J. Differential Equations 246 (2009) 4154–4172

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Well posedness for a higher order modified Camassa–Holm equation

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ARTICLE INFO

Article history:
Received 29 July 2008
Available online 24 December 2008

MSC:
35Q53

Keywords:
Modified Camassa–Holm equation
Nonlinear dispersive equations
Harmonic analysis
Bilinear estimates
Well posedness
Cauchy problem
Sobolev spaces

ABSTRACT

We show that the Cauchy problem for a higher order modification of the Camassa–Holm equation is locally well posed for initial data in the Sobolev space $H^s(\mathbb{R})$ for $s > s'$, where $1/4 \leqslant s' < 1/2$ and the value of s' depends on the order of equation. Employing harmonic analysis methods we derive the corresponding bilinear estimate and then use a contraction mapping argument to prove existence and uniqueness of solutions.

1. Introduction

For $m \geqslant 3$ odd, we consider the following initial value problem

$$\partial_t u + \partial_x^m u + \frac{1}{2}\partial_x\big(u^2\big) + \big(1 - \partial_x^2\big)^{-1}\partial_x\left[u^2 + \frac{1}{2}(\partial_x u)^2\right] = 0, \tag{1.1}$$

$$u(x,0) = \varphi(x), \quad t \in \mathbb{R},\ x \in \mathbb{R}. \tag{1.2}$$

Note that Eq. (1.1) without the dispersive term $\partial_x^m u$ is the nonlocal form of the well-known Camassa–Holm equation, which was derived by Camassa and Holm [5] as a model of water waves (see also Fuchssteiner and Fokas [7]).

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2008.12.001

Himonas and Misiołek show in [9] that the periodic initial value problem for Eq. (1.1) is locally well posed for sufficiently small initial data in the Sobolev space $H^s(\mathbb{T})$, $s > (5-m)/4$, by modifying techniques developed by Bourgain (see [1]) to prove well posedness for the periodic initial value problem for the KdV equation for initial data in $H^s(\mathbb{T})$, $s \geqslant 0$. Large data for the case $m = 3$ was considered in [10]. Kenig, Ponce, and Vega (see [11,12]) further developed the techniques introduced by Bourgain, proving local well posedness for the KdV equation for initial data in $H^s(\mathbb{T})$, $s \geqslant -1/2$, in the periodic case, and in $H^s(\mathbb{R})$, $s > -3/4$, in the non-periodic case. Each motivated by the work of Kenig, Ponce, and Vega, Byers shows in [3] that the non-periodic initial value problem for Eq. (1.1) with $m = 3$ is locally well posed in $H^s(\mathbb{R})$ for $s > 1/4$, and Gorsky shows in [8] local well posedness of the corresponding periodic problem in $H^s(\mathbb{T})$ for $s \geqslant 1/2$. All of the aforementioned works make use of so-called Bourgain spaces. In this work, we will use the following Bourgain spaces.

Definition 1.1. For any $s, b \in \mathbb{R}$, $X^{s,b}$ denotes the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$\|u\|_{s,b} = \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \left(1+|\xi|\right)^{2s} \left(1+\left|\tau - \xi^m\right|\right)^{2b} \left|\widehat{u}(\xi,\tau)\right|^2 d\xi\, d\tau \right)^{1/2}.$$

Next, we state the main result of this paper.

Theorem 1.2. *Let $m \geqslant 3$ be an odd integer. If $s > \frac{1}{2} \cdot \frac{m^2-3m+1}{m^2-3m+2}$ and $\varphi \in H^s(\mathbb{R})$, then there exists $b > \frac{1}{2}$ such that the initial value problem* (1.1)–(1.2) *has a unique local solution in the space $X^{s,b}$.*

Note that by the nature of our proof of Theorem 1.2 (a contraction mapping argument), we also will have Lipschitz dependence on initial data, thus establishing local well posedness (stronger than in the sense of Hadamard) for the initial value problem (1.1)–(1.2) in $H^s(\mathbb{R})$, $s > \frac{1}{2} \cdot \frac{m^2-3m+1}{m^2-3m+2}$. In the case $m = 3$, Wang and Cui have shown in [14] that the initial value problem (1.1)–(1.2) is globally well posed in $H^s(\mathbb{R})$, $s > \frac{5\sqrt{7}-10}{4}$, using the I-method of Colliander, Keel, Staffilani, Takaoka, and Tao (see, e.g., [6]); it is of interest to see what global well posedness results can be shown using this method for $m > 3$ odd.

It is not known whether or not the initial value problem (1.1)–(1.2) is locally well posed for any $s < \frac{1}{2} \cdot \frac{m^2-3m+1}{m^2-3m+2}$. Byers showed in [3] that, in the case $m = 3$, the bilinear estimate upon which the proof of Theorem 1.2 relies does not hold for any $s < 1/4$; it remains to be seen whether or not the bilinear estimate fails for $s < \frac{1}{2} \cdot \frac{m^2-3m+1}{m^2-3m+2}$ when $m > 3$ odd.

Finally, for some results concerning the periodic and non-periodic initial value problems for equations of the form

$$\partial_t u + \partial_x^m u + F\left(u, \partial_x u, \ldots, \partial_x^{m-1} u\right) = 0$$

with certain restrictions on F, see [2] and [13]. Note that Eq. (1.1) does not fall into the classes of equations addressed in [2] and [13] due to its nonlocal nonlinearity. In fact, studying the interaction between dispersion and nonlocal nonlinearities has been part of our motivation for this work.

This paper is structured as follows. In Section 2, we prove Theorem 1.2 by reformulating the initial value problem (1.1)–(1.2) as an integral equation and then showing that this integral equation defines a contraction on one of the $X^{s,b}$ spaces using the corresponding bilinear estimate. Finally, in Section 3 we prove this bilinear estimate using techniques from harmonic analysis.

2. Proof of Theorem 1.2

In this section, we establish Theorem 1.2, focusing on the case of $s \in (\frac{1}{2} \cdot \frac{m^2-3m+1}{m^2-3m+2}, \frac{1}{2})$. Specifically, we prove the following.

Theorem 2.1. *Let $m \geqslant 3$ be an odd integer. If $s \in (\frac{1}{2} \cdot \frac{m^2-3m+1}{m^2-3m+2}, \frac{1}{2})$, then there exist $b > \frac{1}{2}$ and $r > 0$ such that the initial value problem* (1.1)–(1.2) *has a unique local solution in the ball $B(0, r) \subset X^{s,b}$. The value of r can be made arbitrarily large by seeking a solution with a sufficiently small existence time.*

We first reformulate the initial value problem (1.1)–(1.2) as an integral equation. Then, we derive the estimate that will be used to show that this integral equation defines a contraction on one of the $X^{s,b}$ spaces. Using that estimate we prove Lemma 2.6, thereby completing the proof of Theorem 1.2.

Integral equation. Defining

$$w = \frac{1}{2}\partial_x\big(u^2\big) + \big(1-\partial_x^2\big)^{-1}\partial_x\left[u^2 + \frac{1}{2}(\partial_x u)^2\right], \tag{2.1}$$

the initial value problem (1.1)–(1.2) can be recast as the integral equation

$$u(x,t) = W(t)\varphi(x) - \int_0^t W(t-t')w(x,t')\,dt', \tag{2.2}$$

where $W(t)\varphi(x) := \int_{\mathbb{R}} \widehat{\varphi}(\xi)e^{i(\xi x+\xi^m t)}\,d\xi$. Notice that (2.2) can be written in the form

$$u(x,t) = \int_{\mathbb{R}} \widehat{\varphi}(\xi)e^{i(\xi x+\xi^m t)}\,d\xi + i\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{\tau-\xi^m}\big[e^{i(\xi x+\tau t)} - e^{i(\xi x+\xi^m t)}\big]\widehat{w}(\xi,\tau)\,d\xi\,d\tau. \tag{2.3}$$

Also note that

$$\widehat{w}(\xi,\tau) = \frac{i}{8\pi^2}\left(\xi + \frac{2\xi}{1+\xi^2}\right)\widehat{u} * \widehat{u}(\xi,\tau) + \frac{i}{8\pi^2}\frac{\xi}{1+\xi^2}\widehat{\partial_x u} * \widehat{\partial_x u}(\xi,\tau). \tag{2.4}$$

A priori estimate. In the sequel, let $\psi = \psi(t) \in C_0^\infty(-1,1)$ be a cut-off function with $0 \leqslant \psi \leqslant 1$ and $\psi(t) \equiv 1$ for $|t| < 1/2$.

Proposition 2.2. *Let T be the map defined by*

$$Tu(x,t) = \psi(t)W(t)\varphi(x) - \psi(t)\int_0^t W(t-t')w(x,t')\,dt',$$

where

$$w = \frac{1}{2}\partial_x\big(u^2\big) + \big(1-\partial_x^2\big)^{-1}\partial_x\left[u^2 + \frac{1}{2}(\partial_x u)^2\right].$$

Given $s \in (\frac{1}{2} \cdot \frac{m^2-3m+1}{m^2-3m+2}, \frac{1}{2})$, if $b \in (\frac{1}{2}, \frac{1}{2}+\beta]$, $b' \in [\frac{1}{2}-\beta, \frac{1}{2})$, and $b + b' \leqslant 1$, where $\beta = \beta(s)$ is as in Theorem 3.1*, then there is a $c > 0$ such that*

$$|||Tu|||_{s,b} \leqslant c\big(|||u|||_{s,b'}^2 + \|\varphi\|_{H^s}\big) \tag{2.5}$$

for all $u \in X^{s,b'}$.

The proof of Proposition 2.2 relies on the bilinear estimate (Theorem 3.1) and on the following proposition:

Proposition 2.3. *If $b > 1/2$, then there exists $c > 0$ such that*

$$\left|\!\left|\!\left| \psi(t)\int_0^t W(t-t')w(x,t')\,dt' \right|\!\right|\!\right|_{s,b} \leqslant c|\!|\!|w|\!|\!|_{s,b-1}.$$

The proof of Proposition 2.3 in the case $m = 3$ appears in [4]. The general case can be handled in the same fashion.

Proof of Proposition 2.2. For the estimate on the $X^{s,b}$-norm of the linear term

$$f(x,t) \doteq \psi(t)W(t)\varphi(x) = \psi(t)\int_{\mathbb{R}} \widehat{\varphi}(\xi)e^{i(\xi x+\xi^m t)}\,d\xi,$$

note that $\widehat{f}^x(\xi,t) = \psi(t)\widehat{\varphi}(\xi)e^{i\xi^m t}$, thus

$$\widehat{f}(\xi,\tau) = \widehat{\varphi}(\xi)\int_{\mathbb{R}} \psi(t)e^{i(\xi^m-\tau)t}\,dt = \widehat{\varphi}(\xi)\widehat{\psi}\big(\tau-\xi^m\big).$$

Therefore,

$$\begin{aligned}
|\!|\!|\psi(t)W(t)\varphi(x)|\!|\!|_{s,b}^2 &= \int_{\mathbb{R}}\int_{\mathbb{R}} \big(1+\xi^2\big)^s\big(1+\big|\tau-\xi^m\big|\big)^{2b}\big|\widehat{\varphi}(\xi)\widehat{\psi}\big(\tau-\xi^m\big)\big|^2\,d\xi\,d\tau\\
&= \int_{\mathbb{R}} \big(1+\xi^2\big)^s\big|\widehat{\varphi}(\xi)\big|^2\int_{\mathbb{R}}\big(1+\big|\tau-\xi^m\big|\big)^{2b}\big|\widehat{\psi}\big(\tau-\xi^m\big)\big|^2\,d\tau\,d\xi\\
&= c_\psi\|\varphi\|_{H^s}^2.
\end{aligned}$$

For the Duhamel term, we apply the bilinear estimate (Theorem 3.1) and Proposition 2.3, yielding

$$\left|\!\left|\!\left| \psi(t)\int_0^t W(t-\tau)w(x,\tau)\,d\tau \right|\!\right|\!\right|_{s,b} \lesssim |\!|\!|w|\!|\!|_{s,b-1} \lesssim |\!|\!|u|\!|\!|_{s,b'}^2,$$

which completes the proof of Proposition 2.2. □

Since $b' < b$, Proposition 2.2 implies that

$$|\!|\!|Tu|\!|\!|_{s,b} \leqslant c\big(|\!|\!|u|\!|\!|_{s,b}^2 + \|\varphi\|_{H^s}\big) \tag{2.6}$$

for all $u \in X^{s,b}$. Using estimate (2.6), one can show that there exists an $r > 0$ such that, if $\|\varphi\|_{H^s}$ is sufficiently small, then T is a contraction on the closed ball $\bar{B}(0,r) \subset X^{s,b}$. Then an application of the contraction mapping theorem almost establishes Theorem 2.1 for small initial data. Note that uniqueness of the fixed point of T is only established on the ball $\bar{B}(0,r)$. To complete the proof of the theorem, that is, to remove the restriction on the size of the initial data and to show uniqueness on balls of arbitrarily large radii, we will appropriately shrink the existence time of the solutions that we seek. To this end, for $0 < \delta < 1$, define

$$\psi_\delta(t) = \psi\left(\frac{t}{\delta}\right)$$

and define

$$T_\delta u = \psi_\delta T(\psi_\delta u). \tag{2.7}$$

To obtain the desired estimate on $|||T_\delta u|||_{s,b}$, we will use the following two lemmas. (See [4] for proofs.)

Lemma 2.4. *If $s \in \mathbb{R}$ and $b > 1/2$, then there is a $c > 0$ such that for any $u \in X^{s,b}$ and $\theta = \theta(t) \in H^b(\mathbb{R})$*

$$|||\theta u|||_{s,b} \leqslant c\|\theta\|_{H^b}|||u|||_{s,b}.$$

Lemma 2.5. *Let $b > \frac{1}{2}$, $0 \leqslant b' \leqslant b$ and $s \in \mathbb{R}$. Then there exists $c > 0$ such that for any $u \in X^{s,b}$ and $\theta = \theta(t) \in H^b(\mathbb{R})$*

$$|||\theta u|||_{s,b'} \leqslant c\|\theta\|_{L^2}^{1-b'/b}\|\theta\|_{H^b}^{b'/b}|||u|||_{s,b}.$$

By (2.5) and the above two lemmas,

$$\begin{aligned}|||T_\delta u|||_{s,b} &\lesssim \|\psi_\delta\|_{H^b}\big|\big|\big|T(\psi_\delta u)\big|\big|\big|_{s,b}\\ &\lesssim \|\psi_\delta\|_{H^b}\big(|||\psi_\delta u|||_{s,b'}^2 + \|\varphi\|_{H^s}\big)\\ &\lesssim \|\psi_\delta\|_{H^b}\big(\|\psi_\delta\|_{L^2}^{2(1-b'/b)}\|\psi_\delta\|_{H^b}^{2b'/b}|||u|||_{s,b}^2 + \|\varphi\|_{H^s}\big).\end{aligned}$$

To complete the estimate of $|||T_\delta u|||_{s,b}$, note $\widehat{\psi_\delta}(\tau) = \delta\widehat{\psi}(\delta\tau)$, so that

$$\begin{aligned}\|\psi_\delta\|_{H^b}^2 &= \int_{\mathbb{R}}\big(1+\tau^2\big)^b\big|\delta\widehat{\psi}(\delta\tau)\big|^2\,d\tau = \delta^{1-2b}\int_{\mathbb{R}}\big(\delta^2+\widetilde{\tau}^2\big)^b\big|\widehat{\psi}(\widetilde{\tau})\big|^2\,d\widetilde{\tau}\\ &\leqslant \delta^{1-2b}\|\psi\|_{H^b}^2.\end{aligned}$$

Thus $\|\psi_\delta\|_{L^2}^2 = \|\psi_\delta\|_{H^0}^2 \leqslant \delta\|\psi\|_{L^2}^2$ and

$$\begin{aligned}\|\psi_\delta\|_{L^2}^{2(1-b'/b)}\|\psi_\delta\|_{H^b}^{2b'/b} &\leqslant \delta^{1-b'/b}\|\psi\|_{L^2}^{2(1-b'/b)}\delta^{(1-2b)b'/b}\|\psi\|_{H^b}^{2b'/b}\\ &\leqslant \delta^{1-2b'}\|\psi\|_{H^b}^2.\end{aligned}$$

So there exists $c = c(s,b,\psi) > 0$ such that

$$|||T_\delta u|||_{s,b} \leqslant c\delta^{\frac{1}{2}-b}\big(\delta^{1-2b'}|||u|||_{s,b}^2 + \|\varphi\|_{H^s}\big) \tag{2.8}$$

for all $u \in X^{s,b}$ for s, b, and b' as in Theorem 3.1. Choosing $b' = \frac{1}{2} - \beta$, where β is as in Theorem 3.1, estimate (2.8) becomes

$$|||T_\delta u|||_{s,b} \leqslant c\delta^{\frac{1}{2}-b}\big(\delta^{2\beta}|||u|||_{s,b}^2 + \|\varphi\|_{H^s}\big). \tag{2.9}$$

We now will use estimate (2.9) to complete the proof of Theorem 2.1 by proving the following lemma.

Lemma 2.6. *Let $c > 0$ be the constant in* (2.9)*. Choose $\delta \in (0, 1]$ such that*

$$\|\varphi\|_{H^s} < \frac{1}{16c^2}\left(\frac{1}{\delta}\right)^{1-2b+2\beta}. \tag{2.10}$$

Let

$$r = \frac{1}{4c}\left(\frac{1}{\delta}\right)^{\frac{1}{2}-b+2\beta}. \tag{2.11}$$

Then T_δ is a contraction on the closed ball $\bar{B}(0,r) = \{u \in X^{s,b}\colon |||u|||_{s,b} \leqslant r\}$.

Remark. Note that since b can be chosen so that $b < \frac{1}{2} + \beta$, by choosing δ in the above lemma small enough, we can ensure not only that (2.10) holds but also that r is as large as we like.

Proof of Lemma 2.6. By estimate (2.9) and the choice of δ, for all $u \in \bar{B}(0,r)$,

$$\begin{aligned}
|||T_\delta u|||_{s,b} &\leqslant c\left(\frac{1}{\delta}\right)^{b-\frac{1}{2}}\left(\left(\frac{1}{\delta}\right)^{-2\beta}\frac{1}{16c^2}\left(\frac{1}{\delta}\right)^{1-2b+4\beta} + \frac{1}{16c^2}\left(\frac{1}{\delta}\right)^{1-2b+2\beta}\right)\\
&= \frac{1}{8c}\left(\frac{1}{\delta}\right)^{\frac{1}{2}-b+2\beta}\\
&= \frac{1}{2}r,
\end{aligned}$$

so that $T_\delta : \bar{B}(0,r) \to \bar{B}(0,r)$.

To estimate $|||T_\delta u - T_\delta v|||_{s,b}$, first note that, as in the estimate of $|||T_\delta u|||_{s,b}$ above,

$$\begin{aligned}
|||T_\delta u - T_\delta v|||_{s,b} &= \big|\big|\big|\psi_\delta\big(T(\psi_\delta u) - T(\psi_\delta v)\big)\big|\big|\big|_{s,b}\\
&\lesssim \delta^{\frac{1}{2}-b}\big|\big|\big|T(\psi_\delta u) - T(\psi_\delta v)\big|\big|\big|_{s,b}.
\end{aligned}$$

Next note that

$$\begin{aligned}
T(\psi_\delta u) - T(\psi_\delta v) &= -\psi(t)\int_0^t W(t-t')\big(w_{\psi_\delta u}(x,t') - w_{\psi_\delta v}(x,t')\big)\,dt'\\
&= -\psi(t)\int_0^t W(t-t')w_{fg}(x,t')\,dt',
\end{aligned}$$

where $f = \psi_\delta(u+v)$ and $g = \psi_\delta(u-v)$. Then, following the proof of Proposition 2.2, we can show that

$$\big|\big|\big|T(\psi_\delta u) - T(\psi_\delta v)\big|\big|\big|_{s,b} \lesssim \big|\big|\big|\psi_\delta(u+v)\big|\big|\big|_{s,b'}\big|\big|\big|\psi_\delta(u-v)\big|\big|\big|_{s,b'}.$$

Finally, as in the estimate of $|||T_\delta u|||_{s,b}$ above,

$$\big|\big|\big|\psi_\delta(u+v)\big|\big|\big|_{s,b'}\big|\big|\big|\psi_\delta(u-v)\big|\big|\big|_{s,b'} \lesssim \delta^{1-2b'}|||u+v|||_{s,b}|||u-v|||_{s,b}.$$

Putting all of the above together with, again, $b' = \frac{1}{2} - \beta$, where β is as in Theorem 3.1, we have

$$|||T_\delta u - T_\delta v|||_{s,b} \leqslant c\delta^{\frac{1}{2}-b}\big(\delta^{2\beta}|||u+v|||_{s,b}|||u-v|||_{s,b}\big).$$

Thus, for all $u, v \in \bar{B}(0,r)$, we have

$$\begin{aligned}|||T_\delta u - T_\delta v|||_{s,b} &\leqslant 2cr\delta^{\frac{1}{2}-b+2\beta}|||u-v|||_{s,b}\\ &= \frac{1}{2}|||u-v|||_{s,b},\end{aligned}$$

completing the proof of the lemma, which in turn completes the proof of Theorem 2.1. □

3. Bilinear estimate

In this section, we state and prove the bilinear estimate used in the proof of Theorem 2.1.

Theorem 3.1. *Given $s \in (\frac{1}{2} \cdot \frac{m^2-3m+1}{m^2-3m+2}, \frac{1}{2})$, there exists $\beta > 0$ such that if $b \in (\frac{1}{2}, \frac{1}{2}+\beta]$, $b' \in [\frac{1}{2}-\beta, \frac{1}{2})$, and $b + b' \leqslant 1$, then there exists $c > 0$ such that for all $f, g \in X^{s,b'}$*

$$|||w_{fg}|||_{s,b-1} \leqslant c|||f|||_{s,b'}|||g|||_{s,b'}, \tag{3.1}$$

where w_{fg} satisfies

$$\widehat{w}_{fg}(\xi,\tau) \simeq \left(\xi + \frac{2\xi}{1+\xi^2}\right)\widehat{f} * \widehat{g}(\xi,\tau) + \frac{\xi}{1+\xi^2}\widehat{\partial_x f} * \widehat{\partial_x g}(\xi,\tau). \tag{3.2}$$

Notice that $|\xi + \frac{2\xi}{1+\xi^2}| = |\xi||\frac{\xi^2+3}{\xi^2+1}| \leqslant 3|\xi|$, so the estimates for the term

$$\left(\xi + \frac{2\xi}{1+\xi^2}\right)\widehat{f} * \widehat{g}(\xi,\tau) \tag{3.3}$$

can be handled as in [1]. So, to establish Theorem 3.1, we will prove the following proposition.

Proposition 3.2. *Given $s \in (\frac{1}{2} \cdot \frac{m^2-3m+1}{m^2-3m+2}, \frac{1}{2})$, there exists $\beta > 0$ such that if $b \in (\frac{1}{2}, \frac{1}{2}+\beta]$, $b' \in [\frac{1}{2}-\beta, \frac{1}{2})$, and $b + b' \leqslant 1$, then there exists $c > 0$ such that for all $f, g \in X^{s,b'}$*

$$|||w_{fg}|||_{s,b-1} \leqslant c|||f|||_{s,b'}|||g|||_{s,b'}, \tag{3.4}$$

where w_{fg} satisfies

$$\widehat{w}_{fg}(\xi,\tau) \simeq \frac{\xi}{1+\xi^2}\widehat{\partial_x f} * \widehat{\partial_x g}(\xi,\tau). \tag{3.5}$$

Proof. Define

$$c_u(\xi,\tau) = \big(1+\xi^2\big)^{s/2}\big(1+\big|\tau-\xi^m\big|\big)^{b'}\big|\widehat{u}(\xi,\tau)\big|. \tag{3.6}$$

Observe that

$$\|\!|u|\!\|_{s,b'} = \left(\int_{\mathbb{R}} \int_{\mathbb{R}} c_u(\xi,\tau)^2 \right)^{1/2} = \big\| c_u(\xi,\tau) \big\|_{L^2_\tau L^2_\xi}. \tag{3.7}$$

By (3.6),

$$\begin{aligned}
\big|\widehat{w}_{fg}(\xi,\tau)\big| &= \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\xi}{\xi^2+1} \widehat{\partial_x f}(\xi-\xi_1,\tau-\tau_1)\widehat{\partial_x g}(\xi_1,\tau_1)\,d\xi_1\,d\tau_1 \right| \\
&= \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\xi}{\xi^2+1} \xi_1(\xi-\xi_1)\widehat{f}(\xi-\xi_1,\tau-\tau_1)\widehat{g}(\xi_1,\tau_1)\,d\xi_1\,d\tau_1 \right| \\
&\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\xi||\xi-\xi_1||\xi_1|}{(\xi^2+1)(1+|\xi-\xi_1|^2)^{\frac{s}{2}}(1+|\xi_1|^2)^{\frac{s}{2}}} \\
&\quad \cdot \frac{c_f(\xi-\xi_1,\tau-\tau_1)}{(1+|\tau-\tau_1-(\xi-\xi_1)^m|)^{b'}} \cdot \frac{c_g(\xi_1,\tau_1)}{(1+|\tau_1-\xi_1^m|)^{b'}}\,d\xi_1\,d\tau_1.
\end{aligned} \tag{3.8}$$

We want to show that

$$\|\!|w_{fg}|\!\|_{s,b-1} \lesssim \|\!|f|\!\|_{s,b'} \|\!|g|\!\|_{s,b'},$$

that is, we want to show that

$$\left(\int_{\mathbb{R}} \int_{\mathbb{R}} \big(1+\xi^2\big)^s \big(1+\big|\tau-\xi^m\big|\big)^{2(b-1)} \big|\widehat{w}_{fg}(\xi,\tau)\big|^2\,d\xi\,d\tau \right)^{1/2} \lesssim \|\!|f|\!\|_{s,b'} \|\!|g|\!\|_{s,b'}. \tag{3.9}$$

Notice that the left-hand side of (3.9) is

$$\left\| \frac{(1+\xi^2)^{\frac{s}{2}}|\widehat{w}_{fg}(\xi,\tau)|}{(1+|\tau-\xi^m|)^{1-b}} \right\|_{L^2_\tau L^2_\xi},$$

so, by (3.7), what we want to show is that

$$\left\| \frac{(1+\xi^2)^{\frac{s}{2}}|\widehat{w}_{fg}(\xi,\tau)|}{(1+|\tau-\xi^m|)^{1-b}} \right\|_{L^2_\tau L^2_\xi} \lesssim \|c_f\|_{L^2_\tau L^2_\xi} \|c_g\|_{L^2_\tau L^2_\xi}. \tag{3.10}$$

Let

$$\mathcal{D} = \big\{ (\xi,\tau,\xi_1,\tau_1) \in \mathbb{R}^4\colon\ \big|\tau-\tau_1-(\xi-\xi_1)^m\big| \leqslant \big|\tau_1-\xi_1^m\big| \big\}.$$

Note that

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{(\mathbb{R}^4-\mathcal{D})}(\xi,\tau,\xi_1,\tau_1) \frac{\xi}{\xi^2+1} \widehat{\partial_x f}(\xi-\xi_1,\tau-\tau_1)\widehat{\partial_x g}(\xi_1,\tau_1)\,d\xi_1\,d\tau_1 \\
&\quad = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{\mathcal{D}}(\xi,\tau,\xi_1,\tau_1) \frac{\xi}{\xi^2+1} \widehat{\partial_x f}(\xi_1,\tau_1)\widehat{\partial_x g}(\xi-\xi_1,\tau-\tau_1)\,d\xi_1\,d\tau_1.
\end{aligned}$$

Thus, in proving (3.10) we can restrict our attention to the set $\mathcal{D}$.

By (3.8),

$$\begin{aligned}
&\left\| \frac{(1+\xi^2)^{\frac{s}{2}} |\widehat{w}_{fg}(\xi,\tau)|}{(1+|\tau-\xi^m|)^{1-b}} \right\|_{L^2_\tau L^2_\xi} \\
&\quad \lesssim \left\| \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \frac{1}{(1+|\tau-\xi^m|)^{1-b}} \cdot \frac{|\xi||\xi-\xi_1|^{1-s}|\xi_1|^{1-s}}{(1+|\xi|)^{2-s}} \right. \\
&\qquad \left. \cdot \frac{c_f(\xi-\xi_1,\tau-\tau_1)}{(1+|\tau-\tau_1-(\xi-\xi_1)^m|)^{b'}} \cdot \frac{c_g(\xi_1,\tau_1)}{(1+|\tau_1-\xi_1^m|)^{b'}} \, d\xi_1 \, d\tau_1 \right\|_{L^2_\tau L^2_\xi} \\
&\quad = \left\| \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} Q(\xi,\tau,\xi_1,\tau_1) c_f(\xi-\xi_1,\tau-\tau_1) c_g(\xi_1,\tau_1) \, d\xi_1 \, d\tau_1 \right\|_{L^2_\tau L^2_\xi},
\end{aligned}$$

where $Q = Q(\xi,\tau,\xi_1,\tau_1)$ is given by

$$Q \doteq \frac{|\xi||\xi-\xi_1|^{1-s}|\xi_1|^{1-s}}{(1+|\xi|)^{2-s}(1+|\tau-\xi^m|)^{1-b}(1+|\tau_1-\xi_1^m|)^{b'}(1+|\tau-\tau_1-(\xi-\xi_1)^m|)^{b'}}.$$

Thus what we want to show is

$$\begin{aligned}
&\left\| \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \chi_{\mathcal{D}}(\xi,\tau,\xi_1,\tau_1) Q(\xi,\tau,\xi_1,\tau_1) c_f(\xi-\xi_1,\tau-\tau_1) c_g(\xi_1,\tau_1) \, d\xi_1 \, d\tau_1 \right\|_{L^2_\tau L^2_\xi} \\
&\quad \lesssim \|c_f\|_{L^2_\tau L^2_\xi} \|c_g\|_{L^2_\tau L^2_\xi},
\end{aligned}$$

which we will write as

$$\left\| \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \chi_{\mathcal{D}} \cdot Q \cdot c_f(\xi-\xi_1,\tau-\tau_1) c_g(\xi_1,\tau_1) \, d\xi_1 \, d\tau_1 \right\|_{L^2_\tau L^2_\xi} \lesssim \|c_f\|_{L^2_\tau L^2_\xi} \|c_g\|_{L^2_\tau L^2_\xi}. \tag{3.11}$$

To prove (3.11), we will first show it for $s = 1$ so that we can make a simplifying assumption in the proof for $s \in (\frac{1}{2} \cdot \frac{m^2-3m+1}{m^2-3m+2}, \frac{1}{2})$. If $s = 1$, then

$$Q = \frac{|\xi|}{(1+|\xi|)(1+|\tau-\xi^m|)^{1-b}(1+|\tau_1-\xi_1^m|)^{b'}(1+|\tau-\tau_1-(\xi-\xi_1)^m|)^{b'}},$$

so that, in this case (3.11) follows from Lemma 3.6. Otherwise, notice that if $|\xi_1| \leqslant 1$ or $|\xi-\xi_1| \leqslant 1$, then

$$\frac{|\xi-\xi_1|^{1-s}|\xi_1|^{1-s}}{(1+|\xi|)^{2-s}} \leqslant \frac{1}{(1+|\xi|)},$$

which reduces establishing (3.11) to establishing it for the case $s = 1$. So to proceed we define sets $\mathcal{A} \subset \mathcal{D}$ and $\mathcal{B} \subset \mathcal{D}$ as follows:

$$\begin{aligned}
\mathcal{A} &= \left\{ (\xi,\tau,\xi_1,\tau_1) \in \mathcal{D}\colon\ \left|\tau_1 - \xi_1^m\right| \leqslant \left|\tau - \xi^m\right|,\ |\xi_1| \geqslant 1,\ |\xi-\xi_1| \geqslant 1 \right\}, \\
\mathcal{B} &= \left\{ (\xi,\tau,\xi_1,\tau_1) \in \mathcal{D}\colon\ \left|\tau_1 - \xi_1^m\right| \geqslant \left|\tau - \xi^m\right|,\ |\xi_1| \geqslant 1,\ |\xi-\xi_1| \geqslant 1 \right\}.
\end{aligned}$$

By the above discussion, to show (3.11) it suffices to show

$$\left\| \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{\mathcal{A}} \cdot Q \cdot c_f(\xi-\xi_1, \tau-\tau_1) c_g(\xi_1, \tau_1)\, d\xi_1\, d\tau_1 \right\|_{L^2_\tau L^2_\xi} \lesssim \|c_f\|_{L^2_\tau L^2_\xi} \|c_g\|_{L^2_\tau L^2_\xi} \tag{3.12}$$

and

$$\left\| \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{\mathcal{B}} \cdot Q \cdot c_f(\xi-\xi_1, \tau-\tau_1) c_g(\xi_1, \tau_1)\, d\xi_1\, d\tau_1 \right\|_{L^2_\tau L^2_\xi} \lesssim \|c_f\|_{L^2_\tau L^2_\xi} \|c_g\|_{L^2_\tau L^2_\xi}. \tag{3.13}$$

For (3.12), first apply Hölder's inequality:

$$\begin{aligned}
&\left\| \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{\mathcal{A}} \cdot Q \cdot c_f(\xi-\xi_1, \tau-\tau_1) c_g(\xi_1, \tau_1)\, d\xi_1\, d\tau_1 \right\|_{L^2_\tau L^2_\xi} \\
&\quad \leqslant \Big\| \|\chi_{\mathcal{A}} \cdot Q\|_{L^2_{\xi_1} L^2_{\tau_1}} \big\| c_f(\xi-\xi_1, \tau-\tau_1) c_g(\xi_1, \tau_1)\big\|_{L^2_{\xi_1} L^2_{\tau_1}} \Big\|_{L^2_\tau L^2_\xi} \\
&\quad \leqslant \sup_{\xi,\tau} \|\chi_{\mathcal{A}} \cdot Q\|_{L^2_{\xi_1} L^2_{\tau_1}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} c_f(\xi-\xi_1, \tau-\tau_1)^2 c_g(\xi_1, \tau_1)^2\, d\xi_1\, d\tau_1\, d\xi\, d\tau \right)^{1/2} \\
&\quad = \sup_{\xi,\tau} \|\chi_{\mathcal{A}} \cdot Q\|_{L^2_{\xi_1} L^2_{\tau_1}} \|c_f\|_{L^2_\xi L^2_\tau} \|c_g\|_{L^2_\xi L^2_\tau}.
\end{aligned} \tag{3.14}$$

For (3.13), first note that

$$\begin{aligned}
&\left\| \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{\mathcal{B}} \cdot Q \cdot c_f(\xi-\xi_1, \tau-\tau_1) c_g(\xi_1, \tau_1)\, d\xi_1\, d\tau_1 \right\|_{L^2_\tau L^2_\xi} \\
&\quad = \sup_{\|d(\xi,\tau)\|_{L^2_\xi L^2_\tau}=1} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{\mathcal{B}} \cdot Q \cdot d(\xi,\tau) c_f(\xi-\xi_1, \tau-\tau_1) c_g(\xi_1, \tau_1)\, d\xi_1\, d\tau_1\, d\xi\, d\tau \right| \\
&\quad \leqslant \sup_{\substack{\|d(\xi,\tau)\|_{L^2_\xi L^2_\tau}=1 \\ d(\xi,\tau)\geqslant 0}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{\mathcal{B}} \cdot Q \cdot d(\xi,\tau) c_f(\xi-\xi_1, \tau-\tau_1) c_g(\xi_1, \tau_1)\, d\xi_1\, d\tau_1\, d\xi\, d\tau,
\end{aligned}$$

then apply Hölder's inequality:

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{\mathcal{B}} \cdot Q \cdot d(\xi,\tau) c_f(\xi-\xi_1, \tau-\tau_1) c_g(\xi_1, \tau_1)\, d\xi_1\, d\tau_1\, d\xi\, d\tau \\
&\quad \leqslant \left\| \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{\mathcal{B}} \cdot Q \cdot d(\xi,\tau) c_f(\xi-\xi_1, \tau-\tau_1)\, d\xi\, d\tau \right\|_{L^2_{\xi_1} L^2_{\tau_1}} \big\| c_g(\xi_1, \tau_1) \big\|_{L^2_{\xi_1} L^2_{\tau_1}} \\
&\quad \leqslant \sup_{\xi_1,\tau_1} \|\chi_{\mathcal{B}} \cdot Q\|_{L^2_\xi L^2_\tau} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} d(\xi,\tau)^2 c_f(\xi-\xi_1, \tau-\tau_1)^2\, d\xi_1\, d\tau_1\, d\xi\, d\tau \right)^{1/2} \|c_g\|_{L^2_\xi L^2_\tau} \\
&\quad = \sup_{\xi_1,\tau_1} \|\chi_{\mathcal{B}} \cdot Q\|_{L^2_\xi L^2_\tau} \|d\|_{L^2_\xi L^2_\tau} \|c_f\|_{L^2_\xi L^2_\tau} \|c_g\|_{L^2_\xi L^2_\tau},
\end{aligned}$$

so that

$$\begin{aligned}&\left\| \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{\mathcal{B}} \cdot Q \cdot c_f(\xi - \xi_1, \tau - \tau_1) c_g(\xi_1, \tau_1)\, d\xi_1\, d\tau_1 \right\|_{L^2_\tau L^2_\xi} \\ &\quad \leqslant \sup_{\xi_1, \tau_1} \| \chi_{\mathcal{B}} \cdot Q \|_{L^2_\xi L^2_\tau} \|c_f\|_{L^2_\xi L^2_\tau} \|c_g\|_{L^2_\xi L^2_\tau}. \end{aligned} \tag{3.15}$$

By (3.14) and (3.15), to show (3.12) and (3.13) it suffices to show:

$$\sup_{\xi, \tau} \| \chi_{\mathcal{A}} \cdot Q \|_{L^2_{\xi_1} L^2_{\tau_1}} < \infty \tag{3.16}$$

and

$$\sup_{\xi_1, \tau_1} \| \chi_{\mathcal{B}} \cdot Q \|_{L^2_\xi L^2_\tau} < \infty. \tag{3.17}$$

Lemmas 3.7 and 3.8 below establish (3.16) and (3.17), respectively, completing the proof of Proposition 3.2. □

Finally, we prove the three lemmas used in the above proof of the bilinear estimate. The proofs rely on Lemmas 3.3–3.5, the proofs of which are omitted.

Lemma 3.3. *Let $m \geqslant 3$ be an odd integer. If $\frac{1}{m-1} < \ell < 1$, then*

$$\int_{\mathbb{R}} \frac{dx}{(1 + |\alpha - x^{m-1}|)^\ell} \lesssim \frac{1}{(1 + |\alpha|)^{\ell - \frac{1}{m-1}}}. \tag{3.18}$$

Lemma 3.4. *If $\frac{1}{2} < \ell < 1$, then*

$$\int_{\mathbb{R}} \frac{dx}{(1 + |x - \alpha|)^\ell (1 + |x - \beta|)^\ell} \lesssim \frac{1}{(1 + |\alpha - \beta|)^{2\ell - 1}}. \tag{3.19}$$

Lemma 3.5. *For an odd integer $m \geqslant 3$, let $d_m(\xi, \xi_1) = -\xi^m + \xi_1^m + (\xi - \xi_1)^m$. Then there exist positive constants c_m and C_m such that*

$$\left| d_m(\xi, \xi_1) \right| \geqslant c_m |\xi_1|^{m-3} \left| \xi \xi_1 (\xi - \xi_1) \right| \quad \forall \xi, \xi_1 \in \mathbb{R}; \tag{3.20}$$

$$\left| d_m(\xi, \xi_1) \right| \geqslant c_m |\xi|^{m-3} \left| \xi \xi_1 (\xi - \xi_1) \right| \quad \forall \xi, \xi_1 \in \mathbb{R}; \tag{3.21}$$

$$\left| d_m(\xi, \xi_1) \right| \leqslant C_m |\xi_1|^{m-3} \left| \xi \xi_1 (\xi - \xi_1) \right| \quad \text{if } |\xi| \leqslant |\xi_1|. \tag{3.22}$$

Lemma 3.6. *If $b \in (\frac{1}{2}, 1)$ and $b' \in (\frac{m}{4(m-1)}, \frac{1}{2})$, then there exists $c > 0$ such that for all $\xi, \tau \in \mathbb{R}$*

$$\iint \frac{\xi^2\, d\xi_1\, d\tau_1}{(1 + |\xi|)^2 (1 + |\tau - \xi^m|)^{2(1-b)} (1 + |\tau_1 - \xi_1^m|)^{2b'} (1 + |\tau - \tau_1 - (\xi - \xi_1)^m|)^{2b'}} \leqslant c.$$

Proof. By (3.19),

$$\begin{aligned}&\int \frac{d\tau_1}{(1 + |\tau_1 - \xi_1^m|)^{2b'} (1 + |\tau - \tau_1 - (\xi - \xi_1)^m|)^{2b'}} \\ &\quad \lesssim \frac{1}{(1 + |\tau - (\xi - \xi_1)^m - \xi_1^m|)^{4b' - 1}}.\end{aligned}$$

So

$$\iint \frac{\xi^2\,d\xi_1\,d\tau_1}{(1+|\xi|)^2(1+|\tau-\xi^m|)^{2(1-b)}(1+|\tau_1-\xi_1^m|)^{2b'}(1+|\tau-\tau_1-(\xi-\xi_1)^m|)^{2b'}}$$
$$\lesssim \frac{\xi^2}{(1+|\xi|)^2(1+|\tau-\xi^m|)^{2(1-b)}}\int_{\mathbb{R}} \frac{d\xi_1}{(1+|\tau-(\xi-\xi_1)^m-\xi_1^m|)^{4b'-1}}.$$

Since

$$\tau-(\xi-\xi_1)^m-\xi_1^m=\big(1-2^{m-1}\big)\xi\left(\xi_1-\frac{\xi}{2}\right)^{m-1}+\tau-\frac{\xi^m}{2^{m-1}},$$

we can write

$$\big|\tau-(\xi-\xi_1)^m-\xi_1^m\big|=\frac{1}{2^{m-1}}\big|2^{m-1}\tau-\xi^m-\big(2^{m-1}-1\big)\xi(2\xi_1-\xi)^{m-1}\big|.$$

So to estimate $\int_{\mathbb{R}}\frac{d\xi_1}{(1+|\tau-(\xi-\xi_1)^m-\xi_1^m|)^{4b'-1}}$ we make the substitution

$$u=\big[\big(2^{m-1}-1\big)\xi\big]^{\frac{1}{m-1}}(2\xi_1-\xi),\qquad du=2\big[\big(2^{m-1}-1\big)\xi\big]^{\frac{1}{m-1}}\,d\xi_1$$

and apply (3.18) to yield

$$\int_{\mathbb{R}} \frac{d\xi_1}{(1+|\tau-(\xi-\xi_1)^m-\xi_1^m|)^{4b'-1}}$$
$$\lesssim \int_{\mathbb{R}} \frac{d\xi_1}{(1+|2^{m-1}\tau-\xi^m-(2^{m-1}-1)\xi(2\xi_1-\xi)^{m-1}|)^{4b'-1}}$$
$$\simeq \frac{1}{|\xi|^{\frac{1}{m-1}}}\int_{\mathbb{R}} \frac{du}{(1+|2^{m-1}\tau-\xi^m-u^{m-1}|)^{4b'-1}}$$
$$\lesssim \frac{1}{|\xi|^{\frac{1}{m-1}}(1+|2^{m-1}\tau-\xi^m|)^{4b'-1-\frac{1}{m-1}}}.$$

Combining this with the previous estimate yields

$$\iint \frac{\xi^2\,d\xi_1\,d\tau_1}{(1+|\xi|)^2(1+|\tau-\xi^m|)^{2(1-b)}(1+|\tau_1-\xi_1^m|)^{2b'}(1+|\tau-\tau_1-(\xi-\xi_1)^m|)^{2b'}}$$
$$\lesssim \frac{\xi^2}{(1+|\xi|)^2(1+|\tau-\xi^m|)^{2(1-b)}}\cdot\frac{1}{|\xi|^{\frac{1}{m-1}}(1+|2^{m-1}\tau-\xi^m|)^{4b'-1-\frac{1}{m-1}}}$$
$$\leqslant \frac{1}{(1+|\tau-\xi^m|)^{2(1-b)}(1+|2^{m-1}\tau-\xi^m|)^{4b'-\frac{m}{m-1}}}$$
$$\leqslant 1,$$

with the last inequality holding since $b'\geqslant\frac{m}{4(m-1)}$ and $b<1$. $\square$

Lemma 3.7. *If $s \in (\frac{1}{2} \cdot \frac{m^2-3m+1}{m^2-3m+2}, \frac{1}{2})$, $b \in (\frac{1}{2}, \frac{2m-1}{2m(m-1)} + \frac{m-2}{m}s]$, and $b' \in [\frac{m^2+2m-1}{4m(m-1)} - \frac{s}{m}, \frac{1}{2})$, then there exists $c > 0$ such that for all $\xi, \tau \in \mathbb{R}$*

$$\iint_A \frac{(1+|\xi|)^{-2(1-s)}\xi^2|\xi_1(\xi-\xi_1)|^{2(1-s)}\,d\xi_1\,d\tau_1}{(1+|\tau-\xi^m|)^{2(1-b)}(1+|\tau_1-\xi_1^m|)^{2b'}(1+|\tau-\tau_1-(\xi-\xi_1)^m|)^{2b'}} \leqslant c,$$

where

$$A(\xi,\tau) = \left\{(\xi_1,\tau_1) \in \mathbb{R}^2:\ |\xi_1| \geqslant 1,\ \left|\tau-\tau_1-(\xi-\xi_1)^m\right| \leqslant \left|\tau_1-\xi_1^m\right| \leqslant \left|\tau-\xi^m\right|\right\}.$$

Proof. By (3.19),

$$\int \frac{d\tau_1}{(1+|\tau_1-\xi_1^m|)^{2b'}(1+|\tau-\tau_1-(\xi-\xi_1)^m|)^{2b'}} \lesssim \frac{1}{(1+|\tau-(\xi-\xi_1)^m-\xi_1^m|)^{4b'-1}}.$$

Also observe that in A

$$\begin{aligned}\left|\tau-(\xi-\xi_1)^m-\xi_1^m\right| &= \left|\tau_1-\xi_1^m+\tau-\tau_1-(\xi-\xi_1)^m\right| \\ &\leqslant \left|\tau_1-\xi_1^m\right| + \left|\tau-\tau_1-(\xi-\xi_1)^m\right| \leqslant 2\left|\tau-\xi^m\right|,\end{aligned}$$

so, by (3.21),

$$\begin{aligned}\left|\xi\xi_1(\xi-\xi_1)\right| &\lesssim \frac{1}{|\xi|^{m-3}}\left|-\xi^m+\xi_1^m+(\xi-\xi_1)^m\right| \\ &= \frac{|\tau-\xi^m-(\tau-(\xi-\xi_1)^m-\xi_1^m)|}{|\xi_1|^{m-3}} \\ &\leqslant \frac{3|\tau-\xi^m|}{|\xi_1|^{m-3}},\end{aligned}$$

which implies (since $|\xi_1| \geqslant 1$)

$$\begin{aligned}\frac{(1+|\xi|)^{-2(1-s)}\xi^2|\xi_1(\xi-\xi_1)|^{2(1-s)}}{(1+|\tau-\xi^m|)^{2(1-b)}} &= \frac{\xi^{2s}|\xi\xi_1(\xi-\xi_1)|^{2(1-s)}}{(1+|\xi|)^{2(1-s)}(1+|\tau-\xi^m|)^{2(1-b)}} \\ &\lesssim \frac{\xi^{2s}|\tau-\xi^m|^{2(1-s)}}{(1+|\xi|)^{2(1-s)}(1+|\tau-\xi^m|)^{2(1-b)}} \\ &\leqslant \frac{\xi^{2s}(1+|\tau-\xi^m|)^{2(b-s)}}{(1+|\xi|)^{2(1-s)}}.\end{aligned}$$

Thus

$$\begin{aligned}&\iint_A \frac{(1+|\xi|)^{-2(1-s)}\xi^2|\xi_1(\xi-\xi_1)|^{2(1-s)}\,d\xi_1\,d\tau_1}{(1+|\tau-\xi^m|)^{2(1-b)}(1+|\tau_1-\xi_1^m|)^{2b'}(1+|\tau-\tau_1-(\xi-\xi_1)^m|)^{2b'}} \\ &\quad\lesssim \frac{\xi^{2s}(1+|\tau-\xi^m|)^{2(b-s)}}{(1+|\xi|)^{2(1-s)}} \int_{\mathbb{R}} \frac{d\xi_1}{(1+|\tau-(\xi-\xi_1)^m-\xi_1^m|)^{4b'-1}}.\end{aligned}$$

Estimating $\int_{\mathbb{R}} \frac{d\xi_1}{(1+|\tau-(\xi-\xi_1)^m-\xi_1^m|)^{4b'-1}}$ as in the proof of Lemma 3.6 yields

$$\begin{aligned}
&\iint\limits_A \frac{(1+|\xi|)^{-2(1-s)}\xi^2|\xi_1(\xi-\xi_1)|^{2(1-s)}\,d\xi_1\,d\tau_1}{(1+|\tau-\xi^m|)^{2(1-b)}(1+|\tau_1-\xi_1^m|)^{2b'}(1+|\tau-\tau_1-(\xi-\xi_1)^m|)^{2b'}} \\
&\quad\lesssim \frac{\xi^{2s}(1+|\tau-\xi^m|)^{2(b-s)}}{(1+|\xi|)^{2(1-s)}}\cdot\frac{1}{|\xi|^{\frac{1}{m-1}}(1+|2^{m-1}\tau-\xi^m|)^{4b'-1-\frac{1}{m-1}}} \\
&\quad= \frac{|\xi|^{2s-\frac{1}{m-1}}(1+|\xi|)^{2(s-1)}}{(1+|\tau-\xi^m|)^{2(s-b)}(1+|2^{m-1}\tau-\xi^m|)^{4b'-\frac{m}{m-1}}} \\
&\quad\leqslant \frac{(1+|\xi|)^{4s-\frac{2m-1}{m-1}}}{(1+|\tau-\xi^m|)^{2(s-b)}(1+|2^{m-1}\tau-\xi^m|)^{4b'-\frac{m}{m-1}}} \\
&\quad\simeq \frac{(1+|\xi^m|)^{\frac{4s}{m}-\frac{2m-1}{m(m-1)}}}{(1+|\tau-\xi^m|)^{2(s-b)}(1+|2^{m-1}\tau-\xi^m|)^{4b'-\frac{m}{m-1}}} \\
&\quad\leqslant c,
\end{aligned}$$

by the assumptions on s, b, and b'. □

Lemma 3.8. *If $s\in(\frac12\cdot\frac{m^2-3m+1}{m^2-3m+2},\frac12)$, $b\in(\frac12,\frac12+\beta]$, $b'\in[\frac12-\beta,\frac12)$ and $b+b'\leqslant 1$, where $\beta=\frac13\min\{\frac12-s,\frac{m-2}{m}s-\frac{1}{2m(m-1)}\}$, then there exists $c>0$ such that for all $\xi_1,\tau_1\in\mathbb{R}$ with $|\xi_1|\geqslant 1$*

$$\iint\limits_B \frac{(1+|\xi|)^{-2(1-s)}\xi^2|\xi_1(\xi-\xi_1)|^{2(1-s)}\,d\xi\,d\tau}{(1+|\tau_1-\xi_1^m|)^{2b'}(1+|\tau-\xi^m|)^{2(1-b)}(1+|\tau-\tau_1-(\xi-\xi_1)^m|)^{2b'}}\leqslant c,$$

where

$$B(\xi_1,\tau_1)=\bigl\{(\xi,\tau)\in\mathbb{R}^2\colon\ |\xi-\xi_1|\geqslant 1,\ \bigl|\tau-\tau_1-(\xi-\xi_1)^m\bigr|\leqslant\bigl|\tau_1-\xi_1^m\bigr|,\ \bigl|\tau-\xi^m\bigr|\leqslant\bigl|\tau_1-\xi_1^m\bigr|\bigr\}.$$

Proof. By the assumption $b+b'\leqslant 1$ and (3.19),

$$\begin{aligned}
&\int\frac{d\tau}{(1+|\tau-\xi^m|)^{2(1-b)}(1+|\tau-\tau_1-(\xi-\xi_1)^m|)^{2b'}} \\
&\quad\leqslant\int\frac{d\tau}{(1+|\tau-\xi^m|)^{2b'}(1+|\tau-\tau_1-(\xi-\xi_1)^m|)^{2b'}} \\
&\quad\lesssim\frac{1}{(1+|\tau_1+(\xi-\xi_1)^m-\xi^m|)^{1-4\beta}}.
\end{aligned}$$

Observe that in B

$$\begin{aligned}
\bigl|\tau_1+(\xi-\xi_1)^m-\xi^m\bigr|&=\bigl|\tau-\xi^m-\bigl(\tau-\tau_1-(\xi-\xi_1)^m\bigr)\bigr| \\
&\leqslant\bigl|\tau-\xi^m\bigr|+\bigl|\tau-\tau_1-(\xi-\xi_1)^m\bigr|\leqslant 2\bigl|\tau_1-\xi_1^m\bigr|.
\end{aligned}$$

On the other hand

$$\bigl|\tau_1+(\xi-\xi_1)^m-\xi^m\bigr|=\bigl|\tau_1-\xi_1^m+d_m(\xi,\xi_1)\bigr|,$$

where $d_m(\xi,\xi_1)=-\xi^m+\xi_1^m+(\xi-\xi_1)^m$.

Thus, in B,

$$\left|d_m(\xi, \xi_1)\right| \leqslant 3\left|\tau_1 - \xi_1^m\right|, \tag{3.23}$$

so, by (3.20),

$$\left|\xi \xi_1(\xi - \xi_1)\right| \lesssim \frac{|\tau_1 - \xi_1^m|}{|\xi_1|^{m-3}}, \tag{3.24}$$

which implies

$$\begin{aligned} \frac{\xi^2 |\xi_1(\xi - \xi_1)|^{2(1-s)}}{(1+|\xi|)^{2(1-s)}} &= \frac{\xi^2 |\xi \xi_1(\xi - \xi_1)|^{2(1-s)}}{|\xi|^{2(1-s)}(1+|\xi|)^{2(1-s)}} \\ &\lesssim \frac{1}{(1+|\xi|)^{2-4s}} \left(\frac{|\tau_1 - \xi_1^m|}{|\xi_1|^{m-3}}\right)^{2(1-s)}. \end{aligned}$$

Therefore

$$\begin{aligned} &\iint\limits_B \frac{(1+|\xi|)^{-2(1-s)} \xi^2 |\xi_1(\xi - \xi_1)|^{2(1-s)}\, d\xi\, d\tau}{(1+|\tau_1 - \xi_1^m|)^{2b'}(1+|\tau - \xi^m|)^{2b'}(1+|\tau - \tau_1 - (\xi - \xi_1)^m|)^{2b'}} \\ &\quad \lesssim \frac{(1+|\tau_1 - \xi_1^m|)^{2(1-s)-2b'}}{|\xi_1|^{2(1-s)(m-3)}} \int\limits_{B'} \frac{d\xi}{(1+|\xi|)^{2-4s}(1+|\tau_1 + (\xi - \xi_1)^m - \xi^m|)^{1-4\beta}} \\ &\quad \doteq I(B'), \end{aligned}$$

where

$$B' = B'(\xi_1, \tau_1) = \left\{\xi \in \mathbb{R}\colon\ |\xi - \xi_1| \geqslant 1,\ \left|d_m(\xi, \xi_1)\right| \leqslant 3\left|\tau_1 - \xi_1^m\right|\right\}.$$

To proceed, we split B' into subsets B_1 and B_2, where

$$B_1 \doteq \left\{\xi \in B'\colon\ \left|d_m(\xi, \xi_1)\right| \leqslant \frac{1}{2}\left|\tau_1 - \xi_1^m\right|\right\}$$

and

$$\begin{aligned} B_2 &\doteq \left\{\xi \in B'\colon\ \frac{1}{2}\left|\tau_1 - \xi_1^m\right| \leqslant \left|d_m(\xi, \xi_1)\right|\right\} \\ &= \left\{\xi \in B'\colon\ \frac{1}{2}\left|\tau_1 - \xi_1^m\right| \leqslant \left|d_m(\xi, \xi_1)\right| \leqslant 3\left|\tau_1 - \xi_1^m\right|\right\}. \end{aligned}$$

In B_1 we have

$$\begin{aligned} \left|\tau_1 + (\xi - \xi_1)^m - \xi^m\right| &= \left|\tau_1 - \xi_1^m + d_m(\xi, \xi_1)\right| \\ &\geqslant \left|\tau_1 - \xi_1^m\right| - \left|d_m(\xi, \xi_1)\right| \geqslant \frac{1}{2}\left|\tau_1 - \xi_1^m\right| \end{aligned}$$

and, by (3.24), since $|\xi_1| \geqslant 1$ and $|\xi - \xi_1| \geqslant 1$,

$$|\xi| \leqslant |\xi||\xi_1||\xi - \xi_1| \lesssim \frac{1}{|\xi_1|^{m-3}}\left|\tau_1 - \xi_1^m\right|.$$

Hence

$$\begin{aligned}
I(B_1) &= \frac{(1+|\tau_1 - \xi_1^m|)^{2(1-s)-2b'}}{|\xi_1|^{2(1-s)(m-3)}} \int_{B_1} \frac{d\xi}{(1+|\xi|)^{2-4s}(1+|\tau_1 + (\xi-\xi_1)^m - \xi^m|)^{1-4\beta}} \\
&\lesssim \frac{(1+|\tau_1 - \xi_1^m|)^{2(1-s)-2b'+4\beta-1}}{|\xi_1|^{2(1-s)(m-3)}} \int_{|\xi| \lesssim \frac{|\tau_1 - \xi_1^m|}{|\xi_1|^{m-3}}} \frac{d\xi}{(1+|\xi|)^{2-4s}} \\
&\leqslant \left(1+\left|\tau_1 - \xi_1^m\right|\right)^{2(1-s)-2b'+4\beta-1} \int_{|\xi| \lesssim |\tau_1 - \xi_1^m|} \frac{d\xi}{(1+|\xi|)^{2-4s}} \\
&\lesssim \left(1+\left|\tau_1 - \xi_1^m\right|\right)^{2(1-s)-2b'+4\beta-1}\left(1+\left|\tau_1 - \xi_1^m\right|\right)^{4s-1} \\
&= \left(1+\left|\tau_1 - \xi_1^m\right|\right)^{2s-2b'+4\beta} \\
&\leqslant \left(1+\left|\tau_1 - \xi_1^m\right|\right)^{2s-1+6\beta} \\
&\leqslant 1
\end{aligned}$$

with the last inequality holding by the assumption on β.

To bound $I(B_2)$, we will use the following:

$$\frac{1+|\tau_1 - \xi_1^m|}{1+|2^{m-1}\tau_1 - \xi_1^m|} \lesssim 1 + \min\left\{\left|\xi^m\right|, \left|\xi_1^m\right|\right\}. \tag{3.25}$$

To establish (3.25), let

$$C \doteq \max\{1, C_m\},$$

where C_m is the constant in Lemma 3.5.

If $|\xi_1| \leqslant 2^{m-1}C|\xi|$, then $|\xi_1^m| \lesssim \min\{|\xi^m|, |\xi_1^m|\}$. So for (3.25) to hold in this case, it suffices for the left-hand side of (3.25) to be bounded by $1 + |\xi_1^m|$, which it is since

$$\begin{aligned}
1 + \left|\tau_1 - \xi_1^m\right| &\leqslant 1 + \left|2^{m-1}\tau_1 - \xi_1^m\right| + \left(2^{m-1} - 1\right)\left|\xi_1^m\right| \\
&\lesssim 1 + \left|2^{m-1}\tau_1 - \xi_1^m\right| + \left|\xi_1^m\right| \\
&\leqslant \left(1 + \left|2^{m-1}\tau_1 - \xi_1^m\right|\right)\left(1 + \left|\xi_1^m\right|\right).
\end{aligned}$$

On the other hand, if $|\xi_1| > 2^{m-1}C|\xi|$, then

$$|\xi_1| > |\xi| \quad \text{and} \quad |\xi_1| > 2^{m-1}C_m|\xi|.$$

Then, since $\frac{1}{2}|\tau_1 - \xi_1^m| \leqslant |d_m(\xi, \xi_1)|$ in B_2 and by (3.22),

$$\begin{aligned}
2^{m-2}\left|\tau_1-\xi_1^m\right| &\leqslant 2^{m-1}\left|d_m(\xi,\xi_1)\right| \\
&\leqslant 2^{m-1}C_m|\xi_1|^{m-3}\left|\xi\xi_1(\xi-\xi_1)\right| \\
&\leqslant \left(1+\frac{1}{2^{m-1}C}\right)|\xi_1|^m.
\end{aligned}$$

Thus

$$2^{m-1}\left|\tau_1-\xi_1^m\right| \leqslant \left(2+\frac{1}{2^{m-2}C}\right)|\xi_1|^m \leqslant \frac{2^{m-1}+1}{2^{m-2}}|\xi_1|^m.$$

So

$$\begin{aligned}
\left|2^{m-1}\tau_1-\xi_1^m\right| &\geqslant \left(2^{m-1}-1\right)\left|\xi_1^m\right| - 2^{m-1}\left|\tau_1-\xi_1^m\right| \\
&\geqslant \left(2^{m-1}-1\right)\frac{2^{m-1}\cdot 2^{m-2}}{2^{m-1}+1}\left|\tau_1-\xi_1^m\right| - 2^{m-1}\left|\tau_1-\xi_1^m\right| \\
&\geqslant \frac{4}{5}\left|\tau_1-\xi_1^m\right|
\end{aligned}$$

since $m \geqslant 3$. Thus

$$\frac{1+|\tau_1-\xi_1^m|}{1+|2^{m-1}\tau_1-\xi_1^m|} \lesssim 1,$$

which completes the proof of (3.25).

By (3.25)

$$\frac{1}{(1+|\xi|)^{2-4s}} \lesssim \left[\frac{1+|2^{m-1}\tau_1-\xi_1^m|}{1+|\tau_1-\xi_1^m|}\right]^{(2-4s)/m} \leqslant \frac{(1+|2^{m-1}\tau_1-\xi_1^m|)^{1/m}}{(1+|\tau_1-\xi_1^m|)^{(2-4s)/m}},$$

hence

$$\begin{aligned}
I(B_2) &= \frac{(1+|\tau_1-\xi_1^m|)^{2(1-s)-2b'}}{|\xi_1|^{2(1-s)(m-3)}}\int_{B_2}\frac{d\xi}{(1+|\xi|)^{2-4s}(1+|\tau_1+(\xi-\xi_1)^m-\xi^m|)^{1-4\beta}} \\
&\lesssim \frac{(1+|\tau_1-\xi_1^m|)^{2(1-s)-2b'}}{|\xi_1|^{2(1-s)(m-3)}}\cdot\frac{(1+|2^{m-1}\tau_1-\xi_1^m|)^{1/m}}{(1+|\tau_1-\xi_1^m|)^{(2-4s)/m}}\int_{B_2}\frac{d\xi}{(1+|\tau_1+(\xi-\xi_1)^m-\xi^m|)^{1-4\beta}} \\
&\leqslant \left(1+\left|\tau_1-\xi_1^m\right|\right)^{2(1-s)-2b'-(2-4s)/m}\left(1+\left|2^{m-1}\tau_1-\xi_1^m\right|\right)^{1/m}\int_{B_2}\frac{d\xi}{(1+|\tau_1+(\xi-\xi_1)^m-\xi^m|)^{1-4\beta}}.
\end{aligned}$$

To estimate $\int_{B_2}\frac{d\xi}{(1+|\tau_1+(\xi-\xi_1)^m-\xi^m|)^{1-4\beta}}$, notice that, as in the proof of Lemma 3.6,

$$\left|\tau_1+(\xi-\xi_1)^m-\xi^m\right| = \frac{1}{2^{m-1}}\left|2^{m-1}\tau_1-\xi_1^m-\left(2^{m-1}-1\right)\xi_1(2\xi-\xi_1)^{m-1}\right|,$$

then make the substitution

$$u=\left[\left(2^{m-1}-1\right)\xi_1\right]^{\frac{1}{m-1}}(2\xi-\xi_1), \qquad du = 2\left[\left(2^{m-1}-1\right)\xi_1\right]^{\frac{1}{m-1}}d\xi$$

and apply (3.18) to yield

$$\begin{aligned}
&\int\limits_{B_2} \frac{d\xi}{(1+|\tau_1+(\xi-\xi_1)^m-\xi^m|)^{1-4\beta}} \\
&\lesssim \int\limits_{\mathbb{R}} \frac{d\xi}{(1+|2^{m-1}\tau_1-\xi_1^m-(2^{m-1}-1)\xi_1(2\xi-\xi_1)^{m-1}|)^{1-4\beta}} \\
&\simeq \frac{1}{|\xi_1|^{\frac{1}{m-1}}} \int\limits_{\mathbb{R}} \frac{du}{(1+|2^{m-1}\tau_1-\xi_1^m-u^{m-1}|)^{1-4\beta}} \\
&\lesssim \frac{1}{|\xi_1|^{\frac{1}{m-1}}(1+|2^{m-1}\tau_1-\xi_1^m|)^{\frac{m-2}{m-1}-4\beta}} \\
&\lesssim \frac{1}{(1+|\xi_1^m|)^{\frac{1}{m(m-1)}}(1+|2^{m-1}\tau_1-\xi_1^m|)^{\frac{m-2}{m-1}-4\beta}}.
\end{aligned}$$

Combining this with the above estimate gives

$$\begin{aligned}
I(B_2) &\lesssim \frac{(1+|\tau_1-\xi_1^m|)^{2(1-s)-2b'-(2-4s)/m}}{(1+|\xi_1^m|)^{\frac{1}{m(m-1)}}(1+|2^{m-1}\tau_1-\xi_1^m|)^{\frac{m^2-3m+1}{m(m-1)}-4\beta}} \\
&\leqslant \frac{(1+|\tau_1-\xi_1^m|)^{2\beta+(1-\frac{2}{m})(1-2s)}}{(1+|\xi_1^m|)^{\frac{1}{m(m-1)}}(1+|2^{m-1}\tau_1-\xi_1^m|)^{\frac{m^2-3m+1}{m(m-1)}-4\beta}} \\
&\leqslant c,
\end{aligned}$$

by the assumptions on s and β. $\square$

Acknowledgment

The author would like to thank Alex Himonas for helpful suggestions, including the suggestion of the problem considered in this work.

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