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Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth ☆

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ABSTRACT

In this paper we concern with the multiplicity and concentration of positive solutions for the semilinear Kirchhoff type equation

$$\begin{cases} -\left(\varepsilon^2 a + b\varepsilon \displaystyle\int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + M(x)u = \lambda f(u) + |u|^4 u, & x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad u > 0, & x \in \mathbb{R}^3, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, a, b are positive constants and $\lambda > 0$ is a parameter, and f is a continuous superlinear and subcritical nonlinearity. Suppose that $M(x)$ has at least one minimum. We first prove that the system has a positive ground state solution u_ε for $\lambda > 0$ sufficiently large and $\varepsilon > 0$ sufficiently small. Then we show that u_ε converges to the positive ground state solution of the associated limit problem and concentrates to a minimum point of $M(x)$ in certain sense as $\varepsilon \to 0$. Moreover, some further properties of the ground state solutions are also studied. Finally, we investigate the relation between the number of positive solutions and the topology of the set of the global minima of the potentials by minimax theorems and the Ljusternik–Schnirelmann theory.

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1. Introduction and main results

In present paper, we investigate the multiplicity and concentration of positive solutions to a class of semilinear Kirchhoff type equation

$$(KH)_\varepsilon \quad \begin{cases} -\left(\varepsilon^2 a + b\varepsilon \int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + M(x)u = \lambda f(u) + |u|^4 u, & x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad u > 0, & x \in \mathbb{R}^3, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, a, b are positive constants, $\lambda > 0$ is a real parameter, and f is a continuous superlinear and subcritical nonlinearity.

In $(KH)_\varepsilon$, if $\varepsilon = 1$ and $M(x) = 0$, some mathematicians considered the following problem

$$\begin{cases} -\left(a + b \int_{\Omega} |\nabla u|^2\right)\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain. Such problems are often referred to as being nonlocal because of the presence of the term $(\int_\Omega |\nabla u|^2)\Delta u$ which implies that the equation in (1.1) is no longer a pointwise identity. This phenomenon provokes some mathematical difficulties, which make the study of such a class of problem particularly interesting. On the other hand, we have its physical motivation. Indeed, this problem is related to the stationary analogue of the equation

$$\begin{cases} u_{tt} - \left(a + b \int_{\Omega} |\nabla u|^2\right)\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where u denotes the displacement, $f(x,u)$ the external force and b the initial tension while a is related to the intrinsic properties of the string (such as Young's modulus). Equations of this type were first proposed by Kirchhoff in [23] to describe the transversal oscillations of a stretched string, particularly, taking into account the subsequent change in string length caused by oscillations. Problem (1.2) began to call attention of several researchers mainly after the work of Lions [28], where a functional analysis approach was proposed to attack it. We have to point out that nonlocal problems also appear in other fields as biological systems, where u describes a process which depends on the average of itself (for example, population density). See, for example, [24,25,27] and the references therein.

The solvability of the Kirchhoff type equations (1.1) and (1.2) has been well studied in general dimension by various authors; for example, see [25–27,29–33] and the references therein. In [30], Arosio and Panizzi studied the Cauchy–Dirichlet type problem related to (1.2) in the Hadamard sense as a special case of an abstract second-order Cauchy problem in a Hilbert space. Ma and Rivera [31] obtained positive solutions of such problems by using variational methods. Perera and Zhang [32] obtained a nontrivial solution of (1.1) via Yang index and critical group. He and Zou [33] obtained infinitely many solutions by using the local minimum methods and the fountain theorems. Recently, when f is a continuous superlinear nonlinearity with critical growth, the paper [26] proved the existence of positive solution for (1.1). More recently, the paper [27] considered Eq. (1.1) with concave and convex nonlinearities by using Nehari manifold and fibering map methods, and obtained the existence of multiple positive solutions.

We note that if $a = 1$, $b = 0$, $\mathbb{R}^3$ and $\lambda f(u) + |u|^{2^*-2}u$ are replaced by $\mathbb{R}^N$ and $f(x,u)$, respectively, $(KH)_\varepsilon$ is reduced to

$$(SH)_\varepsilon \quad \begin{cases} -\varepsilon^2 \Delta u + M(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^3), \quad u > 0 & x \in \mathbb{R}^N. \end{cases}$$

Eq. $(SH)_\varepsilon$ arises in different models. For example, they are involved with the existence of standing waves of the nonlinear Schrödinger equations

$$(SH)'_\varepsilon \quad i\varepsilon \frac{\partial \psi}{\partial t} = -\varepsilon^2 \Delta \psi + \big(M(x) + E\big)\psi - f\big(x, |\psi|\big)\psi.$$

Here a standing wave of $(SH)'_\varepsilon$ is a solution of the form $\psi(x,t) = u(x)e^{-iEt/\hbar}$, $u(x) \in \mathbb{R}$, where u is a solution of $(SH)_\varepsilon$. The existence and concentration behavior of the positive solutions of $(SH)_\varepsilon$ have been extensively studied in recent years, see for instance, [34–36,7,37] and the reference therein. Recently, He and Zou [3] considered the following equation

$$(SP)_\varepsilon \quad \begin{cases} -\Big(\varepsilon^2 a + b\varepsilon \int\limits_{\mathbb{R}^3} |\nabla u|^2\Big)\Delta u + V(x)u = f(u), & x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad u > 0 & x \in \mathbb{R}^3, \end{cases}$$

where f is a C^1 and subcritical function such that

$$\frac{f(s)}{s^3} \text{ is increasing on } (0,\infty), \qquad 0 < \mu F(s) = \mu \int_0^s f(t)\,dt \leqslant sf(s), \quad \mu > 4,$$

$$f'(s)s^2 - 3f(s)s \geqslant Cs^\sigma, \quad \sigma \in (4,6),\ C > 0, \quad \text{and} \quad f(s) = o\big(s^3\big) \quad \text{as } s \to 0. \tag{1.3}$$

By using Ljusternik–Schnirelmann theory (see [10]) and minimax methods, the author obtained the multiplicity of positive solutions, which concentrate on the minima of $V(x)$ as $\varepsilon \to 0$. This phenomenon of concentration is very interesting for both mathematicians and physicians. Moreover, as far as we know, the existence and concentration behavior of the positive solutions to $(KH)_\varepsilon$ with critical growth have not ever been studied by variational methods. So in this paper we shall fill this gap. Precisely, the goal of this paper is the following three points: (i) To find a family of positive ground state solutions for $(KH)_\varepsilon$ with some properties, such as concentration, exponent decay etc.; we also investigate the relation between the number of solutions and the topology of the set of the global minima of the potentials by minimax theorems and the Ljusternik–Schnirelmann theory, and some concentration phenomenon of positive solutions are also obtained. (ii) We obtain the sufficient conditions for the nonexistence of positive ground state solutions. (iii) We treat the critical case for $(KH)_\varepsilon$, i.e., the nonlinearity is allowed to be critical growth. Furthermore, the conditions on f are more general than [3]. Both of those will depend on the Nehari manifold methods [2] and minimax methods.

Before stating our theorems, we first give some assumptions.

$(\mathcal{C}_0)$ $M \in C(\mathbb{R}^3, \mathbb{R})$ such that $M_\infty = \liminf_{|x|\to\infty} M(x) > M_0 = \inf_{x\in\mathbb{R}^3} M(x) > 0$.
$(\mathcal{C}_1)$ $M \in C(\mathbb{R}^3, \mathbb{R})$ such that $0 < \inf_{x\in\mathbb{R}^3} M(x) = M^\infty = \liminf_{|x|\to\infty} M(x)$ and $M(x) \not\equiv M^\infty$.

The hypothesis $(\mathcal{C}_0)$ was first introduced by Rabinowitz [7] in the study of a nonlinear Schrödinger equation with the nonlinearity subcritical growth. In this paper, we shall also assume that $M_\infty < \infty$. This condition is made only for simplicity, since it is irrelevant to the goal of our paper. Actually, it is even easier to consider potentials which are large at infinity, since the energy space embeds compactly into Lebesgue spaces.

For the nonlinearity we assume f satisfies the following conditions:

($\mathcal{F}_1$) $f \in C(\mathbb{R}^3)$, $f(t) = o(t^3)$ as $t \to 0$, $f(t)t > 0$ for all $t \neq 0$ and $f(t) = 0$ for all $t \leqslant 0$;
($\mathcal{F}_2$) $\frac{f(t)}{t^3}$ is strictly increasing on interval $(0, \infty)$;
($\mathcal{F}_3$) $|f(t)| \leqslant c(1 + |t|^{p-1})$ for some $c > 0$, where $4 < p < 6$.

It follows from the conditions of ($\mathcal{F}_1$)–($\mathcal{F}_2$) that

$$F(u) > 0, \qquad 4F(u) < f(u)u, \quad \forall u \neq 0, \tag{1.4}$$

where $F(u) = \int_0^u f(s)\,ds$. Set

$$\mathcal{M} := \{x \in \mathbb{R}^3\colon M(x) = M_0\}.$$

Without loss of generality, below we assume $0 \in \mathcal{M}$, that is, $M(0) = M_0$. The limit problem associated with $(KH)_\varepsilon$ reads as

$$(\mathcal{H}_{M_0}) \quad -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + M_0 u = \lambda f(u) + |u|^4 u, \quad u \in H^1(\mathbb{R}^3).$$

Let

$$\mathcal{K}_{\varepsilon,\lambda}(u) := \frac{1}{2}\int_{\mathbb{R}^3} \left(a\varepsilon^2|\nabla u|^2 + M(x)|u|^2\right) + \frac{b\varepsilon}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 - \lambda\int_{\mathbb{R}^3} F(u) - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6$$

denote the energy function associated to Eq. $(KH)_\varepsilon$. Set

$$\ell_\varepsilon = \inf\{\mathcal{K}_{\varepsilon,\lambda}(u)\colon u \neq 0 \text{ is a solution of } (KH)_\varepsilon\}.$$

If $u^0 > 0$, and u^0 solves $(KH)_\varepsilon$, we say u^0 is a positive solution. A positive solution u^0 with $\ell_\varepsilon = \mathcal{K}_{\varepsilon,\lambda}(u^0)$ is called a positive ground state solution. Let $\mathcal{L}'_\varepsilon$ denote the set of all positive ground state solutions of Eq. $(KH)_\varepsilon$. We recall that, if Y is a closed subset of a topological space X, the Ljusternik–Schnirelmann category $cat_X(Y)$ is the least number of closed and contractible sets in X which cover Y.

Our main results are as follows.

Theorem 1.1. *Suppose that the assumptions ($\mathcal{C}_0$) and ($\mathcal{F}_1$)–($\mathcal{F}_3$) are satisfied. Then there exist $\lambda^* > 0$ and $\varepsilon^* > 0$ such that for each $\lambda \in [\lambda^*, \infty)$ and $\varepsilon \in (0, \varepsilon^*)$, we have that*

(i) *$(KH)_\varepsilon$ has one positive ground state solution u_ε in $H^1(\mathbb{R}^3)$;*
(ii) *$\mathcal{L}'_\varepsilon$ is compact in $H^1(\mathbb{R}^3)$;*
(iii) *there exists a maximum point x_ε of u_ε such that $\lim_{\varepsilon\to 0} dist(x_\varepsilon, \mathcal{M}) = 0$, and for any sequences of such x_ε, $h_\varepsilon(x) = u_\varepsilon(\varepsilon x + x_\varepsilon)$ uniformly converges to a positive ground state solution of $(\mathcal{H}_{M_0})$, as $\varepsilon \to 0$, where $u_\varepsilon \in \mathcal{L}'_\varepsilon$ denotes one of these positive ground state solutions;*
(iv) *$\lim_{|x|\to\infty} u_\varepsilon(x) = 0$, $\lim_{|x|\to\infty} |\nabla u_\varepsilon(x)| = 0$ and $u_\varepsilon \in C^{1,\sigma}_{loc}(\mathbb{R}^3)$ with $\sigma \in (0, 1)$. Furthermore, there exist constants $C, c > 0$ such that $|u_\varepsilon(x)| \leqslant Ce^{-\frac{c}{\varepsilon}|x - x_\varepsilon|}$ for all $x \in \mathbb{R}^3$.*

Theorem 1.2. *Let the assumptions $(\mathcal{C}_0)$ and $(\mathcal{F}_1)$–$(\mathcal{F}_3)$ be satisfied. Then for each $\delta > 0$, there exist $\varepsilon_\delta > 0$ and λ^* such that for any $\varepsilon \in (0, \varepsilon_\delta)$ and $\lambda \in [\lambda^*, \infty)$, $(KH)_\varepsilon$ has at least $\mathrm{cat}_{\mathcal{M}_\delta}(\mathcal{M})$ positive solutions. Furthermore, if u_ε denotes one of these positive solutions and $\sigma_\varepsilon \in \mathbb{R}^3$ such that $u_\varepsilon(\sigma_\varepsilon) = \max_{x\in\mathbb{R}^3} u_\varepsilon(x)$, then one sees that*

(i) $\lim_{\varepsilon\to 0} M(\sigma_\varepsilon) = M_0$;
(ii) $\lim_{|x|\to\infty} u_\varepsilon(x) = 0$, $\lim_{|x|\to\infty} |\nabla u_\varepsilon(x)| = 0$ *and* $u_\varepsilon \in C^{1,\sigma}_{loc}(\mathbb{R}^3)$ *with* $\sigma \in (0, 1)$. *Furthermore, there exist constants* $C, c > 0$ *such that* $|u_\varepsilon(x)| \leqslant Ce^{-\frac{c}{\varepsilon}|x-\sigma_\varepsilon|}$ *for all* $x \in \mathbb{R}^3$.

Theorem 1.3. *If the assumptions $(\mathcal{C}_1)$ and $(\mathcal{F}_1)$–$(\mathcal{F}_3)$ hold, then for each $\varepsilon > 0$ and $\lambda > 0$, we have that $(KH)_\varepsilon$ has no positive ground state solution.*

Before going to prove our main results, some remarks on these results are in order: (i) To our best knowledge, there is no result on the existence and concentration of positive ground state solutions for Kirchhoff type equation with critical growth on $\mathbb{R}^3$. At present paper we are first devoted to proving the existence of positive ground state solutions of $(KH)_\varepsilon$. Then we also obtain the multiplicity and concentration of positive solutions for Kirchhoff type equation with critical growth. Moreover, some properties for the positive ground state solution of $(KH)_\varepsilon$ are also obtained. (ii) We obtain sufficient conditions for the nonexistence of positive ground state solution. (iii) Obviously, in the present paper the conditions on f are weaker than the previous papers [3] (see (1.3)).

The proof is based on variational method. By comparing with the previous works, the main difficulties in proving our theorems is the lack of compactness. As we shall see, Eq. $(KH)_\varepsilon$ can be viewed as a Schrödinger equation coupled with a nonlocal term. The competing effect of the nonlocal term with the nonlinearity $f(u)$ and the lack of compactness of the embedding of prevents us from using the variational methods in a standard way. Precisely, since the embeddings $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ $(\forall p \in (2,6))$ and $H^1(\mathbb{R}^3) \hookrightarrow L^6_{loc}(\mathbb{R}^3)$ are not compact, we cannot use the variational methods in a standard way. In the later section, we shall show that the key to make up the global compactness $(H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ $(p \in (2,6)))$ is the limit problem $(\mathcal{H}_{M_0})$. Unfortunately, no information on the ground state solution for the Kirchhoff equations can be found in the existing references. As a consequence, we should carefully investigate the limit problem in the Section 3. To remedy the local compactness $(H^1(\mathbb{R}^3) \hookrightarrow L^6_{loc}(\mathbb{R}^3))$, we should give some new estimates for the ground state level for the energy functional. On the other hand, in the previous paper [3], since f is a C^1 function, it follows that $\mathcal{K}_{\varepsilon,\lambda} \in C^2$ and $\mathcal{D}_\varepsilon \in C^1$, where $\mathcal{D}_\varepsilon$ is Nehari manifold given by

$$\mathcal{D}_\varepsilon = \left\{u \in H^1\left(\mathbb{R}^3\right) \setminus \{0\}\colon \mathcal{K}'_{\varepsilon,\lambda}(u)u = 0\right\}.$$

From these properties of $\mathcal{K}_{\varepsilon,\lambda}$ and $\mathcal{D}_\varepsilon$, one can easily deduce that critical points of $\mathcal{K}_{\varepsilon,\lambda}$ on $\mathcal{D}_\varepsilon$ are critical points of $\mathcal{K}_{\varepsilon,\lambda}$ on $H^1(\mathbb{R}^3)$. Furthermore, one can use the standard Ljusternik–Schnirelmann category theory on $\mathcal{D}_\varepsilon$ directly (see [10,20]). However, in present paper we cannot obtain these properties, since f is just a continuous functional, and $\mathcal{D}_\varepsilon$ is only a continuous submanifolds of $H^1(\mathbb{R}^N)$. To overcome this difficulty, we should carefully study the elementary properties for $\mathcal{D}_\varepsilon$ as in [2]. By doing this we can reduce variational problem for indefinite functional to minimax problem on a manifold and find positive solutions for $(KH)_\varepsilon$.

For the proof of our theorems, we shall consider an equivalent system to $(KH)_\varepsilon$. For this purpose, making the change of variable $\varepsilon y = x$, we can rewrite $(KH)_\varepsilon$ as the following equivalent equation

$$(\mathcal{H}_\varepsilon) \quad -\left(a + b\int\limits_{\mathbb{R}^3} |\nabla u|^2\right)\Delta u + M(\varepsilon x)u = \lambda f(u) + |u|^4 u, \quad u > 0,\ u \in H^1\left(\mathbb{R}^3\right).$$

Thus, our theorems for $(KH)_\varepsilon$ are equivalent to the following results for $(\mathcal{H}_\varepsilon)$:

(a) If the assumptions $(\mathcal{C}_0)$ and $(\mathcal{F}_1)$–$(\mathcal{F}_4)$ hold, then there exist $\lambda^* > 0$ and $\varepsilon^* > 0$ such that for each $\lambda \in [\lambda^*, \infty)$ and $\varepsilon \in (0, \varepsilon^*)$, we have that
 (i) $(\mathcal{H}_\varepsilon)$ has a positive ground state solution $u_\varepsilon \in H^1(\mathbb{R}^3)$;
 (ii) the set of all positive ground state solutions of $(\mathcal{H}_\varepsilon)$ is compact in $H^1(\mathbb{R}^3)$;
 (iii) there exists a maximum point y_ε of u_ε such that $\lim_{\varepsilon\to 0} dist(\varepsilon y_\varepsilon, \mathcal{M}) = 0$, and for any sequence of such y_ε, $k_\varepsilon(x) = u_\varepsilon(x + y_\varepsilon)$ converges in E to a ground state solution of $(\mathcal{H}_{M_0})$, where u_ε denotes the positive ground state solution of $(\mathcal{H}_\varepsilon)$;
 (iv) $\lim_{|x|\to\infty} u_\varepsilon(x) = 0$, $\lim_{|x|\to\infty} |\nabla u_\varepsilon(x)| = 0$ and $u_\varepsilon \in C^{1,\sigma}_{loc}(\mathbb{R}^3)$ with $\sigma \in (0, 1)$. Furthermore, there exist constants $C, c > 0$ such that $|u_\varepsilon(x)| \leqslant Ce^{-c|x-y_\varepsilon|}$ for all $x \in \mathbb{R}^3$.

 Moreover, if the assumptions $(\mathcal{C}_1)$ and $(\mathcal{F}_1)$–$(\mathcal{F}_3)$ hold, then $(\mathcal{H}_\varepsilon)$ has no positive ground state solution for all $\lambda > 0$ and $\varepsilon > 0$.

(b) If the assumptions $(\mathcal{C}_0)$ and $(\mathcal{F}_1)$–$(\mathcal{F}_4)$ hold, then for each $\delta > 0$, there exist $\varepsilon_\delta > 0$ and λ^* such that for any $\varepsilon \in (0, \varepsilon_\delta)$ and $\lambda \in [\lambda^*, \infty)$, then $(\mathcal{H})_\varepsilon$ has at least $cat_{\mathcal{M}_\delta}(\mathcal{M})$ positive solutions. Furthermore, if u_ε denotes one of these positive solutions and $\sigma_\varepsilon \in \mathbb{R}^3$ such that $u_\varepsilon(\sigma_\varepsilon) = \max_{x\in\mathbb{R}^3} u_\varepsilon(x)$, then one sees that
 (i) $\lim_{\varepsilon\to 0} M(\sigma_\varepsilon) = M_0$;
 (ii) $\lim_{|x|\to\infty} u_\varepsilon(x) = 0$, $\lim_{|x|\to\infty} |\nabla u_\varepsilon(x)| = 0$ and $u_\varepsilon \in C^{1,\sigma}_{loc}(\mathbb{R}^3)$ with $\sigma \in (0, 1)$. Furthermore, there exist constants $C, c > 0$ such that $|u_\varepsilon(x)| \leqslant Ce^{-\frac{c}{\varepsilon}|x-\sigma_\varepsilon|}$ for all $x \in \mathbb{R}^3$.

2. Variational setting

In order to establish the variational setting for $(\mathcal{H}_\varepsilon)$, we need give more notations:

– $L^p \equiv L^p(\mathbb{R}^3)$ is the usual Lebesgue space endowed with the norm

$$|u|_p^p = \int\limits_{\mathbb{R}^3} |u|^p < \infty \quad \text{for } 1 \leqslant p < \infty, \qquad |u|_\infty = \sup_{x\in\mathbb{R}^3} \big|u(x)\big|;$$

– Let $H^1(\mathbb{R}^3)$ be the usual Sobolev space endowed with the standard scalar product and norm

$$(u, v) = \int\limits_{\mathbb{R}^3} (\nabla u \nabla v + uv), \qquad \|u\|^2 = \int\limits_{\mathbb{R}^3} \big(|\nabla u|^2 + |u|^2\big);$$

– $E = H^1(\mathbb{R}^3)$ and $S = B_1(0) = \{u \in E\colon\ \|u\| = 1\}$;
– The letters c, C, C_i will be indiscriminately used to denote various positive constants whose exact values are irrelevant.

For any $\varepsilon > 0$, let $E_\varepsilon = \{u \in H^1(\mathbb{R}^3)\colon \int_{\mathbb{R}^3} M(\varepsilon x)u^2 < \infty\}$ denote the Hilbert space endowed with inner product

$$(u, v)_\varepsilon = \int\limits_{\mathbb{R}^3} \nabla u \nabla v + M(\varepsilon x)uv, \quad \text{for } u, u \in E_\varepsilon,$$

and the induced norm denoted by $\|u\|_\varepsilon^2 = (u, u)_\varepsilon$. Clearly, $\|\cdot\|_\varepsilon$ and $\|\cdot\|$ are equivalent norms for $\varepsilon > 0$ and $M_\infty < \infty$. Now on E_ε we define the functional

$$\mathcal{T}_{\varepsilon,\lambda}(u) = \frac{1}{2} \int\limits_{\mathbb{R}^{3N}} \big(a|\nabla u|^2 + M(\varepsilon x)|u|^2\big) + \frac{b}{4}\bigg(\int\limits_{\mathbb{R}^3} |\nabla u|^2\bigg)^2 - \lambda \int\limits_{\mathbb{R}^3} F(u) - \frac{1}{6}\int\limits_{\mathbb{R}^3} |u|^6 \quad \text{for } u \in E_\varepsilon.$$

Obviously, $\mathcal{T}_{\varepsilon,\lambda} \in C^1(E_\varepsilon, \mathbb{R})$ and a standard argument shows that critical points of $\mathcal{T}_{\varepsilon,\lambda}$ are solutions of $(\mathcal{H}_\varepsilon)$ (see [3,26,27]).

We shall use the Nehari methods to find critical points for $\mathcal{T}_{\varepsilon,\lambda}$. The Nehari manifold corresponding to $\mathcal{T}_{\varepsilon,\lambda}$ is defined by

$$\mathcal{N}_\varepsilon = \left\{u \in E_\varepsilon \setminus \{0\}\colon\ \mathcal{T}'_{\varepsilon,\lambda}(u)u = 0\right\}.$$

Thus for $u \in \mathcal{N}_\varepsilon$, one sees that

$$\int_{\mathbb{R}^3} \left(a|\nabla u|^2 + M_\varepsilon(x)|u|^2\right) + b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 = \lambda \int_{\mathbb{R}^3} f(u)u + \int_{\mathbb{R}^3} |u|^6, \tag{2.1}$$

where $M_\varepsilon(x) = M(\varepsilon x)$. This implies that for $u \in \mathcal{N}_\varepsilon$

$$\mathcal{T}_{\varepsilon,\lambda}|_{\mathcal{N}_\varepsilon} = \frac{1}{4}\int_{\mathbb{R}^3} \left(a|\nabla u|^2 + M_\varepsilon(x)|u|^2\right) + \lambda \int_{\mathbb{R}^3} \left(\frac{1}{4} f(u)u - F(u)\right) + \frac{1}{12}\int_{\mathbb{R}^3} |u|^6. \tag{2.2}$$

In the following we shall prove some elementary properties for $\mathcal{N}_\varepsilon$. To do this, we first need to prove some properties for the functional $\mathcal{T}_{\varepsilon,\lambda}$.

Lemma 2.1. *Under the assumptions of $(\mathcal{C}_0)$ and $(\mathcal{F}_1)$–$(\mathcal{F}_3)$, we have that for $\lambda > 0$ and $\varepsilon > 0$*

(i) *$\mathcal{T}'_{\varepsilon,\lambda}$ maps bounded sets in E_ε into bounded sets in E_ε;*
(ii) *$\mathcal{T}'_{\varepsilon,\lambda}$ is weakly sequentially continuous in E_ε;*
(iii) *$\mathcal{T}_{\varepsilon,\lambda}(t_n u_n) \to -\infty$ as $t_n \to \infty$, where $u_n \in \mathcal{E}$, and $\mathcal{E} \subset E_\varepsilon \setminus \{0\}$ is a compact subset.*

Proof. (i) Let $\{u_n\}$ denote the boundedness sequence of E_ε. Then for each $\varphi \in E_\varepsilon$ one deduces from $(\mathcal{C}_0)$, $(\mathcal{F}_1)$ and $(\mathcal{F}_3)$ that

$$\begin{aligned}\mathcal{T}'_{\varepsilon,\lambda}(u_n)\varphi &= \int_{\mathbb{R}^3} \left(a\nabla u_n \nabla\varphi + M_\varepsilon(x)u_n\varphi\right) + b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)\int_{\mathbb{R}^3} \nabla u_n\nabla\varphi + \lambda\int_{\mathbb{R}^3} f(u_n)\varphi + \int_{\mathbb{R}^3} |u_n|^5\varphi \\ &\leqslant c\|u_n\|^2|\varphi|_2^2 + \|u_n\|^3|\varphi|_2^2 + c + c\|u_n\|^p|\varphi|_2^2 + \|u_n\|^5|\varphi|_2^2 \leqslant c.\end{aligned}$$

(ii) To prove the conclusion (ii), one can refer to [1]; here we omit the details.

(iii) Finally, we prove the conclusion (iii). Without loss of generality, we may assume that $\|u\|_\varepsilon = 1$ for each $u \in \mathcal{E}$. For $u_n \in \mathcal{E}$, after passing to a subsequences, we obtain that $u_n \to u \in S_\varepsilon := \{u \in E_\varepsilon\colon\ \|u\| = 1\}$. It is clear that

$$\begin{aligned}\mathcal{T}_{\varepsilon,\lambda}(t_n u_n) &= \frac{t_n^2}{2}\int_{\mathbb{R}^3} \left(a|\nabla u_n|^2 + M_\varepsilon(x)|u_n|^2\right) + \frac{bt_n^4}{4}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^2 - \lambda\int_{\mathbb{R}^3} F(t_n u_n) - \frac{t_n^6}{6}\int_{\mathbb{R}^3} |u_n|^6 \\ &\leqslant t_n^4\left(\frac{\int_{\mathbb{R}^3} a|\nabla u_n|^2 + M_\varepsilon(x)|u_n|^2}{2t_n^2} + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^2 - t_n^2\int_{\mathbb{R}^3} |u_n|^6\right) \to -\infty,\end{aligned}$$

as $n \to \infty$. $\square$

Now we are ready to prove some elementary properties for $\mathcal{N}_\varepsilon$.

Lemma 2.2. *Under the assumptions of Lemma 2.1, for $\lambda > 0$ and $\varepsilon > 0$ we have that*

(i) *for all $u \in S_\varepsilon$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\varepsilon$. Moreover, $m_\varepsilon(u) = t_u u$ is the unique maximum of $\mathcal{T}_{\varepsilon,\lambda}$ on E_ε, where $S_\varepsilon = \{u \in E_\varepsilon \colon \|u\|_\varepsilon = 1\}$;*
(ii) *the set $\mathcal{N}_\varepsilon$ is bounded away from 0. Furthermore, $\mathcal{N}_\varepsilon$ is closed in E_ε;*
(iii) *there is $\alpha > 0$ such that $t_u \geqslant \alpha$ for each $u \in S_\varepsilon$ and for each compact subset $\mathcal{W} \subset S_\varepsilon$, there exists $C_{\mathcal{W}} > 0$ such that $t_u \leqslant C_{\mathcal{W}}$, for all $u \in \mathcal{W}$;*
(iv) *$\mathcal{N}_\varepsilon$ is a regular manifolds diffeomorphic to the sphere of E_ε;*
(v) *$c_\varepsilon = \inf_{\mathcal{N}_\varepsilon} \mathcal{T}_{\varepsilon,\lambda} \geqslant \rho > 0$ and $\mathcal{T}_{\varepsilon,\lambda}$ is bounded below on $\mathcal{N}_\varepsilon$, where $\rho > 0$ is independent of ε;*
(vi) *m_ε is a bounded mapping. Moreover, if $u_n \rightharpoonup u$, then we have $m_\varepsilon(u_n) \rightharpoonup m_\varepsilon(u)$.*

Proof. (i) For each $u \in S_\varepsilon$ and $t > 0$, we define $h(t) = \mathcal{T}_{\varepsilon,\lambda}(tu)$. It is easy to verify that $h(0) = 0$, $h(t) < 0$ for $t > 0$ large. Moreover, we claim that $h(t) > 0$ for $t > 0$ small. Indeed, from the conditions $(\mathcal{F}_1)$–$(\mathcal{F}_3)$, we deduce that for each $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\big|f(u)\big| \leqslant \epsilon|u| + C_\epsilon|u|^{p-1} \quad \text{and} \quad \big|F(u)\big| \leqslant \epsilon|u|^2 + C_\epsilon|u|^p, \quad p \in (4,6). \tag{2.3}$$

It follows that

$$\begin{aligned} h(t) = \mathcal{T}_{\varepsilon,\lambda}(tu) &= \frac{t^2}{2}\int_{\mathbb{R}^3}\big(a|\nabla u|^2 + |u|^2\big) + \frac{bt^4}{4}\bigg(\int_{\mathbb{R}^3}|\nabla u|^2\bigg)^2 - \lambda\int_{\mathbb{R}^3}F(tu) - \frac{t^6}{6}\int_{\mathbb{R}^3}|u|^6 \\ &\geqslant \frac{ct^2}{2}\|u\|_\varepsilon^2 - \frac{ct^4}{4}\|u\|_\varepsilon^4 - \epsilon\lambda t^2|u|_2^2 - t^p c\lambda C_\epsilon|u|_p^p - ct^6|u|_6^6 \\ &\geqslant \frac{t^2}{2}\|u\|_\varepsilon^2 - ct^4\|u\|_\varepsilon^4 - ct^2\epsilon\|u\|_\varepsilon^2 - cC_\epsilon t^p\|u\|_\varepsilon^p - ct^6\|u\|_\varepsilon^6. \end{aligned}$$

Since $p > 4$, we prove that $h(t) > 0$ for $t > 0$ small. Therefore, $\max_{t>0} h(t)$ is achieved at a $t = t_u > 0$ so that $h'(t_u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$. Suppose that there exist $t_u' > t_u > 0$ such that $t_u' u, t_u u \in \mathcal{N}_\varepsilon$. Then one has that

$$\begin{gathered} t_u^2\|u\|_\varepsilon^2 + t_u^4\bigg(\int_{\mathbb{R}^3}|\nabla u|^2\bigg)^2 = \lambda\int_{\mathbb{R}^3}f(t_u u)t_u u + t_u^6\int_{\mathbb{R}^3}|u|^6 \quad \text{and} \\ \big(t_u'\big)^2\|u\|_\varepsilon^2 + \big(t_u'\big)^4\bigg(\int_{\mathbb{R}^3}|\nabla u|^2\bigg)^2 = \lambda\int_{\mathbb{R}^3}f\big(t_u' u\big)t_u' u + \big(t_u'\big)^6\int_{\mathbb{R}^3}|u|^6. \end{gathered} \tag{2.4}$$

Then we see that

$$\bigg(\frac{1}{(t_u')^2} - \frac{1}{t_u^2}\bigg)\|u\|_\varepsilon^2 = \lambda\int_{\mathbb{R}^3}\bigg(\frac{f(t_u' u)}{(t_u' u)^3} - \frac{f(t_u u)}{(t_u u)^3}\bigg)u^4 + \big(\big(t_u'\big)^2 - t_u^2\big)\int_{\mathbb{R}^3}|u|^6,$$

which makes no sense in view of $(\mathcal{F}_2)$ and $t_u' > t_u > 0$. So the conclusion (i) follows.

(ii) For $u \in \mathcal{N}_\varepsilon$, we infer from (2.1) and (2.3) that

$$\|u\|_\varepsilon^2 \leqslant \epsilon|u|_\varepsilon^2 + C_\epsilon|u|_p^p + |u|_6^6 \leqslant c\epsilon\|u\|_\varepsilon^2 + cC_\epsilon\|u\|_\varepsilon^p + \|u\|_\varepsilon^6.$$

So for some $\kappa > 0$, we get that

$$\|u\|_\varepsilon \geqslant \kappa > 0. \tag{2.5}$$

Now we prove the set $\mathcal{N}_\varepsilon$ is closed in E_ε. Let $\{u_n\} \subset \mathcal{N}_\varepsilon$ such that $u_n \to u$ in E_ε. In the following we shall prove that $u \in \mathcal{N}_\varepsilon$. By Lemma 2.1, we have that $\mathcal{T}'_{\varepsilon,\lambda}(u_n)$ is bounded, then we infer from

$$\mathcal{T}'_{\varepsilon,\lambda}(u_n)u_n - \mathcal{T}'_{\varepsilon,\lambda}(u)u = \big(\mathcal{T}'_{\varepsilon,\lambda}(u_n) - \mathcal{T}'_{\varepsilon,\lambda}(u)\big)u - \mathcal{T}'_{\varepsilon,\lambda}(u_n)(u_n - u) \to 0, \quad \text{as } n \to \infty,$$

that $\mathcal{T}'_{\varepsilon,\lambda}(u)u = 0$. Moreover, it follows from (2.5) that $\|u\|_\varepsilon = \lim_{n\to\infty} \|u_n\|_\varepsilon \geqslant \kappa > 0$. So $u \in \mathcal{N}_\varepsilon$.

(iii) For $\{u_n\} \subset E_\varepsilon \setminus \{0\}$, there exist t_{u_n} such that $t_{u_n}u_n \in \mathcal{N}_\varepsilon$. By the conclusion (ii), one sees that $\|t_{u_n}u_n\|_\varepsilon = t_{u_n}\|u_n\|_\varepsilon \geqslant \kappa > 0$. It is impossible to have that $t_{u_n} \to 0$, as $n \to \infty$. To prove $t_u \leqslant C_{\mathcal{W}}$, for all $u \in \mathcal{W} \subset S_\varepsilon$. We argue by contraction. Suppose there exists $\{u_n\} \subset \mathcal{W} \subset S_\varepsilon$ such that $t_n = t_{u_n} \to \infty$. Since $\mathcal{W}$ is compact, there exists $u \in \mathcal{W}$ such that $u_n \to u$ in E_ε and $u_n(x) \to u(x)$ a.e. on $\mathbb{R}^3$ after passing to a subsequence. Then Lemma 2.1 implies that $\mathcal{T}_{\varepsilon,\lambda}(t_nu_n) \to -\infty$ as $n \to \infty$. However, from (2.2) we deduce that $\mathcal{T}_{\varepsilon,\lambda}(t_nu_n) \geqslant 0$. This is a contradiction.

(iv) Define the mappings $\hat{m}_\varepsilon : E_\varepsilon \setminus \{0\} \to \mathcal{N}_\varepsilon$ and $m_\varepsilon : S_\varepsilon \to \mathcal{N}_\varepsilon$ by setting

$$\hat{m}_\varepsilon(u) = t_u u \quad \text{and} \quad m_\varepsilon = \hat{m}_\varepsilon|_{S_\varepsilon}. \tag{2.6}$$

By the conclusions (i)–(iii), we know that the conditions of Proposition 3.1 in [2] are satisfied. So the mapping m_ε is a homeomorphism between S_ε and $\mathcal{N}_\varepsilon$, and the inverse of m_ε is given by

$$\check{m}_\varepsilon(u) = m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon}. \tag{2.7}$$

Thus $\mathcal{N}_\varepsilon$ is a regular manifolds diffeomorphic to the sphere of E_ε.

(v) For $\varepsilon > 0$, $\lambda > 0$, $s > 0$ and $u \in E_\varepsilon \setminus \{0\}$, it follows from (2.3) that

$$\begin{aligned}
\mathcal{T}_{\varepsilon,\lambda}(su) &= \frac{s^2}{2}\int_{\mathbb{R}^3}\big(a|\nabla u|^2 + M_\varepsilon(x)|u|^2\big) + \frac{s^4 b}{4}\bigg(\int_{\mathbb{R}^3}|\nabla u|^2\bigg)^2 - \lambda\int_{\mathbb{R}^3}F(su) - \frac{s^6}{6}\int_{\mathbb{R}^3}|u|^6 \\
&\geqslant \frac{cs^2}{2}\|u\|_\varepsilon^2 - cs^4\|u\|_\varepsilon^4 - s^2c\epsilon\|u\|_\varepsilon^2 - s^pcC_\epsilon\|u\|_\varepsilon^p - cs^6\|u\|_\varepsilon^6 \\
&= \frac{cs^2}{2}(1-\epsilon)\|u\|_\varepsilon^2 - cs^4\|u\|_\varepsilon^4 - s^pcC_\epsilon\|u\|_\varepsilon^p - cs^6\|u\|_\varepsilon^6.
\end{aligned}$$

So there is $\rho > 0$ such that $\mathcal{T}_{\varepsilon,\lambda}(su) \geqslant \rho > 0$ for $s > 0$ small. On the other hand, we deduce from the conclusions (i)–(iii) that

$$c_\varepsilon = \inf_{\mathcal{N}_\varepsilon}\mathcal{T}_{\varepsilon,\lambda}(u) = \inf_{w\in E_\varepsilon\setminus\{0\}}\max_{s>0}\mathcal{T}_{\varepsilon,\lambda}(sw) = \inf_{w\in S_\varepsilon}\max_{s>0}\mathcal{T}_{\varepsilon,\lambda}(sw). \tag{2.8}$$

So we get that $c_\varepsilon \geqslant \rho > 0$ and $\mathcal{T}_{\varepsilon,\lambda}|_{\mathcal{N}_\varepsilon} \geqslant \rho > 0$.

(vi) Assume by contradiction that for $\{u_n\} \subset S_\varepsilon$, we have that $m_\varepsilon(u_n) = t_nu_n = t_{u_n}u_n \to \infty$, and so t_n. As in the proof of the conclusion (iii), we can prove that $\mathcal{T}_{\varepsilon,\lambda}(t_nu_n) \to -\infty$ as $n \to \infty$. However, $\mathcal{T}_{\varepsilon,\lambda}(t_nu_n) \geqslant 0$. A contradiction. So m is a bounded mapping. Assume that $\{u_n\}$ is bounded, and so $m_\varepsilon(u_n) = t_nu_n = t_{u_n}u_n$. Without loss of generality, we can assume that $u_n \rightharpoonup u$, $t_n \to t_u$ and $m_\varepsilon(u_n) \rightharpoonup m_\varepsilon(u)$. For each $\varphi \in E_\varepsilon$ one has

$$\big(m(u_n) - t_uu, \varphi\big) = (t_nu_n - t_uu, \varphi) = \big((t_n - t_u)u + t_n(u_n - u), \varphi\big) \to 0, \quad \text{as } n \to \infty.$$

So $m_\varepsilon(u) = t_uu$. This ends the proof. $\square$

Now we shall consider the functionals $\hat{\Upsilon}_{\varepsilon,\lambda} : E_\varepsilon \setminus \{0\} \to \mathbb{R}$ and $\Upsilon_{\varepsilon,\lambda} : S_\varepsilon \to \mathbb{R}$ defined by

$$\hat{\Upsilon}_{\varepsilon,\lambda} = \mathcal{T}_{\varepsilon,\lambda}\big(\hat{m}_\varepsilon(u)\big) \quad \text{and} \quad \Upsilon_{\varepsilon,\lambda} = \hat{\Upsilon}_{\varepsilon,\lambda}|_S,$$

where $\hat{m}_\varepsilon(u) = t_u u$ is given in (2.6). As in [2], we have the following lemma.

Lemma 2.3. *(See Corollary 3.3 of [2].) Under the assumptions of Lemma 2.1, for $\lambda > 0$ and $\varepsilon > 0$ we have that*

(i) $\Upsilon_{\varepsilon,\lambda} \in C^1(S_\varepsilon, \mathbb{R})$, *and*

$$\Upsilon'_{\varepsilon,\lambda}(w)z = \big\|m_\varepsilon(w)\big\|_\varepsilon \mathcal{T}'_{\varepsilon,\lambda}\big(m_\varepsilon(w)\big)z \quad \text{for } z \in \mathcal{T}_w(S_\varepsilon) = \big\{h \in E_\varepsilon\colon\ (w,h)_\varepsilon = 0\big\};$$

(ii) $\{w_n\}$ *is a Palais–Smale sequence for* $\Upsilon_{\varepsilon,\lambda}$ *if and only if* $\{m_\varepsilon(w_n)\}$ *is a Palais–Smale sequence for* $\mathcal{T}_{\varepsilon,\lambda}$. *If* $\{u_n\} \subset \mathcal{N}_\varepsilon$ *is a bounded Palais–Smale sequence for* $\mathcal{T}_{\varepsilon,\lambda}$, *then* $\check{m}_\varepsilon(u_n)$ *is a Palais–Smale sequence for* $\Upsilon_{\varepsilon,\lambda}$, *where* $\check{m}_\varepsilon(u)$ *is given in* (2.7);

(iii)

$$\inf_{S_\varepsilon} \Upsilon_{\varepsilon,\lambda} = \inf_{\mathcal{N}_\varepsilon} \mathcal{T}_{\varepsilon,\lambda} = c_\varepsilon.$$

Moreover, $z \in S_\varepsilon$ *is a critical point of* $\Upsilon_{\varepsilon,\lambda}$ *if and only if* $m_\varepsilon(u)$ *is a critical point of* $\mathcal{T}_{\varepsilon,\lambda}$, *and the corresponding critical values coincide.*

3. The autonomous system

In this section we shall prove some properties of the ground state solution of the limit equation. Precisely, for each $\mu > 0$ and $\lambda > 0$, we concern with the following equation

$$(\mathcal{H}_\mu) \quad -\bigg(a + b\int_{\mathbb{R}^3} |\nabla u|^2\bigg)\Delta u + \mu u = \lambda f(u) + |u|^4 u, \quad u > 0,\ u \in H^1\big(\mathbb{R}^3\big).$$

For any $\mu > 0$, let $E_\mu = \{u \in H^1(\mathbb{R}^3)\colon\ \int_{\mathbb{R}^3} \mu u^2 < \infty\}$ be the Hilbert space endowed the inner product

$$(u, v)_\mu = \int_{\mathbb{R}^3} \nabla u \nabla v + \mu u v, \quad \text{for } u, v \in E_\mu,$$

and correspondingly the norm denoted by $\|u\|_\mu^2 = \int_{\mathbb{R}^3} |\nabla u|^2 + \mu|u|^2$. Then we see the energy functional corresponding to $(\mathcal{H}_\mu)$ is denoted by

$$\mathcal{T}_{\mu,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^3} \big(a|\nabla u|^2 + \mu|u|^2\big) + \frac{b}{4}\bigg(\int_{\mathbb{R}^3} |\nabla u|^2\bigg)^2 - \lambda \int_{\mathbb{R}^3} F(u) - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 \quad \text{for all } u \in E_\mu.$$

As in Section 2, in order to find the critical points for the functional $\mathcal{T}_{\mu,\lambda}$, we also use the Nehari manifold methods. The Nehari manifold corresponding to $\mathcal{T}_{\mu,\lambda}$ is defined by

$$\mathcal{N}_\mu = \big\{u \in E_\mu \setminus \{0\}\colon\ \mathcal{T}'_{\mu,\lambda}(u)u = 0\big\}.$$

Thus for $u \in \mathcal{N}_\mu$, one sees that

$$\int_{\mathbb{R}^3} \left(a|\nabla u|^2 + \mu|u|^2\right) + b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 = \lambda \int_{\mathbb{R}^3} f(u)u + \int_{\mathbb{R}^3} |u|^6. \tag{3.1}$$

This implies that for $u \in \mathcal{N}_\mu$

$$\mathcal{T}_{\mu,\lambda}|_{\mathcal{N}_\mu} = \frac{1}{4}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + \mu|u|^2\right) + \lambda \int_{\mathbb{R}^3} \left(\frac{1}{4} f(u)u - F(u)\right) + \frac{1}{12}\int_{\mathbb{R}^3} |u|^6. \tag{3.2}$$

Similar to Lemma 2.2, we know that $\mathcal{N}_\mu$ has the following elementary properties.

Lemma 3.1. *Under the assumptions of Lemma 2.1, for $\lambda > 0$ and $\mu > 0$ we have that*

(i) *for all $u \in S_\mu := \{u \in E_\mu\colon \|u\|_\mu = 1\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\mu$. Moreover, $m_\mu(u) = t_u u$ is the unique maximum of $\mathcal{T}_{\mu,\lambda}$ on E_μ;*
(ii) *the set $\mathcal{N}_\mu$ is bounded away from 0. Furthermore, $\mathcal{N}_\mu$ is closed in E_μ;*
(iii) *there is $\delta > 0$ such that $t_u \geqslant \delta$ for each $u \in S_\mu$ and for each compact subset $\mathcal{W} \subset S_\mu$, there exists $C_{\mathcal{W}} > 0$ such that $t_u \leqslant C_{\mathcal{W}}$, for all $u \in \mathcal{W}$;*
(iv) *$\mathcal{N}_\mu$ is a regular manifolds diffeomorphic to the sphere of E_μ;*
(v) *$c_\mu = \inf_{\mathcal{N}_\mu} \mathcal{T}_{\mu,\lambda} > 0$ and $\mathcal{T}_{\mu,\lambda}|_{\mathcal{N}_\mu}$ is bounded below by some positive constant;*
(vi) *m_μ is a bounded map. Moreover, if $u_n \rightharpoonup u$, then we have $m_\mu(u_n) \rightharpoonup m_\mu(u)$.*

From the conclusion (i) of Lemma 3.1, we know that for each $u \in E_\mu \setminus \{0\}$, there exists unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\mu$. So we define the mapping $\hat{m}_\mu : E_\mu \setminus \{0\} \to \mathcal{N}_\mu$ by $\hat{m}_\mu(u) = t_u u$. Clearly, $m_\mu = \hat{m}_\mu|_{S_\mu}$. Let

$$\hat{\Upsilon}_{\mu,\lambda} : E_\mu \setminus \{0\} \to \mathbb{R}, \qquad \hat{\Upsilon}_{\mu,\lambda}(w) := \mathcal{T}_{\mu,\lambda}\big(\hat{m}_\mu(w)\big) \quad \text{and} \quad \Upsilon_{\mu,\lambda} := \hat{\Upsilon}_{\mu,\lambda}|_{S_\mu}.$$

If the inverse of the mapping m_μ to S_μ is given by

$$\check{m}_\mu = m_\mu^{-1} : \mathcal{N}_\mu \to S_\mu, \qquad \check{m}_\mu = \frac{u}{\|u\|},$$

then we have the following lemma.

Lemma 3.2. *(See Corollary 3.3 of [2].) Under the assumptions of Lemma 2.1, for $\lambda > 0$ and $\varepsilon > 0$ we have that*

(i) *$\Upsilon_{\mu,\lambda} \in C^1(S_\mu, \mathbb{R})$, and*

$$\Upsilon'_{\mu,\lambda}(w)z = \big\|m_\mu(w)\big\|_\mu \mathcal{T}'_{\mu,\lambda}\big(m_\mu(w)\big)z \quad \text{for } z \in \mathcal{T}_w S_\mu = \big\{h \in E_\mu\colon (w,h)_\mu = 0\big\};$$

(ii) *$\{w_n\}$ is a Palais–Smale sequence for $\Upsilon_{\mu,\lambda}$ if and only if $\{m_\mu(w_n)\}$ is a Palais–Smale sequence for $\mathcal{T}_{\mu,\lambda}$. If $\{u_n\} \subset \mathcal{N}_\mu$ is a bounded Palais–Smale sequence for $\mathcal{T}_{\mu,\lambda}$, then $\check{m}_\mu(u_n)$ is a Palais–Smale sequence for $\Upsilon_{\mu,\lambda}$, where $\check{m}_\mu(u) = m_\mu^{-1}(u) = \frac{u}{\|u\|_\mu}$;*
(iii)

$$\inf_{S_\mu} \Upsilon_{\mu,\lambda} = \inf_{\mathcal{N}_\mu} \mathcal{T}_{\mu,\lambda} = c_\mu.$$

Moreover, $z \in S_\mu$ is a critical point of $\Upsilon_{\mu,\lambda}$ if and only if $m_\mu(u)$ is a critical point of $\mathcal{T}_{\mu,\lambda}$, and the corresponding critical values coincide.

Remark 3.3. By Lemma 3.1, we note that the infimum of $\mathcal{T}_{\mu,\lambda}$ over $\mathcal{N}_\mu$ has the following minimax characterization:

$$0 < c_\mu = \inf_{z\in\mathcal{N}_\mu} \mathcal{T}_{\mu,\lambda}(z) = \inf_{w\in E_\mu\setminus\{0\}} \max_{s>0} \mathcal{T}_{\mu,\lambda}(sw) = \inf_{w\in S_\mu} \max_{s>0} \mathcal{T}_{\mu,\lambda}(sw). \tag{3.3}$$

Similar to [3], one can easily prove the following mountain pass geometry of functional $\mathcal{T}_{\mu,\lambda}$.

Lemma 3.4 *(Mountain Pass Geometry). The functional $\mathcal{T}_{\mu,\lambda}$ satisfies the following conditions*:

(i) *There exist positive constants β, α such that $\mathcal{T}_{\mu,\lambda}(u) \geqslant \beta$ for $\|u\|_\mu = \alpha$.*
(ii) *There exists $e \in E_\mu$ with $\|e\| > \alpha$ such that $\mathcal{T}_{\mu,\lambda}(e) < 0$.*

From Lemma 3.4, by using the Ambrosetti–Rabinowitz Mountain Pass Theorem without $(PS)_c$ condition (see [4,5]), it follows that there exists a $(PS)_c$-sequence $\{u_n\} \subset E_\mu$ such that

$$\mathcal{T}_{\mu,\lambda}(u_n) \to c'_\mu = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \mathcal{T}_{\mu,\lambda}\big(\gamma(t)\big) \quad \text{and} \quad \mathcal{T}'_{\mu,\lambda}(u_n) \to 0, \tag{3.4}$$

where $\Gamma = \{\gamma \in C(E_\mu, \mathbb{R})\colon\ \mathcal{T}_{\mu,\lambda}(\gamma(0)) = 0,\ \mathcal{T}_{\mu,\lambda}(\gamma(1)) < 0\}$. As in Proposition 3.11 of [7] (also see [6]), we shall use the following equivalent characterization of c'_μ, which is more adequate to our purpose, given by

$$c'_\mu = \inf_{u\in E_\mu\setminus\{0\}} \max_{t>0} \mathcal{T}_{\mu,\lambda}(tu) = c_\mu. \tag{3.5}$$

Here in the last equality we used (3.3). Now we have the following estimates for c_μ.

Lemma 3.5. *If the conditions $(\mathcal{C}_0)$ and $(\mathcal{F}_1)$–$(\mathcal{F}_3)$ hold, then there exists $\lambda^* > 0$ such that for any $0 < \mu \leqslant M_\infty$, the number c_μ satisfies*

$$0 < c_\mu < \frac{1}{3}(a\mathcal{S})^{\frac{3}{2}} + \frac{1}{12} b^3 \mathcal{S}^6,$$

where $\mathcal{S}$ is the best Sobolev constant, namely

$$\mathcal{S} = \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^3), u\neq 0} \frac{\int_{\mathbb{R}^3} |\nabla u|^2}{(\int_{\mathbb{R}^3} u^6)^{\frac{1}{3}}}.$$

Proof. For $w \in E_\mu \setminus \{0\}$, it follows from Lemma 3.1 that there exists $t_\lambda > 0$ such that

$$\max_{t\geqslant 0} \mathcal{T}_{\mu,\lambda}(tw) = \mathcal{T}_{\mu,\lambda}(t_\lambda w).$$

Hence

$$t_\lambda^2 \int_{\mathbb{R}^3} \big(a|\nabla w|^2 + \mu|w|^2\big) + t_\lambda^4 b \Big(\int_{\mathbb{R}^3} |\nabla w|^2\Big)^2 = \lambda \int_{\mathbb{R}^3} f(t_\lambda w) t_\lambda w + t_\lambda^6 \int_{\mathbb{R}^3} |w|^6. \tag{3.6}$$

From (3.6) we infer that

$$\int_{\mathbb{R}^3} \left(a|\nabla w|^2 + \mu |w|^2\right) + t_\lambda^2 b \left(\int_{\mathbb{R}^3} |\nabla w|^2 \right)^2 \geqslant t_\lambda^4 \int_{\mathbb{R}^3} |w|^6. \tag{3.7}$$

So t_λ is bounded. Thus, for the sequence $\lambda_n \to \infty$ as $n \to \infty$, there exist $t_0 \geqslant 0$ such that $t_{\lambda_n} \to t_0$. Consequently, one sees that

$$t_{\lambda_n}^2 \int_{\mathbb{R}^3} \left(a|\nabla w|^2 + \mu |w|^2\right) + t_{\lambda_n}^4 b \left(\int_{\mathbb{R}^3} |\nabla w|^2 \right)^2 \leqslant c, \quad \forall n \in \mathbb{N}, \tag{3.8}$$

and so

$$\lambda_n \int_{\mathbb{R}^3} f(t_{\lambda_n} w) t_{\lambda_n} w + t_{\lambda_n}^6 \int_{\mathbb{R}^3} |w|^6 \leqslant c, \quad \forall n \in \mathbb{N}. \tag{3.9}$$

Therefore, if $t_0 > 0$, it follows from Fatou's lemma that

$$\lim_{n\to\infty} \lambda_n \int_{\mathbb{R}^3} f(t_{\lambda_n} w) t_{\lambda_n} w + t_{\lambda_n}^6 \int_{\mathbb{R}^3} |w|^6 = +\infty. \tag{3.10}$$

This contradicts with (3.9). Thus, we conclude that $t_0 = 0$. Set $w = e$. We consider the path $\tilde{\gamma}(t) = te$ for $t \in [0,1]$, then $\tilde{\gamma} \in \Gamma$. Moreover, from (3.5), we infer that

$$0 < c_\mu \leqslant \max_{t\in[0,1]} \mathcal{T}_{\mu,\lambda}\left(\tilde{\gamma}(t)\right) = \mathcal{T}_{\mu,\lambda}(t_\lambda e) \leqslant \frac{t_\lambda^2}{2} \int_{\mathbb{R}^3} \left(a|\nabla e|^2 + \mu |e|^2\right) + \frac{b t_\lambda^4}{4} \left(\int_{\mathbb{R}^3} |\nabla e|^2 \right)^2. \tag{3.11}$$

In this way, if λ is large enough, we derive that

$$\frac{t_\lambda^2}{2} \int_{\mathbb{R}^3} \left(a|\nabla e|^2 + \mu |e|^2\right) + \frac{b t_\lambda^4}{4} \left(\int_{\mathbb{R}^3} |\nabla e|^2 \right)^2 < \frac{1}{3}(a\mathcal{S})^{\frac{3}{2}} + \frac{1}{12} b^3 \mathcal{S}^6,$$

which implies that

$$0 < c_\mu < \frac{1}{3}(a\mathcal{S})^{\frac{3}{2}} + \frac{1}{12} b^3 \mathcal{S}^6. \qquad \square \tag{3.12}$$

To prove the compactness of the minimize sequence for $\mathcal{T}_{\mu,\lambda}$, we need the following lemma, and the details of the proof one can refer to [9–11].

Lemma 3.6. *(See Lions [9].) Let $r > 0$, $q \in [2, 2^*]$. If $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |u|^q \, dx = 0,$$

then we have $u_n \to 0$ in $L^p(\mathbb{R}^3)$ for $p \in (2, 2^*)$. Moreover, if $q = 2^*$, $u_n \to 0$ in $L^p(\mathbb{R}^3)$ for $p \in (2, 2^*]$. Here $2^* = \frac{2N}{N-2}$ if $N \geqslant 3$ and $2^* = \infty$ if $N = 1, 2$.

Now we are ready to study the minimize sequence for $\mathcal{T}_{\mu,\lambda}$.

Lemma 3.7. *Let $\{u_n\} \subset \mathcal{N}_\mu$ be a minimizing sequence for $\mathcal{T}_{\mu,\lambda}$. Then $\{u_n\}$ is bounded. Moreover, there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that*

$$\liminf_{n\to\infty} \int\limits_{B_r(y_n)} |u_n|^2 \geqslant \delta > 0,$$

where $B_r(y_n) = \{y \in \mathbb{R}^3\colon |y - y_n| \leqslant r\}$ for each $n \in \mathbb{N}$.

Proof. We first prove the boundedness of $\{u_n\}$. Arguing by contradiction, suppose that there exists a sequence $\{u_n\} \subset \mathcal{N}_\mu$ such that $\|u_n\|_\mu \to \infty$ and $\mathcal{T}_{\mu,\lambda}(u_n) \to c_\mu$. Let $z_n = \frac{u_n}{\|u_n\|_\mu}$. Then $z_n \rightharpoonup z$ and $z_n(x) \to z_n(x)$ a.e. in $\mathbb{R}^3$ after passing to a subsequence. Moreover, we have either $\{z_n\}$ is vanishing, i.e.,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int\limits_{B_r(y)} |z_n|^6 = 0 \tag{3.13}$$

or non-vanishing, i.e., there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\lim_{n\to\infty} \int\limits_{B_r(y_n)} |z_n|^6 \geqslant \delta > 0. \tag{3.14}$$

As in [8], we shall show neither (3.13) nor (3.14) takes place and this will provide the desired contradiction.

If $\{z_n\}$ is vanishing, Lemma 3.6 implies $z_n \to 0$ in $L^p(\mathbb{R}^3)$ for $p \in (2, 6]$. Therefore from (2.3) we deduce that $\int_{\mathbb{R}^3} F(Kz_n) \to 0$ as $n \to \infty$ for each $K \in \mathbb{R}$. So we infer from Lemma 3.1 that for $\lambda > 0$ and $\mu > 0$

$$\begin{aligned} c_\mu + o(1) &\geqslant \mathcal{T}_{\mu,\lambda}(u_n) \geqslant \mathcal{T}_{\mu,\lambda}(Kz_n) \\ &= \frac{K^2}{2} \int\limits_{\mathbb{R}^3} \left(a|\nabla z_n|^2 + \mu|z_n|^2\right) + \frac{bK^4}{4}\left(\int\limits_{\mathbb{R}^3} |\nabla z_n|^2\right)^2 - \lambda \int\limits_{\mathbb{R}^3} F(Kz_n) - \frac{K^6}{6} \int\limits_{\mathbb{R}^3} |z_n|^6 \\ &\geqslant \frac{cK^2}{2} - \lambda \int\limits_{\mathbb{R}^3} F(Kz_n) - \frac{K^6}{6} \int\limits_{\mathbb{R}^3} |z_n|^6 \to \frac{cK^2}{2}, \end{aligned}$$

as $n \to \infty$. Now we arrive a contradiction if K is large enough. Hence non-vanishing must hold. It follows from (2.3) that

$$\int\limits_{\mathbb{R}^3} F(u_n) \leqslant c\epsilon \|u_n\|_\mu^2 + cC_\epsilon \|u_n\|_\mu^p. \tag{3.15}$$

So from (3.14) and (3.15) we infer that for n large

$$0 \leqslant \frac{\mathcal{T}_{\mu,\lambda}(u_n)}{\|u_n\|_\mu^6} = -\frac{1}{6} \int\limits_{\mathbb{R}^3} |z_n|^6 + o(1) \leqslant -\frac{1}{6} \sup_{y\in\mathbb{R}^3} \int\limits_{B_r(y)} |z_n|^6 + o(1) < 0,$$

a contradiction.

Next we prove the latter conclusion of this lemma. Since $\{u_n\}$ is bounded, if

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_r(y)}|u_n|^2=0,$$

then from Lemma 3.6 we deduce that $u_n\to 0$ in $L^p(\mathbb{R}^3)$ for $p\in(2,6)$. We infer from Lemma 2.2 and (2.3) that $\int_{\mathbb{R}^3}F(u_n)\to 0$, as $n\to\infty$. Moreover, it follows from $\mathcal{T}'_{\mu,\lambda}(u_n)u_n=0$ that

$$\int_{\mathbb{R}^3}\big(a|\nabla u_n|^2+\mu|u_n|^2\big)+b\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2\Big)^2=\int_{\mathbb{R}^3}u_n^6+o(1). \tag{3.16}$$

Assume that $\int_{\mathbb{R}^3}(a|\nabla u_n|^2+\mu|u_n|^2)\to\ell_1\geqslant 0$ and $b(\int_{\mathbb{R}^3}|\nabla u_n|^2)^2\to\ell_2\geqslant 0$. We claim that $\ell_1>0$ if and only if $\ell_2>0$. In fact, if $\ell_2>0$, from

$$b\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2\Big)^2\leqslant c\Big(\int_{\mathbb{R}^3}\big(a|\nabla u_n|^2+\mu|u_n|^2\big)\Big)^2$$

we derive that $0<\ell_2\leqslant\ell_1^2$. Conversely, for $\ell_1>0$, if $\ell_2=0$, then we deduce from (3.16) and

$$\Big(\int_{\mathbb{R}^3}|u_n|^6\Big)^{\frac{1}{3}}\leqslant cb\Big(\int_{\mathbb{R}^3}\big(|\nabla u_n|^2\big)\Big)^2$$

that $0\leqslant(\ell_1+\ell_2)^{\frac{1}{3}}\leqslant\ell_2=0$. So we obtain $\ell_1=0$, this a contradiction. Thus we prove the claim.

If $\ell_1>0$, one has $\ell_2>0$. Since $\mathcal{T}_{\mu,\lambda}(u_n)\to c_\mu>0$, then one sees that

$$\frac{1}{2}\int_{\mathbb{R}^3}\big(a|\nabla u_n|^2+\mu|u_n|^2\big)+\frac{b}{4}\Big(\int_{\mathbb{R}^3}|\nabla u_n|^2\Big)^2-\frac{1}{6}\int_{\mathbb{R}^3}u_n^6\to c_\mu.$$

Thus we obtain $c_\mu=\frac{\ell_1}{3}+\frac{\ell_2}{12}$. On the other hand, since $\ell_1\geqslant a\mathcal{S}(\ell_1+\ell_2)^{\frac{1}{3}}$ and $\ell_2\geqslant b\mathcal{S}^2(\ell_1+\ell_2)^{\frac{2}{3}}$, we deduce that

$$\ell_1\geqslant(a\mathcal{S})^{\frac{3}{2}}\quad\text{and}\quad\ell_2\geqslant b^3\mathcal{S}^6.$$

So $c_\mu=\frac{\ell_1}{3}+\frac{\ell_2}{12}\geqslant\frac{1}{3}(a\mathcal{S})^{\frac{3}{2}}+\frac{1}{12}b^3\mathcal{S}^6$. This contradicts with Lemma 3.5. Therefore $\ell_1=\ell_2=0$, this contradicts with the conclusion (ii) of Lemma 3.1. □

Let us now state the main results for the limit problem $(\mathcal{H}_\mu)$.

Theorem 3.8. *Let the assumptions Lemma 2.1 be satisfied. Then there exists $\lambda^*>0$ such that for each $\lambda\in[\lambda^*,\infty)$ and $\mu>0$, we have the following conclusions hold*:

(i) *$(\mathcal{H}_\mu)$ has at least one positive ground state solution u_μ in $E_\mu=H^1(\mathbb{R}^3)$;*
(ii) *$\lim_{|x|\to\infty}u_\mu(x)=0$, $\lim_{|x|\to\infty}|\nabla u_\mu(x)|=0$ and $u_\mu\in C^{1,\sigma}_{loc}$ with $\sigma\in(0,1)$. Furthermore, there exist $C,c>0$ such that $u_\mu(x)\leqslant Ce^{-c|x-x_\mu|}$, where $u_\mu(x_\mu)=\max_{x\in\mathbb{R}^N}u_\mu(x)$;*
(iii) *$\mathcal{L}_\mu$ is compact in $H^1(\mathbb{R}^3)$, where $\mathcal{L}_\mu$ denotes the set of all least energy positive solutions of $(\mathcal{H}_\mu)$.*

Proof. (i) From the conclusion (v) of Lemma 3.1 we know that $c_\mu > 0$ for each $\mu > 0$. Moreover, if $u_0 \in \mathcal{N}_\mu$ satisfies $\mathcal{T}_{\mu,\lambda}(u_0) = c_\mu$, then $\check{m}_\mu(u_0)$ is a minimizer of $\Upsilon_{\mu,\lambda}$ and therefore a critical point of $\Upsilon_{\mu,\lambda}$, so that u_0 is a critical point of $\mathcal{T}_{\mu,\lambda}$ by Lemma 3.2. It remains to show that there exists a minimizer u of $\mathcal{T}_{\mu,\lambda}|_{\mathcal{N}_\mu}$. By Ekeland's variational principle [10], there exists a sequence $\{v_n\} \subset S_\mu$ such that $\Upsilon_{\mu,\lambda}(v_n) \to c_\mu$ and $\Upsilon'_{\mu,\lambda}(v_n) \to 0$ as $n \to \infty$. Set $u_n = m_\mu(v_n) \in \mathcal{N}_\mu$ for all $n \in \mathbb{N}$. Then $\mathcal{T}_{\mu,\lambda}(u_n) \to c_\mu$ and $\mathcal{T}'_{\mu,\lambda}(u_n) \to 0$ as $n \to \infty$. By Lemma 3.7, we know that $\{u_n\}$ is bounded there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\lim_{n\to\infty} \int\limits_{B_r(y_n)} |u_n|^2 \geqslant \delta > 0.$$

So we can choose $r' > r > 0$ and a sequence $\{y_n\} \subset \mathbb{Z}^3$ such that

$$\lim_{n\to\infty} \int\limits_{B_{r'}(y_n)} |u_n|^2 \geqslant \frac{\delta}{2} > 0. \tag{3.17}$$

Using Φ_μ and $\mathcal{N}_\mu$ are invariant under translations, we may select that $\{y_n\}$ is bounded in $\mathbb{Z}^3$. So $u_n \rightharpoonup u \neq 0$ and $\mathcal{T}'_{\mu,\lambda}(u) = 0$.

It remains to show that $\mathcal{T}_{\mu,\lambda}(u) = c_\mu$. If $\lambda > 0$, since $\{u_n\}$ is bounded, by (1.4) and Fatou's lemma we get that

$$\begin{aligned}
c_\mu &= \liminf_{n\to\infty}\left(\mathcal{T}_{\mu,\lambda}(u_n) - \frac{1}{4}\mathcal{T}'_{\mu,\lambda}(u_n)u_n\right) \\
&= \liminf_{n\to\infty}\left(\frac{1}{4}\int\limits_{\mathbb{R}^3}\left(|\nabla u_n|^2 + \mu|u_n|^2\right) + \lambda\int\limits_{\mathbb{R}^3}\left(\frac{1}{4}f(u_n)u_n - F(u_n)\right)\right) + \frac{1}{12}\int\limits_{\mathbb{R}^3}|u_n|^6 \\
&\geqslant \frac{1}{4}\int\limits_{\mathbb{R}^3}\left(|\nabla u|^2 + \mu|u|^2\right) + \lambda\int\limits_{\mathbb{R}^3}\left(\frac{1}{4}f(u)u - F(u)\right) + \frac{1}{12}\int\limits_{\mathbb{R}^3}|u|^6 \\
&= \mathcal{T}_{\mu,\lambda}(u) - \frac{1}{2}\mathcal{T}'_{\mu,\lambda}(u)u = \mathcal{T}_{\mu,\lambda}(u).
\end{aligned}$$

Hence $\mathcal{T}_{\mu,\lambda}(u) \leqslant c_\mu$. The reverse inequality follows from the definition of c_μ since $u \in \mathcal{N}_\mu$. So we prove that $\mathcal{T}_{\mu,\lambda}(u) = c_\mu$.

Let us note that all the calculations above can be repeated word by word, replacing $\mathcal{T}_{\mu,\lambda}$ with the functional

$$\mathcal{T}^+_{\mu,\lambda}(u) = \frac{1}{2}\int\limits_{\mathbb{R}^3}\left(a|\nabla u|^2 + \mu|u|^2\right) + \frac{b}{4}\left(\int\limits_{\mathbb{R}^3}|\nabla u|\right)^2 - \lambda\int\limits_{\mathbb{R}^3}F\left(u^+\right) - \frac{1}{6}\int\limits_{\mathbb{R}^3}\left(u^+\right)^6,$$

where $u^+ = \max\{u, 0\}$ is the positive part of u. In this way we find a ground state solution $u \in H^1(\mathbb{R}^3)$ of the equation

$$-\left(a + b\int\limits_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u + \mu u + = \lambda f\left(u^+\right) + \left(u^+\right)^5. \tag{3.18}$$

In (3.18), using $u^- = \max\{-u, 0\}$ as a text function and integrating by parts, we obtain

$$\int_{\mathbb{R}^3} \left(a\left|\nabla u^-\right|^2 + \left|u^-\right|^2\right) + b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right) \int_{\mathbb{R}^3} \nabla u \nabla u^- - \lambda \int_{\mathbb{R}^3} f\left(u^+\right) u^- = 0.$$

So we have that

$$\int_{\mathbb{R}^3} \left(a\left|\nabla u^-\right|^2 + \left|u^-\right|^2\right) + b\left(\int_{\mathbb{R}^3} |\nabla u|^2\right) \int_{\mathbb{R}^3} \left|\nabla u^-\right|^2 = 0.$$

Thus $u^- = 0$, and $u \geqslant 0$ is a solution of $(\mathcal{H}_\mu)$. Therefore, from Harnack's inequality (see [14]), we infer that $u > 0$ for all $x \in \mathbb{R}^3$. This finishes the proof of the conclusion (i).

(ii) Using the arguments of [15] (also see [13,12]), we have that $u \in L^t(\mathbb{R}^3)$ for $t \in [2, \infty]$. Let $A = a + b \int_{\mathbb{R}^3} |\nabla u|^2$ and

$$g(x, u) = \frac{1}{A}\left(\lambda f(u) + u^5 - \mu u\right).$$

From (2.3), we infer that

$$\left|g(x, u)\right| \leqslant c\left(|u| + |u|^{p-1} + |u|^5\right).$$

It follows that

$$\left|g(x, u)\right|_{L^\tau(B_{2\rho})} \leqslant c\left(|u|_{L^\tau(B_{2\rho})} + |u|^{p-1}_{L^{(p-1)\tau}(B_{2\rho})} + |u|^5_{L^{5\tau}(B_{2\rho})}\right), \tag{3.19}$$

where $3 < \tau < 6$, $4 < p < 6$ and $B_{2\rho} = \{x \in \mathbb{R}^3\colon\ |x - x_0| \leqslant 2\rho,\ x_0 \in \mathbb{R}^3\}$. Using $(\mathcal{H}_\mu)$, we conclude that $\Delta u \in L^\tau(B_{2\rho})$, for all $3 < \tau < 6$. By the Calderon–Zygmund inequality (see Theorem 9.9 of [16]), we conclude that $u \in W^{2,\tau}(B_{2\rho})$. Next, by the interior L^p-estimates we have

$$\|u\|_{W^{2,\tau}(B_\rho)} \leqslant c\left(|u|_{L^\tau(B_{2\rho})} + \left|g(x, u)\right|_{L^\tau(B_{2\rho})}\right). \tag{3.20}$$

From (3.19) and (3.20), we deduce that

$$\|u\|_{W^{2,\tau}(B_\rho)} \leqslant c\left(|u|_{L^\tau(B_{2\rho})} + |u|^{p-1}_{L^{(p-1)\tau}(B_{2\rho})} + |u|^5_{L^{5\tau}(B_{2\rho})}\right),$$

where $B_\rho = B_\rho(x_0)$. Since $\tau > 3$, by Sobolev imbedding theorem (see [14]) one has

$$\|u\|_{C^{1,\sigma}(\bar{B}_\rho)} \leqslant c\left(|u|_{L^\tau(B_{2\rho})} + |u|^{p-1}_{L^{(p-1)\tau}(B_{2\rho})} + |u|^5_{L^{5\tau}(B_{2\rho})}\right),$$

where $\sigma \in (0, 1)$. Letting $|x_0| \to \infty$, we conclude that $\|u\|_{C^{1,\sigma}(\bar{B}_\rho)} \to 0$. Therefore, we get that $\lim_{|x|\to\infty} u(x) = 0$, $\lim_{|x|\to\infty} |\nabla u(x)| = 0$ and $u \in C^{1,\sigma}_{loc} \cap L^\infty(\mathbb{R}^3)$ for $0 < \sigma < 1$.

Next we shall prove that $u(x) \leqslant Ce^{-c|x - x_\mu|}$, where $u(x_\mu) = \max_{x\in\mathbb{R}^3} u(x)$. To do this, we develop a contradiction argument related to the one introduced in [14] (also see [17]). We fix $\alpha \in (0, \sqrt{\frac{\mu}{A}})$

and let $\eta = \mu - \alpha^2 A$, where $0 < a < A = a + b\int_{\mathbb{R}^3} |\nabla u|^2 < \infty$. Since $u(x) \to 0$ as $|x| \to \infty$, we conclude that there is $R > 0$ such that

$$\frac{\lambda f(u(x)) + (u(x))^5}{u(x)} \leqslant \eta, \quad \forall |x| \geqslant R. \tag{3.21}$$

Let $m(x) = Ge^{-\alpha(|x-x_\mu|-R)}$, where $G = \max\{|u(x)|: |x - x_\mu| = R\}$. For $K > R$, let us define the set

$$\Pi_K = \{x \in \mathbb{R}^3: R < |x - x_\mu| < K, u(x) > m(x), u(x) > 0\}.$$

In the following we shall prove that Π_K is empty. Suppose, by contradiction, that this is not so. Thus Π_K is a nonempty open set and in it we have

$$\Delta(m - u) = \left(\alpha^2 - \frac{2\alpha}{|x|}\right)m(x) + \frac{1}{A}\left(\lambda f(u) + u^5 - \mu u\right).$$

Moreover, we infer from (3.21) that

$$\Delta(m - u) \leqslant \alpha^2 m(x) + \frac{1}{A}\left(\frac{\lambda f(u) + u^5}{u} - \mu\right)u \leqslant \alpha^2\left(m(x) - u\right). \tag{3.22}$$

From the definition of Π_K, we deduce from (3.22) that $\Delta(m - u) < 0$ in Π_K. Using the maximum principle, we conclude that

$$m(x) - u \geqslant \min_{\partial \Pi_K}(m - u).$$

Since $|x - x_\mu| = R$ does not belong to the boundary of Π_K, we have

$$m(x) - u \geqslant \min\Big\{0, \min_{|x-x_\mu|=K}\big(m(x) - u(x)\big)\Big\}.$$

Now, letting $K \to \infty$, and using the fact that u decays to 0 at ∞, we have that, for each fixed $|x - x_\mu| > R$, $m(x) - u(x) \geqslant 0$, contradicting the definition of Π_K. So, the Π_K is empty, i.e., for $|x - x_\mu| > R$ such that $u > 0$, we obtain $u(x) \leqslant m(x)$. That is, $u(x) \leqslant Ce^{c|x-x_\mu|}$ for $C, c > 0$.

(iii) Let the bounded sequence $\{u_n\} \subset \mathcal{L}_\mu \cap \mathcal{N}_\mu$ such that $\mathcal{T}_{\mu,\lambda}(u_n) = c_\mu$ and $\mathcal{T}'_{\mu,\lambda}(u_n) = 0$. Without loss of generality we assume that $u_n \rightharpoonup u$ in E_μ. As in the proof of the conclusion (i), one can easily prove that $\{u_n\}$ is non-vanishing, i.e.,

$$\lim_{n\to\infty} \int_{B_r(y_n)} |u_n|^2 \geqslant \frac{\delta}{2} > 0.$$

By the invariance of $\mathcal{T}_{\mu,\lambda}$ and $\mathcal{N}_\mu$ under translations of the form $u \mapsto u(\cdot - k)$ with $k \in \mathbb{Z}^3$, we may assume that $\{y_n\}$ is bounded in $\mathbb{Z}^3$. So $u_n \rightharpoonup u \neq 0$ and $\mathcal{T}'_{\mu,\lambda}(u) = 0$. Moreover, repeating arguments as in the proof of the conclusion (i), one sees that $\mathcal{T}_{\mu,\lambda}(u) = c_\mu$. So it follows that

$$
\begin{aligned}
c_\mu = \mathcal{T}_{\mu,\lambda}(u) &= \mathcal{T}_{\mu,\lambda}(u) - \frac{1}{4}\big(\mathcal{T}'_{\mu,\lambda}(u), u\big) \\
&= \frac{1}{4}\int_{\mathbb{R}^3}\big(a|\nabla u|^2 + \mu|u|^2\big) + \lambda\int_{\mathbb{R}^3}\left(\frac{1}{4}f(u)u - F(u)\right) + \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_{\mathbb{R}^3}|u|^{2^*} \\
&\leqslant \liminf_{n\to\infty}\left[\frac{1}{4}\int_{\mathbb{R}^3}\big(a|\nabla u_n|^2 + \mu|u_n|^2\big) + \lambda\int_{\mathbb{R}^3}\left(\frac{1}{2}f(u_n)u_n - F(u_n)\right) + \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_{\mathbb{R}^3}|u_n|^{2^*}\right] \\
&= \liminf_{n\to\infty}\left(\mathcal{T}_{\mu,\lambda}(u_n) - \frac{1}{4}\mathcal{T}'_{\mu,\lambda}(u_n)u_n\right) = c_\mu. \qquad (3.23)
\end{aligned}
$$

From (3.23) and $\lambda > 0$, we deduce that $\lim_{n\to\infty}\int_{\mathbb{R}^3}(a|\nabla u_n|^2 + \mu|u_n|^2) = \int_{\mathbb{R}^3}(a|\nabla u|^2 + \mu|u|^2)$. That is, $u_n \to u$ in $H^1(\mathbb{R}^3)$. □

Remark 3.9. We point out that our arguments in this section applies equally well to the case of periodic potentials, namely, equation

$$
(\mathcal{H}_V) \quad -\left(a + b\int_{\mathbb{R}^3}|\nabla u|^2\right)\Delta u + V(x)u = \lambda f(u) + |u|^4 u, \quad u > 0,\ u \in H^1\big(\mathbb{R}^3\big),
$$

where $V(x)$ is a positive continues function and periodic in each variables. Using translation invariance of the problem the same proof is still valid. Thus if f satisfies the assumptions of Theorem 3.8, then the conclusions of Theorem 3.8 hold.

Lemma 3.10. *Under the assumptions of Lemma 2.1, we have that $c_{\mu_1} > c_{\mu_2}$ for $\mu_1 > \mu_2$.*

Proof. For $\mu_1, \mu_2 > 0$, one sees that $E_{\mu_1} = E_{\mu_2} = E$. Let $u_1 \in \mathcal{N}_{\mu_1}$ be such that

$$
c_{\mu_1} = \mathcal{T}_{\mu_1,\lambda}(u_1) = \max_{w\in E_{\mu_1}} \mathcal{T}_{\mu_1,\lambda}(w).
$$

On the other hand, let $u_2 \in E_{\mu_2}$ be such that

$$
\mathcal{T}_{\mu_2,\lambda}(u_2) = \max_{w\in E_{\mu_2}} \mathcal{T}_{\mu_2,\lambda}(w).
$$

Therefore one sees

$$
\begin{aligned}
c_{\mu_1} &\geqslant \mathcal{T}_{\mu_1,\lambda}(u_2) = \mathcal{T}_{\mu_2,\lambda}(u_2) + (\mu_1 - \mu_2)\int_{\mathbb{R}^3} u_2^2 \\
&\geqslant c_{\mu_2} + (\mu_1 - \mu_2)\int_{\mathbb{R}^3} u_2^2 > c_{\mu_2}. \qquad \square
\end{aligned}
$$

4. A compactness condition

In this section we shall prove some compactness results for the functional $\mathcal{T}_{\varepsilon,\lambda}$. Precisely, we shall show that any minimizing sequence of $\mathcal{T}_{\varepsilon,\lambda}$ has a strongly convergent subsequence in E_ε. We begin with the following lemma.

Lemma 4.1. *Under the assumptions of* $(\mathcal{C}_0)$ *and* $(\mathcal{F}_1)$–$(\mathcal{F}_3)$, *we have that*

(i) $c_\varepsilon \geqslant c_{M_0}$ *for all* $\varepsilon > 0$;
(ii) $c_\varepsilon \to c_{M_0}$ *as* $\varepsilon \to 0$.

Proof. (i) Since M is a bounded function, it is easily to check that for all $\varepsilon > 0$ and $\mu > 0$, $E_\varepsilon = E_\mu = H^1(\mathbb{R}^3)$. To prove the first conclusion, we argue by contradiction, assume that $c_\varepsilon < c_{M_0}$ for some $\varepsilon > 0$. By the definition of c_ε, we can choose an $e \in E_\varepsilon \setminus \{0\}$ such that $\max_{s>0} \mathcal{T}_{\varepsilon,\lambda}(se) < c_{M_0}$. Again by the definition of c_{M_0}, we know that $c_{M_0} \leqslant \max_{s>0} \mathcal{T}_{M_0,\lambda}(se)$. Since $M_\varepsilon(x) \geqslant M_0$, $\mathcal{T}_{\varepsilon,\lambda}(u) \geqslant \mathcal{T}_{M_0,\lambda}(u)$ for all $u \in E_\varepsilon$, and we get

$$c_{M_0} > \max_{s>0} \mathcal{T}_{\varepsilon,\lambda}(se) \geqslant \max_{s>0} \mathcal{T}_{M_0,\lambda}(se) \geqslant c_{M_0},$$

a contradiction.

(ii) Set $M^0(x) = M(x) - M_0$ and $M^0_\varepsilon(x) = M^0(\varepsilon x)$. Then we see

$$\mathcal{T}_{\varepsilon,\lambda}(u) = \mathcal{T}_{M_0,\lambda}(u) + \int_{\mathbb{R}^3} M^0_\varepsilon(x) u^2.$$

Let $u \in \mathcal{N}_{M_0}$ be such that $c_{M_0} = \mathcal{T}_{M_0,\lambda}(u) = \max_{w \in E_{M_0} \setminus \{0\}} \mathcal{T}_{M_0,\lambda}(w)$. We take $u_1 \in E_\varepsilon \setminus \{0\}$ such that

$$c_\varepsilon \leqslant \mathcal{T}_{\varepsilon,\lambda}(u_1) = \max_{s>0} \mathcal{T}_{\varepsilon,\lambda}(su) = \mathcal{T}_{M_0,\lambda}(u_1) + \int_{\mathbb{R}^3} M^0_\varepsilon(x) u_1^2. \tag{4.1}$$

Obviously, for each $\epsilon > 0$ we can choose $R > 0$ such that

$$\int_{|x|>R} M^0_\varepsilon(x) |u_1|^2 < c\epsilon. \tag{4.2}$$

Moreover, since $0 \in \mathcal{M}$, one has

$$\int_{|x| \leqslant R} M^0_\varepsilon(x) |u_1|^2 \to 0, \quad \text{as } \varepsilon \to 0. \tag{4.3}$$

Substituting (4.2) and (4.3) into (4.1), we deduce that

$$\int_{\mathbb{R}^3} M^0_\varepsilon(x) u_1^2 \to 0, \quad \text{as } \varepsilon \to 0.$$

Therefore, we get

$$\begin{aligned} c_\varepsilon &\leqslant \mathcal{T}_{M_0,\lambda}(u_1) + o(1) \leqslant \max_{w \in E_m \setminus \{0\}} \mathcal{T}_{M_0,\lambda}(w) + o(1) \\ &= \mathcal{T}_{M_0,\lambda}(u) + o(1) = c_{M_0} + o(1). \end{aligned}$$

Furthermore, it follows from the conclusion (i) that

$$c_{M_0} \leqslant \lim_{\varepsilon \to 0} c_\varepsilon \leqslant \lim_{\varepsilon \to 0} \mathcal{T}_{\varepsilon,\lambda}(u_1) = \mathcal{T}_{M_0,\lambda}(u_1) \leqslant \mathcal{T}_{M_0,\lambda}(u) = c_{M_0}.$$

Hence, we obtain $c_\varepsilon \to c_{M_0}$ as $\varepsilon \to 0$. $\square$

Form $(\mathcal{C}_1)$, we know that $M_0 < M_\infty$. So we can choose $\xi > 0$ such that

$$M_0 < \xi < M_\infty.$$

Therefore we first prove the following lemma.

Lemma 4.2. *Suppose that the assumptions of $(\mathcal{C}_0)$ and $(\mathcal{F}_1)$–$(\mathcal{F}_3)$ hold. Let $\{u_n\} \subset \mathcal{N}_\varepsilon$ such that $\mathcal{T}_{\varepsilon,\lambda}(u_n) \to c$ with $c \leqslant c_\xi$ and $u_n \rightharpoonup 0$ in E_ε, then one of the following conclusions holds*

(i) *$u_n \to 0$ in E_ε;*
(ii) *there exists a sequence $y_n \in \mathbb{R}^3$ and constants $r, \delta > 0$ such that*

$$\liminf_{n \to \infty} \int_{B_r(y_n)} u_n^2 \geqslant \delta.$$

Proof. Suppose that (ii) does not occur, i.e., there exists $r > 0$ such that

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^3} \int_{B_r(y)} u_n^2 = 0.$$

Then by Lemma 3.6, we deduce that $u_n \to 0$ in $L^t(\mathbb{R}^3)$ for $t \in (2,6)$. So from $\mathcal{T}'_{\varepsilon,\lambda}(u_n)u_n = 0$, we infer that

$$\int_{\mathbb{R}^3} \left(a|\nabla u_n|^2 + M(\varepsilon x)u_n^2\right) + b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^2 = \int_{\mathbb{R}^3} u_n^6 + o(1).$$

Assume that $\int_{\mathbb{R}^3}(a|\nabla u_n|^2 + M(\varepsilon x)u_n^2) \to \ell_1 \geqslant 0$ and $b(\int_{\mathbb{R}^3} |\nabla u_n|^2)^2 \to \ell_2 \geqslant 0$. Since $c \leqslant c_\kappa$, by using the same arguments as in Lemma 3.7, one can easily check that $\ell_1 > 0$ if and only if $\ell_2 > 0$. Moreover, if $\ell_1 > 0$ or $\ell_2 > 0$, one can obtain the contradiction. Thus $\ell_1 = \ell_2 = 0$. $\square$

Lemma 4.3. *Let the assumptions of $(\mathcal{C}_0)$ and $(\mathcal{F}_1)$–$(\mathcal{F}_3)$ be satisfied. If $\{u_n\} \subset \mathcal{N}_\varepsilon$ such that $\mathcal{T}_{\varepsilon,\lambda}(u_n) \to c$ with $c \leqslant c_\xi$ and $u_n \rightharpoonup 0$ in E_ε, we have that $u_n \to 0$ in E_ε for $\varepsilon > 0$ small.*

Proof. Let $\{u_n\} \subset \mathcal{N}_\varepsilon$ such that

$$\mathcal{T}_{\varepsilon,\lambda}(u_n) \to c \quad \text{and} \quad \mathcal{T}'_{\varepsilon,\lambda}(u_n)u_n = 0. \tag{4.4}$$

We choose $(t_n) \subset (0,\infty)$ such that $\{t_n u_n\} \subset \mathcal{N}_{M_\infty}$.

If $u_n \nrightarrow 0$ in E_ε, we first claim that the sequence $\{t_n\}$ such that $\limsup_{n\to\infty} t_n \leqslant 1$. Assume by contradiction, there exist $\sigma > 0$ and a subsequence still denoted by $\{t_n\}$ such that $t_n \geqslant 1+\sigma$ for all $n \in \mathbb{N}$. By Lemma 3.1 we see that $\{t_n\}$ is bounded and, from $\mathcal{T}'_{\varepsilon,\lambda}(u_n)u_n = o(1)$, one has

$$\int_{\mathbb{R}^3} \big(a|\nabla u_n|^2 + M(\varepsilon x)|u_n|^2\big) + b\bigg(\int_{\mathbb{R}^3} |\nabla u_n|^2\bigg)^2 = \lambda \int_{\mathbb{R}^3} f(u_n)u_n + \int_{\mathbb{R}^3} u_n^6 + o(1). \tag{4.5}$$

Moreover, since $\{t_n u_n\} \subset \mathcal{N}_{M_\infty}$, then we see

$$t_n^2 \int_{\mathbb{R}^3} \big(a|\nabla u_n|^2 + M_\infty |u_n|^2\big) + t_n^4 b\bigg(\int_{\mathbb{R}^3} |\nabla u_n|^2\bigg)^2 = \lambda \int_{\mathbb{R}^3} f(t_n u_n)t_n u_n + t_n^6 \int_{\mathbb{R}^3} |u_n|^6. \tag{4.6}$$

Combining (4.5) and (4.6), we obtain that

$$\begin{aligned} o(1) + \bigg(\frac{1}{t_n^2} - 1\bigg) \int_{\mathbb{R}^3} a|\nabla u_n|^2 + \int_{\mathbb{R}^3} \bigg(\frac{M_\infty}{t_n^2} - M(\varepsilon x)\bigg) u_n^2 \\ = \lambda \int_{\mathbb{R}^3} \bigg(\frac{f(t_n u_n)}{t_n^3 u_n^3} - \frac{f(u_n)}{u_n^3}\bigg) u_n^4 + \big(t_n^2 - 1\big) \int_{\mathbb{R}^3} u_n^6. \end{aligned} \tag{4.7}$$

By condition $(\mathcal{C}_0)$ and $t_n > 1$, for any $\epsilon > 0$, there exists $G = G(\epsilon) > 0$ such that

$$M(\varepsilon x) \geqslant M_\infty - \epsilon > \frac{M_\infty}{t_n^2} - \epsilon \quad \text{for any } |x| \geqslant G. \tag{4.8}$$

Since $\|u_n\|_\varepsilon \leqslant C$, $u_n \rightharpoonup 0$ in $L^2(B_G(0))$ and $t_n u_n \geqslant u_n$, we deduce from the condition $(\mathcal{F}_2)$ that

$$\big((1+\sigma)^2 - 1\big) \int_{\mathbb{R}^3} u_n^6 \leqslant c\epsilon. \tag{4.9}$$

Since $u_n \nrightarrow 0$ in E_ε, it follows from Lemma 4.2 that there exists a sequence $y'_n \in \mathbb{R}^3$ and constants $r', \delta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_{r'}(y'_n)} u_n^2 \geqslant \delta > 0.$$

Thus we can choose a sequence $y_n \in \mathbb{Z}^3$ and constants $r > r' > 0$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} u_n^2 \geqslant \delta > 0. \tag{4.10}$$

If we set $v_n(x) = u(x + y_n)$, then there exists a function v that, up to a subsequence, $v_n \rightharpoonup v$ in E_ε, $v_n \to v$ in $L^2_{loc}(\mathbb{R}^3)$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^3$. Moreover, by (4.10), there exists a subset Ω in $\mathbb{R}^3$ with positive measure such that $v \neq 0$ a.e. in Ω. It follows from (4.9) and Fatou's lemma that

$$0 < \big((1+\sigma)^2 - 1\big) \int_\Omega v^6 \leqslant c\epsilon \tag{4.11}$$

for any $\epsilon > 0$, which yields a contradiction.

In the sequel, we shall prove that the case of $\limsup_{n\to\infty} t_n \leqslant 1$ cannot be happened. Then we obtain a contradiction and $u_n \to 0$ in E_ε. To do this, we distinguish the following two cases: $\limsup_{n\to\infty} t_n = 1$ and $\limsup_{n\to\infty} t_n < 1$.

(a) $\limsup_{n\to\infty} t_n = 1$.

In this case, there exists a subsequence, still denoted by $\{t_n\}$ such that $t_n \to 1$ as $n \to \infty$. Hence,

$$o(1) + c_\xi \geqslant \mathcal{T}_{\varepsilon,\lambda}(u_n) \geqslant \mathcal{T}_{\varepsilon,\lambda}(u_n) + c_{M_\infty} - \mathcal{T}_{M_\infty,\lambda}(t_n u_n). \tag{4.12}$$

It is clear that

$$\begin{aligned}\mathcal{T}_{\varepsilon,\lambda}(u_n) - \mathcal{T}_{M_\infty,\lambda}(t_n u_n) &= \frac{1}{2}\int_{\mathbb{R}^3} \big(1-t_n^2\big)|\nabla u_n|^2 + b\big(1-t_n^4\big)\bigg(\int_{\mathbb{R}^3} |\nabla u_n|^2\bigg)^2 + \big(1-t_n^6\big)\int_{\mathbb{R}^3} |u_n|^6 \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^3} M(\varepsilon x)u_n^2 - \frac{t_n^2}{2}\int_{\mathbb{R}^3} M_\infty u_n^2 + \lambda \int_{\mathbb{R}^3} \big(F(t_n u_n) - F(u_n)\big).\end{aligned} \tag{4.13}$$

From the boundedness of $\{u_n\}$ and (4.8), we infer that

$$\mathcal{T}_{\varepsilon,\lambda}(u_n) - \mathcal{T}_{M_\infty,\lambda}(t_n u_n) \geqslant o(1) - c\epsilon + \lambda \int_{\mathbb{R}^3} \big(F(t_n u_n) - F(u_n)\big) = o(1) - c\epsilon \tag{4.14}$$

by using the mean value theorem and the Lebesgue theorem. Taking the limit of the above inequality (4.12), we have $c_\xi \geqslant c_{M_\infty}$. On the other hand, from Lemma 4.1, we deduce that $c_\xi < c_{M_\infty}$. This is a contradiction.

(b) $\limsup_{n\to\infty} t_n < 1$.

In this case, we may suppose, without loss of generality, $t_n < 1$ for all $n \in \mathbb{N}$. From (4.8), $\{t_n u_n\} \subset \mathcal{N}_{M_\infty}$, $u_n \to 0$ in $L^2_{loc}(\mathbb{R}^3)$ and $\|u_n\|_\varepsilon \leqslant c$, we see that

$$\begin{aligned}c_{M_\infty} \leqslant \mathcal{T}_{M_\infty,\lambda}(t_n u_n) &= \mathcal{T}_{\varepsilon,\lambda}(t_n u_n) + \frac{t_n^2}{2}\int_{\mathbb{R}^3} \big(M_\infty - M(\varepsilon x)\big)u_n^2 \\ &\leqslant \mathcal{T}_{\varepsilon,\lambda}(u_n) + \epsilon c \leqslant c_\xi + \epsilon c + o(1).\end{aligned} \tag{4.15}$$

Let $n \to \infty$, we get $c_\xi \geqslant c_{M_\infty}$. This contradicts with $c_\xi < c_{M_\infty}$. $\square$

Lemma 4.4. *Under the assumptions of $(\mathcal{C}_0)$ and $(\mathcal{F}_1)$–$(\mathcal{F}_3)$, we have that if $\{u_n\} \subset \mathcal{N}_\varepsilon$ such that $\mathcal{T}_{\varepsilon,\lambda}(u_n) \to c$ with $0 < c \leqslant c_\xi < c_{M_\infty}$, then $\{u_n\}$ has a convergent subsequence in E_ε.*

Proof. Let $\{u_n\} \subset \mathcal{N}_\varepsilon$ such that

$$\mathcal{T}_{\varepsilon,\lambda}(u_n) \to c \quad \text{and} \quad \mathcal{T}'_{\varepsilon,\lambda}(u_n)u_n = 0.$$

Similar to Lemma 3.7, one can easily check that $\{u_n\}$ is bounded. So there exists $u \in E_\varepsilon$ such that $u_n \rightharpoonup u$ in E_ε. Moreover, u is a critical point of $\mathcal{R}'_\varepsilon$. Set $w_n = u_n - u$. By Brezis–Lieb Lemma (see [10]), we have

$$\int_{\mathbb{R}^3} |\nabla w_n|^2 = \int_{\mathbb{R}^3} |\nabla u_n|^2 - \int_{\mathbb{R}^3} |\nabla u|^2 + o(1)$$

and

$$\left(\int_{\mathbb{R}^3} |\nabla w_n|^2\right)^2 = \left(\int_{\mathbb{R}^3} |\nabla u_n|^2\right)^2 - \left(\int_{\mathbb{R}^3} |\nabla u|^2\right)^2 + o(1).$$

Moreover, as in [1,18,19] that $\mathcal{T}_{\varepsilon,\lambda}(w_n) = \mathcal{T}_{\varepsilon,\lambda}(u_n) - \mathcal{T}_{\varepsilon,\lambda}(u) + o(1)$ and $\mathcal{T}'_{\varepsilon,\lambda}(w_n) \to 0$ as $n \to \infty$. It follows from $\mathcal{T}'_{\varepsilon,\lambda}(u) = 0$ and (1.4) that

$$\mathcal{T}_{\varepsilon,\lambda}(u) = \mathcal{T}_{\varepsilon,\lambda}(u) - \frac{1}{4}\mathcal{T}'_{\varepsilon,\lambda}(u)u = \frac{1}{4}\int_{\mathbb{R}^3} \left(a|\nabla u|^2 + M(\varepsilon x)u^2\right) + \lambda \int_{\mathbb{R}^3} \left(\frac{1}{4} f(u)u - F(u)\right) + \int_{\mathbb{R}^3} |u|^6 \geqslant 0.$$

So we deduce that $\mathcal{T}_{\varepsilon,\lambda}(w_n) = \mathcal{T}_{\varepsilon,\lambda}(u_n) - \mathcal{T}_{\varepsilon,\lambda}(u) + o(1) \to c - y$ as $n \to \infty$, where $y = \mathcal{T}_{\varepsilon,\lambda}(u) \geqslant 0$. Thus it follows from $c_1 = c - y \leqslant c \leqslant c_\xi$ and Lemma 4.3 that $w_n = u_n - u \to 0$ in E_ε. □

Now we are in a position to prove that $(\mathcal{H}_\varepsilon)$ has a positive ground state solution.

Lemma 4.5. *Under the assumptions of $(\mathcal{C}_0)$ and $(\mathcal{F}_1)$–$(\mathcal{F}_3)$, we have that c_ε is attained for all small $\varepsilon > 0$.*

Proof. It follows from the conclusion (v) of Lemma 2.2 that $c_\varepsilon \geqslant \rho > 0$ for each $\varepsilon > 0$. Moreover, if $u_\varepsilon \in \mathcal{N}_\varepsilon$ satisfies $\mathcal{T}_{\varepsilon,\lambda}(u_\varepsilon) = c_\varepsilon$, then $\check{m}_\varepsilon(u_\varepsilon)$ is a minimizer of $\Upsilon_{\varepsilon,\lambda}$ and therefore a critical point of $\Upsilon_{\varepsilon,\lambda}$, so that u_ε is a critical point of $\mathcal{T}_{\varepsilon,\lambda}$ by Lemma 2.3. It remains to show that there exists a minimizer u_ε of $\mathcal{T}_\varepsilon|_{\mathcal{N}_\varepsilon}$. By Ekeland's variational principle [10], there exists a sequence $\{\nu_n\} \subset S_\varepsilon$ such that $\Upsilon_{\varepsilon,\lambda}(\nu_n) \to c_\varepsilon$ and $\Upsilon'_{\varepsilon,\lambda}(\nu_n) \to 0$ as $n \to \infty$. Set $w_n = m_\varepsilon(\nu_n) \in \mathcal{N}_\varepsilon$ for all $n \in \mathbb{N}$. Then from Lemma 2.3 again, we deduce that $\mathcal{T}_{\varepsilon,\lambda}(w_n) \to c_\varepsilon$, $\mathcal{T}'_{\varepsilon,\lambda}(w_n)w_n = 0$ and $\mathcal{T}'_{\varepsilon,\lambda}(w_n) \to 0$ as $n \to \infty$. So $\{w_n\}$ is a $(PS)_{c_\varepsilon}$-sequence for $\mathcal{T}_{\varepsilon,\lambda}$. By Lemma 4.2, we know that $c_\varepsilon \leqslant c_\xi$ for $\varepsilon > 0$ small. Thus from Lemma 4.4, we infer that $u_n = w_n - w \to 0$ in E_ε. Therefore we prove that $w \in \mathcal{N}_\varepsilon$ and $\mathcal{T}_{\varepsilon,\lambda}(w) = c_\varepsilon$. □

Let $\mathcal{L}_\varepsilon$ denote the set of all positive ground state solutions of $(\mathcal{H}_\varepsilon)$. Similar to the conclusion (iii) of Theorem 3.8, one has the following lemma.

Lemma 4.6. *Suppose that the assumptions of Theorem 1.1 are satisfied. Then $\mathcal{L}_\varepsilon$ is compact in $H^1(\mathbb{R}^3)$ for all small $\varepsilon > 0$.*

Proof. Let the boundedness sequence $\{u_n\} \subset \mathcal{L}_\varepsilon \cap \mathcal{N}_\varepsilon$ such that $\mathcal{T}_{\varepsilon,\lambda}(u_n) = c_\mu$ and $\mathcal{T}'_{\varepsilon,\lambda}(u_n) = 0$. Without loss of generality we assume that $u_n \rightharpoonup u \in E_\varepsilon$. Then it follows from the weakly continuous of $\mathcal{T}'_\varepsilon$ that $\mathcal{T}'_\varepsilon(u) = 0$. Set $w_n = u_n - u$. As in Lemma 4.6, we can prove that $w_n \to 0$ in $H^1(\mathbb{R}^3)$. □

5. Multiplicity and concentration of positive solutions

In this section, we are in a position to give the proof of the main results. We first prove that $(\mathcal{H}_\varepsilon)$ has existence of multiple positive solutions. To do this, we shall make good use of the ground state solution of $(\mathcal{H}_{M_0})$. Precisely, let w be a ground state solution of problem $(\mathcal{H}_{M_0})$ and ϕ be a smooth nonincreasing function defined in $[0,\infty)$ such that $\phi(s)=1$ if $0\leqslant s\leqslant \frac{1}{2}$ and $\phi(s)=0$ if $s\geqslant 1$. For any $y\in\mathcal{M}$, we define

$$\psi_{\varepsilon,y}(x)=\phi\big(|\varepsilon x-y|\big)w\left(\frac{\varepsilon x-y}{\varepsilon}\right). \tag{5.1}$$

Then there exists $t_\varepsilon>0$ such that $\max_{t\geqslant 0}\mathcal{T}_{\varepsilon,\lambda}(t\psi_{\varepsilon,y})=\mathcal{T}_{\varepsilon,\lambda}(t_\varepsilon\psi_{\varepsilon,y})$. We define $\gamma_\varepsilon:\mathcal{M}\to\mathcal{N}_\varepsilon$ by $\gamma_\varepsilon(y)=t_\varepsilon\psi_{\varepsilon,y}$. By the construction, $\gamma_\varepsilon(y)$ has a compact support for any $y\in\mathcal{M}$.

Lemma 5.1. *Under the assumptions of $(\mathcal{C}_0)$ and $(\mathcal{F}_1)$–$(\mathcal{F}_3)$, we have that the function γ_ε such that $\lim_{\varepsilon\to 0}\mathcal{T}_{\varepsilon,\lambda}(\gamma_\varepsilon(y))=c_{M_0}$.*

Proof. Suppose by contradiction that there exist some $\delta_0>0$, $\{y_n\}\subset\mathcal{M}$ and $\varepsilon_n\to 0$ such that

$$\big|\mathcal{T}_{\varepsilon,\lambda}\big(\gamma_\varepsilon(y_n)\big)-c_{M_0}\big|\geqslant\delta_0. \tag{5.2}$$

Now we first claim that $\lim_{n\to\infty}t_{\varepsilon_n}=1$. Indeed, by the definition of t_{ε_n} and the conclusion (v) of Lemma 2.2 we know that there exists $\rho>0$ such that

$$\begin{aligned}0<\rho&\leqslant\int_{\mathbb{R}^3}\big(a\big|\nabla(t_{\varepsilon_n}\psi_{\varepsilon_n,y_n})\big|^2+M(\varepsilon_n x)|t_{\varepsilon_n}\psi_{\varepsilon,y_n}|^2\big)+b\left(\int_{\mathbb{R}^3}\big|\nabla(t_{\varepsilon_n}\psi_{\varepsilon_n,y_n})\big|^2\right)\\&=\lambda\int_{\mathbb{R}^3}f(t_{\varepsilon_n}\psi_{\varepsilon_n,y_n})t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}+\int_{\mathbb{R}^3}|t_{\varepsilon_n}\psi_{\varepsilon_n,y_n}|^6.\end{aligned} \tag{5.3}$$

We infer from $(\mathcal{F}_1)$ and $(\mathcal{F}_3)$ that for each $\epsilon>0$, there exists $C_\epsilon>0$ such that

$$f(s)\leqslant\epsilon s+C_\epsilon s^{p-1}. \tag{5.4}$$

From (5.3) and (5.4), we deduce that t_{ε_n} cannot go zero, that is to say, $t_{\varepsilon_n}\geqslant t_0>0$ for some $t_0>0$. If $t_{\varepsilon_n}\to\infty$ as $n\to\infty$, it follows from the boundedness of ψ_{ε_n,y_n} that

$$\begin{aligned}&\frac{1}{t_{\varepsilon_n}^2}\int_{\mathbb{R}^3}\big(\big|\nabla(\psi_{\varepsilon_n,y_n})\big|^2+K(\varepsilon_n x)|\psi_{\varepsilon_n,y_n}|^2\big)+b\left(\int_{\mathbb{R}^3}\big|\nabla(t_{\varepsilon_n}\psi_{\varepsilon_n,y_n})\big|^2\right)\\&=\lambda\int_{\mathbb{R}^3}\frac{f(t_{\varepsilon_n}\psi_{\varepsilon_n,y_n})}{(t_{\varepsilon_n}\psi_{\varepsilon_n,y_n})^3}(\psi_{\varepsilon_n,y_n})^4+t_{\varepsilon_n}^2\int_{\mathbb{R}^3}|\psi_{\varepsilon_n,y_n}|^6\\&\geqslant t_{\varepsilon_n}^2\int_{\mathbb{R}^3}|\psi_{\varepsilon_n,y_n}|^6=t_{\varepsilon_n}^2\int_{\mathbb{R}^3}\big(\phi\big(|\varepsilon_n z|\big)w(z)\big)^6\\&\geqslant t_{\varepsilon_n}^2\int_{B_{\frac{1}{2}}(0)}w(z)^6\to\infty\quad\text{as } n\to\infty.\end{aligned} \tag{5.5}$$

However, the left side of the inequality (5.5) tends to $b(\int_{\mathbb{R}^3}|\nabla w|^2)^2$. This is absurd. Hence, we obtain that $0 < t_0 \leqslant t_{\varepsilon_n} \leqslant C$. Without loss of generality we assume that $t_{\varepsilon_n} \to h$. Next we shall prove that $h = 1$. In fact, by using Lebesgue's theorem, one can verify that

$$\lim_{n\to\infty}\|\psi_{\varepsilon_n,y_n}\|_{\varepsilon_n}^2 = \|w\|_{M_0}^2, \qquad \lim_{n\to\infty}\int\limits_{\mathbb{R}^3} F(\psi_{\varepsilon_n,y_n}) = \int\limits_{\mathbb{R}^3} F(w) \quad \text{and}$$

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^3} f(\psi_{\varepsilon_n,y_n})\psi_{\varepsilon_n,y_n} = \int\limits_{\mathbb{R}^3} f(w)w. \tag{5.6}$$

So it follows from (5.3) that

$$\frac{1}{h^2}\int\limits_{\mathbb{R}^3}\big(a|\nabla w|^2 + M_0|w|^2\big) + b\bigg(\int\limits_{\mathbb{R}^3}|\nabla w|^2\bigg)^2 = \lambda\int\limits_{\mathbb{R}^3}\frac{f(hw)}{(hw)^3}w^4 + h^2\int\limits_{\mathbb{R}^3}|w|^6. \tag{5.7}$$

Furthermore, since w is a ground state of $(\mathcal{H}_{M_0})$, then one sees

$$\int\limits_{\mathbb{R}^3}\big(a|\nabla w|^2 + M_0|w|^2\big) + b\bigg(\int\limits_{\mathbb{R}^3}|\nabla w|^2\bigg)^2 = \lambda\int\limits_{\mathbb{R}^3} f(w)w + \int\limits_{\mathbb{R}^3}|w|^6. \tag{5.8}$$

Combining (5.7) and (5.8), we conclude that

$$\left(\frac{1}{h^2} - 1\right)\int\limits_{\mathbb{R}^3}\big(a|\nabla w|^2 + K_0|w|^2\big) = \lambda\int\limits_{\mathbb{R}^3}\left(\frac{f(hw)}{(hw)^3} - \frac{f(w)}{w^3}\right)w^4 + \big(h^2 - 1\big)\int\limits_{\mathbb{R}^3} w^6. \tag{5.9}$$

Thus we deduce from $(\mathcal{F}_2)$ that $h = 1$. On the other hand,

$$\begin{aligned}\mathcal{T}_{\varepsilon_n,\lambda}\big(\gamma_{\varepsilon_n}(y_n)\big) = {} & \frac{t_{\varepsilon_n}^2}{2}\int\limits_{\mathbb{R}^3}\big(a\big|\nabla\big(\phi\big(|\varepsilon_n z|\big)w\big)\big|^2 + M(\varepsilon_n z + y_n)\big|\phi\big(|\varepsilon_n z|\big)w\big|^2\big) - \frac{t_n^2}{6}\int\limits_{\mathbb{R}^3}\big|\phi\big(|\varepsilon_n z|\big)w\big|^6 \\ & - \lambda\int\limits_{\mathbb{R}^3} F\big(t_{\varepsilon_n}\phi\big(|\varepsilon_n z|\big)w\big) + \frac{bt_n^2}{4}\bigg(\int\limits_{\mathbb{R}^3}\big|\nabla\big(\phi\big(|\varepsilon_n z|\big)w\big)\big|^2\bigg)^2.\end{aligned} \tag{5.10}$$

Let $n \to \infty$ in (5.10), we infer from Lebesgue's theorem that $\mathcal{T}_{\varepsilon_n}(\gamma_{\varepsilon_n}(y_n)) = \mathcal{T}_{M_0}(w) = c_{M_0}$. This contradicts with (5.2). $\square$

For each $\delta > 0$, let $\varrho = \varrho(\delta)$ be such that $\mathcal{M}_\delta \subset B_\varrho(0)$. Let $\chi : \mathbb{R}^3 \to \mathbb{R}^3$ be define by $\chi(x) = x$ for $|x| \leqslant \varrho$ and $\chi(x) = \frac{\varrho x}{|x|}$ for $|x| \geqslant \varrho$. Finally, let us define $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}$ by

$$\beta_\varepsilon(u) = \frac{\int_{\mathbb{R}^3}\chi(\varepsilon x)u^2\,dx}{\int_{\mathbb{R}^3}u^2\,dx}.$$

As in the proof of Lemma 5.1, it is easy to see that

$$\beta_\varepsilon\big(\gamma_\varepsilon(y)\big)=\frac{\int_{\mathbb{R}^3}\chi(\varepsilon x)\gamma_\varepsilon(y)^2\,dx}{\int_{\mathbb{R}^3}\gamma_\varepsilon(y)^2\,dx}=\frac{\int_{\mathbb{R}^3}\chi(\varepsilon x+y)|w(x)\phi(|\varepsilon x|)|^2\,dx}{\int_{\mathbb{R}^3}|w(x)\phi(|\varepsilon x|)|^2\,dx}$$
$$=y+\frac{\int_{\mathbb{R}^3}(\chi(\varepsilon x+y)-y)|w(x)\phi(|\varepsilon x|)|^2\,dx}{\int_{\mathbb{R}^3}|w(x)\phi(|\varepsilon x|)|^2\,dx}=y+o(1)$$

as $\varepsilon\to 0$, uniformly for $y\in\mathcal{N}_\varepsilon$. So we conclude that $\lim_{\varepsilon\to 0}\beta_\varepsilon(\gamma_\varepsilon(y))=y$ uniformly for $y\in\mathcal{N}_\varepsilon$.

Next we shall prove some concentration phenomena for the positive ground state solutions of $(\mathcal{H}_\varepsilon)$. Before doing these, we start with the following preliminary lemma.

Lemma 5.2. *Suppose that the assumptions of Theorem 1.1 are satisfied. Let $u_n\subset\mathcal{N}_{M_0}$ be a sequence satisfying $\mathcal{T}_{M_0,\lambda}(u_n)\to c_{M_0}$. Then either $\{u_n\}$ has a subsequence strongly convergent in $H^1(\mathbb{R}^3)$ or there exists $\{y_n\}\subset\mathbb{R}^3$ such that the sequence $w_n(x)=u_n(x+y_n)$ converges strongly in $H^1(\mathbb{R}^3)$. In particular, there exists a minimizer of c_{M_0}.*

Proof. By Lemma 3.7, we know that $\{u_n\}$ is a bounded sequence. From Lemma 2.3, $v_n=\check{m}_\varepsilon(u_n)$ is a minimizer sequence of $\Upsilon_{\varepsilon,\lambda}$. By Ekeland's variational principle [10], we may assume that $\Upsilon_{\varepsilon,\lambda}(v_n)\to c_{M_0}$ and $\Upsilon'_{\varepsilon,\lambda}(v_n)\to 0$. So it follows that

$$\mathcal{T}_{M_0,\lambda}(u_n)\to c_{M_0},\qquad \mathcal{T}'_{M_0,\lambda}(u_n)\to 0\quad\text{and}\quad \mathcal{T}'_{M_0,\lambda}(u_n)u_n=0,\tag{5.11}$$

where $u_n=m_\varepsilon(v_n)$. Hence, for some subsequence, still denoted by $\{u_n\}$, we may assume that there exists an $u\in H^1(\mathbb{R}^3)$ such that $u_n\rightharpoonup u$ in $H^1(\mathbb{R}^3)$. In the following we distinguish the following two cases:

(a) If $u\neq 0$, in this case we deduce from $u\in\mathcal{N}_{M_0}$ and (1.4) that

$$\begin{aligned}
c_{M_0}&\leqslant\mathcal{T}_{M_0,\lambda}(u)=\mathcal{T}_{M_0,\lambda}(u)-\frac14\big(\mathcal{T}'_{M_0,\lambda}(u),u\big)\\
&=\frac14\int_{\mathbb{R}^3}\big(a|\nabla u|^2+M_0|u|^2\big)+\lambda\int_{\mathbb{R}^3}\Big(\frac14 f(u)u-F(u)\Big)+\frac1{12}\int_{\mathbb{R}^3}|u|^6\\
&\leqslant\liminf_{n\to\infty}\Big[\frac14\int_{\mathbb{R}^3}\big(a|\nabla u_n|^2+M_0|u_n|^2\big)+\lambda\int_{\mathbb{R}^3}\Big(\frac14 f(u_n)u_n-F(u_n)\Big)+\frac1{12}\int_{\mathbb{R}^3}|u_n|^6\Big]\\
&=\liminf_{n\to\infty}\Big(\mathcal{T}_{M_0,\lambda}(u_n)-\frac14\mathcal{T}'_{M_0,\lambda}(u_n)u_n\Big)\leqslant c_{M_0}.
\end{aligned}\tag{5.12}$$

Thus, by (5.12) we deduce that $\lim_{n\to\infty}\int_{\mathbb{R}^3}(a|\nabla u_n|^2+M_0|u_n|^2)=\int_{\mathbb{R}^3}(a|\nabla u|^2+M_0|u|^2)$. That is, $u\to u$ in $H^1(\mathbb{R}^3)$.

(b) $u=0$. As in Lemma 3.7, we have that there exist $\{y_n\}\subset\mathbb{R}^3$, $r,\delta>0$ such that

$$\liminf_{n\to\infty}\int_{B_r(y_n)}u_n^2\geqslant\delta.\tag{5.13}$$

We set $w_n(x)=u_n(x+y_n)$, then $\|w_n\|_{M_0}=\|u_n\|_{M_0}$, $\mathcal{T}_{M_0,\lambda}(w_n)\to c_{M_0}$ and $\mathcal{T}'_{M_0,\lambda}(w_n)w_n=0$. It is clear that there exists $w\in H^1(\mathbb{R}^3)$ with $w\neq 0$ such that $w_n\rightharpoonup w$ in $H^1(\mathbb{R}^3)$. Then the proof follows from the arguments used in case of $u\neq 0$. $\square$

Lemma 5.3. *Under the assumptions of Theorem 1.1, one has that there is a maximum point y_ε of u_ε such that $\mathrm{dist}(x_\varepsilon, \mathcal{M}) \to 0$ where $x_\varepsilon = \varepsilon y_\varepsilon$, u_ε denotes the positive ground sate solutions of $(\mathcal{H}_\varepsilon)$ and $0 \in \mathcal{M} = \{x \in \mathbb{R}^3\colon\ M(x) = M_0\}$. Moreover, for such y_ε, we have that $v_\varepsilon(x) = u_\varepsilon(x + y_\varepsilon)$ converges in $H^1(\mathbb{R}^3)$ to a positive ground state solution of $(\mathcal{H}_{M_0})$, as $\varepsilon \to 0$.*

Proof. Let $\varepsilon_j \to 0$, $u_j \in \mathcal{L}_{\varepsilon_j}$ such that $\mathcal{T}_{\varepsilon_j,\lambda}(u_j) = c_{\varepsilon_j}$ and $\mathcal{T}'_{\varepsilon_j,\lambda}(u_j) = 0$. Clearly, $\{u_j\} \subset \mathcal{N}_{\varepsilon_j}$. Using the same arguments as in Lemma 3.7, one easily check $\{u_j\}$ is bounded in $H^1(\mathbb{R}^3)$. So we can assume that $u_j \rightharpoonup u$ in $H^1(\mathbb{R}^3)$. Moreover, since $\mathcal{T}_{\varepsilon_j,\lambda}(u_j) = c_{\varepsilon_j} \to c_{M_0}$ as $j \to \infty$ according to Lemma 4.1, then we have $c_{\varepsilon_j} \leqslant c_{M_\infty}$ for j large. Thus similar to the proof of Lemma 3.7, we can prove that there exist $r, \delta > 0$ and a sequence $\{y'_j\} \subset \mathbb{R}^3$ such that

$$\liminf_{j\to\infty} \int_{B_r(y'_j)} u_j^2 \geqslant \delta > 0. \tag{5.14}$$

For $\{y_j\} \subset \mathbb{R}^3$ such that

$$u_j(y_j) = \max_{y\in\mathbb{R}^3} u_j(y), \quad \forall j.$$

We claim that there is $\kappa > 0$ (independent of j) such that

$$u_j(y_j) \geqslant \kappa > 0, \quad \text{uniformly for all } j \in \mathbb{N}. \tag{5.15}$$

Assume by contradiction that $u_j(y_j) \to 0$ as $j \to \infty$. We deduce from (5.14) that

$$0 < \delta \leqslant \int_{B_r(y'_j)} u_j^2 \leqslant c u_j(y_j)^2 \to 0 \quad \text{as } j \to \infty.$$

This is a contradiction. As in Theorem 3.8, one can easily check that $u_j \in C^{1,\sigma}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ for each $j \in \mathbb{N}$. So it follows from (5.14)–(5.15) (see [11]) that there exist $R > r > 0$ and $\delta' > 0$ such that

$$\liminf_{j\to\infty} \int_{B_R(y_j)} |u_j|^2 \geqslant \delta' > 0.$$

Set

$$v_j(x) = u_j(x + y_j) \quad \text{and} \quad \hat{M}_{\varepsilon_j}(x) = M\big(\varepsilon_j(x + y_j)\big).$$

Then along a subsequence we have $v_j \rightharpoonup v \neq 0$ in $H^1(\mathbb{R}^3)$ and $v_j \to v$ in $L^p_{loc}(\mathbb{R}^3)$ ($p \in (2,6)$). We first claim that $v_j \to v \neq 0$ in $H^1(\mathbb{R}^3)$. In fact, according to Lemma 3.1, we choose $t_j > 0$ such that $m_{M_0}(v_j) = t_j v_j \in \mathcal{N}_{M_0}$. Set $\tilde{v}_j = t_j v_j$. It follows from $(\mathcal{C}_0)$, $u_j \in \mathcal{N}_{\varepsilon_j}$ and Lemma 4.1 that

$$\begin{aligned}\mathcal{T}_{M_0,\lambda}(\tilde{v}_j) &\leqslant \frac{1}{2}\int_{\mathbb{R}^3} \big(|\nabla \tilde{v}_j|^2 + \hat{M}_{\varepsilon_j}(x)|\tilde{v}_j|^2\big) + \frac{b}{4}\bigg(\int_{\mathbb{R}^3} |\nabla \tilde{v}_j|^2\bigg)^2 - \lambda \int_{\mathbb{R}^3} F(\tilde{v}_j) - \frac{1}{6}\int_{\mathbb{R}^3} \tilde{v}_j^6 \\ &= \mathcal{T}_{\varepsilon_j,\lambda}(t_j u_j) \leqslant \mathcal{T}_{\varepsilon_j,\lambda}(u_j) = c_{M_0} + o(1).\end{aligned}$$

Note that $\mathcal{T}_{M_0,\lambda}(\tilde{v}_j) \geqslant c_{M_0}$, thus $\lim_{j\to\infty} \mathcal{T}_{M_0,\lambda}(\tilde{v}_j) = c_{M_0}$. From the conclusion (vi) of Lemma 3.1, we infer that t_j is bounded. Without loss of generality we can assume that $t_j \to t \geqslant 0$. If $t = 0$, we have that $\tilde{v}_j = t_j v_j \to 0$ in view of the boundedness of v_j, and hence $\mathcal{T}_{M_0,\lambda}(\tilde{v}_j) \to 0$ as $j \to \infty$, which contradicts $c_{M_0} > 0$. So, $t > 0$ and the weak limit of $\tilde{v}_j$ is different from zero. Let $\tilde{v}$ be the weak limit of $\tilde{v}_j$ in $H^1(\mathbb{R}^3)$. Since $t_n \to t > 0$ and $v_n \rightharpoonup v \neq 0$, we have from the uniqueness of the weak limit that $\tilde{v} = tv \neq 0$ and $\tilde{v} \in \mathcal{N}_{M_0}$. From Lemma 5.2, $\tilde{v}_j \to \tilde{v}$ in $H^1(\mathbb{R}^3)$, and so, $v_j \to v$ in $H^1(\mathbb{R}^3)$. This proves the claim for $v_j \to v \neq 0$ in $H^1(\mathbb{R}^3)$.

Obviously, v_j solves

$$(\mathcal{H}_\varepsilon^v) \quad -\left(a + b\int_{\mathbb{R}^3} |\nabla v_j|^2\right)\Delta v_j + \hat{M}_{\varepsilon_j}(x) v_j = \lambda f(v_j) + v_j^5 \quad \text{in } \mathbb{R}^3.$$

Correspondingly, the energy functional is denoted by

$$\begin{aligned} H_{\varepsilon_j}(v_j) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(a|\nabla v_j|^2 + \hat{M}_{\varepsilon_j}(x) v_j^2\right) + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v_j|^2\right)^2 - \lambda \int_{\mathbb{R}^3} F(v_j) + \frac{1}{6}\int_{\mathbb{R}^3} v_j^6 \\ &= \mathcal{T}_{\varepsilon_j,\lambda}(u_j) = c_{\varepsilon_j}. \end{aligned}$$

We next show that $\{\varepsilon_j y_j\}$ is bounded. Assume by contradiction that $\varepsilon_j |y_j| \to \infty$. Without loss of generality assume $M(\varepsilon_j y_j) \to \tilde{M}^\infty$. Clearly, $M_0 < \tilde{M}^\infty$ by $(\mathcal{C}_0)$. For each $\eta \in C_0^\infty(\mathbb{R}^3)$, we deduce from $v_j \to v$ in $H^1(\mathbb{R}^3)$ that

$$\begin{aligned} \lim_{j\to\infty} H'_{\varepsilon_j}(v_j)\eta &= \lim_{j\to\infty} \int_{\mathbb{R}^3} \left[\left(a\nabla v_j \nabla \eta + \hat{M}_{\varepsilon_j}(x) v_j \eta\right) + b\left(\int_{\mathbb{R}^3} |\nabla v_j|^2\right)\left(\int_{\mathbb{R}^3} \nabla v_j \nabla \eta\right)\right] \\ &\quad - \lambda \int_{\mathbb{R}^3} f(v_j)\eta - \lim_{j\to\infty} \int_{\mathbb{R}^3} v_j^6 \eta \\ &= \int_{\mathbb{R}^3} \left[\left(a\nabla v \nabla \eta + \tilde{M}^\infty v\eta\right) - \lambda \int_{\mathbb{R}^3} f(v)\eta + b\left(\int_{\mathbb{R}^3} |\nabla v|^2\right)\left(\int_{\mathbb{R}^3} \nabla v \nabla \eta\right) - \int_{\mathbb{R}^3} v^5 \eta\right] \\ &= 0. \end{aligned}$$

Thus, v solves

$$(\mathcal{H}_{\tilde{M}^\infty}) \quad -\left(a + b\int_{\mathbb{R}^3} |\nabla v|^2\right)\Delta v + \tilde{M}^\infty v = \lambda f(v) + v^5 \quad \text{in } \mathbb{R}^3.$$

We denote the energy functional by

$$H_\infty(v) = \frac{1}{2}\int_{\mathbb{R}^3} a|\nabla v|^2 + \tilde{M}^\infty v^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla v|^2\right)^2 - \lambda \int_{\mathbb{R}^3} F(v) - \frac{1}{6}\int_{\mathbb{R}^3} v^6 \geqslant c_{\tilde{M}^\infty}.$$

Remark that since $M_0 < \tilde{M}^\infty$ one has $c_{\tilde{M}^\infty} > c_{M_0}$ by Lemma 4.1. Moreover, since $H'_{\varepsilon_j}(v_j)v_j = \mathcal{T}'_{\varepsilon_j,\lambda}(u_j)u_j = 0$, it follows from Fatou's lemma and (1.4) that

$$\lim_{j\to\infty} c_{\varepsilon_j} = \lim_{j\to\infty} H_{\varepsilon_j}(v_j) = \lim_{j\to\infty}\left(H_{\varepsilon_j}(v_j)(v_j) - \frac{1}{4}H_{\varepsilon_j}(v_j)'(v_j)v_j\right)$$
$$\geqslant \liminf_{j\to\infty}\left[\frac{1}{4}\int_{\mathbb{R}^3}\left(a|v_j|^2 + \hat{M}_{\varepsilon_j}(x)|v_j|^2\right) + \lambda\int_{\mathbb{R}^3}\left(\frac{1}{4}f(v_j)v_j - F(v_j)\right) + \frac{1}{6}\int_{\mathbb{R}^3}|v_j|^6\right]$$
$$\geqslant \frac{1}{4}\int_{\mathbb{R}^3}\left(a|v_j|^2 + M^{\infty}|v_j|^2\right) + \lambda\int_{\mathbb{R}^3}\left(\frac{1}{4}f(v)v - F(v)\right) + \frac{1}{6}\int_{\mathbb{R}^3}|v|^6 = H_\infty(v). \tag{5.16}$$

Consequently, we infer from (5.16) that

$$c_{M_0} < c_{\tilde{M}^\infty} \leqslant H_\infty(v) \leqslant \lim_{j\to\infty} c_{\varepsilon_j} = c_{M_0},$$

a contradiction. Thus $\{\varepsilon_j y_j\}$ is bounded. Hence, we can assume $x_j = \varepsilon_j y_j \to x_0$. Then v solves

$$(\mathcal{P}_{K^0}) \quad -\left(a + b\int_{\mathbb{R}^3}|\nabla v|^2\right)\Delta v + M(x_0)v = \lambda f(v) + v^5 \quad \text{in } \mathbb{R}^3.$$

It follows from $M(x_0) \geqslant M_0$ that

$$H_0(v) = \frac{1}{2}\int_{\mathbb{R}^3}\left(a|\nabla v|^2 + M(x_0)|v|^2\right) + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2\right)^2 - \lambda\int_{\mathbb{R}^3}F(v) - \frac{1}{6}\int_{\mathbb{R}^3}v^6 \geqslant c_{M(x_0)} \geqslant c_{M_0}.$$

Similar to (5.16), one gets

$$c_{M_0} = \lim_{j\to\infty} c_{\varepsilon_j} \geqslant P_0(v) \geqslant c_{M_0}.$$

This implies that $P_0(v) = c_{M_0}$, and hence $M(x_0) = M_0$. So by Lemma 4.1, $x_0 \in \mathcal{M}$. □

Now we study the exponent decay for the ground state solution.

Lemma 5.4. *Under the assumptions of Theorem 1.1, if u_ε is a positive ground state solution of $(\mathcal{H}_\varepsilon)$, one has that for each $\varepsilon > 0$ small, $\lim_{|x|\to\infty} u_\varepsilon(x) = 0$, $\lim_{|x|\to\infty} |\nabla u_\varepsilon(x)| = 0$ and $u_\varepsilon \in C^{1,\sigma}_{loc}(\mathbb{R}^3)$ for $\sigma \in (0,1)$. Furthermore, there exist $C, c > 0$ such that $u_\varepsilon(x) \leqslant Ce^{-c|x-y_\varepsilon|}$, where $u_\varepsilon(y_\varepsilon) = \max_{x\in\mathbb{R}^3} u_\varepsilon(x)$.*

Proof. As in the proof of the conclusion (ii) of Theorem 3.8, we known that for each $\varepsilon > 0$ small, $\lim_{|x|\to\infty} u_\varepsilon(x) = 0$, $\lim_{|x|\to\infty} |\nabla u_\varepsilon(x)| = 0$ and $u_\varepsilon \in C^{1,\sigma}_{loc}(\mathbb{R}^3)$ for $\sigma \in (0,1)$. In the following we shall prove the exponent decay for the positive solution of u_ε. Let $\varepsilon_j \to 0$, $u_j \in \mathcal{L}_{\varepsilon_j}$ such that $\mathcal{T}_{\varepsilon_j,\lambda}(u_j) = c_{\varepsilon_j}$ and $\mathcal{T}'_{\varepsilon_j,\lambda}(u_j) = 0$. As in the proof of Lemma 5.2, we have that $v_j = u_j(x + y_j)$ such that

$$\left(\mathcal{R}^v_\varepsilon\right) \quad -\left(a + b\int_{\mathbb{R}^3}|\nabla v_j|^2\right)\Delta v_j + \hat{M}_{\varepsilon_j}(x)v_j = \lambda f(v_j) + |v_j|^4 v_j \quad \text{in } \mathbb{R}^3$$

and $v_j \to v \neq 0$ in $H^1(\mathbb{R}^3)$ where $u_j(y_j) = \max_{y\in\mathbb{R}^3} u_j(y)$. Let $A_j = a + b\int_{\mathbb{R}^3} |\nabla v_j|^2$. Then it follows that $0 < a \leqslant A_j \leqslant c$ and Eq. $(\mathcal{R}^v_\varepsilon)$ equivalent to

$$\left(\mathcal{R}^v_{\varepsilon,1}\right) \quad -\Delta v_j + \frac{\hat{M}_{\varepsilon_j}(x)}{A_j} v_j = \frac{\lambda}{A_j} f(v_j) + \frac{1}{A_j}|v_j|^4 v_j \quad \text{in } \mathbb{R}^3.$$

So we deduce from Proposition 2 of [6] that $v_j \in L^t(\mathbb{R}^3)$ for all $t \geqslant 2$ and

$$|v_j|_t \leqslant N_t \|v_j\|, \tag{5.17}$$

where N_t does not depend on j. Then we infer from $v_j \to v \neq 0$ in $H^1(\mathbb{R}^3)$ that

$$\lim_{R\to\infty}\left(\int\limits_{|x|\geqslant R} \left(v_j^2 + v_j^6\right)\right) = 0, \quad \text{uniformly for } j \in \mathbb{N}. \tag{5.18}$$

Let $h_j(x) = \frac{1}{A_j}(\lambda f(v_j) + v_j^5)$. It follows from (5.17) that for $t > 3$

$$|h_j|_t \leqslant C, \quad \text{for all } j \in \mathbb{N}.$$

Thus by Proposition 3 in [6] (also see Theorem 8.17 in [14]), we infer that for all $y \in \mathbb{R}^3$

$$\sup_{B_1(y)} v_j \leqslant c\left(|v_j|_{L^2(B_2(y))} + |h_j|_{L^t(B_2(y))}\right). \tag{5.19}$$

This implies that $|v_j|_\infty$ is uniformly bounded. Furthermore, combining the limit (5.18) with inequality (5.19) we reach

$$\lim_{|x|\to\infty} v_j(x) = 0 \quad \text{uniformly for all } j \in \mathbb{N}.$$

Form this we deduce that there is $\varepsilon_0 > 0$ such that

$$\lim_{|x|\to\infty} v_\varepsilon(x) = 0 \quad \text{uniformly for all } \varepsilon \in (0, \varepsilon_0].$$

So by using the same arguments as in the proof of the conclusion (ii) of Theorem 3.8, we know that there exist $C, \delta > 0$ (independent of ε) such that

$$v_\varepsilon(x) \leqslant Ce^{-\delta|x|},$$

where $v_\varepsilon = u_\varepsilon(x + y_\varepsilon)$ and $u_\varepsilon(y_\varepsilon) = \max_{y\in\mathbb{R}^3} u_\varepsilon$. Thus, the conclusions of this lemma hold. □

To prove the concentration phenomenon for the positive solutions of $(\mathcal{H}_\varepsilon)$, we need the following results.

Lemma 5.5. *Under the assumptions of Theorem 1.1 or Theorem 1.2, one has that if $\varepsilon_n \to 0$ and $\{u_n\} \subset \mathcal{N}_{\varepsilon_n}$ such that $\mathcal{T}_{\varepsilon_n,\lambda}(u_n) \to c_{M_0}$, then there exists a sequence $\{y_n\} \subset \mathbb{R}^3$ such that $\tilde{y}_n = \varepsilon_n y_n \to y \in \mathcal{M}$.*

Proof. By using the same arguments as in Lemma 3.7, we can prove that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Moreover, there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\liminf_{n\to\infty} \int_{B_r(y_n)} u_n^2 \geqslant \delta > 0.$$

Let $v_n = u_n(x + y_n)$. Then it follows that $v_n \rightharpoonup v \neq 0$ in $H^1(\mathbb{R}^3)$ and $v_n(x) \to v(x)$ a.e., on $\mathbb{R}^3$. According to Lemma 3.1, we choose $t_n > 0$ such that $m_{M_0}(v_n) = t_n v_n \in \mathcal{N}_{M_0}$. Set $\tilde{v}_n = t_n v_n$. It follows from $(\mathcal{C}_0)$, $u_n \in \mathcal{N}_{\varepsilon_n}$ and Lemma 3.2 that

$$\begin{aligned}\mathcal{T}_{M_0,\lambda}(\tilde{v}_n) &\leqslant \frac{t_n^2}{2}\int_{\mathbb{R}^3}\left(a|\nabla\tilde{v}_n|^2 + \hat{M}_{\varepsilon_n}(x)|\tilde{v}_n|^2\right) + \frac{t_n^4 b}{4}\left(\int_{\mathbb{R}^3}|\nabla\tilde{v}_n|^2\right)^2 - \lambda\int_{\mathbb{R}^3}F(t_n\tilde{v}_n) - \frac{t_n^6}{6}\int_{\mathbb{R}^3}\tilde{v}_n^6 \\ &= \mathcal{T}_{\varepsilon_n,\lambda}(t_n u_n) \leqslant \mathcal{T}_{\varepsilon_n,\lambda}(u_n) = c_{M_0} + o(1).\end{aligned}$$

Note that $\mathcal{T}_{M_0,\lambda}(\tilde{v}_n) \geqslant c_{M_0}$, thus $\lim_{n\to\infty}\mathcal{T}_{M_0,\lambda}(\tilde{v}_n) = c_{M_0}$. From the conclusion (vi) of Lemma 3.1, we infer that t_n is bounded. Without loss of generality we can assume that $t_n \to t \geqslant 0$. If $t = 0$, we have that $\tilde{v}_n = t_n v_n \to 0$ in view of the boundedness of v_n, and hence $\mathcal{T}_{M_0,\lambda}(\tilde{v}_n) \to 0$ as $n \to \infty$, which contradicts $c_{M_0} > 0$. So, $t > 0$ and the weak limit of $\tilde{v}_n$ is different from zero. Let $\tilde{v}$ be the weak limit of $\tilde{v}_n$ in $H^1(\mathbb{R}^3)$. Since $t_n \to t > 0$ and $v_n \rightharpoonup v \neq 0$, we have from the uniqueness of the weak limit that $\tilde{v} = tv \neq 0$. Moreover, it follows from Lemma 5.2 that $\tilde{v} \in \mathcal{N}_{M_0}$.

We claim that $\{\tilde{y}_n\}$ is bounded. Indeed, suppose by contradiction that $|\tilde{y}_n| \to \infty$. It follows from $\tilde{v}_n, \tilde{v} \in \mathcal{N}_{M_0}$ and $M_0 < M_\infty$ that

$$\begin{aligned}c_{M_0} &= \mathcal{T}_{M_0,\lambda}(\tilde{v}) < \mathcal{T}_{M_\infty,\lambda}(\tilde{v}) \\ &= \frac{1}{4}\int_{\mathbb{R}^3} a|\nabla\tilde{v}|^2 + \int_{\mathbb{R}^3}\left(\frac{1}{2}M_\infty - \frac{1}{4}M_0\right)|\tilde{v}|^2 + \lambda\int_{\mathbb{R}^3}\left(\frac{1}{4}f(\tilde{v})\tilde{v} - F(\tilde{v})\right) + \frac{1}{12}\int_{\mathbb{R}^3}|\tilde{v}|^6 \\ &\leqslant \liminf_{n\to\infty}\left[\frac{1}{4}\int_{\mathbb{R}^3} a|\nabla\tilde{v}_n|^2 + \int_{\mathbb{R}^3}\left(\frac{1}{2}M(\varepsilon_n x + \tilde{y}_n) - \frac{1}{4}M_0\right)|\tilde{v}_n|^2\right. \\ &\quad \left. + \lambda\int_{\mathbb{R}^3}\left(\frac{1}{4}f(\tilde{v}_n)\tilde{v}_n - F(\tilde{v}_n)\right) + \int_{\mathbb{R}^3}|\tilde{v}_n|^6\right] \\ &= \liminf_{n\to\infty}\mathcal{T}_{\varepsilon_n,\lambda}(\tilde{v}_n) = \liminf_{n\to\infty}\mathcal{T}_{\varepsilon_n,\lambda}(t_n v_n) = \liminf_{n\to\infty}\mathcal{T}_{\varepsilon_n,\lambda}(t_n u_n) \\ &\leqslant \liminf_{n\to\infty}\mathcal{T}_{\varepsilon_n,\lambda}(u_n) = c_{M_0}.\end{aligned}$$

This is impossible. So $\{\tilde{y}_n\}$ is bounded. Without loss of generality we may assume that $\tilde{y}_n \to y$. If $y \notin M$, then $M(y) > M_0$ and we obtain a contradiction by the same arguments made above. So, $y \in \mathcal{M}$ and the conclusion follows. $\square$

Let $\omega(\varepsilon)$ be any positive function tending to 0 as $\varepsilon \to 0$ and let

$$\Sigma_\varepsilon = \left\{u \in \mathcal{N}_\varepsilon \colon \mathcal{T}_{\varepsilon,\lambda}(u) \leqslant c_{M_0} + \omega(\varepsilon)\right\}.$$

For any $y \in \mathcal{M}$, we deduce from Lemma 5.1 that $\omega(\varepsilon) = |\mathcal{T}_{\varepsilon,\lambda}(\gamma_\varepsilon(y)) - c_{M_0}| \to 0$ as $\varepsilon \to 0^+$. Thus $\gamma_\varepsilon(y) \in \Sigma_\varepsilon$ and $\Sigma_\varepsilon \neq \emptyset$ for $\varepsilon > 0$.

Lemma 5.6. *Suppose that the assumptions of Theorem 1.1 or Theorem 1.2 are satisfied. Then for any $\delta > 0$, there holds that $\lim_{\varepsilon\to 0}\sup_{u\in\Sigma_\varepsilon} dist(\beta_\varepsilon(u), \mathcal{M}_\delta) = 0$.*

Proof. Let $\{\varepsilon_n\} \subset \mathbb{R}^+$ be such that $\varepsilon_n \to 0$. By definition, there exists $\{u_n\} \subset \Sigma_{\varepsilon_n}$ such that $dist(\beta_{\varepsilon_n}(u_n), \mathcal{M}_\delta) = \sup_{u\in\Sigma_{\varepsilon_n}} dist(\beta_{\varepsilon_n}(u), \mathcal{M}_\delta) + o(1)$. From this we know that it suffices to find a sequence $\{\tilde{y}_n\} \subset \mathcal{M}$ satisfying $|\beta_{\varepsilon_n} - \tilde{y}_n| = o(1)$. From $\mathcal{T}_{M_0,\lambda}(tu_n) \leqslant \mathcal{T}_{\varepsilon,\lambda}(tu_n)$ for $t \geqslant 0$ and $\{u_n\} \subset \Sigma_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we obtain that $c_{M_0} \leqslant c_{\varepsilon_n} \leqslant \mathcal{T}_{\varepsilon_n,\lambda}(u_n) \leqslant c_{M_0} + \omega(\varepsilon_n)$. This leads to $\mathcal{T}_{\varepsilon_n,\lambda}(u_n) \to c_{M_0}$. By Lemma 5.5 one sees that there exists a sequence $\{y_n\} \subset \mathbb{R}^3$ such that $\tilde{y}_n = y_n\varepsilon_n \in \mathcal{M}_\delta$ for n sufficiently large. Hence

$$\beta_{\varepsilon_n}(u_n) = \tilde{y}_n + \frac{\int_{\mathbb{R}^3} (\chi(\varepsilon_n z + \tilde{y}_n) - \tilde{y}_n) u_n^2(z + \tilde{y}_n)\, dz}{\int_{\mathbb{R}^3} u_n^2(z + \tilde{y}_n)\, dz}.$$

Since $\varepsilon_n z + \tilde{y}_n \to y \in \mathcal{M}$, we have that $\beta_{\varepsilon_n}(u_n) = \tilde{y}_n + o(1)$ and then the sequence $\{\tilde{y}_n\}$ is what we need. □

Lemma 5.7. *Suppose that the assumptions of Theorem 1.1 or Theorem 1.2 are satisfied. If u_n such that $\mathcal{T}_{\varepsilon_n,\lambda}(u_n) \to c_{K_0}$ and there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that $\liminf_{n\to\infty} \int_{B_r(y_n)} u_n^2 \geqslant \delta > 0$, $v_n(x) = u_n(x + y_n)$ satisfies the following problem*

$$(\mathcal{H}_\varepsilon^1) \quad -\Big(a + b\int_{\mathbb{R}^3} |\nabla v_n|^2\Big)\Delta v_n + \hat{M}_{\varepsilon_n}(x) v_n = \lambda f(v_n) + |v_n|^4 v_n \quad \text{in } \mathbb{R}^3,$$

where $\hat{M}_{\varepsilon_n}(x) = M(\varepsilon_n x + \varepsilon_n y_n)$ and y_n is given in Lemma 5.3. Then we have that $v_n \to v$ in $H^1(\mathbb{R}^3)$ with $v \neq 0$, $v_n \in L^\infty(\mathbb{R}^3)$ and $\|v_n\|_{L^\infty(\mathbb{R}^3)} \leqslant C$ for all $n \in \mathbb{N}$. Furthermore, $\lim_{|x|\to\infty} v_n(x) = 0$ uniformly for $n \in \mathbb{N}$ and $v_n(x) \leqslant ce^{-c|x-y_n|}$.

Proof. Since v_n satisfies Eq. $(\mathcal{H}_\varepsilon^1)$, we know that $\mathcal{T}'_{\varepsilon_n,\lambda}(v_n) = 0$. Moreover, $\mathcal{T}_{\varepsilon_n,\lambda}(u_n) \to c_{M_0}$. So by using the same arguments as in Lemma 5.4, one can obtain the conclusion of this lemma. Here we omit the details. □

Proof of Theorems 1.1. Going back to $(KH)_\varepsilon$ with the variable substitution: $x \mapsto \frac{x}{\varepsilon}$. Lemma 4.5 implies that $(KH)_\varepsilon$ has at least one positive ground state solution $u_\varepsilon \in H^1(\mathbb{R}^3)$ for all $\varepsilon > 0$ small. The conclusions (ii) and (iii) follow from Lemmas 4.6 and 5.3 respectively. Finally, it follows from Lemma 5.4 that the conclusion (iv) of Theorem 1.1 holds. □

Next we shall prove Theorem 1.2, before doing this, we should use the following result for critical points involving Ljusternik–Schnirelmann category. For the details of the proof one can sees [4,20].

Theorem 5.8. *Let $\mathcal{U}$ be a $C^{1,1}$ complete Riemannian manifold (modelled on a Hilbert space). Assume that $h \in C^1(\mathcal{U}, \mathbb{R})$ bounded from below and satisfies $-\infty < \inf_{\mathcal{U}} h < d < k < +\infty$. Moreover, suppose that h satisfies Palais–Smale condition on the sublevel $\{u \in \mathcal{U}\colon h(u) \leqslant k\}$ and that d is not a critical level for h. Then*

$$\#\big\{u \in h^d\colon \nabla h(u) = 0\big\} \geqslant cat_{h^d}\big(h^d\big).$$

With a view to apply Theorem 5.8, the following abstract lemma provides a very useful tool in that it relates the topology of some sublevel of a functional to the topology of some subset of the space $\mathbb{R}^3$. For the proof, an easy application of the definitions of category and of homotopic equivalence between maps, we refer to [21,22,20].

Lemma 5.9. *Let $\mathcal{I}$, $\mathcal{I}_1$, $\mathcal{I}_2$ be closed sets with $\mathcal{I}_1 \subset \mathcal{I}_2$; let $\pi : \mathcal{I} \to \mathcal{I}_2$, $\psi : \mathcal{I}_1 \to \mathcal{I}$ be two continuous maps such that $\pi \circ \psi$ is homotopically equivalent to the embedding $j : \mathcal{I}_1 \to \mathcal{I}_2$. Then $cat_{\mathcal{I}}(\mathcal{I}) \geqslant cat_{\mathcal{I}_2}(\mathcal{I}_1)$.*

To prove Theorem 1.2, we first show that $(\mathcal{H}_\varepsilon)$ has at least $cat_{\mathcal{M}_\delta}(\mathcal{M})$ positive solutions. Since $\mathcal{N}_\varepsilon$ is not a C^1-submanifold of E_ε, we cannot apply Theorem 5.8 directly. Fortunately, from Lemma 2.2, we know that the mapping m_ε is a homeomorphism between $\mathcal{N}_\varepsilon$ and S_ε, and S_ε is a C^1-submanifold of E_ε. So we can apply Theorem 5.8 to $\Upsilon_{\varepsilon,\lambda}(w) = \mathcal{T}_{\varepsilon,\lambda}(\hat{m}_\varepsilon(w))|_{S_\varepsilon} = \mathcal{T}_{\varepsilon,\lambda}(m_\varepsilon(w))$, where $\Upsilon_{\varepsilon,\lambda}$ is given in Lemma 2.3.

Define $\gamma_{\varepsilon,1}(y) = m_\varepsilon^{-1}(t_\varepsilon \psi_{\varepsilon,y}) = m_\varepsilon^{-1}(\gamma_\varepsilon(y)) = \frac{t_\varepsilon \psi_{\varepsilon,y}}{\|t_\varepsilon \psi_{\varepsilon,y}\|} = \frac{\psi_{\varepsilon,y}}{\|\psi_{\varepsilon,y}\|}$ for $y \in \mathcal{M}$. It follows from Lemma 5.1 that

$$\lim_{\varepsilon\to 0} \Upsilon_{\varepsilon,\lambda}\big(\gamma_{\varepsilon,1}(y)\big) = \lim_{\varepsilon\to 0} \mathcal{T}_{\varepsilon,\lambda}\big(\gamma_\varepsilon(y)\big) = c_{M_0}. \tag{5.20}$$

Furthermore, we set

$$\Sigma_{\varepsilon,1} := \big\{ w \in S_\varepsilon \colon \Upsilon_{\varepsilon,\lambda}(w) \leqslant c_{M_0} + \omega(\varepsilon) \big\}, \tag{5.21}$$

where $\omega(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$. It follows from (5.20) that $\omega(\varepsilon) = |\Upsilon_{\varepsilon,\lambda}(\gamma_{\varepsilon,1}(y)) - c_{M_0}| \to 0$ as $\varepsilon \to 0^+$. Thus, $\gamma_{\varepsilon,1}(y) \in \Sigma_{\varepsilon,1}$ and $\Sigma_{\varepsilon,1} \neq \emptyset$ for any $\varepsilon > 0$. Recall that $\Sigma_\varepsilon := \{u \in \mathcal{N}_\varepsilon \colon \mathcal{T}_{\varepsilon,\lambda}(u) \leqslant c_{M_0} + \omega(\varepsilon)\}$. From Lemmas 2.2–2.3, 5.1 and 5.6, we know that for any $\varepsilon > 0$ sufficiently small, the diagram

$$\mathcal{M} \xrightarrow{\gamma_\varepsilon} \Sigma_\varepsilon \xrightarrow{m_\varepsilon^{-1}} \Sigma_{\varepsilon,1} \xrightarrow{m_\varepsilon} \Sigma_\varepsilon \xrightarrow{\beta_\varepsilon} \mathcal{M}_\delta \tag{5.22}$$

is well define. By the arguments in the paragraph just before Lemma 5.2, we see that

$$\lim_{\varepsilon\to 0} \beta_\varepsilon\big(\gamma_\varepsilon(y)\big) = y \quad \text{uniformly in } y \in \mathcal{M}. \tag{5.23}$$

For $\varepsilon > 0$ small enough, we denote $\beta_\varepsilon(\gamma_\varepsilon(y)) = y + \nu(y)$ for $y \in \mathcal{M}$, where $|\nu(y)| < \frac{\delta}{2}$ uniformly for $y \in \mathcal{M}$. Define $H(t, y) = y + (1 - t)\nu(y)$. Then $H : [0, 1] \times \mathcal{O} \to \mathcal{M}_\delta$ is continuous. Obviously, $H(0, y) = \beta_\varepsilon(\gamma_\varepsilon(y))$, $H(1, y) = y$ for all $y \in \mathcal{M}$. Let $\gamma_{\varepsilon,1} = m_\varepsilon^{-1} \circ \gamma_\varepsilon$ and $\beta_{\varepsilon,1} = \beta_\varepsilon \circ m_\varepsilon$. Thus we obtain that the composite mapping $\beta_{\varepsilon,1} \circ \gamma_{\varepsilon,1} = \beta_\varepsilon \circ \gamma_\varepsilon$ is homotopic to the inclusion mapping $id : \mathcal{M} \to \mathcal{M}_\delta$. So it follows from Lemma 5.9 that

$$cat_{\Sigma_{\varepsilon,1}}(\Sigma_{\varepsilon,1}) \geqslant cat_{\mathcal{M}_\delta}(\mathcal{M}). \tag{5.24}$$

On the other hand, let us choose a function $\omega(\varepsilon) > 0$ such that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$ and such that $(c_{M_0} + \omega(\varepsilon))$ is not a critical level for $\mathcal{T}_{\varepsilon,\lambda}$. For $\varepsilon > 0$ small enough, we deduce from Lemma 4.5 that $\mathcal{T}_{\varepsilon,\lambda}$ satisfies the Palais–Smale condition in Σ_ε. By the conclusion (ii) of Lemma 2.3, we infer that $\Upsilon_{\varepsilon,\lambda}$ satisfies the Palais–Smale condition in $\Sigma_{\varepsilon,1}$. So it follows from Theorem 5.8 that $\Upsilon_{\varepsilon,\lambda}$ has at least $cat_{\Sigma_{\varepsilon,1}}(\Sigma_{\varepsilon,1})$ critical point on $\Sigma_{\varepsilon,1}$. By the conclusion (iii) of Lemma 2.3, we conclude that $\mathcal{T}_{\varepsilon,\lambda}$ has at least $cat_{\mathcal{M}_\delta}(\mathcal{M})$ critical points.

Proof of Theorem 1.2. From above arguments we know that $(\mathcal{H}_\varepsilon)$ has at least $cat_{\mathcal{M}_\delta}(\mathcal{M})$ positive solutions. Going back to $(KH)_\varepsilon$ with the variable substitution: $x \mapsto \frac{x}{\varepsilon}$. We obtain that $(KH)_\varepsilon$ has at least $cat_{\mathcal{M}_\delta}(\mathcal{M})$ positive solutions. In the following we shall prove the concentration phenomena for positive solutions. Let u_{ε_n} denote a positive solution of $(\mathcal{H}_{\varepsilon_n})$. Then $v_n(x) = u_n(x + y_n)$ is a solution of the problem

$$-\Bigg(a + b\int\limits_{\mathbb{R}^3} |\nabla v_n|^2\Bigg)\Delta v_n + \hat{M}_{\varepsilon_n}(x)v_n = \lambda f(v_n) + |v_n|^4 v_n \quad \text{in } \mathbb{R}^3,$$

where $\hat{M}_{\varepsilon_n}(x) = M(\varepsilon_n x + \varepsilon_n y_n)$ and y_n is given in Lemma 5.5. Furthermore, up to a subsequence, it follows from Lemma 5.5 that $v_n \to v$ and $\tilde{y}_n = \varepsilon_n y_n \to y \in \mathcal{M}$. We claim that there exists a $\delta > 0$ such that $\|v_n\|_{L^\infty(\mathbb{R}^3)} \geqslant \delta > 0$. Indeed, suppose that $\|v_n\|_{L^\infty(\mathbb{R}^3)} \to 0$. We deduce from $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$ that for each $\beta > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$

$$\frac{f(\|v_n\|_{L^\infty(\mathbb{R}^3)})}{\|v_n\|_{L^\infty(\mathbb{R}^3)}} < \beta \quad \text{and} \quad \|v_n\|_{L^\infty(\mathbb{R}^3)} < \beta.$$

Hence, by $(\mathcal{F}_2)$, we see that

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla v_n|^2 + M_0|v_n|^2 &\leqslant \lambda \int_{\mathbb{R}^3} \frac{f(\|v_n\|_{L^\infty(\mathbb{R}^3)})}{\|v_n\|_{L^\infty(\mathbb{R}^3)}} v_n^2 + \int_{\mathbb{R}^3} v_n^6 \\ &\leqslant \beta\lambda \int_{\mathbb{R}^3} v_n^2 + \|v_n\|^4_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} v_n^2 \leqslant c\beta.\end{aligned}$$

This implies that $\|v_n\|_{M_0} = 0$ for $n \geqslant n_0$, which is impossible because $v_n \to v$ in $H^1(\mathbb{R}^3)$ and $v \neq 0$ by Lemma 5.7. Then the claim is true. Let k_n be the global maximum of v_n, we infer from Lemma 5.7 and the claim above, we see that $\{k_n\} \subset B_R(0)$ for some $R > 0$. Thus, the global maximum of u_{ε_n} given by $z_n = y_n + k_n$ satisfies $\varepsilon_n z_n = \tilde{y}_n + \varepsilon_n k_n$. Since $\{k_n\}$ is bounded, it follows that $\varepsilon_n z_n \to y \in \mathcal{M}$. Moreover, since the function $h_\varepsilon(x) = u_\varepsilon(\frac{x}{\varepsilon})$ is a positive solution of $(KH)_\varepsilon$, then the maximum point σ_ε and z_ε of h_ε and u_ε respectively, satisfy the equality $\sigma_\varepsilon = \varepsilon z_\varepsilon$. So we have that $\lim_{\varepsilon \to 0} M(\sigma_\varepsilon) = \lim_{n\to\infty} M(\varepsilon_n z_n) = M_0$. Finally, from the above arguments and Lemma 5.7, it follows the boundedness of $\{k_n\}$ that $u_n(x) \leqslant ce^{-c|x-z_n+k_n|} \leqslant ce^{-c|x-z_n|}$. So we conclude that u_ε satisfies the conclusion (ii) of Theorem 1.2. $\square$

Proof of Theorem 1.3. Since for each $\varepsilon > 0$, we have $E = H^1(\mathbb{R}^3) = E_\varepsilon$. Therefore, to prove the conclusion, we first claim that $c_\varepsilon = c_{M^\infty}$ for each $\varepsilon > 0$. In fact, as in Lemma 4.2, since $M(x) \leqslant M^\infty$, one can easily check that $c_\varepsilon \geqslant c_{M^\infty}$. So, in order to prove $c_{M^\infty} = c_\varepsilon$, it suffices to show that

$$c_{M^\infty} \leqslant c_\varepsilon. \tag{5.25}$$

By Theorem 3.8, we know that there exists $e \in S_{M^\infty} = \{u \in H^1(\mathbb{R}^3)\colon \|u\|_{M^\infty} = 1\}$ and $s_0 > 0$ such that $u_0 = m_{M^\infty}(e) = s_0 e$ is a positive ground state solution of $(\mathcal{H}_{M^\infty})$. Moreover, $m_{M^\infty}(e)$ is the unique global maximum of $\mathcal{T}_{M^\infty,\lambda}$ on E. Set $w_n = e(\cdot - y_n)$, where $y_n \in \mathbb{R}^3$ and $|y_n| \to \infty$ as $n \to \infty$. Then by Lemma 2.2, it follows that for each n, $m_\varepsilon(w_n) = \hat{m}_\varepsilon(w_n) \in \mathcal{N}_\varepsilon$ is the unique global maximum of $\mathcal{T}_{\varepsilon,\lambda}$ on E. Therefore, we get

$$\begin{aligned} c_\varepsilon &\leqslant \mathcal{T}_{\varepsilon,\lambda}\big(m_\varepsilon(w_n)\big) \\ &= \frac{1}{2}\int_{\mathbb{R}^3} \big(a\big|\nabla m_\varepsilon(w_n)\big|^2 + M_\varepsilon(x)\big|m_\varepsilon(w_n)\big|^2\big) + \frac{b}{4}\bigg(\int_{\mathbb{R}^3} \big|\nabla m_\varepsilon(w_n)\big|^2\bigg)^2 \\ &\quad - \lambda \int_{\mathbb{R}^3} F\big(m(w_n)\big) - \frac{1}{6}\int_{\mathbb{R}^3} \big|m_\varepsilon(w_n)\big|^6 \\ &= \frac{1}{2}\int_{\mathbb{R}^3} \big(a\big|\nabla m_\varepsilon(e)\big|^2 + M(\varepsilon x + \varepsilon y_n)\big|m_\varepsilon(e)\big|^2\big) + \frac{b}{4}\bigg(\int_{\mathbb{R}^3} \big|\nabla m_\varepsilon(e)\big|^2\bigg)^2 \end{aligned}$$

$$
\begin{aligned}
&-\lambda \int_{\mathbb{R}^3} F\big(m(e)\big) - \frac{1}{6}\int_{\mathbb{R}^3} \big|m_\varepsilon(e)\big|^6 \\
&= \mathcal{T}_{M^\infty,\lambda}\big(m_\varepsilon(e)\big) + \int_{\mathbb{R}^3} \big(M(\varepsilon x+\varepsilon y_n) - M^\infty\big) m_\varepsilon^2(e) \\
&\leqslant c_{M^\infty} + \int_{\mathbb{R}^3} \big(M(\varepsilon x+\varepsilon y_n) - M^\infty\big) m_\varepsilon^2(e).
\end{aligned}
\tag{5.26}
$$

It is clear that for each $\epsilon > 0$, there exists $R > 0$ such that

$$
\int_{|x|\geqslant R} \big(M^\infty - M(\varepsilon x+\varepsilon y_n)\big)\big(m_\varepsilon(e)\big)^2 \leqslant c\epsilon. \tag{5.27}
$$

Moreover, we conclude from Lebesgue's dominated convergence theorem that

$$
\begin{aligned}
\lim_{n\to\infty} \int_{|x|<R} \big(M^\infty - M(\varepsilon x+\varepsilon y_n)\big)\big(m_\varepsilon(e)\big)^2 &= \int_{|x|<R} \Big(M^\infty - \lim_{n\to\infty} M(\varepsilon x+\varepsilon y_n)\Big)\big(m_\varepsilon(e)\big)^2 \\
&\leqslant \int_{|x|<R} \Big(M^\infty - \liminf_{n\to\infty} M(\varepsilon x+\varepsilon y_n)\Big)\big(m_\varepsilon(e)\big)^2 = 0.
\end{aligned}
\tag{5.28}
$$

So it follows from (5.26)–(5.28) that $c_{M^\infty} = c_\varepsilon$ for $\varepsilon > 0$.

Finally, assume, seeking a contradiction, for some $\varepsilon_0 > 0$ that there exists $0 < \hat{u} \in \mathcal{N}_{\varepsilon_0}$ such that $c_{\varepsilon_0} = \mathcal{T}_{\varepsilon_0,\lambda}(\hat{u})$. From the conclusion (iv) of Lemma 2.2, we deduce that there exists $\hat{e} \in S_{\varepsilon_0}$ such that $\hat{u} = m_{\varepsilon_0}(\hat{e}) = s_1\hat{e}$, where $s_1 > 0$. From Lemma 2.2 again, we infer that $m_{\varepsilon_0}(\hat{e}) = \hat{m}_{\varepsilon_0}(\hat{e})$ is the unique global maximum of $\mathcal{T}_{\varepsilon_0,\lambda}$ on E. We first have that $c_{M^\infty} \leqslant \mathcal{T}_{M^\infty,\lambda}(m_{M^\infty}(\hat{e})) = \max_{u\in E} \mathcal{T}_{M^\infty,\lambda}(u)$. On the other hand, by $(\mathcal{C}_1)$, it follows that $M(x) \geqslant M^\infty$ for all $x \in \mathbb{R}^3$ and $\mathcal{T}_{M^\infty,\lambda}(u) \leqslant \mathcal{T}_{\varepsilon_0,\lambda}(u)$ for each $u \in E$. Thus, $c_{M^\infty} \leqslant \mathcal{T}_{M^\infty,\lambda}(m_{M^\infty}(\hat{e})) \leqslant \mathcal{T}_{\varepsilon_0,\lambda}(m_{M^\infty}(\hat{e})) \leqslant \mathcal{T}_{\varepsilon_0,\lambda}(m_{\varepsilon_0}(\hat{e})) = c_{\varepsilon_0} = c_{M^\infty}$. This implies $c_{M^\infty} = \mathcal{T}_{M^\infty,\lambda}(m_{M^\infty}(\hat{e})) = \mathcal{T}_{\varepsilon_0,\lambda}(m_{M^\infty}(\hat{e}))$. Moreover, $u^\infty = m_{M^\infty}(\hat{e})$ satisfies

$$
(\mathcal{H}_{M^\infty}) \quad -\Big(a + b\int_{\mathbb{R}^3} \big|\nabla u^\infty\big|^2\Big)\Delta u^\infty + M^\infty u^\infty = \lambda f\big(u^\infty\big) + \big|u^\infty\big|^4 u^\infty \quad \text{in } \mathbb{R}^3.
$$

As in the proof of the conclusion (i) of Theorem 3.8, one can easily check that $u^\infty(x) > 0$ in $\mathbb{R}^3$. However, one has

$$
\mathcal{T}_{M^\infty,\lambda}\big(u^\infty\big) = \mathcal{T}_{\varepsilon_0,\lambda}\big(u^\infty\big) + \int_{\mathbb{R}^3} \big(M^\infty - M(\varepsilon_0 x)\big)\big(u^\infty\big)^2. \tag{5.29}
$$

Furthermore, we deduce from $(\mathcal{C}_1)$ that

$$
\int_{\mathbb{R}^3} \big(M^\infty - M(\varepsilon_0 x)\big)\big(u^\infty\big)^2 < 0. \tag{5.30}
$$

Thus, $\mathcal{T}_{M^\infty,\lambda}(u^\infty) < \mathcal{T}_{\varepsilon_0,\lambda}(u^\infty)$. This is a contradiction. $\square$

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