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p-Harmonic functions with boundary data having jump discontinuities and Baernstein's problem

Anders Björn

Department of Mathematics, Linköpings universitet, SE-581 83 Linköping, Sweden

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ABSTRACT

For p-harmonic functions on unweighted $\mathbf{R}^2$, with $1 < p < \infty$, we show that if the boundary values f has a jump at an (asymptotic) corner point z_0, then the Perron solution Pf is asymptotically $a + b \arg(z - z_0) + o(|z - z_0|)$. We use this to obtain a positive answer to Baernstein's problem on the equality of the p-harmonic measure of a union G of open arcs on the boundary of the unit disc, and the p-harmonic measure of $\overline{G}$. We also obtain various invariance results for functions with jumps and perturbations on small sets. For $p > 2$ these results are new also for continuous functions. Finally we look at generalizations to $\mathbf{R}^n$ and metric spaces.

E-mail address: anbjo@mai.liu.se.

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1. Introduction

Let $\Omega \subset \mathbf{R}^2$ be a nonempty bounded open set and $1 < p < \infty$. Then a boundary point $x_0 \in \partial\Omega$ is *regular* if

$$\lim_{\Omega \ni y \to x_0} Pf(y) = f(x_0) \tag{1.1}$$

for all $f \in C(\partial\Omega)$, where Pf is the Perron solution of the Dirichlet problem for p-harmonic functions. (See Section 2 for notation and definitions.)

If x_0 is regular, then it is fairly easy to see that (1.1) holds for all bounded resolutive $f : \partial\Omega \to \mathbf{R}$ which are continuous at x_0. This means that Pf near x_0 (essentially) only depends on f near x_0, as long as f is continuous at x_0. It is natural to ask if the requirement that f is continuous at x_0 is essential. Which leads us to the following question.

Open problem 1.1. Let $x_0 \in \partial\Omega$ be regular. Assume that $f, h : \partial\Omega \to \mathbf{R}$ are bounded and resolutive and that $f = h$ in $B(x_0, \delta) \cap \partial\Omega$. (Here the ball $B(x_0, \delta) := \{x \in \mathbf{R}^2\colon\ |x - x_0| < \delta\}$.) Does it then follow that

$$\lim_{\Omega \ni y \to x_0} \bigl(Pf(y) - Ph(y)\bigr) = 0?$$

If the answer is positive, then a simple approximation argument makes it possible to replace the requirement that $f = h$ in $B(x_0, \delta) \cap \partial\Omega$ by the assumption that

$$\lim_{\partial\Omega \ni x \to x_0} \bigl(f(x) - h(x)\bigr) = 0. \tag{1.2}$$

If x_0 is not regular then, by definition, we can find f continuous and h constant so that (1.2) fails, and thus it is essential to require x_0 to be regular.

In the linear case, $p = 2$, the conclusion follows directly: just observe that $Pf - Ph = P(f - h)$ and that $f - h$ is continuous at x_0.

In the nonlinear case this question is much harder. In this paper we make a first attempt at answering this question by looking at functions f with jump discontinuities, see Sections 4–7 for the precise statements of our result. (The most general results are Theorems 6.1 and 7.2.) This question is just as relevant for p-harmonic functions on weighted $\mathbf{R}^n$ and more generally on metric spaces, however the technique used here only works in unweighted $\mathbf{R}^n$.

As a particular application we obtain the following result.

Theorem 1.2. *Let $1 < p < \infty$ and let $\Omega = \mathbf{D}$ be the unit disc in the complex plane. Let $G \subset \partial\mathbf{D}$ be the union of m open arcs. Then*

$$\omega_{a,p}(G) = \omega_{a,p}(\overline{G}) \quad \textit{for all } a \in \mathbf{D}.$$

Here $\omega_{a,p}(G) := P\chi_G(a)$ denotes the *p-harmonic measure*. Recall that for $p \neq 2$ the p-harmonic measure, being a nonlinear analogue of the harmonic measure, is *not* a measure in the usual sense.

Baernstein [3, p. 548] asked if Theorem 1.2 holds. (Strictly speaking he states this question for the case $m = 2$.) For the linear case $p = 2$ the positive answer is well known and easy to obtain. In Björn–Björn–Shanmugalingam [14], Baernstein's problem was answered, in the affirmative, for $1 < p \leqslant 2$ and also for $m = 1$ when $p > 2$.

Using the results on jump discontinuities we are now able to give a positive answer to Baernstein's problem for all $1 < p < \infty$, see Section 8 for the proof and similar results for more general sets Ω. For other similar results in various situations see Björn–Björn–Shanmugalingam [14,15].

In the results on jump discontinuities it turns out that the actual value of the boundary function f at the jump point plays no role. We exploit this to give some perturbation results with perturbations on countable sets which are new for $p > n$. In particular we obtain the following result, which is new for $p > n$. Recall that for $p > n$ the capacity of singleton sets is positive. An important part of the theorem below is that h is resolutive, which is far from obvious and again was not known for $p > n$ earlier. Note also that we do not require h to be continuous on $\partial\Omega \setminus E$.

Theorem 1.3. *Let $\Omega \subset \mathbf{R}^n$ be a nonempty bounded open set and $1 < p < \infty$. Let further $f \in C(\partial\Omega)$ and $h = f$ on $\partial\Omega \setminus E$, where E is a finite or countable set of points with exterior rays* (*see Definition* 5.3)*. Then $Ph = Pf$.*

Example 6.4 shows that some geometric condition on the points in E is needed, and it also shows that one cannot replace the exterior ray condition by assuming that all points in E are regular.

Theorem 2.9 below (due to Heinonen–Kilpeläinen–Martio [20]) shows that all bounded semicontinuous functions are resolutive if Ω is regular, a result which we generalize both for regular and semiregular sets in Section 3. In Theorem 7.3 we show that a large number of semicontinuous functions with jump discontinuities are resolutive also on irregular sets. In Section 8 we also obtain new resolutivity results for characteristic functions on irregular sets in connection with the Baernstein problem.

The outline of the paper is as follows. In Section 2 we give some background on p-harmonic functions and Perron solutions. In Section 3 we improve upon the comparison principle between sub- and superharmonic functions. We also take the opportunity to give an alternative definition of Perron solutions and prove some results for it. (Note that the results in Section 3 as well as Lemma 2.10 hold also in metric spaces under the usual assumptions.) In Sections 4 and 5 we study Perron solutions for functions with jump discontinuities at a corner point and at an asymptotic corner point, respectively. These results make it possible to also treat certain perturbations and we pursue this in Sections 6 and 7, where we also obtain some new resolutivity and uniqueness results. In Section 8 we turn to the Baernstein problem, and generalizations of it, whereas in Section 9 we look at generalizations to higher dimensions and metric spaces. We also show that the asymptotic corner condition can be replaced by an asymptotic logarithmic spiral condition, and similarly the exterior ray condition can be replaced by an exterior logarithmic spiral condition.

While this paper was close to be finished the author became aware of the fact that Kim [24] independently has obtained some of the results in this paper, including Theorem 1.2. After this paper was originally submitted, Kim [25] continued his studies and, in particular, answered Open problem 9.8 in the affirmative for unweighted $\mathbf{R}^n$. His probabilistic-game theoretic proof uses the connection between p-harmonicity and tug-of-war with noise discovered by Peres–Sheffield [30].

2. Preliminaries and notation

Let now $\Omega \subset \mathbf{R}^n$ be a nonempty bounded open set. A continuous function $u : \Omega \to \mathbf{R}$ is *p-harmonic* on Ω if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\, dx = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega),$$

where ∇u is the distributional gradient of u, and dx is Lebesgue measure.

If f belongs to the Sobolev space $W^{1,p}(\Omega)$, then there is a unique p-harmonic function $Hf = H_\Omega f$ on Ω such that $f - Hf \in W_0^{1,p}(\Omega)$. Recall also that $E \Subset \Omega$ if $\bar{E}$ is a compact subset of Ω.

Definition 2.1. A function $u : \Omega \to (-\infty, \infty]$ is *superharmonic* in Ω if

(i) u is lower semicontinuous;
(ii) u is not identically ∞ in any component of Ω;

(iii) for every nonempty open set $\Omega' \Subset \Omega$ and all functions $v \in \operatorname{Lip}(\mathbf{R}^n)$, we have $H_{\Omega'} v \leqslant u$ in Ω' whenever $v \leqslant u$ on $\partial\Omega'$.

A function $u : \Omega \to (-\infty, \infty]$ is *hyperharmonic* in Ω if (i) and (iii) are satisfied.

Moreover, $u : \Omega \to [-\infty, \infty)$ is *subharmonic* if $-u$ is superharmonic, and *hypoharmonic* if $-u$ is hyperharmonic.

This definition of superharmonicity is the one usually used in metric space papers on p-harmonic functions, and, by Theorem 6.1 in Björn [4], it is equivalent to the definition used, e.g., in Heinonen–Kilpeläinen–Martio [20].

Now we are ready to define Perron solutions.

Definition 2.2. Let $V \subset \mathbf{R}^n$ be a nonempty bounded open set. Given a function $f : \partial V \to \overline{\mathbf{R}}$, let $\mathcal{U}_f(V)$ be the set of all superharmonic functions u on V bounded from below such that

$$\liminf_{V \ni y \to x} u(y) \geqslant f(x) \quad \text{for all } x \in \partial V.$$

Define the *upper Perron solution* of f by

$$\overline{P}_V f(x) = \inf_{u \in \mathcal{U}_f(V)} u(x), \quad x \in V.$$

Similarly, let $\mathcal{L}_f(V)$ be the set of all subharmonic functions u on V bounded from above such that

$$\limsup_{V \ni y \to x} u(y) \leqslant f(x) \quad \text{for all } x \in \partial V,$$

and define the *lower Perron solution* of f by

$$\underline{P}_V f(x) = \sup_{u \in \mathcal{L}_f(V)} u(x), \quad x \in V.$$

If $\overline{P}_V f = \underline{P}_V f$, then we let $P_V f := \overline{P}_V f$ and f is said to be *resolutive*.

If $V = \Omega$ we usually drop V from the notation and write, e.g., Pf.

It follows from the comparison principle, Theorem 7.6 in Heinonen–Kilpeläinen–Martio [20] (or Theorem 3.1 below) that always $\overline{P} f \geqslant \underline{P} f$. It is also important to observe that $\overline{P} f$ is p-harmonic unless it is identically $\pm\infty$ in some component, see Theorem 9.2 in [20]. Furthermore we have the following result.

Theorem 2.3. *(See Björn–Björn–Shanmugalingam [13].) Assume that $f \in C(\partial\Omega)$ and $h = f$ q.e. Then f and h are resolutive and*

$$Pf = Ph.$$

Moreover $U = Pf$ is the unique bounded p-harmonic function such that

$$\lim_{\Omega \ni y \to x} U(y) = f(x) \quad \text{for q.e. } x \in \partial\Omega.$$

Here and below, quasieverywhere (q.e.) means with the exception of a set of C_p-capacity zero, where C_p denotes the Sobolev capacity associated with the Sobolev space $W^{1,p}$. It is important to observe that points have zero capacity in (unweighted) $\mathbf{R}^n$ if and only if $1 < p \leqslant n$. Moreover, the capacity C_p is countably subadditive so that countable sets have capacity zero if $1 < p \leqslant n$. (Note that if $p > n$ then the requirement $h = f$ q.e. is nothing but requiring that $h \equiv f$.)

We will also need the Newtonian space $N^{1,p}(\mathbf{R}^n)$ which is the set of all quasicontinuous functions $f \in W^{1,p}(\mathbf{R}^n)$ (defined everywhere). A function f is *quasicontinuous* if for every $\varepsilon > 0$ there is an open set U with $C_p(U) < \varepsilon$ such that $f|_{\mathbf{R}^n \setminus U}$ is continuous. Note that if $f \in N^{1,p}(\mathbf{R}^n)$ and $h = f$ q.e., then $h \in N^{1,p}(\mathbf{R}^n)$. See, e.g., Björn–Björn–Shanmugalingam [13] or Björn–Björn [11] for the usual definition of $N^{1,p}$ and [11] for this characterization, which is a consequence of the main results in Kilpeläinen [21] and Björn–Björn–Shanmugalingam [16].

The following result will be of use to us.

Theorem 2.4. *(See Björn–Björn–Shanmugalingam [13].) Assume that $f \in N^{1,p}(\mathbf{R}^n)$ and $h = f$ q.e. Then f and h are resolutive and*

$$Pf = Hf = Ph = Hh.$$

Note that it is important to work with $N^{1,p}(\mathbf{R}^n)$. If we merely assume that $f \in W^{1,p}(\mathbf{R}^n)$ and the Lebesgue measure of $\partial\Omega$ is zero, then we can define f arbitrarily on $\partial\Omega$. While Hf is independent of the representative of f, Pf is highly dependent thereon. If f is required to belong to $N^{1,p}(\mathbf{R}^n)$ we have less freedom, and no more than that Pf remains independent of the choice of representative, by the theorem above.

As already mentioned, $x_0 \in \partial\Omega$ is *regular* if

$$\lim_{\Omega \ni y \to x_0} Pf(y) = f(x_0) \quad \text{for all } f \in C(\partial\Omega).$$

The set Ω is *regular* if all its boundary points are regular.

The following fact is important.

Theorem 2.5 *(The Kellogg property). The set of all irregular points on $\partial\Omega$ has capacity zero.*

See Theorem 9.11 in Heinonen–Kilpeläinen–Martio [20] or Theorem 3.9 in Björn–Björn–Shanmugalingam [12] for a proof (the latter in metric spaces).

We will also need the Wiener criterion, which was obtained by Maz'ya [29] and Kilpeläinen–Malý [22] (for unweighted $\mathbf{R}^n$).

Theorem 2.6 *(The Wiener criterion). The point $x_0 \in \partial\Omega$ is regular if and only if*

$$\int_0^1 \left(\frac{\operatorname{cap}_p(B(x_0,t) \setminus \Omega, B(x_0,2t))}{\operatorname{cap}_p(B(x_0,t), B(x_0,2t))} \right)^{1/(p-1)} \frac{dt}{t} < \infty.$$

Here cap_p is the variational capacity defined on p. 27 in Heinonen–Kilpeläinen–Martio [20].

The irregular boundary points can be divided into two types.

Definition 2.7. An irregular boundary point $x_0 \in \partial\Omega$ is *semiregular* if

$$\lim_{\Omega \ni y \to x_0} Pf(y) \quad \text{exists for all } f \in C(\partial\Omega),$$

and *strongly irregular* if for every $f \in C(\partial\Omega)$ there is a sequence $\{y_j\}_{j=1}^\infty$ in Ω such that

$$y_j \to x_0 \quad \text{and} \quad Pf(y_j) \to f(x_0), \quad \text{as } j \to \infty.$$

The set Ω is *semiregular* if all its boundary points are regular or semiregular.

In Björn [8, Theorem 2.1] it was shown that an irregular boundary point is either semiregular or strongly irregular. It was also shown in [8] that the set S of semiregular points is the largest relatively open subset of $\partial\Omega$ with zero capacity.

Note that in view of the Kellogg property (Theorem 2.5) most boundary points are regular. However, Theorem 4.1 in [8] shows that the sets of semiregular and of strongly irregular points can be quite large.

We will use the following result implicitly several times. It does not seem to have been recorded before in the nonlinear literature.

Proposition 2.8. *Let S be the set of semiregular boundary points with respect to Ω, and let $x_0 \in \partial\Omega \setminus S$. Then x_0 is regular with respect to Ω if and only if it is regular with respect to $\Omega \cup S$.*

This is a straightforward consequence of the Wiener criterion (and the fact that $C_p(S) = 0$), but we would like to provide a proof not depending on the Wiener criterion. Our proof has the advantage that it is valid also for arbitrary metric spaces (under the usual assumptions, see Section 9.5), for which we do not know if the Wiener criterion holds true, see however J. Björn [19].

Our proof can also be easily modified to show the same consequence for regularity for quasiminimizers. Note that semiregularity and semiregularity for quasiminimizers coincide, by Björn [8]. For more on boundary regularity for quasiminimizers see Ziemer [31], J. Björn [18], Martio [28], Björn [5, 8,9], Björn–Björn [10] and Björn–Martio [17].

Proof of Proposition 2.8. By Björn [8], $\widetilde{\Omega} := \Omega \cap S$ is open and $C_p(S) = 0$.

Let $\tilde{f} \in C(\partial\widetilde{\Omega})$ and let f be any continuous extension of $\tilde{f}$ to $\partial\Omega$. Let further $u = P_{\widetilde{\Omega}}\tilde{f}|_\Omega$. Then

$$\lim_{\Omega \ni y \to x} u(y) = \lim_{\widetilde{\Omega} \ni y \to x} P_{\widetilde{\Omega}}\tilde{f}(y) = \tilde{f}(x) = f(x) \tag{2.1}$$

for q.e. $x \in \partial\widetilde{\Omega}$. As $C_p(S) = 0$, (2.1) holds for q.e. $x \in \partial\Omega$. Hence, by Corollary 6.2 in Björn–Björn–Shanmugalingam [13], $u = Pf$, or in other words

$$Pf = P_{\widetilde{\Omega}}\tilde{f}|_\Omega.$$

If x_0 is regular with respect to Ω, then

$$\lim_{\widetilde{\Omega} \ni y \to x_0} P_{\widetilde{\Omega}}\tilde{f}|_\Omega(y) = \lim_{\Omega \ni y \to x_0} Pf(y) = f(x_0) = \tilde{f}(x_0)$$

for all $\tilde{f} \in C(\partial\widetilde{\Omega})$, and thus x_0 is regular with respect to $\widetilde{\Omega}$.

Conversely, if x_0 is regular with respect to $\widetilde{\Omega}$, then

$$\lim_{\Omega \ni y \to x_0} Pf(y) = \lim_{\widetilde{\Omega} \ni y \to x_0} P_{\widetilde{\Omega}}\tilde{f}|_\Omega(y) = \tilde{f}(x_0) = f(x_0)$$

for all $f \in C(\partial\Omega)$, showing that x_0 is regular with respect to Ω. $\square$

Let us also recall the following result.

Theorem 2.9. *Assume that Ω is regular. Then all bounded semicontinuous functions are resolutive.*

This is Proposition 9.31 in Heinonen–Kilpeläinen–Martio [20]. We generalize this both for regular and semiregular sets in Proposition 3.6 and Corollary 3.7.

Lemma 2.10. *Let $x_0 \in \partial\Omega$ be a regular boundary point. Let further $f, h : \partial\Omega \to \overline{\mathbf{R}}$ be such that $f = h$ q.e. Then the following are true*:

(a) *If f is lower semicontinuous at x_0 and bounded from below on $\partial\Omega$, then*

$$\liminf_{\Omega \ni y \to x_0} \underline{P}h(y) \geqslant f(x_0).$$

(b) *If f is continuous at x_0 and bounded on $\partial\Omega$, then*

$$\lim_{\Omega \ni y \to x_0} \underline{P}h(y) = \lim_{\Omega \ni y \to x_0} \overline{P}h(y) = f(x_0).$$

This is very slight, but sometimes useful, improvement upon Proposition 7.1 and Corollary 7.2 in Björn–Björn–Shanmugalingam [13].

Proof of Lemma 2.10. (a) We can find a continuous function $k : \partial\Omega \to \overline{\mathbf{R}}$ such that $k \leqslant f$ and $k(x_0) = f(x_0)$. Let further $k' = \min\{k, h\}$. Then $k' = k$ q.e. By Theorem 2.3, $Pk' = Pk$. As x_0 is regular we have that

$$\liminf_{\Omega \ni y \to x_0} \underline{P}h(y) \geqslant \liminf_{\Omega \ni y \to x_0} Pk'(y) = \liminf_{\Omega \ni y \to x_0} Pk(y) = f(x_0).$$

(b) This follows by applying (a) to both h and $-h$, and using the fact that $\underline{P}h \leqslant \overline{P}h$. $\square$

3. Comparison lemmas

We will need an improvement of the comparison principle in Theorem 7.6 in Heinonen–Kilpeläinen–Martio [20].

Theorem 3.1. *Let S be the set of semiregular boundary points with respect to Ω. Assume that u is superharmonic and that v is subharmonic in Ω. Assume further that u is locally bounded from below and that v is locally bounded from above in $\widetilde{\Omega} := \Omega \cup S$, in the sense that if $G \Subset \widetilde{\Omega}$, then u is bounded from below and v from above in $G \cap \Omega$. If*

$$\liminf_{\Omega \ni y \to x} \big(u(y) - v(y)\big) \geqslant 0 \quad \textit{for all } x \in \partial\Omega \setminus S, \tag{3.1}$$

in particular if

$$\infty \neq \limsup_{\Omega \ni y \to x} v(y) \leqslant \liminf_{\Omega \ni y \to x} u(y) \neq -\infty \quad \textit{for all } x \in \partial\Omega \setminus S, \tag{3.2}$$

then $v \leqslant u$ in Ω.

Assuming that the inequality (3.2) holds for all $x \in \partial\Omega$, this result is Theorem 7.6 in Heinonen–Kilpeläinen–Martio [20] (in metric spaces it was obtained by Kinnunen–Martio [26, Theorem 7.2]). Their proof also covers the case when (3.1) holds for all $x \in \partial\Omega$ (this is true also for the metric space version).

In Björn–Björn–Shanmugalingam [13, Section 10] it was asked if $v \leqslant u$ in Ω whenever u and v are bounded super- and subharmonic functions, respectively, and (3.2) holds for some set S with zero capacity. We do not know the answer to that question, but at least our theorem is some slight progress in this direction.

Observe that the boundedness assumptions for u and v on S are necessary: Let $\Omega = B(0,1) \setminus \{0\} \subset \mathbf{R}^2$ and $1 < p < 2$. Then $S = \{0\}$. Let $u(x) = |x|^{(p-2)/(p-1)}$ and $v(x) \equiv 1$. Then u is p-harmonic and in particular subharmonic in Ω, see Example 7.47 in [20], whereas v is p-harmonic and in particular superharmonic in Ω, giving a counterexample to Theorem 3.1 without the boundedness assumptions.

Proof of Theorem 3.1. By the comments after Theorem 3.1 in Björn [8], $C_p(S) = 0$ and $\widetilde{\Omega}$ is open. Let, for $x \in \widetilde{\Omega}$,

$$\tilde{u}(x) = \operatorname*{ess\,lim\,inf}_{\Omega \ni y \to x} u(y) \quad \text{and} \quad \tilde{v}(x) = \operatorname*{ess\,lim\,sup}_{\Omega \ni y \to x} v(y).$$

Theorem 6.3 in Björn [6] shows that $\tilde{u}$ is superharmonic in G for any open $G \Subset \widetilde{\Omega}$, from which it follows that $\tilde{u}$ is superharmonic in $\widetilde{\Omega}$. Similarly $\tilde{v}$ is subharmonic in $\widetilde{\Omega}$.

It follows immediately that

$$\liminf_{\Omega \ni y \to x} \bigl(\tilde{u}(y) - \tilde{v}(y)\bigr) \geqslant 0 \quad \text{for all } x \in \partial\widetilde{\Omega}.$$

The rest of the proof is the same as the proof of Theorem 7.2 in Kinnunen–Martio [26], for the reader's convenience we repeat it here (slightly modified).

Let $\Omega_1 \Subset \Omega_2 \Subset \cdots \Subset \widetilde{\Omega} = \bigcup_{k=1}^\infty \Omega_k$ and let $\varepsilon > 0$. Then there is $k > 1/\varepsilon$ such that $\tilde{v} \leqslant \tilde{u} + \varepsilon$ on $\partial\Omega_k$. As $\tilde{v}$ is upper semicontinuous (and does not take the value ∞) there is a decreasing sequence $\{\varphi_j\}_{j=1}^\infty$, $\varphi_j \in \operatorname{Lip}(\overline{\Omega}_k)$, such that $\varphi_j \to \tilde{v}$ in $\overline{\Omega}_k$.

Since $\tilde{u} + \varepsilon$ is lower semicontinuous, the compactness of $\partial\Omega_k$ shows that there is i so that $\varphi_i \leqslant \tilde{u} + \varepsilon$ on $\partial\Omega_k$. By (iii) in the definition of superharmonicity, $H_{\Omega'}\varphi_i \leqslant \tilde{u} + \varepsilon$ in Ω_k. Similarly $\tilde{v} \leqslant H_{\Omega'}\varphi_i \leqslant \tilde{u} + \varepsilon$ in Ω_k.

Letting $\varepsilon \to 0$ (and hence $k \to \infty$) completes the proof. $\square$

We can now define an alternative Perron solution.

Definition 3.2. Let $V \subset \mathbf{R}^n$ be a nonempty bounded open set and S be the set of semiregular boundary points in V. Given a function $f : \partial V \to \overline{\mathbf{R}}$, let $\widetilde{\mathcal{U}}_f(V)$ be the set of all superharmonic functions u on V bounded from below such that

$$\liminf_{V \ni y \to x} u(y) \geqslant f(x) \quad \text{for all } x \in \partial V \setminus S.$$

Define

$$\overline{R}_V f(x) = \inf_{u \in \widetilde{\mathcal{U}}_f(V)} u(x), \quad x \in V.$$

Similarly, let $\widetilde{\mathcal{L}}_f(V)$ be the set of all subharmonic functions u on V bounded from above such that

$$\limsup_{V \ni y \to x} u(y) \leqslant f(x) \quad \text{for all } x \in \partial V \setminus S,$$

and define

$$\underline{R}_V f(x) = \sup_{u \in \widetilde{\mathcal{L}}_f(V)} u(x), \quad x \in V.$$

If $\overline{R}_V f = \underline{R}_V f$, then we let $R_V f := \overline{R}_V f$ and f is said to be *R-resolutive*.

If $V = \Omega$ we usually drop V from the notation and write, e.g., Rf.

It follows directly from Theorem 3.1 that

$$\underline{R} f \leqslant \overline{R} f \tag{3.3}$$

for arbitrary functions f. We also trivially have $\underline{P} f \leqslant \underline{R} f \leqslant \overline{R} f \leqslant \overline{P} f$. In particular f is R-resolutive if f is resolutive. Moreover $\overline{R} f$ is p-harmonic unless it is identically $\pm\infty$ in some component, the proof of this being similar to the proof of Theorem 9.2 in Heinonen–Kilpeläinen–Martio [20].

In fact, if $u \in \widetilde{\mathcal{U}}_f$ then u has a unique superharmonic extension U to the open set $\Omega \cup S$, by Theorem 6.3 in Björn [6], and $U \in \mathcal{U}_f(\Omega \cup S)$. Conversely if $U \in \mathcal{U}_f(\Omega \cup S)$, then $U|_\Omega \in \widetilde{\mathcal{U}}_f$. Thus $\overline{R} f = (\overline{P}_{\Omega \cup S} f)|_\Omega$ from which (3.3) and the p-harmonicity of $\overline{R} f$ also follows.

Note that in the linear case $P = Q = R$, where the Q-Perron solutions were defined in Björn–Björn–Shanmugalingam [13, Definition 10.1]. We do not know if this is true in the nonlinear case, see however the discussion in Section 10 in [13].

Proposition 3.3. *If f and h are resolutive and $f = h$ on $\partial\Omega \setminus S$, where S is the set of semiregular boundary points with respect to Ω, then $Pf = Ph$.*

Proof. We have $Pf = Rf = Rh = Ph$. □

Proposition 3.4. *Let S be the set of semiregular boundary points in Ω. The following are equivalent*:

(a) $\overline{P} f = \overline{P} h$ *for all f and h such that $f = h$ on $\partial\Omega \setminus S$*;
(b) $\overline{P} f = \overline{R} f$ *for all functions f*;
(c) $\overline{P} f = \overline{P} h$ *for all nonnegative f and h such that $f = h$ on $\partial\Omega \setminus S$*;
(d) $\overline{P} f = \overline{R} f$ *for all nonnegative functions f.*

We do not know if the corresponding statements for bounded f and h are equivalent. That would however follow from a positive answer to the following problem, which was stated as Problem 10.7 in Björn–Björn–Shanmugalingam [13].

Open problem 3.5. Is it true that $\overline{P} f = \lim_{m\to\infty} \overline{P} \min\{f, m\}$ for all functions f?

In the proof below and later on we use the notation $f_+ := \max\{f, 0\}$.

Proof of Proposition 3.4. (a) $\Rightarrow$ (c) This is trivial.

(c) $\Rightarrow$ (d) Let $h = f\chi_{\partial\Omega\setminus S}$ and let $u \in \widetilde{\mathcal{U}}_f$. As $0 \in \widetilde{\mathcal{L}}_f$ we have $u \geqslant 0$ and hence $u \in \mathcal{U}_h$. Taking infimum over all such u we see that $\overline{P} f = \overline{P} h \leqslant \overline{R} f$. The converse inequality is trivial.

$\neg$(b) $\Rightarrow$ $\neg$(d) There is f and $z \in \Omega$ such that $\overline{P} f(z) > \overline{R} f(z)$. In particular there is $u \in \widetilde{\mathcal{U}}_f$ such that $u(z) < \overline{P} f(z)$. Let $M = \inf_\Omega u > -\infty$ and $h = (f - M)_+$. Then $u - M \in \widetilde{\mathcal{U}}_h$ and $\overline{R} h(z) \leqslant u(z) - M < \overline{P} f(z) - M \leqslant \overline{P} h(z)$.

(b) $\Rightarrow$ (a) We have $\overline{P} f = \overline{R} f = \overline{R} h = \overline{P} h$. □

We can now give a generalization of Theorem 2.9 to semiregular sets.

Proposition 3.6. *Assume that Ω is semiregular and let S be the set of semiregular boundary points. Let $f, h : \partial\Omega \to \overline{\mathbf{R}}$ be such that $f|_{\partial\Omega\setminus S}$ is upper semicontinuous and bounded from above, and $h = f$ on $\partial\Omega \setminus S$. Then*

$$\overline{P}h = \overline{P}f. \tag{3.4}$$

If furthermore one of the following conditions is satisfied:

(a) f *is bounded*;
(b) $\overline{P}f(x) > -\infty$ *for all* $x \in \Omega$;
(c) $\overline{P}f \equiv -\infty$ *in* Ω;
(d) Ω *is connected*;

then

$$Rh = Rf = \overline{P}h = \overline{P}f. \tag{3.5}$$

Note that for (3.5) it is necessary to have some condition on f as in (a)–(d), and this is so even in the case when Ω is regular. If (b) fails, then we must have $\mathcal{L}_f \subset \widetilde{\mathcal{L}}_f = \emptyset$ and thus $\underline{P}f = \underline{R}f \equiv -\infty$. So if also (c) fails, then $\underline{R}f \equiv \underline{P}f \equiv -\infty \not\equiv \overline{P}f$, and (3.5) does not hold. That this can indeed happen is easy to see: Let Ω consist of two components Ω_1 and Ω_2 with disjoint closures and let $f = -\infty\chi_{\partial\Omega_1}$. Then $\overline{P}f = \overline{R}f = -\infty\chi_{\Omega_1}$, whereas $\mathcal{L}_f = \widetilde{\mathcal{L}}_f = \emptyset$ and $\underline{P}f = \underline{R}f \equiv -\infty$, showing that the conclusion is false in this case. We can therefore conclude that for a given function f, (3.5) holds if and only if either (b) or (c) holds.

This anomaly would not have taken place had we defined the lower Perron solutions using hypoharmonic functions.

Corollary 3.7. *Assume that Ω is regular. Let $f : \partial\Omega \to \overline{\mathbf{R}}$ be upper semicontinuous and bounded from above. Assume further that one of the following conditions is satisfied*:

(a) f *is bounded*;
(b) $\overline{P}f(x) > -\infty$ *for all* $x \in \Omega$;
(c) $\overline{P}f \equiv -\infty$ *in* Ω;
(d) Ω *is connected.*

Then f is resolutive.

As with Proposition 3.6 we need some condition as in (a)–(d) (in fact the conclusion holds if and only if either (b) or (c) holds). Again this anomaly would not have taken place had we defined the lower Perron solutions using hypoharmonic functions.

Assuming (a) this is Proposition 9.31 in Heinonen–Kilpeläinen–Martio [20] (for weighted $\mathbf{R}^n$). Assuming (d) it is Corollary 7.4 in Björn–Björn–Shanmugalingam [13] (in metric spaces).

Proof of Corollary 3.7. In this case we have $Pf = Rf$. $\square$

Proof of Proposition 3.6. Without loss of generality we can assume that $f \leqslant h$ on S. Hence $\overline{P}f \leqslant \overline{P}h$.

Let us first prove (3.4). Assume that $\overline{P}f(x_0) < \overline{P}h(x_0)$ for some $x_0 \in \Omega$. Then we can find $a \in \mathbf{R}$ and $\delta > 0$ such that

$$\overline{P}f(x_0) < a < a + \delta < \overline{P}h(x_0).$$

Thus there is $u \in \mathcal{U}_{f+\delta}$ with $u(x_0) < a + \delta$. Extend u to $\partial\Omega$ by

$$u(y) = \liminf_{\Omega \ni x \to y} u(x), \quad y \in \partial\Omega,$$

making $u > f$ and lower semicontinuous on $\partial\Omega$. Let further

$$k_j(y) = \inf\{u(z) + jd(z, y)\colon z \in \partial\Omega\}, \quad y \in \partial\Omega, \ j = 1, 2, \dots .$$

Then

$$C(\partial\Omega) \ni k_j \nearrow u, \quad \text{as } j \to \infty.$$

As f is upper semicontinuous,

$$G_j = \{z \in \partial\Omega\colon f(z) < k_j(z)\}$$

is open. Moreover $\partial\Omega = \bigcup_{j=1}^{\infty} G_j$, since $f < u$ on $\partial\Omega$. By the compactness of $\partial\Omega$ there is some J such that $\partial\Omega = G_J$, or in other words

$$f < k_J \leqslant u \quad \text{on } \partial\Omega.$$

Let $k = k_J + \infty\chi_S$. Then, by Theorem 2.3,

$$\overline{P}h(x_0) \leqslant \overline{P}k(x_0) = \overline{P}k_J(x_0) \leqslant u(x_0) < a + \delta < \overline{P}h(x_0),$$

a contradiction. Hence $\overline{P}f \equiv \overline{P}h$, and (3.4) is proved.

Let us turn to (3.5). Observe first that (a) $\Rightarrow$ (b). Also if (d) holds, then either (b) or (c) holds. Thus we only need to consider the two cases (b) and (c).

Assume first that (b) holds. Let $M = \sup_{\partial\Omega \setminus S} f$,

$$k = \begin{cases} f, & \text{on } \partial\Omega \setminus S, \\ -\infty, & \text{on } S, \end{cases} \quad \text{and} \quad u = \begin{cases} M, & \text{on } \partial\Omega \setminus S, \\ \infty, & \text{on } S. \end{cases}$$

Since $\overline{P}f \leqslant Pu \equiv M$, by Theorem 2.3, we see that $\overline{P}f$ is p-harmonic in Ω. As S is relatively open, k is upper semicontinuous, and thus Lemma 2.10 shows that

$$\limsup_{\partial\Omega \ni y \to x} \overline{P}f(y) \leqslant k(x) = f(x) \quad \text{for } x \in \partial\Omega \setminus S.$$

Therefore $\overline{P}f \in \widetilde{\mathcal{L}}_f$ showing that

$$\overline{P}f \leqslant \underline{R}f \leqslant \overline{R}f \leqslant \overline{P}f.$$

Thus, using (3.4), we obtain that

$$Rh = Rf = \overline{P}f = \overline{P}h$$

as required.

Assume finally that (c) holds. Then

$$-\infty \leqslant \underline{R}f \leqslant \overline{R}f \leqslant \overline{P}f \equiv -\infty.$$

So, again using (3.4),

$$Rh = Rf = \overline{P}f = \overline{P}h$$

concluding the proof. □

4. Jump discontinuities at a corner

Our aim now is to study the boundary behaviour for p-harmonic functions with a jump at a corner point. We start with the simplest situation in which we have a true corner and the function is constant on both rays towards the corner, this is handled in Theorem 4.2. The next step, Theorem 4.3, is to just allow limits at the two rays, but still keeping the true corner. In the next section we allow for approximate corners, which we will call asymptotic corners.

Let in this section $0 < \alpha \leqslant 2\pi$ be fixed and

$$\Omega = \{re^{i\theta}\colon\ 0 < r < 1 \text{ and } 0 < \theta < \alpha\}.$$

We use complex notation for simplicity.

Lemma 4.1. *Let $A \in \mathbf{R}$ and $u(z) = u(re^{i\theta}) = \tilde{u}(\theta)$, $0 < \theta < \alpha$, be a p-harmonic function in Ω which is constant on rays starting at the origin. Assume further that*

$$\lim_{\theta \to 0+} u(re^{i\theta}) = 0 \quad \text{and} \quad \lim_{\theta \to \alpha-} u(re^{i\theta}) = A \quad \text{for } 0 < r < 1.$$

Then $u(re^{i\theta}) = A\theta/\alpha$.

For $p > 2$ and under the assumption that $u \in C^2(\Omega)$, this was obtained in Aronsson [1] together with other "quasiradial" solutions. It follows from Aronsson–Lindqvist [2, Corollary on p. 161], that any p-harmonic function u which is constant on rays is real-analytic, which combined with the result in [1] proves the lemma for $p > 2$. Here we give a more elementary proof which also generalizes to other situations, see Sections 9.4 and 9.5.

Proof of Lemma 4.1. Let $\varphi \in C_0^\infty((0, \alpha))$ and $\psi \in C_0^\infty((0, 1))$, then $\Phi(re^{i\theta}) := \varphi(\theta)\psi(r) \in C_0^\infty(\Omega)$. As u is p-harmonic, we have

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \Phi \, dx = 0,$$

where ∇u is the distributional gradient of u. Changing to polar coordinates and observing that $\partial u/\partial r = 0$, gives

$$0 = \int_0^\alpha \int_0^1 \left|r\tilde{u}'(\theta)\right|^{p-2} r\tilde{u}'(\theta) r\varphi'(\theta) r \, dr \, d\theta = \int_0^\alpha \left|\tilde{u}'(\theta)\right|^{p-2} \tilde{u}'(\theta)\varphi'(\theta) \, d\theta \int_0^1 r^{p+1} \, dr,$$

which shows that the first integral in the right-hand side must equal 0. As this holds for all $\varphi \in C_0^\infty((0, \alpha))$, it means that $\tilde{u}$ is a one-dimensional p-harmonic function on $(0, \alpha)$ with boundary values 0 and A. By the uniqueness of the solution of the Dirichlet problem (which follows from Theorem 2.3) $\tilde{u}(\theta) = A\theta/\alpha$. □

Theorem 4.2. *Let $A \in \mathbf{R}$ and $U(re^{i\theta}) = A\theta/\alpha$ for $0 < r < 1$ and $0 < \theta < 2\pi$. If $\alpha = 2\pi$ we require that $A = 0$. Let $f : \partial\Omega \to \mathbf{R}$ be bounded and such that $f(t) = 0$ and $f(te^{i\alpha}) = A$ for $0 < t < 1$. Then*

$$\lim_{\Omega\ni z\to 0} \big(\underline{P} f(z) - U(z)\big) = \lim_{\Omega\ni z\to 0} \big(\overline{P} f(z) - U(z)\big) = 0. \tag{4.1}$$

Moreover, we have the following radial limits

$$\lim_{t\to 0+} \underline{P} f\big(te^{i\beta}\big) = \lim_{t\to 0+} \overline{P} f\big(te^{i\beta}\big) = \frac{A\beta}{\alpha} \quad \text{for } 0 < \beta < \alpha. \tag{4.2}$$

Proof. Let us first observe that (4.2) follows from (4.1) as U has the same radial limits.

Let $M = \sup_{\partial\Omega} |f| < \infty$ and $u = \underline{P}h$, where

$$h\big(re^{i\theta}\big) = \begin{cases} 0, & \text{if } \theta = 0 \text{ and } 0 < r < 1, \\ A, & \text{if } \theta = \alpha \text{ and } 0 < r < 1, \\ -M, & \text{if } r = 0 \text{ or } r = 1. \end{cases}$$

Let further $0 < \rho < 1$ and

$$v(z) = \liminf_{\Omega\ni w\to z} u(\rho w), \quad z \in \overline{\Omega}.$$

As u is continuous on Ω we have $v(z) = u(\rho z)$ for $z = re^{i\theta}$, $0 < \theta < \alpha$, $0 < r \leqslant 1$. Moreover, Lemma 2.10 yields that $v(t) = 0$ and $v(te^{i\alpha}) = A$ for $0 < t \leqslant 1$.

By the definition of v we have $v \in \mathcal{U}_v$ so that $\overline{P}v \leqslant v$. Moreover for any $\varphi \in \mathcal{L}_h$, let $\widetilde{\varphi}(z) = \varphi(\rho z)$. Then $\widetilde{\varphi} \in \mathcal{L}_v$, and hence $\underline{P}v \geqslant \sup_{\widetilde{\varphi}} \widetilde{\varphi} = v$. Thus, v is resolutive and $v = Pv$. As $v \geqslant h$ on $\partial\Omega$ (by the maximum principle), we have $v \geqslant u$ in Ω. Also since $U \in \mathcal{U}_h$ we have $U \geqslant u$ in Ω and hence, as U is constant on rays starting at the origin, $U \geqslant v$ in Ω.

As $u(\rho z) = v(z) \geqslant u(z)$, $z \in \Omega$, for all $0 < \rho < 1$, we see that $r \mapsto u(re^{i\theta})$ is a decreasing function, and the limit $\lim_{t\to 0+} u(te^{i\theta})$ exists for $0 < \theta < \alpha$.

Let for nonnegative integers j, $v_j(z) = u(2^{-j}z)$, $z \in \Omega$. As we have seen, v_j is an increasing sequence of p-harmonic functions, which obviously is bounded by M. It follows from Harnack's convergence theorem (see Theorem 6.14 in Heinonen–Kilpeläinen–Martio [20]) that $V = \lim_{j\to\infty} v_j$ is a p-harmonic function, and moreover $v_j \to V$ locally uniformly (see the proof of Theorem 6.14 in [20]). Since $u \leqslant V \leqslant U$, we have

$$\lim_{\theta\to 0+} V\big(re^{i\theta}\big) = 0 \quad \text{and} \quad \lim_{\theta\to \alpha-} V\big(re^{i\theta}\big) = A \quad \text{for } 0 < r < 1. \tag{4.3}$$

As $V(re^{i\theta}) = \lim_{t\to 0+} u(te^{i\theta})$ is independent of r, for $0 < \theta < \alpha$, Lemma 4.1 shows that $V = U$.

Let now $0 < \varepsilon < 1$. Then we can find $0 < \beta < \varepsilon\alpha/\max\{|A|, 1\}$ such that

$$\left|u\left(\frac{1}{2}e^{i\theta}\right)\right| < \varepsilon \quad \text{for } 0 < \theta < \beta \quad \text{and} \quad \left|u\left(\frac{1}{2}e^{i\theta}\right) - A\right| < \varepsilon \quad \text{for } \alpha - \beta < \theta < \alpha.$$

Since $u \leqslant v_j \leqslant U$ it follows that

$$\left|v_j\left(\frac{1}{2}e^{i\theta}\right)\right| < \varepsilon \quad \text{for } 0 < \theta < \beta \quad \text{and} \quad \left|v_j\left(\frac{1}{2}e^{i\theta}\right) - A\right| < \varepsilon \quad \text{for } \alpha - \beta < \theta < \alpha \tag{4.4}$$

for all j.

As $v_j \to U$ locally uniformly in Ω, we can find J such that $v_J > U - \varepsilon$ on $\{\frac{1}{2}e^{i\theta}:\ \beta \leqslant \theta \leqslant \alpha - \beta\}$. Together with (4.4) this shows that $U - \varepsilon < v_J \leqslant U$ on $\{\frac{1}{2}e^{i\theta}:\ 0 < \theta < \alpha\}$.

As $r \mapsto u(re^{i\theta})$ is a decreasing function, for each θ, this shows that $U(re^{i\theta}) - \varepsilon \leqslant u(re^{i\theta}) \leqslant U(re^{i\theta})$ for $0 < r < 2^{-(J+1)}$ and $0 < \theta < \alpha$. Hence

$$\liminf_{\Omega \ni z \to 0} \big(\underline{P} f(z) - U(z)\big) \geqslant \liminf_{\Omega \ni z \to 0} \big(u(z) - U(z)\big) \geqslant -\varepsilon.$$

Letting $\varepsilon > 0$ shows that

$$\liminf_{\Omega \ni z \to 0} \big(\underline{P} f(z) - U(z)\big) \geqslant 0.$$

Similarly one obtains that

$$\limsup_{\Omega \ni z \to 0} \big(\overline{P} f(z) - U(z)\big) \leqslant 0.$$

As $\underline{P} f \leqslant \overline{P} f$, (4.1) follows. □

Let us next weaken the assumptions by allowing for two different limits along the two rays instead of constant values along them. In Theorem 5.2 we improve upon this (without referring to Theorem 4.3) and we could therefore have omitted Theorem 4.3. However, when generalizing to higher dimensions, in Section 9.4, we can directly generalize Theorem 4.3 and its proof, but for Theorem 5.2 it is not at all clear how to approximate asymptotic cones with cones getting the necessary estimates working.

Theorem 4.3. *Let $A \in \mathbf{R}$ and $U(re^{i\theta}) = A\theta/\alpha$ for $0 < r < 1$ and $0 < \theta < 2\pi$. If $\alpha = 2\pi$ we require that $A = 0$. Let $f : \partial\Omega \to \mathbf{R}$ be bounded and such that $\lim_{t\to 0+} f(t) = 0$ and $\lim_{t\to 0+} f(te^{i\alpha}) = A$. Then*

$$\lim_{\Omega \ni z \to 0} \big(\underline{P} f(z) - U(z)\big) = \lim_{\Omega \ni z \to 0} \big(\overline{P} f(z) - U(z)\big) = 0. \tag{4.5}$$

Moreover, we have the following radial limits

$$\lim_{t\to 0+} \underline{P} f\big(te^{i\beta}\big) = \lim_{t\to 0+} \overline{P} f\big(te^{i\beta}\big) = \frac{A\beta}{\alpha} \quad \textit{for } 0 < \beta < \alpha. \tag{4.6}$$

Proof. Let us first observe that (4.6) follows from (4.5) as U has the same radial limits.

Let $\varepsilon > 0$. Then there is $r > 0$ such that

$$f(t) < \varepsilon \quad \text{and} \quad f\big(te^{i\alpha}\big) < A + \varepsilon \quad \text{for } 0 < t < r.$$

Let $M = \sup_{\partial\Omega} |f| < \infty$, $\Omega_r = r\Omega$ and

$$h\big(\rho e^{i\theta}\big) = \begin{cases} 0, & \text{if } \theta = 0 \text{ and } 0 < \rho < r, \\ A, & \text{if } \theta = \alpha \text{ and } 0 < \rho < r, \\ M, & \text{if } \rho = 0 \text{ or } \rho = r. \end{cases}$$

Let $u \in \mathcal{U}_h(\Omega_r)$ and

$$v = \begin{cases} \min\{u, M\}, & \text{in } \Omega_r, \\ M, & \text{in } \Omega \setminus \Omega_r, \end{cases}$$

which is superharmonic in Ω by the pasting Lemma 7.28 in Heinonen–Kilpeläinen–Martio [20]. Then $v + \varepsilon \in \mathcal{U}_f(\Omega)$ and in particular $u + \varepsilon \geqslant v + \varepsilon \geqslant \overline{P} f$ in Ω_r. Taking infimum over all $u \in \mathcal{U}_h(\Omega_r)$ we see that $\overline{P}_{\Omega_r} h + \varepsilon \geqslant \overline{P} f$ in Ω_r.

By Theorem 4.2 (applied to $z \mapsto h(z/r)$), we have

$$\limsup_{\Omega \ni z \to 0} \big(\overline{P} f(z) - U(z)\big) \leqslant \limsup_{\Omega \ni z \to 0} \big(\overline{P}_{\Omega_r} h(z) - U(z)\big) + \varepsilon = \varepsilon.$$

Letting $\varepsilon \to 0$ shows that

$$\limsup_{\Omega \ni z \to 0} \big(\overline{P} f(z) - U(z)\big) \leqslant 0.$$

The proof that

$$\liminf_{\Omega \ni z \to 0} \big(\underline{P} f(z) - U(z)\big) \geqslant 0$$

is similar (or follows by applying the above to $-f$). The inequality $\underline{P} f \leqslant \overline{P} f$ now completes the proof. □

5. Asymptotic corners

We now want to generalize Theorem 4.3 to boundary points which are locally asymptotically corners. Let us make the following definition. Assume in Sections 5–8 that $\Omega \subset \mathbf{R}^2$ is a nonempty bounded open set.

Definition 5.1. A boundary point $z_0 \in \partial\Omega$ is an *asymptotic corner point* with directions α and β if $\beta < \alpha < \beta + 2\pi$ and for every $\varepsilon > 0$ there is $\delta > 0$ so that

$$\begin{aligned} &\big\{z_0 + \rho e^{i\theta} \colon 0 < \rho < \delta \text{ and } \beta + \varepsilon < \theta < \alpha - \varepsilon\big\} \\ &\qquad \subset \Omega \cap B(z_0, \delta) \subset \big\{z_0 + \rho e^{i\theta} \colon 0 < \rho < \delta \text{ and } \beta - \varepsilon < \theta < \alpha + \varepsilon\big\}. \end{aligned} \tag{5.1}$$

We also say that a function f has a *jump* at z_0 with limits $a \in \mathbf{R}$ and $A \in \mathbf{R}$ if

$$\lim_{\substack{\partial\Omega \ni z \to z_0 \\ |\arg(z - z_0)e^{-i\beta}| < \gamma}} f(z) = a \quad \text{and} \quad \lim_{\substack{\partial\Omega \ni z \to z_0 \\ |\arg(z - z_0)e^{-i\alpha}| < \gamma}} f(z) = A,$$

where $\gamma = \frac{1}{4} \min\{\alpha - \beta, 2\pi + \beta - \alpha\}$.

Note that if the boundary $\partial\Omega$ is C^1 at $z_0 \in \partial\Omega$, then z_0 is an asymptotic corner point (with angle $\alpha - \beta = \pi$). Note also that we allow jumps to have zero height, so f being continuous at z_0 is considered as a jump (of zero height).

Theorem 5.2. *Let $0 < \alpha < 2\pi$ and $A \in \mathbf{R}$. Assume that $0 \in \partial\Omega$ is an asymptotic corner point with directions 0 and α. Let $f : \partial\Omega \to \mathbf{R}$ be a bounded function which has a jump at 0 with limits 0 and A. Let further $U(re^{i\theta}) = A\theta/\alpha$ for $r > 0$ and $-\gamma < \theta < \alpha + \gamma$, where $\gamma = \frac{1}{4} \min\{\alpha, 2\pi - \alpha\}$. Then*

$$\lim_{\Omega \ni z \to 0} \bigl(\underline{P} f(z) - U(z)\bigr) = \lim_{\Omega \ni z \to 0} \bigl(\overline{P} f(z) - U(z)\bigr) = 0. \tag{5.2}$$

Moreover, we have the following radial limits

$$\lim_{t \to 0+} \underline{P} f\bigl(te^{i\beta}\bigr) = \lim_{t \to 0+} \overline{P} f\bigl(te^{i\beta}\bigr) = \frac{A\beta}{\alpha} \quad \textit{for } 0 < \beta < \alpha. \tag{5.3}$$

Proof. Let us first observe that (5.3) follows from (5.2) as U has the same radial limits.

We may assume that $A \geqslant 0$. Let $M = \sup_{\partial\Omega} |f|$. If $M = 0$ then $f \equiv 0$ and the theorem is trivial. We may thus assume that $M > 0$.

Let $0 < \varepsilon < \min\{\gamma, M/(1 + 2A/\alpha)\}$ and let $\delta > 0$ be so small that (5.1) holds,

$$\bigl|f(z)\bigr| < \varepsilon \quad \text{for } z = \rho e^{i\theta} \in \partial\Omega \text{ with } 0 < \rho < \delta \text{ and } |\theta| < \gamma$$

and

$$\bigl|f(z) - A\bigr| < \varepsilon \quad \text{for } z = \rho e^{i\theta} \in \partial\Omega \text{ with } 0 < \rho < \delta \text{ and } |\theta - \alpha| < \gamma.$$

Let further $\widetilde{\Omega} = \{re^{i\theta}\colon\ 0 < r < \delta$ and $-\varepsilon < \theta < \alpha + \varepsilon\}$ and

$$h\bigl(re^{i\theta}\bigr) = \begin{cases} A - \varepsilon, & \text{if } 0 < r < \delta \text{ and } \theta = \alpha + \varepsilon, \\ -a, & \text{if } 0 < r < \delta \text{ and } \theta = -\varepsilon, \\ -M, & \text{if } r = 0 \text{ or } r = \delta, \end{cases}$$

where $a = \varepsilon + 2A\varepsilon/\alpha$. Let also $\widetilde{U}(re^{i\theta}) = A(\theta - \varepsilon)/\alpha - \varepsilon$ for $0 < r < \delta$ and $-\varepsilon < \theta < \alpha + \varepsilon$. Here we have chosen $\widetilde{U}$ and a so that $\widetilde{U}(re^{i\varepsilon}) = -\varepsilon$ and $\widetilde{U}(re^{-i\varepsilon}) = -a$ for $0 < r < \delta$, and $\widetilde{U} \leqslant f$ on $\widetilde{\Omega} \cap \partial\Omega$. Let $v \in \mathcal{L}_h(\widetilde{\Omega})$ and

$$\tilde{v} = \begin{cases} \max\{v, -M\}, & \text{in } \widetilde{\Omega}, \\ -M, & \text{in } \Omega \setminus \widetilde{\Omega}, \end{cases}$$

which is subharmonic in $\widetilde{\Omega} \cup \Omega$ by the pasting Lemma 7.28 in Heinonen–Kilpeläinen–Martio [20]. As $\widetilde{U} \in \mathcal{U}_h(\widetilde{\Omega})$ we have that $v \leqslant \widetilde{U}$ in $\widetilde{\Omega}$. Hence $\tilde{v} \in \mathcal{L}_f(\Omega)$ and in particular $\tilde{v} \leqslant \underline{P} f$ in Ω.

Let $V = \underline{P}_{\widetilde{\Omega}} h = \sup_{v \in \mathcal{L}_h(\widetilde{\Omega})} v \leqslant \underline{P} f$ in $\widetilde{\Omega} \cap \Omega$. By Theorem 4.2,

$$\lim_{\widetilde{\Omega} \ni z \to 0} \bigl(V(z) - \widetilde{U}(z)\bigr) = 0.$$

It follows that

$$\liminf_{\Omega \ni z \to 0} \bigl(\underline{P} f(z) - U(z)\bigr) \geqslant \liminf_{\Omega \ni z \to 0} \bigl(V(z) - \widetilde{U}(z) + \widetilde{U}(z) - U(z)\bigr) = -\frac{A\varepsilon}{\alpha} - \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain that

$$\liminf_{\Omega \ni z \to 0} \bigl(\underline{P} f(z) - U(z)\bigr) \geqslant 0.$$

Similarly

$$\liminf_{\Omega \ni z \to 0} \bigl(\overline{P} f(z) - U(z)\bigr) \leqslant 0.$$

As $\underline{P} f \leqslant \overline{P} f$ this completes the proof. $\square$

The reader may have noticed that in Theorems 4.2, 4.3 and 5.2, the actual value $f(0)$ played no role in the conclusion. We want to exploit this fact. So far we have assumed f to be bounded, but in fact for $f(0)$ this is not essential. Theorems 4.2, 4.3 and 5.2 are local in nature, but we may of course apply them at several different boundary points. For this reason we want to, and can, allow f to be unbounded on a countable set of jump points, without changing the conclusion. Combining this with Lemma 2.10 we can in fact also allow for an exceptional set of capacity zero. Thus we are in quite different situations when $p > 2$ and $p \leqslant 2$: In the former case, $p > 2$, we allow for an exceptional set (where f is allowed to be infinite) which is countable and thus of positive capacity (unless empty). In the latter case, $p \leqslant 2$, we allow for a set of capacity zero, which in particular includes any countable set. We formulate such results in Sections 6 and 7, respectively.

Such perturbation results are of interest also when the jump is zero, i.e. when the boundary values are continuous apart from the value at the boundary point under consideration. In such situations our approximation arguments become simpler and we can be more general by not requiring the boundary point to be an asymptotic corner, it is enough to have an exterior ray at the point. We now formulate this for one boundary point and a bounded function f, corresponding to the situation in Theorem 5.2. In the next two sections we extend this to unbounded perturbations and combine it with perturbations at jump points.

Note that if $p \leqslant 2$, when points have zero capacity, then Theorem 5.4 below follows from Lemma 2.10. Thus the interest lies primarily in the case when $p > 2$. This result is also the essential ingredient in the proof of Theorem 1.3 (for $n = 2$).

Definition 5.3. A boundary point $z_0 \in \partial\Omega$ is an *exterior ray point* with direction α if there is $\delta > 0$ such that

$$\big\{z_0 + re^{i\alpha}\colon\ 0 \leqslant r < \delta\big\} \subset \mathbf{R}^2 \setminus \Omega. \tag{5.4}$$

Theorem 5.4. *Assume that* 0 *is an exterior ray point. Let* $f : \partial\Omega \to \mathbf{R}$ *be a bounded function satisfying*

$$\lim_{\partial\Omega \ni z \to 0} f(z) = 0.$$

Then

$$\lim_{\Omega \ni z \to 0} \underline{P} f(z) = \lim_{\Omega \ni z \to 0} \overline{P} f(z) = 0. \tag{5.5}$$

Proof. Let $M = \sup_{\partial\Omega} |f|$. If $M = 0$ then $f \equiv 0$ and the theorem is trivial. We may thus assume that $M > 0$ and may also assume that 0 is an exterior ray point with direction 0.

Let $0 < \varepsilon < M$ and let $\delta > 0$ be so small that (5.4) holds and

$$\big|f(z)\big| < \varepsilon \quad \text{for } z \in \partial\Omega \cap \big(B(0,\delta) \setminus \{0\}\big).$$

Let $\widetilde{\Omega} = \{\rho e^{i\theta}\colon\ 0 < \rho < \delta \text{ and } 0 < \theta < 2\pi\}$ and

$$h\big(re^{i\theta}\big) = \begin{cases} -\varepsilon, & \text{if } 0 < r < \delta \text{ and } \theta = 0, \\ -M, & \text{if } r = 0 \text{ or } r = \delta. \end{cases}$$

Let also $\widetilde{U} \equiv -\varepsilon$. Let $v \in \mathcal{L}_h(\widetilde{\Omega})$ and

$$\tilde{v} = \begin{cases} \max\{v, -M\}, & \text{in } \widetilde{\Omega}, \\ -M, & \text{in } \Omega \setminus \widetilde{\Omega}, \end{cases}$$

which is subharmonic in $\widetilde{\Omega} \cup \Omega$ by the pasting Lemma 7.28 in Heinonen–Kilpeläinen–Martio [20]. As $\widetilde{U} \in \mathcal{U}_h(\widetilde{\Omega})$ we have that $v \leqslant \widetilde{U}$ in $\widetilde{\Omega}$. Hence $\tilde{v} \in \mathcal{L}_f(\Omega)$ and in particular $\tilde{v} \leqslant \underline{P} f$ in Ω.

Let $V = \underline{P}_{\widetilde{\Omega}} h = \sup_{v \in \mathcal{L}_h(\widetilde{\Omega})} v \leqslant \underline{P} f$ in $\widetilde{\Omega}$. By Theorem 4.2,

$$\lim_{\widetilde{\Omega} \ni z \to 0} \big(V(z) - \widetilde{U}(z)\big) = 0.$$

It follows that

$$\liminf_{\Omega \ni z \to 0} \underline{P} f(z) \geqslant \liminf_{\Omega \ni z \to 0} \big(V(z) - \widetilde{U}(z) + \widetilde{U}(z)\big) = -\varepsilon.$$

Letting $\varepsilon \to 0$, we obtain that

$$\liminf_{\Omega \ni z \to 0} \underline{P} f(z) \geqslant 0.$$

Similarly

$$\limsup_{\Omega \ni z \to 0} \overline{P} f(z) \leqslant 0.$$

As $\underline{P} f \leqslant \overline{P} f$ this completes the proof. □

6. Perturbations, the case $2 < p < \infty$

As already mentioned our aim now is to improve upon Theorem 5.2 by allowing the boundary function to be unbounded or even infinite on a countable set. The cases $p > 2$ and $p < 2$ are quite different and in this section we concentrate on the former case, leaving the latter case to the next section. We therefore assume that $2 < p < \infty$ in this section. In particular all points have positive capacity and are thus regular, by the Kellogg property (Theorem 2.5).

Theorem 6.1. *Let $0 < \alpha < 2\pi$ and $A \in \mathbf{R}$. Assume that $0 \in \partial\Omega$ is an asymptotic corner point with directions 0 and α. Let $f : \partial\Omega \to \mathbf{R}$ be a bounded function which has a jump at 0 with limits 0 and A. Let further $U(re^{i\theta}) = A\theta/\alpha$ for $r > 0$ and $-\gamma < \theta < \alpha + \gamma$, where $\gamma = \frac{1}{4}\min\{\alpha, 2\pi - \alpha\}$.*

Let $E \subset \partial\Omega$ be a finite or countable set of exterior ray points, and h be a function such that $h = f$ on $\partial\Omega \setminus E$. Then

$$\lim_{\Omega \ni z \to 0} \big(\underline{P} h(z) - U(z)\big) = \lim_{\Omega \ni z \to 0} \big(\overline{P} h(z) - U(z)\big) = 0. \tag{6.1}$$

Moreover, we have the following radial limits

$$\lim_{t \to 0+} \underline{P} h\big(te^{i\beta}\big) = \lim_{t \to 0+} \overline{P} h\big(te^{i\beta}\big) = \frac{A\beta}{\alpha} \quad \text{for } 0 < \beta < \alpha. \tag{6.2}$$

Note that in general it is not known if $\overline{P} f = \lim_{m \to \infty} \overline{P} \min\{f, m\}$, see Open problem 3.5, which makes it necessary to use the induction in the proof below.

Proof of Theorem 6.1. As before the radial limits (6.2) follow from the limits (6.1).

Let $\varepsilon > 0$ be small enough. Then we can find a function $k \geqslant f$ on $\partial\Omega$ such that k is continuous on $\partial\Omega \setminus \{0\}$ and has a jump at 0 with limits 0 and A, and $\max\{A, 0\} \leqslant k(0) < \infty$. Note in particular that k is upper semicontinuous at 0.

Let $z_0 \in \Omega$, let $Z \ni z_0$ be a set containing one point from each component of Ω, and let $z_0 \in Z_0 \subset Z_1 \subset \cdots \subset Z = \bigcup_{j=1}^{\infty} Z_j$ be an increasing sequence of finite sets. Let also $\{x_j\}_{j=0}^{\infty}$ be a sequence of points in $E \cup \{0\}$ such that each point in $E \cup \{0\}$ appears infinitely many times.

We want to construct an increasing sequence $\{k_j\}_{j=1}^{\infty}$ of bounded functions on $\partial\Omega$ such that for each nonnegative integer j,

(i) $k_0 = k$;
(ii) $k_{j+1} - k_j \in C(\partial\Omega)$;
(iii) $k_j \leqslant k_{j+1} \leqslant k_j + 1$;
(iv) $Pk_{j+1}(z) \leqslant Pk_j(z) + 2^{-j}\varepsilon$ for $z \in Z_j$;
(v) $k_{j+1}(x_j) = k_j(x_j) + 1$.

We proceed by induction and assume that k_j has been constructed for some nonnegative integer j. (The initial step is of course to let $k_0 = k$.)

Let

$$\tilde{k}_j = k_j + 2\chi_{\{x_j\}}.$$

By Theorem 5.2 (applied to both k_j and $\tilde{k}_j$), we have

$$\lim_{\Omega \ni y \to 0} \bigl(Pk_j(y) - P\tilde{k}_j(y)\bigr) = 0.$$

Theorem 5.4 (applied to both k_j and $\tilde{k}_j$) yields

$$\lim_{\Omega \ni y \to x_j} P\tilde{k}_j(y) = \lim_{\Omega \ni y \to x_j} Pk_j(y) = k_j(x_j), \quad \text{if } x_j \neq 0.$$

On the other hand, by Lemma 2.10 (applied to both k_j and $\tilde{k}_j$),

$$\lim_{\Omega \ni y \to x} P\tilde{k}_j(y) = \lim_{\Omega \ni y \to x} Pk_j(y) = k_j(x) \quad \text{for all } x \in \partial\Omega \setminus \{0, x_j\}.$$

(Recall that all boundary points are regular.) By Theorem 3.1, $P\tilde{k}_j \equiv Pk_j$.

Hence, for $z \in Z_j$ we can find $u_z \in \mathcal{U}_{\tilde{k}_j}$ such that

$$u_z(z) < P\tilde{k}_j(z) + \frac{\varepsilon}{2^j} = Pk_j(z) + \frac{\varepsilon}{2^j}.$$

Then $u := \min_{z \in Z_j} u_z \in \mathcal{U}_{\tilde{k}_j}$ as Z_j is finite. Extend u to $\partial\Omega$ by

$$u(x) = \liminf_{\Omega \ni y \to x} u(y), \quad x \in \partial\Omega.$$

Then u is lower semicontinuous on $\overline{\Omega}$.

As u is lower semicontinuous, k_j is upper continuous, and $u(x_j) \geqslant \tilde{k}_j(x_j) = k_j(x_j) + 2$, there is $0 < r < 1$ such that

$$u > k_j + 1 \quad \text{on } B(x_j, r) \cap \partial\Omega.$$

Let

$$k_{j+1}(x) = k_j(x) + \left(1 - \frac{|x - x_j|}{r}\right)_+, \quad x \in \partial\Omega.$$

Then $u \geqslant k_{j+1}$ on $\partial\Omega$. Hence $u \in \mathcal{U}_{k_{j+1}}$ and

$$Pk_{j+1}(z) \leqslant u(z) < Pk_j(z) + \frac{\varepsilon}{2^j} \quad \text{for } z \in Z_j.$$

That the other requirements on k_{j+1} are fulfilled is clear. We have therefore completed the construction of the sequence $\{k_j\}_{j=0}^\infty$.

It follows directly that $\{Pk_j\}_{j=0}^\infty$ is an increasing sequence of p-harmonic functions in Ω. Let $v = \lim_{j\to\infty} Pk_j$. For $z \in Z$ and $j > m_z := \inf\{j \geqslant 0\colon\ z \in Z_j\}$ we have

$$Pk_j(z) < Pk_{m_z}(z) + \varepsilon \sum_{k=m_z}^{j-1} 2^{-j} < Pk_{m_z}(z) + 2\varepsilon,$$

and thus

$$v(z) \leqslant Pk_{m_z}(z) + 2\varepsilon < \infty.$$

Harnack's convergence theorem (see Theorem 6.14 in Heinonen–Kilpeläinen–Martio [20]) shows that v is p-harmonic in Ω. We next want to show that $v \in \mathcal{U}_h$. For $x \in \partial\Omega \setminus (E \cup \{0\})$ we have

$$\liminf_{\Omega\ni y\to x} v(y) \geqslant \liminf_{\Omega\ni y\to x} Pk(y) = k(x) \geqslant h(x).$$

On the other hand, for $x \in E \setminus \{0\}$,

$$\liminf_{\Omega\ni y\to x} v(y) \geqslant \lim_{j\to\infty} \liminf_{\Omega\ni y\to x} Pk_j(y) = \lim_{j\to\infty} k_j(x) = \infty.$$

Also

$$\liminf_{\Omega\ni y\to 0} v(y) \geqslant \lim_{j\to\infty} \liminf_{\Omega\ni y\to 0} Pk_j(y) = \lim_{j\to\infty} k_j(0) - k(0) + \min\{A, 0\} = \infty.$$

Thus $v \in \mathcal{U}_h$. In particular

$$\overline{P}h(z_0) \leqslant v(z_0) \leqslant Pk(z_0) + 2\varepsilon.$$

Letting $\varepsilon \to 0$ shows that $\overline{P}h(z_0) \leqslant Pk(z_0)$, and as $z_0 \in \Omega$ was arbitrary we find that

$$\overline{P}h \leqslant Pk \quad \text{in } \Omega. \tag{6.3}$$

It follows that

$$\limsup_{\Omega\ni z\to 0} \big(\underline{P}h(z) - U(z)\big) \leqslant \limsup_{\Omega\ni z\to 0} \big(\overline{P}h(z) - U(z)\big) \leqslant \limsup_{\Omega\ni z\to 0} \big(Pk(z) - U(z)\big) = 0,$$

by Theorem 5.2. Applying this also to $-h$ gives (6.1) completing the proof. $\square$

Let us next generalize Theorem 5.4 in a similar way.

Theorem 6.2. *Assume that $0 \in \partial\Omega$ is an exterior ray point. Let $f : \partial\Omega \to \mathbf{R}$ be a bounded function which is continuous at* 0*. Let $E \subset \partial\Omega$ be a finite or countable set of exterior ray points, and h be a function such that $h = f$ on $\partial\Omega \setminus E$. Then*

$$\lim_{\Omega \ni z \to 0} \underline{P}h(z) = \lim_{\Omega \ni z \to 0} \overline{P}h(z) = f(0).$$

Proof. The proof is the same as the proof of Theorem 6.1, we just need to replace the usage of Theorem 5.2 by the use of Theorem 5.4 (twice). $\square$

We can now obtain an invariance result for certain perturbations on finite or countable sets.

Theorem 6.3. *Let $E_1 \subset \partial\Omega$ be a finite or countable set of asymptotic corner points and $E_2 \subset \partial\Omega$ be a finite or countable set of exterior ray points. Let f be a bounded function on $\partial\Omega$ which is continuous at all points in $\partial\Omega \setminus E_1$ and which has jumps at all points in E_1.*

Let further h be a function on $\partial\Omega$ such that $f = h$ on $\partial\Omega \setminus (E_1 \cup E_2)$. Then

$$Ph = Pf.$$

In particular both f and h are resolutive.

Note that we do not require h to be continuous on $\partial\Omega \setminus (E_1 \cup E_2)$.

Proof of Theorem 6.3. By Theorem 5.2, we have

$$\lim_{\Omega \ni y \to x} \bigl(\overline{P}f(y) - \underline{P}f(y)\bigr) = 0 \quad \text{for } x \in E_1.$$

On the other hand, Lemma 2.10 shows that

$$\lim_{\Omega \ni y \to x} \overline{P}f(y) = \lim_{\Omega \ni y \to x} \underline{P}f(y) = f(x) \quad \text{for } x \in \partial\Omega \setminus E_1.$$

It thus follows from Theorem 3.1 that $\overline{P}f = \underline{P}f$ and thus f is resolutive.

By Theorem 6.1, we have

$$\lim_{\Omega \ni y \to x} \bigl(\overline{P}f(y) - \overline{P}h(y)\bigr) = 0 \quad \text{for } x \in E_1.$$

Let next $x \in \Omega \setminus E_1$. As f is continuous at x and bounded we can find a function $\varphi \in C(\partial\Omega)$ such that $\varphi \geqslant f$ on $\partial\Omega$ and $\varphi(x) = f(x)$. Let also $\psi = \max\{\varphi, h\}$. Fix $z_0 \in \Omega$ and let $\varepsilon > 0$. By the proof of Theorem 6.2 (we let $k = \varphi$) there is $v \in \mathcal{U}_\psi$ such that $v(z_0) \leqslant P\varphi(z_0) + 2\varepsilon$. Letting $\varepsilon \to 0$ and varying z_0 shows that $\overline{P}h \leqslant \overline{P}\psi \leqslant P\varphi$.

Lemma 2.10 shows that

$$\limsup_{\Omega \ni y \to x} \overline{P}h(y) \leqslant \lim_{\Omega \ni y \to x} P\varphi(y) = \varphi(x) = f(x) = \lim_{\Omega \ni y \to x} \overline{P}f(y).$$

Hence Theorem 3.1 shows that $\overline{P}h \leqslant \overline{P}f$.

Applying this also to $-h$ and $-f$ concludes the proof. $\square$

Example 6.4. Let $\Omega = \{z \in \mathbf{C}\colon\ 0 < |z| < 1\}$ and let $f \in C(\partial\Omega)$. Let further $E \subset \partial\Omega \setminus \{0\}$ be a finite or countable set and $h = f$ on $\partial\Omega \setminus E$. Theorem 6.3 shows that $Ph \equiv Pf$.

A natural question is if this may be true also if $E = \{0\}$. Clearly 0 is not an exterior ray point and Theorem 6.3 is not at our disposal. The reason is that the conclusion is not true in this case. Let $f \equiv 0$ and $h = \chi_{\{0\}}$. As 0 is a regular boundary point, by the Kellogg property (Theorem 2.5), we have $\lim_{\Omega \ni y \to 0} Ph(y) = 1$, while $Pf \equiv 0$, showing that $Pf \not\equiv Ph$.

The conclusion is that it is necessary to have some type of geometric condition on the points in E_1 and E_2 in Theorem 6.3.

As a consequence of Theorem 6.3 we have the following uniqueness result.

Theorem 6.5. *Let $E_1 \subset \partial\Omega$ be a finite or countable set of asymptotic corner points, $E_2 \subset \partial\Omega$ be a finite or countable set of exterior ray points, and $E = E_1 \cup E_2$. Let f be a bounded function on $\partial\Omega$ which is continuous at all points in $\partial\Omega \setminus E_1$ and which has jumps at all points in E_1.*

Let further h be a function on $\partial\Omega$ such that $f = h$ on $\partial\Omega \setminus E$. Let finally V be a bounded p-harmonic function in Ω. Then $V = Ph$ if and only if

$$\lim_{\Omega \ni y \to x} V(y) = h(x) \quad \textit{for } x \in \partial\Omega \setminus E. \tag{6.4}$$

Of course a particular application is to let $h \equiv f$.

Proof of Theorem 6.5. Assume first that $V = Ph$. By Theorem 6.3, $V = Pf$. Thus Lemma 2.10 shows that

$$\lim_{\Omega \ni y \to x} V(y) = f(x) = h(x) \quad \text{for } x \in \partial\Omega \setminus E.$$

Assume conversely that (6.4) holds. Let $\varphi = f - \infty\chi_E$ and $\psi = f + \infty\chi_E$. Then $V \in \mathcal{U}_\varphi$ and $V \in \mathcal{L}_\psi$ so that, by Theorem 6.3,

$$V \leqslant P\psi = Ph = P\varphi \leqslant V. \qquad \square$$

7. Perturbations, the case $1 < p < 2$

In this section we assume that $1 < p < 2$ (apart from in the proof at the very end of this section) and our aim is to obtain similar results to those in Section 6 for this case. In particular all points have zero capacity. Theorems 7.2 and 7.3 are easy to obtain for $p = 2$ using linearity. However the proofs given here are not valid for $p = 2$ as Lemma 7.1 fails in this case.

Lemma 7.1. *Let $0 < \alpha < 2\pi$ and*

$$f_\alpha(re^{i\theta}) = \begin{cases} \theta/\alpha, & 0 \leqslant \theta \leqslant \alpha \textit{ and } r > 0, \\ \frac{2\pi - \theta}{2\pi - \alpha}, & \alpha \leqslant \theta < 2\pi \textit{ and } r > 0, \\ 0, & r = 0. \end{cases}$$

Then $f_\alpha \in N^{1,p}(\mathbf{D})$, where $\mathbf{D}$ is the unit disc in the complex plane $\mathbf{C} = \mathbf{R}^2$.

Proof. Let $M = \max\{1/\alpha, 1/(2\pi - \alpha)\}$. Then $|\nabla f_\alpha(re^{i\theta})| \leqslant M/r$, $r > 0$, from which it is easy to see that $f_\alpha \in W^{1,p}(\mathbf{D})$. As $f_\alpha|_{\mathbf{D}\setminus\{0\}}$ is continuous and $C_p(\{0\}) = 0$, f_α is quasicontinuous. Hence $f_\alpha \in N^{1,p}(\mathbf{D})$. $\square$

Theorem 7.2. *Let $0 < \alpha < 2\pi$ and $A \in \mathbf{R}$. Assume that $0 \in \partial\Omega$ is an asymptotic corner point with directions 0 and α. Let $f : \partial\Omega \to \mathbf{R}$ be a bounded function which has a jump at 0 with limits 0 and A. Let further $U(re^{i\theta}) = A\theta/\alpha$ for $r > 0$ and $-\gamma < \theta < \alpha + \gamma$, where $\gamma = \frac{1}{4}\min\{\alpha, 2\pi - \alpha\}$.*

Let $E \subset \partial\Omega$ be a set with $C_p(E) = 0$, and h be a function such that $h = f$ on $\partial\Omega \setminus E$. Then

$$\lim_{\Omega \ni z \to 0} \big(\underline{P}h(z) - U(z)\big) = \lim_{\Omega \ni z \to 0} \big(\overline{P}h(z) - U(z)\big) = 0. \tag{7.1}$$

Moreover, we have the following radial limits

$$\lim_{t \to 0+} \underline{P}h\big(te^{i\beta}\big) = \lim_{t \to 0+} \overline{P}h\big(te^{i\beta}\big) = \frac{A\beta}{\alpha} \quad \text{for } 0 < \beta < \alpha. \tag{7.2}$$

Proof. As before the radial limits (7.2) follow from the limits (7.1). We may also assume that $0 \in E$.

Let $\varepsilon > 0$ be small enough. Then we can find a function $k \geqslant f$ on $\partial\Omega$ such that k is continuous on $\partial\Omega \setminus \{0\}$ and has a jump at 0 with limits 0 and A, and $k(0) = 0$.

It follows that $k - A\eta f_\alpha \in C(\partial\Omega)$, where f_α is given by Lemma 7.1 and $\eta(z) = \min\{2 - |z|, 1\}_+$. Thus there is a Lipschitz function φ on $\mathbf{R}^2$, with compact support, such that $k - A\eta f_\alpha \leqslant \varphi \leqslant k - A\eta f_\alpha + \varepsilon$ on $\partial\Omega$.

Let $\psi = \varphi + A\eta f_\alpha$ and $\tilde{\psi} = \psi + \infty\chi_E$. Then $k \leqslant \psi \leqslant k + \varepsilon$ on $\partial\Omega$ and $\psi \in N^{1,p}(\mathbf{R}^2)$. By Theorem 2.4, $P\tilde{\psi} = P\psi$. Moreover $\tilde{\psi} \geqslant h$ on $\partial\Omega$. We conclude that

$$\begin{aligned}\limsup_{\Omega \ni z \to 0} \big(\overline{P}h(z) - U(z)\big) &\leqslant \limsup_{\Omega \ni z \to 0} \big(P\tilde{\psi}(z) - U(z)\big) = \limsup_{\Omega \ni z \to 0} \big(P\psi(z) - U(z)\big) \\ &\leqslant \limsup_{\Omega \ni z \to 0} \big(\overline{P}k(z) - U(z)\big) + \varepsilon \leqslant \varepsilon,\end{aligned}$$

where the last inequality follows from Theorem 5.2. Letting $\varepsilon \to 0$ yields

$$\limsup_{\Omega \ni z \to 0} \big(\underline{P}h(z) - U(z)\big) \leqslant \limsup_{\Omega \ni z \to 0} \big(\overline{P}h(z) - U(z)\big) = 0.$$

Applying this also to $-h$ gives (7.1), completing the proof. □

Theorem 7.3. *Let $E \subset \partial\Omega$ be a finite or countable set of asymptotic corner points. Let f be a bounded function on $\partial\Omega$ which is continuous at all points in $\partial\Omega \setminus E$ and which has jumps at all points in E.*

Let further h be a function on $\partial\Omega$ such that $f = h$ on $\partial\Omega \setminus \widetilde{E}$, where $C_p(\widetilde{E}) = 0$. Then

$$Ph = Pf.$$

In particular both f and h are resolutive.

Note that the boundary behaviour of $Pf = Ph$ is described by Theorem 7.2 for $x_0 \in E$. If $x_0 \in \partial\Omega \setminus E$ then $\lim_{\Omega \ni y \to x_0} Pf(y) = f(x_0)$ if x_0 is regular (with respect to Ω), the limit $\lim_{\Omega \ni y \to x_0} Pf(y)$ exists, if x_0 is semiregular, and there is a sequence $\Omega \ni y_j \to x_0$ such that $\lim_{j \to \infty} Pf(y_j) = f(x_0)$, if x_0 is strongly irregular. See Björn [9, Theorem 8.7] for a proof of a somewhat stronger statement.

Proof of Theorem 7.3. We may assume that $0 \leqslant f \leqslant 1$. Fix a nonnegative integer j, and let $E_j \subset E$ be the set of points at which f has a jump A of size $2^{-j-1} < |A| \leqslant 2^{-j}$. Then E_j is a finite set, as if not then E_j would have a limit point $x \in \partial\Omega$ at which f would neither be continuous nor have a jump. (We of course have the possibility that $E_j = \emptyset$.)

Let $x_{j,1},\dots,x_{j,N_j}$ be the points in E_j. For each $k=1,\dots,N_j$ we can find a function $f_{j,k}(z)=A_{j,k}f_{\alpha_{j,k}}(e^{i\theta_{j,k}}(z-x_{j,k}))$ such that $f-f_{j,k}$ is continuous at $x_{j,k}$, where $2^{-j-1}<|A_{j,k}|\leqslant 2^{-j}$, $\alpha_{j,k}$ and $\theta_{j,k}$ are chosen appropriately and $f_{\alpha_{j,k}}$ is the function in Lemma 7.1. Let $d_j=\min\{|x-y|\colon x,y\in E_j$ and $x\neq y\}$ if E_j has at least two points, and let $d_j=1$ otherwise. Let further

$$f_j(z)=\sum_{k=1}^{N_j} f_{j,k}(z)\min\left\{2-\frac{5|z-x_{j,k}|}{d_j},1\right\}_+.$$

Note that the terms in the sum have disjoint support and thus $0\leqslant|f_j|\leqslant 2^{-j}$. Lemma 7.1 shows that $f_j\in N^{1,p}(\mathbf{R}^2)$ (with compact support). It is now easy to see that $\rho:=f-\sum_{j=0}^{\infty}f_j\in C(\partial\Omega)$.

Let $\varepsilon>0$. Then we can find a nonnegative integer N such that $2^{-N}<\varepsilon$. Moreover there is a Lipschitz function φ on $\mathbf{R}^2$, with compact support, such that $\rho\leqslant\varphi\leqslant\rho+\varepsilon$ on $\partial\Omega$. Let $\psi=\varphi+\sum_{j=0}^{N}f_j$ and $\tilde\psi=\psi+\infty\chi_{\widetilde E}$. Then $\psi-2\varepsilon\leqslant f\leqslant\psi+2\varepsilon$ and $h\leqslant\tilde\psi+2\varepsilon$. Moreover $\psi\in N^{1,p}(\mathbf{R}^2)$ (with compact support).

We thus get that

$$P\psi-2\varepsilon\leqslant \underline{P}f\leqslant\overline{P}f\leqslant P\psi+2\varepsilon.$$

Letting $\varepsilon\to 0$ shows that f is resolutive.

By Theorem 2.4, $P\tilde\psi=P\psi$ from which we see that

$$\overline{P}h\leqslant P\tilde\psi+2\varepsilon=P\psi+2\varepsilon\leqslant Pf+4\varepsilon.$$

Again letting $\varepsilon\to 0$ shows that $\overline{P}h\leqslant Pf$. Applying this also to $-h$ concludes the proof. $\square$

We also have the following uniqueness result.

Theorem 7.4. *Let $E\subset\partial\Omega$ be a finite or countable set of asymptotic corner points. Let f be a bounded function on $\partial\Omega$ which is continuous at all points in $\partial\Omega\setminus E$ and which has jumps at all points in E.*

Let further h be a function on $\partial\Omega$ such that $f=h$ on $\partial\Omega\setminus\widetilde E$, where $C_p(\widetilde E)=0$. Let finally V be a bounded p-harmonic function in Ω. Then $V=Ph$ if and only if

$$\lim_{\Omega\ni y\to x}V(y)=h(x)\quad\textit{for q.e. } x\in\partial\Omega. \tag{7.3}$$

Of course a particular application is to let $h\equiv f$.

Proof of Theorem 7.4. Assume first that $V=Ph$. By Theorem 7.3, $V=Pf$. Thus Lemma 2.10 shows that

$$\lim_{\Omega\ni y\to x}V(y)=f(x)=h(x)\quad\text{for } x\in\partial\Omega\setminus(E\cup\widetilde E).$$

As $C_p(E\cup\widetilde E)=0$, we have shown that (7.3) holds.

Assume conversely that (7.3) holds. There is then a set $E'\supset E\cup\widetilde E$ such that $C_p(E')=0$ and

$$\lim_{\Omega\ni y\to x}V(y)=h(x)\quad\text{for } x\in\partial\Omega\setminus E'.$$

Let $\varphi=f-\infty\chi_{E'}$ and $\psi=f+\infty\chi_{E'}$.

Then $V \in \mathcal{U}_\varphi$ and $V \in \mathcal{L}_\psi$ so that, by Theorem 7.3,

$$V \leqslant P\psi = Ph = P\varphi \leqslant V. \qquad \square$$

We can now also give a proof of Theorem 1.3 when $n = 2$.

Proof of Theorem 1.3 when $n = 2$. For $p > 2$ this is a special case of Theorem 6.3, whereas for $p < 2$ it is a special case of Theorem 7.3, which holds, by linearity, also for $p = 2$. $\square$

8. Baernstein's problem

We are now prepared to fully answer the question of Baernstein, in the affirmative.

Proof of Theorem 1.2. Let $f = \chi_G$ and $h = \chi_{\overline{G}}$. That f and h are resolutive follows from Theorem 2.9, but we also obtain it directly from our proof. As $\underline{P}f \leqslant \underline{P}h \leqslant \overline{P}h$ and $\underline{P}f \leqslant \overline{P}f \leqslant \overline{P}h$ it is enough to show that $\underline{P}f \geqslant \overline{P}h$.

Lemma 2.10 shows that

$$\lim_{\Omega \ni y \to x} \underline{P}f(y) = \lim_{\Omega \ni y \to x} \overline{P}h(y) = 1 \quad \text{for } x \in G,$$

and that

$$\lim_{\Omega \ni y \to x} \underline{P}f(y) = \lim_{\Omega \ni y \to x} \overline{P}h(y) = 0 \quad \text{for } x \in \partial\mathbf{D} \setminus \overline{G}.$$

A simple application of Theorem 5.2 (with $\alpha = \pi$) shows that

$$\lim_{\Omega \ni y \to x} \big(\underline{P}f(y) - \overline{P}h(y)\big) = 0 \quad \text{for } x \in \overline{G} \setminus G.$$

By Theorem 3.1, it follows that $\underline{P}f \geqslant \overline{P}h$. (Note that $S = \emptyset$ in this case.) $\square$

Theorem 8.1. *Let $G \subset \partial\Omega$ be relatively open and such that $\partial_{\partial\Omega} G$ is a finite set consisting only of asymptotic corner points* (*with respect to Ω*)*. Then*

$$\omega_{a,p}(G; \Omega) = \omega_{a,p}(\overline{G}; \Omega) \quad \textit{for all } a \in \Omega.$$

Note that it follows that χ_G and $\chi_{\overline{G}}$ are resolutive, which does not follow from Theorem 2.9 when Ω is not regular.

The proof of Theorem 1.2 directly works also to prove this result in the case when Ω is regular, and thus whenever $p > 2$. Using the full power of Theorem 3.1, the proof is easily modified to also handle the case when Ω is semiregular.

When Ω is not semiregular we do not have a suitable comparison lemma available to give a similar proof. Nevertheless the result is true: it is a special case of Theorem 7.3 in this case. Note that the proof in this case is quite different.

Proof of Theorem 8.1. If $p > 2$, then we can either apply the proof of Theorem 1.2 or observe that this result is a special case of Theorem 6.3.

If $p < 2$, then this is a special case of Theorem 7.3.

For $p = 2$ this is well known and easy to obtain using linearity. $\square$

9. Generalizations

9.1. Logarithmic spirals

Let $\varphi, \psi : (0,1] \to \mathbf{R}$ be given by

$$\varphi(t) = a \log t \quad \text{and} \quad \psi(t) = \varphi(t) + b \quad \text{for } 0 < t \leqslant 1,$$

where $a \in \mathbf{R}$ and $0 < b < 2\pi$. Let further

$$\Omega = \{re^{i\theta}\colon\ 0 < r < 1 \text{ and } \varphi(r) < \theta < \psi(r)\}.$$

Then Ω is a selfsimilar spiral in the sense that

$$\Omega \cap B(0,\eta) = \{\eta x e^{i\varphi(\eta)}\colon\ x \in \Omega\} \quad \text{for } 0 < \eta \leqslant 1.$$

Let now $f : \partial\Omega \to \mathbf{R}$ be a bounded function such that

$$\lim_{t\to 0+} f\big(te^{i\varphi(t)}\big) = 0 \quad \text{and} \quad \lim_{t\to 0+} f\big(te^{i\psi(t)}\big) = A.$$

The arguments in the proofs of Theorems 4.2 and 4.3 show that $\overline{P}f$ tends to a p-harmonic function U which is independent of the radius in the sense that there is a function $F : [0,b] \to \mathbf{R}$ such that

$$U\big(te^{i\theta}\big) = F\big(\theta - \varphi(t)\big) \quad \text{for } 0 < t < 1 \text{ and } \varphi(t) < \theta < \psi(t).$$

This is done under the requirement that there is a unique such p-harmonic function U.

As p-harmonic functions on (unweighted) $\mathbf{R}^n$ are $C^{1,\alpha}_{\mathrm{loc}}$, by Lewis [27, Theorem 1] we see that $F \in C^1(\Omega)$ and moreover takes the boundary values $F(0) = 0$ and $F(b) = A$ continuously. Furthermore, by Aronsson–Lindqvist [2, Corollary on p. 161] the set

$$\begin{aligned} E &= \{te^{i\theta}\colon\ \nabla U\big(te^{i\theta}\big) = 0,\ 0 < t < 1 \text{ and } \varphi(t) < \theta < \psi(t)\} \\ &= \{te^{i\theta}\colon\ F'\big(\theta - \varphi(t)\big) = 0,\ 0 < t < 1 \text{ and } \varphi(t) < \theta < \psi(t)\} \end{aligned}$$

is either discrete or all of Ω, and U is real-analytic outside E. By the logarithmic symmetry we see that E is either empty or all of Ω, and the latter happens only when $A = 0$ and $U \equiv 0$. Hence U is real-analytic in Ω and $F \in C^2(\Omega)$. Inserting the expression for U into the p-harmonic equation $\operatorname{div}(|\nabla U|^{p-2}\nabla U) = 0$ and simplifying leads to the following equation for F,

$$(p-1)\big(a^2+1\big)F''(t) + (p-2)aF'(t) = 0 \quad \text{for } 0 < t < b.$$

(For this calculation it may be helpful to use some program like Maple.) Solving this equation we see that $F'(t) = Ce^{kt}$ for $t \in (0,b)$, where $C \in \mathbf{R}$ and

$$k = -\frac{a}{a^2+1}\,\frac{p-2}{p-1}.$$

Integrating and solving for the boundary values $F(0) = 0$ and $F(b) = A$ shows that

$$U\bigl(te^{i\theta}\bigr) = \begin{cases} A\frac{e^{k(\theta-\varphi(t))}-1}{e^{kb}-1}, & \text{if } k \neq 0, \\ A\frac{\theta-\varphi(t)}{b}, & \text{if } k = 0, \end{cases} \tag{9.1}$$

for $0 < t < 1$ and $\varphi(t) < \theta < \psi(t)$. Moreover U is the unique p-harmonic function with this invariance and these boundary values.

Proceeding as in the proofs of Theorems 4.2 and 4.3 we find that

$$\lim_{\Omega \ni y \to 0} \bigl(\overline{P} f(y) - U(y)\bigr) = \lim_{\Omega \ni y \to 0} \bigl(\underline{P} f(y) - U(y)\bigr) = 0.$$

It is straightforward to generalize all our other results in Sections 4–8 so that we can allow for asymptotic logarithmic spiral points wherever we have assumed for asymptotic corner points. Similarly we can allow for exterior logarithmic spiral points. Note that the case $a = 0$ corresponds to a corner and is consistent with our earlier results.

9.2. *Other selfsimilar situations*

Let $\varphi, \psi : (0, 1] \to \mathbf{R}$ be continuous functions and $0 < \tau < 1$ be such that

$$\varphi(t) < \psi(t) < \varphi(t) + 2\pi, \qquad \varphi(\tau t) = \varphi(t) \quad \text{and} \quad \psi(\tau t) = \psi(t) \quad \text{for } 0 < t \leqslant 1.$$

Let further

$$\Omega = \bigl\{re^{i\theta}\colon\ 0 < r < 1 \text{ and } \varphi(t) < \theta < \psi(t)\bigr\}$$

and $f : \partial\Omega \to \mathbf{R}$ be a bounded function such that

$$\lim_{t \to 0+} f\bigl(te^{i\varphi(t)}\bigr) = 0 \quad \text{and} \quad \lim_{t \to 0+} f\bigl(te^{i\psi(t)}\bigr) = A.$$

Then most of the arguments in the proofs of Theorems 4.2 and 4.3 can also be applied in this situation. There is only one part missing: We would need that there is a unique bounded p-harmonic function u in Ω with $U(\tau z) = U(z)$, $z \in \Omega$, and with boundary values 0 and A (in Perron sense) on $\{re^{i\theta}\colon\ 0 < r < 1 \text{ and } \theta = \varphi(t)\}$ and $\{re^{i\theta}\colon\ 0 < r < 1 \text{ and } \theta = \psi(t)\}$, respectively. The author does not know how to obtain such a uniqueness result. With such a result it would have been possible to show that

$$\lim_{\Omega \ni z \to 0} \bigl(\overline{P} f(z) - U(z)\bigr) = \lim_{\Omega \ni z \to 0} \bigl(\underline{P} f(z) - U(z)\bigr) = 0.$$

However, in this case it is easy to see that in general we would not have radial limits. Spiralling versions would probably also be obtainable.

9.3. *Unbounded Ω*

It is possible to consider similar questions in unbounded sets $\Omega \subset \mathbf{R}^2$, and in particular obtaining results when the boundary function has a jump at infinity. However, e.g., Theorems 2.3 and 2.4 have only been obtained for bounded Ω and in order to generalize many of the results in this paper to unbounded Ω it is necessary first to generalize Theorems 2.3 and 2.4. We leave such generalizations to another paper.

9.4. Higher dimensions

Let G be a nonempty open subset of the sphere $\mathbf{S}^{n-1} = \{x \in \mathbf{R}^n: |x| = 1\}$, $n \geqslant 3$, such that $C_p(\mathbf{S}^{n-1} \setminus G; \mathbf{S}^{n-1}) > 0$ (where $C_p(\cdot; \mathbf{S}^{n-1})$ is the Sobolev capacity with respect to the space $\mathbf{S}^{n-1}$) and

$$\Omega = \bigl\{r\theta \in \mathbf{R}^n: 0 < r < 1 \text{ and } \theta \in G\bigr\}.$$

Let further $\varphi \in C(\partial_{\mathbf{S}^{n-1}} G)$ and f be a bounded function on $\partial\Omega$ such that $f(r\gamma) = \varphi(\gamma)$ when $0 < r < 1$ and $\gamma \in \partial_{\mathbf{S}^{n-1}} G$. (We thus allow f to be arbitrary uniformly bounded on the rest of $\partial\Omega$.) Let further $\psi = P_G\varphi$, i.e. the Perron solution on G when G is considered as a subspace of the sphere $\mathbf{S}^{n-1}$. This sphere is a particular example of the metric spaces on which the theory of p-harmonic functions has been considered. Note in particular that continuous functions are resolutive in this case, by Theorem 2.3 (which is available). For the theory of p-harmonic functions on metric spaces we refer the reader to Björn–Björn–Shanmugalingam [13] or Björn–Björn [11]. Let also $U(r\theta) = \psi(\theta)$, $0 < r < 1$ and $\theta \in G$.

Theorem 9.1. *Let the notation be as above. Let $h : \partial\Omega \to \mathbf{R}$ be bounded and such that*

$$\lim_{\partial\Omega \ni x \to 0} \bigl(f(x) - h(x)\bigr) = 0.$$

Then we have the following radial limits

$$\lim_{t\to 0+} \underline{P}h(t\theta) = \lim_{t\to 0+} \overline{P}h(t\theta) = \psi(\theta) \quad \textit{for } \theta \in G. \tag{9.2}$$

If G is semiregular then moreover

$$\lim_{\Omega \ni z \to 0} \bigl(\underline{P}h(z) - U(z)\bigr) = \lim_{\Omega \ni z \to 0} \bigl(\overline{P}h(z) - U(z)\bigr) = 0. \tag{9.3}$$

The author does not know if (9.3) holds when G is not semiregular.

The main ideas in the proof of this result are essentially the same as those in the proofs in Section 4. However there are some additional complications, especially when G is not a regular subset of $\mathbf{S}^{n-1}$. So let us explain how to proceed.

We first need the following lemmas.

Lemma 9.2. *Let the notation be as above and let $u(r\theta) = \tilde{u}(\theta)$, $0 < r < 1$, $\theta \in G$, be a bounded p-harmonic function in Ω which is constant on rays starting at the origin. Assume further that*

$$\lim_{G \ni \theta \to \gamma} \tilde{u}(\theta) = \varphi(\gamma) \quad \textit{for q.e. } \gamma \in \partial_{\mathbf{S}^{n-1}} G,$$

where q.e. is taken with respect to the C_p-capacity on $\mathbf{S}^{n-1}$. Then $u = U$.

Proof. The proof is similar to the proof of Lemma 4.1: We change to polar coordinates, and see that $\tilde{u}$ is p-harmonic on G. By Theorem 2.3 (which is available), $\tilde{u}$ is the unique bounded p-harmonic function having boundary values φ q.e. on $\partial_{\mathbf{S}^{n-1}} G$. As this also holds for ψ we must have $\tilde{u} = \psi$ and thus $u = U$. $\square$

Lemma 9.3. *Let $\Omega \subset \mathbf{R}^n$, $n \geqslant 1$, be a nonempty bounded open set and $\widetilde{\Omega} = \Omega \times (0, 1) \subset \mathbf{R}^{n+1}$. Let further $x_0 \in \partial\Omega$ and $0 < t_0 < 1$. Then x_0 is regular with respect to Ω if and only if (x_0, t_0) is regular with respect to $\widetilde{\Omega}$.*

This lemma is well known to all experts in the field, but we have not been able to find a good reference.

Proof of Lemma 9.3. Assume first that x_0 is regular. Then there exists a weak barrier u at x_0, i.e. a positive superharmonic function u in Ω such that $\lim_{\Omega \ni y \to x_0} u(y) = 0$, by Theorem 3.1 in Kilpeläinen–Lindqvist [23] or Theorem 1.1 in Björn [7]. Let now $\tilde{u}(x,t) = u(x)$, $x \in \Omega$, $0 < t < 1$. Then $\tilde{u}$ is a weak barrier at (x_0, t_0) with respect to $\widetilde{\Omega}$, and thus (x_0, t_0) is regular with respect to $\widetilde{\Omega}$ by either of the above mentioned theorems.

Conversely, assume that x_0 is irregular. Then there is $f \in C(\partial\Omega)$ such that $Pf(y) \nrightarrow f(x_0)$, as $\Omega \ni y \to x_0$. Let now

$$\tilde{f}(x,t) = \begin{cases} f(x), & \text{if } x \in \partial\Omega \text{ and } 0 \leqslant t \leqslant 1, \\ Pf(x), & \text{if } x \in \Omega \text{ and } t = 0 \text{ or } t = 1. \end{cases}$$

Let us show that

$$P_{\widetilde{\Omega}} \tilde{f}(x,t) = Pf(x) \quad \text{when } x \in \Omega \text{ and } 0 < t < 1. \tag{9.4}$$

Let $u \in \mathcal{U}_f$ and $\tilde{u}(x,t) = u(x)$, $x \in \Omega$, $0 < t < 1$. Then $\tilde{u} \in \mathcal{U}_{\tilde{f}}(\widetilde{\Omega})$, and thus $\overline{P}_{\widetilde{\Omega}} \tilde{f}(x,t) \leqslant \tilde{u}(x,t) = u(x)$, $x \in \Omega$, $0 < t < 1$. Taking infimum over all $u \in \mathcal{U}_f$ shows that $\overline{P}_{\widetilde{\Omega}} \tilde{f}(x,t) \leqslant Pf(x)$ when $x \in \Omega$ and $0 < t < 1$. Similarly we can show that $\underline{P}_{\widetilde{\Omega}} \tilde{f}(x,t) \geqslant Pf(x)$, when $x \in \Omega$ and $0 < t < 1$, from which we conclude that (9.4) holds.

As $\tilde{f}$ is continuous at (x_0, t_0) but $P_{\widetilde{\Omega}} \tilde{f}(y) \nrightarrow \tilde{f}(x_0, t_0)$, as $\widetilde{\Omega} \ni y \to (x_0, t)$, we conclude from Lemma 2.10 that (x_0, t_0) is irregular with respect to $\widetilde{\Omega}$. $\square$

Proof of Theorem 9.1. Let us first consider the case $h \equiv f$.

Let $M = \sup_{\partial\Omega} |f| < \infty$ and

$$E = \{r\gamma\colon\ 0 < r < 1 \text{ and } \gamma \in \partial_{\mathbf{S}^{n-1}} G \text{ is irregular with respect to } G\}.$$

Let us show that E will exactly be the set of irregular boundary points with respect to Ω. First of all, if $x = \theta \in \partial\Omega$ with $\theta \in \overline{G}$, then x is regular by the cone condition, Theorem 6.31 in Heinonen–Kilpeläinen–Martio [20] (or the Wiener criterion, Theorem 2.6). If $x = r\gamma \in \partial\Omega$ with $0 < r < 1$ and $\gamma \in \partial G$, then locally around x we can take a biLipschitz mapping to a situation as in Lemma 9.3. Using Lemma 9.3, the Wiener criterion, that regularity is a local condition, and that the capacities are only distorted in a bounded way by the biLipschitz mapping, we see that x is regular if and only if γ is regular with respect to G. Here we do not only need the Wiener criterion on (unweighted) $\mathbf{R}^n$, as given in Theorem 2.6, but we also need it on $\mathbf{S}^{n-1}$ where it looks the same and is a special case of the Wiener criterion obtained by J. Björn [19, Theorem 1.1] for Cheeger p-harmonic functions on metric spaces. Observe that on $\mathbf{S}^{n-1}$ (and on $\mathbf{R}^n$) Cheeger p-harmonic functions coincide with the usual p-harmonic functions.

Finally a simple scaling shows that the variational capacity

$$\operatorname{cap}_p\bigl(B(0,\delta) \setminus \Omega, B(0,2\delta)\bigr) = C\delta^{n-p}, \quad 0 < \delta < 1,$$

for some constant $C \geqslant 0$. As $C_p(\mathbf{S}^{n-1} \setminus G; \mathbf{S}^{n-1}) > 0$, the Kellogg property (Theorem 2.5) shows that there is a regular point $\gamma_0 \in \partial G$ with respect to G, and hence $r\gamma_0$, $0 < r < 1$, is regular with respect to Ω. If C were 0, then all points in $\{r\gamma\colon\ 0 \leqslant r \leqslant 1 \text{ and } \gamma \in \partial G\}$ would be semiregular (see the discussion after Definition 2.7), a contradiction. Hence $C > 0$ and it follows from the Wiener criterion (Theorem 2.6) that 0 is regular.

The Kellogg property (Theorem 2.5) now shows that $C_p(E)=0$. We then let $u=\underline{P}k$, where

$$k(r\gamma)=\begin{cases}\varphi(\gamma), & \text{if } \gamma\in\partial_{\mathbf{S}^{n-1}}G\setminus E \text{ and } 0<r<1,\\ -M, & \text{otherwise on } \partial\Omega.\end{cases}$$

Let further $0<\rho<1$ and

$$v(z)=\liminf_{\Omega\ni w\to z} u(\rho w),\quad z\in\overline{\Omega}.$$

As u is continuous on Ω we have $v(z)=u(\rho z)$ for $z=r\theta$, $\theta\in G$, $0<r\leqslant 1$. Moreover, Lemma 2.10 yields that $v(r\gamma)=\varphi(\gamma)$ for $\gamma\in\partial_{\mathbf{S}^{n-1}}G\setminus E$ and $0<r<1$.

By the definition of v we have $v\in\mathcal{U}_v$ so that $\overline{P}v\leqslant v$. Moreover for any $\varphi\in\mathcal{L}_k$, let $\widetilde{\varphi}(z)=\varphi(\rho z)$. Then $\widetilde{\varphi}\in\mathcal{L}_v$, and hence $\underline{P}v\geqslant\sup_{\widetilde{\varphi}}\widetilde{\varphi}=v$. Thus, v is resolutive and $v=Pv$. As $v\geqslant k$ on $\partial\Omega$ (by the maximum principle), we have $v\geqslant u$ in Ω. Also as $U\in\mathcal{U}_k$ we have $U\geqslant u$ in Ω and hence, as U is constant on rays starting at the origin, $U\geqslant v$ in Ω.

Since $u(\rho z)=v(z)\geqslant u(z)$, $z\in\Omega$, for all $0<\rho<1$, we see that $r\mapsto u(r\theta)$ is a decreasing function, and the limit $\lim_{t\to 0+}u(t\theta)$ exists, if $\theta\in G$.

Let for nonnegative integers j, $v_j(z)=u(2^{-j}z)$, $z\in\Omega$. As we have seen v_j is an increasing sequence of p-harmonic functions, which obviously is bounded by M. It follows from Harnack's convergence theorem (see Theorem 6.14 in Heinonen–Kilpeläinen–Martio [20]) that $V=\lim_{j\to\infty}v_j$ is a p-harmonic function, and moreover $v_j\to V$ locally uniformly (see the proof of Theorem 6.14 in [20]). As $u\leqslant V\leqslant U$, we have

$$\lim_{G\ni\theta\to\gamma} V\left(\frac{1}{2}\theta\right)=\varphi(\gamma),\quad \text{if } \gamma\in\partial_{\mathbf{S}^{n-1}}G\setminus E. \tag{9.5}$$

Since $V(r\theta)=\lim_{j\to\infty}u(2^{-j}r\theta)=\lim_{t\to 0+}u(t\theta)$ is independent of r, for $\theta\in G$, Lemma 9.2 shows that $V=U$. We have thus shown (9.2) for $h\equiv f$.

Assume now that G is semiregular and let S be the set of semiregular points with respect to G on $\mathbf{S}^{n-1}$. Then S is the largest relatively open subset of $\partial_{\mathbf{S}^{n-1}}G$ with zero capacity, see the comments after Definition 2.7. It follows that E is a relatively open subset of $\partial\Omega$. By Theorem 6.2 in Björn [6], u and U have p-harmonic extensions (also called u and U) to $\widetilde{\Omega}:=\Omega\cup E$. Also let $u(r\gamma)=U(r\gamma)=\varphi(\gamma)$ for $0<r<1$ and $\gamma\in\partial_{\mathbf{S}^{n-1}}G\setminus S$. Then u and U are continuous on the compact set $\{\frac{1}{2}\theta\colon\ \theta\in\overline{G}\}$, and thus uniformly continuous there, and equal on $\{\frac{1}{2}\gamma\colon\ \gamma\in\partial_{\mathbf{S}^{n-1}}G\setminus S\}$.

Let $\varepsilon>0$. Then there is $\delta>0$ such that

$$\left|u\left(\frac{1}{2}\theta\right)-U\left(\frac{1}{2}\theta\right)\right|<\varepsilon\quad \text{for } \theta\in G\cup S \text{ with } \operatorname{dist}(\theta,\partial_{\mathbf{S}^{n-1}}G\setminus S)<\delta.$$

Extend now v_j to $\widetilde{\Omega}$ by Theorem 6.2 in Björn [6]. Harnack's convergence theorem shows that we still have $v_j\to U$ locally uniformly in $\widetilde{\Omega}$. We also have $u\leqslant v_j\leqslant U$ in $\widetilde{\Omega}$. We can thus find J such that $v_J>U-\varepsilon$ on

$$\left\{\frac{1}{2}\theta\colon\ \theta\in G\cup S \text{ and } \operatorname{dist}(\theta,\partial_{\mathbf{S}^{n-1}}G\setminus S)\geqslant\delta\right\}.$$

Thus $U-\varepsilon<v_J\leqslant U$ on $\{\frac{1}{2}\theta\colon\ \theta\in G\}$.

As $r\mapsto u(r\theta)$ is a decreasing function, for each $\theta\in G$, this shows that $U(r\theta)-\varepsilon\leqslant u(r\theta)\leqslant U(r\theta)$ for $0<r<2^{-(J+1)}$ and $\theta\in G$. Hence

$$\liminf_{\Omega\ni z\to 0}\big(\underline{P}f(z)-U(z)\big)\geqslant\liminf_{\Omega\ni z\to 0}\big(u(z)-U(z)\big)\geqslant-\varepsilon.$$

Letting $\varepsilon > 0$ shows that

$$\liminf_{\Omega \ni z \to 0} \bigl(\underline{P} f(z) - U(z)\bigr) \geqslant 0.$$

Similarly one obtains that

$$\limsup_{\Omega \ni z \to 0} \bigl(\overline{P} f(z) - U(z)\bigr) \leqslant 0.$$

As $\underline{P} f \leqslant \overline{P} f$, (9.3) follows for $h \equiv f$.

To prove (9.2) and (9.3) for a general h we can now proceed exactly as in the proof of Theorem 4.3. $\square$

It is not easy to formulate and prove a reasonable generalization of Theorem 5.2 to higher dimensions. However for Theorem 5.4 we have the following generalization.

Theorem 9.4. *Let $n-1 < p < \infty$ and Ω be a bounded open subset of $\mathbf{R}^n$. Assume that there is $\delta > 0$ so that $0 \in \partial\Omega$ and*

$$E := \bigl\{(x_1, 0, \ldots, 0)\colon\ 0 \leqslant x_1 \leqslant \delta\bigr\} \subset \mathbf{R}^n \setminus \Omega.$$

Let $f : \partial\Omega \to \mathbf{R}$ be a bounded function satisfying

$$\lim_{\partial\Omega \ni z \to 0} f(z) = 0.$$

Then

$$\lim_{\Omega \ni z \to 0} \underline{P} f(z) = \lim_{\Omega \ni z \to 0} \overline{P} f(z) = 0.$$

Note that for the conclusion to hold it is necessary that 0 is a regular boundary point. Moreover, 0 is regular with respect to $B(0,\delta) \setminus E$ if and only if $p > n-1$. To see this first observe that $C_p(E) > 0$ if and only if $p > n-1$ by Theorems 2.26 and 2.27 in Heinonen–Kilpeläinen–Martio [20]. It thus follows by a simple scaling argument that the variational capacity

$$\operatorname{cap}_p\bigl(E, B(0, 2\delta)\bigr) = C\delta^{n-p},$$

where $C > 0$ if and only if $p > n-1$. Moreover $\operatorname{cap}_p(B(0,\delta), B(0,2\delta)) = C'\delta^{n-p}$, where $C' > 0$. Thus the Wiener criterion (Theorem 2.6) shows that x_0 is regular if and only if $p > n-1$.

Proof of Theorem 9.4. Observe first that $C_p(\{1\}; \mathbf{S}^{n-1}) > 0$ as $p > n-1$. Thus Theorem 9.1 is at our disposal. The proof is now the same as the proof of Theorem 5.4 except for using Theorem 9.1 instead of Theorem 4.2. $\square$

We can also generalize Theorem 6.2.

Theorem 9.5. *Let $n-1 < p < \infty$ and let Ω be a nonempty bounded open subset of $\mathbf{R}^n$. Assume that $0 \in \partial\Omega$ is an exterior ray point.*

Let $f : \partial\Omega \to \mathbf{R}$ be a bounded function which is continuous at 0*. Let $E \subset \partial\Omega$ be a finite or countable set consisting of exterior ray points, and h be a function such that $h = f$ on $\partial\Omega \setminus E$. Then*

$$\lim_{\Omega \ni z \to 0} \underline{P}h(z) = \lim_{\Omega \ni z \to 0} \overline{P}h(z) = f(0).$$

Here *exterior ray point* is defined similarly to the $\mathbf{R}^2$ case.

Note that the theorem is false for $1 < p \leqslant n-1$, or more precisely the conclusion fails whenever 0 is not regular. (If 0 is regular and $1 < p \leqslant n-1$ the conclusion follows as in the proof below for $n-1 < p \leqslant n$.)

Proof of Theorem 9.5. For $p > n$ the proof is similar to the proof of Theorem 6.1 (with obvious modifications).

For $n-1 < p \leqslant n$ the result is a special case of Lemma 2.10, as 0 is regular by the proof of Theorem 9.4 and all points have zero capacity. □

We are now able to prove Theorem 1.3.

Proof of Theorem 1.3. If $1 < p \leqslant n$, then $C_p(E) = 0$ and the result follows from Theorem 2.3. Assume therefore that $n < p < \infty$. By Theorem 9.5, we have

$$\lim_{\Omega \ni y \to x} \overline{P}f(y) = \lim_{\Omega \ni y \to x} \underline{P}f(y) = \lim_{\Omega \ni y \to x} \overline{P}h(y) = \lim_{\Omega \ni y \to x} \underline{P}h(y) = f(x) \quad \text{for } x \in E.$$

On the other hand, Lemma 2.10 shows that

$$\lim_{\Omega \ni y \to x} \overline{P}f(y) = \lim_{\Omega \ni y \to x} \underline{P}f(y) = f(x) \quad \text{for } x \in \Omega \setminus E.$$

Arguing as in the proof of Theorem 6.3, we see that also

$$\lim_{\Omega \ni y \to x} \overline{P}h(y) = \lim_{\Omega \ni y \to x} \underline{P}h(y) = \lim_{\Omega \ni y \to x} \overline{P}f(y) \quad \text{for } x \in \Omega \setminus E. \tag{9.6}$$

(Observe that we do not require h to be continuous on $\Omega \setminus E$, and hence cannot apply Lemma 2.10 directly to obtain (9.6).) It thus follows from Theorem 3.1 that $\overline{P}f = \underline{P}f = \overline{P}h = \underline{P}h$ and in particular f and h are resolutive. □

We also have the following uniqueness result corresponding to Theorem 6.5 with $E_1 = \emptyset$.

Theorem 9.6. *Let Ω be a nonempty bounded open subset of $\mathbf{R}^n$. Let also $E \subset \partial\Omega$ be a finite or countable set of exterior ray points. Let $f \in C(\partial\Omega)$ and let h be a function on $\partial\Omega$ such that $f = h$ on $\partial\Omega \setminus E$. Let finally V be a bounded p-harmonic function. Then $V = Ph$ if and only if*

$$\lim_{\Omega \ni y \to x} V(y) = h(x) \quad \textit{for } x \in \partial\Omega \setminus E. \tag{9.7}$$

Of course a particular application is to let $h \equiv f$.

For $1 < p \leqslant n$, Theorem 2.3 gives a stronger result.

Proof of Theorem 9.6. For $1 < p \leqslant n$ this is a special case of Theorem 2.3, so assume that $p > n$.

Assume first that $V = Ph$. By Theorem 1.3, $V = Pf$. Thus Lemma 2.10 shows that

$$\lim_{\Omega \ni y \to x} V(y) = f(x) = h(x) \quad \text{for } x \in \partial\Omega \setminus E.$$

(Recall that all points are regular as $p > n$.)

Assume conversely that (9.7) holds. Let $\varphi = f - \infty\chi_E$ and $\psi = f + \infty\chi_E$. Then $V \in \mathcal{U}_\varphi$ and $V \in \mathcal{L}_\psi$ so that, by Theorem 1.3,

$$V \leqslant P\psi = Ph = P\varphi \leqslant V. \qquad \square$$

9.5. Weighted $\mathbf{R}^n$ *and metric spaces*

For the main results in this paper it is essential that we work with unweighted $\mathbf{R}^n$. More generally one can consider $\mathcal{A}$-harmonic functions on weighted $\mathbf{R}^n$ or p-harmonic functions on complete metric spaces equipped with doubling measures supporting weak $(1, p)$-Poincaré inequalities. For the necessary definitions and basic theory we refer to Heinonen–Kilpeläinen–Martio [20] for weighted $\mathbf{R}^n$, and, e.g., to Björn–Björn–Shanmugalingam [13] or Björn–Björn [11] for metric spaces.

Our proofs of Proposition 2.8, Lemma 2.10 and the results in Section 3 work just as well in metric spaces, and for $\mathcal{A}$-harmonic functions on weighted $\mathbf{R}^n$.

If $\mathcal{A}$ and w are constant on rays from the origin at least near the origin, then it is possible to obtain the results in Sections 4, 5 and 9.4 for $\mathcal{A}$-harmonic functions on $\mathbf{R}^n$ weighted by $w\,dx$, but this is a rather special situation. To solve the Baernstein problem in this case requires such an assumption at all boundary points, or at least at all points in $\partial_{\partial\Omega} G$. We have refrained from doing this.

Let us discuss perturbation results like Theorem 1.3 in more general situations. We let Ω be a nonempty bounded open set in (weighted) $\mathbf{R}^n$ or a metric space X with the usual assumptions, see the references above. (If X is bounded we need to require that $C_p(X \setminus \Omega) > 0$, which is automatic if X is unbounded, as otherwise Ω has no potential-theoretic boundary.) Let us say that a boundary point $x_0 \in \partial\Omega$ is a *bounded* (*unbounded*) *perturbation point* if $Ph = Pf$ whenever $f \in C(\partial\Omega)$, h is *bounded* (*arbitrary*) on $\partial\Omega$, and $h = f$ on $\partial\Omega \setminus \{x_0\}$.

By Theorem 2.3 any point with zero capacity is an unbounded perturbation point. In particular, by the Kellogg property all irregular boundary points are unbounded perturbation points. In unweighted $\mathbf{R}^n$ exterior ray points are unbounded perturbation points by Theorem 1.3. On the other hand, Example 6.4 shows that not all regular boundary points are bounded perturbation points.

In fact if we let S be the set of semiregular boundary points and $x_0 \in \partial\Omega$ is a regular point which is isolated in $\partial\Omega \setminus S$, which holds if and only if $C_p(\{x_0\}) > 0$ and $C_p(B \setminus \{x_0\}) = 0$ for some ball $B = B(x_0, \delta)$, then x_0 is not a bounded perturbation point. (This condition can also equivalently be formulated by requiring that x_0 is isolated in the set of regular boundary points, see Björn [8].) In fact, let h be an arbitrary bounded function on $\partial\Omega$. Then we can find a continuous function k such that $k \geqslant h$ on $\partial\Omega \setminus B$ and $k(x_0) = h(x_0)$. By Lemma 2.10 we have

$$\limsup_{\Omega \ni y \to x_0} \underline{P}h(y) \leqslant \limsup_{\Omega \ni y \to x_0} \overline{P}h(y) \leqslant k(x_0) = h(x_0).$$

The converse inequality is proved similarly and thus

$$\lim_{\Omega \ni y \to x_0} \underline{P}h(y) = \lim_{\Omega \ni y \to x_0} \overline{P}h(y) = h(x_0).$$

Letting $f \equiv 0$ and $h = \chi_{\{x_0\}}$ now shows that x_0 is not a bounded perturbation point.

Example 9.7. Let $n \geqslant 3$, $1 < p \leqslant n-1$, $-n < \delta < p-n$ and consider weighted $\mathbf{R}^n$ with the measure $d\mu(x) = |x|^\delta\, dx$ and $\Omega = B(0,1) \setminus E$, where $E = \{(x_1, 0, \dots, 0) \in \mathbf{R}^n\colon\ 0 \leqslant x_1 \leqslant 1\}$. Then $C_p(\{0\}) > 0$ by Example 2.22 in Heinonen–Kilpeläinen–Martio [20]. On the other hand $C_p(E \setminus \{0\}) = 0$ by Theorem 2.27 in [20]. Thus x_0 is not a bounded perturbation point by the above. On the other hand x_0 is an exterior ray point. Hence exterior ray points need not be bounded perturbation points in weighted $\mathbf{R}^n$, in contrast to unweighted $\mathbf{R}^n$.

Open problem 9.8. Is it true that any regular point which is not isolated among the regular boundary points is a bounded perturbation point?

Remark 9.9. After this paper was submitted, Kim [25] answered this open problem in the affirmative for unweighted $\mathbf{R}^n$.

Observe that except for exterior ray points (and exterior logarithmic spiral points, see Section 9.1) in unweighted $\mathbf{R}^n$ we have no examples of perturbation points with positive capacity. Let us nevertheless draw some general conclusions.

Theorem 9.10. *Assume that $\Omega \subset X$ is regular. Let E be a finite or countable set of bounded perturbation points with respect to Ω. Let further $f \in C(\partial\Omega)$ and $h = f$ on $\partial\Omega \setminus E$. Then $Ph = Pf$.*

In particular bounded and unbounded perturbations are the same in this case.

Proof of Theorem 9.10. The proof is fairly similar to the proof of Theorem 6.1. Just start the induction by letting $k_0 = f$, ignore the parts treating the jump discontinuity at 0, and finish the proof after (6.3) in the obvious way. $\square$

In semiregular sets we have the following corresponding results for R-Perron solutions.

Theorem 9.11. *Assume that $\Omega \subset X$ is semiregular. Let E be a finite or countable set of bounded perturbation points with respect to Ω. Let further $f \in C(\partial\Omega)$ and $h = f$ on $\partial\Omega \setminus E$. Then $Rh = Rf = Pf$.*

Proof. Again we proceed as in the proof of Theorem 6.1, starting the induction by letting $k_0 = f$ and ignoring the parts treating the jump discontinuity at 0. We let $z_0 \in \Omega$ and $\varepsilon > 0$. We obtain an increasing sequence of continuous functions $\{k_j\}_{j=0}^\infty$ such that $v = \lim_{j\to\infty} Pk_j$ is p-harmonic, and $v(z_0) < Pf(z_0) + 2\varepsilon$.

We next want to show that $v \in \widetilde{\mathcal{U}}_h$. Let S be the set of semiregular boundary points. For $x \in \partial\Omega \setminus (E \cup S)$ we have

$$\liminf_{\Omega\ni y\to x} v(y) \geqslant \liminf_{\Omega\ni y\to x} Pf(y) = f(x) = h(x).$$

On the other hand, for $x \in E \setminus S$,

$$\liminf_{\Omega\ni y\to x} v(y) \geqslant \lim_{j\to\infty} \liminf_{\Omega\ni y\to x} Pk_j(y) = \lim_{j\to\infty} k_j(x) = \infty \geqslant h(x).$$

Thus $v \in \widetilde{\mathcal{U}}_h$ and

$$\overline{R}h(z_0) \leqslant v(z_0) < Pf(z_0) + 2\varepsilon.$$

Letting $\varepsilon \to 0$ shows that $\overline{R}h(z_0) \leqslant Pf(z_0)$, and as $z_0 \in \Omega$ was arbitrary we find that $\overline{R}h \leqslant Pf$ in Ω. Similarly $\underline{R}h \geqslant Pf$ and thus $Rh = Pf = Rf$. $\square$

Theorem 9.12. *Let $\Omega \subset X$ be a bounded nonempty open set such that $C_p(X \setminus \Omega) > 0$. Let $f \in C(\partial\Omega)$, E_1 be a finite set of bounded perturbation points, $C_p(E_2) = 0$ and $h : \partial\Omega \to \overline{\mathbf{R}}$ be such that $h = f$ on $\partial\Omega \setminus (E_1 \cup E_2)$ and $\sup_{E_1} |h| < \infty$. Then $Ph = Pf$.*

Observe that this result holds without any regularity assumption on Ω. The author does not know if Theorems 9.10 and 9.11 hold without regularity assumption, not even if E is a singleton set. Observe that in Theorem 9.12 we only consider perturbations which are bounded on E_1, whereas in Theorems 9.10 and 9.11 unbounded perturbations are considered.

Proof of Theorem 9.12. Let

$$k = \begin{cases} f, & \text{on } \partial\Omega \setminus E_1, \\ h, & \text{on } E_1. \end{cases}$$

Let further $z_0 \in \Omega$ and $\varepsilon > 0$. We proceed as in the induction in the proof of Theorem 6.1, with $k_0 = f$ and $E = E_1$. After a finite number of steps we find a function $\tilde{k} \in C(\partial\Omega)$ such that $k \leqslant \tilde{k}$ and $P\tilde{k}(z_0) < Pf(z_0) + 2\varepsilon$.

By Theorem 2.3, $Pu = P\tilde{k}$, where $u = \tilde{k} + \infty\chi_{E_2}$. As $u \geqslant h$ we have

$$\overline{P}h(z_0) \leqslant Pu(z_0) < Pf(z_0) + 2\varepsilon.$$

Letting $\varepsilon \to 0$ shows that $\overline{P}h(z_0) \leqslant Pf(z_0)$. As $z_0 \in \partial\Omega$ was arbitrary, we have $\overline{P}h \leqslant Pf$ in Ω. Similarly $\underline{P}h \geqslant Pf$ and thus $Ph = Pf$ in Ω. □

If we define perturbation sets similarly, then the set S of all semiregular points is the largest relatively open (bounded or unbounded) perturbation set, by Theorem 3.1 in Björn [8].

9.6. *Other equations*

The methods used in this paper should be applicable also to some other elliptic equations than the p-harmonic equation. For the method used in Section 4 it is important that the equation is invariant under dilations, i.e. if u is a solution then also $x \mapsto u(\tau x)$, $0 < \tau < 1$, is a solution. It is also important to have uniqueness as in Lemma 4.1. For the spiralling results in Section 9.1 it is essential to have rotational invariance, but that is not used in Section 4. In particular it seems that for ∞-harmonic functions the method and results should hold. We leave the details for ∞-harmonic functions as well as other equations to the future.

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References

[1] G. Aronsson, Construction of singular solutions to the p-harmonic equation and its limit equation for $p = \infty$, Manuscripta Math. 56 (1986) 135–158.
[2] G. Aronsson, P. Lindqvist, On p-harmonic functions in the plane and their stream functions, J. Differential Equations 74 (1988) 157–178.
[3] A. Baernstein II, Comparison of p-harmonic measures of subsets of the unit circle, St. Petersburg Math. J. 9 (1998) 543–551.
[4] A. Björn, Characterizations of p-superharmonic functions on metric spaces, Studia Math. 169 (2005) 45–62.
[5] A. Björn, A weak Kellogg property for quasiminimizers, Comment. Math. Helv. 81 (2006) 809–825.
[6] A. Björn, Removable singularities for bounded p-harmonic and quasi(super)harmonic functions on metric spaces, Ann. Acad. Sci. Fenn. Math. 31 (2006) 71–95.

[7] A. Björn, Weak barriers in nonlinear potential theory, Potential Anal. 27 (2007) 381–387.

[8] A. Björn, A regularity classification of boundary points for p-harmonic functions and quasiminimizers, J. Math. Anal. Appl. 338 (2008) 39–47.

[9] A. Björn, Cluster sets for Sobolev functions and quasiminimizers, J. Anal. Math., in press.

[10] A. Björn, J. Björn, Approximations by regular sets and Wiener solutions in metric spaces, Comment. Math. Univ. Carolin. 48 (2007) 343–355.

[11] A. Björn, J. Björn, Nonlinear Potential Theory in Metric Spaces, in preparation.

[12] A. Björn, J. Björn, N. Shanmugalingam, The Dirichlet problem for p-harmonic functions on metric spaces, J. Reine Angew. Math. 556 (2003) 173–203.

[13] A. Björn, J. Björn, N. Shanmugalingam, The Perron method for p-harmonic functions, J. Differential Equations 195 (2003) 398–429.

[14] A. Björn, J. Björn, N. Shanmugalingam, A problem of Baernstein on the equality of the p-harmonic measure of a set and its closure, Proc. Amer. Math. Soc. 134 (2006) 509–519.

[15] A. Björn, J. Björn, N. Shanmugalingam, Sobolev extensions of Hölder continuous and characteristic functions on metric spaces, Canad. J. Math. 59 (2007) 1135–1153.

[16] A. Björn, J. Björn, N. Shanmugalingam, Quasicontinuity of Newton–Sobolev functions and density of Lipschitz functions on metric spaces, Houston J. Math. 34 (2008) 1197–1211.

[17] A. Björn, O. Martio, Pasting lemmas and characterizations of boundary regularity for quasiminimizers, Results Math. 55 (2009) 265–279.

[18] J. Björn, Boundary continuity for quasiminimizers on metric spaces, Illinois J. Math. 46 (2002) 383–403.

[19] J. Björn, Wiener criterion for Cheeger p-harmonic functions on metric spaces, in: Potential Theory in Matsue, in: Adv. Stud. Pure Math., vol. 44, Mathematical Society of Japan, Tokyo, 2006, pp. 103–115.

[20] J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, 2nd ed., Dover, Mineola, NY, 2006.

[21] T. Kilpeläinen, A remark on the uniqueness of quasi continuous functions, Ann. Acad. Sci. Fenn. Math. 23 (1998) 261–262.

[22] T. Kilpeläinen, J. Malý, The Wiener test and potential estimates for quasilinear elliptic equations, Acta Math. 172 (1994) 137–161.

[23] T. Kilpeläinen, P. Lindqvist, Nonlinear ground states in irregular domains, Indiana Univ. Math. J. 49 (2000) 325–331.

[24] S. Kim, private communication, January 2009.

[25] S. Kim, A Baernstein problem of p-harmonic measures and an invariance of p-harmonic functions under boundary perturbations using tug-of-war with noise, PhD thesis, New York University, New York, 2010.

[26] J. Kinnunen, O. Martio, Nonlinear potential theory on metric spaces, Illinois J. Math. 46 (2002) 857–883.

[27] J.L. Lewis, Regularity of the derivatives of solutions to certain degenerate elliptic equations, Indiana Univ. Math. J. 32 (1983) 849–858.

[28] O. Martio, Boundary behavior of quasiminimizers and Dirichlet finite PWB solutions in the borderline case, in: Rep. Math., vol. 440, University of Helsinki, Helsinki, 2006.

[29] V.G. Maz'ya, On the continuity at a boundary point of solutions of quasi-linear elliptic equations, Vestnik Leningrad. Univ. Mat. Mekh. Astronom. 25 (13) (1970) 42–55 (Russian); English transl.: Vestnik Leningrad Univ. Math. 3 (1976) 225–242.

[30] Y. Peres, S. Sheffield, Tug-of-war with noise: a game-theoretic view of the p-Laplacian, Duke Math. J. 145 (2008) 91–120.

[31] W.P. Ziemer, Boundary regularity for quasiminima, Arch. Ration. Mech. Anal. 92 (1986) 371–382.