - (\partial_1 \Phi)^{\pm}_{z,\rho/2,\mathbb{T}_{\omega^{2\sigma}}}|^2 \\ \leq c \int_{B_{\rho/2}(z)} \mathbb{T}_{\omega^{2\sigma}} |\partial_1 \Phi - (\partial_1 \psi(\rho, \cdot))^{\pm}_{z,\rho/2,\mathbb{T}_{\omega^{2\sigma}}}|^2 \\ \leq c \int_{B_{\rho/2}(z)} \mathbb{T}_{\omega^{2\sigma}} |\partial_1 \psi(\rho, \cdot) - (\partial_1 \psi(\rho, \cdot))^{\pm}_{z,\rho/2,\mathbb{T}_{\omega^{2\sigma}}}|^2 + c \int_{B_{\rho/2}(z)} \mathbb{T}_{\omega^{2\sigma}} |\partial_1 \phi(\rho, \cdot)|^2 \\ \leq c \int_{B_\rho(z)} \mathbb{T}_{\omega^{2\sigma}} |\partial_k \Phi - (\partial_k \Phi)_{z,\rho,\mathbb{T}_{\omega^{2\sigma}}}|^2 + c \int_{B_\rho(z)} \mathbb{T}_{\omega^{2\sigma}} |\nabla \phi(\rho, \cdot)|^2 \\ \leq c \int_{B_\rho(z)} \mathbb{T}_{\omega^{2\sigma}} |\partial_k \Phi - (\partial_k \Phi)_{z,\rho,\mathbb{T}_{\omega^{2\sigma}}}|^2 + c\rho^{n+2\mu} \mathbf{M}_4,$$

where c is independent of ω, σ, z, ρ. For $0<\rho<r<r^*$, by (4.44),

$$
\begin{aligned}
&|\frac{\rho}{2}|^{-(n+2\mu)} \int\limits_{B_{\rho/2}(z)} \mathcal{P}_{\omega^{2\sigma}} |\partial_1 \Phi - (\partial_1 \Phi)^{\pm}_{z,\rho/2,\mathcal{P}_{\omega^{2\sigma}}}|^2 \\
&\leq c r^{-(n+2\mu)} \int\limits_{B_r(z)} \mathcal{P}_{\omega^{2\sigma}} |\partial_k \Phi - (\partial_k \Phi)_{z,r,\mathcal{P}_{\omega^{2\sigma}}}|^2 dy + c\mathbf{M}_4,
\end{aligned}
\tag{4.47}
$$

where c is independent of $\omega, \sigma, z, r, \rho$. Here $k \in \{2, \cdots, n\}$.

Step 4. By (4.44) and (4.47), there is a $r^* < \ell/2$ such that, for $0<\rho<r^*$,

$$
\rho^{-(n+2\mu)} \int\limits_{B_\rho(z)} \mathcal{P}_{\omega^{2\sigma}} |\nabla \Phi - (\nabla \Phi)^{\pm}_{z,\rho,\mathcal{P}_{\omega^{2\sigma}}}|^2 \leq c\mathbf{M}_4,
\tag{4.48}
$$

where c is independent of ω, σ, z, ρ. In the following argument, r^* is fixed.

Step 5. Suppose $z \in \mathcal{I}_{1-\ell}(0) \times [-\frac{r^*}{2}, \frac{r^*}{2}]$, $0<\rho<\frac{r^*}{2}$, $B_\rho(z) \cap \mathcal{I}_{1-\ell}(0) \neq \emptyset$. If $z_* \in B_\rho(z) \cap \mathcal{I}_{1-\ell}(0)$, then, by (4.48),

$$
\begin{aligned}
&\rho^{-(n+2\mu)} \int\limits_{B_\rho(z)} \mathcal{P}_{\omega^{2\sigma}} |\nabla \Phi - (\nabla \Phi)^{\pm}_{z,\rho,\mathcal{P}_{\omega^{2\sigma}}}|^2 \\
&\leq c|2\rho|^{-(n+2\mu)} \int\limits_{B_{2\rho}(z_*)} \mathcal{P}_{\omega^{2\sigma}} |\nabla \Phi - (\nabla \Phi)^{\pm}_{z_*,2\rho,\mathcal{P}_{\omega^{2\sigma}}}|^2 \leq c\mathbf{M}_4,
\end{aligned}
$$

where c is independent of ω, σ, z, ρ.

Suppose $z \in \mathcal{I}_{1-\ell}(0) \times [-\frac{r^*}{2}, \frac{r^*}{2}]$, $0<\rho<\frac{r^*}{2}$, $B_\rho(z) \cap \mathcal{I}_{1-\ell}(0) = \emptyset$ or suppose $z \in \mathcal{I}_{1-\ell}(0) \times ([-r^*, r^*] \setminus [-\frac{r^*}{2}, \frac{r^*}{2}])$, $0<\rho<\frac{r^*}{2}$, (4.48) holds by following the arguments in Steps 1–4 above. So the lemma is proved. □

Remark 4.2. By Theorem 6.1.1 [19], Remark 4.1, Lemma 4.9, partition of unity, and a similar argument as that in Lemma 4.7, we obtain,

If A1–A3, $\mathbb{Q} \in \mathcal{L}^{2,n+2\mu}(Y_f) \cup \mathcal{L}^{2,n+2\mu}(Y_m)$, $\mathbb{G} \in L^{2,n+2\mu-2}(Y)$, $\sigma \in [0,2]$, and $\mu \in (0,1)$, any solution of

$$
-\nabla \cdot (\mathbb{E}_{\omega^2,1} \nabla \Phi + \mathbb{Q}) = \mathbb{G} \quad in\ Y
$$

satisfies

$$
\begin{aligned}
[\nabla \Phi]_{C^{0,\mu}(\overline{\mathbf{S} \setminus Y_m})} + \omega^\sigma [\nabla \Phi]_{C^{0,\mu}(\overline{Y_m})} &\leq c(\|\mathbb{E}_{\omega^\sigma,1} \Phi\|_{L^2(Y)} \\
&+ \|\mathbb{E}_{\omega^{\sigma-2},1} \mathbb{Q}\|_{\mathcal{L}^{2,n+2\mu}(Y_f) \cup \mathcal{L}^{2,n+2\mu}(Y_m)} + \|\mathbb{E}_{\omega^{\sigma-2},1} \mathbb{G}\|_{L^{2,n+2\mu-2}(Y)}),
\end{aligned}
$$

where c is independent of ω, σ. Here $Y_m \subset\subset \mathbf{S} \subset\subset Y$ and see (2.4) for $\mathbb{E}_{\omega,1}$.

Remark 4.2 for $\sigma = 0$, Poincaré inequality, and energy method imply

Corollary 4.2. *If A1–A3, any solution of*

$$-\nabla\cdot(\mathbb{E}_{\omega^2,1}\nabla\Phi)=0 \quad \text{in } Y$$

satisfies $\|\nabla\Phi\|_{L^\infty(\mathbf{S})}\le c\|\Phi\|_{L^2(Y_f)}$, *where* c *is independent of* ω. *Here* $Y_m\subset\subset\mathbf{S}\subset\subset Y$ *and see (2.4) for* $\mathbb{E}_{\omega,1}$.

Suppose $\omega\in(0,1]$, $\vec{\mathbf{n}}_y$ is a unit normal vector on ∂Y_m, and $\vec{e}_i$ $(i=1,\cdots,n)$ is the unit vector in the i-th direction in $\mathbb{R}^n$. Let $\mathbb{X}^{(i)}_{\omega,1}\in H^1_{per}(\mathbb{R}^n)$ satisfy

$$\begin{cases}\nabla\cdot(\mathbb{E}_{\omega^2,1}(\nabla\mathbb{X}^{(i)}_{\omega,1}+\vec{e}_i))=0 \quad \text{in } Y,\\ \int_{Y_f}\mathbb{X}^{(i)}_{\omega,1}(y)dy=0,\end{cases}$$

and let $\mathbb{X}^{(i)}_{0,1}\in H^1_{per}(\mathcal{O}^1_f)\cup H^1(\mathcal{O}^1_m)$ satisfy

$$\begin{cases}\nabla\cdot(\mathbb{E}_{0,1}(\nabla\mathbb{X}^{(i)}_{0,1}+\vec{e}_i))=0 & \text{in } Y_f,\\ \mathbb{E}_{0,1}(\nabla\mathbb{X}^{(i)}_{0,1}+\vec{e}_i)\cdot\vec{\mathbf{n}}_y=0 & \text{on } \partial Y_m,\\ \mathbb{X}^{(i)}_{0,1}(x)=0 & \text{in } Y_m,\\ \int_{Y_f}\mathbb{X}^{(i)}_{0,1}(y)dy=0.\end{cases}$$

Here $H^1_{per}(\mathcal{O}^1_f)\equiv\{\psi|_{\mathcal{O}^1_f}\mid \psi\in H^1_{per}(\mathbb{R}^n)\}$ and see (2.4) for $\mathbb{E}_{\omega^2,1}$. By Lax–Milgram Theorem [11], Poincaré inequality, and energy method, the solution $\mathbb{X}^{(i)}_{\omega,1}$ for $\omega\in[0,1]$ is uniquely solvable and $\|\mathbb{X}^{(i)}_{\omega,1}\|_{H^1(Y)}\le c$, where c is a constant independent of ω. By Theorem 6.30 [11] and Remark 4.2,

$$\|\nabla\mathbb{X}^{(i)}_{\omega,1}\|_{L^\infty(Y)}\le c(n,Y_m) \qquad \text{for } \omega\in[0,1]. \tag{4.49}$$

Define $\mathbb{X}_{\omega,1}\equiv(\mathbb{X}^{(1)}_{\omega,1},\cdots,\mathbb{X}^{(n)}_{\omega,1})$ and $\mathbb{X}_{\omega,\epsilon}(x)\equiv\epsilon\mathbb{X}_{\omega,1}(\frac{x}{\epsilon})$ for $\omega\in[0,1],\epsilon\in(0,1]$. Denote by Ξ_ω for $\omega\in[0,1]$ a $n\times n$ matrix function whose (k,i)-component is $\partial_k\mathbb{X}^{(i)}_{\omega,1}$. By remark in pages 17–19, 94–95 [13],

$$\begin{cases}\mathcal{K}_\omega\equiv\int_Y\mathbb{E}_{\omega^2,1}(I+\Xi_\omega(y))dy\\ \widehat{\mathcal{K}}_\omega\equiv\int_Y\mathbb{E}_{\omega^\tau,1}dy\end{cases} \quad \text{for } \omega\in[0,1] \tag{4.50}$$

are symmetric positive definite matrices depending only on ω. Here I is the identity matrix. By energy method and (4.49), it is not difficult to see, for $\omega \in [0,1]$,

$$\begin{cases} d_1 I \le \mathcal{K}_\omega, \widehat{\mathcal{K}}_\omega \le d_2 I \quad \text{where } d_1, d_2 \text{ are positive constants,} \\ \mathcal{K}_\omega \text{ is a continuous function of } \omega. \end{cases} \tag{4.51}$$

5. Proof of Lemma 3.1

We now derive the Hölder estimate for the solution of (3.1). The estimate in the interior region is in §§5.1, and the estimate around the boundary region is in §§5.2. In this section,

$$\begin{cases} \text{A1–A4 and } \sigma \in [2-\tau, 1] \text{ are assumed,} \\ \text{Notations in (2.4) are used.} \end{cases} \tag{5.1}$$

5.1. Interior estimate

For convenience we let $\overline{B_1(0)} \subset \Omega$.

Lemma 5.1. *For any $\delta > 0$, there are $\theta_1, \theta_2 \in (0,1)$ (depending on $\delta, \mathbf{K}, Y_f$) with $\theta_1 < \theta_2^2$ and there is a $\epsilon_0 \in (0,1)$ (depending on $\theta_1, \theta_2, \delta, \mathbf{K}$) such that if*

$$-\nabla \cdot (\mathbf{K}_{\omega^2,\nu} \nabla \phi) + \gamma^2 \mathbf{K}_{\omega^\tau,\nu}\, \phi = \mathbb{G} \qquad \text{in } B_1(0), \tag{5.2}$$

and if

$$\begin{cases} \omega \in (0,1],\ \nu \in (0,\epsilon_0),\ \theta \in [\theta_1, \theta_2],\ \gamma \in (0,\infty), \\ \|\Pi_\nu \phi|_{\Omega_f^\nu}\|_{L^2(B_1(0))},\ \|\omega \phi \mathcal{X}_{\Omega_m^\nu}\|_{L^2(B_1(0))} \le 1, \\ \epsilon_0^{-1} \|\mathbb{G}\|_{L^2(B_1(0))},\ \|\mathbf{K}_{\omega^{\sigma-2},\nu} \mathbb{G}\|_{L^2(B_1(0))} \le 1, \end{cases} \tag{5.3}$$

then

$$\begin{cases} \max\{1, \gamma^{2\mu}\} \displaystyle\fint_{B_\theta(0)} \left| \Pi_\nu \phi|_{\Omega_f^\nu} - (\Pi_\nu \phi|_{\Omega_f^\nu})_{0,\theta,I} \right|^2 dx \le \theta^{2\mu}, \\ \max\{1, \gamma^{2\mu}\} \displaystyle\fint_{B_\theta(0) \cap \Omega_m^\nu} \omega^{2\sigma} \left| \phi - (\Pi_\nu \phi|_{\Omega_f^\nu})_{0,\theta,I} \right|^2 dx \le \theta^{2\mu}, \end{cases} \tag{5.4}$$

where $\mu \equiv \frac{\delta}{n+\delta}$. See (2.4) for $\mathbf{K}_{\omega,\nu}$, §2 for the extension operator Π_ν, (2.1) for $(\Pi_\nu \phi|_{\Omega_f^\nu})_{0,\theta,I}$, and I is the identity matrix.

Proof. Consider the following problem

$$-\nabla \cdot (\mathcal{K}_\omega \nabla \phi) + \gamma^2 \widehat{\mathcal{K}}_\omega\, \phi = 0 \qquad \text{in } B_{4/5}(0), \tag{5.5}$$

where $\omega \in [0,1]$, $\mathcal{K}_\omega, \widehat{\mathcal{K}}_\omega$ are defined in (4.50), and $\gamma \in [0,\infty)$. Any solution ϕ of (5.5) satisfies, by Theorem 9.11 [11] and (4.51),

$$\max\{1,\gamma\}\|\phi\|_{C^1(\overline{B_{1/2}(0)})} \le c\|\phi\|_{L^2(B_{4/5}(0))},$$

where c only depends on $n, \mathcal{K}_\omega$. If $\check{\mu}$ satisfies $\mu < \check{\mu} < 1$, Theorem 1.2 in page 70 [10] implies

$$\max\{1,\gamma^2\} \fint_{B_\theta(0)} |\phi - (\phi)_{0,\theta,I}|^2 dx \le \theta^{2\check{\mu}} \fint_{B_{4/5}(0)} \phi^2 dx \tag{5.6}$$

for sufficiently small θ (depending on $\delta, \mathcal{K}_\omega$). Let us fix $\theta_1, \theta_2 \in (0, \frac{1}{2})$ so that $\theta_1 < \theta_2^2$ and (5.6) holds for any $\theta \in [\theta_1, \theta_2]$.

Now we claim $(5.4)_1$. If not, there is a sequence $\{\omega_\nu, \theta_\nu, \gamma_\nu, \phi_\nu, \mathbb{G}_\nu\}$ satisfying (5.2) and

$$\begin{cases} \nu \to 0, \quad \omega_\nu \to \omega \in [0,1], \quad \theta_\nu \to \theta \in [\theta_1,\theta_2], \quad \gamma_\nu \to \gamma \in [0,\infty], \\ \|\Pi_\nu \phi_\nu|_{\Omega_f^\nu}\|_{L^2(B_1(0))}, \ \|\omega_\nu \phi_\nu \mathcal{X}_{\Omega_m^\nu}\|_{L^2(B_1(0))} \le 1, \\ \|\mathbf{K}_{\omega_\nu^{\sigma-2},\nu} \mathbb{G}_\nu\|_{L^2(B_1(0))} \le 1, \\ \lim_{\nu\to 0} \|\mathbb{G}_\nu\|_{L^2(B_1(0))} = 0, \\ \max\{1,\gamma_\nu^{2\mu}\} \fint_{B_{\theta_\nu}(0)} \left|\Pi_\nu\phi_\nu|_{\Omega_f^\nu} - (\Pi_\nu\phi_\nu|_{\Omega_f^\nu})_{0,\theta_\nu,I}\right|^2 dx > \theta_\nu^{2\mu}. \end{cases} \tag{5.7}$$

Let $\eta \in C_0^\infty(B_1(0))$ be a bell-shaped function satisfying $\eta \in [0,1]$ and $\eta = 1$ in $B_{4/5}(0)$. We first multiply (5.2) by $\phi_\nu \eta^2$ and use integration by parts, then repeat the same process except replacing $\phi_\nu\eta^2$ by $\gamma_\nu^2 \phi_\nu \eta^2$ to get

$$\max\{1,\gamma_\nu\}\|\mathbf{K}_{\omega_\nu,\nu}\nabla\phi_\nu, \mathbf{K}_{\omega_\nu^{\tau/2},\nu}\gamma_\nu\phi_\nu\|_{L^2(B_{4/5}(0))} \le c, \tag{5.8}$$

where c is independent of $\nu, \gamma_\nu, \omega_\nu$.
Suppose $\gamma = \infty$ in $(5.7)_1$. (5.8) implies $\gamma_\nu^2 \|\Pi_\nu\phi_\nu|_{\Omega_f^\nu}\|_{L^2(B_{4/5}(0))} \le c$. So

$$0 < \theta^{2\mu} = \lim_{\nu\to 0} \theta_\nu^{2\mu} \le \lim_{\nu\to 0} \max\{1,\gamma_\nu^{2\mu}\} \fint_{B_{\theta_\nu}(0)} \left|\Pi_\nu\phi_\nu|_{\Omega_f^\nu} - (\Pi_\nu\phi_\nu|_{\Omega_f^\nu})_{0,\theta_\nu,I}\right|^2 dx = 0,$$

which is impossible.
Suppose $\gamma < \infty$ in $(5.7)_1$. By (5.8), compactness principle and by tracing the proof of Theorem 2.3 [2], we can extract a subsequence (same notation for subsequence) such that

$$\begin{cases} \Pi_\nu\phi_\nu|_{\Omega_f^\nu} \to \phi & \text{in } L^2(B_{4/5}(0)) \text{ strongly}, \\ \mathbf{K}_{\omega_\nu^2,\nu}\nabla\phi_\nu \to \mathcal{K}_\omega \nabla\phi & \text{in } L^2(B_{4/5}(0)) \text{ weakly}, \\ \gamma_\nu^2 \mathbf{K}_{\omega_\nu^\tau,\nu}\phi_\nu \to \gamma^2 \widehat{\mathcal{K}}_\omega \phi & \text{in } L^2(B_{4/5}(0)) \text{ weakly}. \end{cases} \tag{5.9}$$

See (4.50)–(4.51) for $\mathcal{K}_\omega, \widehat{\mathcal{K}}_\omega$. Moreover, the ϕ in (5.9) satisfies (5.5). Equations (5.6)–(5.9) then imply

$$\theta^{2\mu}=\lim_{\nu\to 0}\theta_\nu^{2\mu}\le\lim_{\nu\to 0}\max\{1,\gamma_\nu^{2\mu}\}\fint_{B_{\theta_\nu}(0)}\left|\Pi_\nu\phi_\nu|_{\Omega_f^\nu}-(\Pi_\nu\phi_\nu|_{\Omega_f^\nu})_{0,\theta_\nu,I}\right|^2dx$$
$$=\max\{1,\gamma^{2\mu}\}\fint_{B_\theta(0)}|\phi-(\phi)_{0,\theta,I}|^2dx\le\theta^{2\check{\mu}}\fint_{B_{4/5}(0)}\phi^2dx.$$

If θ_2 is small enough, then we get contradiction. Therefore we prove $(5.4)_1$.

Set $\zeta\equiv\theta^{-\mu}(\Pi_\nu\phi|_{\Omega_f^\nu}-(\Pi_\nu\phi|_{\Omega_f^\nu})_{0,\theta,I})$ and $\rho\equiv\theta^{-\mu}(\phi-(\Pi_\nu\phi|_{\Omega_f^\nu})_{0,\theta,I})$. (5.2) implies, for any $\psi\in C_0^\infty(\nu(Y_m+j))$ with $\nu(Y_m+j)\subset B_\theta(0)\cap\Omega_m^\nu$ and $j\in\mathbb{Z}^n$,

$$\int_{\nu(Y_m+j)}(\rho-\zeta)\omega^2\nabla\cdot(\mathbf{K}\nabla\psi)-\int_{\nu(Y_m+j)}(\rho-\zeta)\omega^\tau\gamma^2\mathbf{K}\psi$$
$$=\int_{\nu(Y_m+j)}\omega^2\mathbf{K}\nabla\zeta\nabla\psi+\theta^{-\mu}(\omega^\tau\mathbf{K}\gamma^2\Pi_\nu\phi|_{\Omega_f^\nu}-\mathbb{G})\psi. \tag{5.10}$$

If ψ is the solution of

$$\begin{cases}\omega^2\nabla\cdot(\mathbf{K}\nabla\psi)-\omega^\tau\gamma^2\mathbf{K}\psi=\rho-\zeta & \text{in } \nu(Y_m+j),\\ \psi=0 & \text{on } \nu(\partial Y_m+j),\end{cases} \tag{5.11}$$

then

$$\frac{c_1}{\nu}\|\psi\|_{L^2(\nu(Y_m+j))}\le\|\nabla\psi\|_{L^2(\nu(Y_m+j))}\le\frac{c_2}{\omega^2}\min\{\nu,\frac{\omega^{1-\frac{\tau}{2}}}{\gamma}\}\|\rho-\zeta\|_{L^2(\nu(Y_m+j))},$$

where c_1,c_2 are independent of ν. (5.8), $\sigma+\tau-2\ge 0$, and (5.10)–(5.11) imply

$$\int_{\nu(Y_m+j)}\omega^{2\sigma}|\rho-\zeta|^2\le c\left|\min\{\nu,\omega^{1-\frac{\tau}{2}}\gamma^{-1}\}\right|^2\Big(\int_{\nu(Y_m+j)}\omega^{2\sigma}|\nabla\zeta|^2$$
$$+\int_{\nu(Y_m+j)}\theta^{-2\mu}\nu^2\big(\left|\omega^{\sigma+\tau-2}\gamma^2\Pi_\nu\phi|_{\Omega_f^\nu}\right|^2+\left|\omega^{\sigma-2}\mathbb{G}\right|^2\big)\Big). \tag{5.12}$$

Summing (5.12) over all $\nu(Y_m+j)\subset B_\theta(0)\cap\Omega_m^\nu$ for $j\in\mathbb{Z}^n$, we obtain $(5.4)_2$ if ϵ_0 is small enough. $\square$

Lemma 5.2. *For any $\delta>0$, there are $\theta_1,\theta_2\in(0,1)$ (depending on $\delta,\mathbf{K},Y_f$) with $\theta_1<\theta_2^2$ and there is a $\epsilon_0>0$ (depending on $\theta_1,\theta_2,\delta,\mathbf{K}$) such that if*

$$-\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon}\nabla\Phi)+\Lambda^2\mathbf{K}_{\omega^\tau,\epsilon}\Phi=G \qquad \text{in } B_1(0), \tag{5.13}$$

if $\omega\in(0,1]$, $\epsilon\in(0,\epsilon_0)$, $\theta\in[\theta_1,\theta_2]$, $\Lambda\in(0,\infty)$, $3-\mu-\frac{n}{2}>0$, and if k satisfying $\epsilon/\theta^k\le\epsilon_0$, then

$$\begin{cases} \max\{1, |\theta^{k-1}\Lambda|^{2\mu}\} \fint_{B_{\theta^k}(0)} \left|\Pi_\epsilon \Phi|_{\Omega_f^\epsilon} - (\Pi_\epsilon \Phi|_{\Omega_f^\epsilon})_{0,\theta^k,I}\right|^2 dx \le \theta^{2k\mu} J^2, \\ \max\{1, |\theta^{k-1}\Lambda|^{2\mu}\} \fint_{B_{\theta^k}(0)\cap\Omega_m^\epsilon} \omega^{2\sigma} \left|\Phi - (\Pi_\epsilon \Phi|_{\Omega_f^\epsilon})_{0,\theta^k,I}\right|^2 dx \le \theta^{2k\mu} J^2, \end{cases} \tag{5.14}$$

where $\mu \equiv \frac{\delta}{n+\delta}$, $J \equiv \frac{1}{\epsilon_0}\big(\|\Pi_\epsilon \Phi|_{\Omega_f^\epsilon}, \omega\Phi\mathcal{X}_{\Omega_m^\epsilon}\|_{L^2(B_1(0))} + \|\mathbf{K}_{\omega^{\sigma-2},\epsilon} G\|_{L^{n+\delta}(B_1(0))} + \widetilde{\mathcal{H}}\big)$, *I is the identity matrix, and*

$$\widetilde{\mathcal{H}} = \begin{cases} 0 & \text{if } n = 2 \\ \|\mathbf{K}_{\omega^{\frac{-\tau(n+2)}{2n}},\epsilon} G\|_{L^{\frac{2n}{n-2}}(B_1(0))} & \text{if } n \ge 3 \end{cases} \cdot$$

Proof. Let c denote a constant independent of $\omega, \epsilon, \Lambda$. Let $\eta \in C_0^\infty(B_1(0))$ be a bell-shaped function satisfying $\eta \in [0, 1]$ and $\eta = 1$ in $B_{4/5}(0)$. We first multiply (5.13) by $\Phi\eta^2$ and use integration by parts, then repeat the same process except replacing $\Phi\eta^2$ by $\Lambda^2\Phi\eta^2$ to see

$$\max\{1, \Lambda\}\|\mathbf{K}_{\omega,\epsilon}\nabla\Phi, \Lambda\mathbf{K}_{\omega^{\frac{\tau}{2}},\epsilon}\Phi\|_{L^2(B_{4/5}(0))} \le c\|\mathbf{K}_{\omega,\epsilon}\Phi, \mathbf{K}_{1/\omega,\epsilon}G\|_{L^2(B_1(0))}. \tag{5.15}$$

Suppose $n \ge 3$ and $s + \frac{\tau}{2} - 1 \ge 0$, we multiply (5.13) by $\Phi - \Pi_\epsilon \Phi|_{\Omega_f^\epsilon}$ as well as use integration by parts, Sobolev imbedding Theorem [11], and (5.15) to see

$$\begin{aligned} \|\mathbf{K}_{\omega^s,\epsilon}\Phi\|_{L^{\frac{2n}{n-2}}(B_{4/5}(0))} &\le c\|\mathbf{K}_{\omega^s,\epsilon}\Phi, \mathbf{K}_{\omega^s,\epsilon}\nabla\Phi\|_{L^2(B_{4/5}(0))} \\ &\le c\|\mathbf{K}_{\omega,\epsilon}\Phi, \mathbf{K}_{1/\omega,\epsilon}G, \mathbf{K}_{\epsilon\omega^{s-2},\epsilon}G\|_{L^2(B_1(0))}. \end{aligned} \tag{5.16}$$

Suppose $n \ge 3$ and $q = \frac{2n}{n-2}$, we multiply (5.13) by $|\Lambda^2\Phi|^{q-2}\Lambda^2\Phi\eta^2$ and use integration by parts and (5.16) for $s = 2 - \tau(1 - 1/q)$ to see

$$\|\mathbf{K}_{\omega^{\frac{\tau}{q}},\epsilon}\Lambda^2\Phi\|_{L^q(B_{4/5}(0))} \le c\|\mathbf{K}_{\omega^{\tau(\frac{1}{q}-1)},\epsilon}G,\ \mathbf{K}_{\omega^{2-\tau(1-1/q)},\epsilon}\Phi\|_{L^q(B_1(0))} \le c\epsilon_0 J.$$

So we obtain, by Theorem 2.1 [1],

$$\Lambda^2\|\Pi_\epsilon\Phi|_{\Omega_f^\epsilon}\|_{L^{\frac{2n}{n-2}}(B_{4/5}(0))} \le c\epsilon_0 J \qquad \text{if } n \ge 3. \tag{5.17}$$

Proof of this lemma is done by induction. For $k = 1$, we define $\phi \equiv \frac{\Phi}{J}$, $\mathbb{G} \equiv \frac{G}{J}$, $\gamma \equiv \Lambda$. Then they satisfy (5.2) and (5.3) with $\nu = \epsilon$. By Lemma 5.1,

$$\begin{cases} \max\{1, \gamma^{2\mu}\} \fint_{B_\theta(0)} \left|\Pi_\nu\phi|_{\Omega_f^\nu} - (\Pi_\nu\phi|_{\Omega_f^\nu})_{0,\theta,I}\right|^2 dx \le \theta^{2\mu}, \\ \max\{1, \gamma^{2\mu}\} \fint_{B_\theta(0)\cap\Omega_m^\nu} \omega^{2\sigma} \left|\phi - (\Pi_\nu\phi|_{\Omega_f^\nu})_{0,\theta,I}\right|^2 dx \le \theta^{2\mu}. \end{cases}$$

This implies (5.14) for $k = 1$. Suppose (5.14) holds for some k satisfying $\epsilon/\theta^k \le \epsilon_0$, we define

$$\begin{cases} \phi(x) \equiv J^{-1}\theta^{-k\mu}\big(\Phi(\theta^k x) - (\Pi_\epsilon \Phi|_{\Omega_f^\epsilon})_{0,\theta^k,I}\big) \\ \mathbb{G}(x) \equiv J^{-1}\theta^{k(2-\mu)}\big(G(\theta^k x) - \Lambda^2 \mathbf{K}_{\omega^\tau,\epsilon/\theta^k}(\Pi_\epsilon \Phi|_{\Omega_f^\epsilon})_{0,\theta^k,I}\big) \text{ in } B_1(0). \\ \gamma \equiv \theta^k \Lambda \end{cases}$$

Then

$$-\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon/\theta^k} \nabla \phi) + \gamma^2 \mathbf{K}_{\omega^\tau,\epsilon/\theta^k}\, \phi = \mathbb{G} \quad \text{in } B_1(0).$$

By (5.15) and (5.17),

$$\Lambda^2(\Pi_\epsilon \Phi|_{\Omega_f^\epsilon})_{0,\theta^k,I} \le \begin{cases} \Lambda^2 \theta^{-k} \|\Pi_\epsilon \Phi|_{\Omega_f^\epsilon}\|_{L^2(B_{4/5}(0))} \le c\theta^{-k}\epsilon_0 J & \text{if } n=2, \\ \Lambda^2 \theta^{k(1-\frac{n}{2})} \|\Pi_\epsilon \Phi|_{\Omega_f^\epsilon}\|_{L^{\frac{2n}{n-2}}(B_{4/5}(0))} \le c\theta^{k(1-\frac{n}{2})}\epsilon_0 J & \text{if } n \ge 3. \end{cases}$$

Which implies, by induction, small θ_2, (5.1), and $3-\mu-\frac{n}{2} > 0$,

$$\begin{cases} \gamma \in (0,\infty), \\ \|\Pi_{\epsilon/\theta^k}\phi|_{\Omega_f^\epsilon/\theta^k}\|_{L^2(B_1(0))},\ \|\omega\phi\mathcal{X}_{\Omega_m^\epsilon/\theta^k}\|_{L^2(B_1(0))} \le 1, \\ \|\epsilon_0^{-1}\mathbb{G}\|_{L^{n+\delta}(B_1(0))},\ \|\mathbf{K}_{\omega^{\sigma-2},\epsilon/\theta^k}\mathbb{G}\|_{L^{n+\delta}(B_1(0))} \le 1. \end{cases}$$

By Lemma 5.1 (take $\nu = \epsilon/\theta^k$), we obtain

$$\begin{cases} \max\{1,\gamma^{2\mu}\} \fint_{B_\theta(0)} \left|\Pi_{\epsilon/\theta^k}\phi|_{\Omega_f^\epsilon/\theta^k} - (\Pi_{\epsilon/\theta^k}\phi|_{\Omega_f^\epsilon/\theta^k})_{0,\theta,I}\right|^2 dx \le \theta^{2\mu}, \\ \max\{1,\gamma^{2\mu}\} \fint_{B_\theta(0)\cap\Omega_m^\epsilon/\theta^k} \omega^{2\sigma} \left|\phi - (\Pi_{\epsilon/\theta^k}\phi|_{\Omega_f^\epsilon/\theta^k})_{0,\theta,I}\right|^2 dx \le \theta^{2\mu}. \end{cases} \tag{5.18}$$

Note, by Theorem 2.1 [1],

$$\begin{cases} \fint_{B_\theta(0)} \left|\Pi_{\epsilon/\theta^k}\phi|_{\Omega_f^\epsilon/\theta^k} - (\Pi_{\epsilon/\theta^k}\phi|_{\Omega_f^\epsilon/\theta^k})_{0,\theta,I}\right|^2 dx \\ \qquad = \fint_{B_{\theta^{k+1}}(0)} \dfrac{\left|\Pi_\epsilon \Phi|_{\Omega_f^\epsilon} - (\Pi_\epsilon\Phi|_{\Omega_f^\epsilon})_{0,\theta^{k+1},I}\right|^2}{J^2\,\theta^{2k\mu}} dx, \\ \fint_{B_\theta(0)\cap\Omega_m^\epsilon/\theta^k} \left|\phi - (\Pi_{\epsilon/\theta^k}\phi|_{\Omega_f^\epsilon/\theta^k})_{0,\theta,I}\right|^2 dx \\ \qquad = \fint_{B_{\theta^{k+1}}(0)\cap\Omega_m^\epsilon} \dfrac{\left|\Phi - (\Pi_\epsilon\Phi|_{\Omega_f^\epsilon})_{0,\theta^{k+1},I}\right|^2}{J^2\,\theta^{2k\mu}} dx. \end{cases} \tag{5.19}$$

Equations (5.18)–(5.19) imply the inequality (5.14) for $k+1$ case. □

Lemma 5.3. *For any $\delta > 0$, there is a $\epsilon_* \in (0,1)$ (depending on $\delta, \mathbf{K}, Y_f$) such that if $\omega \in (0,1]$, $\epsilon \in (0,\epsilon_*)$, $\Lambda \in (0,\infty)$, and $3 - \mu - \frac{n}{2} > 0$, then any solution of (5.13) satisfies*

$$\max\{1, \Lambda^{\mu/2}\}\big([\Phi]_{C^{0,\mu/2}(\overline{B_{1/2}(0)\cap\Omega_f^\epsilon})} + \omega^\sigma [\Phi]_{C^{0,\mu/2}(\overline{B_{1/2}(0)\cap\Omega_m^\epsilon})}\big) \le cJ, \tag{5.20}$$

where c is a constant independent of $\omega, \epsilon, \Lambda$. See Lemma 5.2 for μ, J.

Proof. Let $\theta_1, \theta_2, \epsilon_0$ be same as in Lemma 5.2, define $\epsilon_* \equiv \epsilon_0\theta_2/2$, and let $\epsilon \le \epsilon_*$. Denote by c a constant independent of $\omega, \epsilon, \Lambda$. From (5.15) and (5.17) in the proof of Lemma 5.2, we know

$$\Lambda^2(\Pi_\epsilon \Phi|_{\Omega_f^\epsilon})_{0,\frac{2\epsilon}{\epsilon_0},I} \le \begin{cases} c\Lambda^2\epsilon^{-1}\|\Pi_\epsilon\Phi|_{\Omega_f^\epsilon}\|_{L^2(B_{\frac{4}{5}}(0))} \le c\epsilon^{-1}\epsilon_0 J & \text{if } n=2, \\ c\Lambda^2\epsilon^{1-\frac{n}{2}}\|\Pi_\epsilon\Phi|_{\Omega_f^\epsilon}\|_{L^{\frac{2n}{n-2}}(B_{\frac{4}{5}}(0))} \le c\epsilon^{1-\frac{n}{2}}\epsilon_0 J & \text{if } n \ge 3. \end{cases} \tag{5.21}$$

Because of $\theta_1 < \theta_2^2$, for any $r \in [\epsilon/\epsilon_0, \theta_2]$, there are $\theta \in [\theta_1, \theta_2]$ and $k \in \mathbb{N}$ satisfying $r = \theta^k$. Lemma 5.2 implies, for any $r \in [\epsilon/\epsilon_0, \theta_2]$,

$$\begin{cases} \max\{1, |r\Lambda|^{2\mu}\} \displaystyle\fint_{B_r(0)} \Big|\Pi_\epsilon\Phi|_{\Omega_f^\epsilon} - (\Pi_\epsilon\Phi|_{\Omega_f^\epsilon})_{0,r,I}\Big|^2 dx \le r^{2\mu}J^2, \\ \max\{1, |r\Lambda|^{2\mu}\} \displaystyle\fint_{B_r(0)\cap\Omega_m^\epsilon} \omega^{2\sigma}\Big|\Phi - (\Pi_\epsilon\Phi|_{\Omega_f^\epsilon})_{0,r,I}\Big|^2 dx \le r^{2\mu}J^2. \end{cases} \tag{5.22}$$

Now we define

$$\begin{cases} \phi(x) \equiv J^{-1}\epsilon^{-\mu}\big(\Phi(\epsilon x) - (\Pi_\epsilon\Phi|_{\Omega_f^\epsilon})_{0,2\epsilon/\epsilon_0,I}\big) \\ \mathbb{G}(x) \equiv J^{-1}\epsilon^{2-\mu}\big(G(\epsilon x) - \Lambda^2\mathbf{K}_{\omega^\tau,1}(x)(\Pi_\epsilon\Phi|_{\Omega_f^\epsilon})_{0,2\epsilon/\epsilon_0,I}\big) \text{ in } B_{2/\epsilon_0}(0). \\ \gamma \equiv \epsilon\Lambda \end{cases}$$

Then

$$-\nabla\cdot(\mathbf{K}_{\omega^2,1}\nabla\phi) + \gamma^2\mathbf{K}_{\omega^\tau,1}\,\phi = \mathbb{G} \qquad \text{in } B_{2/\epsilon_0}(0).$$

Take $r = \frac{2\epsilon}{\epsilon_0}$ in (5.22) and employ (5.21) to get

$$\begin{cases} \gamma \in (0,\infty), \\ \|\mathbf{K}_{\omega^\sigma,1}\phi\|_{L^2(B_{2/\epsilon_0}(0))} + \|\mathbf{K}_{\omega^{\sigma-2},1}\mathbb{G}\|_{L^{n+\delta}(B_{2/\epsilon_0}(0))} \le c. \end{cases}$$

By Lemma 4.7 and Theorem 2.2 in page 296 [14],

$$\max\{1,\gamma\}\big([\phi]_{C^{0,\mu}(\overline{B_{1/\epsilon_0}(0)\cap\Omega_f^\epsilon/\epsilon})} + \omega^\sigma[\phi]_{C^{0,\mu}(\overline{B_{1/\epsilon_0}(0)\cap\Omega_m^\epsilon/\epsilon})}\big) \le c. \tag{5.23}$$

(5.23) implies that $(5.22)_1$ also holds for $r \le \epsilon/\epsilon_0$. So $(5.22)_1$ holds for $r \le \theta_2$. Which implies

$$\max\{1, |r\Lambda|^\mu\} \fint_{B_r(0)} \Big|\Pi_\epsilon\Phi|_{\Omega_f^\epsilon} - (\Pi_\epsilon\Phi|_{\Omega_f^\epsilon})_{0,r,I}\Big|^2 dx \le r^{2\mu}J^2 \qquad \text{for } r \le \theta_2. \tag{5.24}$$

Next we shift the origin of the coordinate system to any point $z \in B_{1/2}(0)$ and repeat above argument to see that (5.24) with 0 replaced by any $z \in B_{1/2}(0)$ also holds for $r \in (0, \theta_2)$. Together with Theorem 1.2 in page 70 [10], we obtain the Hölder estimate of $\Pi_\epsilon \Phi$ in $B_{1/2}(0)$. Hölder estimate of Φ in $B_{1/2}(0) \cap \overline{\Omega_m^\epsilon}$ is from (5.23). So (5.20) is proved. □

Remark 5.1. Let $\delta, \epsilon_*, \omega, \tau, \sigma, \Lambda$ be same as in Lemma 5.3. By Lemma 4.7, we know that if $\epsilon \in [\epsilon_*, 1]$, any solution of (5.13) satisfies (5.20). Together with Lemma 5.3, we know that any solution of (5.13) satisfies (5.20) if $\epsilon \in (0, 1]$.

5.2. Boundary estimate

Assume $0 \in \partial\Omega$. By A1, there exists a $C^{1,0}$ function $\Psi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\begin{cases} \Psi(0) = 0, \\ B_1(0) \cap \Omega/s = B_1(0) \cap \{(y_1, \hat{y}) \in \mathbb{R}^n \mid s y_1 > \Psi(s\hat{y})\} \quad \text{for any } s \in (0, 1]. \end{cases} \tag{5.25}$$

Define $B_1(0) \cap \Omega/s \equiv B_1(0) \cap \{(y_1, \hat{y}) \in \mathbb{R}^n \mid y_1 > 0\} \qquad \text{for } s = 0.$

Lemma 5.4. *For any $\delta > 0$, there are $\tilde{\theta}_1, \tilde{\theta}_2 \in (0, 1)$ (depending on $\delta, \mathbf{K}, Y_f, \partial\Omega$) satisfying $\tilde{\theta}_1 < \tilde{\theta}_2^2$ and there is a $\tilde{\epsilon}_0 > 0$ (depending on $\tilde{\theta}_1, \tilde{\theta}_2, \delta, \mathbf{K}, \partial\Omega$) satisfying $\tilde{\epsilon}_0 < \epsilon_0$ (ϵ_0 is that in Lemma 5.1) such that if*

$$\begin{cases} -\nabla \cdot (\mathbb{K}_{\omega^2,\epsilon,s} \nabla\phi) + \gamma^2 \mathbb{K}_{\omega^\tau,\epsilon,s}\, \phi = \mathbb{G} & \textit{in } B_1(0) \cap \Omega/s, \\ \phi = 0 & \textit{on } B_1(0) \cap \partial\Omega/s, \end{cases} \tag{5.26}$$

and if

$$\begin{cases} \omega, s \in (0, 1],\ \frac{\epsilon}{s} \in (0, \tilde{\epsilon}_0),\ \theta \in [\tilde{\theta}_1, \tilde{\theta}_2],\ \gamma \in (0, \infty), \\ \|\Pi_{\epsilon/s}\phi|_{\Omega_f^\epsilon/s}\|_{L^2(B_1(0))},\ \|\omega\phi\mathcal{X}_{\Omega_m^\epsilon/s}\|_{L^2(B_1(0))} \le 1, \\ \frac{1}{\tilde{\epsilon}_0}\|\mathbb{G}\|_{L^2(B_1(0)\cap\Omega/s)},\ \|\mathbb{K}_{\omega^{\sigma-2},\epsilon,s}\mathbb{G}\|_{L^2(B_1(0)\cap\Omega/s)} \le 1, \end{cases}$$

then

$$\begin{cases} \max\{1, \gamma^{2\mu}\} \displaystyle\fint_{B_\theta(0)\cap\Omega/s} \left|\Pi_{\epsilon/s}\phi|_{\Omega_f^\epsilon/s}\right|^2 dx \le \theta^{2\mu}, \\ \max\{1, \gamma^{2\mu}\} \displaystyle\fint_{B_\theta(0)\cap\Omega_m^\epsilon/s} \omega^{2\sigma}\ \phi^2 dx \le \theta^{2\mu}, \end{cases} \tag{5.27}$$

where $\mu \equiv \frac{\delta}{n+\delta}$. See (2.4) for $\mathbb{K}_{\omega,\epsilon,s}$.

Proof. Consider the following problem

$$\begin{cases} -\nabla \cdot (\mathcal{K}_\omega \nabla\phi) + \gamma^2 \widehat{\mathcal{K}}_\omega\, \phi = 0 & \text{in } B_{4/5}(0) \cap \Omega/t, \\ \phi = 0 & \text{on } B_{4/5}(0) \cap \partial\Omega/t, \end{cases} \tag{5.28}$$

where $t, \omega \in [0,1]$, $\gamma \in [0,\infty)$, and $\mathcal{K}_\omega, \widehat{\mathcal{K}}_\omega$ are defined in (4.50). Any solution ϕ of (5.28) satisfies, by Theorem 9.13 [11] and (4.51),

$$\max\{1,\gamma\}\|\phi\|_{C^1(\overline{B_{1/2}(0)\cap\Omega/t})} \le c\|\phi\|_{L^2(B_{4/5}(0)\cap\Omega/t)}, \tag{5.29}$$

where c is a constant depending on $\mathcal{K}_\omega, \partial\Omega$ but independent of t. If $\check{\mu}$ satisfies $\mu < \check{\mu} < 1$, by (5.29),

$$\max\{1,\gamma^2\} \fint_{B_\theta(0)\cap\Omega/t} \phi^2 dx \le \theta^{2\check{\mu}} \fint_{B_{4/5}(0)\cap\Omega/t} \phi^2 dx \tag{5.30}$$

for sufficiently small θ (depending on $\delta, \mathcal{K}_\omega, \partial\Omega$). Fix $\tilde{\theta}_1, \tilde{\theta}_2 \in (0, \frac{1}{2})$ such that $\tilde{\theta}_1 < \tilde{\theta}_2^2$ and (5.30) holds for any $\theta \in [\tilde{\theta}_1, \tilde{\theta}_2]$.

We claim $(5.27)_1$. If not, there is a sequence $\{\omega_\epsilon, s_\epsilon, \theta_\epsilon, \gamma_\epsilon, \phi_\epsilon, \mathbb{G}_\epsilon\}$ satisfying (5.26) and

$$\begin{cases} \omega_\epsilon, s_\epsilon \to \omega, s \in [0,1], \quad \epsilon, \epsilon/s_\epsilon \to 0, \quad \theta_\epsilon \to \theta \in [\tilde{\theta}_1, \tilde{\theta}_2], \quad \gamma_\epsilon \to \gamma \in [0,\infty], \\ \|\Pi_{\epsilon/s_\epsilon}\phi_\epsilon|_{\Omega^\epsilon_f/s_\epsilon}\|_{L^2(B_1(0))}, \ \|\omega_\epsilon\phi_\epsilon\mathcal{X}_{\Omega^\epsilon_m/s_\epsilon}\|_{L^2(B_1(0))} \le 1, \\ \|\mathbb{K}_{\omega_\epsilon^{\sigma-2},\epsilon,s_\epsilon}\mathbb{G}_\epsilon\|_{L^2(B_1(0)\cap\Omega/s_\epsilon)} \le 1, \\ \lim\limits_{\epsilon\to 0}\|\mathbb{G}_\epsilon\|_{L^2(B_1(0)\cap\Omega/s_\epsilon)} = 0, \\ \max\{1,\gamma_\epsilon^{2\mu}\} \fint_{B_{\theta_\epsilon}(0)\cap\Omega/s_\epsilon} \left|\Pi_{\epsilon/s_\epsilon}\phi_\epsilon|_{\Omega^\epsilon_f/s_\epsilon}\right|^2 dx > \theta_\epsilon^{2\mu}. \end{cases} \tag{5.31}$$

Let $\eta \in C_0^\infty(B_1(0))$ be a bell-shaped function satisfying $\eta \in [0,1]$ and $\eta = 1$ in $B_{4/5}(0)$. We multiply (5.26) by $\phi_\epsilon\eta^2$ and use integration by parts, then repeat the same process except replacing $\phi_\epsilon\eta^2$ by $\gamma_\epsilon^2\phi_\epsilon\eta^2$ to see

$$\max\{1,\gamma_\epsilon\}\|\mathbb{K}_{\omega_\epsilon,\epsilon,s_\epsilon}\nabla\phi_\epsilon, \ \gamma_\epsilon\mathbb{K}_{\omega_\epsilon^{\tau/2},\epsilon,s_\epsilon}\phi_\epsilon\|_{L^2(B_{4/5}(0)\cap\Omega/s_\epsilon)} \le c, \tag{5.32}$$

where c is a constant independent of $\epsilon, \omega_\epsilon, \gamma_\epsilon, s_\epsilon$.

Suppose $\gamma = \infty$ in (5.31)$_1$. (5.32) implies $\gamma_\epsilon^2\|\Pi_{\epsilon/s_\epsilon}\phi_\epsilon|_{\Omega^\epsilon_f/s_\epsilon}\|_{L^2(B_{4/5}(0)\cap\Omega/s_\epsilon)} \le c$. So

$$0 < \theta^{2\mu} = \lim_{\epsilon\to 0}\theta_\epsilon^{2\mu} \le \lim_{\epsilon\to 0}\max\{1,\gamma_\epsilon^{2\mu}\} \fint_{B_{\theta_\epsilon}(0)\cap\Omega/s_\epsilon} \left|\Pi_{\epsilon/s_\epsilon}\phi_\epsilon|_{\Omega^\epsilon_f/s_\epsilon}\right|^2 dx = 0,$$

which is impossible.

Suppose $\gamma < \infty$ in (5.31)$_1$. By (5.32), compactness principle and by tracing the proof of Theorem 2.3 [2], we can extract a subsequence (same notation for subsequence) such that

$$\begin{cases} \Pi_{\epsilon/s_\epsilon}\phi_\epsilon|_{\Omega^\epsilon_f/s_\epsilon} \to \phi & \text{in } L^2(B_{4/5}(0)\cap\Omega/s) \text{ strongly}, \\ \mathbb{K}_{\omega_\epsilon^2,\epsilon,s_\epsilon}\nabla\phi_\epsilon \to \mathcal{K}_\omega\nabla\phi & \text{in } L^2(B_{4/5}(0)\cap\Omega/s) \text{ weakly}, \\ \gamma_\epsilon^2\mathbb{K}_{\omega_\epsilon^\tau,\epsilon,s_\epsilon}\phi_\epsilon \to \gamma^2\widehat{\mathcal{K}}_\omega\phi & \text{in } L^2(B_{4/5}(0)\cap\Omega/s) \text{ weakly}. \end{cases} \tag{5.33}$$

See (4.50)–(4.51) for $\mathcal{K}_\omega, \widehat{\mathcal{K}}_\omega$. In (5.33), function ϕ satisfies (5.28) with $t = s$. By (5.30)–(5.31) and (5.33), we conclude

$$\theta^{2\mu} = \lim_{\epsilon\to 0} \theta_\epsilon^{2\mu} \le \lim_{\epsilon\to 0} \max\{1, \gamma_\epsilon^{2\mu}\} \fint_{B_{\theta_\epsilon}(0)\cap\Omega/s_\epsilon} \left|\Pi_{\epsilon/s_\epsilon}\phi_\epsilon|_{\Omega^\epsilon_f/s_\epsilon}\right|^2 dx$$
$$= \max\{1, \gamma^{2\mu}\} \fint_{B_\theta(0)\cap\Omega/s} \phi^2 dx \le \theta^{2\check{\mu}} \fint_{B_{4/5}(0)\cap\Omega/s} \phi^2 dx. \tag{5.34}$$

But (5.34) is impossible if we take $\tilde{\theta}_2$ small enough. Therefore, there is a $\tilde{\epsilon}_0$ such that $(5.27)_1$ holds for $\epsilon/s \le \tilde{\epsilon}_0$. Clearly, $\tilde{\epsilon}_0$ can be chosen so that $\tilde{\epsilon}_0 < \epsilon_0$ (see Lemma 5.1 for ϵ_0). Proof of $(5.27)_2$ is similar to the proof of $(5.4)_2$, so we skip it. □

Lemma 5.5. *For any $\delta > 0$, there are $\tilde{\theta}_1, \tilde{\theta}_2 \in (0,1)$ (depending on $\delta, \mathbf{K}, Y_f, \partial\Omega$) satisfying $\tilde{\theta}_1 < \tilde{\theta}_2^2$ and there is a $\tilde{\epsilon}_0 > 0$ (depending on $\tilde{\theta}_1, \tilde{\theta}_2, \delta, \mathbf{K}, \partial\Omega$) satisfying $\tilde{\epsilon}_0 < \epsilon_0$ (ϵ_0 is that in Lemma 5.2) such that if*

$$\begin{cases} -\nabla\cdot(\mathbf{K}_{\omega^2,\epsilon}\nabla\Phi) + \Lambda^2\mathbf{K}_{\omega^\tau,\epsilon}\Phi = G & \text{in } B_1(0)\cap\Omega, \\ \Phi = 0 & \text{on } B_1(0)\cap\partial\Omega, \end{cases} \tag{5.35}$$

and if $\omega \in (0,1]$, $\epsilon \in (0, \tilde{\epsilon}_0)$, $\theta \in [\tilde{\theta}_1, \tilde{\theta}_2]$, $\Lambda \in (0,\infty)$, and k satisfying $\epsilon/\theta^k \le \tilde{\epsilon}_0$, then

$$\begin{cases} \max\{1, |\theta^{k-1}\Lambda|^{2\mu}\} \displaystyle\fint_{B_{\theta^k}(0)\cap\Omega} \left|\Pi_\epsilon\Phi|_{\Omega^\epsilon_f}\right|^2 dx \le \theta^{2k\mu}\tilde{J}^2, \\ \max\{1, |\theta^{k-1}\Lambda|^{2\mu}\} \displaystyle\fint_{B_{\theta^k}(0)\cap\Omega^\epsilon_m} \omega^{2\sigma}\,\Phi^2 dx \le \theta^{2k\mu}\tilde{J}^2, \end{cases} \tag{5.36}$$

where $\tilde{J} \equiv \frac{1}{\tilde{\epsilon}_0}(\|\Pi_\epsilon\Phi|_{\Omega^\epsilon_f}, \omega\Phi\mathcal{X}_{\Omega^\epsilon_m}\|_{L^2(B_1(0))} + \|\mathbf{K}_{\omega^{\sigma-2},\epsilon}G\|_{L^{n+\delta}(B_1(0)\cap\Omega)})$ and $\mu \equiv \frac{\delta}{n+\delta}$.

Proof. The proof is similar to that of Lemma 5.2 and is done by induction on k. For $k = 1$, (5.36) is deduced from Lemma 5.4 with $s = 1$. Suppose (5.36) holds for some k with $\epsilon/\theta^k \le \tilde{\epsilon}_0$, we define

$$\begin{cases} \phi(x) \equiv \tilde{J}^{-1}\theta^{-k\mu}\Phi(\theta^k x) \\ \mathbb{G}(x) \equiv \tilde{J}^{-1}\theta^{k(2-\mu)}G(\theta^k x) & \text{in } B_1(0)\cap\Omega/\theta^k. \\ \gamma \equiv \theta^k\Lambda \end{cases}$$

Then

$$\begin{cases} -\nabla\cdot(\mathbb{K}_{\omega^2,\epsilon,\theta^k}\nabla\phi) + \gamma^2\mathbb{K}_{\omega^\tau,\epsilon,\theta^k}\,\phi = \mathbb{G} & \text{in } B_1(0)\cap\Omega/\theta^k, \\ \phi = 0 & \text{on } B_1(0)\cap\partial\Omega/\theta^k. \end{cases}$$

Following the argument of Lemma 5.2 and employing Lemma 5.4 with $s = \theta^k$, we obtain (5.36) with $k+1$ in place of k. □

Lemma 5.6. *For any $\delta > 0$, there is a $\tilde{\epsilon}_* \in (0,1)$ (depending on $\delta, \mathbf{K}, Y_f, \partial\Omega$) such that if $\omega \in (0,1]$, $\epsilon \in (0, \tilde{\epsilon}_*]$, $\Lambda \in (0,\infty)$, and $3 - \mu - \frac{n}{2} > 0$, then any solution of (5.35) satisfies*

$$\max\{1, \Lambda^{\mu/4}\}\big([\Phi]_{C^{0,\mu/4}(\overline{B_{1/2}(0)\cap\Omega_f^\epsilon})} + \omega^\sigma [\Phi]_{C^{0,\mu/4}(\overline{B_{1/2}(0)\cap\Omega_m^\epsilon})}\big) \le c\tilde{J}_*, \tag{5.37}$$

where c is independent of $\omega, \epsilon, \Lambda$. See Lemma 5.5 for μ. $\tilde{J}_*$ is defined as

$$\tilde{J}_* \equiv \frac{1}{\tilde{\epsilon}_*}\big(\|\Pi_\epsilon \Phi|_{\Omega_f^\epsilon}, \omega\Phi\mathcal{X}_{\Omega_m^\epsilon}\|_{L^2(B_1(0))} + \|\mathbf{K}_{\omega^{\sigma-2},\epsilon} G\|_{L^{n+\delta}(B_1(0)\cap\Omega)} + \widetilde{\mathcal{H}}_*\big),$$

$$\widetilde{\mathcal{H}}_* \equiv \begin{cases} 0 & \text{if } n = 2 \\ \|\mathbf{K}_{\omega^{\frac{-\tau(n+2)}{2n}},\epsilon} G\|_{L^{\frac{2n}{n-2}}(B_1(0)\cap\Omega)} & \text{if } n \ge 3 \end{cases}.$$

Proof. Let $\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\epsilon}_0, \tilde{J}$ be same as in Lemma 5.5, set $\tilde{\epsilon}_* \equiv \min\{\tilde{\epsilon}_0\tilde{\theta}_2/3, \epsilon_*\}$ where ϵ_* is the one in Lemma 5.3, and let $\epsilon \le \tilde{\epsilon}_*$. Denote by c a constant independent of $\omega, \epsilon, \Lambda$. Arguing as (5.15) and (5.17) in the proof of Lemma 5.2, we see

$$\begin{cases} \Lambda^2 \|\Pi_\epsilon \Phi|_{\Omega_f^\epsilon}\|_{L^2(B_{4/5}(0)\cap\Omega)} \le c\tilde{\epsilon}_*\tilde{J}_* & \text{if } n \ge 2, \\ \Lambda^2 \|\Pi_\epsilon \Phi|_{\Omega_f^\epsilon}\|_{L^{\frac{2n}{n-2}}(B_{4/5}(0)\cap\Omega)} \le c\tilde{\epsilon}_*\tilde{J}_* & \text{if } n \ge 3. \end{cases} \tag{5.38}$$

For any $x \in B_{\tilde{\theta}_2/3}(0) \cap \Omega$, define $\xi(x) \equiv |x - x_0|$ where $x_0 \in \partial\Omega$ satisfying $|x - x_0| = \min_{y\in\partial\Omega}|x - y|$. Then we have either case (1) $\xi(x) > \frac{2\epsilon}{3\tilde{\epsilon}_0}$ or case (2) $\xi(x) \le \frac{2\epsilon}{3\tilde{\epsilon}_0}$.

For case (1). Because of $\tilde{\theta}_1 < \tilde{\theta}_2^2$, for any $r \in [\epsilon/\tilde{\epsilon}_0, \tilde{\theta}_2]$, there are $\theta \in [\tilde{\theta}_1, \tilde{\theta}_2]$ and $k \in \mathbb{N}$ satisfying $r = \theta^k$. Since $\xi(x) \in [\frac{2\epsilon}{3\tilde{\epsilon}_0}, \frac{\tilde{\theta}_2}{3}]$, by Lemma 5.5,

$$\begin{cases} \max\{1, |r\Lambda|^{2\mu}\} \displaystyle\fint_{B_r(x_0)\cap\Omega} \Big|\Pi_\epsilon \Phi|_{\Omega_f^\epsilon}\Big|^2 dy \le r^{2\mu}\tilde{J}^2 \\ \max\{1, |r\Lambda|^{2\mu}\} \displaystyle\fint_{B_r(x_0)\cap\Omega_m^\epsilon} \omega^{2\sigma}\ \Phi^2 dy \le r^{2\mu}\tilde{J}^2 \end{cases} \text{for } r \in [\tfrac{3}{2}\xi(x), \tilde{\theta}_2].$$

So, for $s \in [\frac{\xi(x)}{2}, \frac{\tilde{\theta}_2}{3}]$,

$$\begin{cases} \max\{1, |s\Lambda|^{2\mu}\} \displaystyle\fint_{B_s(x)\cap\Omega} \Big|\Pi_\epsilon \Phi|_{\Omega_f^\epsilon} - (\Pi_\epsilon \Phi|_{\Omega_f^\epsilon})_{x,s,I}\Big|^2 dy \le cs^{2\mu}\tilde{J}_*^2, \\ \max\{1, |s\Lambda|^{2\mu}\} \displaystyle\fint_{B_s(x)\cap\Omega_m^\epsilon} \omega^{2\sigma}\big|\Phi - (\Pi_\epsilon \Phi|_{\Omega_f^\epsilon})_{x,s,I}\big|^2 dy \le cs^{2\mu}\tilde{J}_*^2, \end{cases} \tag{5.39}$$

where I is the identity. Next we shift the coordinate system such that x is located at the origin as well as define

$$\begin{cases} \phi(y) \equiv \tilde{J}_*^{-1}\, \xi^{-\mu}(x) \left(\Phi(\xi(x)y) - (\Pi_\epsilon \Phi|_{\Omega_f^\epsilon})_{x,\xi(x),I} \right) \\ \mathbb{G}(y) \equiv \tilde{J}_*^{-1}\, \xi^{2-\mu}(x) \left(G(\xi(x)y) - \Lambda^2 \mathbf{K}_{\omega^\tau,\epsilon/\xi(x)}(y)(\Pi_\epsilon \Phi|_{\Omega_f^\epsilon})_{x,\xi(x),I} \right) \text{ in } B_1(x). \\ \gamma \equiv \xi(x)\Lambda \end{cases}$$

Then

$$-\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon/\xi(x)} \nabla\phi) + \gamma^2 \mathbf{K}_{\omega^\tau,\epsilon/\xi(x)}\, \phi = \mathbb{G} \quad \text{ in } B_1(x). \tag{5.40}$$

Take $s = \xi(x) < 1$ in (5.39) and use (5.38) to see

$$\begin{cases} \gamma \in (0,\infty), \\ \|\Pi_{\epsilon/\xi(x)}\phi|_{\Omega_f^\epsilon/\xi(x)}, \omega\phi\mathcal{X}_{\Omega_m^\epsilon/\xi(x)}\|_{L^2(B_1(x))} + \|\mathbf{K}_{\omega^{\sigma-2},\epsilon/\xi(x)}\mathbb{G}\|_{L^{n+\delta}(B_1(x))} \le c, \\ \|\mathbf{K}_{\omega^{\frac{-\tau(n+2)}{2n}},\epsilon/\xi(x)}\mathbb{G}\|_{L^{\frac{2n}{n-2}}(B_1(x))} \le c \qquad \text{when } n \ge 3. \end{cases}$$

Apply Lemma 5.3 to (5.40) to obtain

$$\max\{1,\gamma^{\mu/2}\}\big([\phi]_{C^{0,\mu/2}(\overline{B_{1/2}(x)\cap\Omega_f^\epsilon/\xi(x)})} + \omega^\sigma [\phi]_{C^{0,\mu/2}(\overline{B_{1/2}(x)\cap\Omega_m^\epsilon/\xi(x)})}\big) \le c. \tag{5.41}$$

Which implies

$$\max\{1,|s\Lambda|^\mu\} \fint_{B_s(x)} \left|\Pi_\epsilon\Phi|_{\Omega_f^\epsilon} - (\Pi_\epsilon\Phi|_{\Omega_f^\epsilon})_{x,s,I}\right|^2 dy \le cs^\mu \tilde{J}_*^2 \quad \text{for } s < \tfrac{\xi(x)}{2}. \tag{5.42}$$

For case (2). Because of $\tilde{\theta}_1 < \tilde{\theta}_2^2$, for any $r \in [\epsilon/\tilde{\epsilon}_0, \tilde{\theta}_2]$, there are $\theta \in [\tilde{\theta}_1, \tilde{\theta}_2]$ and $k \in \mathbb{N}$ satisfying $r = \theta^k$. By Lemma 5.5,

$$\begin{cases} \max\{1,|r\Lambda|^{2\mu}\} \fint_{B_r(x_0)\cap\Omega} \left|\Pi_\epsilon\Phi|_{\Omega_f^\epsilon}\right|^2 dy \le cr^{2\mu}\tilde{J}^2 \\ \max\{1,|r\Lambda|^{2\mu}\} \fint_{B_r(x_0)\cap\Omega_m^\epsilon} \omega^{2\sigma}\ \Phi^2 dy \le cr^{2\mu}\tilde{J}^2 \end{cases} \quad \text{for } r \in [\epsilon/\tilde{\epsilon}_0, \tilde{\theta}_2]. \tag{5.43}$$

This implies, for $s \in [\frac{\epsilon}{3\tilde{\epsilon}_0}, \frac{\tilde{\theta}_2}{3}]$,

$$\begin{cases} \max\{1,|s\Lambda|^{2\mu}\} \fint_{B_s(x)\cap\Omega} \left|\Pi_\epsilon\Phi|_{\Omega_f^\epsilon} - (\Pi_\epsilon\Phi|_{\Omega_f^\epsilon})_{x,s,I}\right|^2 dy \le cs^{2\mu}\tilde{J}_*^2, \\ \max\{1,|s\Lambda|^{2\mu}\} \fint_{B_s(x)\cap\Omega_m^\epsilon} \omega^{2\sigma} \left|\Phi - (\Pi_\epsilon\Phi|_{\Omega_f^\epsilon})_{x,s,I}\right|^2 dy \le cs^{2\mu}\tilde{J}_*^2. \end{cases} \tag{5.44}$$

Again we shift the coordinate system such that x is located at the origin. Define

$$\begin{cases} \phi(y) \equiv \tilde{J}_*^{-1}\epsilon^{-\mu}\big(\Phi(\epsilon y) - (\Pi_\epsilon \Phi|_{\Omega_f^\epsilon})_{x,\epsilon/\tilde{\epsilon}_0,I}\big) \\ \mathbb{G}(y) \equiv \tilde{J}_*^{-1}\epsilon^{2-\mu}(G(\epsilon y) - \Lambda^2 \mathbb{K}_{\omega^\tau,\epsilon,\epsilon}(y)(\Pi_\epsilon \Phi|_{\Omega_f^\epsilon})_{x,\epsilon/\tilde{\epsilon}_0,I}) \\ \gamma \equiv \epsilon\Lambda \\ \phi_b \equiv -\tilde{J}_*^{-1}\epsilon^{-\mu}(\Pi_\epsilon \Phi|_{\Omega_f^\epsilon})_{x,\epsilon/\tilde{\epsilon}_0,I} \end{cases} \text{in } B_{\frac{1}{\tilde{\epsilon}_0}}(x) \cap \Omega/\epsilon.$$

Then these functions satisfy

$$\begin{cases} -\nabla \cdot (\mathbb{K}_{\omega^2,\epsilon,\epsilon} \nabla \phi) + \gamma^2 \mathbb{K}_{\omega^\tau,\epsilon,\epsilon}\, \phi = \mathbb{G} & \text{in } B_{\frac{1}{\tilde{\epsilon}_0}}(x) \cap \Omega/\epsilon, \\ \phi = \phi_b & \text{on } B_{\frac{1}{\tilde{\epsilon}_0}}(x) \cap \partial\Omega/\epsilon. \end{cases}$$

By (5.43)$_1$, ϕ_b is a constant independent of $\omega, \epsilon, \Lambda$. Take $s = \frac{\epsilon}{\tilde{\epsilon}_0}$ in (5.44) and use (5.38) to see

$$\begin{cases} \gamma \in (0, \infty), \\ \|\mathbb{K}_{\omega^\sigma,\epsilon,\epsilon}\phi\|_{L^2(B_{1/\tilde{\epsilon}_0}(x)\cap\Omega/\epsilon)} + \|\mathbb{K}_{\omega^{\sigma-2},\epsilon,\epsilon}\mathbb{G}\|_{L^{n+\delta}(B_{1/\tilde{\epsilon}_0}(x)\cap\Omega/\epsilon)} \\ \qquad +\|\phi_b\|_{W^{1,n+\delta}(B_{1/\tilde{\epsilon}_0}(x)\cap\Omega/\epsilon)} \le c. \end{cases}$$

By Lemma 4.7,

$$\max\{1, \gamma\}\big([\phi]_{C^{0,\mu}(\overline{B_{1/2\tilde{\epsilon}_0}(x)\cap\Omega_f^\epsilon/\epsilon})} + \omega^\sigma [\phi]_{C^{0,\mu}(\overline{B_{1/2\tilde{\epsilon}_0}(x)\cap\Omega_m^\epsilon/\epsilon})}\big) \le c. \tag{5.45}$$

(5.45) imply (5.44)$_1$ holds for $s \le \frac{\epsilon}{2\tilde{\epsilon}_0}$.

The Hölder estimate of $\Pi_\epsilon\Phi$ follows from (5.39)$_1$, (5.42), (5.44)$_1$, (5.45), and Theorem 1.2 in page 70 [10]. The Hölder estimate of Φ in $B_{1/2}(0) \cap \overline{\Omega_m^\epsilon}$ is from (5.41) and (5.45). □

Remark 5.2. Let $\delta, \tilde{\epsilon}_*, \omega, \tau, \sigma, \Lambda$ be same as in Lemma 5.6. By Lemma 4.7, we know that if $\epsilon \in [\epsilon_*, 1]$, any solution of (5.35) satisfies (5.37). Together with Lemma 5.6, any solution of (5.35) satisfies (5.37) if $\epsilon \in (0, 1]$.

Lemma 3.1 follows from partition of unity, Remark 5.1, and Remark 5.2.

Finally we give another local estimate for elliptic equations. Its proof is done by employing Remark 4.1 as well as by modifying the arguments of Lemma 3.10 [20] and Lemmas 5.1–5.6. The proof is skipped.

Lemma 5.7. *Under A1–A3, there is a constant c (independent of ω, ϵ) such that any solution of*

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \Phi) = 0 & \textit{in } B_1(0) \cap \Omega \\ \Phi = 0 & \textit{on } B_1(0) \cap \partial\Omega \end{cases}$$

satisfies

$$[\Phi]_{C^{0,\mu}(\overline{B_{1/2}(0)\cap\Omega})} \le c\|\mathbf{K}_{\omega^2,\epsilon}\Phi\|_{L^2(B_1(0)\cap\Omega)} \qquad \textit{for any } \mu \in (0, 1).$$

In particular, if $B_1(0) \subset \Omega$, then

$$\|\nabla\Phi\|_{L^\infty(B_{1/2}(0))} \le c\|\mathbf{K}_{\omega^2,\epsilon}\Phi\|_{L^2(B_1(0))}.$$

6. Proof of Lemma 3.3

This is done by modifying the argument of Theorem 2.1 [20], so we only sketch its proof. A1–A3 are assumed and notations in (2.4) are used. By energy method, Poincaré inequality, and duality argument, we see

Lemma 6.1. *Let $\sigma \in [0,2]$ and $Q \in L^2(\Omega)$. The solution of*

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \Phi + Q) = 0 & \text{in } \Omega \\ \Phi = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies

$$\|\mathbf{K}_{\omega^\sigma,\epsilon} \nabla \Phi\|_{L^2(\Omega)} \leq c \|\mathbf{K}_{\omega^{\sigma-2},\epsilon} Q\|_{L^2(\Omega)},$$

where c is a constant independent of ω, ϵ, σ.

Next we need the notation in (5.25).

Lemma 6.2. *Let $\sigma \in [0,2]$, $y_0 \in \Omega$, and $r \in (0,1]$. There is a constant c independent of $\omega, \epsilon, \sigma, y_0, r$ such that any solution of*

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \Phi) = 0 & \text{in } B_1(y_0) \cap \Omega \\ \Phi = 0 & \text{on } B_1(y_0) \cap \partial\Omega \end{cases} \tag{6.1}$$

satisfies

$$|\Phi(t) - \Phi(y)| \leq c|t-y|^\alpha r^{1-\alpha} \mathcal{F}_{t,y,\sigma} \left(\fint_{B_r(y_0)} |\mathbf{K}_{\omega^\sigma,\epsilon} \nabla \Phi|^2 \mathcal{X}_\Omega \, dz \right)^{1/2}, \tag{6.2}$$

where $t, y \in B_{r/2}(y_0) \cap \Omega$, $\alpha \in (0,1)$, and $\mathcal{F}_{t,y,\sigma} \equiv \max\{\mathbf{K}_{1/\omega^\sigma,\epsilon}(t), \mathbf{K}_{1/\omega^\sigma,\epsilon}(y)\}$.

Proof. For case $y_0 = 0$ and $\epsilon/r \geq 1$. If $B_1(0) \subset \Omega/r$, we define $\psi(z) \equiv \Phi(rz) + d$ for $d \in \mathbb{R}$. Then (6.1) implies

$$-\nabla \cdot (\mathbb{K}_{\omega^2,\epsilon,r} \nabla \psi) = 0 \qquad \text{in } B_1(0).$$

See (2.4) for $\mathbb{K}_{\omega^2,\epsilon,r}$. Theorem 9.11 [11], Corollary 4.2, Lemma 4.9, and Remark 4.1 imply

$$\|\mathbb{K}_{\omega^\sigma,\epsilon,r} \nabla \psi\|_{L^\infty(B_{1/4}(0))} \leq c \|\mathbb{K}_{\omega^\sigma,\epsilon,r} \psi\|_{L^2(B_1(0))},$$

where c is independent of $\omega, \epsilon, r, \sigma$. Since d is any number, it is easy to see

$$\|\mathbb{K}_{\omega^\sigma,\epsilon,r} \nabla \psi\|_{L^\infty(B_{1/4}(0))} \leq c \|\mathbb{K}_{\omega^\sigma,\epsilon,r} \nabla \psi\|_{L^2(B_2(0))}, \tag{6.3}$$

where c is independent of $\omega, \epsilon, r, \sigma$. If $0 \in \partial\Omega/r$, we define $\psi(z) \equiv \Phi(rz)$. Then

$$\begin{cases} -\nabla \cdot (\mathbb{K}_{\omega^2,\epsilon,r} \nabla \psi) = 0 & \text{in } B_2(0) \cap \Omega/r, \\ \psi = 0 & \text{in } B_2(0) \cap \partial\Omega/r. \end{cases}$$

Poincaré inequality and Theorem 9.11 [11] imply that (6.3) also holds true.

For case $y_0 = 0$ **and** $\epsilon/r < 1$. (6.2) is a direct consequence of Lemma 5.7 and Poincaré inequality.

For case $y_0 \neq 0$. (6.2) can be obtained by shifting the coordinate system such that y_0 is the origin of the coordinate system and then repeating the above argument. □

Next we have local L^p gradient estimate for elliptic equations.

Lemma 6.3. *Let* $\sigma \in [0, 2]$, $y_0 \in \Omega$, *and* $r \in (0, 1]$. *There is a constant* c *independent of* $\omega, \epsilon, \sigma, y_0, r$ *such that any solution of*

$$\begin{cases} -\nabla \cdot (\mathbf{K}_{\omega^2,\epsilon} \nabla \Phi) = 0 & \textit{in } B_{2r}(y_0) \cap \Omega \\ \Phi = 0 & \textit{on } B_{2r}(y_0) \cap \partial\Omega \end{cases}$$

satisfies

$$\left(\fint_{B_{r/2}(y_0)} |\mathbf{K}_{\omega^\sigma,\epsilon} \nabla \Phi|^p \mathcal{X}_\Omega \, dz \right)^{1/p} \le c \left(\fint_{B_{2r}(y_0)} |\mathbf{K}_{\omega^\sigma,\epsilon} \nabla \Phi|^2 \mathcal{X}_\Omega \, dz \right)^{1/2}, \tag{6.4}$$

where $p \in (2, \infty)$.

Proof. Let c denote a constant independent of $\omega, \epsilon, r, \sigma, y_0$.

For $B_{2r}(y_0) \subset \Omega$ **case.** By translation, we assume $y_0 = 0 \in \Omega$. Let $d \in \mathbb{R}$ and $\psi(z) = \Phi(rz) + d$. Then

$$-\nabla \cdot (\mathbb{K}_{\omega^2,\epsilon,r} \nabla \psi) = 0 \quad \text{in } B_2(0).$$

If $\epsilon/r \le 1$ (resp. $\epsilon/r > 1$), Lemma 5.7 (resp. Theorem 9.11 [11], Corollary 4.2, Lemma 4.9, and Remark 4.1) implies

$$\|\mathbb{K}_{\omega^\sigma,\epsilon,r} \nabla \psi\|_{L^p(B_{1/2}(0))} \le c \|\mathbb{K}_{\omega^\sigma,\epsilon,r} \psi\|_{L^2(B_1(0))} \qquad \text{for } p \in (2, \infty).$$

Since d is arbitrary, by Poincaré inequality.

$$\|\mathbb{K}_{\omega^\sigma,\epsilon,r} \nabla \psi\|_{L^p(B_{1/2}(0))} \le c \|\mathbb{K}_{\omega^\sigma,\epsilon,r} \nabla \psi\|_{L^2(B_2(0))} \qquad \text{for } p \in (2, \infty),$$

which implies (6.4).

For $y_0 \in \partial\Omega$ **case.** If $y \in B_{r/2}(y_0) \cap \Omega$, let $\xi(y)$ denote the distance from y to the boundary $B_{2r}(y_0) \cap \partial\Omega$. Move y to 0 by translation and define $\psi(z) \equiv \Phi(\xi(y)z) - \Phi(y)$. Then ψ satisfies

$$-\nabla \cdot (\mathbb{K}_{\omega^2,\epsilon,\xi(y)} \nabla \psi) = 0 \qquad \text{in } B_1(y) \;\; (\textit{or in } B_1(0)).$$

If $\epsilon/\xi(y) \leq 1$ (resp. $\epsilon/\xi(y) > 1$), Lemma 5.7 (resp. Theorem 9.13 [11] and definition of Ω_m^ϵ) implies

$$|\nabla\psi|(0) \leq c\|\mathbb{K}_{\omega^2,\epsilon,\xi(y)}\psi\|_{L^2(B_{1/2}(y))}. \tag{6.5}$$

By (6.5) and Lemma 6.2,

$$\begin{aligned}
\mathbf{K}_{\omega^\sigma,\epsilon}(y)|\nabla\Phi|(y) &\leq c\frac{\mathbf{K}_{\omega^\sigma,\epsilon}(y)}{\xi(y)}\Big(\fint_{B_{\xi(y)/2}(y)} \mathbf{K}_{\omega^4,\epsilon}(t)\,|\Phi(t)-\Phi(y)|^2\,dt\Big)^{1/2} \\
&\leq c\frac{r^{1-\alpha}}{\xi(y)}\Big(\fint_{B_{\xi(y)/2}(y)} |t-y|^{2\alpha}dt\Big)^{1/2}\Big(\fint_{B_{2r}(y_0)} |\mathbf{K}_{\omega^\sigma,\epsilon}\nabla\Phi|^2\mathcal{X}_\Omega\,dz\Big)^{1/2} \\
&\leq c\Big|\frac{r}{\xi(y)}\Big|^{1-\alpha}\Big(\fint_{B_{2r}(y_0)} |\mathbf{K}_{\omega^\sigma,\epsilon}\nabla\Phi|^2\mathcal{X}_\Omega\,dz\Big)^{1/2}.
\end{aligned} \tag{6.6}$$

Let us take $\alpha \in (0,1)$ such that $(1-\alpha)p < 1$. It is easy to see that (6.4) follows from (5.25) and (6.6). □

Lemma 3.3 is proved by following the argument in pages 73-74 [20] as well as by employing Theorem 4.1 [20] and Lemmas 6.1, 6.3.

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