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# Global conservative solutions for a model equation for shallow water waves of moderate amplitude ☆

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**Abstract**

In this paper, we study the continuation of solutions to an equation for surface water waves of moderate amplitude in the shallow water regime beyond wave breaking (in [11], Constantin and Lannes proved that this equation accommodates wave breaking phenomena). Our approach is based on a method proposed by Bressan and Constantin [2]. By introducing a new set of independent and dependent variables, which resolve all singularities due to possible wave breaking, the evolution problem is rewritten as a semilinear system. Local existence of the semilinear system is obtained as fixed points of a contractive transformation. Moreover, this formulation allows one to continue the solution after collision time, giving a global conservative solution where the energy is conserved for almost all times. Finally, returning to the original variables, we obtain a semigroup of global conservative solutions, which depend continuously on the initial data.


*MSC:* 65M06; 65M12; 35B10; 35Q53

*Keywords:* Shallow water waves of moderate amplitude; Conservative solutions; Wave breaking

## 1. Introduction

This paper is focused on an equation for surface waves of moderate amplitude in the shallow water regime

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$$\begin{cases} u_t + u_x + 6uu_x - 6u^2u_x + 12u^3u_x + u_{xxx} \\ \quad - u_{xxt} + 14uu_{xxx} + 28u_xu_{xx} = 0, \quad t > 0,\ x \in \mathbb{R}, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.1}$$

which arises as an approximation to the Euler equations, modeling the unidirectional propagation of surface water wavers.

The study of water waves is a fascinating subject because the phenomena are familiar and the mathematical problems are various [29]. Since the exact governing equations for water waves have proven to be nearly intractable, the quest for suitable simplified model equations was initiated at the earliest stages of development of hydrodynamics. Until the early twentieth century, the study of water waves was confined almost exclusively to linear theory. Since linearization failed to explain some important aspects, several nonlinear models have been proposed, explaining nonlinear behaviors linking breaking waves and solitary waves [11]. The most prominent example is the Korteweg–de Vries (KdV) equation [24], the only member of the wider family of BBM-type equations that is integrable and relevant for the phenomenon of soliton manifestation [1]. Since KdV and BBM equations do not model breaking waves (wave breaking means that the wave remains bounded but its slope becomes unbounded in finite time), several model equations were proposed to capture this phenomenon, one of the typical models is the Camassa–Holm equation:

$$u_t - u_{xxt} + 2ku_x + 3uu_x = 2u_xu_{xx} + uu_{xx}. \tag{1.2}$$

Eq. (1.2) was first obtained as a bi-Hamiltonian generalization of KdV equation by Fuchssteiner and Fokas [16], and later derived as a model for unidirectional propagation of shallow water over a flat bottom by Camassa and Holm [4]. Similar to the KdV equation, Camassa–Holm equation has also a bi-Hamiltonian structure [16,25] and is completely integrable [4,7,22]. The orbital stability of solitary waves and the stability of the peakons ($k = 0$) for Camassa–Holm equation are considered by Constantin and Strauss [12,13]. The advantage of the Camassa–Holm equation in comparison with the KdV equation lies in the fact that the Camassa–Holm equation has peaked solitons and models the peculiar wave breaking phenomena (cf. [5,8]). Many results have been obtained for waves of small amplitude, but it is also interesting and important to look at large amplitude waves. Departing from an equation derived by Johnson in [23], which at a certain depth below the fluid surface is a Camassa–Holm equation, one can derive a corresponding equation for the free surface valid for waves of moderate amplitude in the shallow water regime. Local well-posedness for initial value problem associated to (1.1) was first established by Constantin and Lannes [11], and then improved using Kato's semigroup approach for quasi-linear equations and an approach due to Kato by Duruk Mutlubas [14]. Recently, Mi and Mu [26] improved the local well-posedness of Eq. (1.1) in Besov space $B^s_{p,r}$ with $1 \leqslant p, r \leqslant +\infty$ and $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$ by the transport equations theory and the classical Friedrichs regularization method. Note that, unlike KdV or CH, Eq. (1.1) does not have a bi-Hamiltonian integrable structure [6]. Nevertheless, the equation possesses solitary wave profiles that resemble those of CH, analyzed in [9], and present similarities with the shape of the solitary waves for the governing equations for water waves considered in [10,13,17], and the orbital stability of solitary waves for this equation was recently obtained in [15,17].

In view of the possible development of singularities in finite time, continuation of the solution beyond wave breaking has been a challenge. Recently, this issue has been discussed for the Camassa–Holm equation [2,3] and for the hyperelastic rod equation in [27,28,30], by introducing a new set of independent and dependent variables. On the other hand, by introducing a coordinate transformation into Lagrangian coordinates, the conservative and dissipative solutions of the Camassa–Holm equation is also studied in [18–21]. For Eq. (1.1), Constantin and Lannes show that singularities can develop only in the form of wave breaking in [11], it is natural to wonder about the behavior of a solution beyond the occurrence of wave breaking.

Motivated by the works of Bressan and Constantin [2,3] to solve the singularities of solutions to the Camassa–Holm equation, we want to know the behavior of a solution for Eq. (1.1) after wave breaking, and establish the existence of a semigroup of global solutions with non-increasing $H^1(\mathbb{R})$ energy. However, the main difficulty is that we here deal with a higher order nonlinearity. To solve the problem, we present an equivalent semilinear system to the problem (1.1) by introducing new variables as follows.

Consider an energy variable $\xi \in \mathbb{R}$, let $\tilde{u}(\xi) \in H^1(\mathbb{R})$ be the initial data of Eq. (1.1) and the non-decreasing map $\xi \mapsto \tilde{y}(\xi)$ be defined as

$$\int_0^{\tilde{y}(\xi)} \left(1 + \tilde{u}_x^2\right) dx = \xi. \tag{1.3}$$

Let $y(t, \xi)$ be a solution to the following problem

$$\frac{\partial}{\partial t} y(t, \xi) = -1 - 14u\big(t, y(t, \xi)\big), \qquad y(0, \xi) = \tilde{y}(\xi), \tag{1.4}$$

and new variables functions $v = v(t, \xi)$ and $q = q(t, \xi)$ be defined as

$$v \doteq 2 \arctan u_x, \qquad q \doteq \left(1 + u_x^2\right) \cdot \frac{\partial y}{\partial \xi}. \tag{1.5}$$

By the new variables defined in (1.3)–(1.5), we can rewrite (1.1) as the following semilinear system (see Section 2.2)

$$\begin{cases} \dfrac{\partial u}{\partial t} = -P_x, \\ \dfrac{\partial v}{\partial t} = \left(2u + 10u^2 - 2u^3 + 3u^4 - P\right)(1 + \cos v) + 14 \sin^2 \dfrac{v}{2}, \\ \dfrac{\partial q}{\partial t} = \left(2u + 10u^2 - 2u^3 + 3u^4 - P - 7\right) \sin v \cdot q, \end{cases} \tag{1.6}$$

where

$$P(t, \xi) = \frac{1}{2} \int_{-\infty}^{\infty} \exp\left\{-\left|\int_{\xi}^{\theta} \cos^2 \frac{v(s)}{2} \cdot q(s)\, ds\right|\right\} \cdot \left[\left(2u + 10u^2 - 2u^3 + 3u^4\right) \cos^2 \frac{v}{2} - 7 \sin^2 \frac{v}{2}\right](\theta) \cdot q(\theta)\, d\theta,$$

$$P_x(t,\xi)=\frac{1}{2}\left(\int_{\xi}^{\infty}-\int_{-\infty}^{\xi}\right)\exp\left\{-\left|\int_{\xi}^{\theta}\cos^2\frac{v(s)}{2}\cdot q(s)\,ds\right|\right\}$$
$$\cdot\left[\left(2u+10u^2-2u^3+3u^4\right)\cos^2\frac{v}{2}-7\sin^2\frac{v}{2}\right](\theta)\cdot q(\theta)\,d\theta.$$

A local solution of (1.6) can be obtained as the fixed point of a contractive transformation. Moreover, exploiting the conservation of energy property, this local solution can be extended globally in time:

**Theorem 1.1.** *If $\tilde{u}\in H^1(\mathbb{R})$, then the Cauchy problem (1.5) with $(\tilde{u}(\tilde{y}(\xi)), 2\arctan\tilde{u}_x(\tilde{y}(\xi)), 1)$ has a unique solution for all $t\geqslant 0$, in the sense of Definition 2.1.*

Remarkably, the new variables allow us to resolve all singularities. Indeed, solutions of the equivalent semilinear system (1.6) can be globally extended in time, even after wave breaking.

To this, we need go back to the original variables $u(t,x)$. It suffices to show that the global solution of (1.6) yields a global conservative solution to (1.1), in the original variables $(t,x)$. Recall

$$y(t,\xi)\doteq\tilde{y}(\xi)-t-14\int_0^t u(\tau,\xi)\,d\tau. \tag{1.7}$$

For each fixed $\xi$, the function $y(t,\xi)$ thus provides a solution to the Cauchy problem

$$\frac{\partial}{\partial t}y(t,\xi)=-1-14u(t,\xi),\qquad y(0,\xi)=\tilde{y}(\xi). \tag{1.8}$$

We claim that a solution of (2.1) can be obtained by setting

$$u(t,x)\doteq u(t,\xi)\quad \text{if } y(t,\xi)=x. \tag{1.9}$$

Then the following theorem shows that the global solution of (1.6) yields a global conservative solution to (1.1).

**Theorem 1.2.** *Let $(u,v,q)$ be a global solution to (1.6) with initial data $(\tilde{u}, 2\arctan\tilde{u}_x, 1)$. Then the pair of functions $u(t,x)$ defined by (1.7)–(1.9) is the global solution to the problem (1.1). Moreover, this solution $u$ enjoys the following property:*

$$\left\|u(t)\right\|^2_{H^1(\mathbb{R})}=\|\tilde{u}\|^2_{H^1(\mathbb{R})}\quad \text{for a.e. } t\geqslant 0. \tag{1.10}$$

*Furthermore, let $\tilde{u}_n$ be a sequence of initial data such that*

$$\|\tilde{u}_n-\tilde{u}\|_{H^1(\mathbb{R})}\to 0. \tag{1.11}$$

*Then the corresponding solutions $u_n(t,x)$ converge to $u(t,x)$ uniformly for $(t,x)$ in any bounded set.*

In Theorem 1.2, for each initial data $\tilde{u} \in H^1(\mathbb{R})$, we construct a global conservative solution to Eq. (1.1). However, we remark that the resulting flow $u(t) = \Psi_t \tilde{u}$ is not a semigroup yet. Indeed, if $t \in \mathcal{S} \doteq \{t \geqslant 0;\ \text{measure}\ \{\xi \in \mathbb{R};\ v(t,\xi) = -\pi\} > 0\}$, the semigroup property fails.

To obtain a semigroup, it is clear that we need to retain some additional information about the solutions. For this purpose, we consider the domain $\mathcal{D}$ consisting of all couples $(u,\mu)$, where $u \in H^1(\mathbb{R})$, while $\mu$ is a positive Radon measure of $\mathbb{R}$ satisfying $d\mu^a = u_x^2\,dx$. In other words, splitting $\mu = \mu^a + \mu^s$ as the sum of an absolutely continuous and a singular part, we require that the absolutely continuous part has a density $u_x^2$ with respect to the Lebesgue measure. We call $\mathcal{M}(\mathbb{R})$ the metric space of all bounded Radon measures on $\mathbb{R}$, endowed with the topology of weak convergence.

Given $(\tilde{u},\tilde{\mu}) \in \mathcal{D}$, we define the map $\xi \mapsto \tilde{y}(\xi)$ by setting

$$\begin{cases} \tilde{y}(\xi) \doteq \sup\{x;\ x + \tilde{\mu}([0,x]) \leqslant \xi\} & \text{if } \xi \geqslant 0, \\ \tilde{y}(\xi) \doteq \inf\{x;\ |x| + \tilde{\mu}([0,x]) \leqslant |\xi|\} & \text{if } \xi < 0. \end{cases}$$

This definition is designed so that for any Borel set $J \subset \mathbb{R}$ we have

$$\tilde{\mu}(J) + \text{measure}(J) = \text{measure}\{\xi \in \mathbb{R};\ \tilde{y}(\xi) \in J\}.$$

Note that this reduces to (1.3) in this case where $\mu$ is absolutely continuous. In all cases, the map $\xi \mapsto \tilde{y}(\xi)$ is Lipschitz continuous with constant 1, and hence it is differentiable almost everywhere. We now solve the system (1.6) with initial data

$$\tilde{u}(\xi) = \tilde{u}(\tilde{y}(\xi)), \qquad q(\xi) \equiv 1,$$
$$\tilde{v}(\xi) = \begin{cases} 2\arctan u_x(\tilde{y}(\xi)) = 2\arctan \tilde{u}_\xi(\xi)\cdot \frac{d\xi}{d\tilde{y}} & \text{if } \frac{d\tilde{y}}{d\xi} > 0, \\ \pi & \text{if } \frac{d\tilde{y}}{d\xi} = 0. \end{cases}$$

In turn, from this solution $(u,v,q)$ we recover a mapping

$$t \mapsto (u(t), \mu_{(t)}) \in H^1(\mathbb{R}) \times \mathcal{M}(\mathbb{R})$$

defined by (1.7) and (1.9) together with

$$\mu_{(t)}([a,b]) = \int_{\{\xi;\ y(t,\xi)\in[a,b]\}} \sin^2\frac{v(t,\xi)}{2}\cdot q(t,\xi)\,d\xi. \tag{1.12}$$

Our last main result is the following.

**Theorem 1.3.** *There exists a continuous semigroup $\Psi : \mathcal{D} \times [0,\infty) \mapsto \mathcal{D}$ whose trajectories $t \mapsto (\tilde{u},\tilde{\mu}) = \Psi(\tilde{u},\tilde{\mu})$ have the following properties*:

(i) *The function $u$ provides a solution to the Cauchy problem (1.1) in the sense of Definition 2.1, while the measures $\{\mu_{(t)},\ t \geqslant 0\}$ provide a measure valued solution $w$ to the linear transport equation with source*

$$w_t - 14(uw)_x = 2\big(2u + 10u^2 - 2u^3 + 3u^4 - P\big)u_x + w_x. \tag{1.13}$$

(ii) *For a.e.* $t \geqslant 0$ *the measure* $\mu_{(t)}$ *is absolutely continuous. Its density with respect to the Lebesgue measure is given by*

$$d\mu_{(t)} = u_x^2(t, \cdot)\,dx. \tag{1.14}$$

(iii) *If* $\tilde{u}_n \to \tilde{u}$ *in* $H^1(\mathbb{R})$ *and* $\tilde{\mu}_{(t)} \rightharpoonup \tilde{u}$ *weakly, then* $u_n(t,x) \to u(t,x)$ *uniformly for* $(t,x)$ *in bounded sets.*

This paper is organized as follows. In Section 2, we introduce a new set of independent and dependent variables, and we obtain Eq. (1.5). In Section 3, a global continuous semigroup of weak conservative solutions to Eq. (1.1) will be constructed, and we prove Theorems 1.1–1.3.

## 2. Preliminary

### 2.1. The basic equations

As usual, we can rewrite Eq. (1.1) as follows:

$$\begin{cases} u_t - u_x - 14uu_x + P_x = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

with

$$P = \frac{1}{2}e^{-|x|} * \big(2u + 10u^2 - 2u^3 + 3u^4 - 7u_x^2\big). \tag{2.2}$$

If $u \in H^1(\mathbb{R})$, we claim that $Q, P \in H^1(\mathbb{R})$. It follows from the generalization of Young's inequality that

$$\begin{aligned}
\big\|P(t,x)\big\|_{L^2(\mathbb{R})} &\leqslant \left\|\frac{1}{2}e^{-|x|} * \big(2u + 10u^2 - 2u^3 + 3u^4 - 7u_x^2\big)\right\|_{L^2(\mathbb{R})} \\
&\leqslant C_1\Bigg(\big\|e^{-|x|}\big\|_{L^1(\mathbb{R})} \cdot \|u\|_{L^2(\mathbb{R})} + \big\|e^{-|x|}\big\|_{L^2(\mathbb{R})} \cdot \big\|u^2\big\|_{L^1(\mathbb{R})} \\
&\quad + \big\|e^{-|x|}\big\|_{L^2(\mathbb{R})} \cdot \big\|u^2\big\|_{L^1(\mathbb{R})}\|u\|_{L^\infty(\mathbb{R})} + \left\|\frac{1}{2}e^{-|x|}\right\|_{L^2(\mathbb{R})} \cdot \big\|u^2\big\|_{L^1(\mathbb{R})}\|u\|^2_{L^\infty(\mathbb{R})} \\
&\quad + \left\|\frac{1}{2}e^{-|x|}\right\|_{L^2(\mathbb{R})} \cdot \big\|u_x^2\big\|_{L^1(\mathbb{R})}\Bigg) \\
&\leqslant C\big[\big(1 + \|u\|_{L^\infty(\mathbb{R})} + \|u\|^2_{L^\infty(\mathbb{R})}\big)\|u\|^2_{H^1(\mathbb{R})} + \|u\|_{H^1(\mathbb{R})}\big].
\end{aligned}$$

On the other hand, due to the Sobolev inequality $\|u\|_{L^\infty(\mathbb{R})} \leqslant \|u\|_{H^1(\mathbb{R})}$. Thus, $P(x) \in L^2(\mathbb{R})$. Similarly, we can obtain $P_x(x) \in L^2(\mathbb{R})$, that is, $P(x) \in H^1(\mathbb{R})$.

**Definition 2.1.** By a solution of the Cauchy problem (2.1) on $[t_1, t_2]$ we mean a Hölder continuous functions $u(x,t)$ defined on $[t_1, t_2] \times \mathbb{R}$ with the following properties. At each fixed $t$ we have $u(t,\cdot) \in H^1(\mathbb{R})$. Moreover, the map $t \mapsto u(t,\cdot)$ is Lipschitz continuous from $[t_1, t_2]$ into $L^2(\mathbb{R})$, satisfying the initial condition and

$$\frac{d}{dt}u = u_x + 14uu_x - P_x, \quad \text{for a.e. } t. \tag{2.3}$$

Here (2.1) is understood as an equality between functions in $L^2(\mathbb{R})$.

For smooth solutions, we have the conservation law

$$E(t) = \int_{\mathbb{R}} \left(u^2 + u_x^2\right) dx = E(0). \tag{2.4}$$

Indeed, differentiating the first equations in (1.1) with respect to $x$ and using the identity $\partial_x^2 p * f = p * f - f$, we get

$$u_{xt} - u_{xx} - 7u_x^2 - 14uu_{xx} - 2u - 10u^2 + 2u^3 - 3u^4 + P = 0.$$

In view of Eq. (2.1) and the above equality, we have

$$\begin{aligned}
\frac{d}{dt}E(t) &= 2\int_{\mathbb{R}} uu_t + u_x u_{xt}\, dx \\
&= 2\int_{\mathbb{R}} u(u_x + 14uu_x - P_x) \\
&\quad + u_x\left(u_{xx} + 7u_x^2 + 14uu_{xx} + 2u + 10u^2 - 2u^3 + 3u^4 - P\right) dx \\
&= 0.
\end{aligned}$$

Thus (2.4) holds.

### 2.2. *A new set of independent and dependent variables*

Let $\tilde{u} \in H^1(\mathbb{R})$ be the initial data. Consider an energy variable $\xi \in \mathbb{R}$, and set the non-decreasing map $\xi \mapsto \tilde{y}(\xi)$ be defined by following

$$\int_0^{\tilde{y}(\xi)} \left(1 + \tilde{u}_x^2\right) dx = \xi. \tag{2.5}$$

Assuming that the solution $u$ to Eq. (2.1) remains Lipschitz continuous for $t \in [0,T]$, we now derive an equivalent system of equations, using the independent variables $(t,\xi)$. Let $t \mapsto y(t,\xi)$ be the characteristic starting at $\tilde{y}(\xi)$, so that

$$\frac{\partial}{\partial t}y(t,\xi) = -1 - 14u\big(t, y(t,\xi)\big), \qquad y(0,\xi) = \bar{y}(\xi). \tag{2.6}$$

Moreover, we write

$$u(t,\xi) \doteq u\big(t, y(t,\xi)\big), \qquad u_x(t,\xi) \doteq u_x\big(t, y(t,\xi)\big),$$
$$P(t,\xi) \doteq P\big(t, y(t,\xi)\big), \qquad P_x(t,\xi) \doteq P_x\big(t, y(t,\xi)\big).$$

The following further variables will be used: $v = v(t,\xi)$ and $q = q(t,\xi)$ defined as

$$v \doteq 2\arctan u_x, \qquad q \doteq \big(1 + u_x^2\big) \cdot \frac{\partial y}{\partial \xi}. \tag{2.7}$$

We stress that $v$ is defined up to multiples of $2\pi$. All subsequent equations involving $v$ are invariant under addition of multiples of $2\pi$. Notice that (2.5) implies

$$q(0,\xi) \equiv 1. \tag{2.8}$$

And we have the identities

$$\frac{1}{1+u_x^2} = \cos^2\frac{v}{2}, \qquad \frac{u_x}{1+u_x^2} = \frac{1}{2}\sin v, \qquad \frac{u_x^2}{1+u_x^2} = \sin^2\frac{v}{2}, \tag{2.9}$$

and

$$\frac{\partial y}{\partial \xi} = \frac{q}{1+u_x^2} = \cos^2\frac{v}{2} \cdot q. \tag{2.10}$$

In view of (2.10) this yields

$$y(t,\theta) - y(t,\xi) = \int_{\xi}^{\theta} \cos^2\frac{v(t,s)}{2} \cdot q(t,s)\,ds. \tag{2.11}$$

Furthermore, we get

$$P(t,\xi) = \frac{1}{2}\int_{-\infty}^{\infty} \exp\big\{-\big|y(t,\xi) - x\big|\big\}\big(2u + 10u^2 - 2u^3 + 3u^4 - 7u_x^2\big)\,dx,$$

$$P_x(t,\xi) = \frac{1}{2}\left(\int_{y(t,\xi)}^{\infty} - \int_{-\infty}^{y(t,\xi)}\right)\exp\big\{-\big|y(t,\xi) - x\big|\big\}\big(2u + 10u^2 - 2u^3 + 3u^4 - 7u_x^2\big)\,dx.$$

In the above formulae, we can perform the change of variables $x = y(t,\theta)$, and write the convolution as an integral over the variable $\theta$. Using the identities (2.9)–(2.11), we thus get an expression for $P$ and $P_x$ in terms of the new variable $\xi$, that is,

$$P(t,\xi)=\frac{1}{2}\int_{-\infty}^{\infty}\exp\left\{-\left|\int_{\xi}^{\theta}\cos^2\frac{v(s)}{2}\cdot q(s)\,ds\right|\right\}$$
$$\cdot\left[\left(2u+10u^2-2u^3+3u^4\right)\cos^2\frac{v}{2}-7\sin^2\frac{v}{2}\right](\theta)\cdot q(\theta)\,d\theta, \tag{2.12}$$

$$P_x(t,\xi)=\frac{1}{2}\left(\int_{\xi}^{\infty}-\int_{-\infty}^{\xi}\right)\exp\left\{-\left|\int_{\xi}^{\theta}\cos^2\frac{v(s)}{2}\cdot q(s)\,ds\right|\right\}$$
$$\cdot\left[\left(2u+10u^2-2u^3+3u^4\right)\cos^2\frac{v}{2}-7\sin^2\frac{v}{2}\right](\theta)\cdot q(\theta)\,d\theta. \tag{2.13}$$

From (2.1) and (2.6), the evolution equation for $u$ in the new variables $(t,\xi)$ takes the form

$$\frac{\partial}{\partial t}u(t,\xi)=u_t+u_y y_t=u_t-u_x-14uu_x=-P_x(t,\xi), \tag{2.14}$$

where $P_x$ is given at (2.13).

Next, we derive an evolution equation for the variable $q$ in (2.7)

$$\int_{\xi_1}^{\xi_2}q(t,\xi)\,d\xi=\int_{y(t,\xi_1)}^{y(t,\xi_2)}\left(1+u_x^2(t,x)\right)dx.$$

(2.6) yields

$$\frac{d}{dt}\int_{\xi_1}^{\xi_2}q(t,\xi)\,d\xi=\int_{y(t,\xi_1)}^{y(t,\xi_2)}\left\{\left(1+u_x^2\right)_t-\left[(1+14u)\left(1+u_x^2\right)\right]_x\right\}dx$$
$$=\int_{y(t,\xi_1)}^{y(t,\xi_2)}\left(4u+20u^2-4u^3+6u^4-2P-14\right)u_x\,dx.$$

Differentiating with respect to $\xi$ we obtain

$$\frac{\partial}{\partial t}q(t,\xi)=\left(4u+20u^2-4u^3+6u^4-2P-14\right)\frac{u_x}{1+u_x^2}\cdot q$$
$$=\left(2u+10u^2-2u^3+3u^4-P-7\right)\sin v\cdot q. \tag{2.15}$$

Finally, using (2.6) and (2.7), we find

$$\begin{aligned}\frac{\partial}{\partial t}v(t,\xi) &= \frac{2}{1+u_x^2}(u_{xt}-u_{xx}-14uu_{xx})\\ &= \frac{2}{1+u_x^2}\big(7u_x^2+2u+10u^2-2u^3+3u^4-P\big)\\ &= 2\big(2u+10u^2-2u^3+3u^4-P\big)\cos^2\frac{v}{2}+14\sin^2\frac{v}{2}. \end{aligned}\tag{2.16}$$

In (2.15) and (2.16), the function $P$ is defined by (2.12).

## 3. Global solutions of the semilinear system

### 3.1. Global solutions of the equivalent semilinear system

Let initial data $\tilde{u}\in H^1(\mathbb{R})$ be given. By Section 2, we can rewrite the corresponding Cauchy problem for the variables $(u,v,q)$ in the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = -P_x,\\ \dfrac{\partial v}{\partial t} = \big(2u+10u^2-2u^3+3u^4-P\big)(1+\cos v)+14\sin^2\dfrac{v}{2},\\ \dfrac{\partial q}{\partial t} = \big(2u+10u^2-2u^3+3u^4-P-7\big)\sin v\cdot q, \end{cases}\tag{3.1}$$

with

$$\begin{cases} u(0,\xi)=\tilde{u}\big(\tilde{y}(\xi)\big),\\ v(0,\xi)=2\arctan\tilde{u}_x\big(\tilde{y}(\xi)\big),\\ q(0,\xi)=1. \end{cases}\tag{3.2}$$

We regard (3.1) with initial data (3.2) as an ordinary differential equation in the Banach space

$$X \doteq H^1(\mathbb{R})\times\big[L^2(\mathbb{R})\cap L^\infty(\mathbb{R})\big]\times L^\infty(\mathbb{R}),$$

with the norm

$$\big\|(u,v,q)\big\|_X \doteq \|u\|_{H^1(\mathbb{R})}+\|v\|_{L^2(\mathbb{R})}+\|v\|_{L^\infty(\mathbb{R})}+\|q\|_{L^\infty(\mathbb{R})}.$$

The solution of the Cauchy problem means that a fixed point of the integral transformation

$$\Upsilon(u,v,q)=(\bar{u},\bar{v},\bar{q}),$$

with

$$\begin{cases} \bar{u}(t,\xi) = \tilde{u}\big(\tilde{y}(\xi)\big) - \int\limits_0^t P_x(\tau,\xi)\,d\tau, \\ \bar{v} = 2\arctan \tilde{u}_x\big(\tilde{y}(\xi)\big) + \int\limits_0^t \big(2u + 10u^2 - 2u^3 + 3u^4 - P\big)(1+\cos v) + 14\sin^2\frac{v}{2}\,d\tau, \\ \bar{q} = 1 + \int\limits_0^t \big(2u + 10u^2 - 2u^3 + 3u^4 - P - 7\big)\sin v \cdot q\,d\tau. \end{cases} \tag{3.3}$$

By showing the local Lipschitz continuity of the right-hand side of (3.1), the local existence of solution will follow from the standard theorem for ordinary differential equations in Banach spaces. Then, we show the conservation of energy property expressed by (2.4). Moreover, we prove that this local solution can be extended globally in time.

**Proof of Theorem 1.1.** The proof consists of several steps.

**Step 1.** We establish the local existence of solution. In order to do this, it suffices to show that the operator determined by the right-hand side of (3.1), which maps $(u, v, q)$ to

$$\Big(-P_x, \big(2u + 10u^2 - 2u^3 + 3u^4 - P\big)(1+\cos v) + 14\sin^2\frac{v}{2}, \\ \big(2u + 10u^2 - 2u^3 + 3u^4 - P - 7\big)\sin v \cdot q\Big),$$

is Lipschitz continuous on every bounded domain $\Omega \in X$ in the form of

$$\Omega = \Big\{(u, v, q);\ \|u\|_{H^1} \leqslant \alpha,\ \|v\|_{L^2} \leqslant \beta,\ \|v\|_{L^\infty} \leqslant \frac{3\pi}{2},\ q(x) \in \big[q^-, q^+\big] \text{ for a.e. } x \in \mathbb{R}\Big\}$$

for any constants $\alpha, \beta, q^-, q^+ > 0$.

Since the uniform bounds on $v$, $q$, and due to the Sobolev inequality $\|u\|_{L^\infty(\mathbb{R})} \leqslant \|u\|_{H^1(\mathbb{R})}$, for $m = 1, 2, 3, 4$, it is clear that the maps

$$u^m, \quad u^m \cos v, \quad \sin^2\frac{v}{2}, \quad \sin v \cdot q$$

are all Lipschitz continuous as maps from $\Omega \mapsto L^2(\mathbb{R})$, and also from $\Omega \mapsto L^\infty(\mathbb{R})$.

Now, we only need to prove the Lipschitz continuity of the maps

$$(u, v, q) \mapsto P,\ P_x, \tag{3.4}$$

defined at (2.12) and (2.13), as maps from $\Omega$ into $H^1(\mathbb{R})$. Then the local existence of a solution to the Cauchy problem (3.1) on some small time interval $[0, T]$ with $T > 0$ can be followed from the standard theory of ordinary differential equations in Banach spaces.

First, we show that

$$\frac{\partial u}{\partial t} \in H^1(\mathbb{R}), \qquad \frac{\partial v}{\partial t} \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}), \qquad \frac{\partial q}{\partial t} \in L^\infty(\mathbb{R}).$$

To this, we first observe that, as long as $|v| \leqslant \frac{3\pi}{2}$, there holds

$$\sin^2 \frac{v}{2} \leqslant \frac{v^2}{4} \leqslant \frac{9\pi^2}{8} \sin^2 \frac{v}{2}.$$

For $(u, v, q) \in \Omega$, by the definition of $\Omega$ and the above inequality we have

$$\begin{aligned} \operatorname{meas}\left\{\xi \in \mathbb{R},\ \left|\frac{v(\xi)}{2}\right| \geqslant \frac{\pi}{4}\right\} &\leqslant \operatorname{meas} E \doteq \left\{\xi \in \mathbb{R}\colon\ \sin^2 \frac{v(\xi)}{2} \geqslant \frac{1}{18}\right\} \\ &\leqslant 18 \int_E \sin^2 \frac{v(\xi)}{2}\, d\xi \leqslant \frac{9}{2}\beta^2. \end{aligned}$$

Therefore, for any $\xi_1 < \xi_2$ we find

$$\int_{\xi_1}^{\xi_2} \cos^2 \frac{v(\xi)}{2} \cdot q(\theta)\, d\xi \geqslant \int_{\xi \in [\xi_1, \xi_2];\ |\frac{v(\xi)}{2}| \leqslant \frac{\pi}{4}} \frac{q^-}{2}\, d\xi \geqslant \left[\frac{\xi_2 - \xi_1}{2} - \frac{9}{2}\beta^2\right] q^-.$$

The above inequality is a key estimate which guarantees that the exponential term in the formulae (2.12)–(2.13) for $P$ and $P_x$ decreases quickly as $\xi - \xi_1 \to \infty$. Introducing the exponentially decaying function

$$\Gamma(\zeta) \doteq \min\left\{1, \exp\left(\frac{9}{2}\beta^2 q^- - \frac{|\zeta|}{2} q^-\right)\right\},$$

an easy computation yields to

$$\|\Gamma\|_{L^1(\mathbb{R})} = \left(\int_{|\zeta| \leqslant 9\beta^2} + \int_{|\zeta| \geqslant 9\beta^2}\right) \Gamma(\zeta)\, d\zeta = 18\beta^2 + \frac{4}{q^-},$$

$$\|\Gamma\|^2_{L^2(\mathbb{R})} = \left(\int_{|\zeta| \leqslant 9\beta^2} + \int_{|\zeta| \geqslant 9\beta^2}\right) \Gamma^2(\zeta)\, d\zeta = 18\beta^2 + \frac{2}{q^-}.$$

Next, we prove that $P(\xi), P_x(\xi) \in H^1(\mathbb{R})$, namely $P(\xi), \partial_\xi P(\xi), P_x(\xi), \partial_\xi P_x(\xi) \in L^2(\mathbb{R})$. We only give the estimate for $P_x(\xi)$ since the estimates for $P(\xi), \partial_\xi P(\xi), P_x(\xi)$ are entirely similar. By the definition (2.13), it follows that

$$\left|P_x(\xi)\right| \leqslant \frac{q^+}{2} \left|\Gamma^* \left[\left(2u + 10u^2 - 2u^3 + 3u^4\right) \cos^2 \frac{v}{2} - 7 \sin^2 \frac{v}{2}\right](\xi)\right|.$$

Therefore, using standard properties of convolutions and Young's inequality we obtain

$$\begin{aligned}\|P_x(\xi)\|_{L^2(\mathbb{R})} &\leqslant \frac{q^+}{2}\|\Gamma\|_{L^1(\mathbb{R})}\left(\|2u+10u^2-2u^3+3u^4\|_{L^2(\mathbb{R})}+\frac{1}{8}\|v^2\|_{L^2(\mathbb{R})}\right)\\ &\leqslant \frac{q^+}{2}\|\Gamma\|_{L^1(\mathbb{R})}\Big(\|u\|_{L^2(\mathbb{R})}+\big(\|u\|_{L^\infty(\mathbb{R})}+\|u\|^2_{L^\infty(\mathbb{R})}\\ &\quad+\|u\|^3_{L^\infty(\mathbb{R})}\big)\|u\|_{L^2(\mathbb{R})}+\frac{1}{8}\|v\|_{L^\infty(\mathbb{R})}\|v\|^2_{L^2(\mathbb{R})}\Big)\\ &<\infty.\end{aligned}$$

Next, differentiating (2.13) we get

$$\begin{aligned}\frac{\partial}{\partial\xi}P_x(\xi) &= -\left[\big(2u(\xi)+10u^2(\xi)-2u^3(\xi)+3u^4(\xi)\big)\cos^2\frac{v(\xi)}{2}-7\sin^2\frac{v(\xi)}{2}\right]q(\xi)\\ &\quad+\frac{1}{2}\left(\int_\xi^\infty-\int_{-\infty}^\xi\right)\exp\left\{-\left|\int_\xi^\theta\cos^2\frac{v(s)}{2}\cdot q(s)\,ds\right|\right\}\cdot\left[\cos^2\frac{v(\xi)}{2}\cdot q(\xi)\right]\\ &\quad\operatorname{sign}(\theta-\xi)\left[\big(2u+10u^2-2u^3+3u^4\big)\cos^2\frac{v}{2}-7\sin^2\frac{v}{2}\right](\theta)\cdot q(\theta)\,d\theta.\end{aligned}\tag{3.5}$$

Thus,

$$\begin{aligned}\left|\frac{\partial}{\partial\xi}P_x(\xi)\right| &\leqslant q^+\left|\big(2u(\xi)+10u^2(\xi)-2u^3(\xi)+3u^4(\xi)\big)\cos^2\frac{v(\xi)}{2}+\frac{v^2(\xi)}{8}\right|\\ &\quad+\frac{q^+}{2}\left|\Gamma^*\left[\big(2u+10u^2-2u^3+3u^4\big)\cos^2\frac{v}{2}-7\sin^2\frac{v}{2}\right](\xi)\right|,\end{aligned}$$

$$\begin{aligned}\left\|\frac{\partial}{\partial\xi}P_x(\xi)\right\|_{L^2(\mathbb{R})} &\leqslant q^+\left(\|2u+10u^2-2u^3+3u^4\|_{L^2(\mathbb{R})}+\frac{1}{8}\|v^2\|_{L^2(\mathbb{R})}\right)\\ &\quad+\frac{q^+}{2}\|\Gamma\|_{L^1(\mathbb{R})}\left(\|2u+10u^2-2u^3+3u^4\|_{L^2(\mathbb{R})}+\frac{1}{8}\|v^2\|_{L^2(\mathbb{R})}\right)\\ &\leqslant\left(q^++\frac{q^+}{2}\|\Gamma\|_{L^1(\mathbb{R})}\right)\Big(\|u\|_{L^2(\mathbb{R})}+\frac{1}{8}\|v\|_{L^\infty(\mathbb{R})}\|v\|^2_{L^2(\mathbb{R})}\\ &\quad+\big(\|u\|_{L^\infty(\mathbb{R})}+\|u\|^2_{L^\infty(\mathbb{R})}+\|u\|^3_{L^\infty(\mathbb{R})}\big)\|u\|_{L^2(\mathbb{R})}\Big)\\ &<\infty.\end{aligned}$$

Thus, $P_x\in H^1(\mathbb{R})$. By analogous estimates for $P(\xi)$, $\partial_\xi P(\xi)$, we have $P\in H^1(\mathbb{R})$.

We just proved that the maps in (3.4) actually take values in $H^1(\mathbb{R})$. To establish their Lipschitz continuity, it suffices to show that their partial derivatives

$$\frac{\partial P}{\partial u}, \quad \frac{\partial P}{\partial v}, \quad \frac{\partial P}{\partial q}, \quad \frac{\partial P_x}{\partial u}, \quad \frac{\partial P_x}{\partial v}, \quad \frac{\partial P_x}{\partial q} \tag{3.6}$$

are uniformly bounded as $(u, v, q)$ range inside the domain $\Omega$. We observe that these derivatives are bounded linear operators from the appropriate spaces into $H^1(\mathbb{R})$. For the sake of illustration, we shall work out the detailed estimates for $\frac{\partial P_x}{\partial u}$. All other derivatives can be estimated by the same methods.

At a given point $(u, v, q,) \in \Omega$, the partial derivative $\frac{\partial P_x}{\partial u} : H^1(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is the linear operator defined by

$$\left[\frac{\partial P_x(u, v, q)}{\partial u} \cdot \hat{u}\right](\xi) = \frac{1}{2}\left(\int_{\xi}^{\infty} - \int_{-\infty}^{\xi}\right) \exp\left\{-\left|\int_{\xi}^{\theta} \cos^2 \frac{v(s)}{2} \cdot q(s)\, ds\right|\right\}$$
$$\cdot \left[\left(2 + 20u - 6u^2 + 12u^3\right) \cos^2 \frac{v}{2}\right](\theta) \cdot q(\theta) \cdot \hat{u}(\theta)\, d\theta.$$

Thus

$$\left\|\frac{\partial P_x(u, v, q)}{\partial u} \cdot \hat{u}\right\|_{L^2(\mathbb{R})} \leqslant q^+ \left\|\Gamma^* \left|2 + 20u - 6u^2 + 12u^3\right|\right\|_{L^2(\mathbb{R})} \|\hat{u}\|_{L^\infty(\mathbb{R})}.$$

Since $\|\hat{u}\|_{L^\infty(\mathbb{R})} \leqslant \|\hat{u}\|_{H^1(\mathbb{R})}$, the above operator norm can thus be estimated as

$$\left\|\frac{\partial P_x(u, v, q)}{\partial u} \cdot \hat{u}\right\|_{L^2} \leqslant 2q^+ \|\Gamma\|_{L^2} + q^+ \|\Gamma\|_{L^1} \left(20\|u\|_{L^2} + 6\|u\|_{L^\infty} \|u\|_{L^2} + 12\|u\|_{L^\infty}^2 \|u\|_{L^2}\right).$$

From (3.5), $\frac{\partial(\partial_\xi P_x)}{\partial u} : H^1(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is the linear operator defined by

$$\begin{aligned}
&\left[\frac{\partial_\xi \partial P_x(u, v, q)}{\partial u} \cdot \hat{u}\right](\xi) \\
&\quad = -2\left[\left(2 + 20u - 6u^2 + 12u^3\right) \cos^2 \frac{v}{2}\right](\xi) \cdot q(\xi) \cdot \hat{u}(\xi) \\
&\qquad + \left(\int_{\xi}^{\infty} - \int_{-\infty}^{\xi}\right) \exp\left\{-\left|\int_{\xi}^{\theta} \cos^2 \frac{v(s)}{2} \cdot q(s)\, ds\right|\right\} \cdot \left[\cos^2 \frac{v(\xi)}{2} \cdot q(\xi)\right] \\
&\qquad \operatorname{sign}(\theta - \xi) \cdot \left[\left(2 + 20u - 6u^2 + 12u^3\right) \cos^2 \frac{v}{2}\right](\theta) q(\theta) \cdot \hat{u}(\theta)\, d\theta.
\end{aligned}$$

Its norm, as an operator from $H^1(\mathbb{R})$ into $L^2(\mathbb{R})$, is thus bounded by

$$\begin{aligned}
&\left\|\frac{\partial_\xi \partial P_x(u, v, q)}{\partial u}\right\|_{L^2(\mathbb{R})} \\
&\quad \leqslant C_1 q^+ \left(1 + \|u\|_{L^2} + \|u\|_{L^\infty} \|u\|_{L^2} + \|u\|_{L^\infty}^2 \|u\|_{L^2}\right) \\
&\qquad + 2\left(q^+\right)^2 \|\Gamma\|_{L^2} + \left(q^+\right)^2 \|\Gamma\|_{L^1} \left(20\|u\|_{L^2} + 6\|u\|_{L^\infty} \|u\|_{L^2} + 12\|u\|_{L^\infty}^2 \|u\|_{L^2}\right).
\end{aligned}$$

Then, we have the boundedness of $\frac{\partial P_x}{\partial u}$: as a linear operator from $H^1(\mathbb{R})$ into $H^1(\mathbb{R})$. The bounds on the other partial derivatives in (3.6) are obtained in an entirely similar way.

Thus, we have showed that the right-hand side of (3.1) is Lipschitz continuous on a neighborhood of the initial data in the space $X$.

**Step 2.** To ensure that the local solution of (3.1) constructed above can be extended to a global solution defined for all $t \geqslant 0$, it suffices to show that the quantity

$$\|u(t)\|_{H^1(\mathbb{R})} + \|v(t)\|_{L^2(\mathbb{R})} + \|v(t)\|_{L^\infty(\mathbb{R})} + \|q(t)\|_{L^\infty(\mathbb{R})} + \left\|\frac{1}{q(t)}\right\|_{L^\infty(\mathbb{R})}$$

remains uniformly bounded on any bounded time interval.

As long as the local solution of (3.1) is defined, we claim that

$$u_\xi = \frac{q}{2}\sin v, \tag{3.7}$$

and

$$\frac{d}{dt}\int_{\mathbb{R}}\left(u^2\cos^2\frac{v}{2} + \sin^2\frac{v}{2}\right)q\,d\xi = 0. \tag{3.8}$$

From (3.1) and (3.5), we get

$$u_{\xi t} = -\frac{\partial}{\partial \xi}\big(P_x(\xi)\big) = \left[\big(2u + 10u^2 - 2u^3 + 3u^4\big)\cos^2\frac{v}{2} - 7\sin^2\frac{v}{2} - P\cos^2\frac{v}{2}\right]q.$$

On the other hand, from (3.7), we have

$$\begin{aligned}
\left(\frac{q}{2}\sin v\right)_t &= \frac{q_t}{2}\sin v + \frac{q}{2}v_t\cos v \\
&= \frac{q}{2}\Big\{\big(2u + 10u^2 - 2u^3 + 3u^4 - P - 7\big)\sin^2 v \\
&\quad + \cos v\left(\big(2u + 10u^2 - 2u^3 + 3u^4 - P\big)(1 + \cos v) + 14\sin^2\frac{v}{2}\right)\Big\} \\
&= \left[\big(2u + 10u^2 - 2u^3 + 3u^4 - P\big)\cos^2\frac{v}{2} - 7\sin^2\frac{v}{2}\right]q.
\end{aligned}$$

Therefore,

$$u_{\xi t} = \left(\frac{q}{2}\sin v\right)_t.$$

Moreover, at the initial time $t = 0$, by (2.9) and (3.2) we get

$$\frac{\partial u}{\partial \xi} = \frac{u_x}{1 + u_x} = \frac{\sin v}{2} = \frac{q}{2}\sin v.$$

Therefore, we obtain that (3.8) remains valid for all times $t \geqslant 0$, as long as the solution is defined.

To prove (3.9), we proceed as follows. From (3.1) and (3.7) we deduce that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}\left(u^2\cos^2\frac{v}{2}+\sin^2\frac{v}{2}\right)q\,d\xi\\
&\quad=\int_{\mathbb{R}} q\left\{2\left(2u+10u^2-2u^3+3u^4-P-7\right)\sin\frac{v}{2}\cos\frac{v}{2}\left(u^2\cos^2\frac{v}{2}+\sin^2\frac{v}{2}\right)-2uP_x\cos^2\frac{v}{2}\right.\\
&\qquad\left.+\left(1-u^2\right)\left[2\left(2u+10u^2-2u^3+3u^4-P\right)\cos^2\frac{v}{2}+14\sin^2\frac{v}{2}\right]\cos\frac{v}{2}\sin\frac{v}{2}\right\}d\xi\\
&\quad=\int_{\mathbb{R}} q\left\{-2P\sin\frac{v}{2}\cos\frac{v}{2}-2uP_x\cos^2\frac{v}{2}+\left(4u+6u^2-4u^3+6u^4\right)\sin\frac{v}{2}\cos\frac{v}{2}\right\}dx.
\end{aligned}$$

On the other hand, the definition of $P$, $P_x$ in (2.12)–(2.13) yields

$$P_\xi=qP_x\cos^2\frac{v}{2},$$

using (3.8) we obtain

$$(uP)_\xi=u_\xi P+uP_\xi=q\left(P\cos\frac{v}{2}\sin\frac{v}{2}+uP_x\cos^2\frac{v}{2}\right),$$

and

$$u^m q\sin\frac{v}{2}\cos\frac{v}{2}=u^m u_\xi=\frac{1}{m}\left(u^3\right)_\xi.$$

Therefore,

$$\frac{d}{dt}\int_{\mathbb{R}}\left(u^2\cos^2\frac{v}{2}+\sin^2\frac{v}{2}\right)q\,d\xi=\int_{\mathbb{R}}\partial_\xi\left\{-2uP+2u^2+2u^3-u^4+\frac{6}{5}u^5\right\}d\xi=0$$

the last equality being justified since $\lim_{|\xi|\to\infty}u(\xi)=0$ as $u\in H^1(\mathbb{R})$, while $P$ is uniformly bounded. This proves (3.9).

We can now rewrite the total energy (2.4) in terms of the new variables. By (3.9), this energy remains constant in time,

$$E(t)=\int_{\mathbb{R}}\left(u^2(t,\xi)\cos^2\frac{v(t,\xi)}{2}+\sin^2\frac{v(t,\xi)}{2}\right)q(t,\xi)\,d\xi=E(0)\doteq E_0,\tag{3.9}$$

for any solution of (3.1)–(3.2).

From (3.8) and (3.10) we obtain the bound

$$\sup_{\xi\in\mathbb{R}}\left|u^2(t,\xi)\right|\leqslant 2\int_{\mathbb{R}}|uu_\xi|\,d\xi\leqslant 2\int_{\mathbb{R}}|u|\cdot\left|\sin\frac{v}{2}\cos\frac{v}{2}\right|q\,d\xi\leqslant E_0.\tag{3.10}$$

This provides a uniform a priori bound on $\|u(t)\|_{L^\infty(\mathbb{R})}$. From (3.10) and the definitions (2.12)–(2.13) it follows

$$\begin{aligned}
&\|P(t)\|_{L^\infty(\mathbb{R})},\ \|P_x(t)\|_{L^\infty(\mathbb{R})}\\
&\quad\leqslant \frac{1}{2}\|e^{-|x|}\|_{L^\infty}\cdot\|10u^2-2u^3+3u^4-7u_x^2\|_{L^1}+\|e^{-|x|}\|_{L^1}\cdot\|u\|_{L^\infty}\\
&\quad\leqslant C_1\big(E_0^{1/2}+E_0+E_0^{3/2}+E_0^2\big). \qquad (3.11)
\end{aligned}$$

Looking at the third equation in (3.1), by (3.11)–(3.12) we deduce that, as long as the solution is defined,

$$|q_t|\leqslant Cq \quad \text{with } C=C\big(E_0^{1/2},E_0,E_0^{3/2},E_0^2\big)\geqslant 0.$$

Since $q(0,\xi)=1$, the previous differential inequality yields

$$e^{-Ct}\leqslant q(t)\leqslant e^{Ct}. \qquad (3.12)$$

By the second equation in (3.1), it is now clear that

$$\|v\|_{L^\infty(\mathbb{R})}\leqslant \exp\{Mt\} \quad \text{with } M=M\big(E_0^{1/2},E_0,E_0^{3/2},E_0^2\big)\geqslant 0.$$

Moreover, the first equation in (3.1) implies

$$\left|\frac{d}{dt}\int_{\mathbb{R}}\big(u^2(t,\xi)\,d\xi\big)\right|\leqslant 2\|u\|_{L^\infty(\mathbb{R})}\|P_x(t)\|_{L^1(\mathbb{R})},$$

$$\left|\frac{d}{dt}\int_{\mathbb{R}}\big(u_\xi^2(t,\xi)\,d\xi\big)\right|\leqslant 2\|u_\xi\|_{L^\infty(\mathbb{R})}\|\partial_\xi P_x(t)\|_{L^1(\mathbb{R})}.$$

From (3.8)–(3.9) and (3.11), we get the uniform bounds of $u, u_\xi$ on bounded time interval, respectively. The estimates on $\|u\|_{H^1(\mathbb{R})}$ will follow from bounds on the $L^1$-norms of $P_x$ and $\partial_\xi P_x$. For this goal, taking $r$ the right-hand side of (3.13) in any bounded time interval, so that $r^{-1}\leqslant q(t)\leqslant r$. From (3.5) we obtain

$$\begin{aligned}
\frac{\partial}{\partial\xi}P_x(\xi)={}&-\left[\big(2u(\xi)+10u^2(\xi)-2u^3(\xi)+3u^4(\xi)\big)\cos^2\frac{v(\xi)}{2}-7\sin^2\frac{v(\xi)}{2}\right]q(\xi)\\
&+\frac{1}{2}\left(\int_\xi^\infty-\int_{-\infty}^\xi\right)\exp\left\{-\left|\int_\xi^\theta\cos^2\frac{v(s)}{2}\cdot q(s)\,ds\right|\right\}\cdot\left[\cos^2\frac{v(\xi)}{2}\cdot q(\xi)\right]\\
&\operatorname{sign}(\theta-\xi)\left[\big(2u+10u^2-2u^3+3u^4\big)\cos^2\frac{v}{2}-7\sin^2\frac{v}{2}\right](\theta)\cdot q(\theta)\,d\theta,
\end{aligned} \qquad (3.13)$$

$$\begin{aligned}
&\left\|\partial_\xi P_x(t)\right\|_{L^1(\mathbb{R})} \\
&\quad \leqslant r\big(2E_0^{1/2}+10E_0+2E_0^{3/2}+3E_0^2\big)+\frac{1}{2}\int_{\mathbb{R}}\exp\left\{-\left|\int_\xi^\theta r^{-1}\cos^2\frac{v(t,s)}{2}\,ds\right|\right\} \\
&\qquad \times\left|\big(2u(t,\theta)+10u^2(t,\theta)-2u^3(t,\theta)+3u^4(t,\theta)\big)\cos^2\frac{v(t,\theta)}{2}-7\sin^2\frac{v(t,\theta)}{2}\right|r\,d\theta \\
&\quad \leqslant r\big(2E_0^{1/2}+10E_0+2E_0^{3/2}+3E_0^2\big)+\|\tilde\Gamma\|_{L^1(\mathbb{R})}r\big(2E_0^{1/2}+10E_0+2E_0^{3/2}+3E_0^2\big),
\end{aligned}$$

where $\tilde\Gamma \doteq \min\{1, \exp(18(2E_0^{1/2}+10E_0+2E_0^{3/2}+3E_0^2)r^{-1}-\frac{|\zeta|}{2}r^{-1})\}$. We can thus repeat the estimates on $\Gamma$ before, and deduce

$$\|\tilde\Gamma\|_{L^1(\mathbb{R})}=72\big(2E_0^{1/2}+10E_0+2E_0^{3/2}+3E_0^2\big)+\frac{4}{r}.$$

Similar calculations show that the $L^1$-norms of $\|P(t)\|_{L^\infty(\mathbb{R})}$ are uniformly bounded. This proves the boundedness of $\|u(t)\|_{H^1(\mathbb{R})}$ for $t$ in bounded interval.

Finally, the second equations in (3.1) imply that

$$\begin{aligned}
\frac{d}{dt}\|v\|_{L^2} &\leqslant 2\big(2\|u\|_{L^2}+10\|u\|_{L^\infty}\|u\|_{L^2}+2\|u\|^2_{L^\infty}\|u\|_{L^2}+3\|u\|^3_{L^\infty}\|u\|_{L^2}+\|P\|_{L^2}\big) \\
&\quad +\frac{7}{2}\|v\|_{L^\infty(\mathbb{R})}\|v\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

By the previous bounds, it is clear that $\|v\|_{L^2(\mathbb{R})}$ remains bounded on bounded intervals of time. This completes the proof that the solution of (3.1) can be extended globally in time. This completes the proof of Theorem 1.1. □

Indeed, the above solution has an important property. Namely, consider the set of times

$$\mathcal{S}\doteq\big\{t\geqslant 0;\ \text{measure}\big\{\xi\in\mathbb{R};\ v(t,\xi)=-\pi\big\}>0\big\}.$$

Then, we claim measure$(\mathcal{S})=0$. Its validity will be proved in next section.

### 3.2. Returning to the original variables

We now show that the global solution of (3.1) yields a global conservative solution to (2.1), in the original variables $(t,x)$, and prove Theorem 1.2. For the reader's convenience, we collect here the basic definition

$$y(t,\xi)\doteq\bar y(\xi)-t-14\int_0^t u(\tau,\xi)\,d\tau. \tag{3.14}$$

For each fixed $\xi$, the function $y(t,\xi)$ thus provides a solution to the Cauchy problem

$$\frac{\partial}{\partial t} y(t,\xi) = -1 - 14u(t,\xi), \qquad y(0,\xi) = \tilde{y}(\xi). \tag{3.15}$$

We claim that a solution of (2.1) can be obtained by setting

$$u(t,x) \doteq u(t,\xi) \quad \text{if } y(t,\xi) = x. \tag{3.16}$$

**Proof of Theorem 1.2.** Using the uniform bound $|u(t,\xi)| \leqslant E_0^{1/2}$ being valid by (3.11), from (3.14) we have the estimate

$$\tilde{y}(\xi) - t - E_0^{1/2} t \leqslant y(t,\xi) \leqslant \tilde{y}(\xi) - t + E_0^{1/2} t, \quad t \geqslant 0. \tag{3.17}$$

For the definition of $\xi$ in (2.5), this yields

$$\lim_{\xi \to \pm\infty} y(t,\xi) = \pm\infty. \tag{3.18}$$

Therefore, the image of the map $(t,\xi) \mapsto (t, y(t,\xi))$ is the entire half-plane $\mathbb{R}^+ \times \mathbb{R}$. Now we claim

$$y_\xi = q \cos^2 \frac{v}{2} \quad \text{for all } t \geqslant 0 \text{ and a.e. } \xi \in \mathbb{R}. \tag{3.19}$$

Indeed, from (3.7) we have

$$\begin{aligned}
\frac{\partial}{\partial t}\left(q \cos^2 \frac{v}{2}\right)(t,\xi) &= -q v_t \sin \frac{v}{2} \cdot \cos \frac{v}{2} + q_t \cos^2 \frac{v}{2} \\
&= -q \sin \frac{v}{2} \cdot \cos \frac{v}{2} \left( \left(2u + 10u^2 - 2u^3 + 3u^4 - P\right)(1 + \cos v) + 14 \sin^2 \frac{v}{2} \right) \\
&\quad + \cos^2 \frac{v}{2} \left(2u + 10u^2 - 2u^3 + 3u^4 - P - 7\right) \sin v \cdot q \\
&= -14 \cos \frac{v}{2} \sin \frac{v}{2} = -7q \sin v = -14 u_\xi(t,\xi).
\end{aligned}$$

On the other hand, (3.14) implies

$$\frac{\partial}{\partial t} y_\xi(t,\xi) = -14 u_\xi(t,\xi).$$

Since the function $x \mapsto 2\arctan \tilde{u}_x(x)$ is measurable, the identity (3.19) holds true for almost every $\xi$ at $t = 0$. By the above calculation it remains true for all $t \geqslant 0$.

Next, we prove the set $\mathcal{S}$ defined in Section 3.1 has measure zero. Indeed, if $v(t_0,\xi) = -\pi$, then $y_\xi(t_0,\xi) = \cos^2 \frac{v(t_0,\xi)}{2} q(t_0,\xi) = 0$. Using (3.8), we get

$$\frac{\partial}{\partial t} y_\xi(t_0,\xi) = -14 u_\xi(t_0,\xi) = -7q \sin v(t_0,\xi) = 0.$$

Furthermore, we have

$$\frac{\partial^2}{\partial t^2} y_\xi(t_0,\xi) = -14 u_{\xi t}(t_0,\xi) = -7(q_t \sin v + q v_t \cos v)(t_0,\xi)$$
$$= 7 q v_t(t_0,\xi) = 98 q(t_0,\xi) > 0.$$

This implies that $t$ satisfying $y_\xi(t,\xi)=0$ is isolated. Thus $t$ satisfying $v(t,\xi)=-\pi$ is also isolated. Since $v \in L^2(\mathbb{R})$, this infers

$$\mathrm{measure}(\mathcal{S}) = 0.$$

From (3.19), we get that $y(t,\xi)$ is non-decreasing. Moreover, if $\xi < \theta$ but $y(t,\xi) = y(t,\theta)$, then

$$\int_\xi^\theta y_\xi(t,s)\,ds = \int_\xi^\theta q(t,s)\cos^2\frac{v}{2}\,ds = 0.$$

Hence $\cos\frac{v}{2} \equiv 0$ throughout the interval of integration. Therefore, by (3.8), we have

$$u(t,\theta) - u(t,\xi) = \int_\xi^\theta \frac{q(t,s)}{2}\sin v(t,s)\,ds = 0.$$

This shows that the map $(t,x) \mapsto u(t,x)$ in (3.16) is well defined for all $t \geqslant 0$ and $x \in \mathbb{R}$.

Next, using (3.8) and (3.19) to change the variable of integration, for every fixed $t$ we get

$$\begin{aligned}&\int_{\mathbb{R}} \big(u^2(t,x) + u_x^2(t,x)\big)\,dx \\ &\quad = \int_{\{\cos v > -1\}} \left(u^2(t,\xi)\cos^2\frac{v(t,\xi)}{2} + \sin^2\frac{v(t,\xi)}{2}\right) q(t,\xi)\,d\xi \leqslant E_0.\end{aligned} \tag{3.20}$$

Since the measure of $\mathcal{S}$ is zero, the equality holds true for almost all $t$.

By Sobolev's inequality, this implies the uniform Hölder continuity with the exponent $\frac{1}{2}$ for $u$ as functions of $x$. By (3.1) and the bounds $\|P_x\|_{L^\infty(\mathbb{R})} \leqslant C(E_0^{1/2}, E_0, E_0^{3/2}, E_0^2)$, we can infer that $u(t, y(t))$ are Hölder continuous with the exponent $\frac{1}{2}$. Indeed

$$\begin{aligned}\big|u(t,y) - u(s,y)\big| &\leqslant \big|u(t,y) - u(t,x)\big| + \big|u(t,x) - u\big(t, y(t,\xi)\big)\big| \\ &\quad + \big|u\big(t,y(t,\xi)\big) - u\big(t,y(s,\xi)\big)\big| \\ &\leqslant E_0^{\frac{1}{2}}|y-x|^{\frac{1}{2}} + E_0^{\frac{1}{2}}\big|y(t,\xi) - y(s,\xi)\big| + \int_s^t \big|P_x(\tau,\xi)\big|\,d\tau \\ &\leqslant C\big(|y-x|^{\frac{1}{2}} + |t-s|^{\frac{1}{2}} + |t-s|\big),\end{aligned}$$

where we choose $\xi \in \mathbb{R}$ such that the characteristic $t \mapsto y(t,\xi)$ passes through the point $(s,x)$. Notice that $u(t,x) \leqslant E_0^{\frac{1}{2}}$. This implies that $u(t,x)$ is uniformly Hölder continuous with the exponent $\frac{1}{2}$.

We now prove the Lipschitz continuity of $u(t,x)$ with values in $L^2(\mathbb{R})$. Consider any interval $[\tau,\tau+h]$. For a given point $x$, we choose $\xi\in\mathbb{R}$ such that the characteristic $t\mapsto y(t,\xi)$ passes through the point $(\tau,x)$. By (3.11), it follows that

$$\begin{aligned}\big|u(\tau+h,y)-u(\tau,y)\big| &\leqslant \big|u(\tau+h,x)-u\big(\tau+h,y(\tau+h,\xi)\big)\big| \\ &\quad+\big|u\big(\tau+h,y(\tau+h,\xi)\big)-u(\tau,x)\big| \\ &\leqslant \sup_{|y-x|\leqslant E_0^{\frac{1}{2}}}\big|u(\tau+h,y)-u(\tau+h,x)\big|+\int_{\tau}^{\tau+h}\big|P_x(t,\xi)\big|\,dt.\end{aligned}$$

Integrating over the whole real line, in view of (2.2), (3.13) and (3.20), we obtain

$$\begin{aligned}&\int_{\mathbb{R}}\big|u(\tau+h,y)-u(\tau,y)\big|^2\,dx \\ &\quad\leqslant 2\int_{\mathbb{R}}\Bigg(\int_{x-E_0^{\frac{1}{2}}}^{x+E_0^{\frac{1}{2}}}\big|u_x(\tau+h,y)\big|\,dy\Bigg)^2\,dx \\ &\quad\quad+2\int_{\mathbb{R}}\Bigg(\int_{\tau}^{\tau+h}\big|P_x(t,\xi)\big|\,dt\Bigg)^2 q(\tau,\xi)\cos^2\frac{v(\tau,\xi)}{2}\,d\xi \\ &\quad\leqslant 8E_0h^2\big\|u_x(\tau+h)\big\|^2_{L^2(\mathbb{R})}+2h\big\|q(\tau)\big\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}\int_{\tau}^{\tau+h}\big\|P_x(t)\big\|^2_{L^2(\mathbb{R})}\,dt\leqslant Ch^2\end{aligned}$$

holds for some constant $C$ depending only on $T$. This clearly implies the Lipschitz continuity of the map $t\mapsto u(t)$, in terms of the $x$-variable. Since $L^2(\mathbb{R})$ is a reflexive space, the left-hand side of (2.3) is well defined for almost all $t\in\mathbb{R}$. Note that we have proved that the right-hand side of (2.3) also lies in $L^2(\mathbb{R})$ for almost all $t\in\mathbb{R}$. To establish the equality between these two sides, we find that

$$\frac{d}{dt}u\big(t,y(t,\xi)\big)=-P_x(t,\xi).$$

On the other hand, recalling measure$(\mathcal{S})=0$, for every $t\notin\mathcal{S}$ the map $\xi\mapsto x(t,\xi)$ is one-to-one. Then, the change of variable formulae (2.13) yields

$$P_x(t,\xi)=\frac{1}{2}\Bigg(\int_{y(t,\xi)}^{\infty}-\int_{-\infty}^{y(t,\xi)}\Bigg)\exp\big\{-\big|y(t,\xi)-x\big|\big\}\big(2u+10u^2-2u^3+3u^4-7u_x^2\big)\,dx.$$

Hence the identity (2.3) is satisfied for almost all $t\geqslant 0$. This implies that $u$ is a global solution of Eq. (1.1) in the sense of Definition 2.1.

From (3.10) and (3.20), we obtain the identity in (1.9) for almost all $t \geqslant 0$.

Finally, let $\tilde{u}_n$ be a sequence of initial data converging to $\tilde{u}$ in $H^1(\mathbb{R})$. In this case, from (3.1) and the boundedness of $u, v, \rho$ we can infer

$$\left\|q_n(t,\xi) - q(t,\xi)\right\|_{L^2(\mathbb{R})} < \infty.$$

Recalling (2.5) and (3.2) at time $t = 0$, this implies

$$\sup_{\xi\in\mathbb{R}}\left|y_n(0,\xi) - y(0,\xi)\right| \to 0,$$
$$\sup_{\xi\in\mathbb{R}}\left|u_n(0,\xi) - u(0,\xi)\right| \to 0,$$
$$\left\|u_n(0,\xi) - u(0,\xi)\right\|_{H^1(\mathbb{R})} \to 0,$$
$$\left\|v_n(0,\xi) - v(0,\xi)\right\|_{L^2(\mathbb{R})} \to 0,$$
$$\left\|q_n(0,\xi) - q(0,\xi)\right\|_{L^2(\mathbb{R})} \to 0.$$

Now from (3.1) and the bounds of $u$, $v$, $q$, we can obtain

$$\begin{aligned}&\frac{d}{dt}\big(\left\|u_n(t,\xi) - u(t,\xi)\right\|_{L^\infty(\mathbb{R})} + \left\|v_n(0,\xi) - v(0,\xi)\right\|_{L^2(\mathbb{R})} + \left\|q_n(0,\xi) - q(0,\xi)\right\|_{L^2(\mathbb{R})}\big)\\ &\quad\leqslant C\big(\left\|u_n(t,\xi) - u(t,\xi)\right\|_{L^\infty(\mathbb{R})} + \left\|v_n(0,\xi) - v(0,\xi)\right\|_{L^2(\mathbb{R})} + \left\|q_n(0,\xi) - q(0,\xi)\right\|_{L^2(\mathbb{R})}\big).\end{aligned}$$

Thus, Gronwall's inequality implies that $u_n(t,\xi) \to u(t,\xi)$, uniformly for $(t,\xi)$ in bounded sets. Returning to the original coordinates, this yields

$$y_n(t,\xi) \to y(t,\xi), \qquad u_n(t,x) \to u(t,x),$$

uniformly on bounded sets since all functions $u, u_n$ are uniformly Hölder continuous. This completes the proof of Theorem 1.2. □

### *3.3. A semigroup property of the conservations*

In this subsection, we prove Theorem 1.3.

**Proof of Theorem 1.3.** Most of the above statements already follow from the analysis in the previous sections. Indeed, we have already proved that the function $u(t,x)$ defined by (3.14)–(3.16) provides a solution to Eq. (2.1).

Recall the set $\mathcal{S}$, and call $\mathcal{S} \subset \mathbb{R}$ the set of times where measure $\{\xi \in \mathbb{R};\ \cos v(t,\xi) = -1\} > 0$. For $t \notin \mathcal{S}$, the measure $\mu_{(t)}$ is precisely the absolutely continuous Radon measure having density $u_x^2$ with respect to the Lebesgue measure. On the other hand, our construction implies that for $t \in \mathcal{S}$, the measure $\mu_{(t)}$ is the weak limit of the measures $\mu_{(s)}$, as $s \to t$, $s \notin \mathcal{S}$. Since the set $\mathcal{S}$ has measure zero, we deduce (1.11). Notice that this equation can be formulated more precisely as

$$\int_{\mathbb{R}} \varphi_t \, d\mu_{(t)}\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_{\mathbb{R}} \varphi_x (1 - 14u) \, d\mu_{(t)} \, dt - \int_{t_1}^{t_2}\int_{\mathbb{R}} \big(2u + 10u^2 - 2u^3 + 3u^4 - P\big) u_x \varphi \, dx \, dt$$

for every $t_2 > t_1 \geqslant 0$ and any function $\varphi \in C^1([0, \infty] \times \mathbb{R})$ with compact support in $(0, \infty) \times \mathbb{R}$. The statement (iii) has been proved in the last paragraph of Section 3.2.

To complete the proof of Theorem 1.3, it now remains to prove the semigroup property:

$$\Psi_t \circ \Psi_s(\tilde{u}, \tilde{\mu}) = \Psi_{t+s}(\tilde{u}, \tilde{\mu}). \tag{3.21}$$

Given $(\tilde{u}, \tilde{\mu}) \in \mathcal{D}$, let

$$U(\tau, \xi) = \big(u(\tau, \xi), v(\tau, \xi), \ q(\tau, \xi)\big) \in \mathbb{R}^3$$

be the unique solution of (3.1) with initial data $(u(0, \xi), v(0, \xi), 1)$, defined for all $\tau \geqslant 0$. Then $\Psi_{t+s}(\tilde{u}, \tilde{\mu})$ is obtained from $U(t + s, \xi)$ via (3.14)–(3.16) and (1.11). To obtain $\Psi_t \circ \Psi_s(\tilde{u}, \tilde{\mu})$, we proceed as follows. For $\tau \geqslant 0$, $\tilde{U}(\tau, \xi) \in \mathbb{R}^3$ is the solution of (3.1) with initial data $(u(0, \xi), v(0, \xi), 1)$. Then, $\Psi_t \circ \Psi_s(\tilde{u}, \tilde{\mu})$ is obtained from $\tilde{U}(\tau, \xi)$ by (3.14)–(3.16) and (1.11). Notice that $U(t + \tau, \xi)$ with $\tau \geqslant 0$ is the solution of (3.1) with initial data $(u(s, \xi), v(s, \xi), q(s, \xi))$. We claim that

$$U_i(t + s, \xi) = \bar{U}(t, \bar{\xi}), \quad i = 1, 2, \tag{3.22}$$

where $\bar{\xi}$ is bi-Lipschitz parametrization of the $\xi$-variable with

$$\frac{d\bar{\xi}}{d\xi} = \frac{q(\tau, \xi)}{\bar{q}(\tau, \xi)} \quad \text{at time } \tau \geqslant 0. \tag{3.23}$$

Indeed, (2.12)–(2.13) and the form of (3.1) confirm the validity of (1.13) in view of the change of variables (3.23). The fact that $\xi \mapsto \bar{\xi}$ is a bi-Lipschitz parametrization follows at once if we notice that the linearity of the third equation in (3.1) with respect to $q$ yields

$$\begin{aligned}\partial_\tau \frac{q(\tau, \xi)}{\bar{q}(\tau, \xi)} = {} & \big[\big(2u + 10u^2 - 2u^3 + 3u^4 - P - 7\big) \sin v \\ & - \big(2\bar{u} + 10\bar{u}^2 - 2\bar{u}^3 + 3\bar{u}^4 - \bar{P} - 7\big) \sin \bar{v}\big] \frac{q(\tau, \xi)}{\bar{q}(\tau, \xi)}\end{aligned}$$

and the factor of $\frac{q(\tau,\xi)}{\bar{q}(\tau,\xi)}$ on the right-hand side is uniformly bounded. We thus established the validity of (1.13). Looking at (3.14)–(3.15) and (1.11), we prove that (3.21) holds true. This completes the proof of Theorem 1.3. □

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