J. Differential Equations 247 (2009) 2485–2494

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Solving the hypergeometric system of Okubo type in terms of a certain generalized hypergeometric function

Werner Balser, Claudia Röscheisen *

Institut für Angewandte Analysis, Universität Ulm, 89069 Ulm, Germany

ARTICLE INFO

Article history:
Received 18 October 2007
Revised 24 July 2009
Available online 27 August 2009

MSC:
34A25
34A30
33C70
33E30

Keywords:
Ordinary differential equations
Power series solutions
Generalized hypergeometric functions

ABSTRACT

We introduce one scalar function f of a complex variable and finitely many parameters, which allows to represent all solutions of the so-called *hypergeometric system of Okubo type* under the assumption that one of the two coefficient matrices has all distinct eigenvalues. In the simplest non-trivial situation, f is equal to the hypergeometric function, while in other more complicated cases it is related, but not equal, to the generalized hypergeometric functions. In general, however, this function appears to be a new higher transcendental one. The coefficients of the power series of f about the origin can be explicitly given in terms of a generalized version of the classical Pochhammer symbol, involving two square matrices that in general do not commute. The function can also be characterized by a Volterra integral equation, whose kernel is expressed in terms of the solutions of another hypergeometric system of lower dimension.

© 2009 Elsevier Inc. All rights reserved.

0. Introduction

Following K. Okubo [12] et al., the linear system of ordinary differential equations of the form

$$(A_0 - t)y' = A_1 y, \quad A_0, A_1 \in \mathbb{C}^{n\times n}, \tag{0.1}$$

is usually referred to as the *hypergeometric system of Okubo type*. The reason for this name is that in the smallest non-trivial situation of $n = 2$, and assuming that A_0 is diagonalizable, one can compute a fundamental solution for this system in terms of hypergeometric functions; for this, one may re-

* Corresponding author.
E-mail addresses: werner.balser@uni-ulm.de (W. Balser), claudia.roescheisen@alumni.uni-ulm.de (C. Röscheisen).

0022-0396/$ – see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.jde.2009.07.036

fer to [2, Section 2.3]. In another case of arbitrary dimension, but with highly restricted coefficient matrices A_0, A_1, one may compute $n-1$ linearly independent solutions in terms of generalized hypergeometric functions [1]. More work in this direction has been done in papers by T. Yokoyama [19–21], Y. Haraoka [5], Haraoka and Yokoyama [6]. Even earlier, T. Sasai [15] showed in dimension $n=4$ that certain cases occur which have solutions related to Appel's function F_3. In general, however, the solutions of this system are believed to be "new" higher transcendental functions. Their global behaviour as well as their relation via Laplace transform to the *confluent hypergeometric system* (see below) have also been investigated by other authors, among them Balser, Jurkat, and Lutz [3,4], M. Kohno [7–10], Kohno and Yokoyama [11], Okubo and Takano [13], Okubo, Takano, and Yoshida [14], and R. Schäfke [16–18].

While a system of the form (0.1) certainly is interesting enough in its own right, it also is important because of its intimate relation, via Laplace transform, with the system

$$zx' = (zA_0 + A_1)x. \tag{0.2}$$

It has been shown by R. Schäfke [18] that the two systems also are related by means of a confluence process, and therefore we refer to (0.2) as the *confluent hypergeometric system*. This system has a singular point of Poincaré rank $r=1$ at infinity, and thus formal solutions will, in general, diverge, but are multisummable in the sense of J. Ecalle.

In this article we investigate the systems (0.1) resp. (0.2) under the sole assumption that the matrix A_0 has an eigenvalue λ whose algebraic multiplicity equals 1. For simplicity of notation we shall in fact assume that $\lambda = 0$ (if not, use a transformation $x = \mathrm{e}^{\lambda z}\tilde{x}$ to make this hold) and

$$A_0 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \Lambda \end{array}\right],$$

with an invertible matrix Λ of dimension $m := n-1$ (for this to hold, we may use another transformation of the form $x = T\tilde{x}$, with a constant invertible matrix T, which may even put Λ into Jordan canonical form, but the latter shall not be used here). A third type of transformation $x = z^{\alpha}\tilde{x}$, with $\alpha \in \mathbb{C}$, can be used to arrange that the element of A_1 in the upper left corner vanishes. So without loss of generality we may restrict ourselves to systems (0.1) resp. (0.2) with A_0 as above, and

$$A_1 = \left[\begin{array}{c|c} 0 & a^{\tau} \\ \hline b & A \end{array}\right],$$

with vectors $a, b \in \mathbb{C}^m$ and a square matrix $A \in \mathbb{C}^{m\times m}$ (and a^{τ} denoting the transpose of a). Under these *normalizing assumptions*, the system (0.2) has a unique formal vector solution $\hat{x}(z)$ consisting of formal power series in z^{-1}, with $\hat{x}(\infty) = e_1$, the first unit vector of the canonical basis in $\mathbb{C}^n$. The series in $\hat{x}(z)$ are of Gevrey order (at most) 1, hence the formal Borel transform of $\hat{x}(z)$, obtained by division of the k-th coefficient of these series by $k!$ and replacing the variable z^{-1} by t, then leads to a solution $y(t)$ of (0.1) which is holomorphic at the origin, although $t=0$ is a (regular) singularity of the system. It is this vector solution – or in fact, its first component – with which this paper is concerned.

In detail, this article is organized as follows: In the first section we define a function of two variables, which later on serves as the kernel in a Volterra integral equation and therefore shall be called the *kernel function*. Next, we introduce one (scalar) function f, defined by its power series representation about the origin, or alternatively, as the unique solution of this integral equation. In case that $n=2$, i.e., $m=1$, this f is nothing but the classical hypergeometric function, and for lack of a better word, we name f a certain *generalized hypergeometric function*, being fully aware that this term is used for other functions as well. In the final section we shall employ f to construct a solution vector of (0.1), which will turn out to be the formal Borel transform of the vector $\hat{x}(z)$ mentioned above. If the matrix A_0 happens to be diagonalizable, then there even is a fundamental solution of (0.1), whose columns, aside from the normalizing transformations described above, all

can be expressed in terms of the one scalar function f. So we claim that this function should be considered as an important one and deserves to be investigated further in order to understand more of its properties!

1. Definition of the kernel function

Given $A, \Lambda \in \mathbb{C}^{m\times m}$, with $\det \Lambda \neq 0$, and writing $s + A$ instead of $sI + A$ for a scalar $s \in \mathbb{C}$, we define

$$(A, \Lambda)_k = \Lambda^{-1}(k-1+A)\cdot\ldots\cdot\Lambda^{-1}A, \quad k \in \mathbb{N}. \tag{1.1}$$

In addition we set $(A, \Lambda)_0 = I$, and then we have

$$\Lambda(A, \Lambda)_{k+1} = (k + A)(A, \Lambda)_k \quad \forall k \geqslant 0.$$

This is nothing but a simple system of first order difference equations, and if $\det(k + A) \neq 0$ for all $k \in \mathbb{N}_0$, then $(A, \Lambda)_k$ is a fundamental solution. In case that Λ and A commute (which is certainly the case when $m = 1$), then

$$(A, \Lambda)_k = \Lambda^{-k}(A)_k, \quad k \in \mathbb{N}_0,$$

where $(A)_k = (k-1+A)\cdot\ldots\cdot A$ is the matrix version of the classical *Pochhammer symbol*. Hence even in general one may regard $(A, \Lambda)_k$ as a Pochhammer symbol for two non-commuting matrices.

Lemma 1. *The power series*

$$Y(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}(A, \Lambda)_k$$

converges for $|t| < r_\lambda := \min\{|\lambda_j|\colon 1 \leqslant j \leqslant m\}$, *with* $\lambda_1, \ldots, \lambda_m$ *denoting the not necessarily distinct eigenvalues of the matrix* Λ. *For these* t *we have*

$$(\Lambda - t)Y'(t) = AY(t). \tag{1.2}$$

The matrix $Y(t)$ *can be continued into the largest region* G *which is starshaped with respect to the origin and does not contain the values* $\lambda_1, \ldots, \lambda_m$. *Moreover, its determinant does not vanish for* $t \in G$.

Proof. Using the submultiplicative matrix norm $\|A\| = \max_j \sum_k |a_{jk}|$, we can estimate

$$\left\|(A, \Lambda)_k\right\| \leqslant \left\|\Lambda^{-1}\right\|^k \cdot \|k-1+A\|\cdot\ldots\cdot\|A\| \leqslant \left\|\Lambda^{-1}\right\|^k \cdot \left(\|A\|\right)_k.$$

Owing to the quotient criterion, convergence of the power series follows for $|t|\|\Lambda^{-1}\| < 1$. Furthermore, the validity of the differential equation can be checked directly. Since $Y(0) = I$, we see that $Y(t)$ is a fundamental solution of the differential equation, which has no other singularities aside from $\lambda_1, \ldots, \lambda_m$ and infinity. Accordingly, $Y(t)$ can be continued into the region G described above, and its determinant does not vanish there. Moreover, the power series expansion for $Y(t)$ converges up to the nearest singularity of (1.2), which completes the proof. □

Note that (1.2) is nothing but the hypergeometric system for the smaller dimension of $m = n - 1$, and $Y(t)$ is a fundamental solution. In case of $m = 1$, the power series expansion for $Y(t)$ is the binomial series, and

$$Y(t) = \left(I - t\Lambda^{-1}\right)^{-A} = \mathrm{e}^{-A\log(I - t\Lambda^{-1})}.$$

The same holds true for general m and commuting matrices A and Λ.

While one may be satisfied with computing *any* fundamental solution of the hypergeometric system, we want to have one, whenever possible, which is expressed in terms of *one scalar function* whose power series expansion can be explicitly given, and/or which can be characterized by a differential, or integral equation. To do so, we consider two vectors $a, b \in \mathbb{C}^m$ and define

$$k(t,u) = a^\tau \int_u^t Y(v)\,\frac{dv}{v} Y^{-1}(u)(u-\Lambda)^{-1} b, \tag{1.3}$$

for values $t, u \in G \setminus \{0\}$, and integration so that the origin is avoided. This *kernel function* is, in general, a multi-valued function, and singular when either one of the variables tends to the origin, as one can see from the identities derived in the next lemma:

Lemma 2. *For $0 < |t|, |u| < r_\lambda$ and $|\arg(t/u)| < \pi$ we have*

$$\int_u^t Y(v)\,\frac{dv}{v} = \log(t/u)(A,\Lambda)_0 + \sum_{k=1}^{\infty} \frac{t^k - u^k}{k!k}(A,\Lambda)_k,$$

when integration is performed along a straight line segment, and the branch of the logarithm is taken accordingly. Moreover, for $|u| < r_\lambda$

$$Y^{-1}(u)(u-\Lambda)^{-1} = -\Lambda^{-1} \sum_{k=0}^{\infty} \frac{u^k}{k!}(A,\Lambda)^{(k)},$$

with $(A,\Lambda)^{(0)} = -I$ and

$$\begin{aligned}(A,\Lambda)^{(k)} &= (-1)^{k-1}(A-1)\Lambda^{-1}\cdot\ldots\cdot(A-k)\Lambda^{-1}\\ &= -(1-A)\Lambda^{-1}\cdot\ldots\cdot(k-A)\Lambda^{-1} \quad (k \geqslant 1).\end{aligned}$$

Proof. The first identity follows by termwise integration, which is allowed, since $Y(v)$ is analytic on the path of integration (see Lemma 1). For the second one, use that $Y(u)$ is a solution of (1.2) to show that $B(u) := Y^{-1}(u)(u-\Lambda)^{-1}$ satisfies

$$B'(u)(u-\Lambda) = B(u)(A-I), \quad u \in G.$$

Using the power series ansatz $B(u) = \sum_{k=0}^{\infty}(u^k/k!)B_k$ and comparing coefficients leads to

$$B_{k+1} = -B_k(-k-1+A)\Lambda^{-1} \quad (k \geqslant 0).$$

From $Y(0) = I$ we conclude that $B_0 = -\Lambda^{-1}$. So by induction we get $B_k = \Lambda^{-1}(A,\Lambda)^{(k)}$ for all $k \geqslant 0$. □

Remark 3. Note that the two Pochhammer-like symbols $(A,\Lambda)_k$ and $(A,\Lambda)^{(k)}$ differ by A being replaced by $1-A$ and interchanging the ordering of the terms in the matrix product! Moreover, observe that a multiplication of the power series for $Y(t)$ and $Y^{-1}(t)(u-\Lambda)^{-1}$ implies

$$k!\Lambda^{-k-1} = \sum_{j=0}^{k} \binom{k}{j} (A,\Lambda)_{k-j}\Lambda^{-1}(A,\Lambda)^{(j)} \quad \forall k \geqslant 0.$$

2. Definition of a new generalized hypergeometric function

For our main result, we define a third kind of matrix products by $(A, \Lambda, a, b)_0 = I$ and for $k \geqslant 1$

$$(A, \Lambda, a, b)_k = \prod_{j=0}^{k-1} \big[\Lambda^{-1}\big(j(j+A) - ba^\tau\big)\big], \tag{2.1}$$

where the, generally non-commuting, matrices in the product are ordered with respect to decreasing index j. In particular, we have that

$$(A, \Lambda, a, b)_1 = -\Lambda^{-1} ba^\tau, \qquad (A, \Lambda, a, b)_2 = -\Lambda^{-1}\big(1 + A - ba^\tau\big)\Lambda^{-1} ba^\tau,$$

and so forth. Note that unlike $(A, \Lambda)_k$ and $(A, \Lambda)^{(k)}$, the term in the product here is a *quadratic function of* j, and the matrices involved, in general, do not commute. If they do, however, then one can express $(A, \Lambda, a, b)_k$ in terms of the Pochhammer-like products. For this, compare the discussion of the case $m = 1$ presented below.

Taking $\gamma \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, assuming for the moment that $a \neq 0$, and writing $\bar{a}$ for the complex conjugate vector, we define

$$f(\gamma; t) = \sum_{k=0}^{\infty} \frac{f_k t^k}{k!(\gamma)_k}, \quad f_k = \|a\|^{-2} a^\tau (A, \Lambda, a, b)_k \bar{a}, \tag{2.2}$$

leaving convergence of this series to be shown in our main theorem. Observe that $f_0 = 1$, while all remaining coefficients tend to zero as $a \to 0$; hence we interpret $f(\gamma; t) \equiv 1$ for $a = 0$.

Aside from the scalar parameter γ, the function f also depends upon the matrices Λ and A and the two vectors a and b, and therefore we shall occasionally also write $f(\Lambda, A, a^\tau, b; \gamma; t)$ to display this dependence. Note, however, that differentiation of f is always meant to be with respect to the variable t.

In the special case of $m = 1$, the parameters Λ, A, a, and b are complex numbers, and with $\alpha = (A + \sqrt{A^2 + 4ba})/2$ and $\beta = (A - \sqrt{A^2 + 4ba})/2$, for any determination of the square root, one can check that

$$f\big(\Lambda, A, a^\tau, b; \gamma; t\big) = F(\alpha, \beta; \gamma; t/\Lambda),$$

with F denoting the classical hypergeometric function. More generally, if one can show existence of two matrices A_1, A_2 which both commute with Λ and satisfy

$$A_1 + A_2 = A, \qquad A_1 A_2 = A_2 A_1 = -ba^\tau,$$

then $j(j + A) - ba^\tau = (j + A_1)(j + A_2)$, which implies that

$$f\big(\Lambda, A, a^\tau, b; \gamma; t\big) = \|a\|^{-2} a^\tau F\big(A_1, A_2; \gamma; t\Lambda^{-1}\big)\bar{a},$$

with a *matrix version of the hypergeometric function* on the right-hand side, defined by its power series expansion

$$F\big(A_1, A_2; \gamma; t\Lambda^{-1}\big) = \sum_{k=0}^{\infty} \frac{t^k}{k!(\gamma)_k} \Lambda^{-k} (A_1)_k (A_2)_k.$$

Regarding the dependence of f upon the parameter γ, note that owing to the Beta integral one can show by termwise integration of the power series that

$$f(\gamma;t) = t^{1-\gamma}(\gamma-1)\int_0^t (t-u)^{\gamma-2} f(1;u)\,du \quad (\operatorname{Re}\gamma > 1) \tag{2.3}$$

while for every γ as above one sees by means of termwise differentiation

$$\gamma t^{\gamma-1} f(\gamma;t) = \frac{d}{dt} t^{\gamma} f(\gamma+1;t). \tag{2.4}$$

According to these two formulas it is sufficient to analyze both convergence of the power series (2.2) as well as analytic properties of $f(\gamma;t)$ for the special case of $\gamma = 1$, in which case we are going to simply write $f(t)$.

Theorem 4. *There exists a unique function f which is holomorphic in the region G and satisfies the integral equation*

$$f(t) = 1 + \int_0^t k(t,u) f(u)\,du. \tag{2.5}$$

The power series expansion about the origin of this function f is given by (2.2) *for $\gamma = 1$. Moreover, for arbitrary γ as above, the series in* (2.2) *converges for $|t| < r_\lambda$, and the function $f(\gamma;t)$ is holomorphic for $t \in G$.*

Proof. The proof of existence and analyticity of f is the standard one for Volterra integral equations, although in our case the kernel has a weak singularity at the origin. We include the arguments here for the sake of completeness: Inserting (1.3) into (2.5) and interchanging the order of integration, which is justified for every function f that is holomorphic in G, we observe that Eq. (2.5) can be rewritten in the form

$$f(t) = 1 + a^\tau \int_0^t Y(v) \int_0^v Y^{-1}(u)(u-\Lambda)^{-1} b f(u)\,du\,\frac{dv}{v}.$$

To show the existence of a solution, we define a sequence of functions by $f_0(t) \equiv 1$ and

$$f_k(t) = 1 + a^\tau \int_0^t Y(v) \int_0^v Y^{-1}(u)(u-\Lambda)^{-1} b f_{k-1}(u)\,du\,\frac{dv}{v} \tag{2.6}$$

for every $k \geqslant 1$. All f_k are holomorphic in G, and the sequence is compactly convergent, as we shall show now: For $\varepsilon, R > 0$, let $G(\varepsilon,R) \subset G$ be the largest closed subset of $|t| \leqslant R$ which is starshaped with respect to the origin and does not contain the ε-neighbourhoods of the values $\lambda_1,\dots,\lambda_m$. For

$$C = C_{\varepsilon,R} := \|a\|\|b\| \max_{v\in G(\varepsilon,R)} \big\|Y(v)\big\| \max_{u\in G(\varepsilon,R)} \big\|Y^{-1}(u)(u-\Lambda)^{-1}\big\|,$$

one shows by induction that

$$\big|f_k(t) - f_{k-1}(t)\big| \leqslant C^k \frac{|t|^k}{k!} \quad \forall k \geqslant 1,\ t \in G(\varepsilon,R).$$

Hence the sequence $f_n(t) = 1 + \sum_{k=1}^{n}(f_k(t) - f_{k-1}(t))$ converges uniformly on $G(\varepsilon, R)$, and therefore its limit $f(t)$ is a holomorphic function on the region G. Furthermore, in (2.6) limit and integral can be interchanged to obtain (2.5). To show uniqueness and find the power series expansion of f, we differentiate the integral equation to see that

$$tf'(t) = a^\tau Y(t) \int_0^t Y^{-1}(u)(u - \Lambda)^{-1} b f(u)\, du, \quad t \in G. \tag{2.7}$$

Abbreviating

$$g(t) = Y(t) \int_0^t Y^{-1}(u)(\Lambda - u)^{-1} b f(u)\, du, \tag{2.8}$$

we see that g also is holomorphic in G, and $g(0) = 0$. Moreover, $tf'(t) = -a^\tau g(t)$, whereas differentiation of g implies $(\Lambda - t)g'(t) = Ag(t) + bf(t)$. Expanding

$$f(t) = \sum_{k=0}^{\infty} \frac{t^k}{(k!)^2} f_k, \qquad g(t) = \sum_{k=1}^{\infty} \frac{t^k}{k!(k-1)!} g_k, \quad |t| < r_\lambda,$$

with f_k, g_k to be determined, we can insert into these identities and compare coefficients to find that $g_1 = \Lambda^{-1} b$, while for $k \geqslant 1$ we have

$$f_k = -a^\tau g_k, \qquad \Lambda g_{k+1} = f_k b + k(k + A)g_k = \big(k(k + A) - ba^\tau\big)g_k.$$

This implies by induction

$$g_k = -(A, \Lambda, a, b)_k \bar{a} \|a\|^{-2} \quad \forall k \geqslant 1,$$

which shows that the f_k are as in (2.2). To complete the proof, note that holomorphy of $f(\gamma; t)$ in G follows from (2.3) for all γ with $\operatorname{Re}\gamma > 1$, and then (2.4) shows that the same holds for all $\gamma \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. □

Remark 5. Since the coefficients f_k and g_k (in the proof above) are uniquely determined by $\Lambda g_{k+1} = f_k b + k(k + A)g_k$, $f_{k+1} = -a^\tau g_{k+1}$, $k \geqslant 0$ and $f_0 = 1$, it can be checked directly that the following formulas are true for $k \geqslant 1$:

$$g_k = \sum_{j=0}^{k-1} \frac{(k-1)!}{j!} \Lambda^{-1}(k - 1 + A) \cdot \ldots \cdot \Lambda^{-1}(j + 1 + A)\Lambda^{-1} b f_j,$$

$$f_k = -a^\tau \sum_{j=0}^{k-1} \frac{(k-1)!}{j!} \Lambda^{-1}(k - 1 + A) \cdot \ldots \cdot \Lambda^{-1}(j + 1 + A)\Lambda^{-1} b f_j.$$

In particular, one can recursively compute the coefficients f_k from the second identity.

It follows from either (2.5) or (2.2) that f does not change if we replace the parameters Λ, A, a^τ, and b by $T^{-1}\Lambda T$, $T^{-1}AT$, $a^\tau T$, and $T^{-1}b$, with an arbitrary invertible matrix T. Formally, this can be expressed as the *transformation law*

$$f\big(T^{-1}\Lambda T, T^{-1}AT, a^\tau T, T^{-1}b; \gamma; t\big) = f\big(\Lambda, A, a^\tau, b; \gamma; t\big). \tag{2.9}$$

Moreover, we conclude from the power series representation a second transformation law:

$$f\big(\Lambda, A, a^\tau, b; \gamma; \lambda t\big) = f\big(\lambda^{-1}\Lambda, A, a^\tau, b; \gamma; t\big), \quad \lambda \in \mathbb{C} \setminus \{0\}. \tag{2.10}$$

The dependence of f upon the scalar parameter γ has been analyzed in (2.3) and (2.4), and while on one hand it suffices to know f for the case of $\gamma = 1$, these formulas involve integration resp. differentiation with respect to the variable t, so that we observe that the (in-)dependence of f upon γ is of a different nature than that on some of the remaining parameters. Altogether, these identities show that the number of parameters upon which the function f depends is, in fact, smaller than one would expect at first glance: one may without loss of generality restrict to a data set with Λ being in Jordan canonical form, and with one eigenvalue of Λ being equal to 1. In addition, any invertible matrix T that commutes with the Jordan form of Λ can be used to normalize the vector a. When restricting to a generic situation of Λ being diagonal and all entries of a being different from zero, then every diagonal matrix commutes with Λ and hence one can arrange that all entries of a are equal to 1. So in this case, the number of independent parameters, aside from γ, is equal to $m - 1 + m^2 + m = m(m+2) - 1$.

3. A solution vector of the n-dimensional hypergeometric system

In addition to the function f, we also define a vector of functions by means of its power series expansion, again restricting to the case of $a \neq 0$ at this time:

$$\begin{aligned} g(\gamma; t) = g\big(\Lambda, A, a^\tau, b; \gamma; t\big) &= \sum_{k=1}^{\infty} \frac{t^k}{(k-1)!(\gamma)_k} g_k, \\ g_k &= -\|a\|^{-2}(A, \Lambda, a, b)_k \bar{a}. \end{aligned} \tag{3.1}$$

In case of $a = 0$, the coefficients g_k are again defined as the limit for $a \to 0$, and it follows that in this case

$$g_k = (k-1)!(A+1, \Lambda)_{k-1}\Lambda^{-1}b \quad \forall k \geqslant 1.$$

Regarding the dependence upon γ, the identities (2.3) and (2.4) hold for g as well, and we also shall write $g(t)$ instead of $g(1; t)$. By termwise differentiation one can show that (formally)

$$\begin{aligned} -tf'(\gamma; t) &= a^\tau g(\gamma; t), \\ (\Lambda - t)g'(\gamma; t) &= \big(A + t^{-1}(1-\gamma)\Lambda\big)g(\gamma; t) + f(\gamma; t)b. \end{aligned} \tag{3.2}$$

From these identities we conclude convergence of the power series (3.1) for $|t| < r_\lambda$, as well as analyticity of $g(\gamma; t)$ in the region G. The function g also satisfies transformation laws, for an arbitrary invertible matrix T, which now read as follows:

$$\begin{aligned} Tg\big(T^{-1}\Lambda T, T^{-1}AT, a^\tau T, T^{-1}b; \gamma; t\big) &= g\big(\Lambda, A, a^\tau, b; \gamma; t\big), \\ g\big(\Lambda, A, a^\tau, b; \gamma; \lambda t\big) &= g\big(\lambda^{-1}\Lambda, A, a^\tau, b; \gamma; t\big), \quad \lambda \in \mathbb{C} \setminus \{0\}. \end{aligned} \tag{3.3}$$

The second identity in (3.2) simplifies for $\gamma = 1$, and it is obvious that the vector $y(t) = (f(t), g(t))^\tau$ *is a solution of the hypergeometric system* (0.1). Since it is the only solution that is holomorphic at the origin with $y(0) = 1$, we conclude that it is the one that was mentioned in the introduction!

In view of (3.2), the function g may be expressed in terms of f, using the technique of variation of constants; in the particular case of $\gamma = 1$, this formula coincides with (2.8). We interpret this formula as saying that g can be expressed in terms of f, making use of solutions of the hypergeometric system

in dimension m. Since both f and g together constitute a solution of the same system in dimension $n = m + 1$, it is fair to say that (2.8) *expresses g in terms of f and some more elementary functions*. More can be said, however: For $m = 1$, using the notation introduced above, one may verify that $g(\gamma; t) = bt\Lambda^{-1}\gamma^{-1}F(1+\alpha, 1+\beta; 1+\gamma; t\Lambda^{-1})$. Hence in this case $g(t)$, aside from an elementary factor, also is a hypergeometric function like $f(t)$, but for different parameters. For larger m, with $a_0(t) \equiv a$, $r_{00}(t) \equiv 0$, $r_{01}(t) = t$, we define recursively for $k \geqslant 0$:

$$\begin{aligned} a^\tau_{k+1}(t) &= a^\tau_k(t)(\Lambda - t)^{-1}A + \big(a^\tau_k(t)\big)', \\ r_{k+1,0}(t) &= r'_{k0}(t) + a^\tau_k(t)(\Lambda - t)^{-1}b, \\ r_{k+1,j}(t) &= r'_{kj}(t) + r_{k,j-1}(t), \quad 1 \leqslant j \leqslant k+1, \\ r_{k+1,k+2}(t) &= r_{k,k+1}(t)\,(= t). \end{aligned} \tag{3.4}$$

Clearly, these are sequences of rational functions in t with poles at the points $\lambda_1, \ldots, \lambda_m$, and polynomials in the parameters a, b, and A. Differentiating $a^\tau_k(t)g(t)$ and using (3.2), for $\gamma = 1$, to eliminate $g'(t)$, one can show inductively that

$$a^\tau_k(t)g(t) = -\sum_{j=0}^{k+1} r_{kj}(t) f^{(j)}(t) \quad \forall k \geqslant 0. \tag{3.5}$$

The first m such identities can be combined into a matrix equation: If $C(t)$ denotes the matrix with rows $a^\tau_0(t), \ldots, a^\tau_{m-1}(t)$, while $R(t)$ is the $m \times (m+1)$-matrix defined by

$$R(t) = \begin{bmatrix} 0 & t & 0 & 0 & \ldots & 0 \\ r_{10}(t) & r_{11}(t) & r_{12}(t) & \ldots & 0 & \\ \vdots & & & & & \vdots \\ r_{m-1,0}(t) & r_{m-1,1}(t) & r_{m-1,2}(t) & \ldots & & r_{m-1,m}(t) \end{bmatrix},$$

then $C(t)g(t) = R(t)F(t)$, with $F(t) = (f(t), f'(t), \ldots, f^{(m)}(t))^\tau$. So we conclude that *whenever $C(t)$ is invertible, we can express a solution vector of* (0.1) *as a linear combination, over the field of rational functions, of one new higher transcendental function f and its first m derivatives*!

To see that the determinant of $C(t)$, regarded as a function of t and all the entries of A, Λ, a, b, does not vanish identically, observe that in the special case of $\Lambda = I$ one has $a_k(t) = (1-t)^{-k}a_k$, with constant vectors a_k satisfying $a_0 = a$, $a^\tau_{k+1} = a^\tau_k(k + A)$, for $k \geqslant 0$. Hence in this case, the determinant of $C(t)$ is a negative power of $1 - t$ times the determinant of the matrix whose rows are equal to $a^\tau, a^\tau A, \ldots, a^\tau A^{m-1}$, which is non-zero if a is equal to the first unit vector, while A is a permutation matrix.

On one hand, the representation $g(t) = C(t)^{-1}R(t)F(t)$ is meaningful only when $\det C(t) \neq 0$, which does not hold for some constellation of a, b, and A. On the other hand, $g(t)$, $F(t)$, and $R(t)$ are entire functions of these parameters, and so the formula necessarily carries over to such cases by taking appropriate limits. We shall illustrate this for the case of $m = 2$, assuming for simplicity that Λ is diagonal, and putting $a^\tau = (a_1, a_2)$, $b^\tau = (b_1, b_2)$, $A = [a_{jk}]$. Then

$$C(t) = \begin{bmatrix} a_1 & a_2 \\ \frac{a_1 a_{11}}{\lambda_1 - t} + \frac{a_2 a_{21}}{\lambda_2 - t} & \frac{a_1 a_{12}}{\lambda_1 - t} + \frac{a_2 a_{22}}{\lambda_2 - t} \end{bmatrix}, \qquad R(t) = \begin{bmatrix} 0 & t & 0 \\ \frac{a_1 b_1}{\lambda_1 - t} + \frac{a_2 b_2}{\lambda_2 - t} & 1 & t \end{bmatrix}.$$

Hence we obtain for the two components $g_j(t)$ of the vector $g(t)$

$$g_1(t) = \frac{-a_2\left(\frac{a_1 b_1}{\lambda_1 - t} + \frac{a_2 b_2}{\lambda_2 - t}\right) f(t) + \left(t\left(\frac{a_1 a_{12}}{\lambda_1 - t} + \frac{a_2 a_{22}}{\lambda_2 - t}\right) - a_2\right) f'(t) - a_2 t f''(t)}{a_1 \frac{a_1 a_{12} - a_2 a_{11}}{\lambda_1 - t} + a_2 \frac{a_1 a_{22} - a_2 a_{21}}{\lambda_2 - t}},$$

$$g_2(t) = \frac{a_1\left(\frac{a_1 b_1}{\lambda_1 - t} + \frac{a_2 b_2}{\lambda_2 - t}\right) f(t) - \left(t\left(\frac{a_1 a_{11}}{\lambda_1 - t} + \frac{a_2 a_{21}}{\lambda_2 - t}\right) - a_1\right) f'(t) + a_1 t f''(t)}{a_1 \frac{a_1 a_{12} - a_2 a_{11}}{\lambda_1 - t} + a_2 \frac{a_1 a_{22} - a_2 a_{21}}{\lambda_2 - t}}.$$

It is worthwhile to mention that, for general $m \geqslant 1$, the identity $g(t) = C(t)^{-1} R(t) F(t)$ may be inserted into (3.5), for $k = m$, to obtain a differential equation for f, which is of order $m + 1$. It is quite possible, however, that f may satisfy a much simpler differential equation which is meaningful even when $C(t)$ fails to be invertible, but we have not been able to obtain one!

References

[1] W. Balser, Explicit evaluation of the Stokes multipliers and central connection coefficients for certain systems of linear differential equations, Math. Nachr. 138 (1988) 131–144.
[2] W. Balser, Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations, Springer-Verlag, New York, 2000.
[3] W. Balser, W.B. Jurkat, D.A. Lutz, On the reduction of connection problems for differential equations with an irregular singular point to ones with only regular singularities; Part I, SIAM J. Math. Anal. 12 (1981) 691–721.
[4] W. Balser, W.B. Jurkat, D.A. Lutz, On the reduction of connection problems for differential equations with an irregular singular point to ones with only regular singularities; Part II, SIAM J. Math. Anal. 19 (1988) 398–443.
[5] Y. Haraoka, Integral representations of solutions of differential equations free from accessory parameters, Adv. Math. 169 (2002) 187–240.
[6] Y. Haraoka, T. Yokoyama, Construction of rigid local systems and integral representations of their sections, Math. Nachr. 279 (2006) 255–271.
[7] M. Kohno, The convergence condition of a series appearing in connection problems and the determination of Stokes' multipliers, Publ. RIMS Kyoto Univ. 3 (1968) 337–350.
[8] M. Kohno, On the calculation of the approximate values of Stokes' multipliers, Publ. Res. Inst. Math. Sci. Ser. A 4 (1968/1969) 277–297.
[9] M. Kohno, A two point connection problem for n-th order single linear ordinary differential equations with an irregular singular point of rank two, Jpn. J. Math. 40 (1971) 11–62.
[10] M. Kohno, A two point connection problem for general linear ordinary differential equations, Hiroshima Math. J. 4 (1974) 293–338.
[11] M. Kohno, T. Yokoyama, A central connection problem for a normal system of linear differential equations, Hiroshima Math. J. 14 (1984) 257–263.
[12] K. Okubo, Connection problems for systems of linear differential equations, in: Japan–United States Seminar on Ordinary Differential and Functional Equations, Kyoto, 1971, in: Lecture Notes in Math., vol. 243, Springer-Verlag, Berlin, 1971, pp. 238–248.
[13] K. Okubo, K. Takano, Generalized hypergeometric functions, in: K. Okubo, On the Group of Fuchsian Equations, 1981, Progress Report for Grant-in-Aid, Project Number 454034, The Ministry of Education, Science and Culture, Japan.
[14] K. Okubo, K. Takano, S. Yoshida, A connection problem for the generalized hypergeometric equation, Funkcial. Ekvac. 31 (1988) 483–495.
[15] T. Sasai, On a monodromy group and irreducible conditions of a fourth order Fuchsian differential system of Okubo type, J. Reine Angew. Math. 299/300 (1978) 38–50.
[16] R. Schäfke, Über das globale analytische Verhalten der Lösungen der über die Laplace-Transformation zusammenhängenden Differentialgleichungen $tx'(t) = (A + tB)x$ und $(s - B)v' = (\rho - A)v$, Dissertation, Essen, 1979.
[17] R. Schäfke, Über das globale Verhalten der Normallösungen von $x'(t) = (B + t^{-1}A)x(t)$, und zweier Arten von assoziierten Funktionen, Math. Nachr. 121 (1985) 123–145.
[18] R. Schäfke, Confluence of several regular singular points into an irregular singular point, J. Dyn. Control Syst. 4 (1998) 401–424.
[19] T. Yokoyama, On connection formulas for a fourth order hypergeometric system, Hiroshima Math. J. 15 (1985) 297–320.
[20] T. Yokoyama, Characterization of connection coefficients for hypergeometric systems, Hiroshima Math. J. 17 (1987) 219–233.
[21] T. Yokoyama, On the structure of connection coefficients for hypergeometric systems, Hiroshima Math. J. 18 (1988) 309–339.