Cauchy problem for the ellipsoidal BGK model for polyatomic particles

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Abstract

We establish the existence and uniqueness of mild solutions for the polyatomic ellipsoidal BGK model, which is a relaxation type kinetic model describing the evolution of polyatomic gaseous system at the mesoscopic level.

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1. Introduction

The derivation of the celebrated Boltzmann equations relies heavily on the assumption that the gas consists of monatomic particles, which is not the case for most of the realistic gases. Efforts to derive Boltzmann type kinetic models soon confront with the difficulty that it is virtually impossible to write the pre- and post-collision velocities in an explicit form, since polyatomic molecules can possess arbitrarily complicated structures. In search of tractable model equation for polyatomic gases that avoids such difficulties, a BGK type model was suggested as a generalization of the ellipsoidal BGK model [2,6,7,9,32,36]:

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$$\partial_t f + v \cdot \nabla_x f = A_{v,\theta}(\mathcal{M}_{v,\theta}(f) - f)$$
$$f(0, x, v, I) = f_0(x, v, I). \tag{1.1}$$

Unlike the monatomic case, a new variable I related to the internal energy due to the rotational and vibrational motions of the molecules is introduced so that the velocity distribution function $f(t, x, v, I)$ represents the number density on $(x, v) \in \mathbb{T}_x^3 \times \mathbb{R}_v^3$ at time t with internal energy $I^{2/\delta} \in \mathbb{R}^+$, where δ is the number of degrees of freedom except for the translational motion. We consider the fixed collision frequency $A_{v,\theta} = 1/(1 - v + v\theta)$ throughout this paper. Two relaxation parameters $-1/2 < v < 1$ and $0 \leq \theta \leq 1$ are chosen in such a way that Prandtl number and the second viscosity coefficient computed through the Chapmann–Enskog expansion, agrees with the physical data. (See [1,8,11,36].)

The polyatomic Gaussian $\mathcal{M}_{v,\theta}(f)$ reads

$$\mathcal{M}_{v,\theta}(f) = \frac{\rho \Lambda_\delta}{\sqrt{\det(2\pi \mathcal{T}_{v,\theta})}(T_\theta)^{\frac{\delta}{2}}} \exp\left(-\frac{1}{2}(v - U)^\top \mathcal{T}_{v,\theta}^{-1}(v - U) - \frac{I^{\frac{2}{\delta}}}{T_\theta}\right) \tag{1.2}$$

with normalizing factor

$$\Lambda_\delta^{-1} = \int\limits_{\mathbb{R}^+} \exp(-I^{\frac{2}{\delta}}) dI.$$

The macroscopic local density $\rho(t, x)$, bulk velocity $U(t, x)$, stress tensor $\Theta(t, x)$ and internal energy $E_\delta(t, x)$ are defined respectively by

$$\rho(t, x) = \int\limits_{\mathbb{R}^3 \times \mathbb{R}^+} f(t, x, v, I) dv dI$$

$$U(t, x) = \frac{1}{\rho} \int\limits_{\mathbb{R}^3 \times \mathbb{R}^+} v f(t, x, v, I) dv dI$$

$$\Theta(t, x) = \frac{1}{\rho} \int\limits_{\mathbb{R}^3 \times \mathbb{R}^+} f(t, x, v, I)(v - U(t, x)) \otimes (v - U(t, x)) dv dI \tag{1.3}$$

$$E_\delta(t, x) = \int\limits_{\mathbb{R}^3 \times \mathbb{R}^+} \left(\frac{1}{2}|v - U(t, x)|^2 + I^{\frac{2}{\delta}}\right) f(t, x, v, I) dv dI.$$

We split the internal energy E_δ into the internal energy from the translational motion E_{tr} and the one from the non-translational motion $E_{I,\delta}$:

$$E_{tr} = \int\limits_{\mathbb{R}^3 \times \mathbb{R}^+} \frac{1}{2}|v - U|^2 f dv dI,$$

$$E_{I,\delta} = \int\limits_{\mathbb{R}^3 \times \mathbb{R}^+} I^{\frac{2}{\delta}} f dv dI,$$

and define the corresponding temperatures T_δ, T_{tr} and $T_{I,\delta}$ by the equi-partition principle:

$$E_\delta = \frac{3+\delta}{2}\rho T_\delta, \quad E_{tr} = \frac{3}{2}\rho T_{tr}, \quad E_{I,\delta} = \frac{\delta}{2}\rho T_{I,\delta}.$$

Note that T_δ is a convex combination of T_{tr} and $T_{I,\delta}$:

$$T_\delta = \frac{3}{3+\delta}T_{tr} + \frac{\delta}{3+\delta}T_{I,\delta}. \tag{1.4}$$

Then, the relaxation temperature T_θ and the corrected temperature tensor $\mathcal{T}_{v,\theta}$ are defined as follows:

$$T_\theta = \theta T_\delta + (1-\theta)T_{I,\delta},$$
$$\mathcal{T}_{v,\theta} = \theta T_\delta Id + (1-\theta)\{(1-v)T_{tr}Id + v\Theta\}. \tag{1.5}$$

The relaxation operator satisfies the following cancellation properties:

$$\int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^+} (\mathcal{M}_{v,\theta}(f)-f)dxdvdI = 0$$

$$\int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^+} v(\mathcal{M}_{v,\theta}(f)-f)dxdvdI = 0 \tag{1.6}$$

$$\int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^+} \left(\frac{1}{2}|v|^2 + I^{\frac{2}{\delta}}\right)(\mathcal{M}_{v,\theta}(f)-f)dxdvdI = 0,$$

yielding the conservation of mass, momentum and energy respectively. The entropy dissipation for the polyatomic gas was proved by Andries and Perthame et al. [2]. (See also [9,28].)

$$\frac{d}{dt}\int\limits_{\mathbb{T}^3\times\mathbb{R}^3\times\mathbb{R}^+} f(t)\ln f(t)dxdvdI \leq 0.$$

2. Main result

Definition 2.1. Let $T > 0$. $f \in C_+([0,T]; \|\cdot\|_{L_q^\infty})$ is said to be a mild solution for (1.1) if it satisfies

$$f(t,x,v,I) = e^{-A_{v,\theta}t}f_0(x-vt,v,I) + A_{v,\theta}\int\limits_0^t e^{-A_{v,\theta}(t-s)}\mathcal{M}_{v,\theta}(f)(x-(t-s)v,v,s,I)ds,$$

where the weighted norm $\|\cdot\|_{L_q^\infty}$ is defined by

$$\|f(t)\|_{L_q^\infty} = ess\sup_{x,v,I}|f(t,x,v,I)(1+|v|^2+I^{\frac{2}{\delta}})^{\frac{q}{2}}|.$$

Our main result is as follows:

Theorem 2.2. *Let $0 < \theta \le 1$, $-1/2 < \nu < 1$, $\delta > 0$ and $q > 5 + \delta$. Suppose there exist positive constants C_u, C_l and C_1 such that*

$$\|f_0\|_{L_q^\infty} < C_u, \qquad \int_{\mathbb{R}^3 \times \mathbb{R}^+} f_0(x - vt, v, I)\, dv dI \ge C_l > 0.$$

Then, for any final time $T > 0$, there exists a unique mild solution $f \in C_+([0, T]; \|\cdot\|_{L_q^\infty})$ for (1.1) such that

(1) *f is bounded on $t \in [0, T)$ as*

$$\|f(t)\|_{L_q^\infty} \le e^{C_1 t} \|f_0\|_{L_q^\infty}.$$

(2) *There exist positive constants C_{T, f_0}, $C_{T, f_0, \delta}$ and $C_{T, f_0, \delta, q}$ such that*

$$\rho(x, t) \ge C_{T, f_0},$$
$$T_\delta(x, t) \ge C_{T, f_0, \delta},$$
$$\rho(x, t) + |U(x, t)| + T_\delta(x, t) \le C_{T, f_0, \delta, q}.$$

(3) *Conservation laws of mass, momentum and energy hold:*

$$\frac{d}{dt} \int_{\mathbb{T}^3 \times \mathbb{R}^3 \times \mathbb{R}^+} f\left(1, v, \frac{1}{2}|v|^2 + I^{\frac{2}{\delta}}\right) dx dv dI = 0.$$

(4) *H-theorem holds:*

$$\frac{d}{dt} \int_{\mathbb{T}^3 \times \mathbb{R}^3 \times \mathbb{R}^+} f \ln f\, dx dv dI = \int_{\mathbb{T}^3 \times \mathbb{R}^3 \times \mathbb{R}^+} \left(\mathcal{M}_{\nu, \theta}(f) - f\right) \ln f\, dx dv dI \le 0.$$

Remark 2.3. When $\theta = 0$, all the above estimates break down. Therefore, this case should be considered separately. See Section 7 for the discussion of this case.

The work is largely motivated by [1], and the proof is in the spirit of [30]. The main element of the argument in [30] is the establishment of a set of moment estimates obtained by applying various velocity domain decompositions in the moments of f. Those moments estimates then are used to control the local Maxwellian. The polyatomic feature of the model (1.1), however, gives rise to novel difficulties unobserved in the monatomic case.

First, due to the presence of the variable I related to the internal configuration of the molecules, the decomposition of the domain of integrals now has to be carried out in a combined domain of the velocity v and the internal configuration parameter I. (See Section 2.) An equivalence inequality between the relaxation temperature T_θ, the temperature tensor $\mathcal{T}_{\nu, \theta}$ and T_δ are

also derived, which, combined with the moment estimates, leads to the desired control of the polyatomic Gaussian.

Moreover, all the constants in those moment estimates and the equivalence estimates must be derived explicitly, and in a way that covers the whole range of θ except for 0, since the breakdown of the estimate at $\theta = 0$ cannot be observed otherwise. The case $\theta = 0$ corresponds to the situation where there is no interchange between the energy associated to the internal configuration (vibration and rotation of molecules) and the energy arise from the translational motions of the molecules [47], which means that (1.1) is essentially monatomic in this case, making such breakdown of the polyatomic estimates at $\theta = 0$ a natural one. (See Section 7.) We also mention that this agrees well with the dichotomy at $\theta = 0$ observed in the entropy production estimate of the polyatomic BGK model [27,47].

Ever since it was introduced in [4,42], the BGK model has seen huge applications in engineering and physics. The first mathematical study was carried out by Perthame in [29], where the existence of weak solutions was proven under the assumption of finite mass, momentum, energy and entropy. Perthame and Pulvirenti [30] then considered the class of solution space in which the uniqueness is guaranteed. It was later extended to the whole space [26], and to L^p solutions [50]. The Cauchy problem in the presence of external force or mean field was considered in [5,43,49]. Ukai studied a stationary problem on a bounded interval in [39]. The existence and asymptotic behavior near a global Maxwellian were studied in [3,44,49]. For various macroscopic limits of BGK type models, see [15,23–25,34,35]. Recently, Holway's ellipsoidal generalization of the original BGK model (ES-BGK model) was re-suggested in [2] with the first proof of H-theorem, and studied analytically in a series of paper [8,10,16,27,45–48]. Mathematical study on the polyatomic BGK model is in its initial state. See [9] for the derivation of this model. In [28], the entropy–entropy production estimate was derived. [44] studies the existence in the near-global-polyatomic Maxwellian-regime. A dichotomy in the dissipative estimate was also observed.

For the numerical results of BGK model – monatomic, or polyatomic – we refer to [1,11,17, 18,20,22,31,33,51] and references therein. A nice survey on various mathematical and physical issues on kinetic equations can be found in [12–14,19,32,37,38,40,41].

Following is the notational convention kept throughout this paper:

- Constants, usually denoted by C, are defined generically. Their value may vary line by line but can be computed in principle.
- When necessary, we use $C_{a,b,c,...}$, to show the dependence, not necessarily exclusive, on $a, b, c \cdots$.
- For $\kappa \in \mathbb{R}^3$, $\kappa^\top$ denotes its transpose.
- For symmetric $n \times n$ matrices A and B, $A \leq B$ means $B - A$ is positive definite. That is, $k^\top \{B - A\} k \geq 0$ for all $k \in \mathbb{R}^n$.

The paper is organized as follows: In the following Section 3, we establish several estimates for macroscopic variables. In Section 4, we define our solution space and show that the approximate solutions lie in that space for all steps of iterations. Section 5 is devoted to showing that the relaxation operator is Lipschitz continuous in the solution space. In Section 6, we combine all the previous results to complete the existence proof. The reason why the case $\theta = 0$ should be treated independently is briefly discussed in Section 7. In the appendix, we prove the cancellation property of the relaxation operator.

3. Estimates on macroscopic fields

Lemma 3.1. *Let $\delta > 0$, $-1/2 < \nu < 1$ and $0 < \theta \leq 1$. Suppose $\rho > 0$, $T_{tr} > 0$ and $T_{I,\delta} > 0$. Then temperature tensor $\mathcal{T}_{\nu,\theta}$ and the relaxation temperature T_θ satisfy the following equivalence type estimates:*

$$(1) \ \ \theta T_\delta Id \leq \mathcal{T}_{\nu,\theta} \leq \frac{1}{3} C_\nu \{3 + \delta(1-\theta)\} T_\delta Id,$$

$$(2) \ \ \theta T_\delta \leq T_\theta \leq \frac{1}{\delta} \{\delta + 3(1-\theta)\} T_\delta,$$

where $C_\nu = \max_\nu \{1 - \nu, 1 + 2\nu\}$.

Proof. (1) **(a) Upper bound**: Recalling the definition of $\mathcal{T}_{\nu,\theta}$, we write

$$\rho \mathcal{T}_{\nu,\theta} = \theta \rho T_\delta Id + (1-\theta)\left\{(1-\nu)\rho T_{tr} Id + \nu \rho \Theta\right\}$$

$$= \theta \rho T_\delta Id + (1-\theta)\left\{(1-\nu)\rho T_{tr} Id + \nu \int_{\mathbb{R}^3 \times \mathbb{R}^+} f(v-U) \otimes (v-U) dv dI\right\}.$$

From the identity

$$k^\top \left\{(v-U) \otimes (v-U)\right\} k = \left\{(v-U) \cdot k\right\}^2, \quad \text{for } k \in \mathbb{R}^3,$$

we derive

$$k^\top \{\rho \mathcal{T}_{\nu,\theta}\} k = \theta \rho T_\delta |k|^2 + (1-\theta)\left\{(1-\nu)\rho T_{tr} |k|^2 + \nu \int_{\mathbb{R}^3 \times \mathbb{R}^+} f\left\{(v-U) \cdot k\right\}^2 dv dI\right\}. \tag{3.1}$$

If $0 \leq \nu < 1$, using the Cauchy–Schwartz inequality, we get

$$\int_{\mathbb{R}^3 \times \mathbb{R}^+} f\left\{(v-U) \cdot k\right\}^2 dv dI \leq \int_{\mathbb{R}^3 \times \mathbb{R}^+} f|v-U|^2 |k|^2 dv dI = 3\rho T_{tr} |k|^2,$$

so that

$$k^\top \{\rho \mathcal{T}_{\nu,\theta}\} k \leq \theta \rho T_\delta |k|^2 + (1-\theta)\left\{(1-\nu)\rho T_{tr}|k|^2 + 3\nu\rho T_{tr}|k|^2\right\}$$

$$= \theta \rho T_\delta |k|^2 + (1-\theta)(1+2\nu)\rho T_{tr} |k|^2$$

$$\leq (1+2\nu)\rho \left\{\theta T_\delta + (1-\theta)T_{tr}\right\} |k|^2.$$

In the case of $-1/2 < \nu < 0$, the last term in (3.1) is non-positive. Thus

$$k^\top \{\rho \mathcal{T}_{\nu,\theta}\} k \leq \theta \rho T_\delta |k|^2 + (1-\theta)(1-\nu)\rho T_{tr}|k|^2$$
$$\leq (1-\nu)\rho \left\{ \theta T_\delta |k|^2 + (1-\theta)T_{tr} \right\} |k|^2.$$

Combining these two cases, we arrive at

$$k^\top \{\rho \mathcal{T}_{\nu,\theta}\} k \leq \max\{1-\nu, 1+2\nu\}\rho \{(1-\theta)T_{tr} + \theta T_\delta\} |k|^2. \tag{3.2}$$

Now, we recall (1.4) to see

$$T_\delta = \frac{3}{3+\delta}T_{tr} + \frac{\delta}{3+\delta}T_{I,\delta} \geq \frac{3}{3+\delta}T_{tr}, \tag{3.3}$$

or

$$T_{tr} \leq \frac{3+\delta}{3}T_\delta$$

to derive from (3.2) that

$$k^\top \{\rho \mathcal{T}_{\nu,\theta}\} k \leq \frac{1}{3}\max\{1-\nu, 1+2\nu\}\rho \{3 + \delta(1-\theta)\}T_\delta |k|^2.$$

This implies the desired estimate, since we assumed $\rho > 0$.

(b) Lower bound: Denote the last term in (3.1) by A:

$$A = (1-\nu)\rho T_{tr}|k|^2 + \nu \int_{\mathbb{R}^3 \times \mathbb{R}^+} f\{(v-U)\cdot k\}^2 dv dI.$$

Then, when $0 < \nu < 1$, A satisfies

$$A \geq (1-\nu)\rho T_{tr}|k|^2,$$

whereas we have

$$A \geq (1-\nu)\rho T_{tr}|k|^2 + \nu \left(\int_{\mathbb{R}^3} f|v-U|^2 dv \right) |k|^2 = (1+2\nu)\rho T_{tr}|k|^2,$$

for $-1/2 < \nu \leq 0$. Therefore, we conclude from our assumption on ρ and T_{tr} that $A \geq 0$. Thus, we deduce from (3.1)

$$k^\top \{\rho \mathcal{T}_{\nu,\theta}\} k \geq \theta \rho T_\delta |k|^2 + (1-\theta)A \geq \theta \rho T_\delta |k|^2, \tag{3.4}$$

which gives the desired result.

(2) From the definition of T_δ (1.4), we have

$$T_\delta = \frac{3}{3+\delta} T_{tr} + \frac{\delta}{3+\delta} T_{I,\delta} \geq \frac{\delta}{3+\delta} T_{I,\delta}, \tag{3.5}$$

so that

$$T_{I,\delta} \leq \frac{3+\delta}{\delta} T_\delta.$$

Therefore,

$$\begin{aligned}
T_\theta &= (1-\theta) T_{I,\delta} + \theta T_\delta \\
&\leq (1-\theta) \left(\frac{3+\delta}{\delta} T_\delta \right) + \theta T_\delta \\
&= \frac{1}{\delta} \left\{ \delta + 3(1-\theta) \right\} T_\delta.
\end{aligned}$$

The lower bound comes directly from the definition:

$$T_\theta = (1-\theta) T_{I,\delta} + \theta T_\delta \geq \theta T_\delta. \qquad \square$$

Lemma 3.2. *Assume $\rho > 0$ and $\|f\|_{L_q^\infty} < \infty$. Then we have*

$$\rho \leq C_\delta \|f\|_{L_q^\infty} T_\delta^{\frac{3+\delta}{2}}$$

for

$$C_\delta = 2^{\frac{7}{2}} \pi^2 (3+\delta)^{\frac{1+\delta}{2}} \delta.$$

Proof. We divide the integral domain as

$$\begin{aligned}
\rho &= \int_{\mathbb{R}^3 \times \mathbb{R}^+} f \, dv dI \\
&\leq \int_{\frac{1}{3+\delta}|v-U|^2 + \frac{2}{3+\delta} I^{\frac{2}{\delta}} > R^2} f \, dv dI + \int_{\frac{1}{3+\delta}|v-U|^2 + \frac{2}{3+\delta} I^{\frac{2}{\delta}} \leq R^2} f \, dv dI \tag{3.6} \\
&\equiv I_1 + I_2.
\end{aligned}$$

From the definition of T_δ, we see that

$$I_1 \leq \frac{1}{R^2} \int_{\frac{1}{3+\delta}|v-U|^2+\frac{2}{3+\delta}I^{\frac{2}{\delta}}>R^2} \left(\frac{1}{3+\delta}|v-U|^2 + \frac{2}{3+\delta}I^{\frac{2}{\delta}} \right) f \, dv \, dI$$

$$\leq \frac{1}{R^2} \rho T_\delta.$$

For I_2, we estimate

$$I_2 \leq \left(\int_{\frac{1}{3+\delta}|v-U|^2+\frac{2}{3+\delta}I^{\frac{2}{\delta}}\leq R^2} dv \, dI \right) \|f\|_{L_q^\infty},$$

and make a change of variable:

$$\sqrt{\frac{1}{3+\delta}}(v_1 - U_1) = r \sin\varphi \cos\theta \sin k,$$

$$\sqrt{\frac{1}{3+\delta}}(v_2 - U_2) = r \sin\varphi \sin\theta \sin k,$$

$$\sqrt{\frac{1}{3+\delta}}(v_3 - U_3) = r \cos\varphi \sin k,$$

$$\sqrt{\frac{2}{3+\delta}}I^{\frac{1}{\delta}} = r \cos k,$$

for $0 \leq r \leq R, 0 \leq \varphi \leq \pi, 0 \leq \theta \leq 2\pi, 0 \leq k \leq \frac{\pi}{2}$. The Jacobian

$$J = \frac{\partial(v_1, v_2, v_3, I)}{\partial(r, \varphi, \theta, k)},$$

is computed as

$$|J| = (3+\delta)^{\frac{3}{2}} \left(\frac{3+\delta}{2} \right)^{\frac{\delta}{2}}$$

$$\times \left| \det \begin{pmatrix} \cos\theta \sin\varphi \sin k & r \cos\theta \sin k \cos\varphi & -r \sin\varphi \sin\theta \sin k & r \sin\varphi \cos\theta \cos k \\ \sin\varphi \sin\theta \sin k & r \cos\varphi \sin\theta \sin k & r \sin\varphi \cos\theta \sin k & r \sin\varphi \sin\theta \cos k \\ \cos\varphi \sin k & -r \sin\varphi \sin k & 0 & r \cos\varphi \cos k \\ \delta r^{\delta-1} \cos^\delta k & 0 & 0 & \delta r^\delta \cos^{\delta-1} k \sin k \end{pmatrix} \right|$$

$$= \delta (3+\delta)^{\frac{3}{2}} \left(\frac{3+\delta}{2} \right)^{\frac{\delta}{2}} r^{\delta+2} |\sin\varphi \cos^{\delta-1} k \sin^2 k|,$$

so that

$$I_2 \leq \|f\|_{L_q^\infty} \int_0^{\frac{\pi}{2}} \int_0^\pi \int_0^{2\pi} \int_0^R \delta(3+\delta)^{\frac{3}{2}} \left(\frac{3+\delta}{2}\right)^{\frac{\delta}{2}} r^{\delta+2} |\sin\varphi\cos^{\delta-1}k\sin^2 k| dr d\theta d\varphi dk$$

$$\leq \|f\|_{L_q^\infty} \left\{ (3+\delta)^{\frac{3}{2}} \left(\frac{3+\delta}{2}\right)^{\frac{\delta}{2}} \frac{2\pi^2\delta}{3+\delta} \right\} R^{3+\delta}$$

$$= \|f\|_{L_q^\infty} \left\{ 2^{\frac{2-\delta}{2}} \pi^2 (3+\delta)^{\frac{1+\delta}{2}} \delta \right\} R^{3+\delta}.$$

Thus, (3.6) can be estimated as follows:

$$\rho \leq \frac{1}{R^2}\rho T_\delta + \left\{ 2^{\frac{2-\delta}{2}} \pi^2 (3+\delta)^{\frac{1+\delta}{2}} \delta \right\} R^{3+\delta} \|f\|_{L_q^\infty}.$$

We optimize this by setting

$$R^{5+\delta} = \frac{\rho T_\delta}{\left\{ 2^{\frac{2-\delta}{2}} \pi^2 (3+\delta)^{\frac{1+\delta}{2}} \delta \right\} \|f\|_{L_q^\infty}}$$

to get

$$\rho \leq 2 \left\{ 2^{\frac{2-\delta}{2}} \pi^2 (3+\delta)^{\frac{1+\delta}{2}} \delta \right\}^{\frac{2}{5+\delta}} \{\rho T_\delta\}^{\frac{3+\delta}{5+\delta}},$$

which implies

$$\rho \leq \left\{ 2^{\frac{7}{2}} \pi^2 (3+\delta)^{\frac{1+\delta}{2}} \delta \right\} \|f\|_{L_q^\infty} T_\delta^{\frac{3+\delta}{2}}.$$

This completes the proof.   □

Lemma 3.3. *Assume $\rho > 0$ and $\|f\|_{L_q^\infty} > 0$. Then, for $q > 5 + \delta$, we have*

$$\rho(T_\delta + |U|^2)^{\frac{q-\delta-3}{2}} \leq C_{\delta,q} \|f\|_{L_q^\infty},$$

where constant $C_{\delta,q}$ is given by

$$C_{\delta,q} = \left\{ \frac{2^{\frac{q-2\delta-1}{2}} \pi^2 (3+\delta)^{\frac{q}{2}} \delta}{q-\delta-5} \right\}.$$

Proof. From the definition of T_δ, we write

$$\rho\left(T_\delta + \frac{1}{3+\delta}|U|^2\right) = \int_{\mathbb{R}^3 \times \mathbb{R}^+} \left(\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta}I^{\frac{2}{\delta}}\right) f dv dI.$$

We then split the integral into the following two part as

$$\rho\left(T_\delta + \frac{1}{3+\delta}|U|^2\right) = \int\limits_{\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta}I^{\frac{2}{\delta}} > R^2} \left(\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta}I^{\frac{2}{\delta}}\right) f\,dv\,dI$$

$$+ \int\limits_{\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta}I^{\frac{2}{\delta}} \le R^2} \left(\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta}I^{\frac{2}{\delta}}\right) f\,dv\,dI \qquad (3.7)$$

$$= I_1 + I_2.$$

The estimate for I_2 is simple:

$$I_2 \le R^2 \int\limits_{\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta}I^{\frac{2}{\delta}} \le R^2} f\,dv\,dI \le R^2\rho.$$

For I_1, we extract $\|f\|_{L_q^\infty}$ out of the integral:

$$I_1 \le \int\limits_{\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta}I^{\frac{2}{\delta}} > R^2} \frac{\left(\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta}I^{\frac{2}{\delta}}\right)^{\frac{q}{2}} f}{\left(\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta}I^{\frac{2}{\delta}}\right)^{\frac{q-2}{2}}} \,dv\,dI$$

$$\le \|f\|_{L_q^\infty} \int\limits_{\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta}I^{\frac{2}{\delta}} > R^2} \frac{1}{\left(\frac{1}{3+\delta}|v|^2 + \frac{2}{3+\delta}I^{\frac{2}{\delta}}\right)^{\frac{q-2}{2}}} \,dv\,dI,$$

and use the same change of variable as in the proof of the previous lemma to estimate

$$I_1 \le \|f\|_{L_q^\infty} \int\limits_0^{\frac{\pi}{2}} \int\limits_0^\pi \int\limits_0^{2\pi} \int\limits_R^\infty \frac{\delta(3+\delta)^{\frac{3}{2}}\left(\frac{3+\delta}{2}\right)^{\frac{\delta}{2}} r^{\delta+2}|\sin\varphi\cos^{\delta-1}k\sin^2 k|}{r^{q-2}} \,dr\,d\theta\,d\varphi\,dk$$

$$\le \|f\|_{L_q^\infty} \left\{ \frac{2\pi^2\delta(3+\delta)^{\frac{3}{2}}\left(\frac{3+\delta}{2}\right)^{\frac{\delta}{2}}}{q-\delta-5} \right\} R^{\delta+5-q}$$

$$= \|f\|_{L_q^\infty} \left\{ \frac{2^{\frac{2-\delta}{2}}\pi^2(3+\delta)^{\frac{3+\delta}{2}}\delta}{q-\delta-5} \right\} R^{\delta+5-q}.$$

Inserting these computations into (3.7), we get

$$\rho\left(T_\delta + \frac{1}{3+\delta}|U|^2\right) \le \rho R^2 + \left\{ \frac{2^{\frac{2-\delta}{2}}\pi^2(3+\delta)^{\frac{3+\delta}{2}}\delta}{q-\delta-5} \right\} \|f\|_{L_q^\infty} R^{\delta+5-q}.$$

Now, take

$$R^{\delta+3-q} = \left\{ \frac{q-\delta-5}{2^{\frac{2-\delta}{2}}\pi^2(3+\delta)^{\frac{3+\delta}{2}}\delta} \right\} \frac{\rho}{\|f\|_{L_q^\infty}},$$

to get

$$\rho\left(T_\delta + \frac{1}{3+\delta}|U|^2\right) \le 2\left\{ \frac{2^{\frac{2-\delta}{2}}\pi^2(3+\delta)^{\frac{3+\delta}{2}}\delta}{q-\delta-5} \right\}^{\frac{2}{q-\delta-3}} \rho^{\frac{\delta+5-q}{\delta+3-q}} \|f\|_{L_q^\infty}^{\frac{2}{q-\delta-3}}.$$

This implies

$$\rho(T_\delta + |U|^2)^{\frac{q-\delta-3}{2}} \le \{2(3+\delta)\}^{\frac{q-\delta-3}{2}} \left\{ \frac{2^{\frac{2-\delta}{2}}\pi^2(3+\delta)^{\frac{3+\delta}{2}}\delta}{q-\delta-5} \right\} \|f\|_{L_q^\infty}$$

$$= \left\{ \frac{2^{\frac{q-2\delta-1}{2}}\pi^2(3+\delta)^{\frac{q}{2}}\delta}{q-\delta-5} \right\} \|f\|_{L_q^\infty},$$

which completes the proof. □

Lemma 3.4. *Assume* $\|f\|_{L_q^\infty}$, ρ, $T_\delta > 0$. *Then we have*

$$\frac{\rho|U|^{3+\delta+q}}{[(T_\delta + |U|^2)T_\delta]^{\frac{3+\delta}{2}}} \le C_{\delta,q} \|f\|_{L_q^\infty}$$

where $C_{\delta,q} = 2^{\frac{11+2\delta+2q}{2}}\pi^2(3+\delta)^{2+\delta}\delta$.

Proof. For simplicity, we set

$$A(v, I) = \sqrt{\frac{1}{3+\delta}}|v-U| + \sqrt{\frac{2}{3+\delta}}\, I^{\frac{1}{\delta}}.$$

We split the macroscopic momentum as

$$\rho|U| \le \int\limits_{A(v,I)\le R} f|v|dvdI + \int\limits_{A(v,I)>R} f|v|dvdI$$

$$\equiv I_1 + I_2.$$

By Hölder's inequality,

$$I_1 \le \left\{ \int\limits_{A(v,I)\le R} f dvdI \right\}^{1-\frac{1}{q}} \left\{ \int\limits_{A(v,I)\le R} \left(f^{\frac{1}{q}}|v|\right)^q dvdI \right\}^{\frac{1}{q}}$$

$$\leq \left\{ \int_{\mathbb{R}^3 \times \mathbb{R}^+} f \, dv dI \right\}^{1-\frac{1}{q}} \left\{ \int_{A(v,I) \leq R} f |v|^q \, dv dI \right\}^{\frac{1}{q}}$$

$$\leq \rho^{1-\frac{1}{q}} \|f\|_{L_q^\infty}^{\frac{1}{q}} \left\{ \int_{A(v,I) \leq R} dv dI \right\}^{\frac{1}{q}}.$$

Then, computing similarly as in the previous lemma, we have

$$\int_{A(v,I) \leq R} dv dI \leq \int_{\frac{1}{3+\delta}|v-U|^2 + \frac{2}{3+\delta}I^{2/\delta} \leq R^2} dv dI \leq \left\{ 2^{\frac{2-\delta}{2}} \pi^2 (3+\delta)^{\frac{1+\delta}{2}} \delta \right\} R^{3+\delta}.$$

Therefore, we bound I_1 by

$$\rho^{1-\frac{1}{q}} \|f\|_{L_q^\infty}^{\frac{1}{q}} \left\{ 2^{\frac{2-\delta}{2}} \pi^2 (3+\delta)^{\frac{1+\delta}{2}} \delta \right\}^{\frac{1}{q}} R^{\frac{3+\delta}{q}}.$$

On the other hand, we observe

$$I_2 \leq \frac{1}{R} \int_{A(v,I) > R} f |v| \left\{ \sqrt{\frac{1}{3+\delta}} |v - U| + \sqrt{\frac{2}{3+\delta}} I^{\frac{1}{\delta}} \right\} dv dI.$$

Applying Hölder's inequality again,

$$I_2 \leq \frac{\sqrt{2(3+\delta)}}{R} \left\{ \int_{\mathbb{R}^3 \times \mathbb{R}^+} f \left(\frac{1}{3+\delta} |v|^2 + \frac{2}{3+\delta} I^{\frac{2}{\delta}} \right) dv dI \right\}^{\frac{1}{2}}$$

$$\times \left\{ \int_{\mathbb{R}^3 \times \mathbb{R}^+} f \left(\frac{1}{3+\delta} |v - U|^2 + \frac{2}{3+\delta} I^{\frac{2}{\delta}} \right) dv dI \right\}^{\frac{1}{2}}$$

$$= \frac{\sqrt{2(3+\delta)}}{R} \left\{ \frac{1}{3+\delta} \rho |U|^2 + \rho T_\delta \right\}^{\frac{1}{2}} \{ \rho T_\delta \}^{\frac{1}{2}}.$$

In conclusion,

$$\rho |U| \leq \rho^{1-\frac{1}{q}} \|f\|_{L_q^\infty}^{\frac{1}{q}} \left\{ 2^{\frac{2-\delta}{2}} \pi^2 (3+\delta)^{\frac{1+\delta}{2}} \delta \right\}^{\frac{1}{q}} R^{\frac{3+\delta}{q}} + \frac{\sqrt{2(3+\delta)}}{R} \rho [(|U|^2 + T_\delta) T_\delta]^{\frac{1}{2}}.$$

$$(3.8)$$

The optimizing choice for R then is

$$R^{3+\delta+q} = \frac{[2(3+\delta)]^{\frac{q}{2}} \rho[(|U|^2 + T_\delta)T_\delta]^{\frac{q}{2}}}{\left\{2^{\frac{2-\delta}{2}} \pi^2 (3+\delta)^{\frac{1+\delta}{2}} \delta\right\} \|f\|_{L_q^\infty}},$$

for which the right hand side of (3.8) becomes

$$2\left\{2^{\frac{2-\delta}{2}} \pi^2 (3+\delta)^{\frac{1+\delta}{2}} \delta\right\}^{\frac{1}{3+\delta+q}} \{2(3+\delta)\}^{\frac{3+\delta}{2(3+\delta+q)}} \rho^{\frac{2+\delta+q}{3+\delta+q}} [(|U|^2 + T_\delta)T_\delta]^{\frac{3+\delta}{2(3+\delta+q)}} \|f\|_{L_q^\infty}^{\frac{1}{3+\delta+q}}.$$

This gives

$$\frac{\rho|U|^{3+\delta+q}}{[(|U|^2 + T_\delta)T_\delta]^{\frac{3+\delta}{2}}} \le 2^{3+\delta+q} \{2(3+\delta)\}^{\frac{3+\delta}{2}} \left\{2^{\frac{2-\delta}{2}} \pi^2 (3+\delta)^{\frac{1+\delta}{2}} \delta\right\} \|f\|_{L_q^\infty}$$

$$= 2^{\frac{11+2\delta+2q}{2}} \pi^2 (3+\delta)^{2+\delta} \delta \|f\|_{L_q^\infty}. \qquad \square$$

4. Solution space and approximate scheme

We set up our solution space Ω:

$$\Omega = \left\{f \in C_+\big([0, T]; \|\cdot\|_{L_q^\infty}\big) \mid f \text{ satisfies } (\mathcal{A}1) \text{ and } (\mathcal{A}2)\right\},$$

where properties $(\mathcal{A}1)$ and $(\mathcal{A}2)$ are

- $(\mathcal{A}1)$: There exists a constant $C_1 > 0$ such that

$$\|f(t)\|_{L_q^\infty} \le e^{C_1 t} \|f_0\|_{L_q^\infty}, \quad \text{for } t \in [0, T].$$

- $(\mathcal{A}2)$: There exist positive constants C_{T, f_0}, $C_{T, f_0, \delta}$ and $C_{T, f_0, \delta, q}$ such that

$$(1) \quad \rho(x, t) \ge C_{T, f_0},$$
$$(2) \quad T_\delta(x, t) \ge C_{T, f_0, \delta},$$
$$(3) \quad \rho + |U| + T_\delta \le C_{T, f_0, \delta, q}.$$

We consider the following iteration scheme: $(n \ge 1)$

$$\partial_t f^{n+1} + v \cdot \nabla_x f^{n+1} = A_{v,\theta}\big(\mathcal{M}_{v,\theta}(f^n) - f^{n+1}\big),$$
$$f^{n+1}(0) = f_0. \tag{4.1}$$

We set $f^0 = 0$ and $\mathcal{M}(f^0) = 0$, so that

$$\partial_t f^1 + v \cdot \nabla_x f^1 + A_{v,\theta} f^1 = 0,$$
$$f^1(0) = f_0.$$

Our first goal is to show that $\{f^n\}$ lies in Ω for all $n \geq 0$. We start with the following estimates on the polyatomic Gaussian.

Proposition 4.1. *Suppose $f \in \Omega$, there exists a constant $C_{\mathcal{M}}$ depending on v, δ, θ and q such that*

$$\|\mathcal{M}_{v,\theta}(f)\|_{L_q^\infty} \leq C_{\mathcal{M}}\|f\|_{L_q^\infty}.$$

Remark 4.1. $C_{\mathcal{M}}$ blows up as θ tends to 0. See the end of the proof.

Proof. We will show that $\mathcal{M}_{v,\theta}(f)$, $|v|^q \mathcal{M}_{v,\theta}(f)$ and $I^{\frac{q}{\delta}} \mathcal{M}_{v,\theta}(f)$ are controlled by $\|f\|_{L_q^\infty}$.

(a) **The estimate for $\mathcal{M}_{v,\theta}(f)$:** We first recall Lemma 3.1 to observe

$$\frac{1}{2}(v-U)^\top \mathcal{T}_{v,\theta}^{-1}(v-U) + \frac{I^{\frac{2}{\delta}}}{T_\theta} \geq \frac{3}{2C_v\{3+\delta(1-\theta)\}}\frac{|v-U|^2}{T_\delta} + \frac{I^{\frac{2}{\delta}}}{T_\theta} \geq 0 \qquad (4.2)$$

for $f \in \Omega$. Hence we have

$$\exp\left(-\frac{1}{2}(v-U)^\top \mathcal{T}_{v,\theta}^{-1}(v-U) - \frac{I^{\frac{2}{\delta}}}{T_\theta}\right) \leq 1. \qquad (4.3)$$

Using this and Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned}
\mathcal{M}_{v,\theta}(f) &\leq \frac{\rho \Lambda_\delta}{\sqrt{\det(2\pi \mathcal{T}_{v,\theta})}(T_\theta)^{\frac{\delta}{2}}} \\
&\leq \frac{1}{(2\pi)^{3/2}}\frac{1}{\theta^{\frac{3+\delta}{2}}}\frac{\rho}{T_\delta^{\frac{3+\delta}{2}}} \\
&\leq \frac{1}{(2\pi)^{3/2}}\frac{1}{\theta^{\frac{3+\delta}{2}}}\left\{2^{\frac{7}{2}}(3+\delta)^{\frac{1+\delta}{2}}\pi^2\delta\right\}\|f\|_{L_q^\infty} \\
&\equiv \frac{C_0}{\theta^{\frac{3+\delta}{2}}}\|f\|_{L_q^\infty}.
\end{aligned}$$

(b) **The estimate for $\mathcal{M}_{v,\theta}(f)|v|^q$:** We divide it into the estimates of $|U|^q \mathcal{M}_{v,\theta}(f)$ and $|v-U|^q \mathcal{M}_{v,\theta}(f)$.

(b$_1$) $|U|^q \mathcal{M}_{v,\theta}(f)$: We use (4.3) and Lemma 3.1 to compute

$$|U|^q \mathcal{M}_{v,\theta}(f) \leq |U|^q \frac{\rho \Lambda_\delta}{\sqrt{\det(2\pi \mathcal{T}_{v,\theta})}(T_\theta)^{\frac{\delta}{2}}} \leq \frac{1}{(2\pi)^{3/2}}\frac{1}{\theta^{\frac{3+\delta}{2}}}|U|^q \frac{\rho}{T_\delta^{\frac{3+\delta}{2}}}.$$

We divide this estimate into two cases. In the case of $|U| < T_\delta^{\frac{1}{2}}$, we have from Lemma 3.3 that

$$|U|^q \frac{\rho}{T_\delta^{\frac{3+\delta}{2}}} \leq \rho(T_\delta + |U|^2)^{\frac{q-3-\delta}{2}} \leq \left\{ \frac{2^{\frac{q-2\delta-1}{2}} \pi^2 (3+\delta)^{\frac{q}{2}} \delta}{q - \delta - 5} \right\} \|f\|_{L_q^\infty}.$$

On the other hand, in the case of $|U| \geq T_\delta^{\frac{1}{2}}$, we have from Lemma 3.4 that

$$|U|^q \frac{\rho}{T_\delta^{\frac{3+\delta}{2}}} \leq \frac{\rho |U|^{q+3+\delta}}{|U|^{3+\delta} T_\delta^{\frac{3+\delta}{2}}} \leq 2^{\frac{3+\delta}{2}} \frac{\rho |U|^{q+3+\delta}}{[(T_\delta + |U|^2) T_\delta]^{\frac{3+\delta}{2}}} \leq 2^{\frac{14+3\delta+2q}{2}} \pi^2 (3+\delta)^{2+\delta} \delta \|f\|_{L_q^\infty}.$$

These two estimates give

$$|U|^q \mathcal{M}_{\nu,\theta}(f) \leq \frac{C_1}{\theta^{\frac{3+\delta}{2}}} \|f\|_{L_q^\infty},$$

for

$$C_1 = \left\{ \frac{2^{\frac{q-2\delta-4}{2}} \sqrt{\pi} (3+\delta)^{\frac{q}{2}} \delta}{q - \delta - 5} \right\} + 2^{\frac{11+3\delta+2q}{2}} \sqrt{\pi} (3+\delta)^{2+\delta} \delta.$$

(b$_2$) $|v - U|^q \mathcal{M}_{\nu,\theta}(f)$: From (4.2) and Lemma 3.1, we have

$$|v - U|^q \mathcal{M}_{\nu,\theta}(f)$$

$$\leq \frac{1}{(2\pi)^{3/2}} \frac{1}{\theta^{\frac{3+\delta}{2}}} |v - U|^q \frac{\rho}{T_\delta^{\frac{3+\delta}{2}}} \exp\left(-\frac{3}{2C_\nu \{3 + \delta(1-\theta)\}} \frac{|v-U|^2}{T_\delta} \right)$$

$$= \frac{1}{(2\pi)^{3/2}} \frac{1}{\theta^{\frac{3+\delta}{2}}} T_\delta^{\frac{q}{2}} \frac{\rho}{T_\delta^{\frac{3+\delta}{2}}} \left(\frac{|v-U|^2}{T_\delta} \right)^{\frac{q}{2}} \exp\left(-\frac{3}{2C_\nu \{3 + \delta(1-\theta)\}} \frac{|v-U|^2}{T_\delta} \right)$$

$$\equiv \frac{C_2}{\theta^{\frac{3+\delta}{2}}} \rho T_\delta^{\frac{q-3-\delta}{2}},$$

where

$$C_2 = \frac{1}{(2\pi)^{3/2}} \sup_{x \geq 0} \left(x^{q/2} e^{-x} \right) \left\{ \frac{2C_\nu(3 + \delta(1-\theta))}{3} \right\}^{q/2}.$$

This, combined with Lemma 3.3 implies

$$|v - U|^q \mathcal{M}_{\nu,\theta}(f) \leq \frac{C_2}{\theta^{\frac{3+\delta}{2}}} \rho(T_\delta + |U|^2)^{\frac{q-3-\delta}{2}}$$

$$\leq \frac{C_2}{\theta^{\frac{3+\delta}{2}}} \left\{ \frac{2^{\frac{q-2\delta-1}{2}} \pi^2 (3+\delta)^{\frac{q}{2}} \delta}{q-\delta-5} \right\} \|f\|_{L_q^\infty}$$

$$\equiv \frac{C_3}{\theta^{\frac{3+\delta}{2}}} \|f\|_{L_q^\infty}.$$

(c) **The estimate for** $I^{\frac{q}{\delta}} \mathcal{M}_{v,\theta}(f)$: Again from (4.2), we have

$$\frac{1}{2}(v-U)^\top \mathcal{T}_{v,\theta}^{-1}(v-U) + \frac{I^{\frac{2}{\delta}}}{T_\theta} \geq \frac{\delta}{\delta + 3(1-\theta)} \frac{I^{\frac{2}{\delta}}}{T_\delta},$$

so that $I^{\frac{q}{\delta}} \mathcal{M}_{v,\theta}(f)$ is estimated as follows:

$$I^{\frac{q}{\delta}} \mathcal{M}_{v,\theta}(f) \leq \frac{1}{\sqrt{(2\pi)^3}} I^{\frac{q}{\delta}} \frac{1}{\theta^{\frac{3+\delta}{2}}} \frac{\rho}{T_\delta^{\frac{3+\delta}{2}}} \exp\left(-\frac{\delta}{\delta + 3(1-\theta)} \frac{I^{\frac{2}{\delta}}}{T_\delta}\right)$$

$$= \frac{1}{\sqrt{(2\pi)^3}} \frac{1}{\theta^{\frac{3+\delta}{2}}} T_\delta^{\frac{q}{2}} \frac{\rho}{T_\delta^{\frac{3+\delta}{2}}} \left(\frac{I^{\frac{2}{\delta}}}{T_\delta}\right)^{\frac{q}{2}} \exp\left(-\frac{\delta}{\delta + 3(1-\theta)} \frac{I^{\frac{2}{\delta}}}{T_\delta}\right)$$

$$\equiv \frac{C_4}{\theta^{\frac{3+\delta}{2}}} \rho T_\delta^{\frac{q-3-\delta}{2}},$$

where

$$C_4 = \frac{1}{(2\pi)^{3/2}} \sup_{x \geq 0}(x^{q/2} e^{-x}) \left(\frac{\delta + 3(1-\theta)}{\delta}\right)^{q/2}.$$

Then, in view of Lemma 3.3, we derive

$$I^{\frac{q}{\delta}} \mathcal{M}_{v,\theta}(f) \leq \frac{C_4}{\theta^{\frac{3+\delta}{2}}} \rho (T_\delta + |U|^2)^{\frac{q-3-\delta}{2}}$$

$$\leq \frac{C_4}{\theta^{\frac{3+\delta}{2}}} \left\{ \frac{2^{\frac{q-2\delta-1}{2}} \pi^2 (3+\delta)^{\frac{q}{2}} \delta}{q-\delta-5} \right\} \|f\|_{L_q^\infty}$$

$$\equiv \frac{C_5}{\theta^{\frac{3+\delta}{2}}} \|f\|_{L_q^\infty}.$$

Finally, we combine (a), (b) and (c) to conclude that

$$\|\mathcal{M}_{v,\theta}(f)\|_{L_q^\infty} \leq \frac{C_{v,\delta,\theta,q}}{\theta^{\frac{3+\delta}{2}}} \|f\|_{L_q^\infty},$$

where

$$C_{v,\delta,\theta,q} = C_0 + C_1 + C_3 + C_5.$$

Note that $\max\limits_{0 \le \theta \le 1} C_{v,\delta,\theta,q} < \infty$. □

Proposition 4.2. f^n *lies in* Ω *for all* $n > 0$. *That is,* f^n *satisfies*

- ($\mathcal{A}1$): f^n *is uniformly bounded in* $\| \cdot \|_{L_q^\infty}$

$$\|f^n\|_{L_q^\infty} \le e^{C_1 t} \|f_0\|_{L_q^\infty},$$

  where $C_1 = A_{v,\theta} (C_\mathcal{M} - 1)$.
- ($\mathcal{A}2$): *There exist positive constants* C_{T,f_0}, $C_{T,f_0,\delta}$ *and* $C_{T,f_0,\delta,q}$ *such that*

$$(1) \quad \rho^n(x,t) \ge C_{T,f_0},$$

$$(2) \quad T_\delta^n(x,t) \ge C_{T,f_0,\delta},$$

$$(3) \quad \rho^n + |U^n| + T_\delta^n \le C_{T,f_0,\delta,q}.$$

Proof. We proceed by induction. The properties are trivially satisfied for $n = 0$. Assume that $f^n \in \Omega$. We prove that f^{n+1} also satisfies ($\mathcal{A}1$) and ($\mathcal{A}2$).

($\mathcal{A}1$) We write (4.1) in the mild form:

$$f^{n+1}(t,x,v,I) = e^{-A_{v,\theta} t} f_0(x - vt, v, I)$$

$$+ A_{v,\theta} \int_0^t e^{-A_{v,\theta}(t-s)} \mathcal{M}_{v,\theta}(f^n)(x - (t-s)v, v, s, I) ds$$

and take $\| \cdot \|_{L_q^\infty}$ on both sides,

$$\|f^{n+1}(t)\|_{L_q^\infty} \le e^{-A_{v,\theta} t} \|f_0\|_{L_q^\infty} + A_{v,\theta} \int_0^t e^{-A_{v,\theta}(t-s)} \|\mathcal{M}_{v,\theta}(f^n)(s)\|_{L_q^\infty} ds. \qquad (4.4)$$

Since $f^n \in \Omega$, we can apply Proposition 4.1 to estimate

$$A_{v,\theta} \int_0^t e^{-A_{v,\theta}(t-s)} \|\mathcal{M}_{v,\theta}(f^n)(s)\|_{L_q^\infty} ds \le A_{v,\theta} \int_0^t e^{-A_{v,\theta}(t-s)} C_\mathcal{M} \|f^n(s)\|_{L_q^\infty} ds$$

$$\le A_{v,\theta} \int_0^t e^{-A_{v,\theta}(t-s)} C_\mathcal{M} e^{C_1 s} \|f_0\|_{L_q^\infty} ds$$

$$= \frac{A_{v,\theta} C_\mathcal{M}}{C_1 + A_{v,\theta}} (e^{C_1 t} - e^{-A_{v,\theta} t}) \|f_0\|_{L_q^\infty},$$

where we used $\|f^n\|_{L_q^\infty} \leq e^{C_1 t} \|f_0\|_{L_q^\infty}$. Plugging this estimate into (4.4), we get

$$\|f^{n+1}(t)\|_{L_q^\infty} \leq e^{C_1 t} \|f_0\|_{L_q^\infty},$$

since $(A_{v,\theta} C_{\mathcal{M}})/(C_1 + A_{v,\theta}) = 1$.

($\mathcal{A}2$) By the nonnegativity of polyatomic Gaussian $\mathcal{M}_{v,\theta}(f^n)$, we have from the above mild form

$$f^{n+1} \geq e^{-A_{v,\theta} t} f_0(x - vt, v, I).$$

Integrating in v and I on both sides, and recalling the lower bound assumption imposed on f_0,

$$\rho^{n+1} = \int_{\mathbb{R}^3 \times \mathbb{R}^+} f^{n+1} dv dI \geq e^{-A_{v,\theta} t} \int_{\mathbb{R}^3 \times \mathbb{R}^+} f_0(x - vt, v, I) dv dI \geq C_{f_0} e^{-A_{v,\theta} t}.$$

Hence, combining the above results and Lemma 3.2 gives

$$C_{f_0} e^{-A_{v,\theta} t} \leq \rho^{n+1} \leq C_\delta \|f^{n+1}\|_{L_q^\infty} \left\{ T_\delta^{n+1} \right\}^{\frac{3+\delta}{2}} \leq C_\delta e^{C_1 t} \|f_0\|_{L_q^\infty} \left\{ T_\delta^{n+1} \right\}^{\frac{3+\delta}{2}}.$$

Therefore,

$$T_\delta^{n+1} \geq \left(\frac{C_{f_0} e^{-A_{v,\theta} t}}{C_\delta e^{C_1 t} \|f_0\|_{L_q^\infty}} \right)^{\frac{2}{3+\delta}} \geq C_{T, f_0, \delta}.$$

The estimate ($\mathcal{A}2$) (3) follows immediately from the above lower bound for ρ^{n+1} and Lemma 3.3. This completes the proof. □

5. Lipschitz continuity of $\mathcal{M}_{v,\theta}$

Proposition 5.1. *Let f and g lie in Ω. Then $\mathcal{M}_{v,\theta}$ satisfies the following continuity property:*

$$\|\mathcal{M}_{v,\theta}(f) - \mathcal{M}_{v,\theta}(g)\|_{L_q^\infty} \leq C_{Lip} \|f - g\|_{L_q^\infty}$$

for some constant C_{Lip} depending on T, δ, θ, q and f_0.

Proof. For the proof of this proposition, we set the transitional macroscopic fields $\rho_\eta, U_\eta, \mathcal{T}_{v,\theta\eta}, T_{I,\delta\eta}$:

$$(\rho_\eta, U_\eta, \mathcal{T}_{v,\theta\eta}, T_{I,\delta\eta}) = \eta(\rho_f, U_f, \mathcal{T}_{v,\theta f}, T_{I,\delta f}) + (1 - \eta)(\rho_g, U_g, \mathcal{T}_{v,\theta g}, T_{I,\delta g})$$

and define the transitional polyatomic Gaussian:

$$\mathcal{M}_{v,\theta}(\eta) = \frac{\rho_\eta \Lambda_\delta}{\sqrt{\det(2\pi \mathcal{T}_{v,\theta\eta})}(T_{\theta\eta})^{\frac{\delta}{2}}} \exp\left(-\frac{1}{2}(v - U_\eta)^\top \mathcal{T}_{v,\theta\eta}^{-1}(v - U_\eta) - \frac{I^{\frac{2}{\delta}}}{T_{\theta\eta}} \right).$$

Applying Taylor's theorem, we expand

$$
\begin{aligned}
\mathcal{M}_{v,\theta}(f) - \mathcal{M}_{v,\theta}(g) = {} & (\rho_f - \rho_g) \int_0^1 \frac{\partial \mathcal{M}_{v,\theta}(\eta)}{\partial \rho_\eta} d\eta \\
& + (U_f - U_g) \int_0^1 \frac{\partial \mathcal{M}_{v,\theta}(\eta)}{\partial U_\eta} d\eta \\
& + (\mathcal{T}_{v,\theta f} - \mathcal{T}_{v,\theta g}) \int_0^1 \frac{\partial \mathcal{M}_{v,\theta}(\eta)}{\partial \mathcal{T}_{v,\theta \eta}} d\eta \\
& + (T_{I,\delta f} - T_{I,\delta g}) \int_0^1 \frac{\partial \mathcal{M}_{v,\theta}(\eta)}{\partial T_{I,\delta \eta}} d\eta \\
= {} & I_1 + I_2 + I_3 + I_4.
\end{aligned}
\tag{5.1}
$$

We only consider I_3. Other terms can be treated in a similar and simpler manner. Recalling the definition of $\mathcal{T}_{v,\theta}$, we see that

$$
\begin{aligned}
& \rho_f \mathcal{T}_{v,\theta f} - \rho_g \mathcal{T}_{v,\theta g} \\
& = (1-\theta)\rho_f \{(1-v)T_{trf} Id + v\Theta_f\} + \theta \rho_f T_{\delta f} Id \\
& \quad - (1-\theta)\rho_g \{(1-v)T_{trg} Id + v\Theta_g\} - \theta \rho_g T_{\delta g} Id \\
& = (1-\theta) \int_{\mathbb{R}^3 \times \mathbb{R}^+} f\Big\{ \frac{(1-v)}{3} |v-U_f|^2 Id + v(v-U_f) \otimes (v-U_f) \Big\} dv dI \\
& \quad + \frac{\theta}{3+\delta} \int_{\mathbb{R}^3 \times \mathbb{R}^+} f\big\{ |v-U_f|^2 + 2I^{\frac{2}{\delta}} \big\} Id\, dv dI \\
& \quad - (1-\theta) \int_{\mathbb{R}^3 \times \mathbb{R}^+} g\Big\{ \frac{(1-v)}{3} |v-U_g|^2 Id + v(v-U_g) \otimes (v-U_g) \Big\} dv dI \\
& \quad - \frac{\theta}{3+\delta} \int_{\mathbb{R}^3 \times \mathbb{R}^+} g\big\{ |v-U_g|^2 + 2I^{\frac{2}{\delta}} \big\} Id\, dv dI,
\end{aligned}
$$

which can be rearranged as

$$
\begin{aligned}
& \Big\{ (1-\theta)\frac{1-v}{3} + \frac{\theta}{3+\delta} \Big\} \Big(\int_{\mathbb{R}^3 \times \mathbb{R}^+} (f|v-U_f|^2 - g|v-U_g|^2) dv dI \Big) Id \\
& + (1-\theta)v \int_{\mathbb{R}^3 \times \mathbb{R}^+} \{ f(v-U_f) \otimes (v-U_f) - g(v-U_g) \otimes (v-U_g) \} dv dI
\end{aligned}
$$

$$+ \frac{2\theta}{3+\delta} \left(\int\limits_{\mathbb{R}^3 \times \mathbb{R}^+} (f-g) I^{\frac{2}{\delta}} dv dI \right) Id$$

$$\equiv T_1 + T_2 + T_3.$$

For T_1, we note that

$$\int\limits_{\mathbb{R}^3 \times \mathbb{R}^+} f|v - U_f|^2 - g|v - U_g|^2 dv dI = \int\limits_{\mathbb{R}^3 \times \mathbb{R}^+} (f-g)|v|^2 dv dI - \left(\rho_f |U_f|^2 - \rho_g |U_g|^2 \right).$$

The first term is clearly bounded by $C\|f - g\|_{L_q^\infty}$ $(q > 3)$. For the second term, we compute

$$\left| \rho_f |U_f|^2 - \rho_g |U_g|^2 \right| = \left| \frac{\rho_f^2 |U_f|^2 - \rho_g^2 |U_g|^2}{\rho_f} + \frac{\rho_g^2}{\rho_f} |U_g|^2 - \rho_g |U_g|^2 \right|$$

$$\leq \frac{1}{\rho_f} \left(|\rho_f U_f| + |\rho_g U_g| \right) |\rho_f U_f - \rho_g U_g| + \frac{1}{\rho_f \rho_g} |\rho_g U_g|^2 |\rho_f - \rho_g|.$$

Now, since $f, g \in \Omega$, we have from $(\mathcal{A}1)$

$$|\rho_f U_f| + |\rho_g U_g| \leq C_{\delta,q} \left(\|f\|_{L_q^\infty} + \|g\|_{L_q^\infty} \right) \leq C_{\delta,q} e^{C_1 t} \|f_0\|_{L_q^\infty}$$

and $\rho_f, \rho_g \geq C_{T, f_0}$. Therefore,

$$\left| \rho_f |U_f|^2 - \rho_g |U_g|^2 \right| \leq C_{T, f_0, \delta, q} \left\{ |\rho_f U_f - \rho_g U_g| + |\rho_f - \rho_g| \right\}$$

$$\leq C_{T, f_0, \delta, q} \int\limits_{\mathbb{R}^3 \times \mathbb{R}^+} |f - g|(1 + |v|^2) dv dI$$

$$\leq C_{T, f_0, \delta, q} \|f - g\|_{L_q^\infty}.$$

The estimate for T_2 can be derived from a similar computation using the following identity:

$$\{(1-\theta)v\}^{-1} T_2 = \int\limits_{\mathbb{R}^3 \times \mathbb{R}^+} (f-g)v \otimes v \, dv dI$$

$$- \frac{1}{\rho_f} \rho_f U_f \otimes (\rho_f U_f - \rho_g U_g) - \frac{\rho_g}{\rho_f} (\rho_f U_f - \rho_g U_g) \otimes U_g$$

$$+ \frac{1}{\rho_f \rho_g} (\rho_f - \rho_g) \{\rho_g U_g\} \otimes \{\rho_g U_g\}.$$

We omit the estimate for T_3. What we have shown so far is

$$|\rho_f \mathcal{T}_{v,\theta f} - \rho_g \mathcal{T}_{v,\theta g}| \leq C_{T, f_0, \delta, \theta, q} \|f - g\|_{L_q^\infty}.$$

Therefore,

$$|\mathcal{T}_{v,\theta f} - \mathcal{T}_{v,\theta g}| \leq \frac{1}{\rho_f}|\rho_f \mathcal{T}_{v,\theta f} - \rho_g \mathcal{T}_{v,\theta g}| + \frac{1}{\rho_f}|\rho_f - \rho_g||\mathcal{T}_{v,\theta g}| \leq C_{T,f_0,\delta,\theta,q}\|f - g\|_{L_q^\infty},$$

where we used Lemma 3.1 and Property $(\mathcal{A}2)(1)$ of Ω as

$$\rho_f \geq C_{T,f_0},$$

and

$$|\mathcal{T}_{v,\theta g}^{ij}| \leq \frac{1-\theta}{\rho_g}\left|\int\limits_{\mathbb{R}^3 \times \mathbb{R}^+} g\left\{\frac{(1-v)}{3}|v - U_g|^2 \delta_{ij} + v(v - U_g)_i(v - U_g)_j\right\} dvdI\right|$$

$$+ \frac{\theta}{\rho_g}\left|\int\limits_{\mathbb{R}^3 \times \mathbb{R}^+} g\left\{\frac{1}{3+\delta}\left(|v - U_g|^2 + 2I^{\frac{2}{\delta}}\right)\delta_{ij}\right\} dvdI\right| \tag{5.2}$$

$$\leq (3 + \delta - 2\theta - \delta\theta)T_{\delta g}.$$

We now move on to the estimate of the integral in I_3. A straightforward computation gives

$$\int\limits_0^1 \frac{\partial \mathcal{M}_{v,\theta}(\eta)}{\partial \mathcal{T}_{v,\theta\eta}} d\eta$$

$$= \int\limits_0^1 \frac{1}{2}\left[-\frac{1}{\det \mathcal{T}_{v,\theta\eta}}\frac{\partial \det \mathcal{T}_{v,\theta\eta}}{\partial \mathcal{T}_{v,\theta\eta ij}} + (v - U_\eta)^\top \mathcal{T}_{v,\theta\eta}^{-1}\left(\frac{\partial \mathcal{T}_{v,\theta\eta}}{\partial \mathcal{T}_{v,\theta\eta ij}}\right)\mathcal{T}_{v,\theta\eta}^{-1}(v - U_\eta)\right]\mathcal{M}_{v,\theta}(\eta)d\eta.$$

Before proceeding further, we establish the following claims: For $f, g \in \Omega$, we have

$$(\mathcal{F}_1): \quad \left|(v - U_\eta)^\top \mathcal{T}_{v,\theta\eta}^{-1}\left(\frac{\partial \mathcal{T}_{v,\theta\eta}^{ij}}{\partial \mathcal{T}_{v,\theta\eta}}\right)\mathcal{T}_{v,\theta\eta}^{-1}(v - U_\eta)\right| \leq \frac{|v - U_\eta|^2}{[\theta\{\eta T_{\delta f} + (1-\eta)T_{\delta g}\}]^2}$$

$$(\mathcal{F}_2): \quad \left|\det \mathcal{T}_{v,\theta\eta}\right| \geq \theta^3\{\eta T_{\delta f} + (1-\eta)T_{\delta g}\}^3$$

$$(\mathcal{F}_3): \quad \left|\frac{\partial \det \mathcal{T}_{v,\theta\eta}}{\partial \mathcal{T}_{v,\theta\eta}^{ij}}\right| \leq 2(3 + \delta - 2\theta - \delta\theta)^2\{\eta T_{\delta f} + (1-\eta)T_{\delta g}\}^2.$$

● $(\mathcal{F}_1)$: Let D_{ij} denote a $n \times n$ matrix whose ijth and jith entries are 1 and the remaining entries are 0. Then,

$$\left|X^\top \left(\frac{\partial \mathcal{T}_{v,\theta\eta}}{\partial \mathcal{T}_{v,\theta\eta}^{ij}}\right)Y\right| = |X^\top D_{ij}Y| \leq |X_iY_j + X_jY_i| \leq |X||Y|.$$

Thus we have

$$\left| (v - U_\eta)^\top \mathcal{T}_{v,\theta\eta}^{-1} \left(\frac{\partial \mathcal{T}_{v,\theta\eta}}{\partial \mathcal{T}_{v,\theta\eta}^{ij}} \right) \mathcal{T}_{v,\theta\eta}^{-1} (v - U_\eta) \right| \leq |(v - U_\eta)^\top \mathcal{T}_{v,\theta\eta}^{-1}| \, |\mathcal{T}_{v,\theta\eta}^{-1}(v - U_\eta)|.$$

Now we use Lemma 3.1:

$$\mathcal{T}_{v,\theta\eta} = \eta \mathcal{T}_{v,\theta f} + (1 - \eta) \mathcal{T}_{v,\theta g} \geq \left[\theta \{ \eta T_{\delta f} + (1 - \eta) T_{\delta g} \} \right] Id \tag{5.3}$$

to compute

$$\begin{aligned}
|(v - U_\eta)^\top \mathcal{T}_{v,\theta\eta}^{-1}| &\leq \sup_{|Y| \leq 1} |(v - U_\eta)^\top \mathcal{T}_{v,\theta\eta}^{-1} Y| \\
&\leq \sup_{|Y| \leq 1} \frac{|v - U_\eta| |Y|}{\theta \{ \eta T_{\delta f} + (1 - \eta) T_{\delta g} \}} \\
&\leq \frac{|v - U_\eta|}{\theta \{ \eta T_{\delta f} + (1 - \eta) T_{\delta g} \}}.
\end{aligned}$$

Likewise,

$$|\mathcal{T}_{v,\theta\eta}^{-1}(v - U_\eta)| \leq \frac{|v - U_\eta|}{\theta \{ \eta T_{\delta f} + (1 - \eta) T_{\delta g} \}},$$

which gives the desired estimate.

- $(\mathcal{F}_2)$: By (5.3), we have

$$\det \mathcal{T}_{v,\theta\eta} \geq \theta^3 \{ \eta T_{\delta f} + (1 - \eta) T_{\delta g} \}^3.$$

- $(\mathcal{F}_3)$: We only prove the case: $(i, j) = (1, 2)$. An explicit calculation gives

$$\frac{\partial \det \mathcal{T}_{v,\theta\eta}}{\partial \mathcal{T}_{v,\theta\eta}^{12}} = \mathcal{T}_{v,\theta\eta}^{23} \mathcal{T}_{v,\theta\eta}^{31} - \mathcal{T}_{v,\theta\eta}^{33} \mathcal{T}_{v,\theta\eta}^{21}. \tag{5.4}$$

Then we recall (5.2) to derive

$$|\mathcal{T}_{v,\theta\eta}^{ij}| = |\eta \mathcal{T}_{v,\theta f}^{ij} + (1 - \eta) \mathcal{T}_{v,\theta g}^{ij}| \leq (3 + \delta - 2\theta - \delta\theta)|\eta T_{\delta f} + (1 - \eta) T_{\delta g}|.$$

Therefore, (5.4) leads to

$$\left| \frac{\partial \det \mathcal{T}_{v,\theta\eta}}{\partial \mathcal{T}_{v,\theta\eta}^{12}} \right| \leq 2(3 + \delta - 2\theta - \delta\theta)^2 \{ \eta T_{\delta f} + (1 - \eta) T_{\delta g} \}^2.$$

This ends the proof of the claims.

Using these claims, we estimate the integral in I_3 as $(T_{\delta\eta} = \eta T_{\delta f} + (1 - \eta)T_{\delta g})$.

$$\int_0^1 \frac{\partial \mathcal{M}_{v,\theta}(\eta)}{\partial \mathcal{T}_{v,\theta\eta}} d\eta \leq C_{\delta,\theta} \int_0^1 \left[\frac{1}{T_{\delta\eta}} + \frac{|v - U_\eta|^2}{T_{\delta\eta}^2} \right] \mathcal{M}_{v,\theta}(\eta) d\eta. \tag{5.5}$$

Now, since $f, g \in \Omega$, we have

$$C_{T,f_0,\delta,1} \leq T_{\delta\eta} = \eta T_{\delta f} + (1 - \eta)T_{\delta g} \leq C_{T,f_0,\delta,2},$$

and

$$\mathcal{M}_{v,\theta}(\eta) \leq C_{T,f_0,\delta,q} \frac{\rho_\eta}{\theta^{\frac{3+\delta}{2}} T_{\delta\eta}^{\frac{3+\delta}{2}}}$$

$$\times \exp\left(-\frac{3}{2C_v\{3 + \delta(1-\theta)\}} \frac{|v - U_\eta|^2}{T_{\delta\eta}} \right) \exp\left(-\frac{\delta}{\delta + 3(1-\theta)} \frac{I^{\frac{2}{\delta}}}{T_{\delta\eta}} \right)$$

$$\leq C_{T,f_0,\delta,\theta,q} e^{-C_{T,f_0,\delta,\theta,q}(|v-U_\eta|^2 + I^{2/\delta})}.$$

Therefore, we can proceed further from (5.5) as

$$\int_0^1 \frac{\partial \mathcal{M}_{v,\theta}(\eta)}{\partial \mathcal{T}_{v,\theta\eta}} d\eta \leq C_{T,f_0,\delta,\theta,q} \int_0^1 \left(1 + |v - U_\eta|^2 \right) e^{-C_{T,f_0,\delta,\theta,q}(|v-U_\eta|^2 + I^{2/\delta})} d\eta$$

$$\leq C_{T,f_0,\delta,\theta,q} \int_0^1 e^{-C_{T,f_0,\delta,\theta,q}(|v-U_\eta|^2 + I^{2/\delta})} d\eta$$

$$\leq C_{T,f_0,\delta,\theta,q} \int_0^1 e^{-C_{T,f_0,\delta,\theta,q}(|v|^2 + I^{2/\delta})} d\eta$$

$$\leq C_{T,f_0,\delta,\theta,q} e^{-C_{T,f_0,\delta,\theta,q}(|v|^2 + I^{2/\delta})}.$$

In the last line, we have used

$$|U_\eta| \leq \eta|U_f| + (1 - \eta)|U_g| \leq C_{T,f_0,\delta,q}$$

which holds when $f, g \in \Omega$.

Finally, we turn to the proof of the proposition, which is almost done. Plugging the above inequalities into (5.1), we have

$$|\mathcal{M}_{v,\theta}(f) - \mathcal{M}_{v,\theta}(g)|$$

$$\leq C\Big\{|\rho_f - \rho_g| + |U_f - U_g| + |\mathcal{T}_{v,\theta f} - \mathcal{T}_{v,\theta g}| + |T_{I,\delta f} - T_{I,\delta g}|\Big\}e^{-C(|v|^2 + I^{2/\delta})}$$

$$\leq C\|f - g\|_{L_q^\infty} e^{-C(|v|^2 + I^{2/\delta})}.$$

Multiplying $(1 + |v|^2 + I^{\frac{2}{\delta}})^{\frac{q}{2}}$ and taking supremum on both sides, we get the desired estimate. $\square$

6. Proof of the main theorem

In the mild form, (4.4) reads

$$f^{n+1}(x, v, t, I) = e^{-A_{v,\theta}t} f_0(x - vt, v, I)$$

$$+ A_{v,\theta} \int_0^t e^{-A_{v,\theta}(t-s)} \mathcal{M}_{v,\theta}(f^n)(x - (t-s)v, v, s, I)ds,$$

$$f^n(x, v, t, I) = e^{-A_{v,\theta}t} f_0(x - vt, v, I)$$

$$+ A_{v,\theta} \int_0^t e^{-A_{v,\theta}(t-s)} \mathcal{M}_{v,\theta}(f^{n-1})(x - (t-s)v, v, s, I)ds.$$

Taking difference and applying Proposition 5.1, we obtain

$$\|f^{n+1}(t) - f^n(t)\|_{L_q^\infty} \leq A_{v,\theta} \int_0^t e^{-A_{v,\theta}(t-s)} \|\mathcal{M}_{v,\theta}(f^n(t)) - \mathcal{M}_{v,\theta}(f^{n-1}(t))\|_{L_q^\infty} ds$$

$$\leq A_{v,\theta} C_{Lip} \int_0^t \|f^n(t) - f^{n-1}(t)\|_{L_q^\infty} ds.$$

Iterating this inequality,

$$\|f^{n+1}(t) - f^n(t)\|_{L_q^\infty} \leq A_{v,\theta}^n C_{Lip}^n \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \|f^1(s_n) - f^0(s_n)\|_{L_q^\infty} ds_n \cdots ds_2 ds_1$$

$$\leq A_{v,\theta}^n C_{Lip}^n \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} e^{-A_{v,\theta}s_n} \|f_0\|_{L_q^\infty} ds_n \cdots ds_2 ds_1$$

$$\leq A_{v,\theta}^n C_{Lip}^n \frac{t^n}{n!} \|f_0\|_{L_q^\infty}.$$

This immediately gives for $n > m$

$$\sup_{0 \leq t \leq T} \|f^n(t) - f^m(t)\|_{L_q^\infty} \leq \left(e^{A_{v,\theta} C_{Lip} T} - \sum_{k=0}^{m-1} \frac{\left(A_{v,\theta} C_{Lip} T\right)^k}{k!} \right) \|f_0\|_{L_q^\infty}.$$

Therefore, we conclude that $\{f^n\}$ is a Cauchy sequence and converges to an element, say f, in Ω. It is standard to check that f is the mild solution:

$$f(t, x, v, I) = e^{-A_{v,\theta} t} f_0(x - vt, v, I) + A_{v,\theta} \int_0^t e^{-A_{v,\theta}(t-s)} \mathcal{M}_{v,\theta}(f)(s, x - (t-s)v, v, I) ds.$$

This proves the existence and estimates (1) and (2).

For the proof of conservation laws, we find that Proposition (4.1) and the Lebesgue differentiation theorem give from the above mild form,

$$\frac{d}{dt} f(t, x + tv, v, I) = A_{v,\theta} \{\mathcal{M}_{v,\theta}(f) - f\}(t, x + tv, v, I)$$

for almost all t. Multiplying $1, v, \frac{1}{2}|v|^2 + I^{\frac{2}{\delta}}$ on both sides and integrating with respect to x, v, I, we obtain (3).

The entropy dissipation estimate in the form of (4) was established in [2,8,28] at the formal level. But the lower and upper bounds for f and the macroscopic fields justify all those formal computations given in [2,8,28]. This completes the proof.

7. When $\theta = 0$

Recall that the l.h.s. of equivalence estimates in Lemma 3.1 vanish, and the r.h.s. of the inequality in Proposition 4.1 blows up, as θ tends to zero. Therefore, the argument we've developed so far does not work for $\theta = 0$. We, however, observe that the polyatomic Gaussian $\mathcal{M}_{v,\theta}(f)$ in this case is completely split into the translational internal energy part and the non-translational internal energy part as follows:

$$\mathcal{M}_{v,0}(f) = \frac{\rho \Lambda_\delta}{\sqrt{\det(2\pi \mathcal{T}_{v,0})} T_{I,\delta}^{\frac{\delta}{2}}} e^{-\frac{1}{2}(v-U)^\top \mathcal{T}_{v,0}^{-1}(v-U) - \frac{I^{\frac{2}{\delta}}}{T_{I,\delta}}}$$

$$= \left(\frac{\rho}{\sqrt{\det(2\pi \mathcal{T}_{v,0})}} e^{-\frac{1}{2}(v-U)^\top \mathcal{T}_{v,0}^{-1}(v-U)} \right) \left(\frac{\Lambda_\delta}{T_{I,\delta}^{\delta/2}} e^{-\frac{I^{2/\delta}}{T_{I,\delta}}} \right)$$

in the sense that $\mathcal{T}_{v,0}$ and $T_{I,\delta}$ does not share the common factor T_δ. Now, if we define

$$g(t, x, v) = \int_{\mathbb{R}^+} f(t, x, v, I) dI$$

and integrate (1.1) with respect to I, we get

$$\partial_t g + v \cdot \nabla_x g = A_0 \{\mathcal{M}_v(g) - g\}$$

$$g_0(x, v) = \int_{\mathbb{R}_+} f_0(x, v, I) dI, \tag{7.1}$$

where

$$\mathcal{M}_v(g) \equiv \int_{\mathbb{R}_+} \mathcal{M}_{v,0}(f) dI = \frac{\rho}{\sqrt{\det(2\pi \mathcal{T}_v)}} e^{-\frac{1}{2}(v-U)^\top \mathcal{T}_v^{-1}(v-U)}$$

and

$$\mathcal{T}_v \equiv \mathcal{T}_{v,0} = (1 - v) T_{tr} + v\Theta.$$

Note that ρ, U, $\mathcal{T}_v$ are naturally interpreted as macroscopic fields of g, and hence, so is $\mathcal{M}_v(g)$. This is exactly the ellipsoidal BGK model for monatomic particles [2,8,21]. Relevant existence result for (7.1) can be found in [27,45,46]. Thus, in the case that $\theta = 0$, our problem (1.1) should be understood in the form (7.1). This dichotomy between the two case, $\theta = 0$ and $0 < \theta \le 1$, was also observed in [28,47].

8. Appendix. Conservation laws

In this appendix, we prove the cancellation property (1.6) for reader's convenience. To avoid repetition, we only prove the case of $|v|^2$.

To compute the translational part, we make a change of variable $X = \frac{1}{\sqrt{2}} \mathcal{T}_{v,\theta}^{-\frac{1}{2}} (v - U)$, so that

$$dX = (\sqrt{2})^{-3} \det \mathcal{T}_{v,\theta}^{-\frac{1}{2}} dv = \frac{(\sqrt{\pi})^3 dv}{\sqrt{\det(2\pi \mathcal{T}_{v,\theta})}}.$$

We also find that

$$\frac{1}{2}(v - U)^\top \mathcal{T}_{v,\theta}^{-1}(v - U) = \left\{\frac{1}{\sqrt{2}} \mathcal{T}_{v,\theta}^{-1/2}(v - U)\right\}^\top \left\{\frac{1}{\sqrt{2}} \mathcal{T}_{v,\theta}^{-1/2}(v - U)\right\} = X^\top X = |X|^2,$$

and

$$|v - U|^2 = (v - U)^\top(v - U) = \left(\sqrt{2}\mathcal{T}_{v,\theta}^{1/2} X\right)^\top \left(\sqrt{2}\mathcal{T}_{v,\theta}^{1/2} X\right) = 2X^\top \mathcal{T}_{v,\theta} X.$$

Therefore, we have

$$\int\limits_{\mathbb{R}^3 \times \mathbb{R}^+} \frac{|v - U|^2}{2} \mathcal{M}_{v,\theta}(f) dv dI = \frac{\rho}{(\sqrt{\pi})^3} \int\limits_{\mathbb{R}^3} \{X^\top \mathcal{T}_{v,\theta} X\} e^{-|X|^2} dX$$

$$= \frac{\rho}{(\sqrt{\pi})^3} \int\limits_{\mathbb{R}^3} \left\{ \sum_{ij} X_i X_j \mathcal{T}_{v,\theta}^{ij} \right\} e^{-|X|^2} dX$$

$$= \frac{\rho}{(\sqrt{\pi})^3} \int\limits_{\mathbb{R}^3} \left\{ \sum_{i=1}^{3} X_i^2 \mathcal{T}_{v,\theta}^{ii} \right\} e^{-|X|^2} dX \tag{8.1}$$

$$= \frac{\rho}{(\sqrt{\pi})^3} \sum_{i=1,2,3} \mathcal{T}_{v,\theta}^{ii} \left(\int\limits_{\mathbb{R}^3} X_i^2 e^{-|X|^2} dX \right)$$

$$= \frac{\rho}{2} tr \mathcal{T}_{v,\theta}$$

$$= \frac{3}{2} \rho \{ (1 - \theta) T_{tr} + \theta T_\delta \},$$

where we used

$$\left(\int\limits_{\mathbb{R}^3} X_i^2 e^{-|X|^2} dX \right) = \frac{\sqrt{\pi}^3}{2}.$$

For the non-translational part, one finds

$$\int\limits_{\mathbb{R}^3 \times \mathbb{R}^+} I^{2/\delta} \mathcal{M}_{v,\theta}(f) dv dI$$

$$= \rho \Lambda_\delta \int\limits_{\mathbb{R}^+} \frac{I^{2/\delta}}{T_\theta^{\delta/2}} e^{-\frac{I^{2/\delta}}{T_\theta}} dI \int\limits_{\mathbb{R}^3} \frac{1}{\sqrt{\det(2\pi \mathcal{T}_{v,\theta})}} e^{-\frac{1}{2}(v-U)^\top \mathcal{T}_{v,\theta}^{-1}(v-U)} dv$$

$$= \rho \Lambda_\delta \int\limits_{\mathbb{R}^+} \frac{I^{2/\delta}}{T_\theta^{\delta/2}} e^{-\frac{I^{2/\delta}}{T_\theta}} dI.$$

Let $X = I^{2/\delta}/T_\theta$, then $dI = \frac{\delta}{2} T_\theta^{\delta/2} X^{\delta/2 - 1} dX$, and thus,

$$\int\limits_{\mathbb{R}^3 \times \mathbb{R}^+} I^{2/\delta} \mathcal{M}_{v,\theta}(f) dv dI = \rho \Lambda_\delta \int\limits_{\mathbb{R}^+} \frac{I^{2/\delta}}{T_\theta^{\delta/2}} e^{-\frac{I^{2/\delta}}{T_\theta}} dI$$

$$= \frac{\delta}{2} \rho \Lambda_\delta T_\theta \int\limits_{\mathbb{R}^+} X^{\delta/2} e^{-X} dX \tag{8.2}$$

$$= \frac{\delta}{2} \rho T_\theta,$$

where we have used

$$
\frac{\delta/2\rho T_\theta \int_{\mathbb{R}^+} X^{\delta/2} e^{-X} dX}{\int_{\mathbb{R}^+} e^{-I^{2/\delta}} dI} = \frac{\delta/2\rho T_\theta \int_{\mathbb{R}^+} X^{\delta/2} e^{-X} dX}{\delta/2 \int_{\mathbb{R}^+} X^{\delta/2-1} e^{-X} dX}
$$

$$
= \frac{\rho T_\theta \left[\left(-X^{\delta/2} e^{-X} \right)_{X=0}^{X=\infty} + \delta/2 \int_{\mathbb{R}^+} X^{\delta/2-1} e^{-X} dX \right]}{\int_{\mathbb{R}^+} X^{\delta/2-1} e^{-X} dX}
$$

$$
= \frac{\delta}{2} \rho T_\theta.
$$

Combining (8.2) with (8.1) and recalling the definition of T_δ in (1.4), we get

$$
\int_{\mathbb{R}^3 \times \mathbb{R}^+} \left(\frac{|v-U|^2}{2} + I^{2/\delta} \right) \mathcal{M}_{v,\theta}(f) \, dv \, dI
$$

$$
= \frac{3}{2}\rho\left\{ (1-\theta)T_{tr} + \theta T_\delta \right\} + \frac{\delta}{2}\rho\left\{ (1-\theta)T_{I,\delta} + \theta T_\delta \right\}
$$

$$
= \frac{3+\delta}{2}(1-\theta)\rho\left\{ \frac{3}{3+\delta}T_{tr} + \frac{\delta}{3+\delta}\delta T_{I,\delta} \right\} + \frac{3+\delta}{2}\rho\theta T_\delta
$$

$$
= \frac{3+\delta}{2}\rho T_\delta
$$

$$
= \int_{\mathbb{R}^3 \times \mathbb{R}^+} \left(\frac{|v-U|^2}{2} + I^{2/\delta} \right) f \, dv \, dI.
$$

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