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Quasi-periodic solutions of a non-autonomous wave equations with quasi-periodic forcing ☆

Jianguo Si

School of Mathematics, Shandong University, Jinan, Shandong 250100, People's Republic of China

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ABSTRACT

In this paper, we prove existence of small amplitude quasi-periodic solutions for a non-autonomous, quasi-periodically forced nonlinear wave equations with periodic spatial boundary conditions via KAM theory.

1. Introduction and main results

A fundamental problem in the theory of Hamiltonian partial differential equations (PDEs) is to determine whether the equation possesses time-periodic solutions or time-quasi-periodic solutions. Such a problem can be studied in either the forced or unforced context. In the unforced case, for nonresonant PDEs, a developed existence theory of periodic and quasi-periodic solutions has been established by Kuksin [1], Wayne [2], Craig and Wayne [3], Pöschel [4,5], and Bourgain [6] and references therein. For completely resonant autonomous PDEs, existence of periodic solutions has been proven in [7–13], and the existence of quasi-periodic solutions with two frequencies has been recently

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E-mail addresses: sijgmath@yahoo.com.cn, sijgmath@sdu.edu.cn.

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obtained in [14] and [15]. In particular, Yuan [16] obtained quasi-periodic solutions of the completely resonant wave equations $u_{tt} - u_{xx} + u^3 = 0$ subject to periodic boundary conditions via KAM theory.

Forced problems include an external force in the PDE that is time-periodic or time-quasi-periodic; time-periodic solutions or time-quasi-periodic solutions are then sought. In [17], M. Zhang and the author of this paper investigated the existence of quasi-periodic solutions for quasi-periodically forced nonlinear wave equation

$$u_{tt} = u_{xx} - \mu u - \varepsilon\phi(t)h(u), \quad \mu > 0$$

under Dirichlet boundary conditions by making use of KAM theory. In [18], Jiao and Wang considered the existence of quasi-periodic solutions for quasi-periodically forced Schrödinger equation

$$iu_t = u_{xx} - mu - f(\beta t, x)|u|^2 u$$

under the boundary conditions $u(t,0) = u(t,a\pi) = 0$ by using the KAM method. Periodic solutions for completely resonant wave equations with periodic forcing and Dirichlet boundary conditions was the first time constructed by Rabinowita [21,22] using global variational methods and a Lyapunov–Schmidt decomposition. In this directions, we refer to [23–26] and references therein for a detailed description. More recently, by using the Lyapunov–Schmidt decomposition method, M. Berti and M. Procesi [27] proved existence of small amplitude quasi-periodic solutions with two frequencies $\omega = (\omega_1, \omega_2) = (\omega_1, 1+\varepsilon)$ for the completely resonant wave equations with periodic forcing

$$\begin{cases} u_{tt} - u_{xx} + f(\omega_1 t, u) = 0, \\ u(t,x) = u(t, x+2\pi), \end{cases} \tag{1.1}$$

where the nonlinear forcing term

$$f(\omega_1 t, u) = a(\omega_1 t)u^{2d+1} + O\left(u^{2d+2}\right), \quad d \in \mathbb{N} = \{1, 2, \ldots\}$$

is $2\pi/\omega_1$-periodic in time, when $\omega_1 \in \mathbb{Q}$ or $\omega_1 \in \mathbb{R}\setminus\mathbb{Q}$.

Naturally, we should ask that whether or not there is any quasi-periodic solution with multi-frequencies to (1.1) when the nonlinear term depends on time in a quasi-periodic way. In this paper, we will give an answer to this question under some appropriate hypotheses. Concretely, we are concerned with existence of quasi-periodic solutions for non-autonomous quasi-periodically forced nonlinear wave equation

$$u_{tt} - u_{xx} + \phi(t,\varepsilon)u^3 = 0 \tag{1.2}$$

subject to periodic boundary conditions

$$u(t,x) = u(t, x+2\pi), \tag{1.3}$$

where ε is a small parameter and $\phi(t,\varepsilon)$ is real analytic quasi-periodic function in t with frequency vector $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$.

The result of this paper under review seem to coincided with the result by M. Berti and M. Procesi [27]. However, the strategy of proof is quite different. Our approach is based on KAM theory, while the proof in [27] is based on Lyapunov–Schmidt decomposition. The two techniques are somehow complementary. Compared with the Lyapunov–Schmidt reduction method, the KAM approach has its own advantages. Besides obtaining the existence of quasi-periodic solutions it allows one to construct a local normal form in a neighborhood the obtained solution. This would allow, in principle, to study the dynamics of the PDE in its neighborhood. The results obtained in [27] show that at the first order the quasi-periodic solution of Eq. (1.1) is the superposition of two waves traveling $q_+(\cdot)$

and $q_-(\cdot)$ in opposite directions. Here $q_+(t+x)+q_-(t-x)$ is 2π-periodic solutions of linearized equation

$$u_{tt}-u_{xx}=0,$$

where $q_+(\cdot)$ and $q_-(\cdot)$ are 2π-periodic. While our result shows that (1.2) possesses many quasi-periodic solutions in the neighborhood of a quasi-periodic solution of nonlinear ODE with quasi-periodic forcing:

$$\ddot{x}+\phi(t,\varepsilon)x^3=0.$$

The method used in this paper is based on infinite-dimensional KAM theory as developed by Kuksin [1,31] and Pöschel [4]. Thus the main step is to reduce the equation to a setting where KAM theory for PDE can be applied. This needs to reduce the linear part of Hamiltonian system to constant coefficients by a linear quasi-periodic change of variables with the same basic frequencies as the initial system. However, we cannot guarantee in general such reducibility. A large part of the present paper will be devoted to the proof of reducibility of an infinite-dimensional linear quasi-periodic systems. In general, the question of reducibility of infinite-dimensional linear quasi-periodic systems remains open and very attracting. Such kind of reducibility result for PDE using KAM machinery was first obtained by Bambusi and Graffi [19], later, Yuan [16], and more recent, Eliasson and Kuksin [20]. However, it would seem that the results in [19] and [20] cannot be directly applied to our problems because of the difference in the orders of corresponding morphism of the Hilbert scales between Schrödinger equations and wave equations, and the work in [16] cannot also be directly applied to the case with quasi-periodic forcing.

For our purpose, we first introduce a definition and a hypothesis.

We denote by $Q(\omega)$ the set of real analytic quasi-periodic functions with the frequencies ω.

Definition 1.1. Let $Q_\sigma(\omega)\subset Q(\omega)$ be the set of real analytic functions $f(\vartheta)$ which are bounded on the subset $\Pi_\sigma=\{(\vartheta_1,\dots,\vartheta_m)\in\mathbb{C}^m\colon\ |\mathrm{Im}\,\vartheta_j|\leqslant\sigma\}$, with the supremum norm

$$|f|_\sigma=\sup_{\vartheta\in\Pi_\sigma}\big|f(\vartheta)\big|.$$

Denote by

$$[f]=\frac{1}{(2\pi)^m}\int\limits_{\mathbb{T}^m}F(\vartheta)\,d\vartheta$$

the average of f, where F is the shell function of f.

(H) $\phi(t,\varepsilon)=B+\varepsilon\tilde{\phi}(t)$, $B>0$ and $\tilde{\phi}(t)$ is a real analytic quasi-periodic function in t with frequency vector $\omega=(\omega_1,\omega_2,\dots,\omega_m)$, where $\omega\in D_\Lambda$:

$$D_\Lambda:=\left\{\omega\in\mathbb{R}^m\colon\ \big|\langle k,\omega\rangle\big|\geqslant\frac{\Lambda}{|k|^{m+1}},\ \Lambda>0,\ 0\neq k\in\mathbb{Z}^m\right\}.$$

Define the set

$$\begin{aligned}&A_\gamma=\big\{\alpha\in\mathbb{R}\colon\ \big|\langle k,\omega\rangle+l\alpha\big|>\gamma\big(|k|+|l|\big)^{-(m+1)}\big\},\\&\quad k=(k_1,\dots,k_m)\in\mathbb{Z}^m,\ l\in\mathbb{Z},\qquad |k|+|l|=|k_1|+\cdots+|k_m|+|l|>0,\quad\gamma>0.\end{aligned}$$

The following theorem is the main result of this paper.

Theorem 1.2 *(Main theorem). Assume that* **(H)** *is satisfied. For index set $\tilde{\mathcal{J}} = \{1, 2, \ldots, n\}$ with $n \geqslant 1$, there is small enough positive ε^{**} such that for any $0 < \varepsilon < \varepsilon^{**}$, there are the sets $\mathscr{J} \subset \hat{J} \subset [\pi/T, 3\pi/T]$ and $\Sigma_\varepsilon \subset \Sigma := D_\Lambda \times A_\gamma \times [0,1]^{n+1}$ with $\operatorname{meas} \hat{J} > 0$, $\operatorname{meas} \mathscr{J} > 0$ and $\operatorname{meas}(\Sigma \setminus \Sigma_\varepsilon) \leqslant \varepsilon$, such that for any $\bar{\xi} \in \mathscr{J}$ and $(\omega, \alpha(\bar{\xi}), \tilde{\xi}_0, \tilde{\xi}_1, \ldots, \tilde{\xi}_n) \in \Sigma_\varepsilon$, the nonlinear wave equation* (1.2)+(1.3) *possess a solution of the form*

$$u(t,x) = u_0(t, \bar{\xi}, \varepsilon) + \frac{\varepsilon^{\frac{4}{3}}}{\sqrt[4]{[\bar{V}]}} \bigl(1 + \varepsilon^{2/3} f_0^*\bigl(\tilde{\omega}(\bar{\xi})t, \bar{\xi}, \varepsilon\bigr)\bigr) \sqrt{\frac{\tilde{\xi}_0}{\pi}} \cos\bigl(\hat{\omega}_0 t + \mathcal{O}\bigl(\varepsilon^{\iota}\bigr)\bigr)$$
$$+ \varepsilon \sum_{1 \leqslant j \leqslant n} \frac{1 + \varepsilon^{2/3} f_j^*(\tilde{\omega}(\bar{\xi})t, \bar{\xi}, \varepsilon)}{\sqrt[4]{j^2 + \varepsilon^{2/3}[\bar{V}]}} \sqrt{\frac{2\varepsilon\tilde{\xi}_j}{\pi}} \cos\bigl(\hat{\omega}_j t + \mathcal{O}\bigl(\varepsilon^{\iota}\bigr)\bigr) \cos jx + \mathcal{O}\bigl(\varepsilon^2\bigr), \quad (1.4)$$

where

$$0 < \iota < \frac{7}{2}, \quad T = 4 \int_0^1 \frac{1}{\sqrt{B/2(1-x^4)}}\, dx,$$

$u_0(t, \bar{\xi}, \varepsilon)$ is a non-trivial quasi-periodic solution of the form (2.17) *of* (2.1) *and $f_j^*(\theta, \bar{\xi}, \varepsilon)$ is of period 2π in each component of θ and for $0 \leqslant j \leqslant n$, $\theta \in \Theta(\sigma_0/3)$, $\bar{\xi} \in \mathscr{J}$, we have $|f_j^*(\theta, \bar{\xi}, \varepsilon)| \leqslant C$ (an absolute constant). Furthermore, the obtained solution $u(t,x)$ is quasi-periodic in time t with the frequency vector $\hat{\hat{\omega}} = (\tilde{\omega}(\bar{\xi}), (\hat{\omega}_j)_{0 \leqslant j \leqslant n})$, and there is an absolute constant c such that*

$$\bigl|\hat{\omega}_j - \hat{\omega}_j\bigr| \leqslant c\varepsilon^{\frac{7}{2}},$$

where

$$\hat{\omega}_0 = \varepsilon^{1/3}\sqrt{[\hat{V}]} + \sum_{k=2}^{\infty} \varepsilon^{\frac{2k}{3}} \tilde{\lambda}_{0,k}\bigl(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\bigr)$$

and

$$\hat{\omega}_j = \sqrt{j^2 + \varepsilon^{2/3}[\hat{V}]} + \sum_{k=2}^{\infty} \varepsilon^{\frac{2k}{3}} \tilde{\lambda}_{j,k}\bigl(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\bigr) + \varepsilon^3 \sum_{k_l \in \tilde{\mathcal{J}}} A_{k_j k_l} \tilde{\xi}_l, \quad j, k_j \in \tilde{\mathcal{J}}$$

with $|\tilde{\lambda}_{j,k}(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon)| \leqslant C$,

$$A_{ij} = \begin{cases} \frac{B(-\frac{15}{4} + o(1))}{\pi\sqrt{(i^2+\varepsilon^{2/3}[\bar{V}])(j^2+\varepsilon^{2/3}[\bar{V}])}} + \mathcal{O}(\varepsilon^{2/3}), & i \neq j, \\ \frac{B(-\frac{27}{16} + o(1))}{\pi\sqrt{(i^2+\varepsilon^{3/2}[\bar{V}])(j^2+\varepsilon^{3/2}[\bar{V}])}} + \mathcal{O}(\varepsilon^{3/2}), & i = j, \end{cases}$$

here $C > 0$ is a constant, $\hat{V}$ is defined in (3.2) *and* $\lim_{\varepsilon \to 0} o(1) = 0$.

2. Quasi-periodic solutions of a nonlinear ODE

In this section, we will apply the results in [28] (see also [29]) to show the existence of quasi-periodic solution for the following nonlinear ordinary differential equation with quasi-periodic coefficient

$$\ddot{x} + \phi(t, \varepsilon)x^3 = 0. \tag{2.1}$$

From **(H)**, Eq. (2.1) is equivalent to the system

$$\dot{x} = -y, \qquad \dot{y} = Bx^3 + \varepsilon\tilde{\phi}(t)x^3. \tag{2.2}$$

We have the following lemma.

Lemma 2.1. *For any $\omega \in D_\Lambda$, there exists an ε^* such that for any positive $0 < \varepsilon < \varepsilon^*$ and sufficiently small $\gamma > 0$ there exist a real analytic function $a_0(\alpha) : A_\gamma \to \mathbb{R}$ and a set $\hat{J} \subset [\pi/T, 3\pi/T]$ with $\operatorname{meas} \hat{J} > 0$, such that for $\alpha \in A_\gamma$, $\bar{\xi} \in \hat{J}$ and some $\sigma > 0$, Eq. (2.1) has a quasi-periodic solution $x(t, \bar{\xi}, \varepsilon) \in Q_\sigma(\tilde{\omega})$ with $\tilde{\omega}(\bar{\xi}) = (\omega_1, \omega_2, \ldots, \omega_m, \alpha(\bar{\xi}))$ satisfying $x(t, \bar{\xi}, \varepsilon) = \mathcal{O}(\varepsilon^{1/3})$.*

Proof. Let us consider the auxiliary equation

$$\ddot{x} + Bx^3 = 0, \tag{2.3}$$

which is equivalent to the system

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = Bx^3. \end{cases} \tag{2.4}$$

The system (2.4) is a Hamiltonian system

$$\begin{cases} \dot{x} = -\frac{\partial h}{\partial y}, \\ \dot{y} = \frac{\partial h}{\partial x}, \end{cases} \tag{2.5}$$

with the Hamiltonian

$$h(x, y) = \frac{1}{2}y^2 + \frac{B}{4}x^4. \tag{2.6}$$

Clearly $h > 0$ on $\mathbb{R}^2$ except at the only equilibrium point $(0, 0)$ where $h = 0$. All the solutions of (2.4) are periodic, the periods tending to zero as $h = H_0$ tends to infinity. Let $(C(t), S(t))$ be the solution of the system (2.4) satisfying the initial condition $(C(0), S(0)) = (1, 0)$. Let T be its minimal period, then we have

$$0 < T = 4\int_0^1 \frac{1}{\sqrt{B/2(1 - x^4)}}\,dx < +\infty.$$

From (2.4), one can easily see that there is a $\sigma > 0$ such that $S(t)$ and $C(t)$ are analytic in the strip $\{|\operatorname{Im} t| < \sigma\}$. From (2.3), these analytic functions satisfy

(i) $S(t + T) = S(t)$, $C(t + T) = C(t)$;
(ii) $C'(t) = -S(t)$, $S'(t) = BC^3(t)$;

(iii) $S^2(t) + \frac{B}{2}C^4(t) = \frac{B}{2}$;
(iv) $C(-t) = C(t)$, $S(-t) = -S(t)$.

The action and angle variables are now by the map $\Phi^+ : \mathbb{R}^+ \times \mathbb{S}^1 \mapsto \mathbb{R}^2$, where $(x, y) = \Phi^+(\rho, \varphi)$ with $\rho > 0$ and $\varphi (\mathrm{mod}\, 2\pi)$ is given the formula

$$\Phi^+ : x = c^{1/3}\rho^{1/3}C(\varphi T), \qquad y = c^{2/3}\rho^{2/3}S(\varphi T), \tag{2.7}$$

where $c = 3/(BT)$, and $\mathbb{S}^1 = \mathbb{R}^1/2\pi\mathbb{Z}$. One can easily verify that the transformation Φ^+ is a symplectic diffeomorphism from $\mathbb{R}^+ \times \mathbb{S}^1$ to $\mathbb{R}^2 \setminus \{0\}$. In fact, it follows that $|\frac{\partial(x,y)}{\partial(\rho,\varphi)}| = 1$.

Under this transformation, the system (2.4) is transformed into the simpler form

$$\begin{cases} \dot{\rho} = -\frac{\partial h_0}{\partial \varphi} = 0, \\ \dot{\varphi} = \frac{\partial h_0}{\partial \rho} = \frac{4s}{3}\rho^{1/3}, \end{cases} \tag{2.8}$$

where $h_0(\rho, \varphi, t) = s\rho^{4/3}$ and $s = \frac{1}{4}B \cdot c^{4/3}$.

Eq. (2.2) is equivalent to the system

$$\begin{cases} \dot{x} = -\frac{\partial H}{\partial y}, \\ \dot{y} = \frac{\partial H}{\partial x}, \end{cases} \tag{2.9}$$

where

$$H(x, y, t) = \frac{1}{2}y^2 + \frac{1}{4}Bx^4 + \frac{\varepsilon}{4}\tilde{\phi}(t)x^4.$$

Under the canonical transformation Φ^+, the system (2.9) is transformed into the form

$$\begin{cases} \dot{\rho} = -\frac{\partial h}{\partial \varphi}, \\ \dot{\varphi} = \frac{\partial h}{\partial \rho}, \end{cases} \tag{2.10}$$

where

$$h(\rho, \varphi, t) = s\rho^{4/3} + \frac{\varepsilon}{4}c^{4/3}\rho^{4/3}\tilde{\phi}(t)C^4(\varphi T).$$

We expand around a fixed small value $\rho = \bar{\xi}\varepsilon$ with $\bar{\xi} \in [\frac{1}{2}, \frac{3}{2}]$, and scale the translated variable by $\varepsilon^{\frac{\bar{\sigma}}{4}}$ with

$$\frac{8}{9} < \bar{\sigma} < 1, \tag{2.11}$$

cf. [28] or [29]. This amounts for every $0 < \varepsilon < \varepsilon^*$ to the transformation

$$\Theta_\varepsilon : \left(-\varepsilon^{-\frac{\bar{\sigma}}{4}}\bar{\xi}, \varepsilon^{-\frac{\bar{\sigma}}{4}}\bar{\xi}\right) \times \mathbb{T} \to \mathbb{R}^+ \times \mathbb{T}$$
$$(I, \varphi) \mapsto (\rho, \varphi)$$

defined by $\varrho = \varepsilon(\bar{\xi} + \varepsilon^{\bar{\sigma}} I)$, $\varphi = \varphi$ and turns (2.10) into

$$\begin{cases} \dot{I} = \mathcal{O}(\varepsilon^{\frac{4}{3}-\bar{\sigma}}), \\ \dot{\varphi} = \frac{4}{3}s\varepsilon^{\frac{1}{3}}\bar{\xi}^{\frac{1}{3}} + \mathcal{O}(\varepsilon^{\frac{1}{3}+\frac{3\bar{\sigma}}{4}}) \end{cases} \tag{2.12}$$

with the Hamiltonian

$$G(I, \varphi, t, \varepsilon) = \frac{4}{3}s\varepsilon^{\frac{1}{3}}\bar{\xi}^{1/3} I + H(I, \varphi, t, \varepsilon), \qquad H = \mathcal{O}\big(\varepsilon^{\frac{4}{3}-\bar{\sigma}}\big). \tag{2.13}$$

Replacing the parameter $\bar{\xi}$ by the parameter $a = \frac{4}{3}s\varepsilon^{\frac{1}{3}}\bar{\xi}^{1/3}$ in (2.13), we can obtain the Hamiltonian system with the Hamiltonian

$$G(I, \varphi, t, \varepsilon) = aI + \varepsilon^{\frac{4}{3}-\bar{\sigma}}\hat{H}$$

with $|\hat{H}| < M$.

Let us take $\gamma = K(\varepsilon^*)^{\frac{1}{3}}$ and $\mu = (\varepsilon^*)^{\frac{1}{3}}$, then condition (1.2) in [28] is satisfied if $(\varepsilon^*)^{1-\bar{\sigma}} M < p_0^{m+3} K \delta_0^2$, where p_0 and δ_0 defined in [28]. Consequently, by reducing ε^* we can infinitely decrease K. From argumentations of Section 1 in [28], there exists a real analytic function Γ_∞ such that

$$a_0(\alpha) := \alpha + \Gamma_\infty(\varepsilon, \alpha) = \frac{4}{3}s\varepsilon^{\frac{1}{3}}\bar{\xi}^{1/3}, \quad \alpha \in A_\gamma$$

and $a_0(\alpha)$ is invertible. Denote the inverse function of a_0 by a_0^{-1}, we have

$$\alpha(\bar{\xi}) = a_0^{-1}\left(\frac{4}{3}s\varepsilon^{\frac{1}{3}}\bar{\xi}^{1/3}\right) \tag{2.14}$$

and

$$\alpha'(\bar{\xi}) = \mathcal{O}\big(\varepsilon^{\frac{1}{3}}\big) \neq 0, \quad \text{for } \varepsilon > 0. \tag{2.15}$$

It follows from Lemma 2 in [28] that for $\omega_0 = \frac{2\pi}{T}$, there exists a set

$$\hat{J} = \big\{\bar{\xi} \in [\omega_0/2, 3\omega_0/2]\colon \big|\langle k, \omega\rangle + l\bar{\xi}\big| \geqslant K\omega_0|k|^{-(m+1)}\big\},$$

whose measure tends to ω_0 as $K \to 0$. If $\bar{\xi} \in \hat{J}$, by using Lemma 1 in [28], it follows that the system (2.12) with Hamiltonian (2.13) has quasi-periodic solutions

$$I = v\big(0, \alpha(\bar{\xi})t + \psi_0, \omega t, \alpha(\bar{\xi})\big), \qquad \varphi = \alpha(\bar{\xi})t + \psi_0 + u\big(\alpha(\bar{\xi})t + \psi_0, \omega t, \alpha(\bar{\xi})\big), \tag{2.16}$$

where $\alpha \in A_\gamma$, ψ_0 is an arbitrary constant, and u and v are defined as in [28], as $\varepsilon \ll 1$. Hence, Eq. (2.1) has quasi-periodic solutions

$$x(t, \bar{\xi}, \varepsilon) = c^{1/3}\big[\varepsilon\big(\bar{\xi} + \varepsilon^{\bar{\sigma}} I\big)\big]^{\frac{1}{3}} C(\varphi T), \quad \alpha(\bar{\xi}) \in A_\gamma,\ \bar{\xi} \in \hat{J} \tag{2.17}$$

with frequency vector $\tilde{\omega}(\bar{\xi}) = (\omega_1, \omega_2, \ldots, \omega_m, \alpha(\bar{\xi}))$. Moreover, by (2.17) we have

$$x(t, \bar{\xi}, \varepsilon) = O\big(\varepsilon^{1/3}\big). \qquad \square$$

3. Hamiltonian setting of wave equations

From Lemma 2.1 we know that for every $\varepsilon \in (0, \varepsilon^*)$ Eq. (2.1) has a non-trivial quasi-periodic solution $u_0(t, \bar{\xi}, \varepsilon)$ of the form (2.17) with frequency vector $\tilde{\omega}(\bar{\xi})$ and satisfying $u_0(t, \bar{\xi}, \varepsilon) = \mathcal{O}(\varepsilon^{1/3})$. Taking $u = u_0(t, \bar{\xi}, \varepsilon) + \varepsilon v(t, x)$ in (1.2), we get the following equation

$$v_{tt} - v_{xx} + V\big(\tilde{\omega}(\bar{\xi})t, \bar{\xi}, \varepsilon\big)v + \varepsilon W\big(\tilde{\omega}(\bar{\xi})t, \bar{\xi}, \varepsilon\big)v^2 + \varepsilon^2 \hat{\phi}(\omega t, \varepsilon)v^3 = 0, \tag{3.1}$$

where $V(\tilde{\omega}(\bar{\xi})t, \bar{\xi}, \varepsilon) := 3\hat{\phi}(\omega t, \varepsilon)\bar{u}_0^2(\tilde{\omega}(\bar{\xi})t, \bar{\xi}, \varepsilon)$ and $W(\tilde{\omega}(\bar{\xi})t, \bar{\xi}, \varepsilon) := 3\hat{\phi}(\omega t, \varepsilon)\bar{u}_0(\tilde{\omega}(\bar{\xi})t, \bar{\xi}, \varepsilon)$ are quasi-periodic in time t with frequency vector $\tilde{\omega}(\bar{\xi})$, and $\hat{\phi}$ and $\bar{u}_0$ are the shell functions of ϕ and u_0, respectively. Let us write

$$\hat{V}(\theta, \bar{\xi}, \varepsilon) = 3c^{2/3}\hat{\phi}(\omega t, \varepsilon)\big[\big(\bar{\xi} + \varepsilon^{\bar{\sigma}} I\big)^{\frac{1}{3}} C(\varphi T)\big]^2 \tag{3.2}$$

with $\theta = \tilde{\omega}(\bar{\xi})t \in \mathbb{T}^{m+1}$, and

$$\begin{aligned}\frac{d}{d\bar{\xi}}\hat{V}(\theta, \bar{\xi}, \varepsilon) = 6c^{2/3}\hat{\phi}(\omega t, \varepsilon)\big[\big(\bar{\xi} + \varepsilon^{\bar{\sigma}} I\big)^{\frac{1}{3}} C(\varphi T)\big]\bigg[\frac{1}{3}\big(\bar{\xi} + \varepsilon^{\bar{\sigma}} I\big)^{-\frac{2}{3}}\bigg(1 + \varepsilon^{\bar{\sigma}}\frac{dI}{d\bar{\xi}}\bigg)C(\varphi T)\\ + T\big(\bar{\xi} + \varepsilon^{\bar{\sigma}} I\big)^{\frac{1}{3}} C'(\varphi T)\frac{d\varphi}{d\bar{\xi}}\bigg].\end{aligned}$$

From (2.14) and (2.16), we obtain

$$\lim_{\varepsilon\to 0}\varphi = a_0^{-1}(0)t + \psi_0 + u\big(a_0^{-1}(0)t + \psi_0, \omega t, a_0^{-1}(0)\big) := \varphi_0$$

and

$$\lim_{\varepsilon\to 0}\frac{d}{d\bar{\xi}}\varphi = \lim_{\varepsilon\to 0}\bigg(\alpha'(\bar{\xi})t + \frac{\partial u}{\partial(\alpha(\bar{\xi})t + \psi_0)}\alpha'(\bar{\xi})t + \frac{\partial u}{\partial(\alpha(\bar{\xi}))}\alpha'(\bar{\xi})\bigg) = 0.$$

Therefore, we have

$$\lim_{\varepsilon\to 0}\big[\hat{V}(\theta, \bar{\xi}, \varepsilon)\big] = \frac{1}{(2\pi)^{m+1}}\int_{\mathbb{T}^{m+1}}\lim_{\varepsilon\to 0}\hat{V}(\theta, \bar{\xi}, \varepsilon)\,d\theta = 3B\big(c^{1/3}\bar{\xi}^{1/3}C(\varphi_0 T)\big)^2$$

and

$$\lim_{\varepsilon\to 0}\frac{d}{d\bar{\xi}}\big[\hat{V}(\theta, \bar{\xi}, \varepsilon)\big] = \frac{1}{(2\pi)^{m+1}}\int_{\mathbb{T}^{m+1}}\lim_{\varepsilon\to 0}\frac{d}{d\bar{\xi}}\hat{V}(\theta, \bar{\xi}, \varepsilon)\,d\theta = 2B\big(c^{1/3}C(\varphi_0 T)\big)^2\bar{\xi}^{-1/3}.$$

Thus, there exists $0 < \varepsilon_1 < \varepsilon^*$ such that for any $\varepsilon \in (0, \varepsilon_1)$

$$\big[\hat{V}(\theta, \bar{\xi}, \varepsilon)\big] > \frac{3}{2}B\big(c^{1/3}\bar{\xi}^{1/3}C(\varphi_0 T)\big)^2 := I_1 > 0 \tag{3.3}$$

and

$$\frac{\partial}{\partial\bar{\xi}}\big[\hat{V}(\theta, \bar{\xi}, \varepsilon)\big] > B\big(c^{1/3}C(\varphi_0 T)\big)^2\bar{\xi}^{-1/3} := I_2 > 0. \tag{3.4}$$

Let us write

$$\hat{V}(\theta,\bar{\xi},\varepsilon) := \big[\hat{V}(\theta,\bar{\xi},\varepsilon)\big] + \tilde{V}(\theta,\bar{\xi},\varepsilon)$$

with $[\tilde{V}(\theta,\bar{\xi},\varepsilon)] = 0$, and $m_\varepsilon := \varepsilon^{\frac{2}{3}}[\hat{V}(\theta,\bar{\xi},\varepsilon)] > 0$, then

$$V(\theta,\bar{\xi},\varepsilon) = \varepsilon^{\frac{2}{3}}\hat{V}(\theta,\bar{\xi},\varepsilon) = m_\varepsilon + \varepsilon^{\frac{2}{3}}\tilde{V}(\theta,\bar{\xi},\varepsilon).$$

Furthermore, we can show that

$$\tilde{V}(\theta,\bar{\xi},\varepsilon) = \mathcal{O}\big(\varepsilon^{\frac{3\bar{\sigma}}{4}}\big) \tag{3.5}$$

as $\varepsilon \ll 1$. In fact, from (3.2), we have

$$\hat{V}(\theta,\bar{\xi},\varepsilon) = 3c^{2/3}\bar{\xi}^{\frac{2}{3}}\hat{\phi}(\omega t,\varepsilon)C^2(\varphi T) + \mathcal{O}\big(\varepsilon^{\frac{3\bar{\sigma}}{4}}\big)$$

and

$$\big[\hat{V}(\theta,\bar{\xi},\varepsilon)\big] = 3\big(B + \varepsilon[\tilde{\phi}]\big)c^{2/3}\bar{\xi}^{\frac{2}{3}}C^2(\varphi T) + \mathcal{O}\big(\varepsilon^{\frac{3\bar{\sigma}}{4}}\big).$$

Thus, we get

$$\begin{aligned}\tilde{V}(\theta,\bar{\xi},\varepsilon) &= \hat{V}(\theta,\bar{\xi},\varepsilon) - \big[\hat{V}(\theta,\bar{\xi},\varepsilon)\big] \\ &= 3c^{2/3}\bar{\xi}^{\frac{2}{3}}C^2(\varphi T)\big(\varepsilon\big(\tilde{\phi} - [\tilde{\phi}]\big)\big) + \mathcal{O}\big(\varepsilon^{\frac{3\bar{\sigma}}{4}}\big) \\ &= \mathcal{O}(\varepsilon) + \mathcal{O}\big(\varepsilon^{\frac{3\bar{\sigma}}{4}}\big) = \mathcal{O}\big(\varepsilon^{\frac{3\bar{\sigma}}{4}}\big).\end{aligned}$$

We can rewrite Eq. (3.1) as follows

$$\dot{v} = w, \qquad \dot{w} + Av = -\varepsilon^{\frac{2}{3}}\tilde{V}\big(\tilde{\omega}(\bar{\xi})t,\bar{\xi},\varepsilon\big)v - \varepsilon W\big(\tilde{\omega}(\bar{\xi})t,\bar{\xi},\varepsilon\big)v^2 - \varepsilon^2\hat{\phi}(\omega t,\varepsilon)v^3, \tag{3.6}$$

where $A = -d^2/dx^2 + m_\varepsilon, t \in \mathbb{R}$. As is well known, Eq. (3.1) can be studied as an infinite-dimensional Hamiltonian system by taking the phase space to be product of the Sobolev spaces $H_0^1([0,2\pi]) \times L^2([0,2\pi])$ with coordinates v and $w = \partial_t v$. The Hamiltonian for (3.6) is then

$$\begin{aligned}H = \frac{1}{2}\langle w,w\rangle + \frac{1}{2}\langle Av,v\rangle + \frac{1}{2}\varepsilon^{\frac{2}{3}}\tilde{V}\big(\tilde{\omega}(\bar{\xi})t,\bar{\xi},\varepsilon\big)\int_0^{2\pi} v^2\,dx \\ + \frac{1}{3}\varepsilon W\big(\tilde{\omega}(\bar{\xi})t,\bar{\xi},\varepsilon\big)\int_0^{2\pi} v^3\,dx + \frac{1}{4}\varepsilon^2\hat{\phi}(\omega t,\varepsilon)\int_0^{2\pi} v^4\,dx,\end{aligned} \tag{3.7}$$

here $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $L^2([0,2\pi])$.

By a simple computation, the eigenvalues λ_j and eigenfunctions $\phi_j(x) \in L^2[0,2\pi]$ of the operator A with the periodic boundary condition are, respectively,

$$\lambda_j = j^2 + m_\varepsilon, \quad j \in \mathbb{Z}, \quad \text{and} \quad \phi_j(x) = \begin{cases} \frac{1}{\sqrt{\pi}} \cos jx, & j > 0, \\ -\frac{1}{\sqrt{\pi}} \sin jx, & j < 0, \\ \frac{1}{\sqrt{2\pi}}, & j = 0. \end{cases} \tag{3.8}$$

In order to avoid the double eigenvalues we restrict ourselves to find some solutions which are even in x. Note the eigenfunctions $\phi_j(x)'s$ with $j \geqslant 0$ are a complete orthogonal basis of the subspace consisting of all even functions of $L^2(0, 2\pi)$.

We introduce coordinates $q = (q_0, q_1, q_2, \ldots)$, $p = (p_0, p_1, p_2, \ldots)$ through the relations

$$v(t,x) = \sum_{j\geqslant 0} \frac{q_j(t)}{\sqrt[4]{\lambda_j}} \phi_j(x), \qquad \partial_t v(t,x) = \sum_{j\geqslant 0} \sqrt[4]{\lambda_j} p_j(t) \phi_j(x). \tag{3.9}$$

The coordinates are taken from some real Hilbert space:

$$l^{a,s} = l^{a,s}(\mathbb{R}) := \left\{ q = (q_0, q_1, q_2, \ldots),\ q_i \in \mathbb{R},\ i \geqslant 0 \text{ s.t. } \|q\|_{a,s}^2 = |q_0|^2 + \sum_{i\geqslant 1} |q_i|^2 i^{2s} e^{2ai} < \infty \right\}.$$

Below we will assume that $a \geqslant 0$ and $s > 1/2$. One rewrites the Hamiltonian (3.7) in the coordinates (q, p),

$$H = \Lambda + G, \tag{3.10}$$

where

$$\Lambda = \frac{1}{2} \sum_{j\geqslant 0} \sqrt{\lambda_j}\big(p_j^2 + q_j^2\big) + \varepsilon^{\frac{2}{3}} \frac{\tilde{V}(\tilde{\omega}(\bar{\xi})t, \bar{\xi}, \varepsilon)}{\sqrt{\lambda_j}} q_j^2,$$

$$G = \frac{1}{3} \varepsilon W\big(\tilde{\omega}(\bar{\xi})t, \bar{\xi}, \varepsilon\big) \int_0^{2\pi} \bigg(\sum_{j\geqslant 0} \frac{q_j(t)}{\sqrt[4]{\lambda_j}} \phi_j(x) \bigg)^3 dx + \frac{1}{4} \varepsilon^2 \hat{\phi}(\omega t, \varepsilon) \int_0^{2\pi} \bigg(\sum_{j\geqslant 0} \frac{q_j(t)}{\sqrt[4]{\lambda_j}} \phi_j(x) \bigg)^4 dx.$$

The equations of motion are

$$\dot{q_j} = \frac{\partial H}{\partial p_j} = \sqrt{\lambda_j} p_j, \qquad \dot{p_j} = -\frac{\partial H}{\partial q_j} = -\sqrt{\lambda_j} q_j - \varepsilon^{\frac{2}{3}} \frac{\tilde{V}(\theta, \bar{\xi}, \varepsilon)}{\sqrt{\lambda_j}} q_j - \frac{\partial G}{\partial q_j}, \quad j \geqslant 0 \tag{3.11}$$

with respect to the symplectic structure $\sum dq_i \wedge dp_i$ on $l^{a,s} \times l^{a,s}$.

Lemma 3.1. *Let I be an interval and let*

$$t \in I \to \big(q(t), p(t)\big) \equiv \big(\{q_j(t)\}_{j\geqslant 0}, \{p_j(t)\}_{j\geqslant 0}\big)$$

be a real analytic solution of (3.11) *for $a > 0$. Then*

$$v(t,x) = \sum_{j\geqslant 0} \frac{q_j(t)}{\sqrt[4]{\lambda_j}} \phi_j(x)$$

is a classical solution of (3.1) *that is real analytic on $I \times [0, 2\pi]$.*

Proof. From the hypotheses it is easy to see that the sum defining $v(t,x)$ is absolutely convergent in some complex neighborhood of the x-interval $[0,2\pi]$ and some complex disc around a given t in I. Therefore v is real analytic in x and t, and we may differentiate under the summation sign. Since

$$\frac{\partial G}{\partial q_j}=\frac{1}{\sqrt[4]{\lambda_j}}\left(\varepsilon W\big(\tilde{\omega}(\xi)t,\xi,\varepsilon\big)\int_0^{2\pi} v^2\phi_j(x)\,dx+\varepsilon^2\hat{\phi}(\omega t,\varepsilon)\int_0^{2\pi} v^3\phi_j(x)\,dx\right). \tag{3.12}$$

Hence, we have, by (3.11) and (3.12)

$$\begin{aligned} v_{tt}+Av &= \sum_{j\geqslant 0}\frac{\ddot{q}_j(t)}{\sqrt[4]{\lambda_j}}\phi_j(x)+\sum_{j\geqslant 0}\frac{q_j(t)}{\sqrt[4]{\lambda_j}}A\phi_j(x)\\ &= \sum_{j\geqslant 0}\sqrt[4]{\lambda_j}\left(-\sqrt{\lambda_j}q_j-\varepsilon^{\frac{2}{3}}\frac{\tilde{V}(\tilde{\omega}(\bar{\xi})t,\bar{\xi},\varepsilon)}{\sqrt{\lambda_j}}q_j-\frac{\partial G}{\partial q_j}\right)\phi_j(x)+\sum_{j\geqslant 0}\frac{q_j(t)}{\sqrt[4]{\lambda_j}}\lambda_j\phi_j(x)\\ &= -\varepsilon^{\frac{2}{3}}\tilde{V}\big(\tilde{\omega}(\bar{\xi})t,\bar{\xi},\varepsilon\big)v-\varepsilon W\big(\tilde{\omega}(\bar{\xi})t,\bar{\xi},\varepsilon\big)\sum_{j\geqslant 0}\langle v^2,\phi_j\rangle\phi_j-\varepsilon^2\hat{\phi}(\omega t,\varepsilon)\sum_{j\geqslant 0}\langle v^3,\phi_j\rangle\phi_j\\ &= -\varepsilon^{\frac{2}{3}}\tilde{V}\big(\tilde{\omega}(\bar{\xi})t,\bar{\xi},\varepsilon\big)v-\varepsilon W\big(\tilde{\omega}(\bar{\xi})t,\bar{\xi},\varepsilon\big)v^2-\varepsilon^2\hat{\phi}(\omega t,\varepsilon)v^3 \end{aligned}$$

as required. □

One introduces a pair of action-angle variables (J,θ), where $J\in\mathbb{R}^{m+1}$ is canonically conjugate to $\theta=\tilde{\omega}(\bar{\xi})t\in\mathbb{T}^{m+1}$. Then (3.6) can be written as a Hamiltonian system

$$\dot{q_j}=\frac{\partial H}{\partial p_j},\qquad \dot{p_j}=-\frac{\partial H}{\partial q_j},\quad j\geqslant 0,\qquad \dot{\theta}=\tilde{\omega}(\bar{\xi}),\qquad \dot{J}=-\frac{\partial H}{\partial\theta} \tag{3.13}$$

with the Hamiltonian

$$\begin{aligned} H=\big\langle\tilde{\omega}(\bar{\xi}),J\big\rangle+\frac{1}{2}\sum_{j\geqslant 0}\sqrt{\lambda_j}\big(p_j^2+q_j^2\big)+\varepsilon^{\frac{2}{3}}\frac{\tilde{V}(\theta,\bar{\xi},\varepsilon)}{\sqrt{\lambda_j}}q_j^2\\ +\varepsilon G^3(q,\theta,\bar{\xi},\varepsilon)+\varepsilon^2G^4(q,\vartheta), \end{aligned} \tag{3.14}$$

where $\vartheta=\omega t$,

$$G^3(q,\theta,\bar{\xi},\varepsilon)=\sum_{i,j,d\geqslant 0}G^3_{i,j,d}(\theta,\bar{\xi},\varepsilon)q_iq_jq_d \tag{3.15}$$

with

$$G^3_{i,j,d}(\theta,\bar{\xi},\varepsilon)=\frac{1}{3}\frac{W(\theta,\bar{\xi},\varepsilon)}{\sqrt[4]{\lambda_i\lambda_j\lambda_d}}\int_{\mathbb{T}}\phi_i(x)\phi_j(x)\phi_d(x)\,dx \tag{3.16}$$

and

$$G^4(q,\vartheta)=\sum_{i,j,d,l\geqslant 0}G^4_{i,j,d,l}(\vartheta)q_iq_jq_dq_l \tag{3.17}$$

with

$$G^4_{i,j,d,l}(\vartheta)=\frac{1}{4}\frac{\hat{\phi}(\vartheta)}{\sqrt[4]{\lambda_i\lambda_j\lambda_d\lambda_l}}\int\limits_{\mathbb{T}}\phi_i(x)\phi_j(x)\phi_d(x)\phi_l(x)\,dx. \tag{3.18}$$

It is not difficult to verify that, from (3.8),

$$G^3_{i,j,d}(\theta,\bar{\xi},\varepsilon)=0,\quad \text{unless } i\pm j\pm d=0 \tag{3.19}$$

and

$$G^4_{i,j,d,l}(\vartheta)=0,\quad \text{unless } i\pm j\pm d\pm l=0. \tag{3.20}$$

We introduce complex coordinate

$$z_j=\frac{1}{\sqrt{2}}(q_j-\mathrm{i}\,p_j),\qquad \bar{z}_j=\frac{1}{\sqrt{2}}(q_j+\mathrm{i}\,p_j),\quad j\geqslant 0,$$

that live in the now complex Hilbert space:

$$l^{a,s}=l^{a,s}(\mathbb{C}):=\left\{z=(z_0,z_1,z_2,\ldots),z_j\in\mathbb{C},\ j\geqslant 0 \text{ s.t. } \|z\|^2_{a,s}=|z_0|+\sum_{j\geqslant 1}|z_j|^2j^{2s}e^{2aj}<\infty\right\}.$$

This transformation is symplectic with $dq\wedge dp=\sqrt{-1}\,dz\wedge d\bar{z}$, and then (3.14) is changed into

$$H=\tilde{H}_j+\varepsilon G^3(z,\theta,\bar{\xi},\varepsilon)+\varepsilon^2G^4(z,\vartheta) \tag{3.21}$$

where

$$\tilde{H}_j=\langle\tilde{\omega}(\bar{\xi}),J\rangle+\sum_{j\geqslant 0}\sqrt{\lambda_j}z_j\bar{z}_j+\frac{\varepsilon^{\frac{2}{3}}\tilde{V}(\theta,\bar{\xi},\varepsilon)}{4\sqrt{\lambda_j}}(z_j+\bar{z}_j)^2, \tag{3.22}$$

$$\begin{aligned}G^3(z,\theta,\bar{\xi},\varepsilon)={}&G^3_{0,0,0}(\theta,\bar{\xi},\varepsilon)\left(\frac{z_0+\bar{z}_0}{\sqrt{2}}\right)^3\\&+3\sum_{j,d\neq 0}G^3_{0,j,d}(\theta,\bar{\xi},\varepsilon)\left(\frac{z_j+\bar{z}_j}{\sqrt{2}}\right)\left(\frac{z_d+\bar{z}_d}{\sqrt{2}}\right)\left(\frac{z_0+\bar{z}_0}{\sqrt{2}}\right)\\&+\sum_{i,j,d\neq 0}G^3_{i,j,d}(\theta,\bar{\xi},\varepsilon)\left(\frac{z_i+\bar{z}_i}{\sqrt{2}}\right)\left(\frac{z_j+\bar{z}_j}{\sqrt{2}}\right)\left(\frac{z_d+\bar{z}_d}{\sqrt{2}}\right)\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}G^4(z,\vartheta)={}&G^4_{0,0,0,0}(\vartheta)\left(\frac{z_0+\bar{z}_0}{\sqrt{2}}\right)^4\\&+6\sum_{j,d\neq 0}G^4_{0,0,j,d}(\vartheta)\left(\frac{z_j+\bar{z}_j}{\sqrt{2}}\right)\left(\frac{z_d+\bar{z}_d}{\sqrt{2}}\right)\left(\frac{z_0+\bar{z}_0}{\sqrt{2}}\right)^2\end{aligned}$$

$$+4\sum_{j,d,l\neq 0} G^4_{0,j,d,l}(\vartheta)\left(\frac{z_j+\bar z_j}{\sqrt2}\right)\left(\frac{z_d+\bar z_d}{\sqrt2}\right)\left(\frac{z_l+\bar z_l}{\sqrt2}\right)\left(\frac{z_0+\bar z_0}{\sqrt2}\right)$$
$$+\sum_{i,j,d,l\neq 0} G^4_{i,j,d,l}(\vartheta)\left(\frac{z_i+\bar z_i}{\sqrt2}\right)\left(\frac{z_j+\bar z_j}{\sqrt2}\right)\left(\frac{z_d+\bar z_d}{\sqrt2}\right)\left(\frac{z_l+\bar z_l}{\sqrt2}\right). \tag{3.24}$$

4. Reducibility of linear Hamiltonian system (3.22)

In this section, we will be concerned with the reducibility of linear quasi-periodic Hamiltonian system (4.2). Our result shows that system (3.22) can be reduced to constant coefficients for any fixed $\omega \in D_\Lambda$.

4.1. Notations

For given $\sigma > 0$, $r > 0$, we define sequences $\{\sigma_\nu\}$ and $\{r_\nu\}$:

(1) $\sigma_0 = \sigma$, $\sigma_\nu = \sigma_0(1-\tau_\nu)$ with $\tau_0 = 0$ and $\tau_\nu = \frac{\sum_{j=1}^{\nu} j^{-2}}{2\sum_{j=1}^{\infty} j^{-2}}$, $\nu = 1, 2, \ldots$. It is easy to see $\sigma_\nu > \sigma_{\nu+1} > \sigma/2$.

(2) $r_0 = r$, $r_\nu = r_0(1-\tau_\nu)$, $\nu = 1, 2, \ldots$. It is easy to see $r_\nu > r_{\nu+1} > r/2$.

In addition, we need the following notations:

(3) We let

$$\Theta(\sigma) = \left\{\theta = (\theta_1, \ldots, \theta_m, \theta_{m+1}) \in \mathbb{C}^{m+1}/2\pi\mathbb{Z}^{m+1}\colon |\operatorname{Im}\theta| < \sigma\right\}$$

and

$$D^{a,s}(\sigma, r) = \left\{(\theta, J, z, \bar z) \in \mathbb{C}^{m+1}/2\pi\mathbb{Z}^{m+1} \times \mathbb{C}^{m+1} \times l^{a,s} \times l^{a,s}\colon |\operatorname{Im}\theta| < \sigma,\right.$$
$$\left. |J| < r^2,\ \|z\|_{a,s} < r,\ \|\bar z\|_{a,s} < r\right\},$$

where $|\cdot|$ denotes the sup-norm for complex vectors and $l^{a,s}$ denotes now complex Hilbert space. Thus, we get a family of domains:

$$\Theta(\sigma_0) \supset \Theta(\sigma_1) \supset \cdots \supset \Theta(\sigma_\nu) \supset \Theta(\sigma_{\nu+1}) \supset \cdots \supset \Theta\left(\frac{\sigma_0}{2}\right)$$

and

$$D^{a,s}(\sigma_0, r_0) \supset D^{a,s}(\sigma_1, r_1) \supset \cdots \supset D^{a,s}(\sigma_\nu, r_\nu) \supset D^{a,s}(\sigma_{\nu+1}, r_{\nu+1}) \supset \cdots \supset D^{a,s}\left(\frac{\sigma_0}{2}, \frac{r_0}{2}\right).$$

(4) We write $\Theta_l := \Theta(\sigma_l)$, $D^{a,s}_l = D^{a,s}(\sigma_l, r_l)$, $l = 0, 1, \ldots$.

For an one order Whitney smooth function $F(\bar\xi)$ on closed bounded set J^*, we define

$$\|F\|^*_{J^*} = \max\left\{\sup_{\bar\xi\in J^*}|F|,\ \sup_{\bar\xi\in J^*}|\partial_{\bar\xi}F|\right\}.$$

If $F(\bar\xi)$ is a vector function from $\hat J$ to $l^{a,s}(or\mathbb{R}^{m_1\times m_2})$ which is one order Whitney smooth on J^*, we define

$$\|F\|^*_{a,s,J^*} = \left\|\left(\left\|F_i(\bar\xi)\right\|^*_{J^*}\right)_i\right\|_{a,s} \quad \left(\text{or } \|F\|^*_{J^*} = \max_{1\leqslant i_1\leqslant m_1}\sum_{1\leqslant i_2\leqslant m_2}\left(\left\|F_{i_1i_2}(\bar\xi)\right\|^*_{J^*}\right)\right).$$

Let $\tilde{w} = (\theta, J, z, \bar{z}) \in D^{a,s}$, we denote the weighted norm for $\tilde{w}$ by letting

$$|\tilde{w}|_{a,s} = |\theta| + \frac{1}{r^2}|J| + \frac{1}{r}\|z\|_{a,s} + \frac{1}{r}\|\bar{z}\|_{a,s}.$$

If $F(\eta;\bar{\xi})$ is a vector function from $D^{a,s} \times J^*$ to $l^{a,s}(or\mathbb{R}^{m_1\times m_2})$ which is one order Whitney smooth on $\bar{\xi}$, we define

$$\|F\|^*_{a,s,D^{a,s}\times J^*} = \sup_{\eta\in D^{a,s}} \|F\|^*_{a,s,J^*} \quad \Big(\text{or } \|F\|^*_{D^{a,s}\times J^*} = \sup_{\eta\in D^{a,s}} \|F\|^*_{J^*}\Big).$$

To function F, associate a Hamiltonian vector field defined as $X_F = \{F_J, -F_\theta, \mathrm{i}F_{\bar{z}}, -\mathrm{i}F_z\}$, we denote the weighted norm for X_F by letting

$$|X_F|^*_{a,s,D^{a,s}\times J^*} = \|F_J\|^*_{D^{a,s}\times J^*} + \frac{1}{r^2}\|F_\theta\|^*_{D^{a,s}\times J^*} + \frac{1}{r}\|F_{\bar{z}}\|^*_{a,s,D^{a,s}\times J^*} + \frac{1}{r}\|F_z\|^*_{a,s,D^{a,s}\times J^*}.$$

Let $A(\eta;\bar{\xi})$ be an operator from $l_b^{a,s}$ to $l_b^{a,\bar{s}}$ for $(\eta;\bar{\xi}) \in D^{a,s} \times J^*$, we define the operator norm

$$\big\|A(\eta;\bar{\xi})\big\|^{op}_{a,\bar{s},s,D^{a,s}\times J^*} = \sup_{(\eta;\bar{\xi})\in D^{a,s}\times J^*} \sup_{w\neq 0} \frac{\|A(\eta;\bar{\xi})w\|_{a,\bar{s}}}{\|w\|_{a,s}},$$
$$\big\|A(\eta;\bar{\xi})\big\|^{*op}_{a,\bar{s},s,D^{a,s}\times J^*} = \max\big\{\|A\|^{op}_{a,\bar{s},s,D^{a,s}\times J^*}, \|\partial_{\bar{\xi}}A\|^{op}_{a,\bar{s},s,D^{a,s}\times J^*}\big\}.$$

Let $B(\eta;\bar{\xi})$ be an operator from $D^{a,s}$ to $D^{a,\bar{s}}$ for $(\eta;\bar{\xi}) \in D^{a,s} \times J^*$, we define the operator norm

$$\big|B(\eta;\bar{\xi})\big|^{op}_{a,\bar{s},s,D^{a,s}\times J^*} = \sup_{(\eta;\bar{\xi})\in D^{a,s}\times J^*} \sup_{\tilde{w}\neq 0} \frac{|B(\eta;\bar{\xi})\tilde{w}|_{a,\bar{s}}}{|\tilde{w}|_{a,s}},$$
$$\big|B(\eta;\bar{\xi})\big|^{*op}_{a,\bar{s},s,D^{a,s}\times J^*} = \max\big\{|B|^{op}_{a,\bar{s},s,D^{a,s}\times J^*}, |\partial_{\bar{\xi}}B|^{op}_{a,\bar{s},s,D^{a,s}\times J^*}\big\}.$$

In view of (3.5), it follows that $\tilde{V}(\theta,\bar{\xi},\varepsilon)$ there is $C>0$ such that$\tilde{V}(\theta,\bar{\xi},\varepsilon)$ is analytic on Θ_0 and

$$\big|\tilde{V}(\theta,\bar{\xi},\varepsilon)\big|_{\Theta_0\times\hat{J}} < C\varepsilon^{\frac{3\bar{\sigma}}{4}} \tag{4.1}$$

for $\sigma > 0$ and $r > 0$. Let us rewrite Hamiltonian (3.22) as

$$\tilde{H} = H_0 + \varepsilon^{\frac{2}{3}} H_1, \tag{4.2}$$

where

$$H_0 = \big\langle\tilde{\omega}(\bar{\xi}), J\big\rangle + \sum_{j\geqslant 0}\sqrt{\lambda_j}z_j\bar{z}_j \quad \text{and} \quad H_1 = \sum_{j\geqslant 0}\frac{\tilde{V}(\theta,\bar{\xi},\varepsilon)}{4\sqrt{\lambda_j}}(z_j+\bar{z}_j)^2.$$

4.2. Reducibility theorem

Theorem 4.1. *Consider the Hamiltonian $\tilde{H}$ as given by Eq.* (4.2). *Assume that* (4.1) *is satisfied. Then there are a $0<\varepsilon^{**}<\varepsilon^*$, $0<\varrho<1$ and a set $\bar{J}\subset\hat{J}$ with* $\operatorname{meas}\bar{J}\geqslant\operatorname{meas}\hat{J}(1-\mathcal{O}(\varrho))$ *such that for any $0<\varepsilon<\varepsilon^{**}$, $\bar{\xi}\in\bar{J}$ and $\alpha(\bar{\xi})\in A_\gamma$ there is a linear symplectic transformation*

$$\Sigma^\infty:\mathcal{D}^{a,s}(\sigma/2,r/2)\times\bar{J}\to\mathcal{D}^{a,s}(\sigma,r)$$

such that the following statements hold:

(i) *There is some absolute constant $C>0$ such that*

$$\left|\Sigma^\infty-id\right|^*_{a,s+1,\,\mathcal{D}^{a,s}(\sigma/2,r/2)\times\bar{J}}\leqslant C\varepsilon^{2/3},$$

where id is identity mapping.

(ii) *The transformation Σ^∞ changes Hamiltonian* (4.2) *into*

$$\tilde{H}\circ\Sigma^\infty=\langle\tilde{\omega}(\bar{\xi}),\,J\rangle+\sum_{j\geqslant0}\mu_j z_j\bar{z}_j,$$

where

$$\mu_j=\sqrt{\lambda_j}+\sum_{k=1}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{j,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big),\tag{4.3}$$

$\tilde{\lambda}_{j,1}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon)=\frac{[\tilde{V}(\theta,\bar{\xi},\varepsilon)]}{2\sqrt{\lambda_j}}=0$ *and* $|\tilde{\lambda}_{j,k}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon)|\leqslant C$, $k=2,3,\ldots.$

Proof. Step 1. Constructing iterative sequence. We will construct iteratively a series $\{\tilde{H}_l\}$ of Hamiltonian functions of the form

$$\begin{aligned}\tilde{H}_l={}&\langle\tilde{\omega}(\bar{\xi}),\,J\rangle+\sum_{j\geqslant0}\lambda_{j,l}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)z_j\bar{z}_j+\varepsilon^{\frac{2}{3}(l+1)}R_{l+1}\big(z,\bar{z},\theta,\tilde{\omega}(\bar{\xi}),\bar{\xi},\varepsilon\big),\\ &l=0,1,\ldots,\nu.\end{aligned}\tag*{$(E)_l$}$$

where

$$R_{l+1}\big(z,\bar{z},\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)=\sum_{j\geqslant0}\sum_{n_1+n_2=2}\zeta_{j,l,n_1,n_2}\big(\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)z_j^{n_1}\bar{z}_j^{n_2},\quad l=0,1,\ldots,\nu$$

with $\zeta_{j,l,n_1,n_2}(\theta,\xi,\tilde{\omega}(\bar{\xi}),\varepsilon)=\sum_{k\in\mathbb{Z}^{m+1}}\zeta_{j,l,k,n_1,n_2}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon)e^{i\langle k,\theta\rangle}$, $\zeta_{j,0,2,0}=\zeta_{j,0,0,2}=\frac{1}{4\sqrt{\lambda_j}}\tilde{V}(\theta,\bar{\xi},\varepsilon)$ and $\zeta_{j,0,1,1}=\frac{1}{2\sqrt{\lambda_j}}\tilde{V}(\theta,\bar{\xi},\varepsilon)$. Furthermore, the functions $\zeta_{j,l,n_1,n_2}(\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon)$ are analytic on the domain $\Theta_l\times\bar{J}_l$,

$$\begin{aligned}&\zeta_{j,l,n_1,n_2}=\varepsilon^{\frac{3\bar{\sigma}}{4}}\lambda_j^{-1/2}\zeta^*_{j,l,n_1,n_2},\qquad\big\|\zeta^*_{j,l,n_1,n_2}\big\|^*_{\Theta_l\times\bar{J}_l}\leqslant C,\quad n_1,n_2\in\mathbb{N},\\&\quad n_1+n_2=2,\quad l=0,1,\ldots,\nu,\end{aligned}\tag*{$(4.1)_l$}$$

and

$$\lambda_{j,0}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)=\sqrt{\lambda_j},$$

$$\lambda_{j,l}\big(\bar\xi,\tilde\omega(\bar\xi),\varepsilon\big)=\sqrt{\lambda_j}+\sum_{k=1}^{l}\varepsilon^{\frac{2k}{3}}\tilde\lambda_{j,k}\big(\bar\xi,\tilde\omega(\bar\xi),\varepsilon\big),\quad l=2,3,\dots,\nu, \tag{4.2$)_l$}$$

with

$$\tilde\lambda_{j,k}\big(\bar\xi,\tilde\omega(\bar\xi),\varepsilon\big)=\frac{1}{(2\pi)^{m+1}}\int\limits_{\mathbb{T}^{m+1}}\zeta_{j,k-1,1,1}\big(\theta,\bar\xi,\tilde\omega(\bar\xi),\varepsilon\big)\,d\theta,\quad k=1,2,\dots,l. \tag{4.4}$$

Clearly, we have that $\tilde H_0=\tilde H$ for $l=0$. The functions $\zeta_{j,0,n_1,n_2}(\theta,\bar\xi,\tilde\omega(\bar\xi),\varepsilon)$ are analytic on the domain $\Theta_0\times\hat J$ and satisfy $(4.1)_0$, and

$$\tilde\lambda_{j,1}\big(\bar\xi,\tilde\omega(\bar\xi),\varepsilon\big)=\frac{1}{(2\pi)^{m+1}}\int\limits_{\mathbb{T}^{m+1}}\zeta_{j,0,1,1}(\theta,\bar\xi,\varepsilon)\,d\theta=\frac{1}{2\sqrt{\lambda_j}}\big[\tilde V(\theta,\bar\xi,\varepsilon)\big]=0.$$

Step 2. Solving the homological equations. Let

$$\big\langle k,\tilde\omega(\bar\xi)\big\rangle=\sum_{j=1}^{m}k_j\omega_j+k_{m+1}\alpha(\bar\xi),$$

then we can show that following fact: for ε small sufficiently there exist a family of closed subsets $\bar J_l(l=0,\dots,\nu)$

$$\bar J_\nu\subset\cdots\subset\bar J_{l+1}\subset\bar J_l\subset\cdots\subset\bar J_0\subset\hat J\subset[\pi/T,3\pi/T]$$

such that for $\bar\xi\in\bar J_l$,

$$\big|\big\langle k,\tilde\omega(\bar\xi)\big\rangle\pm2\lambda_{0,l}\big(\bar\xi,\tilde\omega(\bar\xi),\varepsilon\big)\big|\geqslant\frac{\varepsilon^{\frac13}\varrho\,\mathrm{meas}\,\hat J}{(1+l^2)(|k|+1)^{m+3}} \tag{4.3$)_l$}$$

and

$$\big|\big\langle k,\tilde\omega(\bar\xi)\big\rangle\pm2\lambda_{j,l}\big(\bar\xi,\tilde\omega(\bar\xi),\varepsilon\big)\big|\geqslant\begin{cases}\frac{\varepsilon^{\frac13}\varrho\,\mathrm{meas}\,\hat J}{(1+l^2)(|k|+1)^{m+3}}, & k_{m+1}\neq0,\\ \frac{\varepsilon^{\frac23}\varrho\,\mathrm{meas}\,\hat J}{(1+l^2)(|k|+1)^{m+3}}, & k_{m+1}=0,\end{cases}\quad j\geqslant1,\ l=0,\dots,\nu, \tag{4.4$)_l$}$$

and

$$\mathrm{meas}\,\bar J_l\geqslant\mathrm{meas}\,\hat J\left(1-C\varrho\sum_{i=0}^{l}\frac{1}{1+i^2}\right), \tag{4.5$)_l$}$$

where C is a constant depending on m and $\bar\xi$. Moreover, let us assume that $\bar J=\bigcap_{l=0}^{\infty}\bar J_l$, then

$$\mathrm{meas}\,\bar J\geqslant\mathrm{meas}\,\hat J\big(1-\mathcal{O}(\varrho)\big)$$

provided that ϱ is small enough. The proof will give out in Lemma 4.2 below.

Let $X_{\mathcal{F}_\nu}$ be the Hamiltonian vector field associated with a function $\mathcal{F}_\nu$:

$$\mathcal{F}_\nu = \varepsilon^{\frac{2}{3}(\nu+1)} \sum_{j\geqslant 0} \sum_{n_1+n_2=2} \beta_{j,\nu,n_1,n_2}\big(\theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) z_j^{n_1} \bar{z}_j^{n_2}$$

with

$$\beta_{j,\nu,n_1,n_2}\big(\theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) = \sum_{k\in\mathbb{Z}^{m+1}} \beta_{j,\nu,k,n_1,n_2}\big(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) e^{\mathrm{i}\langle k,\theta\rangle} \tag{4.5}$$

and $[\beta_{j,\nu,1,1}] = 0$, and let $X^t_{\mathcal{F}_\nu}$ denote its time-t map.

We look for a change of variables S_ν defined in a domain $D^{a,s}_{\nu+1}$ by the time-one map $X^1_{\mathcal{F}_\nu}$ of the Hamiltonian vector field $X_{\mathcal{F}_\nu}$, such that the system $(E)_\nu$ is transformed into the form $(E)_{\nu+1}$ and satisfies $(4.1)_{\nu+1}$, $(4.2)_{\nu+1}$, $(4.3)_{\nu+1}$, $(4.4)_{\nu+1}$ and $(4.5)_{\nu+1}$. In fact, the new Hamiltonian $\tilde{H}_{l+1}$ can be written as

$$\begin{aligned}
\tilde{H}_{\nu+1} := \tilde{H}_\nu \circ X^1_{\mathcal{F}_\nu} = {}& \big\langle \tilde{\omega}(\bar{\xi}), J\big\rangle + \sum_{j\geqslant 0} \lambda_{j,\nu}\big(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) z_j \bar{z}_j \\
&+ \Big\{ \big\langle \tilde{\omega}(\bar{\xi}), J\big\rangle + \sum_{j\geqslant 0} \lambda_{j,\nu}\big(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) z_j \bar{z}_j, \mathcal{F}_\nu \Big\} \\
&+ \varepsilon^{\frac{2}{3}(\nu+1)} R_{\nu+1}\big(z, \bar{z}, \theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) + \big\{ \varepsilon^{\frac{2}{3}(\nu+1)} R_{\nu+1}\big(z, \bar{z}, \theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big), \mathcal{F}_\nu \big\} \\
&+ \int_0^1 (1-t) \big\{ \{\tilde{H}_\nu, \mathcal{F}_\nu\}, \mathcal{F}_\nu \big\} \circ X^t_{\mathcal{F}_\nu}\, dt.
\end{aligned} \tag{4.6}$$

The function F_ν is determined by the homological equation

$$\begin{aligned}
&\Big\{ \big\langle \tilde{\omega}(\bar{\xi}), J\big\rangle + \sum_{j\geqslant 0} \lambda_{j,\nu}\big(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) z_j \bar{z}_j, \mathcal{F}_\nu \Big\} + \varepsilon^{\frac{2}{3}(\nu+1)} R_{\nu+1}\big(z, \bar{z}, \theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) \\
&\quad = \varepsilon^{\frac{2}{3}(\nu+1)} \sum_{j\geqslant 0} [\zeta_{j,\nu,1,1}] z_j \bar{z}_j,
\end{aligned} \tag{4.7}$$

which is equivalent to

$$\begin{cases}
\big\langle \tilde{\omega}(\bar{\xi}), \partial_\theta \beta_{j,\nu,1,1}\big(\theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big)\big\rangle + \zeta_{j,\nu,1,1}\big(\theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) = [\zeta_{j,\nu,1,1}], \\
2\mathrm{i}\,\lambda_{j,\nu}\big(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) \beta_{j,\nu,0,2}\big(\theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) + \big\langle \tilde{\omega}(\bar{\xi}), \partial_\theta \beta_{j,\nu,0,2}\big(\theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big)\big\rangle \\
\quad + \zeta_{j,\nu,0,2}\big(\theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) = 0, \\
-2\mathrm{i}\,\lambda_{j,\nu}\big(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) \beta_{j,\nu,2,0}\big(\theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) + \big\langle \tilde{\omega}(\bar{\xi}), \partial_\theta \beta_{j,\nu,2,0}\big(\theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big)\big\rangle \\
\quad + \zeta_{j,\nu,2,0}\big(\theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) = 0.
\end{cases} \tag{4.8}$$

Inserting (4.5) into (4.8) we get

$$\begin{cases} \mathrm{i}\langle k,\tilde{\omega}(\bar{\xi})\rangle\beta_{j,\nu,k,1,1} = -\zeta_{j,\nu,k,1,1}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big), \quad k\neq 0, \\ \mathrm{i}\left(\langle k,\tilde{\omega}(\bar{\xi})\rangle + 2\lambda_{j,\nu}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon)\right)\beta_{j,\nu,k,0,2}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big) = -\zeta_{j,\nu,k,0,2}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big), \\ \mathrm{i}\left(\langle k,\tilde{\omega}(\bar{\xi})\rangle - 2\lambda_{j,\nu}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right)\beta_{j,\nu,k,2,0}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big) = -\zeta_{j,\nu,k,2,0}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big). \end{cases}$$

Hence, we get

$$\beta_{j,\nu,1,1}\big(\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big) = \sum_{0\neq k\in\mathbb{Z}^{m+1}} \frac{-\zeta_{j,\nu,k,1,1}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon)}{\mathrm{i}\,\langle k,\tilde{\omega}(\bar{\xi})\rangle} e^{\mathrm{i}\,\langle k,\theta\rangle}, \tag{4.9}$$

$$\beta_{j,\nu,0,2}\big(\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big) = \sum_{k\in\mathbb{Z}^{m+1}} \frac{-\zeta_{j,\nu,k,0,2}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon)}{\mathrm{i}(\langle k,\tilde{\omega}(\bar{\xi})\rangle + 2\lambda_{j,\nu}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon))} e^{\mathrm{i}\langle k,\theta\rangle}, \tag{4.10}$$

and

$$\beta_{j,\nu,2,0}\big(\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big) = \sum_{k\in\mathbb{Z}^{m+1}} \frac{-\zeta_{j,\nu,k,2,0}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon)}{\mathrm{i}(\langle k,\tilde{\omega}(\bar{\xi})\rangle - 2\lambda_{j,\nu}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon))} e^{\mathrm{i}\langle k,\theta\rangle}. \tag{4.11}$$

We can get by Cauchy's estimate and $(4.1)_\nu$

$$|\zeta_{j,\nu,k,n_1,n_2}| \leqslant \|\zeta_{j,\nu,n_1,n_2}\|^*_{\Theta_\nu\times\bar{J}_\nu} e^{-|k|\sigma_\nu} \leqslant C\varepsilon^{\frac{3\bar{\sigma}}{4}}\lambda_j^{-1/2} e^{-|k|\sigma_\nu} \tag{4.12}$$

and

$$|\partial_{\bar{\xi}}\zeta_{j,\nu,k,n_1,n_2}| \leqslant \|\zeta_{j,\nu,n_1,n_2}\|^*_{\Theta_\nu\times\bar{J}_\nu} e^{-|k|\sigma_\nu} \leqslant C\varepsilon^{\frac{3\bar{\sigma}}{4}}\lambda_j^{-1/2} e^{-|k|\sigma_\nu}. \tag{4.13}$$

Note that $\alpha\in A_\gamma$, we have

$$\sup_{(\theta;\bar{\xi})\in\Theta_{\nu+1}\times\bar{J}_\nu} |\beta_{j,\nu,1,1}| \leqslant C\varepsilon^{\frac{3\bar{\sigma}}{4}}\gamma^{-1}\lambda_j^{-1/2} \sum_{0\neq k\in\mathbb{Z}^{m+1}} |k|^{m+3} e^{-|k|(\sigma_\nu-\sigma_{\nu+1})}$$

and

$$\sup_{(\theta;\bar{\xi})\in\Theta_{\nu+1}\times\bar{J}_\nu} |\beta_{0,\nu,n_1,n_2}| \leqslant \left(\varepsilon^{\frac{1}{3}}\varrho\frac{2\pi}{T}\right)^{-1}\varepsilon^{\frac{3\bar{\sigma}}{4}}\lambda_0^{-1/2}\big(1+\nu^2\big)\left(1+\sum_{0\neq k\in\mathbb{Z}^{m+1}} |k|^{m+3} e^{-|k|(\sigma_\nu-\sigma_{\nu+1})}\right).$$

Similarly, for $j\geqslant 1$, we get

$$\sup_{(\theta;\bar{\xi})\in\Theta_{\nu+1}\times\bar{J}_\nu} |\beta_{j,\nu,n_1,n_2}| \leqslant \begin{cases} (\varepsilon^{\frac{2}{3}}\varrho\frac{2\pi}{T})^{-1}\varepsilon^{\frac{3\bar{\sigma}}{4}}\lambda_j^{-1/2}(1+\nu^2)(1+\sum_{0\neq k\in\mathbb{Z}^{m+1}} |k|^{m+3} e^{-|k|(\sigma_\nu-\sigma_{\nu+1})}), & k_{m+1}=0, \\ (\varepsilon^{\frac{1}{3}}\varrho\frac{2\pi}{T})^{-1}\varepsilon^{\frac{3\bar{\sigma}}{4}}\lambda_j^{-1/2}(1+\nu^2)(1+\sum_{0\neq k\in\mathbb{Z}^{m+1}} |k|^{m+3} e^{-|k|(\sigma_\nu-\sigma_{\nu+1})}), & k_{m+1}\neq 0, \end{cases}$$

for $n_1=0$, $n_2=2$ or $n_1=2$, $n_2=0$. So, using Lemma 3.3 in [16] (see also [30]), we get, for $(\theta;\bar{\xi})\in\Theta_{\nu+1}\times\bar{J}_\nu$

$$|\beta_{0,\nu,1,1}|, |\beta_{0,\nu,2,0}|, |\beta_{0,\nu,0,2}| \leqslant C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{1}{3}}\lambda_j^{-1/2}(\nu+1)^{4m+8}, \tag{4.14}$$

$$|\beta_{j,\nu,1,1}|, |\beta_{j,\nu,2,0}|, |\beta_{j,\nu,0,2}| \leqslant \begin{cases} C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{2}{3}}\lambda_j^{-1/2}(\nu+1)^{4m+8}, & k_{m+1}=0, \\ C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{1}{3}}\lambda_j^{-1/2}(\nu+1)^{4m+8}, & k_{m+1}\neq 0, \end{cases} \quad j\geqslant 1. \tag{4.15}$$

For $j\geqslant 1$, we have

$$\partial_{\bar{\xi}}\big(\langle k,\tilde{\omega}(\bar{\xi})\rangle\big)\mp 2\partial_{\bar{\xi}}\lambda_{j,\nu}=\begin{cases}\mathcal{O}(\varepsilon^{\frac{2}{3}}), & k_{m+1}=0,\\ \mathcal{O}(\varepsilon^{\frac{1}{3}}), & k_{m+1}\neq 0.\end{cases}$$

Thus, in view of (4.9)–(4.13) and (2.15), we have , for $(\theta;\bar{\xi})\in\Theta_{\nu+1}\times\bar{J}_\nu$,

$$\begin{aligned}
|\partial_{\bar{\xi}}\beta_{j,\nu,1,1}| &\leqslant \sum_{0\neq k\in\mathbb{Z}^{m+1}}\left(\left|\frac{\partial_{\bar{\xi}}\zeta_{j,\nu,k,1,1}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon)}{\langle k,\tilde{\omega}(\bar{\xi})\rangle}\right|+\left|\frac{\zeta_{j,\nu,k,1,1}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon)\partial_{\bar{\xi}}(\langle k,\tilde{\omega}(\bar{\xi})\rangle)}{(\langle k,\tilde{\omega}(\bar{\xi})\rangle)^2}\right|\right)|e^{\mathrm{i}\langle k,\theta\rangle}| \\
&\leqslant C\varepsilon^{\frac{3\bar{\sigma}}{4}}\lambda_j^{-1/2}\left(\sum_{0\neq k\in\mathbb{Z}^{m+1}}|k|^{m+3}e^{-|k|(\sigma_\nu-\sigma_{\nu+1})}+\sum_{0\neq k\in\mathbb{Z}^{m+1}}|k|^{2m+7}e^{-|k|(\sigma_\nu-\sigma_{\nu+1})}\right) \\
&\leqslant C\varepsilon^{\frac{3\bar{\sigma}}{4}}\lambda_j^{-1/2}\big((\nu+1)^{4m+8}+(\nu+1)^{6m+16}\big) \\
&\leqslant C\varepsilon^{\frac{3\bar{\sigma}}{4}}\lambda_j^{-1/2}(\nu+1)^{6m+16}, \quad j\geqslant 0,
\end{aligned} \tag{4.16}$$

$$\begin{aligned}
|\partial_{\bar{\xi}}\beta_{0,\nu,n_1,n_2}| &\leqslant \sum_{k\in\mathbb{Z}^{m+1}}\left(\left|\frac{\partial_{\bar{\xi}}\zeta_{0,\nu,k,n_1,n_2}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon)}{\langle k,\tilde{\omega}(\bar{\xi})\rangle\mp 2\lambda_{0,\nu}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon))}\right|\right. \\
&\quad\left.+\left|\frac{\zeta_{0,\nu,k,n_1,n_2}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon)(\partial_{\bar{\xi}}(\langle k,\tilde{\omega}(\bar{\xi})\rangle)\mp 2\partial_{\bar{\xi}}\lambda_{0,\nu}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon))}{(\langle k,\tilde{\omega}(\bar{\xi})\rangle\mp 2\lambda_{0,\nu}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon))^2}\right|\right)|e^{\mathrm{i}\langle k,\theta\rangle}| \\
&\leqslant C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{1}{3}}\lambda_0^{-1/2}\big(1+\nu^2\big)\left(1+\sum_{0\neq k\in\mathbb{Z}^{m+1}}|k|^{m+3}e^{-|k|(\sigma_\nu-\sigma_{\nu+1})}\right) \\
&\quad+C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{1}{3}}\lambda_0^{-1/2}\big(1+\nu^2\big)^2\left(1+\sum_{0\neq k\in\mathbb{Z}^{m+1}}|k|^{2m+7}e^{-|k|(\sigma_\nu-\sigma_{\nu+1})}\right) \\
&\leqslant C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{1}{3}}\lambda_j^{-1/2}(\nu+1)^{4m+10}+C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{1}{3}}\lambda_0^{-1/2}(\nu+1)^{6m+20} \\
&\leqslant C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{1}{3}}\lambda_0^{-1/2}(\nu+1)^{6m+20}.
\end{aligned} \tag{4.17}$$

Similarly, for $j\geqslant 1$ and $n_1=0$, $n_2=2$ or $n_1=2$, $n_2=0$, we get

$$|\partial_{\bar{\xi}}\beta_{j,\nu,n_1,n_2}|\leqslant\begin{cases}C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{2}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+20}, & k_{m+1}=0,\\ C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{1}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+20}, & k_{m+1}\neq 0.\end{cases} \tag{4.18}$$

In view of (4.15), (4.16),(4.17) and (4.18), we have

$$\|\beta_{0,\nu,1,1}\|^*_{\Theta_{\nu+1}\times\bar J_\nu}, \|\beta_{0,\nu,2,0}\|^*_{\Theta_{\nu+1}\times\bar J_\nu}, \|\beta_{0,\nu,0,2}\|^*_{\Theta_{\nu+1}\times\bar J_\nu} \leqslant C\varepsilon^{\frac{3\bar\sigma}{4}-\frac{1}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+20} \tag{4.19}$$

and

$$\begin{aligned}&\|\beta_{j,\nu,1,1}\|^*_{\Theta_{\nu+1}\times\bar J_\nu},\ \|\beta_{j,\nu,2,0}\|^*_{\Theta_{\nu+1}\times\bar J_\nu},\ \|\beta_{j,\nu,0,2}\|^*_{\Theta_{\nu+1}\times\bar J_\nu}\\ &\quad\leqslant \begin{cases} C\varepsilon^{\frac{3\bar\sigma}{4}-\frac{2}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+20}, & k_{m+1}=0,\\ C\varepsilon^{\frac{3\bar\sigma}{4}-\frac{1}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+20}, & k_{m+1}\neq 0,\end{cases}\quad j\geqslant 1.\end{aligned} \tag{4.20}$$

In view of (4.9),(4.10) and (4.11), we have

$$\partial_\theta\beta_{j,\nu,1,1}\big(\theta,\bar\xi,\tilde\omega(\bar\xi),\varepsilon\big)=\sum_{0\neq k\in\mathbb{Z}^{m+1}}\frac{-\zeta_{j,\nu,k,1,1}(\bar\xi,\tilde\omega(\bar\xi),\varepsilon)}{\langle k,\tilde\omega(\bar\xi)\rangle}e^{\mathrm{i}\langle k,\theta\rangle}\cdot k,$$

$$\partial_\theta\beta_{j,\nu,0,2}\big(\theta,\bar\xi,\tilde\omega(\bar\xi),\varepsilon\big)=\sum_{k\in\mathbb{Z}^{m+1}}\frac{-\zeta_{j,\nu,k,0,2}(\bar\xi,\tilde\omega(\bar\xi),\varepsilon)}{\langle k,\tilde\omega(\bar\xi)\rangle+2\lambda_{j,\nu}(\bar\xi,\tilde\omega(\bar\xi),\varepsilon)}e^{\mathrm{i}\langle k,\theta\rangle}\cdot k,$$

$$\partial_\theta\beta_{j,\nu,2,0}\big(\theta,\bar\xi,\tilde\omega(\bar\xi),\varepsilon\big)=\sum_{k\in\mathbb{Z}^{m+1}}\frac{-\zeta_{j,\nu,k,2,0}(\bar\xi,\tilde\omega(\bar\xi),\varepsilon)}{\langle k,\tilde\omega(\bar\xi)\rangle-2\lambda_{j,\nu}(\bar\xi,\tilde\omega(\bar\xi),\varepsilon)}e^{\mathrm{i}\langle k,\theta\rangle}\cdot k,$$

$$\partial_{\theta\theta}\beta_{j,\nu,1,1}\big(\theta,\bar\xi,\tilde\omega(\bar\xi),\varepsilon\big)=\sum_{0\neq k\in\mathbb{Z}^{m+1}}\frac{-\zeta_{j,\nu,k,1,1}(\bar\xi,\tilde\omega(\bar\xi),\varepsilon)}{\langle k,\tilde\omega(\bar\xi)\rangle}e^{\mathrm{i}\langle k,\theta\rangle}\cdot \mathrm{i}kk^T,$$

$$\partial_{\theta\theta}\beta_{j,\nu,0,2}\big(\theta,\bar\xi,\tilde\omega(\bar\xi),\varepsilon\big)=\sum_{k\in\mathbb{Z}^{m+1}}\frac{-\zeta_{j,\nu,k,0,2}(\bar\xi,\tilde\omega(\bar\xi),\varepsilon)}{\langle k,\tilde\omega(\bar\xi)\rangle+2\lambda_{j,\nu}(\bar\xi,\tilde\omega(\bar\xi),\varepsilon)}e^{\mathrm{i}\langle k,\theta\rangle}\cdot \mathrm{i}kk^T,$$

$$\partial_{\theta\theta}\beta_{j,\nu,2,0}\big(\theta,\bar\xi,\tilde\omega(\bar\xi),\varepsilon\big)=\sum_{k\in\mathbb{Z}^{m+1}}\frac{-\zeta_{j,\nu,k,2,0}(\bar\xi,\tilde\omega(\bar\xi),\varepsilon)}{\langle k,\tilde\omega(\bar\xi)\rangle-2\lambda_{j,\nu}(\bar\xi,\tilde\omega(\bar\xi),\varepsilon)}e^{\mathrm{i}\langle k,\theta\rangle}\cdot \mathrm{i}kk^T,$$

where k is a $m+1$ column vector and kk^T is a $m+1\times m+1$ matrix. Similar to above discussion, we get the following estimates

$$\begin{aligned}&\|\partial_\theta\beta_{0,\nu,1,1}\|^*_{\Theta_{\nu+1}\times\bar J_\nu},\ \|\partial_\theta\beta_{0,\nu,2,0}\|^*_{\Theta_{\nu+1}\times\bar J_\nu},\ \|\partial_\theta\beta_{0,\nu,0,2}\|^*_{\Theta_{\nu+1}\times\bar J_\nu}\\ &\quad\leqslant C\varepsilon^{\frac{3\bar\sigma}{4}-\frac{1}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+22},\end{aligned} \tag{4.21}$$

$$\begin{aligned}&\|\partial_\theta\beta_{j,\nu,1,1}\|^*_{\Theta_{\nu+1}\times\bar J_\nu},\ \|\partial_\theta\beta_{j,\nu,2,0}\|^*_{\Theta_{\nu+1}\times\bar J_\nu},\ \|\partial_\theta\beta_{j,\nu,0,2}\|^*_{\Theta_{\nu+1}\times\bar J_\nu}\\ &\quad\leqslant \begin{cases} C\varepsilon^{\frac{3\bar\sigma}{4}-\frac{2}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+22}, & k_{m+1}=0,\\ C\varepsilon^{\frac{3\bar\sigma}{4}-\frac{1}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+22}, & k_{m+1}\neq 0,\end{cases}\quad j\geqslant 1,\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}&\|\partial_{\theta\theta}\beta_{0,\nu,1,1}\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu},\ \|\partial_{\theta\theta}\beta_{0,\nu,2,0}\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu},\ \|\partial_{\theta\theta}\beta_{0,\nu,0,2}\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu}\\&\quad\leqslant C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{1}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+24},\end{aligned}\tag{4.23}$$

$$\begin{aligned}&\|\partial_{\theta\theta}\beta_{j,\nu,1,1}\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu},\ \|\partial_{\theta\theta}\beta_{j,\nu,2,0}\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu},\ \|\partial_{\theta\theta}\beta_{j,\nu,0,2}\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu}\\&\quad\leqslant\begin{cases}C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{2}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+24}, & k_{m+1}=0,\\ C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{1}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+24}, & k_{m+1}\neq 0,\end{cases}\qquad j\geqslant 1.\end{aligned}\tag{4.24}$$

Step 3. The estimates of the flow $X^t_{\mathcal{F}_\nu}$. To get the estimates for the flow $X^t_{\mathcal{F}_\nu}$, we let

$$\begin{gathered}B_{j,\nu}\big(\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)=\begin{pmatrix}2\beta_{j,\nu,2,0}(\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon) & \beta_{j,\nu,1,1}(\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon)\\ \beta_{j,\nu,1,1}(\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon) & 2\beta_{j,\nu,0,2}(\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon)\end{pmatrix},\\ \mathcal{J}=\mathrm{i}\begin{pmatrix}0&1\\-1&0\end{pmatrix}.\end{gathered}\tag{4.25}$$

In view of (4.19)–(4.24), we have

$$\|B_{0,\nu}\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu}\leqslant C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{1}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+20},\tag{4.26}$$

$$\|B_{j,\nu}\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu}\leqslant\begin{cases}C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{2}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+20}, & k_{m+1}=0,\\ C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{1}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+20}, & k_{m+1}\neq 0,\end{cases}\qquad j\geqslant 1,\tag{4.27}$$

$$\|\partial_\theta B_{0,\nu}\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu}\leqslant C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{1}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+22},\tag{4.28}$$

$$\|\partial_\theta B_{j,\nu}\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu}\leqslant\begin{cases}C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{2}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+22}, & k_{m+1}=0,\\ C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{1}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+22}, & k_{m+1}\neq 0,\end{cases}\qquad j\geqslant 1,\tag{4.29}$$

$$\|\partial_{\theta\theta} B_{0,\nu}\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu}\leqslant C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{1}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+24},\tag{4.30}$$

$$\|\partial_{\theta\theta} B_{j,\nu}\|^*_{\Theta_{\nu+1}\times\bar{J}_\nu}\leqslant\begin{cases}C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{2}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+24}, & k_{m+1}=0,\\ C\varepsilon^{\frac{3\bar{\sigma}}{4}-\frac{1}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+24}, & k_{m+1}\neq 0,\end{cases}\qquad j\geqslant 1.\tag{4.31}$$

Moreover, we note that the vector field $X_{\mathcal{F}_\nu}$ is as follows

$$\begin{cases}\dot{\theta}=\frac{\partial\mathcal{F}_\nu}{\partial J}=0,\\ \frac{d}{dt}\begin{pmatrix}z_j\\ \bar{z}_j\end{pmatrix}=\begin{pmatrix}i\frac{\partial\mathcal{F}_\nu}{\partial\bar{z}_j}\\ -i\frac{\partial\mathcal{F}_\nu}{\partial z_j}\end{pmatrix}=\varepsilon^{\frac{2}{3}(\nu+1)}\mathcal{J}B_{j,\nu}\big(\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\cdot\begin{pmatrix}z_j\\ \bar{z}_j\end{pmatrix}, \quad j=0,1,2,\ldots\\ \dot{J}=-\frac{\partial\mathcal{F}_\nu}{\partial\theta}=-\varepsilon^{\frac{2}{3}(\nu+1)}\sum_{j\geqslant 0}\sum_{n_1+n_2=2}\partial_\theta\beta_{j,\nu,n_1,n_2}\big(\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)z_j^{n_1}\bar{z}_j^{n_2}.\end{cases}\tag{4.32}$$

It is easy to see that

$$\big|\varepsilon^{\frac{\nu}{3}+\frac{3\bar{\sigma}}{4}}(\nu+1)^{6m+20}\big|\leqslant C,\qquad \big|\varepsilon^{\frac{\nu}{3}+\frac{3\bar{\sigma}}{4}-\frac{1}{3}}(\nu+1)^{6m+20}\big|\leqslant C,\quad \nu=0,1,\ldots\tag{4.33}$$

as $\varepsilon < 1$, where C is an absolute constant independent on ν, ε. From (4.26) and (4.27), we obtain, for $(\theta, \bar{\xi}) \in \Theta_{\nu+1} \times \bar{J}_\nu$,

$$\left\| \varepsilon^{\frac{2}{3}(\nu+1)} \mathcal{J} B_{j,\nu}\big(\theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) \right\|^*_{\Theta_{\nu+1} \times \bar{J}_\nu} \leqslant C\lambda_j^{-1/2} \varepsilon^{\frac{1}{3}(\nu+1)}. \tag{4.34}$$

It follows from (4.32) and (4.34) that

$$\begin{aligned} \left\| (\mathcal{F}_\nu)_{\bar{z}_j} \right\|^*_{\Theta_{\nu+1} \times \bar{J}_\nu}, \left\| (\mathcal{F}_\nu)_{z_j} \right\|^*_{\Theta_{\nu+1} \times \bar{J}_\nu} &\leqslant \left\| \varepsilon^{\frac{2}{3}(\nu+1)} \mathcal{J} B_{j,\nu}\big(\theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) \right\|^*_{\Theta_{\nu+1} \times \bar{J}_\nu} |z_j| \\ &\leqslant C\lambda_j^{-1/2} \varepsilon^{\frac{1}{3}(\nu+1)} |z_j|. \end{aligned} \tag{4.35}$$

Similarly, from (4.21), (4.22) and (4.32) we get that

$$\left\| (\mathcal{F}_\nu)_\theta \right\|^*_{\Theta_{\nu+1} \times \bar{J}_\nu} \leqslant C\varepsilon^{\frac{1}{3}(\nu+1)} \sum_{j \geqslant 0} \lambda_j^{-1/2} |z_j|^2. \tag{4.36}$$

Integrating the above equation from 0 to t, we get $X^t_{\mathcal{F}_\nu}$:

$$\begin{cases} \theta = \theta^{\mathcal{C}}, \\ \begin{pmatrix} z_j(t) \\ \bar{z}_j(t) \end{pmatrix} = \exp(\varepsilon^{\frac{2}{3}(\nu+1)} \mathcal{J} B_{j,\nu}(\theta^{\mathcal{C}}, \bar{\xi}, \tilde{\omega}, \varepsilon) t) \cdot \begin{pmatrix} z_j(0) \\ \bar{z}_j(0) \end{pmatrix}, \quad j = 0, 1, 2, \ldots \\ J(t) = J(0) - \varepsilon^{\frac{2}{3}(\nu+1)} \int_0^t \sum_{j \geqslant 0} \sum_{n_1+n_2=2} \partial_\theta \beta_{j,\nu,n_1,n_2}(\theta^{\mathcal{C}}, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon) \cdot (z_j(t))^{n_1} \cdot (\bar{z}_j(t))^{n_2}\, dt, \end{cases} \tag{4.37}$$

where $\theta^{\mathcal{C}}$ is a constant vector in $\mathbb{C}^{m+1}/2\pi\mathbb{Z}^{m+1}$ and $(\theta^{\mathcal{C}}, J(0), z(0), \bar{z}(0))$ is the initial value.

We write

$$\begin{aligned} \begin{pmatrix} z_j(t) \\ \bar{z}_j(t) \end{pmatrix} &= \exp\big(\varepsilon^{\frac{2}{3}(\nu+1)} \mathcal{J} B_{j,\nu}\big(\theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) t\big) \begin{pmatrix} z_j(0) \\ \bar{z}_j(0) \end{pmatrix} \\ &= \big(Id + f_{j,\nu}\big(\theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon, t\big)\big) \begin{pmatrix} z_j(0) \\ \bar{z}_j(0) \end{pmatrix}, \quad t \in [0,1], \end{aligned} \tag{4.38}$$

where Id is the unit matrix and

$$f_{j,\nu} := \begin{pmatrix} f_{j,\nu,11} & f_{j,\nu,12} \\ f_{j,\nu,21} & f_{j,\nu,22} \end{pmatrix}.$$

By using of (4.34), therefore, we have

$$\left\| f_{j,\nu}\big(\theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon, t\big) \right\|^*_{\Theta_{\nu+1} \times \bar{J}_\nu} \leqslant C\lambda_j^{-1/2} \varepsilon^{\frac{1}{3}(\nu+1)} \tag{4.39}$$

and

$$\begin{aligned} &\left\| \partial_\theta \big(f_{j,\nu}\big(\theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon, t\big) \cdot z_j\big) \right\|^*_{\Theta_{\nu+1} \times \bar{J}_\nu}, \ \left\| \partial_\theta \big(f_{j,\nu}\big(\theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon, t\big) \cdot \bar{z}_j\big) \right\|^*_{\Theta_{\nu+1} \times \bar{J}_\nu} \\ &\quad \leqslant C\lambda_j^{-1/2} \varepsilon^{\frac{1}{3}(\nu+1)} |z_j|, \quad t \in [0,1]. \end{aligned} \tag{4.40}$$

Let

$$f_{J,\nu}\big(\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon;z,\bar{z},t\big)=-\varepsilon^{\frac{2}{3}(\nu+1)}\int_0^t\sum_{j\geqslant 0}\sum_{n_1+n_2=2}\partial_\theta\beta_{j,\nu,n_1,n_2}\big(\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)z_j^{n_1}(t)\bar{z}_j^{n_2}(t)\,dt,$$

then

$$J(t)=J(0)+f_{J,\nu}\big(\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon;z,\bar{z},t\big),\quad t\in[0,1].\tag{4.41}$$

From (4.32) and (4.36), we get

$$\big\|f_{J,\nu}\big(\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon;z,\bar{z},t\big)\big\|^*_{D^{a,s}_{\nu+1}\times\bar{J}_\nu}\leqslant C\varepsilon^{\frac{1}{3}(\nu+1)}\sum_{j\geqslant 0}\lambda_j^{-1/2}|z_j|^2.\tag{4.42}$$

Furthermore, from (4.22), (4.24) and (4.37), we have

$$\big\|\partial_\theta f_{J\nu}\big(\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon;z,\bar{z},t\big)\big\|^*_{D^{a,s}_{\nu+1}\times\bar{J}_\nu}\leqslant C\varepsilon^{\frac{1}{3}(\nu+1)}\sum_{j\geqslant 0}\lambda_j^{-1/2}|z_j|^2\tag{4.43}$$

and

$$\begin{aligned}&\big\|\partial_{z_j} f_{J\nu}\big(\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon;z,\bar{z},t\big)\big\|^*_{D^{a,s}_{\nu+1}\times\bar{J}_\nu},\ \big\|\partial_{\bar{z}_j} f_{J\nu}\big(\theta,\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon;z,\bar{z},t\big)\big\|^*_{D^{a,s}_{\nu+1}\times\bar{J}_\nu}\\&\quad\leqslant C\lambda_j^{-1/2}\varepsilon^{\frac{1}{3}(\nu+1)}|z_j|.\end{aligned}\tag{4.44}$$

Therefore, by using of (4.35) and (4.36), we obtain that

$$\begin{aligned}|X_{\mathcal{F}_\nu}|^*_{a,s+1,D^{a,s}_{\nu+1}\times\bar{J}_{\nu+1}}&=\frac{1}{r^2_{\nu+1}}\big\|(\mathcal{F}_\nu)_\theta\big\|^*_{D^{a,s}_{\nu+1}\times\bar{J}_{\nu+1}}\\&\quad+\frac{1}{r_{\nu+1}}\big\|(\mathcal{F}_\nu)_z\big\|^*_{a,s+1,D^{a,s}_{\nu+1}\times\bar{J}_{\nu+1}}+\frac{1}{r_{\nu+1}}\big\|(\mathcal{F}_\nu)_{\bar{z}}\big\|^*_{a,s+1,D^{a,s}_{\nu+1}\times\bar{J}_{\nu+1}}\\&\leqslant\frac{1}{r^2_{\nu+1}}C\varepsilon^{\frac{1}{3}(\nu+1)}r^2_{\nu+1}+\frac{1}{r_{\nu+1}}C\varepsilon^{\frac{1}{3}(\nu+1)}r_{\nu+1}+\frac{1}{r_{\nu+1}}C\varepsilon^{\frac{1}{3}(\nu+1)}r_{\nu+1}\\&\leqslant C\varepsilon^{\frac{1}{3}(\nu+1)}.\end{aligned}\tag{4.45}$$

To get the estimates for $X^t_{\mathcal{F}_\nu}$, we consider the integral equation

$$X^t_{\mathcal{F}_\nu}=id+\int_0^t X_{\mathcal{F}_\nu}\circ X^s_{\mathcal{F}_\nu}\,ds,\quad 0\leqslant t\leqslant 1.$$

Hence, we obtain from (4.45)

$$\big|X^1_{\mathcal{F}_\nu}-id\big|^*_{a,s+1,D^{a,s}_{\nu+1}\times\bar{J}_{\nu+1}}\leqslant|X_{\mathcal{F}_\nu}|^*_{a,s+1,D^{a,s}_{\nu+1}\times\bar{J}_{\nu+1}}\leqslant C\varepsilon^{\frac{1}{3}(\nu+1)}.\tag{4.46}$$

We now estimate $DX^1_{\mathcal{F}_\nu} - Id$. Assume the coordinate transformation is of the form

$$X^1_{\mathcal{F}_\nu} : (\theta, J, z, \bar{z}) \mapsto \big(\Theta(\theta), \mathbb{J}\big(\theta, J, z, \bar{z}; \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big), \mathbb{z}\big(\theta, z, \bar{z}; \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big), \bar{\mathbb{z}}\big(\theta, z, \bar{z}; \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big)\big).$$

So, by using of (4.37),

$$DX^1_{\mathcal{F}_\nu} = \begin{pmatrix} Id_{(m+1)\times(m+1)} & 0 & 0 & 0 \\ \frac{\partial \mathbb{J}}{\partial \theta} & Id_{(m+1)\times(m+1)} & \frac{\partial \mathbb{J}}{\partial z} & \frac{\partial \mathbb{J}}{\partial \bar{z}} \\ \frac{\partial \mathbb{z}}{\partial \theta} & 0 & \frac{\partial \mathbb{z}}{\partial z} & \frac{\partial \mathbb{z}}{\partial \bar{z}} \\ \frac{\partial \bar{\mathbb{z}}}{\partial \theta} & 0 & \frac{\partial \bar{\mathbb{z}}}{\partial z} & \frac{\partial \bar{\mathbb{z}}}{\partial \bar{z}} \end{pmatrix},$$

where

$$\frac{\partial \mathbb{J}}{\partial \theta} = \frac{\partial f_{J,\nu}}{\partial \theta}, \qquad \frac{\partial \mathbb{J}}{\partial z} = \frac{\partial f_{J,\nu}}{\partial z} = \left(\frac{\partial f_{J,\nu}}{\partial z_0}, \frac{\partial f_{J,\nu}}{\partial z_1}, \frac{\partial f_{J,\nu}}{\partial z_2}, \dots\right)_{(m+1)\times\infty},$$

and

$$\frac{\partial \mathbb{J}}{\partial \bar{z}} = \frac{\partial f_{J,\nu}}{\partial \bar{z}} = \left(\frac{\partial f_{J,\nu}}{\partial \bar{z}_0}, \frac{\partial f_{J,\nu}}{\partial \bar{z}_1}, \frac{\partial f_{J,\nu}}{\partial \bar{z}_2}, \dots\right)_{(m+1)\times\infty}.$$

Since (4.38) holds, we have

$$\mathbb{z} = \big((1 + f_{0,\nu,11})z_0 + f_{0,\nu,12}\bar{z}_0, (1 + f_{1,\nu,11})z_1 + f_{1,\nu,12}\bar{z}_1, (1 + f_{2,\nu,11})z_2 + f_{2,\nu,12}\bar{z}_2, \dots\big).$$

It follows that

$$\frac{\partial \mathbb{z}}{\partial \theta} = \left(\left(\frac{\partial(f_{0,\nu,11}z_0 + f_{0,\nu,12}\bar{z}_0)}{\partial \theta}, \frac{\partial(f_{1,\nu,11}z_1 + f_{1,\nu,12}\bar{z}_1)}{\partial \theta}, \frac{\partial(f_{2,\nu,11}z_2 + f_{2,\nu,12}\bar{z}_2)}{\partial \theta}, \dots\right)^T\right)_{\infty\times(m+1)},$$

$$\frac{\partial \mathbb{z}}{\partial z} = \begin{pmatrix} 1 + f_{0,\nu,11} & 0 & 0 & \cdots \\ 0 & 1 + f_{1,\nu,11} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{\infty\times\infty},$$

and

$$\frac{\partial \mathbb{z}}{\partial \bar{z}} = \begin{pmatrix} f_{0,\nu,12} & 0 & 0 & \cdots \\ 0 & f_{1,\nu,12} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{\infty\times\infty}.$$

Since (4.38) holds, we have

$$\bar{\mathbb{z}} = \big(f_{0,\nu,21}z_0 + (1 + f_{0,\nu,22})\bar{z}_0, f_{1,\nu,21}z_1 + (1 + f_{1,\nu,22})\bar{z}_1, f_{2,\nu,21}z_2 + (1 + f_{2,\nu,22})\bar{z}_2, \dots\big).$$

It follows that

$$\frac{\partial \bar{\mathbb{z}}}{\partial \theta} = \left(\left(\frac{\partial(f_{0,\nu,21}z_0 + f_{0,\nu,22}\bar{z}_0)}{\partial \theta}, \frac{\partial(f_{1,\nu,21}z_1 + f_{1,\nu,22}\bar{z}_1)}{\partial \theta}, \frac{\partial(f_{2,\nu,21}z_2 + f_{2,\nu,22}\bar{z}_2)}{\partial \theta}, \dots\right)^T\right)_{\infty\times(m+1)},$$

$$\frac{\partial \bar{z}}{\partial z} = \begin{pmatrix} f_{0,\nu,21} & 0 & 0 & \cdots \\ 0 & f_{1,\nu,21} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{\infty\times\infty},$$

and

$$\frac{\partial \bar{z}}{\partial \bar{z}} = \begin{pmatrix} 1+f_{0,\nu,22} & 0 & 0 & \cdots \\ 0 & 1+f_{1,\nu,22} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{\infty\times\infty}.$$

So, from (4.39), (4.40), (4.43) and (4.44), we get, for $0 \neq \tilde{w} = (\theta', J', z', \bar{z}')^T \in D^{a,s}_{\nu+1}$,

$$\left|\left(DX^1_{\mathcal{F}_\nu} - Id\right)\tilde{w}\right|_{a,s+1} \leqslant C\varepsilon^{\frac{1}{3}(\nu+1)}|\tilde{w}|_{a,s},$$

where D is the differentiation operator with respect to $(\theta, J, z, \bar{z})$ and Id is unit matrix. Hence,

$$\left|DX^1_{\mathcal{F}_\nu} - Id\right|^{op}_{a,s+1,s,D^{a,s}_{\nu+1}\times\bar{J}_{\nu+1}} \leqslant C\varepsilon^{\frac{1}{3}(\nu+1)}.$$

Similarly,

$$\left|DX^1_{\mathcal{F}_\nu} - Id\right|^{*op}_{a,s+1,s,D^{a,s}_{\nu+1}\times\bar{J}_{\nu+1}} \leqslant C\varepsilon^{\frac{1}{3}(\nu+1)}. \tag{4.47}$$

Step 4. The estimates of the term $R_{\nu+2}$**.** We now estimate the smaller term $R_{\nu+2}$ and we will finish one cycle of the iteration. Let

$$\tilde{\lambda}_{j,\nu+1}\left(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\right) = [\zeta_{j,\nu,1,1}] = \frac{1}{(2\pi)^{m+1}} \int\limits_{\mathbb{T}^{m+1}} \zeta_{j,\nu,1,1}\left(\theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\right) d\theta$$

and

$$\lambda_{j,\nu+1}\left(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\right) = \lambda_{j,\nu}\left(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\right) + \varepsilon^{\frac{2}{3}(\nu+1)}\tilde{\lambda}_{j,\nu+1}\left(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\right),$$

then it is easy to see that $\lambda_{j,\nu+1}(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon)$ satisfies the conditions $(4.2)_{\nu+1}$. Moreover, from the homological equation (4.7), we know that

$$\tilde{H}_{\nu+1} = \left\langle \tilde{\omega}(\bar{\xi}), J\right\rangle + \sum_{j\geqslant 0} \lambda_{j,\nu+1}\left(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\right) z_j \bar{z}_j + \varepsilon^{\frac{2}{3}(\nu+1)} R_{\nu+2}\left(z, \bar{z}, \theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\right),$$

where

$$R_{\nu+2}\left(z, \bar{z}, \theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\right) = \varepsilon^{-\frac{2}{3}(\nu+2)} \Bigg(\left\{\varepsilon^{\frac{2}{3}(\nu+1)} R_{\nu+1}\left(z, \bar{z}, \theta, \bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\right), \mathcal{F}_\nu\right\} + \int_0^1 (1-t)\left\{\{\tilde{H}_\nu, \mathcal{F}_\nu\}, \mathcal{F}_\nu\right\} \circ X^t_{\mathcal{F}_\nu}\, dt \Bigg).$$

By a direct calculation we can get that

$$R_{\nu+2}\big(z,\bar z,\theta,\bar\xi,\tilde\omega(\bar\xi),\varepsilon\big)=\varepsilon^{\frac{2\nu}{3}}\sum_{j\geqslant 0}\sum_{n_1+n_2=2}\tilde\zeta_{j,\nu+1,n_1,n_2}\big(\theta,\bar\xi,\tilde\omega(\bar\xi),\varepsilon\big)z_j^{n_1}\bar z_j^{n_2},$$

where $\tilde\zeta_{j,\nu+1,n_1,n_2}(\theta,\bar\xi,\tilde\omega(\bar\xi),\varepsilon)$'s are a linear combination of the product of $\beta_{j,\nu,n_1,n_2}(\theta,\bar\xi,\tilde\omega(\bar\xi),\varepsilon)$ and $\zeta_{j,\nu,m_1,m_2}(\theta,\bar\xi,\tilde\omega(\bar\xi),\varepsilon)$'s, with $n_1,n_2,m_1,m_2\in\mathbb{N}$ and $n_1+n_2=2$, $m_1+m_2=2$. Thus, by using of $(4.1)_\nu$, (4.19) and (4.20),

$$\|\tilde\zeta_{0,\nu+1,n_1,n_2}\|^*_{\Theta_{\nu+1}\times\bar J_\nu}\leqslant C\varepsilon^{\frac{3\bar\sigma}{2}-\frac{2}{3}}\lambda_0^{-1/2}(\nu+1)^{6m+20} \tag{4.48}$$

and

$$\|\tilde\zeta_{j,\nu+1,n_1,n_2}\|^*_{\Theta_{\nu+1}\times\bar J_\nu}\leqslant\begin{cases}C\varepsilon^{\frac{3\bar\sigma}{2}-\frac{2}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+20}, & k_{m+1}=0,\\ C\varepsilon^{\frac{3\bar\sigma}{2}-\frac{1}{3}}\lambda_j^{-1/2}(\nu+1)^{6m+20}, & k_{m+1}\neq 0,\end{cases}\qquad j\geqslant 1 \tag{4.49}$$

is true. We can suppose that

$$\zeta_{j,\nu+1,n_1,n_2}:=\varepsilon^{\frac{2\nu}{3}}\tilde\zeta_{j,\nu+1,n_1,n_2},\quad j\geqslant 0.$$

Note that $C\varepsilon^{\frac{2\nu}{3}+\frac{3\bar\sigma}{4}-\frac{1}{3}}(\nu+1)^{6m+20}$, $C\varepsilon^{\frac{2\nu}{3}+\frac{3\bar\sigma}{4}-\frac{2}{3}}(\nu+1)^{6m+20}\leqslant 1$ as $\varepsilon<1$, it follows from (4.48) and (4.49) that

$$\|\zeta_{j,\nu+1,n_1,n_2}\|^*_{\Theta_{\nu+1}\times\bar J_\nu}\leqslant C\varepsilon^{\frac{3\bar\sigma}{4}}\lambda_j^{-1/2},\quad j\geqslant 0.$$

This implies $(E)_{\nu+1}$ is defined in $D^{a,s}_{\nu+1}$ and $\zeta_{j,\nu+1,n_1,n_2}$'s satisfy $(4.1)_{\nu+1}$.

Step 5. The convergence of transformations Σ^N. In view of (4.46) and (4.47), by letting

$$S_\nu=X^1_{\mathcal{F}_\nu}:D^{a,s}_{\nu+1}\times\bar J_{\nu+1}\mapsto D^{a,s}_\nu\times\bar J_\nu \tag{4.50}$$

we have

$$|S_\nu-id|^*_{a,s+1,D^{a,s}_{\nu+1}\times\bar J_{\nu+1}}\leqslant C\varepsilon^{\frac{1}{3}(\nu+1)},\qquad |DS_\nu-Id|^{*op}_{a,s+1,s,D^{a,s}_{\nu+1}\times\bar J_{\nu+1}}\leqslant C\varepsilon^{\frac{1}{3}(\nu+1)}. \tag{4.51}$$

Now we are ready to prove the limiting transformation $S_0\circ S_1\circ\cdots$ convergent to a transformation Σ^∞ and that this transformation integrates Eq. (4.2). For any $\bar\xi\in\bar J$, $N\geqslant 0$, we denote by Σ^N the map

$$\Sigma^N(\cdot;\bar\xi)=S_0(\cdot;\bar\xi)\circ\cdots\circ S_{N-1}(\cdot;\bar\xi):D^{a,s}_N\mapsto D^{a,s}(\sigma,r)$$

as usual, Σ^0 is the identity mapping. From the second inequality of (4.51), we have

$$\big|D\Sigma^N\big|^{*op}_{a,s+1,s,D^{a,s}_N\times\bar J}\leqslant\prod_{\mu=0}^{N-1}|DS_\mu|^{*op}_{a,s+1,s,D^{a,s}_{\mu+1}\times\bar J}\leqslant\prod_{\mu\geqslant 0}\big(1+C\varepsilon^{\frac{1}{3}(\mu+1)}\big)\leqslant 2$$

provided that ε is small enough. Thus, by using the first inequality of (4.51), we have

$$\begin{aligned}\left|\Sigma^{N+1}-\Sigma^{N}\right|^{*}_{a,s+1,D^{a,s}_{N+1}\times\bar J} &\leqslant \left|D\Sigma^{N}\right|^{*op}_{a,s+1,s,D^{a,s}_{N}\times\bar J}\cdot|S_N-id|^{*}_{a,s+1,D^{a,s}_{N+1}\times\bar J}\\ &\leqslant C\varepsilon^{\frac{1}{3}(N+2)}.\end{aligned}$$

So the sequence $\{\Sigma^N\}$ converges uniformly in $D^{a,s}_N$ to an analytic map

$$\Sigma^\infty: D^{a,s}(\sigma/2,r/2)\mapsto D^{a,s}(\sigma,r).$$

We remark that the Hamiltonian (4.2) satisfies (E_ν), $(4.1)_\nu$ and $(4.2)_\nu$ with $\nu=0$, the above iterative procedure can run repeatedly. So

$$\begin{aligned}\mu_j &= \sqrt{\lambda_j}+\frac{\varepsilon^{\frac{2}{3}}}{2\sqrt{\lambda_j}}\big[\tilde V(\theta,\bar\xi,\varepsilon)\big]+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde\lambda_{j,k}(\bar\xi,\tilde\omega,\varepsilon)\\ &=\sqrt{\lambda_j}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde\lambda_{j,k}(\bar\xi,\tilde\omega,\varepsilon),\end{aligned}$$

where $|\tilde\lambda_{j,k}(\bar\xi,\tilde\omega,\varepsilon)|\leqslant C\varepsilon^{\bar\sigma}\lambda_j^{-1/2}, k=2,3,\ldots.$ So (i) and (ii) are obtained. This completes the proof. $\square$

4.3. Proof of the small divisors lemma

Now we show the following the small divisors lemma which have been applied in proving the above reducibility theorem.

Lemma 4.2. *For $k\in\mathbb{Z}^{m+1}$, $j,l\in\mathbb{N}=\{0,1,2,\ldots\}$, there exists a family of closed subsets $\bar J_l (l=0,\ldots,\nu)$*

$$\bar J_\nu\subset\cdots\subset\bar J_{l+1}\subset\bar J_l\subset\cdots\subset\bar J_0\subset\hat J\subset[\pi/T,3\pi/T]$$

such that for $\bar\xi\in\bar J_l$,

$$\left|\langle k,\tilde\omega(\bar\xi)\rangle\pm2\lambda_{0,l}\big(\bar\xi,\tilde\omega(\bar\xi),\varepsilon\big)\right|\geqslant\frac{\varepsilon^{\frac{1}{3}}\varrho\,\mathrm{meas}\,\hat J}{(1+l^2)(|k|+1)^{m+3}}$$

and

$$\left|\langle k,\tilde\omega(\bar\xi)\rangle\pm2\lambda_{j,l}\big(\bar\xi,\tilde\omega(\bar\xi),\varepsilon\big)\right|\geqslant\begin{cases}\frac{\varepsilon^{\frac{1}{3}}\varrho\,\mathrm{meas}\,\hat J}{(1+l^2)(|k|+1)^{m+3}}, & k_{m+1}\neq0,\\ \frac{\varepsilon^{\frac{2}{3}}\varrho\,\mathrm{meas}\,\hat J}{(1+l^2)(|k|+1)^{m+3}}, & k_{m+1}=0,\end{cases}\quad j\geqslant1,\ l=0,\ldots,\nu,$$

and

$$\mathrm{meas}\,\bar J_l\geqslant\mathrm{meas}\,\hat J\left(1-C\varrho\sum_{i=0}^{l}\frac{1}{1+i^2}\right),$$

where C is a constant depending on m and $\bar\xi$. Moreover, let $\bar J=\bigcap_{l=0}^{\infty}\bar J_l$, then

$$\mathrm{meas}\,\bar J\geqslant\mathrm{meas}\,\hat J\big(1-\mathcal{O}(\varrho)\big)\tag{4.52}$$

provided that ϱ is small enough.

Proof. First of all, from (4.4) and (4.13) we get

$$\left|\partial_{\bar{\xi}}\tilde{\lambda}_{j,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right| \leqslant C\varepsilon^{\frac{3\bar{\sigma}}{4}}\lambda_j^{-\frac{1}{2}}.$$

From $(4.2)_l$, it follows that $\lambda_{j,l}(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon)=\sqrt{\lambda_j}+\mathcal{O}(\varepsilon^{\frac{4}{3}})$ and

$$\left|\partial_{\bar{\xi}}\lambda_{j,l}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right| = \frac{\varepsilon^{\frac{2}{3}}\frac{\partial}{\partial\bar{\xi}}[\hat{V}(\theta,\bar{\xi},\varepsilon)]}{2\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}(\theta,\bar{\xi},\varepsilon)]}}+\sum_{k=2}^{l}\varepsilon^{\frac{2k}{3}}\partial_{\bar{\xi}}\tilde{\lambda}_{j,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big).$$

Hence, we get

$$\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle \pm 2\lambda_{j,l}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big) = \begin{cases} \pm 2\sqrt{\lambda_j}+\mathcal{O}(\varepsilon^{\frac{4}{3}})+\langle k,\tilde{\omega}(\bar{\xi})\rangle, & l=2,\dots,\nu, \\ \pm 2\sqrt{\lambda_j}+\langle k,\tilde{\omega}(\bar{\xi})\rangle, & l=0,1. \end{cases} \tag{4.53}$$

Let $k=0$, then

$$\begin{aligned}\left|\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle \pm 2\lambda_{j,l}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right| &\geqslant 2\sqrt{\lambda_j}-C\varepsilon^{\frac{4}{3}} > 2\varepsilon^{\frac{1}{3}}\sqrt{I_1}-C\varepsilon^{\frac{4}{3}} \\ &\geqslant \frac{\varepsilon^{\frac{1}{3}}\varrho\,\mathrm{meas}\,\hat{J}}{1+l^2} \geqslant \frac{\varepsilon^{\frac{2}{3}}\varrho\,\mathrm{meas}\,\hat{J}}{1+l^2}\end{aligned}$$

holds provided that ε and ϱ are small enough.

Now, we let that $k\neq 0$. Note that $\alpha(\bar{\xi})\in A_\gamma$, we have

$$\left|\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle\right| \geqslant \frac{\gamma}{|k|^{m+1}}. \tag{4.54}$$

Hence, for $j=0$, we have

$$\begin{aligned}\left|\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle \pm 2\lambda_{0,l}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right| &\geqslant \left|\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle\right| - 2\varepsilon^{\frac{1}{3}}\sqrt{[\hat{V}]}-\left|\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\right| \\ &\geqslant \frac{\gamma}{|k|^{m+1}}-2\varepsilon^{\frac{1}{3}}\sqrt{[\hat{V}]}-\left|\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\right| \\ &\geqslant \frac{\varepsilon^{\frac{1}{3}}\varrho\,\mathrm{meas}\,\hat{J}}{(1+l^2)(|k|+1)^{m+3}}\end{aligned}$$

when ε small enough. For $j\geqslant 1$, when $k_{m+1}=0$, we assume that

$$\bar{J}^{0\pm}_{j,k,l}=\left\{\bar{\xi}\in\hat{J}\colon\ \left|\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle \pm 2\lambda_{j,l}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right| < \frac{\varepsilon^{\frac{2}{3}}\,\mathrm{meas}\,\hat{J}}{(1+l^2)(|k|+1)^{m+3}}\right\}$$

and

$$\bar{J}^{02}_l=\bigcup_{j\geqslant 1}\bigcup_{k\in\mathbb{Z}^{m+1}}\left(\bar{J}^{0+}_{j,k,l}\cup\bar{J}^{0-}_{j,k,l}\right)$$

and when $k_{m+1}\neq 0$, we assume that

$$\bar{J}^{\pm}_{j,k,l} = \left\{ \bar{\xi} \in \hat{J} \colon \left| \left\langle k, \tilde{\omega}(\bar{\xi}) \right\rangle \pm 2\lambda_{j,l}\left(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\right) \right| < \frac{\varepsilon^{\frac{1}{3}} \operatorname{meas} \hat{J}}{(1+l^2)(|k|+1)^{m+3}} \right\}$$

and

$$\bar{J}^{2}_{l} = \bigcup_{j \geqslant 1} \bigcup_{k \in \mathbb{Z}^{m+1}} \left(\bar{J}^{+}_{j,k,l} \cup \bar{J}^{-}_{j,k,l} \right).$$

We have known that $|\mathcal{O}(\varepsilon^{\frac{4}{3}}) + \langle k, \tilde{\omega}(\bar{\xi}) \rangle| \leqslant 1 + |k||\tilde{\omega}(\bar{\xi})|$ holds when ε small enough. Thus, when $j > 1 + |k||\tilde{\omega}(\bar{\xi})|$, we have, from (4.53)

$$\begin{aligned}
\left| \left\langle k, \tilde{\omega}(\bar{\xi}) \right\rangle \pm 2\lambda_{j,l}\left(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\right) \right| &\geqslant \left| \pm 2\lambda_{j,l}\left(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\right) \right| - \left| \mathcal{O}\left(\varepsilon^{\frac{4}{3}}\right) + \left\langle k, \tilde{\omega}(\bar{\xi}) \right\rangle \right| \\
&\geqslant 2\sqrt{j^2} - \left(1 + |k|\left|\tilde{\omega}(\bar{\xi})\right|\right) \\
&> 2\left(1 + |k|\left|\tilde{\omega}(\bar{\xi})\right|\right) - \left(1 + |k|\left|\tilde{\omega}(\bar{\xi})\right|\right) \\
&> 1 + |k|\left|\tilde{\omega}(\bar{\xi})\right|,
\end{aligned}$$

which implies the set $\bar{J}^{0\pm}_{j,k,l}$ and $\bar{J}^{\pm}_{j,k,l}$ are empty. So, we only need to consider the case $1 \leqslant j \leqslant 1 + [\![|k||\tilde{\omega}(\bar{\xi})|]\!]$ in order to calculate $\bar{J}^{02}_{l}$ and $\bar{J}^{2}_{l}$, where $[\![\cdot]\!]$ stands for the integer part of $\cdot$. For $1 \leqslant j \leqslant 1 + [\![|k||\tilde{\omega}(\bar{\xi})|]\!]$, we let

$$f^{\pm}_{j,k,l}(\bar{\xi}) = \left\langle k, \tilde{\omega}(\bar{\xi}) \right\rangle \pm 2\lambda_{j,l}\left(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\right)$$

and

$$\left\langle k, \tilde{\omega}(\bar{\xi}) \right\rangle = \sum_{i=1}^{m} k_i \omega_i + k_{m+1} \alpha(\bar{\xi}).$$

If $k_{m+1} = 0$, we have

$$\begin{aligned}
\left| \frac{d}{d\bar{\xi}} f^{\pm}_{j,k,l}(\bar{\xi}) \right| &= \left| \frac{d}{d\bar{\xi}} \left(\sum_{i=1}^{m} k_i \omega_i + k_{m+1} \alpha(\bar{\xi}) \right) \pm 2 \frac{d}{d\bar{\xi}} \lambda_{j,l}\left(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\right) \right| \\
&= 2 \left| \frac{d}{d\bar{\xi}} \lambda_{j,l}\left(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\right) \right| \geqslant \left| \frac{\varepsilon^{\frac{2}{3}} \frac{\partial}{\partial \bar{\xi}} [\hat{V}(\theta, \bar{\xi}, \varepsilon)]}{\sqrt{j^2 + \varepsilon^{\frac{2}{3}} [\hat{V}(\theta, \bar{\xi}, \varepsilon)]}} \right| - 2 \left| \sum_{k=2}^{l} \varepsilon^{\frac{2k}{3}} \partial_{\bar{\xi}} \tilde{\lambda}_{j,k}\left(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\right) \right| \\
&\geqslant \frac{\varepsilon^{\frac{2}{3}} I_2}{\sqrt{2(1 + [\![|k||\tilde{\omega}(\bar{\xi})|]\!])^2}} - 2 \sum_{k=2}^{l} \varepsilon^{\frac{2k}{3}} \varepsilon^{\frac{3\bar{\sigma}}{4}} \lambda_j^{-\frac{1}{2}} \geqslant \frac{\varepsilon^{\frac{2}{3}} I_2}{\sqrt{2}(1 + [\![|k||\tilde{\omega}(\bar{\xi})|]\!])} - C_1 \varepsilon^{\frac{4}{3}} \\
&\geqslant C_2(\bar{\xi}) \varepsilon^{\frac{2}{3}}
\end{aligned}$$

provided that ε is small enough. By using Lemma 7.8 in [32], we get

$$\operatorname{meas} \bar{J}^{0\pm}_{j,k,l} \leqslant \frac{2\varrho \operatorname{meas} \hat{J}}{C_2(\bar{\xi})(1+l^2)(|k|+1)^{m+3}}. \tag{4.55}$$

It yields that, from (4.60),

$$\begin{aligned}
\operatorname{meas} \bar{J}_l^{02} &= \operatorname{meas} \bigcup_{0\neq k\in\mathbb{Z}^{m+1}} \bigcup_{j=1}^{1+[\![|k||\omega|]\!]} \left(\bar{J}_{j,k,l}^{+} \cup \bar{J}_{j,k,l}^{-}\right) \\
&\leqslant \sum_{0\neq k\in\mathbb{Z}^{m+1}} \left(1+[\![|k||\omega|]\!]\right) \frac{4\varrho \operatorname{meas} \hat{J}}{C_2(\bar{\xi})(1+l^2)(|k|+1)^{m+3}} \\
&\leqslant \frac{C(\bar{\xi})\varrho \operatorname{meas} \hat{J}}{1+l^2} \sum_{0\neq k\in\mathbb{Z}^{m+1}} \frac{1}{|k|^{m+2}},
\end{aligned}$$

where $C(\bar{\xi})$ is a constant depending on $\bar{\xi}$ only. Let $|k|_\infty := \max\{|k_1|, |k_2|, \dots, |k_{m+1}|\}$. From the inequalities

$$|k|_\infty \leqslant |k| \leqslant (m+1)|k|_\infty, \tag{4.56}$$

and

$$\sum_{|k|_\infty = p} 1 \leqslant 2(m+1)(2p+1)^m, \tag{4.57}$$

we get

$$\begin{aligned}
\operatorname{meas} \bar{J}_l^{02} &\leqslant \frac{C(\bar{\xi})(m+1)\varrho \operatorname{meas} \hat{J}}{1+l^2} \sum_{p=1}^{\infty} (2p+1)^m p^{-(m+2)} \\
&= \frac{C(\bar{\xi})\varrho \operatorname{meas} \hat{J}}{1+l^2}
\end{aligned}$$

by using of the convergence of $\sum_{p=1}^{\infty}(2p+1)^m p^{-(m+2)}$. Letting

$$\bar{J}_0 = \hat{J} \setminus \bar{J}_0^{02}, \quad \text{and} \quad \bar{J}_{l+1} = \bar{J}_l \setminus \bar{J}_{l+1}^{02}, \quad l = 0, 1, \dots, \nu - 1, \tag{4.58}$$

then $(4.39)_l$ and $(4.40)_l$ hold true. let $\bar{J} = \bigcap_{l=0}^{\infty} \bar{J}_l$, then

$$\begin{aligned}
\operatorname{meas} \bar{J} = \lim_{l\to\infty} \operatorname{meas} \bar{J}_l &\geqslant \operatorname{meas} \hat{J} \left(1 - C(\bar{\xi})\varrho \sum_{i=0}^{\infty} \frac{1}{1+i^2}\right) \\
&= \operatorname{meas} \hat{J}\left(1 - \mathcal{O}(\varrho)\right).
\end{aligned}$$

Let $k_{m+1} \neq 0$. From (2.14), we have

$$\left|\alpha'(\bar{\xi})\right| = \frac{\frac{4}{9} s \varepsilon^{\frac{1}{3}} \bar{\xi}^{-\frac{2}{3}}}{|a_0'(\alpha(\bar{\xi}))|} \geqslant \frac{\frac{4}{9} s \varepsilon^{\frac{1}{3}} (\frac{3\omega_0}{2})^{-\frac{2}{3}}}{|a_0'(\alpha(\bar{\xi}))|} = C_3(\bar{\xi})\varepsilon^{\frac{1}{3}} \tag{4.59}$$

with $C_3(\bar{\xi}) := \frac{\frac{4}{9} s (\frac{3\omega_0}{2})^{-\frac{2}{3}}}{|a_0'(\alpha(\bar{\xi}))|} > 0$. Thus, we have

$$\begin{aligned}\left|\frac{d}{d\bar\xi} f^{\pm}_{j,k,l}(\bar\xi)\right| &= \left|\frac{d}{d\bar\xi}\left(\sum_{i=1}^{m} k_i\omega_i + k_{m+1}\alpha(\bar\xi)\right) \pm 2\frac{d}{d\bar\xi}\lambda_{j,l}\big(\bar\xi, \tilde\omega(\bar\xi), \varepsilon\big)\right| \\ &\geqslant |k_{m+1}|\big|\alpha'(\bar\xi)\big| - 2\left|\frac{d}{d\bar\xi}\lambda_{j,l}\big(\bar\xi, \tilde\omega(\bar\xi), \varepsilon\big)\right| \\ &\geqslant |k_{m+1}|C_3(\bar\xi)\varepsilon^{\frac{1}{3}} - \frac{\varepsilon^{\frac{2}{3}}\frac{\partial}{\partial\bar\xi}[\hat V(\theta, \bar\xi, \varepsilon)]}{\sqrt{j^2 + \varepsilon^{\frac{2}{3}}[\hat V(\theta, \bar\xi, \varepsilon)]}} - C_1\varepsilon^{\frac{4}{3}} \\ &\geqslant |k_{m+1}|C_3(\bar\xi)\varepsilon^{\frac{1}{3}} - \varepsilon^{\frac{2}{3}}\frac{\partial}{\partial\bar\xi}\big[\hat V(\theta, \bar\xi, \varepsilon)\big] - C_1\varepsilon^{\frac{4}{3}} \\ &\geqslant C(\bar\xi)\varepsilon^{\frac{1}{3}}.\end{aligned}$$

By the same argument as the above case, we have

$$\operatorname{meas} \bar J^{\pm}_{j,k,l} \leqslant \frac{2\varrho \operatorname{meas} \hat J}{C(\bar\xi)(1+l^2)(|k|+1)^{m+3}}, \tag{4.60}$$

and

$$\operatorname{meas} \bar J^2_l \leqslant \frac{C(\bar\xi)\varrho \operatorname{meas} \hat J}{1+l^2},$$

where $C(\bar\xi)$ is a constant depending on $\bar\xi$ only. Letting

$$\bar J_0 = \hat J \setminus \bar J^2_0, \quad \text{and} \quad \bar J_{l+1} = \bar J_l \setminus \bar J^2_{l+1}, \quad l = 0, 1, \ldots, \nu - 1, \tag{4.61}$$

then $(4.39)_l$ and $(4.40)_l$ hold true. Let $\bar J = \bigcap_{l=0}^{\infty} \bar J_l$, then

$$\begin{aligned}\operatorname{meas} \bar J = \lim_{l\to\infty} \operatorname{meas} \bar J_l &\geqslant \operatorname{meas} \hat J\left(1 - C(\bar\xi)\varrho\sum_{i=0}^{\infty}\frac{1}{1+i^2}\right) \\ &= \operatorname{meas} \hat J\big(1 - \mathcal{O}(\varrho)\big). \qquad \square\end{aligned}$$

4.4. The regularity of perturbation term

Set $G^3_{i,j,d} = G^3_{|i|,|j|,|d|}$ and $G^4_{ijdl} = G^4_{|i||j||d||l|}$. Noting that the transformation Σ^∞ is linear, and from (4.38) and (i) of Theorem 4.1 we get for $j = 0, 1, 2, \ldots$

$$z_j \circ \Sigma^\infty = z_j + \varepsilon^{2/3}\tilde f^*_{j,\infty}(\theta; \bar\xi, \varepsilon) z_j + \varepsilon^{2/3}\tilde f^*_{\infty,j}(\theta; \bar\xi, \varepsilon)\bar z_j$$

where

$$\big\|\tilde f^*_{j,\infty}(\theta; \bar\xi, \varepsilon)\big\|^*_{\Theta(\sigma/2)\times\bar J}, \ \big\|\tilde f^*_{\infty,j}(\theta; \bar\xi, \varepsilon)\big\|^*_{\Theta(\sigma/2)\times\bar J} \leqslant C.$$

For convenience we introduce another coordinates $(\dots, w_{-2}, w_{-1}, w_0, w_1, w_2, \dots)$ in l_b^s by setting $z_0 = w_0$, $\bar{z_0} = w_{-0}$, $z_j = w_j$, $\bar{z_j} = w_{-j}$ where l_b^s consists of all bi-infinite sequence with finite norm

$$\|w\|_{a,s}^2 = |w_0|^2 + |w_{-0}|^2 + \sum_{|j|\geqslant 1}^{\infty} |w_j|^2 |j|^{2s} e^{2a|j|}.$$

Hamiltonian (3.22) is changed into

$$\hat{H} := \tilde{H} \circ \Sigma^{\infty} = \langle \tilde{\omega}(\bar{\xi}), J \rangle + \sum_{j\geqslant 0} \mu_j w_j w_{-j}, \tag{4.62}$$

and

$$(z_0 + \bar{z_0}) \circ \Sigma^{\infty} = S_{11}(\theta, \bar{\xi}, \varepsilon) w_0 + S_{12}(\theta, \bar{\xi}, \varepsilon) w_{-0},$$

where

$$S_{11}(\theta, \bar{\xi}, \varepsilon) := 1 + \varepsilon^{2/3} \tilde{f}^1_{0,\infty}(\theta; \bar{\xi}, \varepsilon), \qquad S_{12}(\theta, \bar{\xi}, \varepsilon) := 1 + \varepsilon^{2/3} \tilde{f}^2_{0,\infty}(\theta; \bar{\xi}, \varepsilon)$$

with

$$\big\| \tilde{f}^1_{0,\infty}(\theta; \bar{\xi}, \varepsilon) \big\|^*_{\Theta(\sigma/2)\times \bar{J}}, \ \big\| \tilde{f}^2_{0,\infty}(\theta; \bar{\xi}, \varepsilon) \big\|^*_{\Theta(\sigma/2)\times \bar{J}} \leqslant C.$$

Moreover, (3.23) and (3.24) are changed, respectively, into

$$\begin{aligned}
\tilde{G}^3 &= G^3(z, \theta, \bar{\xi}, \varepsilon) \circ \Sigma^{\infty} \\
&= \frac{1}{2\sqrt{2}} G^3_{0,0,0}(\theta, \bar{\xi}, \varepsilon) \big[S_{11}(\theta, \bar{\xi}, \varepsilon) w_0 + S_{12}(\theta, \bar{\xi}, \varepsilon) w_{-0} \big]^3 \\
&\quad + \frac{3}{2\sqrt{2}} \big[S_{11}(\theta, \bar{\xi}, \varepsilon) w_0 + S_{12}(\theta, \bar{\xi}, \varepsilon) w_{-0} \big] \sum_{\substack{j \pm d = 0 \\ j,d \neq 0}} \tilde{G}^3_{0,j,d}(\theta, \bar{\xi}, \varepsilon) w_j w_d \\
&\quad + \frac{1}{2\sqrt{2}} \sum_{\substack{j \pm d \pm l = 0 \\ j,d,l \neq 0}} \tilde{G}^3_{j,d,l}(\theta, \bar{\xi}, \varepsilon) w_j w_d w_l
\end{aligned} \tag{4.63}$$

and

$$\begin{aligned}
\tilde{G}^4 = G^4(z, \vartheta) \circ \Sigma^{\infty} &= \frac{1}{4} G^4_{0000}(\theta, \bar{\xi}, \varepsilon) \big[S_{11}(\theta, \bar{\xi}, \varepsilon) w_0 + S_{12}(\theta, \bar{\xi}, \varepsilon) w_{-0} \big]^4 \\
&\quad + \frac{3}{2} \big[S_{11}(\theta, \bar{\xi}, \varepsilon) w_0 + S_{12}(\theta, \bar{\xi}, \varepsilon) w_{-0} \big]^2 \sum_{\substack{j \pm d = 0 \\ j,d \neq 0}} \tilde{G}^4_{00jd}(\theta, \bar{\xi}, \varepsilon) w_j w_d \\
&\quad + \big[S_{11}(\theta, \bar{\xi}, \varepsilon) w_0 + S_{12}(\theta, \bar{\xi}, \varepsilon) w_{-0} \big] \sum_{\substack{j \pm d \pm l = 0 \\ j,d,l \neq 0}} \tilde{G}^4_{0jdl}(\theta, \bar{\xi}, \varepsilon) w_j w_d w_l \\
&\quad + \frac{1}{4} \sum_{\substack{i \pm j \pm d \pm l = 0 \\ i,j,d,l \neq 0}} \tilde{G}^4_{ijdl}(\theta, \bar{\xi}, \varepsilon) w_i w_j w_d w_l,
\end{aligned} \tag{4.64}$$

$$\tilde{G}^3_{i,j,d}(\theta,\bar{\xi},\varepsilon)=G^3_{i,j,d}(\theta,\bar{\xi},\varepsilon)\big(1+\varepsilon^{\frac{2}{3}}G^{3*}_{ijd}(\theta,\bar{\xi},\varepsilon)\big) \tag{4.65}$$

with

$$\big\|G^{3*}_{ijd}(\theta,\bar{\xi},\varepsilon)\big\|^*_{\Theta(\sigma/2)\times\bar{J}}\leqslant C, \tag{4.66}$$

and

$$\tilde{G}^4_{ijdl}(\theta,\bar{\xi},\varepsilon)=G^4_{i,j,d,l}(\vartheta)\big(1+\varepsilon^{\frac{2}{3}}G^{4*}_{ijdl}(\theta,\bar{\xi},\varepsilon)\big), \tag{4.67}$$

and

$$\big\|G^{4*}_{ijdl}(\theta,\bar{\xi},\varepsilon)\big\|^*_{\Theta(\sigma/2)\times\bar{J}}\leqslant C. \tag{4.68}$$

This implies the Hamiltonian (3.21) is changed by the transformation Σ^∞ into

$$H=\hat{H}+\varepsilon\tilde{G}^3+\varepsilon^2\tilde{G}^4. \tag{4.69}$$

Next we consider the regularity of the gradient of $\tilde{G}^3$ and $\tilde{G}^4$. Following Pöschel [5], we have the following lemma.

The following lemma was proved in [5], we only give the result.

Lemma 4.3. *For $a\geqslant 0$ and $s>1/2$, the space $l^{a,s}$ is a Hilbert algebra with respect to convolution of the sequences, $(q*p)_j:=\sum_k q_{j-k}p_k$, and*

$$\|q*p\|_{a,s}\leqslant C\|q\|_{a,s}\|p\|_{a,s}$$

with a constant C depending only on s.

Using the above lemma, we can prove the following lemma.

Lemma 4.4. *For $a\geqslant 0$ and $s>1$, the gradient $\tilde{G}^3_w$ and $\tilde{G}^4_w$ are real analytic for real argument as a map from some neighborhood of origin in $l^{a,s}$ into $l^{a,s+1/2}$, with*

$$\big\|\tilde{G}^3_w\big\|_{a,s+1/2}\leqslant C\|w\|^2_{a,s},\qquad \big\|\tilde{G}^4_w\big\|_{a,s+1/2}\leqslant C\|w\|^3_{a,s} \tag{4.70}$$

uniformly for $(\theta,\bar{\xi})\in\Theta(\sigma/2)\times\hat{J}$, where C is a constant large enough as ε small enough. The Hamiltonian $\tilde{G}^3$ and $\tilde{G}^4$ depend on the "time" $\theta=(\theta_1,\dots,\theta_m,\theta_{m+1})=(\omega_1 t,\dots,\omega_m t,\alpha(\bar{\xi})t)$.

Proof. Due to (4.63), then for $(\theta,\bar{\xi})\in\Theta(\sigma/2)\times\hat{J}$,

$$\begin{aligned}\left|\frac{\partial\tilde{G}^3}{\partial w_0}\right| &\leqslant \frac{3}{2\sqrt{2}}\big|G^3_{0,0,0}(\theta,\bar{\xi},\varepsilon)\big|\big(\big|S_{11}(\theta,\bar{\xi},\varepsilon)\big||w_0|+\big|S_{12}(\theta,\bar{\xi},\varepsilon)\big||w_{-0}|\big)^2\big|S_{11}(\theta,\bar{\xi},\varepsilon)\big|\\ &\quad+\frac{3}{2\sqrt{2}}\big|S_{11}(\theta,\bar{\xi},\varepsilon)\big|\sum_{\substack{j\pm d=0\\ j,d\neq 0}}\big|\tilde{G}^3_{0,j,d}(\theta,\bar{\xi},\varepsilon)\big||w_j||w_d|\\ &\leqslant C\frac{(|w_0|+|w_{-0}|)^2}{\sqrt[4]{\lambda_0\lambda_0\lambda_0}}+C\sum_{\substack{j\pm d=0\\ j,d\neq 0}}\frac{|w_j w_d|}{\sqrt[4]{\lambda_0\lambda_j\lambda_d}}.\end{aligned}$$

Similarly, we have

$$\left|\frac{\partial \tilde{G}^3}{\partial w_{-0}}\right| \leqslant C\frac{(|w_0|+|w_{-0}|)^2}{\sqrt[4]{\lambda_0\lambda_0\lambda_0}} + C\sum_{\substack{j\pm d=0\\ j,d\neq 0}} \frac{|w_j w_d|}{\sqrt[4]{\lambda_0\lambda_j\lambda_d}},$$

and for $l \neq 0$,

$$\begin{aligned}\left|\frac{\partial \tilde{G}^3}{\partial w_l}\right| &\leqslant C\big(|w_0|+|w_{-0}|\big)\sum_{\pm d=l}\big|\tilde{G}^3_{0,l,d}(\theta,\bar{\xi},\varepsilon)\big||w_d| + C\sum_{\pm i\pm j=l}\big|\tilde{G}^3_{i,j,l}(\theta,\bar{\xi},\varepsilon)\big||w_i w_j| \\ &\leqslant C\big(|w_0|+|w_{-0}|\big)\sum_{\pm d=l}\frac{|w_d|}{\sqrt[4]{\lambda_d\lambda_l}} + C\sum_{\pm i\pm j=l}\frac{|w_i w_j|}{\sqrt[4]{\lambda_i\lambda_j\lambda_l}}.\end{aligned}$$

From Lemma 4.3, (4.65) and (3.16), and with $\tilde{w}_j = \frac{|w_j|+|w_{-j}|}{\sqrt[4]{\lambda_j}}$, we have

$$\begin{aligned}\|\tilde{G}^3_w\|^2_{a,s+1/2} &= \big|\tilde{G}^3_{w_0}\big|^2 + \big|\tilde{G}^3_{w_{-0}}\big|^2 + \sum_{|l|\geqslant 1}\big|\tilde{G}^3_{w_l}\big|^2|l|^{2s+1}e^{2a|l|} \\ &\leqslant C\left(\frac{(|w_0|+|w_{-0}|)^2}{\sqrt[4]{\lambda_0\lambda_0\lambda_0}} + \sum_{\substack{j\pm d=0\\ j,d\neq 0}}\frac{|w_j w_d|}{\sqrt[4]{\lambda_0\lambda_j\lambda_d}}\right)^2 \\ &\quad + C\sum_{|l|\geqslant 1}\left(\big(|w_0|+|w_{-0}|\big)\sum_{\pm d=l}\frac{|w_d|}{\sqrt[4]{\lambda_d\lambda_l}} + \sum_{\pm i\pm j=l}\frac{|w_i w_j|}{\sqrt[4]{\lambda_i\lambda_j\lambda_l}}\right)^2|l|^{2s+1}e^{2a|l|} \\ &\leqslant C\left(2\big|(\tilde{w}*\tilde{w})_0\big|^2 + \sum_{|l|\geqslant 1}(\tilde{w}*\tilde{w})_l^2|l|^{2s}e^{2a|l|}\right) \\ &\leqslant C\|\tilde{w}*\tilde{w}\|^2_{a,s} \leqslant C\big(\|\tilde{w}\|^2_{a,s}\big)^2 \leqslant C\big(\|w\|^2_{a,s}\big)^2\end{aligned}$$

as required, where C is a large constant when ε is small enough. Similarly, we can prove that

$$\big\|\tilde{G}^4_w\big\|_{a,s+1/2} \leqslant C\|w\|^3_{a,s}. \qquad \square$$

5. The Birkhoff normal form

In this section, we will transform the Hamiltonian (4.69) into some partial Birkhoff normal form of order four so that it appears, in a sufficiently small neighborhood of the origin, as a small perturbation of some nonlinear integrable system. To this end we have to kill the perturbation $\tilde{G}^3$ and the nonresonant part of the perturbation $\tilde{G}^4$ by Birkhoff normal form.

5.1. Elimination of Hamiltonian $\tilde{G}^3$ via Birkhoff normal form

Consider a Hamilton function

$$
\begin{aligned}
F_3 = {} & \zeta_{30}(\theta;\bar{\xi},\varepsilon)w_0^3 + \zeta_{21}(\theta;\bar{\xi},\varepsilon)w_0^2 w_{-0} + \zeta_{12}(\theta;\bar{\xi},\varepsilon)w_0 w_{-0}^2 + \zeta_{03}(\theta;\bar{\xi},\varepsilon)w_{-0}^3 \\
& + w_0 \sum_{\substack{j\pm d=0\\ j,d\neq 0}} f^3_{0,j,d}(\theta,\bar{\xi},\varepsilon)w_j w_d + w_{-0} \sum_{\substack{j\pm d=0\\ j,d\neq 0}} f^3_{-0,j,d}(\theta,\bar{\xi},\varepsilon)w_j w_d \\
& + \sum_{\substack{j\pm d\pm l=0\\ j,d,l\neq 0}} f^3_{j,d,l}(\theta,\bar{\xi},\varepsilon)w_j w_d w_l
\end{aligned}
$$

with

$$
\zeta_{ij}(\theta,\bar{\xi},\varepsilon) = \sum_{k\in\mathbb{Z}^{m+1}} \zeta_{ij;k}(\bar{\xi},\varepsilon)e^{\mathrm{i}\langle k,\theta\rangle} \tag{5.1}
$$

and

$$
f^3_{j,d,l}(\theta,\bar{\xi},\varepsilon) = \sum_{k\in\mathbb{Z}^{m+1}} f^3_{jdl;k}(\bar{\xi},\varepsilon)e^{\mathrm{i}\langle k,\theta\rangle}. \tag{5.2}
$$

By X_{F_3} denote the time-1 map of the vector field of the Hamiltonian εF_3. Then

$$
\begin{aligned}
H\circ X_{F_3} = {} & \hat{H} + \varepsilon\tilde{G}^3 + \varepsilon\{\hat{H},F_3\} + \varepsilon^2\tilde{G}^4 + \varepsilon^2\big\{\tilde{G}^3,F_3\big\} + \frac{1}{2}\varepsilon^2\big\{\{\hat{H},F_3\},F_3\big\} \\
& + \varepsilon^3\Bigg[\big\{\tilde{G}^4,F_3\big\} + \int_0^1\bigg(\frac{1}{2}(1-s)^2\big\{\big\{\{\hat{H},F_3\},F_3\big\},F_3\big\} + (1-s)\big\{\big\{\tilde{G}^3,F_3\big\},F_3\big\}\bigg)\circ X^s_{F_3}\,ds\Bigg] \\
& + \varepsilon^4\int_0^1 (1-s)\big\{\big\{\tilde{G}^4,F_3\big\},F_3\big\}\circ X^s_{F_3}\,ds,
\end{aligned} \tag{5.3}
$$

where $\{\cdot,\cdot\}$ is Poisson bracket with respect to the symplectic structure $\mathrm{i}\,dz\wedge d\bar{z} + d\theta\wedge dJ$. Now let us write $\mu'_i = \operatorname{sgn} i\cdot\mu_{|i|}$ and compute $\{\hat{H},F_3\}$.

$$
\begin{aligned}
\{\hat{H},F_3\} = {} & \mathrm{i}\left(\frac{\partial\hat{H}}{\partial z}\frac{\partial F_3}{\partial\bar{z}} - \frac{\partial\hat{H}}{\partial\bar{z}}\frac{\partial F_3}{\partial z}\right) - \frac{\partial\hat{H}}{\partial J}\frac{\partial F_3}{\partial\theta} \\
= {} & -\mathrm{i}\,\mu_0\Bigg(3\zeta_{30}(\theta;\bar{\xi},\varepsilon)w_0^3 + \zeta_{21}(\theta;\bar{\xi},\varepsilon)w_0^2 w_{-0} - \zeta_{12}(\theta;\bar{\xi},\varepsilon)w_0 w_{-0}^2 - 3\zeta_{03}(\theta;\bar{\xi},\varepsilon)w_{-0}^3 \\
& + w_0 \sum_{\substack{j\pm d=0\\ j,d\neq 0}} f^3_{0,j,d}(\theta,\bar{\xi},\varepsilon)w_j w_d - w_{-0} \sum_{\substack{j\pm d=0\\ j,d\neq 0}} f^3_{-0,j,d}(\theta,\bar{\xi},\varepsilon)w_j w_d\Bigg) \\
& - \mathrm{i}\,w_0 \sum_{\substack{j\pm d=0\\ j,d\neq 0}} f^3_{0,j,d}(\theta,\bar{\xi},\varepsilon)\big(\mu'_j+\mu'_d\big)w_j w_d \\
& - \mathrm{i}\,w_{-0} \sum_{\substack{j\pm d=0\\ j,d\neq 0}} f^3_{-0,j,d}(\theta,\bar{\xi},\varepsilon)\big(\mu'_j+\mu'_d\big)w_j w_d
\end{aligned}
$$

$$
\begin{aligned}
&- \mathrm{i} \sum_{\substack{j\pm d\pm l=0\\ j,d,l\neq 0}} \big[\big(\mu'_j+\mu'_d+\mu'_l\big) f^3_{j,d,l}(\bar\xi,\varepsilon)\big] w_j w_d w_l \\
&- \left\langle \tilde\omega(\bar\xi), \frac{\partial}{\partial\theta}\zeta_{30}(\theta;\bar\xi,\varepsilon)\right\rangle w_0^3 - \left\langle \tilde\omega(\bar\xi), \frac{\partial}{\partial\theta}\zeta_{21}(\theta;\bar\xi,\varepsilon)\right\rangle w_0^2 w_{-0} \\
&- \left\langle \tilde\omega(\bar\xi), \frac{\partial}{\partial\theta}\zeta_{12}(\theta;\bar\xi,\varepsilon)\right\rangle w_0 w_{-0}^2 - \left\langle \tilde\omega(\bar\xi), \frac{\partial}{\partial\theta}\zeta_{03}(\theta;\bar\xi,\varepsilon)\right\rangle w_{-0}^3 \\
&- w_0 \sum_{\substack{j\pm d=0\\ j,d\neq 0}} \left\langle \tilde\omega(\bar\xi), \frac{\partial}{\partial\theta} f^3_{0,j,d}(\theta,\bar\xi,\varepsilon)\right\rangle w_j w_d \\
&- w_{-0} \sum_{\substack{j\pm d=0\\ j,d\neq 0}} \left\langle \tilde\omega(\bar\xi), \frac{\partial}{\partial\theta} f^3_{-0,j,d}(\theta,\bar\xi,\varepsilon)\right\rangle w_j w_d \\
&- \sum_{\substack{j\pm d\pm l=0\\ j,d,l\neq 0}} \left\langle \tilde\omega(\bar\xi), \frac{\partial}{\partial\theta} f^3_{j,d,l}(\theta,\bar\xi,\varepsilon)\right\rangle w_j w_d w_l.
\end{aligned}
$$

Letting

$$\tilde G^3 + \{\hat H, F_3\} = 0,$$

we get the following homological equations:

$$
\begin{gathered}
\frac{1}{2\sqrt2} G^3_{0,0,0}(\theta,\bar\xi,\varepsilon) S^3_{11}(\theta,\bar\xi,\varepsilon) - 3\mathrm{i}\,\mu_0\zeta_{30}(\theta,\bar\xi,\varepsilon) - \left\langle \tilde\omega(\bar\xi), \frac{\partial}{\partial\theta}\zeta_{30}(\theta;\bar\xi,\varepsilon)\right\rangle = 0, \\
\frac{3}{2\sqrt2} G^3_{0,0,0}(\theta,\bar\xi,\varepsilon) S^2_{11}(\theta,\bar\xi,\varepsilon) S_{12}(\theta,\bar\xi,\varepsilon) - \mathrm{i}\,\mu_0\zeta_{21}(\theta,\bar\xi,\varepsilon) - \left\langle \tilde\omega(\bar\xi), \frac{\partial}{\partial\theta}\zeta_{21}(\theta;\bar\xi,\varepsilon)\right\rangle = 0, \\
\frac{3}{2\sqrt2} G^3_{0,0,0}(\theta,\bar\xi,\varepsilon) S_{11}(\theta,\bar\xi,\varepsilon) S^2_{12}(\theta,\bar\xi,\varepsilon) + \mathrm{i}\,\mu_0\zeta_{12}(\theta,\bar\xi,\varepsilon) - \left\langle \tilde\omega(\bar\xi), \frac{\partial}{\partial\theta}\zeta_{12}(\theta;\bar\xi,\varepsilon)\right\rangle = 0, \\
\frac{1}{2\sqrt2} G^3_{0,0,0}(\theta,\bar\xi,\varepsilon) S^3_{12}(\theta,\bar\xi,\varepsilon) + 3\mathrm{i}\,\mu_0\zeta_{03}(\theta,\bar\xi,\varepsilon) - \left\langle \tilde\omega(\bar\xi), \frac{\partial}{\partial\theta}\zeta_{03}(\theta;\bar\xi,\varepsilon)\right\rangle = 0, \\
\frac{3}{2\sqrt2} S_{11}(\theta;\bar\xi,\varepsilon) \tilde G^3_{0,j,d}(\theta,\bar\xi,\varepsilon) - \mathrm{i}\big(\mu_0+\mu'_j+\mu'_d\big) f^3_{0,j,d}(\theta,\bar\xi,\varepsilon) - \left\langle \tilde\omega(\bar\xi), \frac{\partial}{\partial\theta} f^3_{0,j,d}(\theta,\bar\xi,\varepsilon)\right\rangle = 0, \\
\frac{3}{2\sqrt2} S_{12}(\theta;\bar\xi,\varepsilon) \tilde G^3_{0,j,d}(\theta,\bar\xi,\varepsilon) + \mathrm{i}\big(\mu_0-\mu'_j-\mu'_d\big) f^3_{-0,j,d}(\theta,\bar\xi,\varepsilon) - \left\langle \tilde\omega(\bar\xi), \frac{\partial}{\partial\theta} f^3_{-0,j,d}(\theta,\bar\xi,\varepsilon)\right\rangle = 0, \\
\frac{1}{2\sqrt2} \tilde G^3_{j,d,l}(\theta,\bar\xi,\varepsilon) - \mathrm{i}\big(\mu'_j+\mu'_d+\mu'_l\big) f^3_{j,d,l}(\theta,\bar\xi,\varepsilon) - \left\langle \tilde\omega(\bar\xi), \frac{\partial}{\partial\theta} f^3_{j,d,l}(\theta,\bar\xi,\varepsilon)\right\rangle = 0.
\end{gathered}
$$

Let

$$G^3_{0,0,0}(\theta,\bar\xi,\varepsilon) S^3_{11}(\theta,\bar\xi,\varepsilon) = \sum_{k\in\mathbb{Z}^{m+1}} U_k(\bar\xi,\varepsilon) e^{\mathrm{i}\langle k,\theta\rangle},$$

$$G^3_{0,0,0}(\theta,\bar\xi,\varepsilon)S^2_{11}(\theta,\bar\xi,\varepsilon)S_{12}(\theta,\bar\xi,\varepsilon)=\sum_{k\in\mathbb{Z}^{m+1}}V_k(\bar\xi,\varepsilon)e^{\mathrm{i}\langle k,\theta\rangle},$$

$$G^3_{0,0,0}(\theta,\bar\xi,\varepsilon)S_{11}(\theta,\bar\xi,\varepsilon)S^2_{12}(\theta,\bar\xi,\varepsilon)=\sum_{k\in\mathbb{Z}^{m+1}}W_k(\bar\xi,\varepsilon)e^{\mathrm{i}\langle k,\theta\rangle},$$

$$G^3_{0,0,0}(\theta,\bar\xi,\varepsilon)S^3_{12}(\theta,\bar\xi,\varepsilon)=\sum_{k\in\mathbb{Z}^{m+1}}X_k(\bar\xi,\varepsilon)e^{\mathrm{i}\langle k,\theta\rangle},$$

$$S_{11}(\theta;\bar\xi,\varepsilon)\tilde G^3_{0,j,d}(\theta,\bar\xi,\varepsilon)=\sum_{k\in\mathbb{Z}^{m+1}}Y_k(\bar\xi,\varepsilon)e^{\mathrm{i}\langle k,\theta\rangle},$$

$$S_{12}(\theta;\bar\xi,\varepsilon)\tilde G^3_{0,j,d}(\theta,\bar\xi,\varepsilon)=\sum_{k\in\mathbb{Z}^{m+1}}Z_k(\bar\xi,\varepsilon)e^{\mathrm{i}\langle k,\theta\rangle},$$

$$\tilde G^3_{j,d,l}(\theta,\bar\xi,\varepsilon)=\sum_{k\in\mathbb{Z}^{m+1}}\tilde G^3_{j,d,l;k}(\bar\xi,\varepsilon)e^{\mathrm{i}\langle k,\theta\rangle},$$

then from the above seven equations we get the following formal solutions:

$$\zeta_{30}(\theta,\bar\xi,\varepsilon)=\sum_{k\in\mathbb{Z}^{m+1}}\frac{U_k(\bar\xi,\varepsilon)}{2\sqrt{2}\,\mathrm{i}(3\mu_0+\langle k,\tilde\omega(\bar\xi)\rangle)}e^{\mathrm{i}\langle k,\theta\rangle},\tag{5.4}$$

$$\zeta_{21}(\theta,\bar\xi,\varepsilon)=\sum_{k\in\mathbb{Z}^{m+1}}\frac{3V_k(\bar\xi,\varepsilon)}{2\sqrt{2}\,\mathrm{i}(\mu_0+\langle k,\tilde\omega(\bar\xi)\rangle)}e^{\mathrm{i}\langle k,\theta\rangle},\tag{5.5}$$

$$\zeta_{12}(\theta,\bar\xi,\varepsilon)=\sum_{k\in\mathbb{Z}^{m+1}}\frac{3W_k(\bar\xi,\varepsilon)}{2\sqrt{2}\,\mathrm{i}(-\mu_0+\langle k,\tilde\omega(\bar\xi)\rangle)}e^{\mathrm{i}\langle k,\theta\rangle},\tag{5.6}$$

$$\zeta_{03}(\theta,\bar\xi,\varepsilon)=\sum_{k\in\mathbb{Z}^{m+1}}\frac{X_k(\bar\xi,\varepsilon)}{2\sqrt{2}\,\mathrm{i}(-3\mu_0+\langle k,\tilde\omega(\bar\xi)\rangle)}e^{\mathrm{i}\langle k,\theta\rangle},\tag{5.7}$$

$$f^3_{0,j,d}(\theta,\bar\xi,\varepsilon)=\sum_{k\in\mathbb{Z}^{m+1}}\frac{3Y_k(\bar\xi,\varepsilon)}{2\sqrt{2}\,\mathrm{i}(\mu_0+\mu'_j+\mu'_d+\langle k,\tilde\omega(\bar\xi)\rangle)}e^{\mathrm{i}\langle k,\theta\rangle},\tag{5.8}$$

$$f^3_{-0,j,d}(\theta,\bar\xi,\varepsilon)=\sum_{k\in\mathbb{Z}^{m+1}}\frac{3Z_k(\bar\xi,\varepsilon)}{2\sqrt{2}\,\mathrm{i}(-\mu_0+\mu'_j+\mu'_d+\langle k,\tilde\omega(\bar\xi)\rangle)}e^{\mathrm{i}\langle k,\theta\rangle},\tag{5.9}$$

$$f^3_{i,j,d}(\theta,\bar\xi,\varepsilon)=\sum_{k\in\mathbb{Z}^{m+1}}\frac{\tilde G^3_{ijd;k}(\bar\xi,\varepsilon)}{2\sqrt{2}\,\mathrm{i}(\mu'_i+\mu'_j+\mu'_d+\langle k,\tilde\omega(\bar\xi)\rangle)}e^{\mathrm{i}\langle k,\theta\rangle}.\tag{5.10}$$

Now we show that the convergence of (5.4)–(5.10). First of all, it is easy to see that, for any positive integer N,

$$\left|\pm N\mu_0\pm\left\langle k,\tilde\omega(\bar\xi)\right\rangle\right|\geqslant\frac{\varepsilon^{\frac{2}{3}}\varrho\,\mathrm{meas}\,\hat J}{(|k|+1)^{2m+6}}\tag{5.11}$$

as $\varepsilon\ll 1$. In fact, if $k=0$, then

$$\begin{aligned}\left|\pm N\mu_0 \pm \langle k, \tilde{\omega}(\bar{\xi})\rangle\right| &= \left|\varepsilon^{\frac{1}{3}} N\sqrt{[\hat{V}]} + \mathcal{O}\left(\varepsilon^{\frac{4}{3}}\right)\right| \\ &\geqslant \varepsilon^{\frac{1}{3}}\left(N\sqrt{[\hat{V}]} - \left|\mathcal{O}(\varepsilon)\right|\right) \\ &\geqslant \frac{\varepsilon^{\frac{2}{3}}\varrho \operatorname{meas} \hat{J}}{(|k|+1)^{2m+6}},\end{aligned}$$

and if $k \neq 0$, then

$$\begin{aligned}\left|\pm N\mu_0 \pm \langle k, \tilde{\omega}(\bar{\xi})\rangle\right| &\geqslant \left|\langle k, \tilde{\omega}(\bar{\xi})\rangle\right| - \left|\mathcal{O}\left(\varepsilon^{\frac{1}{3}}\right)\right| - \left|\mathcal{O}\left(\varepsilon^{\frac{4}{3}}\right)\right| \\ &\geqslant \frac{\gamma}{|k|^{m+1}} - \left|\mathcal{O}\left(\varepsilon^{\frac{1}{3}}\right)\right| - \left|\mathcal{O}\left(\varepsilon^{\frac{4}{3}}\right)\right| \\ &\geqslant \frac{\varepsilon^{\frac{2}{3}}\varrho \operatorname{meas} \hat{J}}{(|k|+1)^{2m+6}}\end{aligned}$$

provided that ε is small enough.

From (3.16) we get

$$\left|G^3_{i,j,d}(\theta, \bar{\xi}, \varepsilon)\right| \leqslant C\varepsilon^{\frac{1}{3}}(\lambda_i\lambda_j\lambda_d)^{-\frac{1}{4}},$$

and by the definitions of S_{11} and S_{12}, (4.65), and the Cauchy estimate, we have

$$\begin{gathered}\left|\zeta_{30}(\theta, \bar{\xi}, \varepsilon)\right| \leqslant C\varepsilon^{-\frac{7}{12}}, \qquad \left|\zeta_{21}(\theta, \bar{\xi}, \varepsilon)\right| \leqslant C\varepsilon^{-\frac{7}{12}}, \\ \left|\zeta_{12}(\theta, \bar{\xi}, \varepsilon)\right| \leqslant C\varepsilon^{-\frac{7}{12}}, \qquad \left|\zeta_{03}(\theta, \bar{\xi}, \varepsilon)\right| \leqslant C\varepsilon^{-\frac{7}{12}}, \quad \theta \in \Theta\left(\frac{\sigma_0}{2}\right).\end{gathered}$$

In order to show the convergence of (5.8)–(5.10) we need the following two lemmas which be proved in Appendix A.

Lemma 5.1. *Let j, d be non-zero integers, such that $j \pm d = 0$. For the parameter set $\hat{J}$, there is a set $\bar{J}_0 \subset \hat{J}$ with*

$$\operatorname{meas} \bar{J}_0 \geqslant \operatorname{meas} \hat{J}(1 - \hat{C}\varrho) \tag{5.12}$$

such that, for any $\bar{\xi} \in \bar{J}_0$ and $\varrho > 0$ small enough,

$$\left|\pm\mu_0 + \mu'_j + \mu'_d + \langle k, \tilde{\omega}(\bar{\xi})\rangle\right| \geqslant \frac{\varepsilon^{\frac{2}{3}}\varrho \operatorname{meas} \hat{J}}{(|k|+1)^{2m+6}}, \quad \forall k \in \mathbb{Z}^{m+1}, \tag{5.13}$$

where $\hat{C}$ is a constant depending on $\bar{\xi}$.

Lemma 5.2. *Let i, j, d be non-zero integers, such that $i \pm j \pm d = 0$. For the parameter set $\hat{J}$, there is a set $\bar{J}_1 \subset \hat{J}$ with*

$$\operatorname{meas} \bar{J}_1 \geqslant \operatorname{meas} \hat{J}(1 - \hat{C}\varrho) \tag{5.14}$$

such that, for any $\bar{\xi} \in \bar{J}_1$ and $\varrho > 0$ small enough,

$$\left|\mu'_i+\mu'_j+\mu'_d+\langle k,\tilde{\omega}(\bar{\xi})\rangle\right| \geqslant \frac{\varepsilon^{\frac{2}{3}}\varrho\,\mathrm{meas}\,\hat{J}}{C_*(|k|+1)^{2m+6}}, \quad \forall k\in\mathbb{Z}^{m+1}, \tag{5.15}$$

where $\hat{C}$ is a constant depending on $\bar{\xi}$ and C_* is a constant sufficiently large.

Using (3.16), (4.65), Lemma 5.1 and Lemma 5.2, we get

$$\left|f^3_{0,j,d}(\theta,\bar{\xi},\varepsilon)\right| \leqslant \hat{C}(\lambda_0\lambda_j\lambda_d)^{-\frac{1}{4}}\varepsilon^{-\frac{1}{3}},$$
$$\left|f^3_{-0,j,d}(\theta,\bar{\xi},\varepsilon)\right| \leqslant \hat{C}(\lambda_0\lambda_j\lambda_d)^{-\frac{1}{4}}\varepsilon^{-\frac{1}{3}},$$
$$\left|f^3_{i,j,d}(\theta,\bar{\xi},\varepsilon)\right| \leqslant \hat{C}(\lambda_i\lambda_j\lambda_d)^{-\frac{1}{4}}\varepsilon^{-\frac{1}{3}}, \quad \theta\in\Theta\left(\frac{\sigma_0}{2}\right).$$

Consequently we have found a quasi-periodic function F_3 analytic in $\Theta(\frac{\sigma_0}{2})$ for $\bar{\xi}\in\bar{J}\cap\bar{J}_0\cap\bar{J}_1$ such that $\tilde{G}^3+\{\hat{H},F_3\}=0$. Therefore, we obtain the new Hamiltonian

$$H=\hat{H}+\varepsilon^2\mathcal{G}^4+\varepsilon^3R_{11}+\varepsilon^4R_{12}, \tag{5.16}$$

where

$$\mathcal{G}^4=\tilde{G}^4+\frac{1}{2}\left\{\tilde{G}^3,F_3\right\},$$

$$R_{11}=\left\{\tilde{G}^4,F_3\right\}+\int_0^1\left((1-s)\left\{\left\{\tilde{G}^3,F_3\right\},F_3\right\}-\frac{1}{2}(1-s)^2\left\{\left\{\tilde{G}^3,F_3\right\},F_3\right\}\right)\circ X^s_{F_3}\,ds,$$

$$R_{12}=\int_0^1(1-s)\left\{\left\{\tilde{G}^4,F_3\right\},F_3\right\}\circ X^s_{F_3}\,ds.$$

5.2. Elimination of all terms of degree 4

In this subsection, we will eliminate some resonant terms in $\mathcal{G}^4$. We first give the following three lemmas which will be applied in the sequel and will be proved in Appendix A.

Lemma 5.3. *Let j,d be non-zero integers, such that $j\pm d=0$. For the parameter set $\hat{J}$, there is a set $\bar{J}_2\subset\hat{J}$ with*

$$\mathrm{meas}\,\bar{J}_2\geqslant\mathrm{meas}\,\hat{J}(1-\hat{C}\varrho) \tag{5.17}$$

such that, for any $\bar{\xi}\in\bar{J}_2$ and $\varrho>0$ small enough, if $\mu'_j+\mu'_d\neq0$, then

$$\left|\mu'_j+\mu'_d+\langle k,\tilde{\omega}(\bar{\xi})\rangle\right| \geqslant \frac{\varepsilon^{\frac{2}{3}}\varrho\,\mathrm{meas}\,\hat{J}}{(|k|+1)^{2m+6}}, \quad \forall k\in\mathbb{Z}^{m+1}, \tag{5.18}$$

where $\hat{C}$ is a constant depending on $\bar{\xi}$.

Lemma 5.4. *Let j, d be non-zero integers, such that $j \pm d = 0$. For the parameter set $\hat{J}$, there is a set $\bar{J}_3 \subset \hat{J}$ with*

$$\operatorname{meas} \bar{J}_3 \geqslant \operatorname{meas} \hat{J}(1 - \hat{C}\varrho) \tag{5.19}$$

such that, for any $\bar{\xi} \in \bar{J}_3$ and $\varrho > 0$ small enough,

$$\left|2\mu_0 \pm \left(\mu'_j + \mu'_d\right) \pm \langle k, \tilde{\omega}(\bar{\xi})\rangle\right| \geqslant \frac{\varepsilon^{\frac{2}{3}} \varrho \operatorname{meas} \hat{J}}{(|k|+1)^{2m+6}}, \quad \forall k \in \mathbb{Z}^{m+1}, \tag{5.20}$$

where $\hat{C}$ is a constant depending on $\bar{\xi}$.

Lemma 5.5. *Let i, j, d be non-zero integers, such that $i \pm j \pm d = 0$. For the parameter set $\hat{J}$, there is a set $\bar{J}_4 \subset \hat{J}$ with*

$$\operatorname{meas} \bar{J}_4 \geqslant \operatorname{meas} \hat{J}(1 - \hat{C}\varrho) \tag{5.21}$$

such that, for any $\bar{\xi} \in \bar{J}_4$ and $\varrho > 0$ small enough,

$$\left|\mu_0 \pm \left(\mu'_i + \mu'_j + \mu'_d\right) \pm \langle k, \tilde{\omega}(\bar{\xi})\rangle\right| \geqslant \frac{\varepsilon^{\frac{2}{3}} \varrho \operatorname{meas} \hat{J}}{C_*(|k|+1)^{2m+6}}, \quad \forall k \in \mathbb{Z}^{m+1}, \tag{5.22}$$

where $\hat{C}$ is a constant depending on $\bar{\xi}$ and C_* is a constant sufficiently large.

Let $\mathcal{L}_n = \{(i, j, d, l) \in \mathbb{Z}^4\colon\ 0 \neq \min(|i|, |j|, |d|, |l|) \leqslant n\}$, and $\mathcal{N}_n \subset \mathcal{L}_n$ be the subset of all $(i, j, d, l) \equiv (p, -p, q, -q)$. That is, they are of the form $(p, -p, q, -q)$ or some permutation of it.

Lemma 5.6. *If i, j, d, l are non-zero integers, such that $i \pm j \pm d \pm l = 0$, $(i, j, d, l) \in \mathcal{L}_n \setminus \mathcal{N}_n$ or $(i, j, d, l) \in \mathcal{N}_n$ and $k \neq 0$. Then, for the parameter set $\hat{J}$, there is a subset $\bar{J}_5 \subset \hat{J}$ with*

$$\operatorname{meas} \bar{J}_5 \geqslant \operatorname{meas} \hat{J} \cdot (1 - \hat{C}\varrho) \tag{5.23}$$

such that, for any $\bar{\xi} \in \bar{J}_5$ and $\varrho > 0$ small enough,

$$\left|\mu'_i + \mu'_j + \mu'_d + \mu'_l + \langle k, \tilde{\omega}(\bar{\xi})\rangle\right| \geqslant \frac{\varepsilon^{\frac{2}{3}} \varrho \operatorname{meas} \hat{J}}{C_*(|k|+1)^{2m+6}}, \quad \forall k \in \mathbb{Z}^{m+1} \tag{5.24}$$

where $\hat{C}$ is a constant depending on $\bar{\xi}$ and C_* is a constant sufficiently large.

Let $\underline{J} = \bar{J}_2 \cap \bar{J}_3 \cap \bar{J}_4 \cap \bar{J}_5$ and

$$\mathscr{J} = \bar{J} \cap \underline{J},$$

it is obvious that

$$\operatorname{meas} \mathscr{J} \geqslant \operatorname{meas} \hat{J} \cdot (1 - \hat{C}\varrho).$$

By the above three lemmas, we can prove the following proposition.

Proposition 5.7. *For each finite $n \geqslant 1$, there exists a real analytic, symplectic change of coordinates $X^1_{F_4}$ in some neighborhood of the origin on the complex Hilbert space $l^{a,s}$ such that the Hamiltonian* (5.16) *is changed into*

$$H \circ X^1_{F_4} = \hat{H} + \bar{c}\varepsilon^2 z_0^2 \bar{z}_0^2 + \varepsilon^2 z_0 \bar{z}_0 \sum_{1 \leqslant j \leqslant n} c_j z_j \bar{z}_j + \varepsilon^2 z_0 \bar{z}_0 \sum_{j>n} c_j z_j \bar{z}_j$$
$$+ \varepsilon^2 \tilde{\mathcal{G}} + \varepsilon^2 \hat{\mathcal{G}} + \varepsilon^3 K,$$

where

$$K = R_{11} + \varepsilon R_{22},$$
$$\bar{c} = \frac{3[\phi]}{16[\hat{V}]\pi} \varepsilon^{-\frac{1}{3}} \big(1 + \mathcal{O}\big(\varepsilon^{\frac{2}{3}}\big)\big) + \mathcal{O}\big(\varepsilon^{-\frac{1}{2}}\big),$$
$$c_j = \frac{3[\phi]}{8\sqrt{\lambda_0 \lambda_j}} \big(1 + \mathcal{O}\big(\varepsilon^{\frac{2}{3}}\big)\big) + \mathcal{O}\big(\varepsilon^{-\frac{1}{3}}\big),$$

and

$$\tilde{\mathcal{G}} = \frac{1}{2} \sum_{1 \leqslant \min(i,j) \leqslant n} \tilde{\mathcal{G}}_{ij} |z_i|^2 |z_j|^2$$

with uniquely determined coefficients

$$\tilde{\mathcal{G}}_{ij} = \begin{cases} 24[\mathcal{G}^4_{iijj}] = \frac{Ba_0}{\pi\sqrt{\lambda_i \lambda_j}} + \varpi_{ij}(\bar{\xi}, \varepsilon), & i \neq j, \\ 12[\mathcal{G}^4_{iiii}] = \frac{Bb_0}{\pi \lambda_i} + \varpi_{ij}(\bar{\xi}, \varepsilon), & i = j, \end{cases} \tag{5.25}$$

where

$$a_0 = -\frac{15}{4} + o(1), \qquad b_0 = -\frac{27}{16} + o(1)$$

with $\lim_{\varepsilon \to 0} o(1) = 0$, and $\varpi_{ij}(\bar{\xi}, \varepsilon)$ depends smoothly on $\bar{\xi}$ and ε and there is an absolute constant C such that $\|\varpi_{ij}(\bar{\xi}, \varepsilon)\|^*_{\mathscr{J}} \leqslant C\varepsilon^{2/3}$ for ε small enough, while $\hat{\mathcal{G}}$ is independent on the coordinates in $\{z_0, z_1, \ldots, z_n\}$, and we have

$$|\hat{\mathcal{G}}| = O\big(\|\hat{z}\|^4_{a,s}\big), \qquad |K| = O\big(\|z\|^5_{a,s}\big),$$

uniformly for $|\mathrm{Im}\,\theta| < \sigma/3$, $\bar{\xi} \in \mathscr{J}$, $\hat{z} = (z_{n+1}, z_{n+2}, \ldots)$.

Proof. Let

$$\mathcal{G}^4 = \chi_{40}(\theta, \bar{\xi}, \varepsilon) w_0^4 + \chi_{31}(\theta, \bar{\xi}, \varepsilon) w_0^3 w_{-0} + \chi_{22}(\theta, \bar{\xi}, \varepsilon) w_0^2 w_{-0}^2$$
$$+ \chi_{13}(\theta, \bar{\xi}, \varepsilon) w_0 w_{-0}^3 + \chi_{04}(\theta, \bar{\xi}, \varepsilon) w_{-0}^4$$
$$+ w_0^2 \sum_{\substack{j \pm d = 0 \\ j,d \neq 0}} \bar{\mathcal{G}}^4_{20jd}(\theta, \bar{\xi}, \varepsilon) w_j w_d + w_0 w_{-0} \sum_{\substack{j \pm d = 0 \\ j,d \neq 0}} \bar{\mathcal{G}}^4_{22jd}(\theta, \bar{\xi}, \varepsilon) w_j w_d$$

$$+ w_{-0}^2 \sum_{\substack{j\pm d=0\\ j,d\neq 0}} \bar{\mathcal{G}}^4_{02jd}(\theta,\bar{\xi},\varepsilon) w_j w_d + w_0 \sum_{\substack{j\pm d\pm l=0\\ j,d,l\neq 0}} \hat{\mathcal{G}}^4_{1jdl}(\theta,\bar{\xi},\varepsilon) w_j w_d w_l$$

$$+ w_{-0} \sum_{\substack{j\pm d\pm l=0\\ j,d,l\neq 0}} \hat{\mathcal{G}}^4_{2jdl}(\theta,\bar{\xi},\varepsilon) w_j w_d w_l$$

$$+ \sum_{\substack{i\pm j\pm d\pm l=0\\ i,j,d,l\neq 0}} \mathcal{G}^4_{ijdl}(\theta,\bar{\xi},\varepsilon) w_i w_j w_d w_l \tag{5.26}$$

with

$$\chi_{ij}(\theta,\bar{\xi},\varepsilon) = \sum_{k\in\mathbb{Z}^{m+1}} \chi_{ij,k}(\bar{\xi},\varepsilon) e^{\mathrm{i}\langle k,\theta\rangle},$$

$$\bar{\mathcal{G}}^4_{stjd}(\theta,\bar{\xi},\varepsilon) = \sum_{k\in\mathbb{Z}^{m+1}} \bar{\mathcal{G}}^4_{stjd,k}(\bar{\xi},\varepsilon) e^{\mathrm{i}\langle k,\theta\rangle},$$

$$\hat{\mathcal{G}}^4_{sjdl}(\theta,\bar{\xi},\varepsilon) = \sum_{k\in\mathbb{Z}^{m+1}} \hat{\mathcal{G}}^4_{sjdl,k}(\bar{\xi},\varepsilon) e^{\mathrm{i}\langle k,\theta\rangle},$$

and

$$\mathcal{G}^4_{ijdl}(\theta,\bar{\xi},\varepsilon) = \sum_{k\in\mathbb{Z}^{m+1}} \mathcal{G}^4_{ijdl,k}(\bar{\xi},\varepsilon) e^{\mathrm{i}\langle k,\theta\rangle}.$$

Consider a Hamilton function

$$F_4 = \delta_{40}(\theta,\bar{\xi},\varepsilon) w_0^4 + \delta_{31}(\theta,\bar{\xi},\varepsilon) w_0^3 w_{-0} + \delta_{22}(\theta,\bar{\xi},\varepsilon) w_0^2 w_{-0}^2$$

$$+ \delta_{13}(\theta,\bar{\xi},\varepsilon) w_0 w_{-0}^3 + \delta_{04}(\theta,\bar{\xi},\varepsilon) w_{-0}^4$$

$$+ w_0^2 \sum_{\substack{j\pm d=0\\ j,d\neq 0}} \bar{f}^4_{20jd}(\theta,\bar{\xi},\varepsilon) w_j w_d + w_0 w_{-0} \sum_{\substack{j\pm d=0\\ j,d\neq 0}} \bar{f}^4_{22jd}(\theta,\bar{\xi},\varepsilon) w_j w_d$$

$$+ w_{-0}^2 \sum_{\substack{j\pm d=0\\ j,d\neq 0}} \bar{f}^4_{02jd}(\theta,\bar{\xi},\varepsilon) w_j w_d + w_0 \sum_{\substack{j\pm d\pm l=0\\ j,d,l\neq 0}} \hat{f}^4_{1jdl}(\theta,\bar{\xi},\varepsilon) w_j w_d w_l$$

$$+ w_{-0} \sum_{\substack{j\pm d\pm l=0\\ j,d,l\neq 0}} \hat{f}^4_{2jdl}(\theta,\bar{\xi},\varepsilon) w_j w_d w_l$$

$$+ \sum_{\substack{i\pm j\pm d\pm l=0\\ i,j,d,l\neq 0}} f^4_{ijdl}(\theta,\bar{\xi},\varepsilon) w_i w_j w_d w_l \tag{5.27}$$

with

$$\delta_{ij}(\theta,\bar{\xi},\varepsilon) = \sum_{k\in\mathbb{Z}^{m+1}} \delta_{ij,k}(\bar{\xi},\varepsilon) e^{\mathrm{i}\langle k,\theta\rangle},$$

$$\bar{f}^4_{stjd}(\theta,\bar{\xi},\varepsilon)=\sum_{k\in\mathbb{Z}^{m+1}}\bar{f}^4_{stjd,k}(\bar{\xi},\varepsilon)e^{\mathrm{i}\langle k,\theta\rangle},$$

$$\hat{f}^4_{sjdl}(\theta,\bar{\xi},\varepsilon)=\sum_{k\in\mathbb{Z}^{m+1}}\hat{f}^4_{sjdl,k}(\bar{\xi},\varepsilon)e^{\mathrm{i}\langle k,\theta\rangle},$$

and

$$f^4_{ijdl}(\theta,\bar{\xi},\varepsilon)=\sum_{k\in\mathbb{Z}^{m+1}}f^4_{ijdl,k}(\bar{\xi},\varepsilon)e^{\mathrm{i}\langle k,\theta\rangle},$$

where

$$\delta_{40,k}(\bar{\xi},\varepsilon)=\frac{\chi_{40,k}(\bar{\xi},\varepsilon)}{\mathrm{i}(4\mu_0+\langle k,\tilde{\omega}(\bar{\xi})\rangle)},\quad k\in\mathbb{Z}^{m+1},$$

$$\delta_{31,k}(\bar{\xi},\varepsilon)=\frac{\chi_{31,k}(\bar{\xi},\varepsilon)}{\mathrm{i}(2\mu_0+\langle k,\tilde{\omega}(\bar{\xi})\rangle)},\quad k\in\mathbb{Z}^{m+1},$$

$$\delta_{22,k}(\bar{\xi},\varepsilon)=\begin{cases}\frac{\chi_{22,k}(\bar{\xi},\varepsilon)}{\mathrm{i}\langle k,\tilde{\omega}(\bar{\xi})\rangle}, & k\neq 0,\\ \text{arbitrary constant}, & k=0,\end{cases}$$

$$\delta_{13,k}(\bar{\xi},\varepsilon)=\frac{\chi_{13,k}(\bar{\xi},\varepsilon)}{-\mathrm{i}(2\mu_0-\langle k,\tilde{\omega}(\bar{\xi})\rangle)},\quad k\in\mathbb{Z}^{m+1},$$

$$\delta_{04,k}(\bar{\xi},\varepsilon)=\frac{\chi_{04,k}(\bar{\xi},\varepsilon)}{-\mathrm{i}(4\mu_0-\langle k,\tilde{\omega}(\bar{\xi})\rangle)},\quad k\in\mathbb{Z}^{m+1},$$

$$\bar{f}^4_{20jd,k}(\bar{\xi},\varepsilon)=\frac{\bar{\mathcal{G}}^4_{20jd,k}(\bar{\xi},\varepsilon)}{\mathrm{i}(2\mu_0+\mu'_j+\mu'_d+\langle k,\tilde{\omega}(\bar{\xi})\rangle)},\quad k\in\mathbb{Z}^{m+1},$$

$$\bar{f}^4_{22jd,k}(\bar{\xi},\varepsilon)=\begin{cases}\frac{\bar{\mathcal{G}}^4_{22jd,k}(\bar{\xi},\varepsilon)}{\mathrm{i}(\mu'_j+\mu'_d+\langle k,\tilde{\omega}(\bar{\xi})\rangle)}, & k\neq 0 \text{ or } k=0,\ \mu'_j+\mu'_d\neq 0,\\ \text{arbitrary constant}, & k=0,\ \mu'_j+\mu'_d=0,\end{cases}$$

$$\bar{f}^4_{02jd,k}(\bar{\xi},\varepsilon)=\frac{\bar{\mathcal{G}}^4_{02jd,k}(\bar{\xi},\varepsilon)}{-\mathrm{i}(2\mu_0-\mu'_j-\mu'_d-\langle k,\tilde{\omega}(\bar{\xi})\rangle)},\quad k\in\mathbb{Z}^{m+1},$$

$$\hat{f}^4_{1jdl,k}(\bar{\xi},\varepsilon)=\frac{\hat{\mathcal{G}}^4_{1jdl,k}(\bar{\xi},\varepsilon)}{\mathrm{i}(\mu_0+\mu'_j+\mu'_d+\mu'_l+\langle k,\tilde{\omega}(\bar{\xi})\rangle)},\quad k\in\mathbb{Z}^{m+1},$$

$$\hat{f}^4_{2jdl,k}(\bar{\xi},\varepsilon)=\frac{\hat{\mathcal{G}}^4_{2jdl,k}(\bar{\xi},\varepsilon)}{-\mathrm{i}(\mu_0-\mu'_j-\mu'_d-\mu'_l-\langle k,\tilde{\omega}(\bar{\xi})\rangle)},\quad k\in\mathbb{Z}^{m+1}$$

and

$$
\mathrm{i}\, f^4_{ijdl,k}(\bar{\xi},\varepsilon) = \begin{cases} \frac{\mathcal{G}^4_{ijdl,k}(\bar{\xi},\varepsilon)}{\mu'_i+\mu'_j+\mu'_d+\mu'_l+\langle k,\tilde{\omega}(\bar{\xi})\rangle}, & \text{for } (i,j,d,l)\in\mathcal{L}_n\setminus\mathcal{N}_n \\ & \text{or } (i,j,d,l)\in\mathcal{N}_n \text{ and } k\neq 0, \\ 0, & \text{for } (i,j,d,l)\notin\mathcal{L}_n \text{ or } (i,j,d,l)\in\mathcal{N}_n,\ k=0 \\ & \text{and } \mu'_i+\mu'_j+\mu'_d+\mu'_l\neq 0, \\ \text{arbitrary constant}, & \text{for } (i,j,d,l)\in\mathcal{N}_n,\ k=0 \text{ and } \mu'_i+\mu'_j+\mu'_d+\mu'_l=0. \end{cases} \tag{5.28}
$$

By $X^1_{F_4}$ denotes the time-1 map of the vector field of the Hamiltonian $\varepsilon^2 F_4$. Then

$$
H\circ X^1_{F_4} = \hat{H} + \varepsilon^2\big(\mathcal{G}^4 + \{\hat{H},F_4\}\big) + \varepsilon^3 R_{11} + \varepsilon^4 R_{22},
$$

where R_{11} is defined in (5.16) and

$$
R_{22} = \int_0^1 \big\{\varepsilon R_{11} + \mathcal{G}^4, F_4\big\}\circ X^s_{F_4}\,ds + R_{12}\circ X^s_{F_4} + \int_0^1 (1-s)\big\{\{\hat{H},F_4\},F_4\big\}\circ X^s_{F_4}\,ds, \tag{5.29}
$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket with respect to the symplectic structure $\mathrm{i}\,dz\wedge d\bar{z} + d\vartheta\wedge dJ$. Now let us compute $\mathcal{G}^4+\{\hat{H},F_4\}$.

$$
\begin{aligned}
\mathcal{G}^4+\{\hat{H},F_4\} = {} & \left(\chi_{40}(\theta,\bar{\xi},\varepsilon) - 4\mathrm{i}\,\mu_0\delta_{40}(\theta,\bar{\xi},\varepsilon) - \left\langle\tilde{\omega}(\bar{\xi}), \frac{\partial}{\partial\theta}\delta_{40}(\theta,\bar{\xi},\varepsilon)\right\rangle\right) w_0^4 \\
& + \left(\chi_{31}(\theta,\bar{\xi},\varepsilon) - 2\mathrm{i}\,\mu_0\delta_{31}(\theta,\bar{\xi},\varepsilon) - \left\langle\tilde{\omega}(\bar{\xi}), \frac{\partial}{\partial\theta}\delta_{31}(\theta,\bar{\xi},\varepsilon)\right\rangle\right) w_0^3 w_{-0} \\
& + \left(\chi_{22}(\theta,\bar{\xi},\varepsilon) - \left\langle\tilde{\omega}(\bar{\xi}), \frac{\partial}{\partial\theta}\delta_{22}(\theta,\bar{\xi},\varepsilon)\right\rangle\right) w_0^2 w_{-0}^2 \\
& + \left(\chi_{13}(\theta,\bar{\xi},\varepsilon) + 2\mathrm{i}\,\mu_0\delta_{13}(\theta,\bar{\xi},\varepsilon) - \left\langle\tilde{\omega}(\bar{\xi}), \frac{\partial}{\partial\theta}\delta_{13}(\theta,\bar{\xi},\varepsilon)\right\rangle\right) w_0 w_{-0}^3 \\
& + \left(\chi_{04}(\theta,\bar{\xi},\varepsilon) + 4\mathrm{i}\,\mu_0\delta_{04}(\theta,\bar{\xi},\varepsilon) - \left\langle\tilde{\omega}(\bar{\xi}), \frac{\partial}{\partial\theta}\delta_{04}(\theta,\bar{\xi},\varepsilon)\right\rangle\right) w_{-0}^4 \\
& + w_0^2 \sum_{\substack{j\pm d=0\\ j,d\neq 0}} \Bigg(\bar{\mathcal{G}}^4_{20jd}(\theta,\bar{\xi},\varepsilon) - \mathrm{i}\big(2\mu_0+\mu'_j+\mu'_d\big)\bar{f}^4_{20jd}(\theta,\bar{\xi},\varepsilon) \\
& - \left\langle\tilde{\omega}(\bar{\xi}), \frac{\partial}{\partial\theta}\bar{f}^4_{20jd}(\theta,\bar{\xi},\varepsilon)\right\rangle\Bigg) w_j w_d \\
& + w_0 w_{-0} \sum_{\substack{j\pm d=0\\ j,d\neq 0}} \Bigg(\bar{\mathcal{G}}^4_{22jd}(\theta,\bar{\xi},\varepsilon) - \mathrm{i}\,\big(\mu'_j+\mu'_d\big)\bar{f}^4_{22jd}(\theta,\bar{\xi},\varepsilon) \\
& - \left\langle\tilde{\omega}(\bar{\xi}), \frac{\partial}{\partial\theta}\bar{f}^4_{22jd}(\theta,\bar{\xi},\varepsilon)\right\rangle\Bigg) w_j w_d \\
& + w_{-0}^2 \sum_{\substack{j\pm d=0\\ j,d\neq 0}} \Bigg(\bar{\mathcal{G}}^4_{02jd}(\theta,\bar{\xi},\varepsilon) + \mathrm{i}\big(2\mu_0-\mu'_j-\mu'_d\big)\bar{f}^4_{02jd}(\theta,\bar{\xi},\varepsilon)
\end{aligned}
$$

$$
\begin{aligned}
&-\Big\langle\tilde\omega(\bar\xi),\frac{\partial}{\partial\theta}\bar f^4_{02jd}(\theta,\bar\xi,\varepsilon)\Big\rangle\Big)w_jw_d\\
&+w_0\sum_{\substack{j\pm d\pm l=0\\ j,d,l\neq 0}}\Big(\hat{\mathcal G}^4_{1jdl}(\theta,\bar\xi,\varepsilon)-\mathrm i\big(\mu_0+\mu'_j+\mu'_d+\mu'_l\big)\hat f^4_{1jdl}(\theta,\bar\xi,\varepsilon)\\
&-\Big\langle\tilde\omega(\bar\xi),\frac{\partial}{\partial\theta}\hat f^4_{1jdl}(\theta,\bar\xi,\varepsilon)\Big\rangle\Big)w_jw_dw_l\\
&+w_{-0}\sum_{\substack{j\pm d\pm l=0\\ j,d,l\neq 0}}\Big(\hat{\mathcal G}^4_{2jdl}(\theta,\bar\xi,\varepsilon)+\mathrm i\big(\mu_0-\mu'_j-\mu'_d-\mu'_l\big)\hat f^4_{2jdl}(\theta,\bar\xi,\varepsilon)\\
&-\Big\langle\tilde\omega(\bar\xi),\frac{\partial}{\partial\theta}\hat f^4_{2jdl}(\theta,\bar\xi,\varepsilon)\Big\rangle\Big)w_jw_dw_l\\
&+\sum_{\substack{\sum_{i\pm j\pm d\pm l=0}\\ i,j,d,l\neq 0}}\Big\{\sum_{k\in\mathbb Z^{m+1}}\big[\mathcal G^4_{ijdl,k}(\bar\xi,\varepsilon)-\mathrm i\big(\mu'_i+\mu'_j+\mu'_d+\mu'_l\\
&+\langle k,\tilde\omega(\bar\xi)\rangle\big)f^4_{ijdl,k}(\bar\xi,\varepsilon)\big]e^{\mathrm i\langle k,\theta\rangle}\Big\}w_iw_jw_dw_l\\
={}&\big[\chi_{22}(\theta,\bar\xi,\varepsilon)\big]w_0^2w_{-0}^2+w_0w_{-0}\sum\nolimits_{j\neq 0}\bar{\mathcal G}^4_{22jj,0}(\bar\xi,\varepsilon)w_jw_{-j}\\
&+\sum_{(i,j,d,l)\in\mathcal N_n}\big[\mathcal G^4_{ijdl,0}(\bar\xi,\varepsilon)-\mathrm i\big(\mu'_i+\mu'_j+\mu'_d+\mu'_l\big)f^4_{ijdl,0}(\bar\xi,\varepsilon)\big]w_iw_jw_dw_l\\
&+\sum_{(i,j,d,l)\in\mathcal N_n}\Big\{\sum_{0\neq k\in\mathbb Z^{m+1}}\big[\mathcal G^4_{ijdl,k}(\bar\xi,\varepsilon)-\mathrm i\big(\mu'_i+\mu'_j+\mu'_d+\mu'_l\\
&+\langle k,\tilde\omega(\bar\xi)\rangle\big)f^4_{ijdl,k}(\bar\xi,\varepsilon)\big]e^{\mathrm i\langle k,\theta\rangle}\Big\}w_iw_jw_dw_l\\
&+\sum_{(i,j,d,l)\in\mathcal L_n\setminus\mathcal N_n}\Big\{\sum_{k\in\mathbb Z^{m+1}}\big[\mathcal G^4_{ijdl,k}(\bar\xi,\varepsilon)-\mathrm i\big(\mu'_i+\mu'_j+\mu'_d+\mu'_l\\
&+\langle k,\tilde\omega(\bar\xi)\rangle\big)f^4_{ijdl,k}(\bar\xi,\varepsilon)\big]e^{\mathrm i\langle k,\theta\rangle}\Big\}w_iw_jw_dw_l\\
&+\sum_{(i,j,d,l)\notin\mathcal L_n}\Big\{\sum_{k\in\mathbb Z^{m+1}}\big[\mathcal G^4_{ijdl,k}(\bar\xi,\varepsilon)-\mathrm i\big(\mu'_i+\mu'_j+\mu'_d+\mu'_l\\
&+\langle k,\tilde\omega(\bar\xi)\rangle\big)f^4_{ijdl,k}(\xi,\varepsilon)\big]e^{\mathrm i\langle k,\theta\rangle}\Big\}w_iw_jw_dw_l\\
={}&\big[\chi_{22}(\theta,\bar\xi,\varepsilon)\big]w_0^2w_{-0}^2+w_0w_{-0}\sum\nolimits_{j\neq 0}\bar{\mathcal G}^4_{22jj,0}(\bar\xi,\varepsilon)w_jw_{-j}\\
&+\sum_{(i,j,d,l)\in\mathcal N_n}\big[\mathcal G^4_{ijdl}\big]w_iw_jw_dw_l+\sum_{(i,j,d,l)\notin\mathcal L_n}\mathcal G^4_{ijdl}w_iw_jw_dw_l\\
={}&\big[\chi_{22}(\theta,\bar\xi,\varepsilon)\big]w_0^2w_{-0}^2+w_0w_{-0}\sum_{j\neq 0}\bar{\mathcal G}^4_{22jj,0}(\bar\xi,\varepsilon)w_jw_{-j}
\end{aligned}
$$

$$+\frac{1}{2}\sum_{1\leqslant\min(i,j)\leqslant n}\tilde{\mathcal{G}}_{ij}w_iw_{-i}w_jw_{-j}+\sum_{(i,j,d,l)\notin\mathcal{L}_n}\mathcal{G}^4_{ijdl}w_iw_jw_dw_l$$

$$=\big[\chi_{22}(\theta,\bar{\xi},\varepsilon)\big]w_0^2w_{-0}^2+w_0w_{-0}\sum_{j\neq 0}\overline{\mathcal{G}}^4_{22jj,0}(\bar{\xi},\varepsilon)w_jw_{-j}+\tilde{\mathcal{G}}+\hat{\mathcal{G}}.$$

Re-introducing the notations $z_j,\bar{z}_j$ and counting multiplicities we have

$$\tilde{\mathcal{G}}=\frac{1}{2}\sum_{1\leqslant\min(i,j)\leqslant n}\tilde{\mathcal{G}}_{ij}|z_i|^2|z_j|^2$$

with uniquely determined coefficients

$$\tilde{\mathcal{G}}_{ij}=\begin{cases}24[\mathcal{G}^4_{iijj}]=\frac{Ba_0}{\pi\sqrt{\lambda_i\lambda_j}}+\varpi_{ij}(\bar{\xi},\varepsilon), & i\neq j,\\ 12[\mathcal{G}^4_{iiii}]=\frac{Bb_0}{\pi\lambda_i}+\varpi_{ij}(\bar{\xi},\varepsilon), & i=j,\end{cases}\tag{5.30}$$

where

$$a_0=-\frac{15}{4}+o(1),\qquad b_0=-\frac{27}{16}+o(1)$$

with $\lim_{\varepsilon\to 0}o(1)=0$, and $\varpi_{ij}(\bar{\xi},\varepsilon)$ depends smoothly on $\bar{\xi}$ and ε and there is an absolute constant C such that $\|\varpi_{ij}(\bar{\xi},\varepsilon)\|^*_{\mathscr{J}}\leqslant C\varepsilon^{2/3}$ for ε small enough, while $\hat{\mathcal{G}}$ is independent on the first $n+1$ coordinates. By a direct calculation we get

$$\begin{aligned}\big[\chi_{22}(\theta,\bar{\xi},\varepsilon)\big]&=\frac{1}{(2\pi)^{m+1}}\int_{\mathbb{T}^{m+1}}\bigg[\frac{3}{2}G^4_{0000}(\theta,\bar{\xi},\varepsilon)S^2_{11}(\theta,\bar{\xi},\varepsilon)S^2_{12}(\theta,\bar{\xi},\varepsilon)\\&\quad+\frac{9\sqrt{-1}}{4\sqrt{2}}G^3_{000}(\theta,\bar{\xi},\varepsilon)\big(S^2_{11}(\theta,\bar{\xi},\varepsilon)S_{12}(\theta,\bar{\xi},\varepsilon)\zeta_{12}(\theta,\bar{\xi},\varepsilon)\\&\quad+S^3_{11}(\theta,\bar{\xi},\varepsilon)\zeta_{03}(\theta,\bar{\xi},\varepsilon)\\&\quad-S_{11}(\theta,\bar{\xi},\varepsilon)S^2_{12}(\theta,\bar{\xi},\varepsilon)\zeta_{21}(\theta,\bar{\xi},\varepsilon)-S^3_{12}(\theta,\bar{\xi},\varepsilon)\zeta_{30}(\theta,\bar{\xi},\varepsilon)\big)\bigg]d\theta\\&=\frac{3[\phi]}{16[\hat{V}]\pi}\varepsilon^{-\frac{1}{3}}\big(1+\mathcal{O}\big(\varepsilon^{\frac{2}{3}}\big)\big)+\mathcal{O}\big(\varepsilon^{-\frac{1}{2}}\big)=\mathcal{O}\big(\varepsilon^{-\frac{1}{2}}\big):=\bar{c}\end{aligned}$$

and

$$\begin{aligned}\big[\overline{\mathcal{G}}^4_{22jj}(\theta,\bar{\xi},\varepsilon)\big]&=\frac{1}{(2\pi)^{m+1}}\int_{\mathbb{T}^{m+1}}\bigg[3\tilde{G}^4_{00jj}(\theta,\bar{\xi},\varepsilon)S_{11}(\theta,\bar{\xi},\varepsilon)S_{12}(\theta,\bar{\xi},\varepsilon)\\&\quad+\frac{3\sqrt{-1}}{2\sqrt{2}}\big(S_{11}(\theta,\bar{\xi},\varepsilon)\zeta_{12}(\theta,\bar{\xi},\varepsilon)\tilde{G}^3_{0,j,j}(\theta,\bar{\xi},\varepsilon)\\&\quad+G^3_{000}(\theta,\bar{\xi},\varepsilon)S^2_{11}(\theta,\bar{\xi},\varepsilon)S_{12}(\theta,\bar{\xi},\varepsilon)f^3_{-0,j,j}(\theta,\bar{\xi},\varepsilon)\\&\quad-S_{12}(\theta,\bar{\xi},\varepsilon)\zeta_{21}(\theta,\bar{\xi},\varepsilon)\tilde{G}^3_{0,j,j}(\theta,\bar{\xi},\varepsilon)\end{aligned}$$

$$- G^3_{000}(\theta,\bar{\xi},\varepsilon)S_{11}(\theta,\bar{\xi},\varepsilon)S^2_{12}(\theta,\bar{\xi},\varepsilon)f^3_{0,j,j}(\theta,\bar{\xi},\varepsilon)\big) + \mathcal{O}\big(\varepsilon^{-\frac{1}{3}}\big)\Big]\,d\theta$$

$$= \frac{3[\phi]}{8\sqrt{\lambda_0\lambda_j}}\big(1+\mathcal{O}\big(\varepsilon^{\frac{2}{3}}\big)\big) + \mathcal{O}\big(\varepsilon^{-\frac{1}{3}}\big) = \mathcal{O}\big(\varepsilon^{-\frac{1}{3}}\big) := c_j, \quad j \neq 0.$$

Hence, we have

$$H \circ X^1_{F_4} = \hat{H} + \bar{c}\varepsilon^2 z_0^2\bar{z}_0^2 + \varepsilon^2 z_0\bar{z}_0 \sum_{1\leqslant j\leqslant n} c_j z_j\bar{z}_j + \varepsilon^2 z_0\bar{z}_0\sum_{j>n} c_j z_j\bar{z}_j$$
$$+ \varepsilon^2\tilde{\mathcal{G}} + \varepsilon^2\hat{\mathcal{G}} + \varepsilon^3 K,$$

where $K = R_{11} + \varepsilon R_{22}$. $\square$

We introduce the action-angle variable by setting

$$z_j = \begin{cases} \sqrt{I_j}e^{-\mathrm{i}\hat{\theta}_j}, & 0\leqslant j\leqslant n, \\ z_j, & j\geqslant n+1. \end{cases} \tag{5.31}$$

By the symplectic change (5.31), the normal form becomes

$$\hat{H} + \bar{c}\varepsilon^2 z_0^2\bar{z}_0^2 + \varepsilon^2 z_0\bar{z}_0 \sum_{1\leqslant j\leqslant n} c_j z_j\bar{z}_j + \varepsilon^2 z_0\bar{z}_0\sum_{j>n} c_j z_j\bar{z}_j + \varepsilon^2\tilde{\mathcal{G}}$$
$$= \big\langle\tilde{\omega}(\bar{\xi}), J\big\rangle + \mu_0 I_0 + \bar{c}\varepsilon^2 I_0^2 + \sum_{1\leqslant j\leqslant n}\big(\mu_j + \varepsilon^2 I_0 c_j\big)I_j + \sum_{j>n}\big(\mu_j + \varepsilon^2 I_0 c_j\big)z_j\bar{z}_j$$
$$+ \frac{\varepsilon^2}{2}\langle AI, I\rangle + \varepsilon^2\langle BI, \hat{Z}\rangle,$$

with $I = (I_1,\ldots,I_n)$, $A = (\tilde{\mathcal{G}}_{ij})_{1\leqslant i,j\leqslant n}$, $B = (\tilde{\mathcal{G}}_{ij})_{1\leqslant j\leqslant n<i}$ and $\hat{Z} = (|z_{n+1}|^2, |z_{n+2}|^2,\ldots)$.

Now let us introduce the parameter vector $\tilde{\xi} = (\tilde{\xi}_j)_{0\leqslant j\leqslant n}$ and the new action variable and $\tilde{\rho} = (\tilde{\rho}_j)_{0\leqslant j\leqslant n}$ as follows

$$I_j = \varepsilon\tilde{\xi}_j + \tilde{\rho}_j, \qquad \tilde{\xi}_j\in[0,1], \qquad |\tilde{\rho}_j| < \varepsilon^2, \quad 0\leqslant j\leqslant n.$$

Clearly, $d\hat{\theta}_j \wedge dI_j = d\hat{\theta}_j\wedge d\tilde{\rho}_j$. So the transformation is symplectic. Then the normal form is changed into

$$\hat{H} + \bar{c}\varepsilon^2 z_0^2\bar{z}_0^2 + \varepsilon^2 z_0\bar{z}_0 \sum\nolimits_{1\leqslant j\leqslant n} c_j z_j\bar{z}_j + \varepsilon^2 z_0\bar{z}_0\sum_{j>n} c_j z_j\bar{z}_j + \varepsilon^2\tilde{\mathcal{G}}$$
$$= \big\langle\tilde{\omega}(\bar{\xi}), J\big\rangle + \Big(\mu_0 + 2\bar{c}\varepsilon^3\tilde{\xi}_0 + \varepsilon^3\sum_{1\leqslant j\leqslant n} c_j\tilde{\xi}_j\Big)\tilde{\rho}_0 + \sum_{1\leqslant j\leqslant n}\big(\mu_j + \varepsilon^3\tilde{\xi}_0 c_j\big)\tilde{\rho}_j + \bar{c}\varepsilon^2\tilde{\rho}_0^2$$
$$+ \tilde{\rho}_0\varepsilon^2\sum_{1\leqslant j\leqslant n} c_j\tilde{\rho}_j + \sum_{j>n}\big(\mu_j + \varepsilon^2(\varepsilon\tilde{\xi}_0 + \tilde{\rho}_0)c_j\big)z_j\bar{z}_j + \frac{\varepsilon^3}{2}\sum_{1\leqslant i,j\leqslant n}\tilde{\mathcal{G}}_{ij}\tilde{\rho}_i\tilde{\xi}_j + \frac{\varepsilon^3}{2}\sum_{1\leqslant i,j\leqslant n}\tilde{\mathcal{G}}_{ij}\tilde{\xi}_i\tilde{\rho}_j$$
$$+ \frac{\varepsilon^2}{2}\sum_{1\leqslant i,j\leqslant n}\tilde{\mathcal{G}}_{ij}\tilde{\rho}_i\tilde{\rho}_j + \varepsilon^3\sum_{1\leqslant i\leqslant n<j}\tilde{\mathcal{G}}_{ij}\tilde{\xi}_i|z_j|^2 + \varepsilon^2\sum_{1\leqslant i\leqslant n<j}\tilde{\mathcal{G}}_{ij}\tilde{\rho}_i|z_j|^2.$$

Hence, the total Hamiltonian is

$$\begin{aligned} H = {} & \langle \tilde{\omega}(\bar{\xi}), J \rangle + \check{\omega}_0 \tilde{\rho}_0 + \sum_{1 \leqslant j \leqslant n} \check{\omega}_j \tilde{\rho}_j + \sum_{j > n} \check{\lambda}_j z_j \bar{z}_j \\ & + \frac{\varepsilon^3}{2} \sum_{1 \leqslant i, j \leqslant n} \tilde{\mathcal{G}}_{ij} \tilde{\rho}_i \tilde{\xi}_j + \frac{\varepsilon^3}{2} \sum_{1 \leqslant i, j \leqslant n} \tilde{\mathcal{G}}_{ij} \tilde{\xi}_i \tilde{\rho}_j \\ & + \varepsilon^3 \sum_{1 \leqslant i \leqslant n < j} \tilde{\mathcal{G}}_{ij} \tilde{\xi}_i |z_j|^2 + P, \end{aligned} \tag{5.32}$$

where

$$\begin{gathered} \check{\omega}_0 = \mu_0 + 2\bar{c}\varepsilon^3 \tilde{\xi}_0 + \varepsilon^3 \sum_{1 \leqslant j \leqslant n} c_j \tilde{\xi}_j, \\ \check{\omega}_j = \mu_j + \varepsilon^3 \tilde{\xi}_0 c_j, \quad j = 1, 2, \ldots, n, \\ \check{\lambda}_j = \mu_j + \varepsilon^3 \tilde{\xi}_0 c_j, \quad j > n, \\ P = \varepsilon^2 \check{\mathcal{G}} + \varepsilon^2 \hat{\mathcal{G}} + \varepsilon^3 K \end{gathered}$$

with $\check{\mathcal{G}} = \mathcal{O}(|\tilde{\rho}|^2) + \mathcal{O}(|\tilde{\rho}| \|\hat{Z}\|)$.

Next, we will give out the estimates of the perturbed term P. To this end we need some notations which are taken from [4]. Let $l^{a,s}$ is now the Hilbert space of all complex sequence $w = (\ldots, w_1, w_2, \ldots)$ with

$$\|w\|_{a,s}^2 = \sum_{j \geqslant n+1} |w_j|^2 |j|^{2s} e^{2a|j|} < \infty, \quad a, s > 0.$$

Set $x = (\hat{\theta}_0, \theta) \oplus \hat{\theta}$, with $\hat{\theta} = (\hat{\theta}_j)_{1 \leqslant j \leqslant n}$, $y = (J, \tilde{\rho}_0) \oplus \tilde{\rho}$, $\tilde{\rho} = (\tilde{\rho}_j)_{1 \leqslant j \leqslant n}$, $Z = (z_j)_{j \geqslant n+1}$, and let us introduce the phase space

$$\mathcal{P}^{a,s} = \hat{\mathbb{T}}^{m+n+2} \times \mathbb{C}^{m+n+2} \times l^{a,s} \times l^{a,s} \ni (x, y, Z, \bar{Z})$$

where $\hat{\mathbb{T}}^{m+n+2}$ is the complexiation of the usual $(m+n+2)$-torus $\mathbb{T}^{m+n+2}$. Set

$$D(s', r) := \{(x, y, Z, \bar{Z}) \in \mathcal{P}^{a,s}\colon |\mathrm{Im}\, x| < s',\ |y| < r^2,\ \|Z\|_{a,s} + \|\bar{Z}\|_{a,s} < r\}.$$

We define the weighted phase norms

$$|W|_r = |W|_{\bar{s},r} = |x| + \frac{1}{r^2}|y| + \frac{1}{r}\|Z\|_{a,\bar{s}} + \frac{1}{r}\|\bar{Z}\|_{a,\bar{s}}$$

for $W = (x, y, Z, \bar{Z}) \in \mathcal{P}^{a,\bar{s}}$ with $\bar{s} = s + 1$. Denote by $\underline{\Sigma}$ the parameter set $\mathscr{J} \times [0,1]^{m+n+1}$. For a map $U : D(s', r) \times \underline{\Sigma} \to \mathcal{P}^{a,\bar{s}}$, define its Lipschitz semi-norm $|U|_r^{\mathcal{L}}$:

$$|U|_r^{\mathcal{L}} = \sup_{\hat{\xi} \neq \xi} \frac{|\Delta_{\hat{\xi}\xi} U|_r}{|\hat{\xi} - \xi|},$$

where $\Delta_{\hat{\xi}\xi} U = U(\cdot, \hat{\xi}) - U(\cdot, \xi)$, and where the supremum is taken over $\underline{\Sigma}$. Denote by X_P the vector field corresponding the Hamiltonian P with respect to the symplectic structure $dx \wedge dy + \mathrm{i}\, dZ \wedge d\bar{Z}$, namely,

$$X_P = (\partial_y P, -\partial_x P, \nabla_{\bar{Z}} P, -\nabla_Z P).$$

Lemma 5.8. *The Perturbation $P(x, y, Z, \bar{Z}; \zeta)$ is real analytic for real argument $(x, y, Z, \bar{Z}) \in D(s', r)$ for some given $s', r > 0$, and Lipschitz in the parameters $\xi \in \underline{\Sigma}$, and for each $\xi \in \underline{\Sigma}$ its gradients with respect to $Z, \bar{Z}$ satisfy*

$$\partial_Z P, \partial_{\bar{Z}} P \in \mathcal{A}\big(l^{a,s}, l^{a,s+1/2}\big),$$

where $\mathcal{A}(l^{a,s}, l^{a,s+1/2})$ denotes the class of all maps from some neighborhood of the origin in $l^{a,s}$ into $l^{a,s+1/2}$, which is real analytic in the real and imaginary parts of the complex coordinate Z. In addition, for the perturbed term P we have the following estimates

$$\sup_{D(s',r)\times\underline{\Sigma}} |X_P|_r \leqslant C\varepsilon^{\frac{7}{2}} \qquad \sup_{D(s',r)\times\underline{\Sigma}} |\partial_\xi X_P|_r \leqslant C\varepsilon^{\frac{7}{2}},$$

where $s' = \sigma/3$ and $r = \varepsilon$.

Proof. For $0 \leqslant j \leqslant n$, it follows from (5.31) that $|\partial_{\hat{\theta}_j} w_j| \leqslant C\varepsilon^{\frac{1}{2}}$ and $|\partial_{\tilde{\rho}_j} w_j| \leqslant C\varepsilon^{-\frac{1}{2}}$, where $w_j = z_j$ or $w_j = \bar{z}_j$. From (5.31) and $\|Z\|_{a,s} \leqslant r = \varepsilon$, we get $\|z\|_{a,s} \leqslant C\varepsilon^{\frac{1}{2}}$ where $z = (z_0, \underline{z}) \oplus Z$ with $\underline{z} = (z_1, \dots, z_n)$. In view of $|\breve{\mathcal{G}}| = \mathcal{O}(\varepsilon^4), |\hat{\mathcal{G}}| = \mathcal{O}(\varepsilon^4)$ and $K = \mathcal{O}(\varepsilon^{\frac{5}{2}})$, it follows that $|P| = \mathcal{O}(\varepsilon^{\frac{11}{2}})$ on $D(s', 2r)$. Using Cauchy estimates for $\partial_x P$, $\partial_y P$, $\partial_{\bar{Z}} P$ and $\partial_Z P$, we obtain $|\partial_x P| = \mathcal{O}(\varepsilon^{\frac{11}{2}}), |\partial_y P| = \mathcal{O}(\varepsilon^{\frac{7}{2}}), |\partial_{\bar{Z}} P| = \mathcal{O}(\varepsilon^{\frac{9}{2}}), |\partial_Z P| = \mathcal{O}(\varepsilon^{\frac{9}{2}})$ on $D(s', r)$. Hence, we have $\sup_{D(s,r)\times\underline{\Sigma}} |X_P|_r \leqslant C\varepsilon^{\frac{7}{2}}$. Using again Cauchy estimates with respect to ξ, we also have $\sup_{D(s,r)\times\underline{\Sigma}} |\partial_\xi X_P|_r \leqslant C\varepsilon^{\frac{7}{2}}$. $\square$

6. An infinite-dimensional KAM theorem for partial differential equations

In order to prove our main result (Theorem 1.2), we need to state a KAM theorem which was first proved by Kuksin [1,31]. Also see Pöschel [4]. Here we recite the theorem from [4].

Let us consider the perturbations of a family of linear integrable Hamiltonian

$$H_0 = \sum_{j=1}^{n} \hat{\omega}_j(\xi) y_j + \frac{1}{2} \sum_{j=n+1}^{\infty} \hat{\Omega}_j(\xi)\big(u_j^2 + v_j^2\big),$$

in n-dimensional angle-action coordinates (x, y) and infinite-dimensional Cartesian coordinates (u, v) with symplectic structure

$$\sum_{j=1}^{n} dx_j \wedge dy_j + \sum_{j=n+1}^{\infty} du_j \wedge dv_j.$$

The tangent frequencies $\hat{\omega} = (\hat{\omega}_1, \dots, \hat{\omega}_n)$ and normal ones $\hat{\Omega} = (\hat{\Omega}_{n+1}, \hat{\Omega}_{n+2} \dots)$ depend on n parameters

$$\xi \in \Pi \subset \mathbb{R}^n,$$

with Π a closed bounded set of positive Lebesgue measure.

For each ξ there is an invariant n-torus $\mathcal{T}_0^n = \mathbb{T}^n \times \{0,0,0\}$ with frequencies $\hat{\omega}(\xi)$. In its normal space described by the uv-coordinates the origin is an elliptic fixed point with characteristic frequencies $\hat{\Omega}(\xi)$. Hence $\mathcal{T}_0^n$ is linear stable. The aim is to prove the persistence of a large portion of this family of linearly stable rotational tori under small perturbations $H = H_0 + P$ of H_0. To this end the following assumptions are made.

Assumption A *(Non-degeneracy).* The map $\xi \mapsto \hat{\omega}(\xi)$ is a lipeomorphism between Π and its image, that is, a homomorphism which is Lipschitz continuous in both directions. Moreover, for all integer vectors $(k,l) \in \mathbb{Z}^n \times \mathbb{Z}^\infty$ with $1 \leqslant |l| \leqslant 2$,

$$\operatorname{meas}\bigl\{\xi\colon \bigl\langle k, \hat{\omega}(\xi)\bigr\rangle + \bigl\langle l, \hat{\Omega}(\xi)\bigr\rangle = 0\bigr\} = 0$$

and

$$\bigl\langle l, \hat{\Omega}(\xi)\bigr\rangle \neq 0 \quad \text{on } \Pi,$$

where meas denotes Lebesgue measure for sets, $|l| = \sum_j |l_j|$ for integer vectors, and $\langle\cdot,\cdot\rangle$ is the usual scalar product.

Assumption B *(Spectral asymptotics and the Lipschitz property).* There exist $\varsigma \geqslant 1$ and $\delta < \tau - 1$ such that

$$\hat{\Omega}_j(\xi) = j^\varsigma + \cdots + O\bigl(j^\delta\bigr),$$

where the dots stands for fixed lower order term in j, allowing also negative exponents. More precisely, there exists a fixed, parameter-independent sequence $\tilde{\Omega}$ with $\tilde{\Omega}_j = j^\varsigma + \ldots$ such that the tails $\hat{\Omega}_j - \tilde{\Omega}_j$ give rise to a Lipschitz map

$$\hat{\Omega}_j - \tilde{\Omega}_j : \Pi \to l_\infty^{-\delta},$$

where l_∞^p is the space of all real sequence with finite norm $|w|_p = \sup_j |w_j| j^p$.

Assumption C *(Regularity).* The perturbation $P(x,y,Z,\bar{Z};\xi)$ is real analytic for real argument $(x,y,Z,\bar{Z}) \in D(s,r)$ for given $s,r > 0$, and Lipschitz in the parameters $\xi \in \Pi$, and for each $\xi \in \Pi$ its gradients with respect to $Z, \bar{Z}$ satisfy

$$P_Z, P_{\bar{Z}} \in \mathcal{A}\bigl(l^{a,p}, l^{a,\bar{p}}\bigr), \qquad \begin{cases} \bar{p} \geqslant p, & \text{for } \varsigma > 1, \\ \bar{p} > p, & \text{for } \varsigma = 1, \end{cases}$$

where $\mathcal{A}(l^{a,p}, l^{a,\bar{p}})$ denotes the class of all maps from some neighborhood of the origin in $l^{a,p}$ into $l^{a,\bar{p}}$, which is real analytic in the real and imaginary parts of the complex coordinate Z.

We assume that

$$|\hat{\omega}|_\Pi^{\mathcal{L}} + |\hat{\Omega}|_{-\delta,\Pi}^{\mathcal{L}} \leqslant M < \infty, \qquad \bigl|(\hat{\omega})^{-1}\bigr|_{\hat{\omega}(\Pi)}^{\mathcal{L}} \leqslant L < \infty. \tag{6.1}$$

In addition, we introduce the notations

$$\langle l\rangle_\varsigma = \max\left(1, \left|\sum_j j^\varsigma l_j\right|\right), \qquad A_k = 1 + |k|^\tau,$$

where $\tau > n+1$ is fixed later. Finally, let $\mathcal{Z} = \{(k,l) \neq 0,\ |l| \leqslant 2\} \subset \mathbb{Z}^n \times \mathbb{Z}^\infty$.

We can now state the basic KAM theorem which is attributed to Pöschel [4] (see also [5,31]).

Theorem 6.1. *(See [4, Theorem A].) Suppose $H = H_0 + P$ satisfies Assumptions* A, B *and* C, *and*

$$\epsilon = \sup_{D(s,r)\times\Pi} |X_P|_r + \sup_{D(s,r)\times\Pi} \frac{\alpha}{M} |X_P|_r^{\mathcal{L}} \leqslant \gamma\alpha, \tag{6.2}$$

where $0 < \alpha \leqslant 1$ is a parameter, and γ depends on the parameters described below. Then there is a Cantor set $\Pi_\alpha \subset \Pi$ with $\mathrm{meas}(\Pi \setminus \Pi_\alpha) \to 0$ as $\alpha \to 0$, a Lipschitz continuous family of torus embedding $\Phi : \mathbb{T}^n \times \Pi_\alpha \to \mathcal{P}^{a,\bar{p}}$, and a Lipschitz continuous map $\hat{\hat{\omega}} : \Pi_\alpha \to \mathbb{R}^n$, such that for each $\xi \in \Pi_\alpha$ the map Φ restricted to $\mathbb{T}^n \times \{\xi\}$ is a real analytic embedding of an elliptic rotational torus with frequencies $\hat{\hat{\omega}}(\xi)$ for the Hamiltonian H at ξ.

Each embedding is analytic on $|\mathrm{Im}\, x| < \frac{s}{2}$, and

$$\begin{aligned} |\Phi - \Phi_0|_r + \frac{\alpha}{M} |\Phi - \Phi_0|_r^{\mathcal{L}} &\leqslant \frac{c\epsilon}{\alpha}, \\ |\hat{\hat{\omega}} - \hat{\omega}| + \frac{\alpha}{M} |\hat{\hat{\omega}} - \hat{\omega}|^{\mathcal{L}} &\leqslant c\epsilon, \end{aligned} \tag{6.3}$$

uniformly on that domain and Π_α, where $\Phi_0 : \mathbb{T}^n \times \Pi \to \mathcal{T}_0^n$ is the trivial embedding, and $c \leqslant \gamma^{-1}$ depends on the same parameters as γ.

Moreover, there exist a family of Lipschitz maps $\hat{\omega}_j$ and Λ_j on Π for $0 \leqslant j \in \mathbb{Z}$ satisfying $\hat{\omega}_0 = \hat{\omega}$, $\Lambda_0 = \hat{\Omega}$ and

$$\begin{aligned} |\hat{\omega}_j - \hat{\omega}| + \frac{\alpha}{M} |\hat{\omega}_j - \hat{\omega}|^{\mathcal{L}} &\leqslant c\epsilon, \\ |\Lambda_j - \hat{\Omega}|_{-\delta} + \frac{\alpha}{M} |\Lambda_j - \hat{\Omega}|_{-\delta}^{\mathcal{L}} &\leqslant c\epsilon, \end{aligned}$$

such that $\Pi \setminus \Pi_\alpha \subset \bigcup \mathcal{R}_{k,l}^j(\alpha)$, where

$$\mathcal{R}_{k,l}^j(\alpha) = \left\{ \xi \in \Pi \colon \left| \langle k, \hat{\omega}_j(\xi) \rangle + \langle l, \Lambda_j \rangle \right| \leqslant \alpha \frac{\langle l \rangle_d}{A_k} \right\},$$

and the union is taken over all $j \geqslant 0$ and $(k,l) \in \mathcal{Z}$ such that $|k| > K_0 2^{j-1}$ for $j \geqslant 1$ with a constant $K_0 \geqslant 1$ depending only on n and τ.

Concerning the measure of the "bad" frequency set $\Pi \setminus \Pi_\alpha$, we recite Pöschel's theorem [4]

Theorem 6.2. *(See [4, Theorem D].) Suppose that in Theorem* 6.1 *the unperturbed frequencies are affine functions of the parameters. Then there is a constant $\tilde{c}$ such that*

$$\mathrm{meas}(\Pi \setminus \Pi_\alpha) \leqslant \tilde{c} (\mathrm{diam}\, \Pi)^{n-1} \alpha^{\tilde{\mu}}, \qquad \tilde{\mu} = \begin{cases} 1, & \text{for } \varsigma > 1, \\ \frac{\kappa}{\kappa + 1 - (\varpi/4)}, & \text{for } \varsigma = 1, \end{cases}$$

for all sufficiently small α, where ϖ is any number in $[0, \min(\bar{p} - p, 1))$, and where, in the case $\varsigma = 1$, κ is a positive constant such that

$$\frac{\hat{\Omega}_i - \hat{\Omega}_j}{i - j} = 1 + O\left(j^{-\kappa}\right), \quad i > j \tag{6.4}$$

uniformly on Π.

In order to apply the above theorems to our problem, we need to introduce a new parameter $\bar{\omega}$ below.

For any $\bar{\xi} \in \mathscr{J}$, we have $\alpha(\bar{\xi}) \in A_\gamma$. Hence, for fixed $\omega_- = (\omega_-^1, \omega_-^2, \ldots, \omega_-^m) \in D_\Lambda$ and $\omega_-^{m+1}(\bar{\xi}) \in A_\gamma$ arbitrarily. For

$$\tilde{\omega}(\bar{\xi}) \in \bar{\bar{\Omega}} := \left\{\tilde{\omega}(\bar{\xi}) = \left(\omega_1, \ldots, \omega_m, \alpha(\bar{\xi})\right) \in D_\Lambda \times A_\gamma \;\middle|\; \left|\omega_i - \omega_-^i\right| \leqslant \varepsilon,\ \left|\alpha(\bar{\xi}) - \omega_-^{m+1}(\bar{\xi})\right| \leqslant \varepsilon\right\},$$

we can introduce new parameter $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \ldots, \bar{\omega}_m, \bar{\omega}_{m+1})$ by the following

$$\omega_j = \omega_-^j + \varepsilon^3 \bar{\omega}_j, \qquad \bar{\omega}_j \in [0,1], \quad j = 1, 2, \ldots, m,$$
$$\alpha(\bar{\xi}) = \omega_-^{m+1}(\bar{\xi}) + \varepsilon^3 \bar{\omega}_{m+1}, \quad \bar{\omega}_{m+1} \in [0,1].$$

Hence, the Hamiltonian (5.32) becomes

$$H = \left\langle \hat{\omega}(\xi), \hat{y}\right\rangle + \left\langle \hat{\Omega}(\xi), \hat{Z}\right\rangle + P \tag{6.5}$$

where $\hat{\omega}(\xi) = \tilde{\omega}(\bar{\xi}) \oplus \check{\omega}_0 \oplus \check{\omega}$ with $\check{\omega} = \tilde{\alpha} + \varepsilon^3 A\tilde{\xi}$, $\hat{\Omega}(\xi) = \tilde{\beta} + \varepsilon^3 B\tilde{\xi}$, and $\xi = \bar{\omega} \oplus \tilde{\xi}_0 \oplus \tilde{\xi}$ and $\tilde{\xi} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_n)$,

$$\hat{y} = J \oplus \tilde{\rho}_0 \oplus \tilde{\rho}, \qquad \tilde{\alpha} = (\check{\omega}_1, \ldots, \check{\omega}_n), \qquad \tilde{\beta} = (\check{\lambda}_{n+1}, \check{\lambda}_{n+2}, \ldots).$$

Lemma 6.3. *Let $\Pi = [0,1]^{m+n+2}$. Then we have $X_P \in \mathcal{A}(l^{a,s}, l^{a,s+1/2})$ and*

$$\sup_{D(s,r)\times\Pi} |X_P|_r \leqslant C\varepsilon^{\frac{7}{2}} \qquad \sup_{D(s,r)\times\Pi} |\partial_\zeta X_P|_r \leqslant C\varepsilon^{\frac{7}{2}}.$$

The proof of the above lemma is the same as one of Lemma 5.8.

7. Proof of main theorem

In the following, we will verify Assumptions A, B and C for the above Hamiltonian (6.5). Recalling (5.25), we have

$$A := (\tilde{\mathcal{G}}_{ij})_{1\leqslant i,j\leqslant n} = \begin{pmatrix} \frac{Bb_0}{\pi\lambda_1} + \varpi_{11}(\bar{\xi},\varepsilon) & \frac{Ba_0}{\pi\sqrt{\lambda_1\lambda_2}} + \varpi_{12}(\bar{\xi},\varepsilon) & \cdots & \frac{Ba_0}{\pi\sqrt{\lambda_1\lambda_n}} + \varpi_{1n}(\bar{\xi},\varepsilon) \\ \frac{Ba_0}{\pi\sqrt{\lambda_2\lambda_1}} + \varpi_{21}(\bar{\xi},\varepsilon) & \frac{Bb_0}{\pi\lambda_2} + \varpi_{22}(\bar{\xi},\varepsilon) & \cdots & \frac{Ba_0}{\pi\sqrt{\lambda_2\lambda_n}} + \varpi_{2n}(\bar{\xi},\varepsilon) \\ \cdots & \cdots & \cdots & \cdots \\ \frac{Ba_0}{\pi\sqrt{\lambda_n\lambda_1}} + \varpi_{n1}(\bar{\xi},\varepsilon) & \frac{Ba_0}{\pi\sqrt{\lambda_n\lambda_2}} + \varpi_{n2}(\bar{\xi},\varepsilon) & \cdots & \frac{Bb_0}{\pi\lambda_n} + \varpi_{nn}(\bar{\xi},\varepsilon) \end{pmatrix}_{n\times n},$$

$$B := (\tilde{\mathcal{G}}_{ij})_{1\leqslant j\leqslant n<i} = \begin{pmatrix} \frac{Ba_0}{\pi\sqrt{\lambda_{n+1}\lambda_1}} + \varpi_{n+1,1}(\bar{\xi},\varepsilon) & \cdots & \frac{Ba_0}{\pi\sqrt{\lambda_{n+1}\lambda_n}} + \varpi_{n+1,n}(\bar{\xi},\varepsilon) \\ \frac{Ba_0}{\pi\sqrt{\lambda_{n+2}\lambda_1}} + \varpi_{n+2,1}(\bar{\xi},\varepsilon) & \cdots & \frac{Ba_0}{\pi\sqrt{\lambda_{n+2}\lambda_n}} + \varpi_{n+2,n}(\bar{\xi},\varepsilon) \\ \vdots & \vdots & \vdots \end{pmatrix}_{\infty\times n},$$

and

$$\lim_{\varepsilon\to 0} A = -\frac{3B}{16\pi}\begin{pmatrix} \frac{9}{1\times 1} & \frac{5}{1\times 2} & \cdots & \frac{5}{1\times n} \\ \frac{5}{2\times 1} & \frac{9}{2\times 2} & \cdots & \frac{5}{2\times n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{5}{n\times 1} & \frac{5}{n\times 2} & \cdots & \frac{9}{n\times n} \end{pmatrix} := D,$$

$$\lim_{\varepsilon\to 0} B = -\frac{15B}{4\pi}\begin{pmatrix} \frac{1}{(n+1)\times 1} & \cdots & \frac{1}{(n+1)\times n} \\ \frac{1}{(n+2)\times 1} & \cdots & \frac{1}{(n+2)\times n} \\ \vdots & \vdots & \vdots \end{pmatrix}_{\infty\times n} := \tilde{D}.$$

Setting $\hat{u} = (1, 2, \ldots, n)$ and $\hat{v} = (n+1, n+2, \ldots)$, and defining the matrices

$$\bar{E} := \operatorname{diag}[\hat{u}] \quad \text{and} \quad \bar{F} := \operatorname{diag}[\hat{v}],$$

we can rewrite D and $\tilde{D}$ as

$$D = -\frac{3B}{16\pi}\bar{E}^{-1}\overline{A}\bar{E}^{-1} \quad \text{and} \quad \tilde{D} = -\frac{15B}{4\pi}\bar{F}^{-1}\overline{B}\bar{E}^{-1},$$

where

$$\overline{A} = \begin{pmatrix} 9 & 5 & \cdots & 5 \\ 5 & 9 & \cdots & 5 \\ \cdots & \cdots & \cdots & \cdots \\ 5 & 5 & \cdots & 9 \end{pmatrix}, \qquad \overline{B} = \begin{pmatrix} 1 & \cdots & 1 \\ 1 & \cdots & 1 \\ \vdots & \vdots & \vdots \end{pmatrix}_{\infty\times n}.$$

We know that $\det D \neq 0$ since $\det \overline{A} = 4^{n-1}(4+5n) \neq 0$. Therefore, we get $\det A \neq 0$ provided that $0 < \varepsilon \ll 1$. Moreover, by the definition of $\hat{\omega}$, we get that

$$\frac{\partial \hat{\omega}}{\partial \xi} = \varepsilon^3 \begin{pmatrix} I_{m+1} & 0 & 0 \\ 0 & 2\bar{c} & \mathbb{Y} \\ 0 & \mathbb{Y}^T & A \end{pmatrix}, \quad \text{for } \xi \in \Pi,$$

where I_{m+1} denotes the unit $(m+1)\times(m+1)$-matrix, $\mathbb{Y} = (c_1, c_2, \ldots, c_n)$ and $\mathbb{Y}^T$ denotes the transpose of $\mathbb{Y}$. In view of $\bar{c} = \mathcal{O}(\varepsilon^{-1})$, $c_j = \mathcal{O}(\varepsilon^{-\frac{2}{3}})$ and

$$\begin{pmatrix} 1 & -\mathbb{Y}A^{-1} \\ 0 & I_n \end{pmatrix}\begin{pmatrix} 2\bar{c} & \mathbb{Y} \\ \mathbb{Y}^T & A \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -A^{-1}\mathbb{Y}^T & I_n \end{pmatrix} = \begin{pmatrix} 2\bar{c} - \mathbb{Y}A^{-1}\mathbb{Y}^T & 0 \\ 0 & A \end{pmatrix},$$

we get

$$\det \begin{pmatrix} 2\bar{c} & \mathbb{Y} \\ \mathbb{Y}^T & A \end{pmatrix} \neq 0$$

provided that $0 < \varepsilon \ll 1$. Therefore, the real map $\xi \mapsto \hat{\omega}(\xi)$ is a lipeomorphism between Π and its imagine.

For any $k \in \mathbb{Z}^{m+2+n}$, we write

$$k = (k_1, k_2, k_3), \quad k_1 \in \mathbb{Z}^{m+1},\ k_2 \in \mathbb{Z},\ k_3 \in \mathbb{Z}^n.$$

Let

$$\begin{aligned} \mathcal{Y}(\xi) &= \langle k, \hat{\omega}(\xi)\rangle + \langle l, \hat{\Omega}(\xi)\rangle = \langle k_1, \tilde{\omega}(\bar{\xi})\rangle + k_2\check{\omega}_0 + \langle k_3, \tilde{\alpha}\rangle + \langle k_3, \varepsilon^3 A\tilde{\xi}\rangle + \langle l, \tilde{\beta} + \varepsilon^3 B\tilde{\xi}\rangle, \\ \Delta &:= \{\xi \in \Pi\colon\ \mathcal{Y}(\xi) = 0\}. \end{aligned}$$

We need to prove that $\operatorname{meas}\Delta = 0$. To this end we divide two cases.

Case 1. Let $k_1 = (k^1, k^2, \dots, k^m, k^{m+1}) \neq 0$ and write

$$\langle k_1, \tilde{\omega}(\bar{\xi}) \rangle = \sum_{i=1}^{m} k^i \omega_i + k^{m+1} \alpha(\bar{\xi}),$$

then there exists some $1 \leqslant i_0 \leqslant m$ such that $k^{i_0} \neq 0$. Observe that $\check{\omega}_0$, $\tilde{\alpha}$ and $\tilde{\beta}$ do not involve the parameter $\bar{\omega}$. Then

$$\frac{\partial \mathcal{Y}(\xi)}{\partial \bar{\omega}_{i_0}} = k^{i_0} \varepsilon^3 \neq 0, \quad 0 < \varepsilon \ll 1.$$

This implies meas $\Delta = 0$.

Case 2. Let $k_2 \neq 0$, then

$$\frac{\partial \mathcal{Y}(\xi)}{\partial \tilde{\xi}_0} = 2\bar{c}\varepsilon^3 + \mathcal{O}\big(\varepsilon^{\frac{7}{3}}\big) \neq 0$$

which implies meas $\Delta = 0$.

Case 3. Let $k_1 = k_2 = 0$, then

$$\begin{aligned}
\mathcal{Y}(\xi) &= \langle k_1, \omega \rangle + k_2 \check{\omega}_0 + \langle k_3, \tilde{\alpha} \rangle + \big\langle k_3, \varepsilon^3 A\tilde{\xi} \big\rangle + \big\langle l, \tilde{\beta} + \varepsilon^3 B\tilde{\xi} \big\rangle \\
&= \langle k_3, \tilde{\alpha} \rangle + \big\langle k_3, \varepsilon^3 A\tilde{\xi} \big\rangle + \big\langle l, \tilde{\beta} + \varepsilon^3 B\tilde{\xi} \big\rangle \\
&= \langle k_3, \tilde{\alpha} \rangle + \langle l, \tilde{\beta} \rangle + \varepsilon^3 \big\langle Ak_3 + B^T l, \tilde{\xi} \big\rangle,
\end{aligned}$$

where B^T is the transpose of B. (Note that A is symmetric.) We claim that either $\langle k_3, \tilde{\alpha} \rangle + \langle l, \tilde{\beta} \rangle \neq 0$ or $Ak_3 + B^T l \neq 0$.

Since

$$\lim_{\varepsilon \to 0} \big(Ak_3 + B^T l\big) = Dk_3 + \tilde{D}^T l$$

and

$$\lim_{\varepsilon \to 0} \big(\langle k_3, \tilde{\alpha} \rangle + \langle l, \tilde{\beta} \rangle\big) = \langle k_3, \hat{\alpha} \rangle + \langle l, \hat{\beta} \rangle$$

with $\hat{\alpha} = (1, 2, \dots, n)$ and $\hat{\beta} = (n+1, n+2, \dots)$, it suffices to show that $\langle k_3, \hat{\alpha} \rangle + \langle l, \hat{\beta} \rangle \neq 0$ or $Dk_3 + \tilde{D}^T l \neq 0$. The result is proved in [5] (see Lemma 6 in [5]). Hence, we get that $\langle k_3, \tilde{\alpha} \rangle + \langle l, \tilde{\beta} \rangle \neq 0$ or $Ak_3 + B^T l \neq 0$ as $0 < \varepsilon \ll 1$. Moreover, it is easy to that $\langle l, \hat{\Omega}(\xi) \rangle \neq 0$ as $0 < \varepsilon \ll 1$, with $1 \leqslant |l| \leqslant 2$ and $\xi \in \Pi$. This completes the verification of Assumption A.

Note that $\lambda_j = j^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]$ and

$$\mu_j = \sqrt{\lambda_j} + \sum_{k=2}^{\infty} \varepsilon^{\frac{2k}{3}} \tilde{\lambda}_{j,k}\big(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big),$$

we have $\hat{\Omega}_j = j^{\varsigma} + \cdots$ with $\varsigma = 1$, and $\tilde{\Omega}_j := \hat{\Omega}_j - j$ is a Lipschitz map $\tilde{\Omega} : \Pi \to l_\infty^{-\delta}$ with $\delta = -1$. Thus, Assumption B is fulfilled for $\hat{\Omega}$ with $\delta = -1$ and $\varsigma = 1$. Assumption C can be verified easily fulfilled by Lemma 5.8, letting $\bar{p} = s + 1/2$, $p = s$. Using $\hat{\omega}(\xi) = \tilde{\omega}(\bar{\xi}) \oplus \check{\omega}_0 \oplus (\tilde{\alpha} + \varepsilon^3 A\tilde{\xi})$ and $\hat{\Omega}(\xi) = \tilde{\beta} + \varepsilon^3 B\tilde{\xi}$ we find that (6.1) is satisfied with $M = C_1\varepsilon^2$ and $L = C_2\varepsilon^{-1/2}$.

Now let us verify the smallness condition (6.2) in Theorem 6.1. By letting $\alpha=\varepsilon^{\frac{7}{2}-\iota}$ with $0<\iota<\frac{7}{2}$ fixed and Lemma 6.3, we have

$$\sup_{D(s,r)\times\Pi}|X_P|_r+\sup_{D(s,r)\times\Pi}\frac{\alpha}{M}|X_P|_r^{\mathcal{L}}\leqslant\gamma\alpha,$$

if $0<\varepsilon<\varepsilon^{**}$ with a constant $\varepsilon^{**}=\varepsilon^{**}(\gamma,C)$. This implies the smallness condition (6.2) is satisfied. Next, Let us check the conditions of Theorem 6.2 for the Hamiltonian (6.5). First of all, we remark that $\hat{\omega}(\xi)$ is affine function of the parameter ξ. And by

$$\hat{\Omega}_j=\mu_j+\varepsilon^3(B\tilde{\xi})_j,\quad j\geqslant n+1,$$

we have $\hat{\Omega}_j=j+O(j^{-1})$. Thus, for $i>j$,

$$\frac{\hat{\Omega}_i-\hat{\Omega}_j}{i-j}=1+O\big(j^{-2}\big).$$

This gives $\kappa=2$ in (6.4), and we can choose $\tilde{\mu}=\frac{3}{7-2\iota}$ in Theorem 6.2.

Let us run Theorems 6.1 and 6.2 for Hamiltonian (6.5). Then there is a subset $\Pi_\alpha\subset\Pi$ with

$$\operatorname{meas}(\Pi\setminus\Pi_\alpha)\leqslant\hat{c}L^{n+m+2}M^{n+m+1}(\operatorname{diam}\Pi)^{n+m+1}\alpha^{\tilde{\mu}}\leqslant C\varepsilon^{-1/2}\alpha^{\tilde{\mu}}<\varepsilon,$$

and a Lipschitz continuous family of torus embedding $\Phi:\mathbb{T}^{n+m+2}\times\Pi_\alpha\to\mathcal{P}^{a,s+1/2}$, and a Lipschitz continuous map $\hat{\hat{\omega}}:\Pi_\alpha\to\mathbb{R}^{n+m+2}$, such that for each $\xi\in\Pi_\alpha$ the map Φ restricted to $\mathbb{T}^{n+m+2}\times\{\xi\}$ is a real analytic embedding of an elliptic rotational torus with frequencies $\hat{\hat{\omega}}(\xi)$ for the Hamiltonian H at ξ. Moreover, $|\hat{\hat{\omega}}(\xi)-\hat{\omega}(\xi)|<c\varepsilon^{\frac{7}{2}}$ and (6.3) holds. We return from the parameter set Π to

$$\Pi^*\big(\omega_-,\omega_-^{m+1}\big)=\bar{\bar{\Omega}}\times[0,1]^{n+1}.$$

Let

$$\Pi^*=\bigcup_{(\omega_-,\omega_-^{m+1})\in D_\Lambda\times A_\gamma}\Pi^*\big(\omega_-,\omega_-^{m+1}\big),$$

where ω_-,ω_-^{m+1} are chosen such that $\Pi^*(\omega_-^*,\omega_-^{m+1^*})\cap\Pi^*(\omega_-^{**},\omega_-^{m+1^{**}})=\emptyset$ if $(\omega_-^*,\omega_-^{m+1^*})\neq(\omega_-^{**},\omega_-^{m+1^{**}})$. Hence, we get a subset $\Pi_\alpha^*\subset\Pi^*$ such that

$$\Sigma_\alpha=\Pi_\alpha^*\subset D_\Lambda\times A_\gamma\times[0,1]^{n+1}\subset\Sigma$$

with

$$\operatorname{meas}(\Sigma\setminus\Sigma_\varepsilon)\leqslant\varepsilon.$$

Therefore, for the new parameter set, we have that there are a Lipschitz continuous family of torus embedding $\Phi:\mathbb{T}^{m+n+2}\times\Sigma_\varepsilon\to\mathcal{P}^{a,s+1}$, and a Lipschitz continuous map $\hat{\hat{\omega}}:\Sigma_\varepsilon\to\mathbb{R}^{m+n+2}$, such that for each $\xi\in\Sigma_\varepsilon$ the map Φ restricted to $\mathbb{T}^{m+n+2}\times\{\xi\}$ is a real analytic embedding of an elliptic rotational torus with frequencies $\hat{\hat{\omega}}(\xi)=(\tilde{\omega}(\bar{\xi}),(\hat{\omega}_j)_{0\leqslant j\leqslant n})$ for the Hamiltonian H at ξ. And

$$|\Phi-\Phi_0|_r+\frac{\alpha}{M}|\Phi-\Phi_0|_r^{\mathcal{L}}\leqslant c\varepsilon^{\frac{7}{2}-(\frac{7}{2}-\iota)}=c\varepsilon^{\iota}, \tag{7.1}$$

$$\left|\hat{\hat{\omega}}-\omega^0\right|_r+\frac{\alpha}{M}\left|\hat{\hat{\omega}}(\xi)-\omega^0(\xi)\right|^{\mathcal{L}}\leqslant c\varepsilon^{\frac{7}{2}}, \tag{7.2}$$

where $\omega^0(\xi)=\hat{\omega}(\xi)$ and $\xi=(\omega,\alpha(\bar{\xi}),\tilde{\xi}_0,\tilde{\xi}_1,\dots,\tilde{\xi}_n)$. Therefore, all motions starting from the torus $\Phi(\mathbb{T}^{m+n+2}\times\Sigma_\varepsilon)$ are quasi-periodic with frequencies $\hat{\hat{\omega}}(\xi)$. By (7.1) and (7.2), those motion can written as follows:

$$\tilde{\rho}_0(t)=O\big(\varepsilon^2\big),\qquad \hat{\theta}_0(t)=\hat{\omega}_0t+O\big(\varepsilon^{\iota}\big),$$
$$\tilde{\rho}_j(t)=O\big(\varepsilon^2\big),\qquad \hat{\theta}_j(t)=\hat{\omega}_jt+O\big(\varepsilon^{\iota}\big),\quad j=1,2,\dots,n,$$
$$\big\|Z(t)\big\|_{a,s+1}=O(\varepsilon),\qquad \theta(t)=\tilde{\omega}(\bar{\xi})t,$$

where $Z=(z_j)_{j>n}$ and we have taken the initial phase $\hat{\theta}_j(0)=0$. Returning the original equation (1.2), we may get the solution described in Theorem 1.2. $\square$

Appendix A

In the whole of this section, we denote by $\hat{C}$, the universal constant depending on $\bar{\xi}$ if we do not care its value. Recall (4.3)

$$\mu_j=\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{j,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)=\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big) \tag{A.1}$$

and let

$$\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle=\sum_{i=1}^{m}k_m\omega_m+k_{m+1}\alpha(\bar{\xi}).$$

Proof of Lemma 5.1. Let

$$g_{jd}(\varepsilon,\bar{\xi})=\mu_j'+\mu_d'.$$

Then

$$\begin{aligned}
g_1(\varepsilon,\bar{\xi})&=\pm\mu_0+g_{jd}(\varepsilon,\bar{\xi})+\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle\\
&=\pm\left(\varepsilon^{\frac{1}{3}}\sqrt{[\hat{V}]}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{0,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right)\pm\left(\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{j,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right)\\
&\quad\pm\left(\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{d,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right)+\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle.
\end{aligned} \tag{A.2}$$

Case 1. $k=0$. We have

$$g_1(\varepsilon,\bar{\xi})=\pm\left(j+\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j}\right)\pm\left(d+\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d}\right)\pm\varepsilon^{\frac{1}{3}}\sqrt{[\hat{V}]}+\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big).$$

Case 1.1. If $\pm j \pm d = 0$, then

$$\begin{aligned}\left|g_1(\varepsilon,\bar{\xi})\right| &= \varepsilon^{\frac{1}{3}}\sqrt{[\hat{V}]} - \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j} - \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d} - \left|O\left(\varepsilon^{\frac{4}{3}}\right)\right| \\ &\geqslant C\varepsilon^{\frac{1}{3}} \geqslant \frac{\varepsilon^{\frac{2}{3}}\varrho \operatorname{meas}\hat{J}}{(|k|+1)^{2m+6}}.\end{aligned}$$

If $\pm j \pm d \neq 0$, then

$$\left|g_1(\varepsilon,\bar{\xi})\right| \geqslant 1 - \left|O\left(\varepsilon^{\frac{1}{3}}\right)\right| \geqslant \frac{\varepsilon^{\frac{2}{3}}\varrho \operatorname{meas}\hat{J}}{(|k|+1)^{2m+6}}.$$

It follows that inequality (5.13) holds true in the case $k=0$.

Case 2. $k \neq 0$. It suffices to investigate the following two cases.

Case 2.1. Let $g_{jd}(\varepsilon,\bar{\xi}) = \mu_j + \mu_d$. Assume $j,d>0$ without loss of generality. Then

$$\begin{aligned}g_1(\varepsilon,\bar{\xi}) = g_{jd}(\varepsilon,\bar{\xi}) \pm \mu_0 + \langle k,\tilde{\omega}(\bar{\xi})\rangle &= j + d + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j} + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d} \\ &\quad \pm \varepsilon^{\frac{1}{3}}\sqrt{[\hat{V}]} + \mathcal{O}\left(\varepsilon^{\frac{4}{3}}\right) + \langle k,\tilde{\omega}(\bar{\xi})\rangle.\end{aligned}$$

Let

$$\mathcal{R}^{10}_{jd,k} = \left\{\bar{\xi} \in \hat{J}\colon \left|g_1(\varepsilon,\bar{\xi})\right| < \frac{\varepsilon^{\frac{2}{3}}\varrho \operatorname{meas}\hat{J}}{(|k|+1)^{2m+6}}\right\}$$

and

$$J^{10} = \bigcup_{0\neq k\in\mathbb{Z}^{m+1}} \bigcup_{j,d} \mathcal{R}^{10}_{jd,k}. \tag{A.3}$$

Case 2.1.1. If $k_{m+1}=0$, then we have

$$\begin{aligned}\left|\frac{\partial}{\partial\bar{\xi}} g_1(\varepsilon,\bar{\xi})\right| &\geqslant \left|\frac{\varepsilon^{\frac{1}{3}}\frac{\partial[\hat{V}]}{\partial\bar{\xi}}}{2\sqrt{[\hat{V}]}}\right| - \left|\frac{\varepsilon^{\frac{2}{3}}\frac{\partial}{\partial\bar{\xi}}[\hat{V}]}{2\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}} + \frac{\varepsilon^{\frac{2}{3}}\frac{\partial}{\partial\bar{\xi}}[\hat{V}]}{2\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}} + O\left(\varepsilon^{\frac{4}{3}}\right) + k_{m+1}\alpha'(\bar{\xi})\right| \\ &\geqslant \frac{\varepsilon^{\frac{1}{3}}\frac{\partial[\hat{V}]}{\partial\bar{\xi}}}{2\sqrt{[\hat{V}]}} - C\varepsilon^{\frac{2}{3}} \geqslant \hat{C}\varepsilon^{\frac{2}{3}}.\end{aligned}$$

Case 2.2.2. Let $k_{m+1}\neq 0$. Write $\eta_{\pm}(\bar{\xi}) = k_{m+1}\alpha'(\bar{\xi}) \pm \frac{\varepsilon^{\frac{1}{3}}\frac{\partial[\hat{V}]}{\partial\bar{\xi}}}{2\sqrt{[\hat{V}]}}$ and note that

$$\frac{\varepsilon^{\frac{2}{3}}\frac{\partial}{\partial\bar{\xi}}[\hat{V}]}{2\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}} + \frac{\varepsilon^{\frac{2}{3}}\frac{\partial}{\partial\bar{\xi}}[\hat{V}]}{2\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}} + O\left(\varepsilon^{\frac{4}{3}}\right) = \mathcal{O}\left(\varepsilon^{\frac{2}{3}}\right) \neq 0, \quad \varepsilon \to 0,$$

and $\varepsilon^{-\frac{1}{3}}\eta_{\pm}(\bar{\xi})$ are nonzero constant when $\eta_{\pm}(\bar{\xi}) \neq 0$. Therefore, we get

$$\begin{aligned}\left|\frac{\partial}{\partial\bar{\xi}} g_1(\varepsilon,\bar{\xi})\right| &\geqslant \left|\left|k_{m+1}\alpha'(\bar{\xi}) \pm \frac{\varepsilon^{\frac{1}{3}}\frac{\partial[\hat{V}]}{\partial\bar{\xi}}}{2\sqrt{[\hat{V}]}}\right| - \left|\frac{\varepsilon^{\frac{2}{3}}\frac{\partial}{\partial\bar{\xi}}[\hat{V}]}{2\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}} + \frac{\varepsilon^{\frac{2}{3}}\frac{\partial}{\partial\bar{\xi}}[\hat{V}]}{2\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}} + O\left(\varepsilon^{\frac{4}{3}}\right)\right|\right| \\ &= \varepsilon^{\frac{1}{3}}\left|\left|\varepsilon^{-\frac{1}{3}}\eta_{\pm}(\bar{\xi})\right| - \left|\mathcal{O}\left(\varepsilon^{\frac{1}{3}}\right)\right|\right|\end{aligned}$$

and

$$\left|\frac{\partial}{\partial\bar{\xi}} g_1(\varepsilon,\bar{\xi})\right| \geqslant \begin{cases} |\mathcal{O}(\varepsilon^{\frac{2}{3}})| > 0, & \text{if } \eta_{\pm}(\bar{\xi}) = 0, \\ \hat{C}\varepsilon^{\frac{2}{3}}, & \text{if } \eta_{\pm}(\bar{\xi}) \neq 0 \end{cases} \tag{A.4}$$

provided that ε is small enough. By using Lemma 7.8 in [32], we have

$$\operatorname{meas} R^{10}_{jd,k} \leqslant \frac{2\varrho \operatorname{meas} \hat{J}}{\hat{C}(|k|+1)^{2m+6}}. \tag{A.5}$$

Note that $\operatorname{meas} R^{10}_{jd,k}$ is empty when $\min\{j,d\} > 1 + |k||\tilde{\omega}(\bar{\xi})|$. Therefore,

$$\begin{aligned}\operatorname{meas} J^{10} &= \operatorname{meas} \bigcup_{0\neq k\in\mathbb{Z}^{m+1}} \bigcup_{jd} \mathcal{R}^{10}_{jd,k} \\ &\leqslant \sum_{0\neq k\in\mathbb{Z}^{m+1}} \left(1+|k|\left|\tilde{\omega}(\bar{\xi})\right|\right)^2 \frac{2\varrho \operatorname{meas} \hat{J}}{\hat{C}(|k|+1)^{2m+6}} \\ &\leqslant \hat{C}\varrho \operatorname{meas} \hat{J} \sum_{0\neq k\in\mathbb{Z}^{m+1}} \frac{1}{|k|^{m+2}}.\end{aligned}$$

Let $|k|_\infty := \max\{|k_1|, |k_2|, \ldots, |k_{m+1}|\}$. From the inequalities

$$|k|_\infty \leqslant |k| \leqslant (m+1)|k|_\infty,$$

and

$$\sum_{|k|_\infty = p} 1 \leqslant 2(m+1)(2p+1)^m,$$

we get

$$\begin{aligned}\operatorname{meas} J^{10} &\leqslant \hat{C}(m+1)\varrho \operatorname{meas} \hat{J} \sum_{p=1}^{\infty} (2p+1)^m p^{-(m+2)} \\ &= \hat{C}\varrho \operatorname{meas} \hat{J} \leqslant \hat{C}\varrho \operatorname{meas} \hat{J},\end{aligned}$$

by using of the convergence of $\sum_{p=1}^{\infty}(2p+1)^m p^{-(m+2)}$. Let

$$\bar{J}_0 = \hat{J} \setminus J^{10}.$$

We get

$$\operatorname{meas} \bar{J}_0 \geqslant \operatorname{meas} \hat{J}(1-\hat{C}\varrho). \tag{A.6}$$

Case 2.2. Let $g_{jd}(\varepsilon,\bar{\xi}) = \mu_j - \mu_d$. Assume $j, d > 0$ without loss of generality. Then

$$\begin{aligned} g_1(\varepsilon,\bar{\xi}) = g_{jd}(\varepsilon,\bar{\xi}) \mp \mu_0 + \langle k, \tilde{\omega}(\bar{\xi})\rangle = j - d + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j} - \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d} \\ \pm \varepsilon^{\frac{1}{3}}\sqrt{[\hat{V}]} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big) + \langle k, \tilde{\omega}(\bar{\xi})\rangle. \end{aligned}$$

In view of $j \pm d = 0$, we have only the case $j - d = 0$, that is $\mu_j - \mu_d = 0$. Therefore,

$$\big|g_1(\varepsilon,\bar{\xi})\big| \geqslant \big|\langle k, \tilde{\omega}(\bar{\xi})\rangle\big| - \big|\mathcal{O}\big(\varepsilon^{\frac{1}{3}}\big)\big| \geqslant \frac{\gamma}{|k|^{m+1}} - \hat{C}\varepsilon^{\frac{1}{3}} \geqslant \frac{\varepsilon^{\frac{2}{3}}\varrho \operatorname{meas} \hat{J}}{(|k|+1)^{2m+6}},$$

from which follows that the inequality (5.13) holds true. □

Proof of Lemma 5.2. Let

$$\delta(\mu) = \pm\sqrt{i^2+\mu} \pm \sqrt{j^2+\mu} \pm \sqrt{d^2+\mu}$$

with $\mu = \varepsilon^{\frac{2}{3}}[\hat{V}]$.

For $i \pm j \pm d = 0$, we can show that

$$\big|\delta(\mu)\big| \geqslant \hat{C}\varepsilon^{\frac{2}{3}}. \tag{A.7}$$

In fact, we may restrict to positive integers such that $i \leqslant j \leqslant d$. The condition $i \pm j \pm d = 0$ then reduces to two possibilities, either $i + j - d = 0$ or $i - j + d = 0$. We have to study $\delta(\mu)$ for all possible combinations of plus and minus signs. To do this, we distinguish them according to their number of plus and minus signs.

(1) No minus sign.

$$\begin{aligned} \big|\delta(\mu)\big| &= \big|\sqrt{i^2+\mu} + \sqrt{j^2+\mu} + \sqrt{d^2+\mu}\big| \\ &\geqslant 3\varepsilon^{\frac{1}{3}}\sqrt{[\hat{V}]} \geqslant \hat{C}\varepsilon^{\frac{2}{3}}. \end{aligned}$$

(2) One minus sign. To simplify the notation, let us call

$$\delta_{+-+} = \sqrt{i^2+\mu} - \sqrt{j^2+\mu} + \sqrt{d^2+\mu}.$$

We have the following terms:

$$\delta_{+-+}, \delta_{-++} \geqslant \delta_{++-},$$

so it suffices to study $\delta(\mu) = \delta_{++-}$. We notice that

$$\delta(0) = i + j - d \geqslant 0$$

and

$$\delta'(\mu) = \frac{1}{2\sqrt{i^2+\mu}} + \frac{1}{2\sqrt{j^2+\mu}} - \frac{1}{2\sqrt{d^2+\mu}} \geqslant \frac{1}{2\sqrt{i^2+\mu}} > 0.$$

Thus, we get

$$\delta(\mu) \geqslant \int_0^{\mu} \frac{1}{2\sqrt{i^2+s}}\,ds \geqslant \frac{\mu}{2\sqrt{i^2+\mu}} \geqslant \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{2i\sqrt{2}} \geqslant \hat{C}\varepsilon^{\frac{2}{3}}$$

provided that ε is small enough.

(3) Two and three minus signs. These ones reduce to the case (2) and (1), respectively.

Let us write $g_2(\varepsilon,\bar{\xi}) = \pm\mu_i \pm \mu_j \pm \mu_d$, then we have

$$\big|g_2(\varepsilon,\bar{\xi})\big| \geqslant \hat{C}\varepsilon^{\frac{2}{3}}. \tag{A.8}$$

In fact,

$$\begin{aligned}\big|g_2(\varepsilon,\bar{\xi})\big| &= |\pm\mu_i \pm \mu_j \pm \mu_d| \\ &= \pm\Big(\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\Big) \\ &\quad \pm\Big(\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\Big) \\ &\quad \pm\Big(\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\Big) \\ &\geqslant \big|\delta(\mu)\big| - \big|\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\big| \geqslant \hat{C}\varepsilon^{\frac{2}{3}} - \hat{C}\varepsilon^{\frac{4}{3}} \geqslant \hat{C}\varepsilon^{\frac{2}{3}}.\end{aligned}$$

Therefore, (5.15) is true for $k=0$ provided that ε and ϱ are small enough.

Now let us consider the case $k \neq 0$. We distinguish two cases:

Case 1. $g_2(\varepsilon,\bar{\xi}) = \mu_i + \mu_j + \mu_d$. Let

$$f(\bar{\xi}) = g_2(\varepsilon,\bar{\xi}) + \big\langle k, \tilde{\omega}(\bar{\xi})\big\rangle,$$

$$\mathcal{R}^{11}_{ijd,k} = \left\{\bar{\xi} \in \hat{J}\colon\ \big|f(\bar{\xi})\big| < \frac{\varepsilon^{\frac{2}{3}}\varrho\,\mathrm{meas}\,\hat{J}}{C_*(|k|+1)^{2m+6}}\right\}$$

and

$$J^{11} = \bigcup_{0\neq k\in\mathbb{Z}^{m+1}} \bigcup_{i,j,d} \mathcal{R}^{11}_{ijd,k}. \tag{A.9}$$

We have known that $|\mathcal{O}(\varepsilon^{\frac{4}{3}}) + \langle k, \tilde{\omega}(\bar{\xi})\rangle| \leqslant 1 + |k||\tilde{\omega}(\bar{\xi})|$ holds when ε small enough. Thus, when $i > 1 + |k||\tilde{\omega}(\bar{\xi})|$, we have

$$\begin{aligned}\left|f(\bar{\xi})\right| &= \left|\mu_i+\mu_j+\mu_d+\langle k,\tilde{\omega}(\bar{\xi})\rangle\right| \\ &= \left|\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)+\langle k,\tilde{\omega}(\bar{\xi})\rangle\right| \\ &\geqslant i+j+d-\left|\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)+\langle k,\tilde{\omega}(\bar{\xi})\rangle\right| \\ &\geqslant 2\big(1+|k|\big|\tilde{\omega}(\bar{\xi})\big|\big),\end{aligned}$$

which implies the set $R^{11}_{ijd,k}$ is empty. So, we only need to consider the case $1\leqslant i,j,d\leqslant 1+[\![|k||\tilde{\omega}(\bar{\xi})|]\!]$ in order to calculate meas J^{11}. Since

$$\begin{aligned}f(\bar{\xi})| &= \mu_i+\mu_j+\mu_d+\langle k,\tilde{\omega}(\bar{\xi})\rangle \\ &= \left(\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{i,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right) \\ &\quad+\left(\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{j,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right) \\ &\quad+\left(\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{d,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right) \\ &\quad+\sum_{i=1}^{m}k_i\omega_i+k_{m+1}\alpha(\bar{\xi}),\end{aligned}$$

we get

$$\begin{aligned}\frac{\partial}{\partial\bar{\xi}}f(\bar{\xi}) &= \left(\frac{\varepsilon^{\frac{2}{3}}\frac{\partial[\hat{V}]}{\partial\bar{\xi}}}{2\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\frac{\partial}{\partial\bar{\xi}}\tilde{\lambda}_{i,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right) \\ &\quad+\left(\frac{\varepsilon^{\frac{2}{3}}\frac{\partial[\hat{V}]}{\partial\bar{\xi}}}{2\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\frac{\partial}{\partial\bar{\xi}}\tilde{\lambda}_{j,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right) \\ &\quad+\left(\frac{\varepsilon^{\frac{2}{3}}\frac{\partial[\hat{V}]}{\partial\bar{\xi}}}{2\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\frac{\partial}{\partial\bar{\xi}}\tilde{\lambda}_{d,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right)+k_{m+1}\alpha'(\bar{\xi}).\end{aligned}\tag{A.10}$$

Thus, when $k_{m+1}=0$, we get

$$\begin{aligned}\left|\frac{d}{d\bar{\xi}}f(\bar{\xi})\right| &\geqslant \frac{\varepsilon^{\frac{2}{3}}\frac{\partial[\hat{V}]}{\partial\bar{\xi}}}{2\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}}+\frac{\varepsilon^{\frac{2}{3}}\frac{\partial[\hat{V}]}{\partial\bar{\xi}}}{2\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}}+\frac{\varepsilon^{\frac{2}{3}}\frac{\partial[\hat{V}]}{\partial\bar{\xi}}}{2\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}}-\left|\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\right|-|k_{m+1}|\left|\alpha'(\bar{\xi})\right| \\ &\geqslant \frac{\varepsilon^{\frac{2}{3}}\frac{\partial[\hat{V}]}{\partial\bar{\xi}}}{2\sqrt{(1+[\![|k||\tilde{\omega}(\bar{\xi})|]\!])^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}}-\left|\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\right|\geqslant\frac{\varepsilon^{\frac{2}{3}}I_2}{2\sqrt{2}(1+[\![|k||\tilde{\omega}(\bar{\xi})|]\!])^2}-C\varepsilon^{\frac{4}{3}} \\ &\geqslant \hat{C}\varepsilon^{\frac{2}{3}},\end{aligned}\tag{A.11}$$

and when $k_{m+1} \neq 0$, we get

$$\begin{aligned} \left| \frac{d}{d\bar{\xi}} f(\bar{\xi}) \right| &\geqslant |k_{m+1}| \left| \alpha'(\bar{\xi}) \right| - \left| \mathcal{O}\left(\varepsilon^{\frac{2}{3}}\right) \right| \\ &\geqslant |k_{m+1}| C_3(\bar{\xi}) \varepsilon^{\frac{1}{3}} - C\varepsilon^{\frac{2}{3}} \\ &\geqslant \hat{C}\varepsilon^{\frac{2}{3}}, \end{aligned} \tag{A.12}$$

here $C_3(\bar{\xi})$ is defined in (4.59). Hence, for $1 \leqslant i, j, d \leqslant 1 + [\![|k||\tilde{\omega}(\bar{\xi})|]\!]$, from (A.11) and (A.12) we have

$$\operatorname{meas} R^{11}_{ijd,k} \leqslant \frac{2\varrho \operatorname{meas} \hat{J}}{\hat{C} C_* (|k|+1)^{2m+6}} \tag{A.13}$$

by using Lemma 7.8 in [32]. It follows that from (A.9) and (A.13)

$$\begin{aligned} \operatorname{meas} J^{11} &= \operatorname{meas} \bigcup_{0 \neq k \in \mathbb{Z}^{m+1}} \bigcup_{i,j,d} \mathcal{R}^{11}_{ijd,k} \\ &\leqslant \sum_{0 \neq k \in \mathbb{Z}^{m+1}} \left(1 + \left[\left[|k| \left|\tilde{\omega}(\bar{\xi})\right|\right]\right]\right)^3 \frac{2\varepsilon^{\frac{5}{3}} \varrho \operatorname{meas} \hat{J}}{\hat{C} C_* (|k|+1)^{2m+6}} \\ &\leqslant \hat{C} \varrho \operatorname{meas} \hat{J} \sum_{0 \neq k \in \mathbb{Z}^{m+1}} \frac{1}{|k|^{m+2}}. \end{aligned}$$

Let $|k|_\infty := \max\{|k_1|, |k_2|, \ldots, |k_{m+1}|\}$. From the inequalities

$$|k|_\infty \leqslant |k| \leqslant (m+1)|k|_\infty,$$

and

$$\sum_{|k|_\infty = p} 1 \leqslant 2(m+1)(2p+1)^m,$$

we get

$$\begin{aligned} \operatorname{meas} J^{11} &\leqslant \hat{C}(m+1)\varrho \operatorname{meas} \hat{J} \sum_{p=1}^{\infty} (2p+1)^m p^{-(m+2)} \\ &= \hat{C} \varrho \operatorname{meas} \hat{J} \leqslant \hat{C} \varrho \operatorname{meas} \hat{J} \end{aligned}$$

by using of the convergence of $\sum_{p=1}^{\infty} (2p+1)^m p^{-(m+2)}$.

Case 2. $\hat{g}(\varepsilon, \bar{\xi}) = \mu_i + \mu_j - \mu_d$. Let

$$\begin{aligned} \hat{f}(\bar{\xi}) &= \hat{g}(\varepsilon, \bar{\xi}) + \langle k, \tilde{\omega}(\bar{\xi}) \rangle, \\ \mathcal{R}^{12}_{ijd,k} &= \left\{ \bar{\xi} \in \hat{J} \colon \left| \hat{f}(\bar{\xi}) \right| < \frac{\varepsilon^{\frac{2}{3}} \varrho \operatorname{meas} \hat{J}}{C_* (|k|+1)^{2m+2}} \right\}, \\ \mathcal{R}^{13}_{ijd,k} &= \left\{ \bar{\xi} \in \hat{J} \colon \left| \hat{f}(\bar{\xi}) \right| < \frac{\varepsilon^{\frac{2}{3}} \varrho \operatorname{meas} \hat{J}}{C_* (|k|+1)^{2m+6}} \right\}, \end{aligned}$$

and

$$J^{13} = \bigcup_{0 \neq k \in \mathbb{Z}^{m+1}} \bigcup_{i,j,d} \mathcal{R}^{13}_{ijd,k}. \tag{A.14}$$

Case 2.1. $i + j - d = 0$.

$$\begin{aligned}
\hat{f}(\bar{\xi}) &= \mu_i + \mu_j - \mu_d + \langle k, \tilde{\omega}(\bar{\xi}) \rangle \\
&= \left(i + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + i} + \sum_{k=2}^{\infty} \varepsilon^{\frac{2k}{3}} \tilde{\lambda}_{i,k}\big(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) \right)) \\
&\quad + \left(j + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + j} + \sum_{k=2}^{\infty} \varepsilon^{\frac{2k}{3}} \tilde{\lambda}_{j,k}\big(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) \right) \\
&\quad - \left(d + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + d} + \sum_{k=2}^{\infty} \varepsilon^{\frac{2k}{3}} \tilde{\lambda}_{d,k}\big(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) \right) + \langle k, \tilde{\omega}(\bar{\xi}) \rangle \\
&= \langle k, \tilde{\omega}(\bar{\xi}) \rangle + \mathcal{O}\big(\varepsilon^{\frac{2}{3}}\big).
\end{aligned}$$

This implies the set $\mathcal{R}^{13}_{ijd,k}$ is empty.

Case 2.2. $i - j + d = 0$. If $d > \hat{d} := (C_d \operatorname{meas} \hat{J})^{-1}[\hat{V}](|k|+1)^{m+2}$ for $C_d\sqrt{C_*} = \varrho$, then

$$\left| \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + j} \right|, \left| \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + d} \right| < \frac{\varepsilon^{\frac{2}{3}} \varrho \operatorname{meas} \hat{J}}{\sqrt{C_*}(|k|+1)^{m+2}}.$$

It follows that

$$\begin{aligned}
\mathcal{R}^{12}_{ijd,k} \subset \Bigg\{ \bar{\xi} \in \hat{J} \colon \Bigg| 2i &+ \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + i} + \sum_{k=2}^{\infty} \varepsilon^{\frac{2k}{3}} \tilde{\lambda}_{i,k}\big(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) + \sum_{k=2}^{\infty} \varepsilon^{\frac{2k}{3}} \tilde{\lambda}_{j,k}\big(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) \\
&- \sum_{k=2}^{\infty} \varepsilon^{\frac{2k}{3}} \tilde{\lambda}_{d,k}\big(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) + \langle k, \tilde{\omega}(\bar{\xi}) \rangle \Bigg| < \frac{3\varepsilon^{\frac{2}{3}} \varrho \operatorname{meas} \hat{J}}{\sqrt{C_*}(|k|+1)^{m+2}} \Bigg\} := \mathcal{R}^{121}_{i,k}.
\end{aligned}$$

Let

$$\begin{aligned}
\tilde{f}_1(\bar{\xi}) = 2i &+ \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + i} + \sum_{k=2}^{\infty} \varepsilon^{\frac{2k}{3}} \tilde{\lambda}_{i,k}\big(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) + \sum_{k=2}^{\infty} \varepsilon^{\frac{2k}{3}} \tilde{\lambda}_{j,k}\big(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) \\
&- \sum_{k=2}^{\infty} \varepsilon^{\frac{2k}{3}} \tilde{\lambda}_{d,k}\big(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) + \langle k, \tilde{\omega}(\bar{\xi}) \rangle.
\end{aligned}$$

Then, when $k_{m+1} = 0$,

$$\left|\frac{\partial}{\partial\bar{\xi}}\tilde{f}_1(\bar{\xi})\right| = \left|\varepsilon^{\frac{2}{3}}\frac{\partial}{\partial\bar{\xi}}\left(\frac{[\hat{V}]}{\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+i}\right) + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big) + k_{m+1}\alpha'(\bar{\xi})\right| \geqslant \hat{C}\varepsilon^{\frac{2}{3}},$$

and when $k_{m+1} \neq 0$,

$$\left|\frac{\partial}{\partial\bar{\xi}}\tilde{f}_1(\bar{\xi})\right| \geqslant |k_{m+1}|\big|\alpha'(\bar{\xi})\big| - \big|\mathcal{O}\big(\varepsilon^{\frac{2}{3}}\big)\big| \geqslant |k_{m+1}|C_3(\bar{\xi})\varepsilon^{\frac{1}{3}} - C\varepsilon^{\frac{2}{3}} \geqslant \hat{C}\varepsilon^{\frac{2}{3}}$$

provided that ε is small enough. Thus, we get

$$\operatorname{meas}\mathcal{R}^{121}_{i,k} \leqslant \frac{6\varrho\operatorname{meas}\hat{J}}{\hat{C}\sqrt{C_*}(|k|+1)^{m+2}}.$$

Moreover,

$$\operatorname{meas}\bigcup_k\bigcup_{j=i+d,d>\hat{d}}\mathcal{R}^{121}_{ijd,k} \leqslant \operatorname{meas}\bigcup_k\mathcal{R}^{121}_{i,k} \leqslant \sum_k\frac{6\varrho\operatorname{meas}\hat{J}}{\hat{C}\sqrt{C_*}(|k|+1)^{m+2}} \leqslant \hat{C}\varrho\operatorname{meas}\hat{J}.$$

Now let us consider $d \leqslant \hat{d}$. By $j = i + d$ we can write

$$\begin{aligned}\hat{f}(\bar{\xi}) = {} & \left(j-d+\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{(j-d)^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j-d}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{j-d,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right)\\ & +\left(j+\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{j,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right)\\ & -\left(d+\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{d,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right)+\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle.\end{aligned}$$

Writing

$$\begin{aligned}\tilde{f}_2(\bar{\xi}) = {} & \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{(j-d)^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j-d}+\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j}\\ & -\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{j-d,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{j,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\\ & -\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{d,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)+\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle,\end{aligned}$$

we have

$$\big|\tilde{f}_2(\bar{\xi})\big| = \big|o\big(\varepsilon^{\frac{2}{3}}\big)+\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle\big| \leqslant 1+|k|\big|\tilde{\omega}(\bar{\xi})\big|$$

and

$$\mathcal{R}_{ijd,k}^{13} \subset \left\{ \bar{\xi} \in \hat{J}\colon \left|2(j-d) + \tilde{f}_2(\bar{\xi})\right| < \frac{\varepsilon^{\frac{2}{3}} \varrho \operatorname{meas} \hat{J}}{C_*(|k|+1)^{2m+4}} \right\} := \mathcal{R}_{ijd,k}^{131}.$$

From $j-d = i \leqslant d$, it follows that

$$\frac{1}{j-d} + \frac{1}{j} - \frac{1}{d} > \frac{1}{j} > 0.$$

When $k_{m+1} = 0$, we get

$$\begin{aligned} \left|\frac{\partial}{\partial\bar{\xi}} \hat{f}(\bar{\xi})\right| &= \left| \frac{\varepsilon^{\frac{2}{3}} \frac{\partial}{\partial\bar{\xi}}[\hat{V}]}{2\sqrt{(j-d)^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]}} + \frac{\varepsilon^{\frac{2}{3}} \frac{\partial}{\partial\bar{\xi}}[\hat{V}]}{2\sqrt{j^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]}} - \frac{\varepsilon^{\frac{2}{3}} \frac{\partial}{\partial\bar{\xi}}[\hat{V}]}{2\sqrt{d^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]}} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big) + k_{m+1}\alpha'(\bar{\xi}) \right| \\ &\geqslant \frac{\frac{\partial}{\partial\bar{\xi}}[\hat{V}]}{2} \left(\frac{1}{j-d} + \frac{1}{j} - \frac{1}{d} \right) \varepsilon^{\frac{2}{3}} \geqslant \hat{C} \varepsilon^{\frac{2}{3}} \end{aligned}$$

and when $k_{m+1} \neq 0$, we get

$$\left|\frac{\partial}{\partial\bar{\xi}} \hat{f}(\bar{\xi})\right| \geqslant |k_{m+1}| \left|\alpha'(\bar{\xi})\right| - \left|\mathcal{O}\big(\varepsilon^{\frac{2}{3}}\big)\right| \geqslant |k_{m+1}| C_3(\bar{\xi}) \varepsilon^{\frac{1}{3}} - C\varepsilon^{\frac{2}{3}} \geqslant \hat{C}\varepsilon^{\frac{2}{3}}$$

provided that ε is small enough.

Thus, we have

$$\operatorname{meas} \mathcal{R}_{ijd,k}^{131} \leqslant \frac{2\varrho \operatorname{meas} \hat{J}}{\hat{C} C_*(|k|+1)^{2m+4}}.$$

Further

$$\begin{aligned} \operatorname{meas} \bigcup_k \bigcup_{j=i+d, d \leqslant \hat{d}} \mathcal{R}_{ijd,k}^{13} &\leqslant \sum_k \big((C_d \operatorname{meas} \hat{J})^{-1} [\hat{V}] (|k|+1)^{m+2} \big) \cdot \frac{2\varrho \operatorname{meas} \hat{J}}{\hat{C} C_*(|k|+1)^{2m+4}} \\ &\leqslant \frac{\hat{C}}{\sqrt{C_* \varrho}} \cdot \varrho \operatorname{meas} \hat{J} \sum_k \frac{1}{(|k|)^{m+2}} \leqslant \hat{C} \varrho \operatorname{meas} \hat{J}, \end{aligned}$$

where $\hat{C}$ is small as C_* is large enough.

So, we have

$$\operatorname{meas} J^{13} \leqslant \hat{C} \varrho \operatorname{meas} \hat{J}$$

Let

$$\bar{J}_1 = \hat{J} \setminus \big(J^{11} \cup J^{13}\big).$$

Therefore, we can get

$$\text{meas}\,\bar{J}_1 > \text{meas}\,\hat{J}(1-\hat{C}\varrho)$$

provided that ϱ is small enough. $\square$

Proof of Lemma 5.3. Let

$$g_3(\varepsilon,\bar{\xi}) = \mu'_j + \mu'_d.$$

And assume that $j, d > 0$ without loss of generality. Then by (A.1)

$$\begin{aligned} g_3(\varepsilon,\bar{\xi}) &= \pm\big(\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\big) \pm \big(\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\big) \\ &= \pm\left(j + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j}\right) \pm \left(d + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d}\right) + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big). \end{aligned}$$

Case 1. $k=0$ and $g_3(\varepsilon,\bar{\xi}) \neq 0$.
Case 1.1. $g_3(\varepsilon,\bar{\xi}) = \mu_j + \mu_d$.
In this case, we have

$$\big|g_3(\varepsilon,\bar{\xi}) + \big\langle k, \tilde{\omega}(\bar{\xi})\big\rangle\big| \geqslant |j+d| - \big|\mathcal{O}\big(\varepsilon^{\frac{2}{3}}\big)\big| \geqslant \frac{1}{2},$$

from which follows that inequality (5.18) holds true.
Case 1.2. $g_3(\varepsilon,\bar{\xi}) = \mu_j - \mu_d$.
In this case, we have only $j-d \neq 0$, otherwise $g_3(\varepsilon,\bar{\xi}) = 0$. Therefore, we get

$$\big|g_3(\varepsilon,\bar{\xi}) + \big\langle k, \tilde{\omega}(\bar{\xi})\big\rangle\big| \geqslant |j-d| - \big|\mathcal{O}\big(\varepsilon^{\frac{2}{3}}\big)\big| \geqslant \frac{1}{2}.$$

Other cases can be reduced to case 1.1 or case 1.2.
Case 2. $k \neq 0$.
Case 2.1. If $\pm j \pm d = 0$, then

$$\begin{aligned} \big|g_3(\varepsilon,\bar{\xi}) + \big\langle k, \tilde{\omega}(\bar{\xi})\big\rangle\big| &\geqslant \big|\big\langle k, \tilde{\omega}(\bar{\xi})\big\rangle\big| - \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j} - \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d} - \big|\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\big| \\ &\geqslant \frac{\gamma}{|k|^{m+1}} - \big|\mathcal{O}\big(\varepsilon^{\frac{2}{3}}\big)\big| \geqslant \frac{\varepsilon^{\frac{2}{3}}\varrho\,\text{meas}\,\hat{J}}{(|k|+1)^{2m+6}}, \end{aligned}$$

from which follows also that inequality (5.18) holds true.
Case 2.2. Assume $\pm j \pm d \neq 0$. Let

$$\mathcal{R}^{20}_{jd,k} = \left\{\bar{\xi} \in \hat{J}\colon\ \big|g_3(\varepsilon,\bar{\xi}) + \big\langle k, \tilde{\omega}(\bar{\xi})\big\rangle\big| < \frac{\varepsilon^{\frac{2}{3}}\varrho\,\text{meas}\,\hat{J}}{(|k|+1)^{2m+6}}\right\}$$

and

$$J^{20} = \bigcup_{0\neq k\in\mathbb{Z}^{m+1}} \bigcup_{j,d} \mathcal{R}^{20}_{jd,k}. \tag{A.15}$$

Case 2.2.1. No minus sign. Let

$$f_3(\varepsilon,\bar{\xi}) = g_3(\varepsilon,\bar{\xi}) + \langle k, \tilde{\omega}(\bar{\xi})\rangle.$$

Then, when $k_{m+1} = 0$,

$$\left|\frac{\partial}{\partial\bar{\xi}} f_3(\varepsilon,\bar{\xi})\right| = \left|\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{2\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}} + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{2\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big) + k_{m+1}\alpha'(\bar{\xi})\right| \geqslant \hat{C}\varepsilon^{\frac{2}{3}},$$

when $k_{m+1} \neq 0$,

$$\left|\frac{\partial}{\partial\bar{\xi}} f_3(\varepsilon,\bar{\xi})\right| \geqslant |k_{m+1}|\big|\alpha'(\bar{\xi})\big| - \big|\mathcal{O}\big(\varepsilon^{\frac{2}{3}}\big)\big| \geqslant |k_{m+1}|C_3(\bar{\xi})\varepsilon^{\frac{1}{3}} - C\varepsilon^{\frac{2}{3}} \geqslant \hat{C}\varepsilon^{\frac{2}{3}}.$$

Case 2.2.2. One minus sign. Then, when $k_{m+1} = 0$,

$$\begin{aligned}\left|\frac{\partial}{\partial\bar{\xi}} f_3(\varepsilon,\bar{\xi})\right| &\geqslant \left|\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{2\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}} - \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{2\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}}\right| - \big|\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big) + k_{m+1}\alpha'(\bar{\xi})\big| \\ &\geqslant \hat{C}\varepsilon^{\frac{2}{3}} - \big|\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\big| \geqslant \hat{C}\varepsilon^{\frac{2}{3}}\end{aligned}$$

and when $k_{m+1} \neq 0$,

$$\left|\frac{\partial}{\partial\bar{\xi}} f_3(\varepsilon,\bar{\xi})\right| \geqslant |k_{m+1}|\big|\alpha'(\bar{\xi})\big| - \big|\mathcal{O}\big(\varepsilon^{\frac{2}{3}}\big)\big| \geqslant |k_{m+1}|C_3(\bar{\xi})\varepsilon^{\frac{1}{3}} - C\varepsilon^{\frac{2}{3}} \geqslant \hat{C}\varepsilon^{\frac{2}{3}}.$$

Case 2.2.3. Two minus signs. These ones reduce to the case 2.2.1.
By using Lemma 7.8 in [32], we have

$$\operatorname{meas} R^{20}_{jd,k} \leqslant \frac{2\varrho \operatorname{meas} \hat{J}}{\hat{C}(|k|+1)^{2m+6}}. \tag{A.16}$$

Moreover, note that $R^{20}_{jd,k}$ is empty when $\min\{j,d\} > 1 + |k||\tilde{\omega}(\bar{\xi})|$. Thus, it follows that from (A.15) and (A.16)

$$\begin{aligned}\operatorname{meas} J^{20} &= \operatorname{meas} \bigcup_{0\neq k\in\mathbb{Z}^{m+1}} \bigcup_{jd} \mathcal{R}^{20}_{jd,k} \\ &\leqslant \sum_{0\neq k\in\mathbb{Z}^{m+1}} \big(1+|k|\big|\tilde{\omega}(\bar{\xi})\big|\big)^2 \frac{2\varrho \operatorname{meas} \hat{J}}{\hat{C}(|k|+1)^{2m+6}} \\ &\leqslant \hat{C}\varrho \operatorname{meas} \hat{J} \sum_{0\neq k\in\mathbb{Z}^{m+1}} \frac{1}{|k|^{m+2}}.\end{aligned}$$

Similar to the proof of Lemma 5.1, we get

$$\operatorname{meas} J^{20} \leqslant \hat{C}(m+1)\varrho \operatorname{meas} \hat{J} \sum_{p=1}^{\infty} (2p+1)^m p^{-(m+2)} \leqslant \hat{C}\varrho \operatorname{meas} \hat{J},$$

by using of the convergence of $\sum_{p=1}^{\infty}(2p+1)^m p^{-(m+2)}$. Let

$$\bar{J}_2 = \hat{J} \setminus J^{20}.$$

We get

$$\operatorname{meas} \bar{J}_2 \geqslant \operatorname{meas} \hat{J}(1 - \hat{C}\varrho). \qquad \square \tag{A.17}$$

Proof of Lemma 5.4. Let

$$g_4(\varepsilon, \bar{\xi}) := 2\mu_0 \pm \left(\mu'_j + \mu'_d\right).$$

Then by (A.1)

$$\begin{aligned} g_4(\varepsilon, \bar{\xi}) &= 2\varepsilon^{\frac{1}{3}}\sqrt{[\hat{V}]} \pm \sqrt{j^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} \pm \sqrt{d^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + \mathcal{O}\left(\varepsilon^{\frac{4}{3}}\right) \\ &= 2\varepsilon^{\frac{1}{3}}\sqrt{[\hat{V}]} \pm \left(j + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + j}\right) \pm \left(d + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + d}\right) + \mathcal{O}\left(\varepsilon^{\frac{4}{3}}\right). \end{aligned}$$

Case 1. $k = 0$.
Case 1.1. If $\pm j \pm d \neq 0$, then

$$\left|g_4(\varepsilon, \bar{\xi}) \pm \langle k, \tilde{\omega}(\bar{\xi})\rangle\right| \geqslant |\pm j \pm d| - \left|\mathcal{O}\left(\varepsilon^{\frac{1}{3}}\right)\right| \geqslant \frac{1}{2},$$

from which follows that inequality (5.20) holds true.
Case 1.2. If $\pm j \pm d = 0$, then

$$\left|g_4(\varepsilon, \bar{\xi}) \pm \langle k, \tilde{\omega}(\bar{\xi})\rangle\right| \geqslant 2\varepsilon^{\frac{1}{3}}\sqrt{[\hat{V}]} - \left|\mathcal{O}\left(\varepsilon^{\frac{2}{3}}\right)\right| \geqslant \hat{C}\varepsilon^{\frac{2}{3}},$$

from which follows also that inequality (5.20) holds true.
Case 2. $k \neq 0$.
Case 2.1. If $\pm j \pm d = 0$, then

$$\begin{aligned} \left|g_4(\varepsilon, \bar{\xi}) \pm \langle k, \tilde{\omega}(\bar{\xi})\rangle\right| &\geqslant \left|\langle k, \tilde{\omega}(\bar{\xi})\rangle\right| - \left|\mathcal{O}\left(\varepsilon^{\frac{1}{3}}\right)\right| \\ &\geqslant \frac{\gamma}{|k|^{m+1}} - \left|\mathcal{O}\left(\varepsilon^{\frac{1}{3}}\right)\right| \geqslant \frac{\varepsilon^{\frac{2}{3}}\varrho \operatorname{meas} \hat{J}}{(|k|+1)^{2m+6}}, \end{aligned}$$

from which follows that inequality (5.20) holds true.
Case 2.2. Let $\pm j \pm d \neq 0$. And assume $j, d > 0$ without loss of generality. Let

$$\mathcal{R}^{30}_{jd,k} = \left\{\bar{\xi} \in \hat{J}\colon \left|g_4(\varepsilon, \bar{\xi}) \pm \langle k, \tilde{\omega}(\bar{\xi})\rangle\right| < \frac{\varepsilon^{\frac{2}{3}}\varrho \operatorname{meas} \hat{J}}{(|k|+1)^{2m+6}}\right\}$$

and

$$J^{30} = \bigcup_{0 \neq k \in \mathbb{Z}^{m+1}} \bigcup_{j,d} \mathcal{R}^{30}_{jd,k}. \tag{A.18}$$

Let

$$f_4(\varepsilon,\bar\xi)=g_4(\varepsilon,\bar\xi)\pm\big\langle k,\tilde\omega(\bar\xi)\big\rangle.$$

Then, when $k_{m+1}=0$,

$$\begin{aligned}\left|\frac{\partial}{\partial\bar\xi}f_4(\varepsilon,\bar\xi)\right|&=\left|\varepsilon^{\frac13}\frac{\frac{\partial}{\partial\bar\xi}[\hat V]}{2\sqrt{[\hat V]}}\pm\frac{\varepsilon^{\frac23}[\hat V]}{2\sqrt{j^2+\varepsilon^{\frac23}[\hat V]}}\pm\frac{\varepsilon^{\frac23}[\hat V]}{2\sqrt{d^2+\varepsilon^{\frac23}[\hat V]}}+\mathcal{O}\big(\varepsilon^{\frac43}\big)\pm k_{m+1}\alpha'(\bar\xi)\right|\\&\geqslant\varepsilon^{\frac13}\frac{\frac{\partial}{\partial\bar\xi}[\hat V]}{2\sqrt{[\hat V]}}-\big|\mathcal{O}\big(\varepsilon^{\frac23}\big)\big|\geqslant\hat C\varepsilon^{\frac23},\end{aligned}$$

when $k_{m+1}\neq0$, note that for cases $\mu_j'+\mu_d'=\mu_j\pm\mu_d$ it follows that

$$\pm\frac{\varepsilon^{\frac23}[\hat V]}{2\sqrt{j^2+\varepsilon^{\frac23}[\hat V]}}\pm\frac{\varepsilon^{\frac23}[\hat V]}{2\sqrt{d^2+\varepsilon^{\frac23}[\hat V]}}+\mathcal{O}\big(\varepsilon^{\frac43}\big)=\mathcal{O}\big(\varepsilon^{\frac23}\big)\neq0.$$

Writing $\hat\eta_\pm(\bar\xi):=\pm k_{m+1}\alpha'(\bar\xi)+\varepsilon^{\frac13}\frac{\frac{\partial}{\partial\bar\xi}[\hat V]}{2\sqrt{[\hat V]}}$, we have

$$\begin{aligned}\left|\frac{\partial}{\partial\bar\xi}f_4(\varepsilon,\bar\xi)\right|&\geqslant\left|\left|\pm k_{m+1}\alpha'(\bar\xi)+\varepsilon^{\frac13}\frac{\frac{\partial}{\partial\bar\xi}[\hat V]}{2\sqrt{[\hat V]}}\right|-\left|\pm\frac{\varepsilon^{\frac23}[\hat V]}{2\sqrt{j^2+\varepsilon^{\frac23}[\hat V]}}\pm\frac{\varepsilon^{\frac23}[\hat V]}{2\sqrt{d^2+\varepsilon^{\frac23}[\hat V]}}+\mathcal{O}\big(\varepsilon^{\frac43}\big)\right|\right|\\&\geqslant\varepsilon^{\frac13}\left|\varepsilon^{-\frac13}\big|\hat\eta_\pm(\bar\xi)\big|-\big|\mathcal{O}\big(\varepsilon^{\frac13}\big)\big|\right|\geqslant\hat C\varepsilon^{\frac23},\end{aligned}$$

$$\left|\frac{\partial}{\partial\bar\xi}f_4(\varepsilon,\bar\xi)\right|\geqslant\begin{cases}|\mathcal{O}(\varepsilon^{\frac23})|>0, & \text{if } \hat\eta_\pm(\bar\xi)=0,\\ \hat C\varepsilon^{\frac23}, & \text{if } \hat\eta_\pm(\bar\xi)\neq0\end{cases}\tag{A.19}$$

provided that ε is small enough. By using Lemma 7.8 in [32],

By using Lemma 7.8 in [32], we have

$$\operatorname{meas}R^{30}_{jd,k}\leqslant\frac{2\varrho\operatorname{meas}\hat J}{\hat C(|k|+1)^{2m+6}}.\tag{A.20}$$

Note that $R^{30}_{jd,k}$ is empty when $\min\{j,d\}>1+|k||\tilde\omega(\bar\xi)|$. Therefore, it follows that from (A.18) and (A.20)

$$\begin{aligned}\operatorname{meas}J^{30}&=\operatorname{meas}\bigcup_{0\neq k\in\mathbb{Z}^{m+1}}\bigcup_{jd}\mathcal{R}^{30}_{jd,k}\\&\leqslant\sum_{0\neq k\in\mathbb{Z}^{m+1}}\big(1+|k|\big|\tilde\omega(\bar\xi)\big|\big)^2\frac{2\varrho\operatorname{meas}\hat J}{\hat C(|k|+1)^{2m+6}}\\&\leqslant\hat C\varrho\operatorname{meas}\hat J\sum_{0\neq k\in\mathbb{Z}^{m+1}}\frac{1}{|k|^{m+2}}.\end{aligned}$$

Let

$$\bar{J}_3 = \hat{J} \setminus J^{30}.$$

Similar to the proof of Lemma 5.1, we get

$$\text{meas}\, \bar{J}_3 \geqslant \text{meas}\, \hat{J}(1 - \hat{C}\varrho). \qquad \square \tag{A.21}$$

Proof of Lemma 5.5. Let

$$g_5^{\pm}(\varepsilon, \bar{\xi}) := \mu_0 \pm \left(\mu_i' + \mu_j' + \mu_d'\right) \pm \left\langle k, \tilde{\omega}(\bar{\xi})\right\rangle.$$

Then by (A.1)

$$\begin{aligned} g_5^{\pm}(\varepsilon, \bar{\xi}) &= \varepsilon^{\frac{1}{3}}\sqrt{[\hat{V}]} \pm \left[\pm\sqrt{i^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} \pm \sqrt{j^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} \pm \sqrt{d^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]}\right] + \mathcal{O}\left(\varepsilon^{\frac{4}{3}}\right) \\ &= \varepsilon^{\frac{1}{3}}\sqrt{[\hat{V}]} \pm \left[\pm\left(i + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + i}\right) \pm \left(j + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + j}\right)\right. \\ &\qquad \left.\pm \left(d + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + d}\right)\right] + \mathcal{O}\left(\varepsilon^{\frac{4}{3}}\right) \pm \left\langle k, \tilde{\omega}(\bar{\xi})\right\rangle. \end{aligned}$$

Case 1. $k = 0$.
Case 1.1. If $\pm i \pm j \pm d = 0$, then

$$\left|g_5^{\pm}(\varepsilon, \bar{\xi})\right| \geqslant \varepsilon^{\frac{1}{3}}\sqrt{[\hat{V}]} - \left|\mathcal{O}\left(\varepsilon^{\frac{2}{3}}\right)\right| \geqslant \hat{C}\varepsilon^{\frac{1}{3}} \geqslant \frac{\varepsilon^{\frac{2}{3}}\varrho\, \text{meas}\, \hat{J}}{C_*(|k|+1)^{2m+6}},$$

from which follows that inequality (5.22) holds true.
Case 1.2. If $\pm i \pm j \pm d \neq 0$, then

$$\left|g_5^{\pm}(\varepsilon, \bar{\xi})\right| \geqslant |\pm i \pm j \pm d| - \left|\mathcal{O}\left(\varepsilon^{\frac{1}{3}}\right)\right| \geqslant 1 - \hat{C}\varepsilon^{\frac{1}{3}} \geqslant \frac{\varepsilon^{\frac{2}{3}}\varrho\, \text{meas}\, \hat{J}}{C_*(|k|+1)^{2m+6}},$$

from which follows also that inequality (5.22) holds true.
Case 2. $k \neq 0$. We distinguish two cases:
Case 2.1. Let $g_{ijd}(\varepsilon, \bar{\xi}) = \mu_i + \mu_j + \mu_d$. Assume $i, j, d > 0$ without loss of generality. Let

$$g_5^{\pm}(\varepsilon, \bar{\xi}) = \pm g_{ijd}(\varepsilon, \bar{\xi}) + \mu_0 \pm \left\langle k, \tilde{\omega}(\bar{\xi})\right\rangle,$$

$$\mathcal{R}_{ijd,k}^{41} = \left\{\bar{\xi} \in \hat{J}\colon \left|g_5^{\pm}(\varepsilon, \bar{\xi})\right| < \frac{\varepsilon^{\frac{2}{3}}\varrho\, \text{meas}\, \hat{J}}{C_*(|k|+1)^{2m+6}}\right\}$$

and

$$J^{41} = \bigcup_{0 \neq k \in \mathbb{Z}^{m+1}} \bigcup_{i,j,d} \mathcal{R}_{ijd,k}^{41}. \tag{A.22}$$

We have known that $|\mathcal{O}(\varepsilon^{\frac{4}{3}})+\mu_0\mp\langle k,\tilde{\omega}(\bar{\xi})\rangle|\leqslant 1+|k||\tilde{\omega}(\bar{\xi})|$ holds when ε small enough. Thus, when $\min\{i,j,d\}>1+|k||\tilde{\omega}(\bar{\xi})|$, we have

$$\begin{aligned}
\left|g_5^{\pm}(\varepsilon,\bar{\xi})\right| &= \left|\pm(\mu_i+\mu_j+\mu_d)+\mu_0\pm\langle k,\tilde{\omega}(\bar{\xi})\rangle\right| \\
&= \left|\pm\left(\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}\right)+\mathcal{O}\left(\varepsilon^{\frac{4}{3}}\right)+\mu_0\pm\langle k,\tilde{\omega}(\bar{\xi})\rangle\right| \\
&\geqslant i+j+d-\left|\mathcal{O}\left(\varepsilon^{\frac{4}{3}}\right)+\mu_0\pm\langle k,\tilde{\omega}(\bar{\xi})\rangle\right| \\
&\geqslant 2\left(1+|k|\left|\tilde{\omega}(\bar{\xi})\right|\right),
\end{aligned}$$

which implies the set $R^{41}_{ijd,k}$ is empty. So, we only need to consider the case $1\leqslant i,j,d\leqslant 1+[\![|k||\tilde{\omega}(\bar{\xi})|]\!]$ in order to calculate meas J^{41}. Since

$$\begin{aligned}
g_5^{\pm}(\varepsilon,\bar{\xi}) &= \pm(\mu_i+\mu_j+\mu_d)+\mu_0\mp\langle k,\tilde{\omega}(\bar{\xi})\rangle \\
&= \pm\Bigg[\left(\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{i,k}\left(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\right)\right) \\
&\quad+\left(\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{j,k}\left(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\right)\right) \\
&\quad+\left(\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{d,k}\left(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\right)\right)\Bigg] \\
&\quad+\left(\varepsilon^{\frac{1}{3}}\sqrt{[\hat{V}]}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{0,k}\left(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\right)\right) \\
&\quad\pm\left(\sum_{i=1}^{m}k_i\omega_i+k_{m+1}\alpha(\bar{\xi})\right),
\end{aligned}$$

we get

$$\begin{aligned}
\frac{\partial}{\partial\bar{\xi}}g_5^{\pm}(\varepsilon,\bar{\xi}) &= \pm\Bigg[\frac{\varepsilon^{\frac{2}{3}}\frac{\partial[\hat{V}]}{\partial\bar{\xi}}}{2\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\frac{\partial}{\partial\bar{\xi}}\tilde{\lambda}_{i,k}\left(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\right) \\
&\quad+\frac{\varepsilon^{\frac{2}{3}}\frac{\partial[\hat{V}]}{\partial\bar{\xi}}}{2\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\frac{\partial}{\partial\bar{\xi}}\tilde{\lambda}_{j,k}\left(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\right) \\
&\quad+\frac{\varepsilon^{\frac{2}{3}}\frac{\partial[\hat{V}]}{\partial\bar{\xi}}}{2\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\frac{\partial}{\partial\bar{\xi}}\tilde{\lambda}_{d,k}\left(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\right)\Bigg] \\
&\quad+\varepsilon^{\frac{1}{3}}\frac{\partial}{\partial\bar{\xi}}\sqrt{[\hat{V}]}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\frac{\partial}{\partial\bar{\xi}}\tilde{\lambda}_{0,k}\left(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\right) \\
&\quad\pm k_{m+1}\alpha'(\bar{\xi}).
\end{aligned}\tag{A.23}$$

Thus, when $k_{m+1} = 0$, we have

$$\left|\frac{d}{d\bar{\xi}} g_5^{\pm}(\varepsilon, \bar{\xi})\right| \geqslant \varepsilon^{\frac{1}{3}} \frac{\partial}{\partial \bar{\xi}} \sqrt{[\hat{V}]} - \left|\mathcal{O}\big(\varepsilon^{\frac{2}{3}}\big)\right| \geqslant \hat{C} \varepsilon^{\frac{2}{3}}. \tag{A.24}$$

When $k_{m+1} \neq 0$, similar to the proof of (A.19), we can show that

$$\left|\frac{\partial}{\partial \bar{\xi}} g_5^{\pm}(\varepsilon, \bar{\xi})\right| \geqslant \begin{cases} |\mathcal{O}(\varepsilon^{\frac{2}{3}})| > 0, & \text{if } \hat{\eta}_{\pm}(\bar{\xi}) = 0, \\ \hat{C} \varepsilon^{\frac{2}{3}}, & \text{if } \hat{\eta}_{\pm}(\bar{\xi}) \neq 0 \end{cases} \tag{A.25}$$

provided that ε is small enough. Hence, for $1 \leqslant i, j, d \leqslant 1 + [\![|k||\tilde{\omega}(\bar{\xi})|]\!]$, from (A.24) we have

$$\operatorname{meas} R_{ijd,k}^{41} \leqslant \frac{2\varrho \operatorname{meas} \hat{J}}{\hat{C} C_*(|k|+1)^{2m+6}} \tag{A.26}$$

by using Lemma 7.8 in [32]. It follows that from (A.22) and (A.26)

$$\begin{aligned} \operatorname{meas} J^{41} &= \operatorname{meas} \bigcup_{0 \neq k \in \mathbb{Z}^{m+1}} \bigcup_{i,j,d} \mathcal{R}_{ijd,k}^{41} \\ &\leqslant \sum_{0 \neq k \in \mathbb{Z}^{m+1}} \big(1 + [\![|k| |\tilde{\omega}(\bar{\xi})|]\!]\big)^3 \frac{2\varrho \operatorname{meas} \hat{J}}{\hat{C} C_*(|k|+1)^{2m+6}} \\ &\leqslant \hat{C} \varrho \operatorname{meas} \hat{J} \sum_{0 \neq k \in \mathbb{Z}^{m+1}} \frac{1}{|k|^{m+2}}. \end{aligned}$$

Similar to the proof of Lemma 5.1, we get

$$\operatorname{meas} J^{41} \leqslant \hat{C} \varrho \operatorname{meas} \hat{J}.$$

Case 2.2. Let $g_{ijd}(\varepsilon, \bar{\xi}) = \mu_i + \mu_j - \mu_d$ and

$$\mathcal{R}_{ijd,k}^{42} = \left\{\bar{\xi} \in \hat{J}\colon \left|g_5^{\pm}(\varepsilon, \bar{\xi})\right| < \frac{\varepsilon^{\frac{2}{3}} \varrho \operatorname{meas} \hat{J}}{C_*(|k|+1)^{2m+2}}\right\}$$

and

$$\mathcal{R}_{ijd,k}^{43} = \left\{\bar{\xi} \in \hat{J}\colon \left|g_5^{\pm}(\varepsilon, \bar{\xi})\right| < \frac{\varepsilon^{\frac{2}{3}} \varrho \operatorname{meas} \hat{J}}{C_*(|k|+1)^{2m+6}}\right\}.$$

In view of $i \pm j \pm d = 0$, it suffices to investigate the following two cases:

Case 2.2.1. Let $i + j - d = 0$. From

$$\begin{aligned} g_5^{\pm}(\varepsilon, \bar{\xi}) &= \pm(\mu_i + \mu_j - \mu_d) + \mu_0 \pm \big\langle k, \tilde{\omega}(\bar{\xi}) \big\rangle \\ &= \pm\Bigg[\Bigg(i + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + i} + \sum_{k=2}^{\infty} \varepsilon^{\frac{2k}{3}} \tilde{\lambda}_{i,k}\big(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big)\Bigg) \end{aligned}$$

$$
\begin{aligned}
&+\left(j+\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{j,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right)\\
&-\left(d+\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{d,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right)\Bigg]\\
&+\left(\varepsilon^{\frac{1}{3}}\sqrt{[\hat{V}]}+\sum_{k=2}^{\infty}\varepsilon^{\frac{2k}{3}}\tilde{\lambda}_{0,k}\big(\bar{\xi},\tilde{\omega}(\bar{\xi}),\varepsilon\big)\right)\\
&\pm\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle,
\end{aligned}
$$

we get

$$
\big|g_5^{\pm}(\varepsilon,\bar{\xi})\big|\geqslant\big|\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle\big|-\big|\mathcal{O}\big(\varepsilon^{\frac{1}{3}}\big)\big|\geqslant\frac{\gamma}{|k|^{m+1}}-C\varepsilon^{\frac{1}{3}}.
$$

This implies the set $\mathcal{R}^{43}_{ijd,k}$ is empty.

Case 2.2.2. $i-j+d=0$. And assume $i,j,d>0$ without loss of generality. If $d>\hat{d}:=(C_d\,\mathrm{meas}\,\hat{J})^{-1}[\hat{V}](|k|+1)^{2m+2}$ for $C_d\sqrt{C_*}=\varrho$, then

$$
\left|\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j}\right|,\left|\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d}\right|<\frac{\varepsilon^{\frac{2}{3}}\varrho\,\mathrm{meas}\,\hat{J}}{\sqrt{C_*}(|k|+1)^{2m+2}}.
$$

It follows that

$$
\begin{aligned}
\mathcal{R}^{42}_{ijd,k}\subset{}&\left\{\bar{\xi}\in\hat{J}:\ \left|\pm\left(2i+\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+i}\right)+\varepsilon^{\frac{1}{3}}[\hat{V}]+\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\pm\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle\right|<\frac{3\varepsilon^{\frac{2}{3}}\varrho\,\mathrm{meas}\,\hat{J}}{\sqrt{C_*}(|k|+1)^{2m+2}}\right\}\\
&:=\mathcal{R}^{042}_{ijd,k}.
\end{aligned}
$$

Let

$$
\tilde{f}_{\pm}(\varepsilon,\bar{\xi})=\pm\left(2i+\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+i}\right)+\varepsilon^{\frac{1}{3}}[\hat{V}]+\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\mp\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle,
$$

then when $k_{m+1}=0$ we have

$$
\begin{aligned}
\left|\frac{\partial}{\partial\bar{\xi}}\tilde{f}_{\pm}(\varepsilon,\bar{\xi})\right|&=\left|\pm\varepsilon^{\frac{2}{3}}\frac{\partial}{\partial\bar{\xi}}\left(\frac{[\hat{V}]}{\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+i}\right)+\varepsilon^{\frac{1}{3}}\frac{\partial}{\partial\bar{\xi}}[\hat{V}]+\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\mp k_{m+1}\alpha'(\bar{\xi})\right|\\
&\geqslant\varepsilon^{\frac{1}{3}}\frac{\partial}{\partial\bar{\xi}}[\hat{V}]-\left|\pm\varepsilon^{\frac{2}{3}}\frac{\partial}{\partial\bar{\xi}}\left(\frac{[\hat{V}]}{\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+i}\right)+\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\right|\\
&\geqslant\varepsilon^{\frac{1}{3}}\frac{\partial}{\partial\bar{\xi}}[\hat{V}]-\hat{C}\varepsilon^{\frac{2}{3}}\geqslant\hat{C}\varepsilon^{\frac{2}{3}}
\end{aligned}
$$

and when $k_{m+1}\neq 0$ we have by the same proof as (A.25)

$$\left|\frac{\partial}{\partial\bar\xi}\tilde f_{\pm}(\varepsilon,\bar\xi)\right| \geqslant \begin{cases} |\mathcal{O}(\varepsilon^{\frac{2}{3}})|>0, & \text{if } \hat\eta_{\pm}(\bar\xi)=0,\\ \hat C\varepsilon^{\frac{2}{3}}, & \text{if } \hat\eta_{\pm}(\bar\xi)\neq 0\end{cases} \tag{A.27}$$

provided that ε is small enough. Thus, we get

$$\operatorname{meas}\mathcal{R}^{042}_{ijd,k} \leqslant \frac{2\varrho \operatorname{meas}\hat J}{\hat C\sqrt{C_*}(|k|+1)^{2m+2}}$$

by using Lemma 7.8 in [32].

For $i-j+d=0$ and $d>\hat d$, we have $R^{42}_{ijd,k}\subset\mathcal{R}^{042}_{ijd,k}$. Thus, we get

$$\left(\bigcup_{j=i+d,d>\hat d}\mathcal{R}^{42}_{ijd,k}\right)\subset\mathcal{R}^{042}_{ijd,k}.$$

Note that $\mathcal{R}^{042}_{ijd,k}$ is empty when $i>\frac{1}{2}(1+|k||\tilde\omega(\bar\xi)|)$, if letting $J^{42}=\bigcup_k\bigcup_i\bigcup_{j=i+d,d>\hat d}\mathcal{R}^{42}_{ijd,k}$, then

$$\operatorname{meas}J^{42}\leqslant\sum_k\left(\frac{1}{2}\big(1+|k|\big|\tilde\omega(\bar\xi)\big|\big)\right)\frac{2\varrho\operatorname{meas}\hat J}{\hat C\sqrt{C_*}(|k|+1)^{2m+2}}\leqslant\hat C\varrho\operatorname{meas}\hat J.$$

Now let us consider $d\leqslant\hat d$. By $j=i+d$ we can write

$$\begin{aligned} g_5^{\pm}(\varepsilon,\bar\xi) &= \pm(\mu_i+\mu_j-\mu_d)+\mu_0\pm\langle k,\tilde\omega(\bar\xi)\rangle\\ &=\pm\Bigg[2(j-d)+\frac{\varepsilon^{\frac{2}{3}}[\hat V]}{\sqrt{(j-d)^2+\varepsilon^{\frac{2}{3}}[\hat V]}+(j-d)}+\frac{\varepsilon^{\frac{2}{3}}[\hat V]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat V]}+j}\\ &\quad-\frac{\varepsilon^{\frac{2}{3}}[\hat V]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat V]}+d}\Bigg]+\varepsilon^{\frac{1}{3}}\sqrt{[\hat V]}+\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\pm\langle k,\tilde\omega(\bar\xi)\rangle. \end{aligned}$$

Writing

$$\begin{aligned} \tilde f_5(\bar\xi) &=\pm\Bigg[\frac{\varepsilon^{\frac{2}{3}}[\hat V]}{\sqrt{(j-d)^2+\varepsilon^{\frac{2}{3}}[\hat V]}+(j-d)}+\frac{\varepsilon^{\frac{2}{3}}[\hat V]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat V]}+j}\\ &\quad-\frac{\varepsilon^{\frac{2}{3}}[\hat V]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat V]}+d}\Bigg]+\varepsilon^{\frac{1}{3}}\sqrt{[\hat V]}+\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\pm\langle k,\tilde\omega(\bar\xi)\rangle, \end{aligned}$$

we have

$$\big|\tilde f_5(\bar\xi)\big|=\big|\mathcal{O}\big(\varepsilon^{\frac{1}{3}}\big)\pm\langle k,\tilde\omega(\bar\xi)\rangle\big|\leqslant 1+|k|\big|\tilde\omega(\bar\xi)\big|$$

and

$$\mathcal{R}^{42}_{ijd,k}\subset\left\{\bar\xi\in\hat J\colon\ \big|\pm2(j-d)+\tilde f_5(\bar\xi)\big|<\frac{\varepsilon^{\frac{2}{3}}\varrho\operatorname{meas}\hat J}{C_*(|k|+1)^{2m+3}}\right\}:=\mathcal{R}^{142}_{ijd,k}.$$

Furthermore, we have

$$\left|\pm 2(j-d)+\tilde{f}_5(\bar{\xi})\right| \geqslant 2(j-d)-\left|\tilde{f}_5(\bar{\xi})\right| \geqslant 2(j-d)-\big(1+|k|\left|\tilde{\omega}(\bar{\xi})\right|\big)$$

which implies the set $\mathcal{R}^{131}_{ijd,k}$ is empty when $j-d>\frac{1}{2}(1+|k||\tilde{\omega}(\bar{\xi})|)$. So, we only need to consider the case $j-d\leqslant\frac{1}{2}(1+|k||\tilde{\omega}(\bar{\xi})|)$ and $1\leqslant d\leqslant\hat{d}$. Therefore, when $k_{m+1}=0$ we get

$$\begin{aligned}
\left|\frac{\partial}{\partial\bar{\xi}}g_5^{\pm}(\varepsilon,\bar{\xi})\right| &= \left|\frac{\partial}{\partial\bar{\xi}}\tilde{f}_5(\bar{\xi})\right| \\
&= \Bigg|\pm\Bigg[\frac{\varepsilon^{\frac{2}{3}}\frac{\partial}{\partial\bar{\xi}}[\hat{V}]}{2\sqrt{(j-d)^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}}+\frac{\varepsilon^{\frac{2}{3}}\frac{\partial}{\partial\bar{\xi}}[\hat{V}]}{2\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}}-\frac{\varepsilon^{\frac{2}{3}}\frac{\partial}{\partial\bar{\xi}}[\hat{V}]}{2\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}}\Bigg] \\
&\quad +\varepsilon^{\frac{1}{3}}\frac{\frac{\partial}{\partial\bar{\xi}}[\hat{V}]}{2\sqrt{[\hat{V}]}}+\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\pm k_{m+1}\alpha'(\bar{\xi})\Bigg| \\
&\geqslant \varepsilon^{\frac{1}{3}}\frac{\frac{\partial}{\partial\bar{\xi}}[\hat{V}]}{2\sqrt{[\hat{V}]}}-\hat{C}\varepsilon^{\frac{2}{3}}\geqslant\hat{C}\varepsilon^{\frac{2}{3}}
\end{aligned}$$

and when $k_{m+1}\neq 0$ we have by the same proof as (A.19)

$$\left|\frac{\partial}{\partial\bar{\xi}}g_5^{\pm}(\varepsilon,\bar{\xi})\right|\geqslant\begin{cases}|\mathcal{O}(\varepsilon^{\frac{2}{3}})|>0, & \text{if } \hat{\eta}_{\pm}(\bar{\xi})=0,\\ \hat{C}\varepsilon^{\frac{2}{3}}, & \text{if } \hat{\eta}_{\pm}(\bar{\xi})\neq 0\end{cases}\tag{A.28}$$

provided that ε is small enough. Thus, we have

$$\operatorname{meas}\mathcal{R}^{142}_{ijd,k}\leqslant\frac{2\varrho\operatorname{meas}\hat{J}}{\hat{C}C_*(|k|+1)^{2m+3}}.$$

If letting

$$J^{43}=\bigcup_k\bigcup_{j=i+d,d\leqslant\hat{d}}\mathcal{R}^{42}_{ijd,k},$$

then

$$\operatorname{meas}J^{43}\leqslant\sum_k\big((C_d\operatorname{meas}\hat{J})^{-1}[\hat{V}]\big(|k|+1\big)^{2m+2}\big)\cdot\frac{2\varrho\operatorname{meas}\hat{J}}{\hat{C}C_*(|k|+1)^{2m+3}}\leqslant\hat{C}\varrho\operatorname{meas}\hat{J},$$

where $\hat{C}$ is small as C_* is large enough.

Let

$$\bar{J}_4=\hat{J}\setminus\big(J^{42}\cup J^{43}\big).$$

Therefore, we can get

$$\operatorname{meas}\bar{J}_4>\operatorname{meas}\hat{J}(1-\hat{C}\varrho)$$

provided that ϱ is small enough. □

Proof of Lemma 5.6.

Case 1. Assume $i \pm j \pm d \pm l = 0$, $(i, j, d, l) \in \mathcal{L}_n \setminus \mathcal{N}_n$. Let

$$\hat{\delta}(\mu) = \pm\sqrt{i^2+\mu} \pm \sqrt{j^2+\mu} \pm \sqrt{d^2+\mu} \pm \sqrt{l^2+\mu}$$

with $\mu = \varepsilon^{\frac{2}{3}}[\hat{V}]$. From Lemma 4 in [5], we have

$$\big|\hat{\delta}(\mu)\big| \geqslant \frac{\mu}{(\sqrt{h^2+\mu})^3}, \quad h = \min\big\{|i|, |j|, |d|, |l|\big\}.$$

Let

$$h(\varepsilon, \bar{\xi}) := \pm\mu_i \pm \mu_j \pm \mu_d \pm \mu_l + \big\langle k, \tilde{\omega}(\bar{\xi})\big\rangle$$

and recall (4.3)

$$\mu_j = \sqrt{j^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + \sum_{k=2}^{\infty} \varepsilon^{\frac{2k}{3}} \tilde{\lambda}_{j,k}\big(\bar{\xi}, \tilde{\omega}(\bar{\xi}), \varepsilon\big) = \sqrt{j^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big). \tag{A.29}$$

Case 1.1. Assume $k = 0$. then

$$\big|h(\varepsilon, \bar{\xi})\big| \geqslant \frac{\mu}{(\sqrt{h^2+\mu})^3} - \big|\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\big| \geqslant \frac{\varepsilon^{\frac{2}{3}} \varrho \operatorname{meas} \hat{J}}{C_*(|k|+1)^{2m+6}} \tag{A.30}$$

provided that ε and ϱ are small enough. he inequality (A.30) holds true in the case.

Case 1.2. Assume $k \neq 0$. And assume $i, j, d, l > 0$ and $i = \min\{i, j, d, l\}$ without loss of generality. Then $1 \leqslant i \leqslant n$. We write

$$\big\langle k, \tilde{\omega}(\bar{\xi})\big\rangle = \sum_{\upsilon=1}^{m} k_\upsilon \omega_\upsilon + k_{m+1}\alpha(\bar{\xi}).$$

Case 1.2.1. Assume $f_{jdl}(\varepsilon, \bar{\xi}) = \pm(\mu_j + \mu_d + \mu_l)$. Then

$$\begin{aligned}
h(\varepsilon, \bar{\xi}) &= f_{jdl}(\varepsilon, \bar{\xi}) + \big(\pm\mu_i + \big\langle k, \tilde{\omega}(\bar{\xi})\big\rangle\big) \\
&= \pm\big(\sqrt{j^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + \sqrt{d^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} \\
&\quad + \sqrt{l^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]}\big) + \big(\pm\sqrt{i^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + \big\langle k, \tilde{\omega}(\bar{\xi})\big\rangle\big) + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big) \\
&= \pm\left(j + d + l + \frac{\varepsilon^{\frac{2}{3}} \frac{\partial[\hat{V}]}{\partial\bar{\xi}}}{\sqrt{j^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + j} + \frac{\varepsilon^{\frac{2}{3}} \frac{\partial[\hat{V}]}{\partial\bar{\xi}}}{\sqrt{d^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + d} + \frac{\varepsilon^{\frac{2}{3}} \frac{\partial[\hat{V}]}{\partial\bar{\xi}}}{\sqrt{l^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + l}\right) \\
&\quad + \left(\pm i \pm \frac{\varepsilon^{\frac{2}{3}} \frac{\partial[\hat{V}]}{\partial\bar{\xi}}}{\sqrt{i^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + i} + \big\langle k, \tilde{\omega}(\bar{\xi})\big\rangle\right) + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big).
\end{aligned}$$

Let

$$\mathcal{R}^{21}_{ijdl,k} = \left\{ \bar{\xi} \in \hat{J}\colon \left| h(\varepsilon, \bar{\xi}) \right| < \frac{\varepsilon^{\frac{2}{3}} \varrho \operatorname{meas} \hat{J}}{C_*(|k|+1)^{2m+6}} \right\}$$

and

$$J^{21} = \bigcup_{0 \neq k \in \mathbb{Z}^{m+1}} \bigcup_{i,j,d,l} \mathcal{R}^{21}_{ijdl,k}. \tag{A.31}$$

First of all, we have $|\pm \mu_i + \langle k, \tilde{\omega}(\bar{\xi}) \rangle| \leqslant 1 + \sqrt{1 + (\frac{n}{2})^2} + |k||\tilde{\omega}(\bar{\xi})|$ if $1 \leqslant i \leqslant \frac{n}{2}$. Thus, the set $R^{21}_{ijdl,k}$ is empty when $|f_{jdl}(\varepsilon, \bar{\xi})| \geqslant 2 + \sqrt{1 + (\frac{n}{2})^2} + |k||\tilde{\omega}(\bar{\xi})|$. If $l \geqslant \frac{n}{2} + 2 + |k||\tilde{\omega}(\bar{\xi})|$, $d \geqslant n$ and $j \geqslant n$, then

$$\left| f_{jdl}(\varepsilon, \bar{\xi}) \right| \geqslant 2n + \frac{n}{2} + 2 + |k| \left| \tilde{\omega}(\bar{\xi}) \right| \geqslant 2 + \sqrt{1 + \left(\frac{n}{2} \right)^2} + |k| \left| \tilde{\omega}(\bar{\xi}) \right|$$

as $\varepsilon \ll 1$. So, we only need to consider $1 \leqslant j$, $d < n$, $1 \leqslant l < \frac{n}{2} + 2 + |k||\tilde{\omega}(\bar{\xi})|$ and $\frac{n}{2} < i \leqslant n$ in order to calculate meas J^{21}. Since

$$\begin{aligned} \frac{\partial}{\partial \bar{\xi}} h(\varepsilon, \bar{\xi}) = \pm \left(\frac{\varepsilon^{\frac{2}{3}} \frac{\partial [\hat{V}]}{\partial \bar{\xi}}}{2\sqrt{j^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]}} + \frac{\varepsilon^{\frac{2}{3}} \frac{\partial [\hat{V}]}{\partial \bar{\xi}}}{2\sqrt{d^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]}} + \frac{\varepsilon^{\frac{2}{3}} \frac{\partial [\hat{V}]}{\partial \bar{\xi}}}{2\sqrt{l^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]}} \right) \\ \pm \frac{\varepsilon^{\frac{2}{3}} \frac{\partial [\hat{V}]}{\partial \bar{\xi}}}{2\sqrt{i^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]}} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big) + k_{m+1} \alpha'(\bar{\xi}), \end{aligned}$$

when $k_{m+1} = 0$ we get

$$\begin{aligned} \left| \frac{\partial}{\partial \bar{\xi}} h(\varepsilon, \bar{\xi}) \right| &\geqslant \frac{\varepsilon^{\frac{2}{3}} \frac{\partial [\hat{V}]}{\partial \bar{\xi}}}{2\sqrt{j^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]}} + \frac{\varepsilon^{\frac{2}{3}} \frac{\partial [\hat{V}]}{\partial \bar{\xi}}}{2\sqrt{d^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]}} + \frac{\varepsilon^{\frac{2}{3}} \frac{\partial [\hat{V}]}{\partial \bar{\xi}}}{2\sqrt{l^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]}} \\ &\quad - \frac{\varepsilon^{\frac{2}{3}} \frac{\partial [\hat{V}]}{\partial \bar{\xi}}}{2\sqrt{i^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]}} - \left| \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big) + k_{m+1} \alpha'(\bar{\xi}) \right| \geqslant \frac{\varepsilon^{\frac{2}{3}} \frac{\partial [\hat{V}]}{\partial \bar{\xi}}}{4} \left(\frac{1}{j} + \frac{1}{d} + \frac{1}{l} - \frac{1}{i} \right) \\ &\geqslant \frac{\varepsilon^{\frac{2}{3}} \frac{\partial [\hat{V}]}{\partial \bar{\xi}}}{4} \left(\frac{2}{n} - \frac{1}{i} + \frac{1}{l} \right) \\ &\geqslant \hat{C} \varepsilon^{\frac{2}{3}} \end{aligned}$$

and when $k_{m+1} \neq 0$ we get

$$\left| \frac{\partial}{\partial \bar{\xi}} h(\varepsilon, \bar{\xi}) \right| \geqslant \left| k_{m+1} \alpha'(\bar{\xi}) \right| - \left| \mathcal{O}\big(\varepsilon^{\frac{2}{3}}\big) \right| \geqslant |k_{m+1}| C_3(\bar{\xi}) \varepsilon^{\frac{1}{3}} - C \varepsilon^{\frac{2}{3}} \geqslant \hat{C} \varepsilon^{\frac{2}{3}}.$$

Therefore, we have

$$\operatorname{meas}\mathcal{R}^{21}_{ijdl,k} \leqslant \frac{2\varrho \operatorname{meas}\hat{J}}{\hat{C}C_*(|k|+1)^{2m+6}} \tag{A.32}$$

by using Lemma 7.8 in [32]. It follows that from (A.31) and (A.32)

$$\begin{aligned}\operatorname{meas} J^{21} &= \operatorname{meas} \bigcup_{0\neq k\in\mathbb{Z}^{m+1}} \bigcup_{i,j,d,l} \mathcal{R}^{21}_{ijdl,k} \\ &\leqslant \sum_{0\neq k\in\mathbb{Z}^{m+1}} n^3\left(\frac{n}{2}+2+|k|\left|\tilde{\omega}(\bar{\xi})\right|\right)\frac{2\operatorname{meas}\hat{J}}{\hat{C}C_*(|k|+1)^{2m+6}} \\ &\leqslant \hat{C}\varrho \operatorname{meas}\hat{J} \sum_{0\neq k\in\mathbb{Z}^{m+1}} \frac{1}{|k|^{m+2}}.\end{aligned}$$

Similar to the proof of Lemma 5.1, we get

$$\operatorname{meas} J^{21} \leqslant \hat{C}\varrho \operatorname{meas}\hat{J}.$$

Case 1.2.2. Assume $f_{jdl}(\varepsilon,\bar{\xi}) = \mu_j + \mu_d - \mu_l$. Then $h(\varepsilon,\bar{\xi}) = f_{jdl}(\varepsilon,\bar{\xi}) + (\pm\mu_i + \langle k, \tilde{\omega}(\bar{\xi})\rangle)$. Observe that there is a combination of plus and minus such that $i \pm j \pm d \pm l = 0$. It suffices to consider the following two cases (a) and (b) as follows.

Case (a). If $l = j - d \pm i$, then

$$h(\varepsilon,\bar{\xi}) = \sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]} + \sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]} - \sqrt{l^2+\varepsilon^{\frac{2}{3}}[\hat{V}]} \pm \sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big) + \big\langle k, \tilde{\omega}(\bar{\xi})\big\rangle.$$

From $l = j - d \pm i$, it follows that $l \leqslant j$.

Case (a1). Let $l = j$, then $i = d$. Therefore, we have

(i) $h(\varepsilon,\bar{\xi}) = \langle k, \tilde{\omega}(\bar{\xi})\rangle$ or

(ii) $h(\varepsilon,\bar{\xi}) = \langle k, \tilde{\omega}(\bar{\xi})\rangle + 2\mu_i$.

If $h(\varepsilon,\bar{\xi}) = \langle k, \tilde{\omega}(\bar{\xi})\rangle$, then the inequality (5.24) holds true. Now we consider the second case. Let

$$\mathcal{R}^{3,1}_{i,k} = \left\{\bar{\xi}\in\hat{J}\colon \left|2\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big) + \big\langle k, \tilde{\omega}(\bar{\xi})\big\rangle\right| < \frac{\varepsilon^{\frac{2}{3}}\varrho \operatorname{meas}\hat{J}}{C_*(|k|+1)^{2m+6}}\right\}$$

and

$$J^{3,0} = \bigcup_{0\neq k\in\mathbb{Z}^{m+1}} \bigcup_{i} \mathcal{R}^{3,1}_{i,k}.$$

When $k_{m+1} = 0$, we have

$$\left|\frac{\partial}{\partial\bar{\xi}} h(\varepsilon,\bar{\xi})\right| = \left|\frac{\varepsilon^{\frac{2}{3}}\frac{\partial}{\partial\bar{\xi}}[\hat{V}]}{\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big) + k_{m+1}\alpha'(\bar{\xi})\right| \geqslant \hat{C}\varepsilon^{\frac{2}{3}}$$

and when $k_{m+1} \neq 0$, we have

$$\left|\frac{\partial}{\partial \bar{\xi}} h(\varepsilon, \bar{\xi})\right| = \left|k_{m+1}\alpha'(\bar{\xi})\right| - \left|\mathcal{O}\big(\varepsilon^{\frac{2}{3}}\big)\right| \geqslant |k_{m+1}| C_3(\bar{\xi})\varepsilon^{\frac{1}{3}} - C\varepsilon^{\frac{2}{3}} \geqslant \hat{C}\varepsilon^{\frac{2}{3}}.$$

Therefore, we get

$$\operatorname{meas} \mathcal{R}_{i,k}^{3,1} \leqslant \frac{2\varrho \operatorname{meas} \hat{J}}{\hat{C} C_*(|k|+1)^{2m+6}}.$$

Hence,

$$\begin{aligned}
\operatorname{meas} J^{3,0} &\leqslant \operatorname{meas} \bigcup_{0 \neq k \in \mathbb{Z}^{m+1}} \bigcup_{i} \mathcal{R}_{i,k}^{3,1} \leqslant \sum_{0 \neq k \in \mathbb{Z}^{m+1}} n \frac{2\varrho \operatorname{meas} \hat{J}}{\hat{C} C_*(|k|+1)^{2m+6}} \\
&\leqslant \hat{C}\varrho \operatorname{meas} \hat{J} \sum_{0 \neq k \in \mathbb{Z}^{m+1}} \frac{1}{|k|^{m+2}} \leqslant \hat{C}\varrho \operatorname{meas} \hat{J}.
\end{aligned}$$

Case (a2). Let $l < j$, then $i < d$. Then

$$\begin{aligned}
h(\varepsilon, \bar{\xi}) &= \sqrt{j^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + \sqrt{d^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} - \sqrt{l^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} \pm \sqrt{i^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big) + \big\langle k, \tilde{\omega}(\bar{\xi})\big\rangle \\
&= 2d + [\![i]\!] + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + j} + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + d} - \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{l^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + l} \\
&\quad \pm \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2 + \varepsilon^{\frac{2}{3}}[\hat{V}]} + i} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big) + \big\langle k, \tilde{\omega}(\bar{\xi})\big\rangle,
\end{aligned}$$

where $[\![i]\!]$ is equal to $\pm 2i$ or 0. Obviously, there does not exist any difference in the measure estimate whether $[\![i]\!] = \pm 2i$ or $[\![i]\!] = 0$. Thereby, we only consider the case $[\![i]\!] = 0$ for convenience. Let

$$\mathcal{R}_{ijdl,k}^{3,1,1} = \left\{\bar{\xi} \in \hat{J}\colon \left|h(\varepsilon, \bar{\xi})\right| < \frac{\varepsilon^{\frac{2}{3}}\varrho \operatorname{meas} \hat{J}}{\sqrt{C_*}(|k|+1)^{m+2}}\right\},$$

$$\mathcal{R}_{ijdl,k}^{3,1,2} = \left\{\bar{\xi} \in \hat{J}\colon \left|h(\varepsilon, \bar{\xi})\right| < \frac{\varepsilon^{\frac{2}{3}}\varrho \operatorname{meas} \hat{J}}{C_*(|k|+1)^{2m+5}}\right\},$$

$$\mathcal{R}_{ijdl,k}^{3,1} = \left\{\bar{\xi} \in \hat{J}\colon \left|h(\varepsilon, \bar{\xi})\right| < \frac{\varepsilon^{\frac{2}{3}}\varrho \operatorname{meas} \hat{J}}{C_*(|k|+1)^{2m+6}}\right\}$$

and

$$J^{3,1} = \bigcup_{0 \neq k \in \mathbb{Z}^{m+1}} \bigcup_{i,j,d,l} \mathcal{R}_{ijdl,k}^{3,1}.$$

We consider the following two cases:

Case (a21). If $l > \hat{l} := (C_l \operatorname{meas} \hat{J})^{-1}[\hat{V}](|k|+1)^{m+2}$ for $C_l\sqrt{C_*} = \varrho$, then we have

$$\left|\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j}\right|, \left|\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{l^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+l}\right| < \frac{\varepsilon^{\frac{2}{3}}\varrho \operatorname{meas} \hat{J}}{\sqrt{C_*}(|k|+1)^{m+2}}.$$

It follows that

$$R^{3,1,1}_{ijdl,k} \subset \left\{\bar{\xi} \in \hat{J}\colon \left|2d + \langle k, \tilde{\omega}(\bar{\xi})\rangle + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d} \pm \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+i} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\right| < \frac{3\varepsilon^{\frac{2}{3}}\varrho \operatorname{meas} \hat{J}}{\sqrt{C_*}(|k|+1)^{m+2}}\right\} := R^{3,1,1}_{id,k},$$

and we obtain

$$\left(\bigcup_{l=j-d\pm i}\bigcup_{l>\hat{l}} \mathcal{R}^{3,1,1}_{ijdl,k}\right) \subset \mathcal{R}^{3,1,1}_{id,k}.$$

Writing

$$\tilde{h}(\varepsilon,\bar{\xi}) = 2d + \langle k, \tilde{\omega}(\bar{\xi})\rangle + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d} \pm \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+i} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big),$$

when $k_{m+1} = 0$, we have

$$\left|\frac{\partial}{\partial\bar{\xi}}\tilde{h}(\varepsilon,\bar{\xi})\right| = \left|\frac{\partial}{\partial\bar{\xi}}\left(\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+i}\right) - \frac{\partial}{\partial\bar{\xi}}\left(\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d}\right)\right| - \left|\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big) + k_{m+1}\alpha'(\bar{\xi})\right| \geqslant \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{4}\left(\frac{1}{i}-\frac{1}{d}\right) \geqslant \hat{C}\varepsilon^{\frac{2}{3}}$$

and when $k_{m+1} \neq 0$, we have

$$\left|\frac{\partial}{\partial\bar{\xi}}\tilde{h}(\varepsilon,\bar{\xi})\right| \geqslant \left|k_{m+1}\alpha'(\bar{\xi})\right| - \left|\mathcal{O}\big(\varepsilon^{\frac{2}{3}}\big)\right| \geqslant |k_{m+1}|C_3(\bar{\xi})\varepsilon^{\frac{1}{3}} - C\varepsilon^{\frac{2}{3}} \geqslant \hat{C}\varepsilon^{\frac{2}{3}}.$$

Therefore, we get

$$\operatorname{meas} R^{3,1,1}_{id,k} \leqslant \frac{6\varrho \operatorname{meas} \hat{J}}{\hat{C}\sqrt{C_*}(|k|+1)^{m+2}}.$$

For $d > 1 + |k||\tilde{\omega}(\bar{\xi})|$, it is obviously that the set $\mathcal{R}^{3,1,1}_{id,k}$ is empty. Let

$$J^{3,1,1} = \bigcup_{0\neq k\in\mathbb{Z}^m}\bigcup_{i,d}\bigcup_{l=j-d\pm i}\bigcup_{l>\hat{l}} \mathcal{R}^{3,1,1}_{ijdl,k} \subset \bigcup_{0\neq k\in\mathbb{Z}^m}\bigcup_{i,d} \mathcal{R}^{3,1,1}_{id,k},$$

then

$$\begin{aligned}
\operatorname{meas} J^{3,1,1} &\leqslant \operatorname{meas} \bigcup_{0\neq k\in\mathbb{Z}^{m+1}} \bigcup_{i,d} \mathcal{R}^{3,1,1}_{id,k} \\
&= \operatorname{meas} \bigcup_{0\neq k\in\mathbb{Z}^{m+1}} \bigcup_{1\leqslant i\leqslant n} \bigcup_{1\leqslant d\leqslant 1+|k||\tilde{\omega}(\bar{\xi})|} \mathcal{R}^{3,1,1}_{id,k} \\
&\leqslant \sum_{0\neq k\in\mathbb{Z}^{m+1}} n\big(1+|k|\big|\tilde{\omega}(\bar{\xi})\big|\big)\frac{6\varrho \operatorname{meas} \hat{J}}{\hat{C}\sqrt{C_*}(|k|+1)^{m+3}} \\
&\leqslant \hat{C}\varrho \operatorname{meas} \hat{J} \sum_{0\neq k\in\mathbb{Z}^{m+1}} \frac{1}{|k|^{m+2}} \leqslant \hat{C}\varrho \operatorname{meas} \hat{J}.
\end{aligned}$$

Case (a22). $1 \leqslant l \leqslant \hat{l}$. At this time, by $l = j - d \pm i$, we get

$$\begin{aligned}
\mathcal{R}^{3,1,2}_{ijdl,k} \subset \Bigg\{\bar{\xi} \in \hat{J}\colon \Bigg|&2d + \big\langle k, \tilde{\omega}(\bar{\xi})\big\rangle + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j} + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d} \\
&- \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{l^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+l} \pm \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+i} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\Bigg| < \frac{3\varepsilon^{\frac{2}{3}}\varrho \operatorname{meas} \hat{J}}{C_*(|k|+1)^{2m+5}}\Bigg\} \\
:&= \mathcal{R}^{3,1,2}_{ijd,k}.
\end{aligned}$$

From $l < j$ and $i < d$, it follows that

$$\frac{1}{j} + \frac{1}{d} - \frac{1}{l} \pm \frac{1}{i} \neq 0.$$

Writing

$$\begin{aligned}
\bar{h}(\varepsilon, \bar{\xi}) = 2d + \big\langle k, \tilde{\omega}(\bar{\xi})\big\rangle &+ \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j} + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d} \\
&- \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{l^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+l} \pm \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+i} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big),
\end{aligned}$$

when $k_{m+1} = 0$, we have

$$\begin{aligned}
\left|\frac{\partial}{\partial\bar{\xi}}\bar{h}(\varepsilon, \bar{\xi})\right| = \Bigg|&\frac{\partial}{\partial\bar{\xi}}\left(\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j}\right) + \frac{\partial}{\partial\bar{\xi}}\left(\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d}\right) \\
&- \frac{\partial}{\partial\bar{\xi}}\left(\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{l^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+l}\right) \pm \frac{\partial}{\partial\bar{\xi}}\left(\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+i}\right)\Bigg| \\
&- \big|\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big) + k_{m+1}\alpha'(\bar{\xi})\big| \geqslant \frac{\varepsilon^{\frac{2}{3}}\frac{\partial}{\partial\bar{\xi}}[\hat{V}]}{2}\left|\frac{1}{j} + \frac{1}{d} - \frac{1}{l} \pm \frac{1}{i}\right|
\end{aligned}$$

and when $k_{m+1} \neq 0$, we have

$$\left|\frac{\partial}{\partial \bar{\xi}} \bar{h}(\varepsilon, \bar{\xi})\right| \geqslant \left|k_{m+1}\alpha'(\bar{\xi})\right| - \left|\mathcal{O}\left(\varepsilon^{\frac{2}{3}}\right)\right| \geqslant |k_{m+1}| C_3(\bar{\xi})\varepsilon^{\frac{1}{3}} - C\varepsilon^{\frac{2}{3}} \geqslant \hat{C}\varepsilon^{\frac{2}{3}}.$$

Thus, we get

$$\left|\frac{\partial}{\partial \bar{\xi}} \bar{h}(\varepsilon, \bar{\xi})\right| \geqslant \hat{C}\varepsilon^{\frac{2}{3}}$$

provided that ε is small enough. Therefore, we get

$$\operatorname{meas} \mathcal{R}^{3,1,2}_{ijd,k} < \frac{6\varrho \operatorname{meas} \hat{J}}{\hat{C} C_*(|k|+1)^{2m+5}}.$$

For $d > 1 + |k||\tilde{\omega}(\bar{\xi})|$, it is obviously that the set $\mathcal{R}^{3,1,2}_{ijd,k}$ is empty. Let

$$\begin{aligned}
J^{3,1,2} &= \bigcup_{0\neq k\in\mathbb{Z}^{m+1}} \bigcup_{i,d} \bigcup_{l=j-d\pm i} \bigcup_{1\leqslant l\leqslant \hat{l}} \mathcal{R}^{3,1,2}_{ijdl,k} \\
&\subset \bigcup_{0\neq k\in\mathbb{Z}^{m+1}} \bigcup_{i,d} \bigcup_{1\leqslant l\leqslant \hat{l}} \mathcal{R}^{3,1,2}_{ijd,k},
\end{aligned}$$

then

$$\begin{aligned}
\operatorname{meas} J^{3,1,2} &\leqslant \operatorname{meas} \bigcup_{0\neq k\in\mathbb{Z}^{m+1}} \bigcup_{i,d} \bigcup_{1\leqslant l\leqslant \hat{l}} \mathcal{R}^{3,1,2}_{ijd,k} \\
&= \operatorname{meas} \bigcup_{0\neq k\in\mathbb{Z}^{m+1}} \bigcup_{1\leqslant i\leqslant n} \bigcup_{1\leqslant d\leqslant 1+|k||\tilde{\omega}(\bar{\xi})|} \bigcup_{1\leqslant l\leqslant \hat{l}} \mathcal{R}^{3,1,2}_{ijd,k} \\
&\leqslant \sum_{0\neq k\in\mathbb{Z}^{m+1}} n\left(1+|k|\left|\tilde{\omega}(\bar{\xi})\right|\right)\hat{j}\frac{6\varrho \operatorname{meas} \hat{J}}{\hat{C} C_*(|k|+1)^{2m+5}} \\
&\leqslant \hat{C}\varrho \operatorname{meas} \hat{J},
\end{aligned}$$

where $\hat{C}$ is small as C_* is large enough.

It is obvious that

$$J^{3,1} \subset \left(J^{3,1,1} \cup J^{3,1,2}\right),$$

therefore

$$\operatorname{meas} J^{3,1} \leqslant \operatorname{meas} J^{3,1,1} + \operatorname{meas} J^{3,1,2} \leqslant \hat{C}\varrho \operatorname{meas} \hat{J} + \hat{C}\varrho \operatorname{meas} \hat{J} \leqslant \hat{C}\varrho \operatorname{meas} \hat{J}.$$

Case (b). If $l = j + d \pm i$, then

$$\begin{aligned} h(\varepsilon,\bar{\xi}) &= \sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}-\sqrt{l^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}\pm\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)+\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle \\ &= [|i|]+\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j}+\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d} \\ &\quad -\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{l^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+l}\pm\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+i}+\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)+\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle, \end{aligned}$$

where $[|i|]$ is equal to either $\pm 2i$ or 0. If $[|i|]=0$, then

$$\big|h(\varepsilon,\bar{\xi})\big|\geqslant\big|\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle\big|-\big|\mathcal{O}\big(\varepsilon^{\frac{2}{3}}\big)\big|\geqslant\frac{\gamma}{|k|^{m+1}}-C\varepsilon^{\frac{2}{3}}\geqslant\frac{\varepsilon^{\frac{2}{3}}\varrho\,\mathrm{meas}\,\hat{J}}{C_*(|k|+1)^{2m+6}},$$

which follows that the inequality (5.24) holds true. Now we consider the case $[|i|]=\pm 2i$. Let

$$\begin{aligned} \mathcal{R}^{3,3,1}_{ijdl,k} &= \left\{\bar{\xi}\in\hat{J}\colon\ \big|h(\varepsilon,\bar{\xi})\big|<\frac{\varepsilon^{\frac{2}{3}}\varrho\,\mathrm{meas}\,\hat{J}}{\sqrt[4]{C_*}(|k|+1)^{2m+2}}\right\}, \\ \mathcal{R}^{3,3,2}_{ijdl,k} &= \left\{\bar{\xi}\in\hat{J}\colon\ \big|h(\varepsilon,\bar{\xi})\big|<\frac{\varepsilon^{\frac{2}{3}}\varrho\,\mathrm{meas}\,\hat{J}}{C_*(|k|+1)^{2m+5}}\right\}, \\ \mathcal{R}^{3,3}_{ijdl,k} &= \left\{\bar{\xi}\in\hat{J}\colon\ \big|h(\varepsilon,\bar{\xi})\big|<\frac{\varepsilon^{\frac{2}{3}}\varrho\,\mathrm{meas}\,\hat{J}}{C_*(|k|+1)^{2m+6}}\right\}, \\ J^{3,3} &= \bigcup_{0\neq k\in\mathbb{Z}^{m+1}}\ \bigcup_{i,j,d,l}\mathcal{R}^{3,3}_{ijdl,k}. \end{aligned}$$

We consider the following two cases:

Case (b1). From $l=j+d\pm i$, it follows that $l>j$ and $d>i$. If $j>\hat{j}:=(C_j\,\mathrm{meas}\,\hat{J})^{-1}[\hat{V}](|k|+1)^{2m+2}$ for $C_j\sqrt[4]{C_*}=\varrho$, then we have

$$\left|\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j}\right|,\ \left|\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{l^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+l}\right|<\frac{\varepsilon^{\frac{2}{3}}\varrho\,\mathrm{meas}\,\hat{J}}{\sqrt[4]{C_*}(|k|+1)^{2m+2}}.$$

It follows that, for fixed i,d,

$$\begin{aligned} R^{3,3,1}_{ijdl,k}\subset\Bigg\{\bar{\xi}\in\hat{J}\colon\ &\Bigg|\pm 2i+\big\langle k,\tilde{\omega}(\bar{\xi})\big\rangle+\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d} \\ &\pm\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+i}+\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\Bigg|<\frac{3\varepsilon^{\frac{2}{3}}\varrho\,\mathrm{meas}\,\hat{J}}{\sqrt[4]{C_*}(|k|+1)^{2m+2}}\Bigg\}:=R^{3,3,1}_{id,k}, \end{aligned}$$

and we obtain

$$\left(\bigcup_{l=j+d\pm i}\ \bigcup_{l>\hat{l}}\mathcal{R}^{3,3,1}_{ijdl,k}\right)\subset\mathcal{R}^{3,3,1}_{id,k}.$$

Writing

$$\tilde{h}(\varepsilon,\bar{\xi}) = \pm 2i + \langle k, \tilde{\omega}(\bar{\xi})\rangle + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d} \pm \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+i} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big),$$

when $k_{m+1}=0$, we have

$$\begin{aligned}\left|\frac{\partial}{\partial\bar{\xi}}\tilde{h}(\varepsilon,\bar{\xi})\right| &= \left|\frac{\partial}{\partial\bar{\xi}}\left(\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+i}\right) - \frac{\partial}{\partial\bar{\xi}}\left(\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d}\right)\right| - \left|\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big) + k_{m+1}\alpha'(\bar{\xi})\right| \\ &\geqslant \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{4}\left(\frac{1}{i}-\frac{1}{d}\right) \geqslant \hat{C}\varepsilon^{\frac{2}{3}}\end{aligned}$$

and when $k_{m+1}\neq 0$, we have

$$\left|\frac{\partial}{\partial\bar{\xi}}\tilde{h}(\varepsilon,\bar{\xi})\right| \geqslant \left|k_{m+1}\alpha'(\bar{\xi})\right| - \left|\mathcal{O}\big(\varepsilon^{\frac{2}{3}}\big)\right| \geqslant |k_{m+1}|C_3(\bar{\xi})\varepsilon^{\frac{1}{3}} - C\varepsilon^{\frac{2}{3}} \geqslant \hat{C}\varepsilon^{\frac{2}{3}}$$

provided that ε is small enough. Therefore, we get

$$\operatorname{meas} R^{3,3,1}_{id,k} \leqslant \frac{6\varrho \operatorname{meas}\hat{J}}{\hat{C}\sqrt[4]{C_*}(|k|+1)^{2m+2}}.$$

For $d>i>1+|k||\tilde{\omega}(\bar{\xi})|$, it is obviously that the set $\mathcal{R}^{3,3,1}_{id,k}$ is empty. Let

$$J^{3,3,1} = \bigcup_{0\neq k\in\mathbb{Z}^m}\bigcup_{i,d}\bigcup_{l=j+d\pm i}\bigcup_{j>\hat{j}}\mathcal{R}^{3,3,1}_{ijdl,k} \subset \bigcup_{0\neq k\in\mathbb{Z}^m}\bigcup_{i,d}\mathcal{R}^{3,3,1}_{id,k},$$

then

$$\begin{aligned}\operatorname{meas} J^{3,3,1} &\leqslant \operatorname{meas}\bigcup_{0\neq k\in\mathbb{Z}^{m+1}}\bigcup_{i,d}\mathcal{R}^{3,3,1}_{id,k} \\ &= \operatorname{meas}\bigcup_{0\neq k\in\mathbb{Z}^{m+1}}\bigcup_{1\leqslant i\leqslant n}\bigcup_{1\leqslant d\leqslant 1+|k||\tilde{\omega}(\bar{\xi})|}\mathcal{R}^{3,3,1}_{id,k} \\ &\leqslant \sum_{0\neq k\in\mathbb{Z}^{m+1}} n\big(1+|k|\big|\tilde{\omega}(\bar{\xi})\big|\big)\frac{6\varrho \operatorname{meas}\hat{J}}{\hat{C}\sqrt[4]{C_*}(|k|+1)^{2m+2}} \\ &\leqslant \hat{C}\varrho \operatorname{meas}\hat{J}\sum_{0\neq k\in\mathbb{Z}^{m+1}}\frac{1}{|k|^{m+2}} \\ &\leqslant \hat{C}\varrho \operatorname{meas}\hat{J}.\end{aligned}$$

Case (b2). $1\leqslant j\leqslant\hat{j}$. At this time, by $l=j+d\pm i$, we get

$$\mathcal{R}^{3,3,2}_{ijdl,k} \subset \Bigg\{\bar{\xi} \in \hat{J}\colon \Bigg|\pm 2i + \langle k, \tilde{\omega}(\bar{\xi})\rangle + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j} + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d}$$
$$- \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{l^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+l} \pm \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+i} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big)\Bigg| < \frac{3\varepsilon^{\frac{2}{3}}\varrho \operatorname{meas}\hat{J}}{C_*(|k|+1)^{2m+5}}\Bigg\}$$
$$:= \mathcal{R}^{3,3,2}_{ijd,k}.$$

From $j<l$ and $i<d$, it follows that

$$\frac{1}{j}+\frac{1}{d}-\frac{1}{l}\pm\frac{1}{i}\neq 0.$$

Writing

$$\bar{h}(\varepsilon,\bar{\xi}) = \pm 2i + \langle k, \tilde{\omega}(\bar{\xi})\rangle + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j} + \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d}$$
$$- \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{l^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+l} \pm \frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+i} + \mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big),$$

when $k_{m+1}=0$, we have

$$\left|\frac{\partial}{\partial\bar{\xi}}\bar{h}(\varepsilon,\bar{\xi})\right| = \left|\frac{\partial}{\partial\bar{\xi}}\left(\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{j^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+j}\right) + \frac{\partial}{\partial\bar{\xi}}\left(\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{d^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+d}\right)\right.$$
$$\left. - \frac{\partial}{\partial\bar{\xi}}\left(\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{l^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+l}\right) \pm \frac{\partial}{\partial\bar{\xi}}\left(\frac{\varepsilon^{\frac{2}{3}}[\hat{V}]}{\sqrt{i^2+\varepsilon^{\frac{2}{3}}[\hat{V}]}+i}\right)\right|$$
$$- \left|\mathcal{O}\big(\varepsilon^{\frac{4}{3}}\big) + k_{m+1}\alpha'(\bar{\xi})\right| \geqslant \frac{\varepsilon^{\frac{2}{3}}\frac{\partial}{\partial\bar{\xi}}[\hat{V}]}{4}\left|\frac{1}{j}+\frac{1}{d}-\frac{1}{l}\pm\frac{1}{i}\right|$$

and when $k_{m+1}\neq 0$, we have

$$\left|\frac{\partial}{\partial\bar{\xi}}\bar{h}(\varepsilon,\bar{\xi})\right| \geqslant \left|k_{m+1}\alpha'(\bar{\xi})\right| - \left|\mathcal{O}\big(\varepsilon^{\frac{2}{3}}\big)\right| \geqslant |k_{m+1}|C_3(\bar{\xi})\varepsilon^{\frac{1}{3}} - C\varepsilon^{\frac{2}{3}} \geqslant \hat{C}\varepsilon^{\frac{2}{3}}$$

provided that ε is small enough. Therefore, we get

$$\operatorname{meas}\mathcal{R}^{3,3,2}_{ijd,k} < \frac{6\varrho \operatorname{meas}\hat{J}}{\hat{C}C_*(|k|+1)^{2m+5}}.$$

For $d>i>1+|k||\tilde{\omega}(\bar{\xi})|$, it is obviously that the set $\mathcal{R}^{3,3,2}_{ijd,k}$ is empty. Let

$$J^{3,3,2} = \bigcup_{0\neq k\in\mathbb{Z}^{m+1}}\ \bigcup_{i,d}\ \bigcup_{l=j+d\pm i}\ \bigcup_{1\leqslant j\leqslant\hat{j}} \mathcal{R}^{3,3,2}_{ijdl,k} \subset \bigcup_{0\neq k\in\mathbb{Z}^{m+1}}\ \bigcup_{i,d}\ \bigcup_{1\leqslant j\leqslant\hat{j}} \mathcal{R}^{3,3,2}_{ijd,k},$$

then

$$\begin{aligned}
\operatorname{meas} J^{3,3,2} &\leqslant \operatorname{meas} \bigcup_{0\neq k\in\mathbb{Z}^{m+1}} \bigcup_{i,d} \bigcup_{1\leqslant j\leqslant \hat{j}} \mathcal{R}^{3,3,2}_{ijd,k} \\
&= \operatorname{meas} \bigcup_{0\neq k\in\mathbb{Z}^{m+1}} \bigcup_{1\leqslant i\leqslant n} \bigcup_{1\leqslant d\leqslant 1+|k||\tilde{\omega}(\bar{\xi})|} \bigcup_{1\leqslant j\leqslant \hat{j}} \mathcal{R}^{3,3,2}_{ijd,k} \\
&\leqslant \sum_{0\neq k\in\mathbb{Z}^{m+1}} n\big(1+|k|\big|\tilde{\omega}(\bar{\xi})\big|\big)\hat{j}\,\frac{6\varrho \operatorname{meas} \hat{J}}{\hat{C}C_*(|k|+1)^{2m+5}} \\
&\leqslant \frac{\hat{C}}{\sqrt[4]{C_*^3}\varrho}\cdot \varrho \operatorname{meas} \hat{J} \sum_{0\neq k\in\mathbb{Z}^{m+1}} \frac{1}{(|k|+1)^{m+2}} \\
&\leqslant \hat{C}\varrho \operatorname{meas} \hat{J},
\end{aligned}$$

where $\hat{C}$ is small as C_* is large enough.

It is obvious that

$$J^{3,3} \subset \big(J^{3,3,1} \cup J^{3,3,2}\big),$$

therefore

$$\operatorname{meas} J^{3,3} \leqslant \operatorname{meas} J^{3,3,1} + \operatorname{meas} J^{3,3,2} \leqslant \hat{C}\varrho \operatorname{meas} \hat{J} + \hat{C}\varrho \operatorname{meas} \hat{J} \leqslant \hat{C}\varrho \operatorname{meas} \hat{J}.$$

Case II. Assume $(i, j, d, l) \in \mathcal{N}_n$ and $k \neq 0$. In this case, obviously, we have

$$\big|\mu_i' + \mu_j' + \mu_d' + \mu_l' + \big\langle k, \tilde{\omega}(\bar{\xi})\big\rangle\big| \geqslant \frac{\gamma}{|k|^{m+1}} > \frac{\varepsilon^{\frac{2}{3}}\varrho \operatorname{meas} \hat{J}}{C_*(|k|+1)^{2m+6}}$$

provided ε small enough.

Let

$$\bar{J}_5 = \hat{J} \setminus \big(J^{2,1} \cup J^{3,0} \cup J^{3,1} \cup J^{3,3}\big),$$

then

$$\operatorname{meas} \bar{J}_5 \geqslant \operatorname{meas} \hat{J} \cdot (1 - \hat{C}\varrho).$$

Let

$$\underline{J} = \bar{J}_2 \cap \bar{J}_3 \cap \bar{J}_4 \cap \bar{J}_5,$$

then we get

$$\operatorname{meas} \mathscr{J} = \operatorname{meas}(\bar{J} \cap \underline{J}) \geqslant \operatorname{meas} \hat{J} \cdot (1 - \hat{C}\varrho)$$

provided that ε and ϱ are small enough. $\square$

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