



Stabilizing effect of the power law inflation on isentropic relativistic fluids

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Received 29 May 2017; revised 20 April 2018

Available online 10 May 2018

Abstract

This paper is concerned with the stabilizing effect of the power law inflation on relativistic Eulerian fluids. We prove the global stability of the background solutions to the relativistic fluids including the isothermal gases and generalized Chaplygin gases by the method of conformal transformation when the initial data is a small perturbation to the background solution. We also prove the blowup phenomena of the relativistic Euler fluids including the isothermal gases and polytropic gases when the initial data satisfies suitable assumptions.

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Keywords: Relativistic Euler equations; Power law inflation; Conformal transformation

1. Introduction

In this paper, we study the global stability and blowup phenomena of classical solutions to the following 1 + 3-dimensional relativistic Euler equations in standard coordinates (t, x^1, x^2, x^3)

$$\nabla_\mu T^{\mu\nu} = 0, \quad (1.1)$$

where Greek indice μ takes its values in $\{0, 1, \dots, 3\}$, ∇_μ denotes the covariant derivative with respect to the given metric $g = (g_{\mu\nu})$ and $T^{\mu\nu}$ denotes the energy momentum tensor, whose

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components are given by

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu},$$

in which ρ represents the energy density, $p = p(\rho)$ denotes the fluid pressure. $u = (u^0, u^1, u^2, u^3)$ is the four-velocity, which is future directed ($u^0 > 0$) and timelike vector field normalized by

$$u_\mu u^\mu = g_{\mu\nu} u^\mu u^\nu = -1.$$

Throughout the whole paper, repeated upper and lower indices are always summed over their ranges and the index is raised and lowered by the following given metric

$$g = -dt^2 + a^2(t) \sum_{i=1}^3 (dx^i)^2, \quad (1.2)$$

where $a(t) > 0$ is known as the scale factor. Spacetime metric of form (1.2) is important, since the experimental results indicate that our universe is accelerated expanding and approximately spacial flat. We say the spacetime is accelerated expanding, we mean

$$\frac{d}{dt}a(t) > 0 \quad \text{and} \quad \frac{d^2}{dt^2}a(t) > 0.$$

As is known to all, by the hyperbolic nature of relativistic Euler equations, the classical solution to (1.1) always blowup in finite time no matter how small and smooth the initial data is, especially in Minkowski spacetime. Indeed, so many results have been obtained to study the blowup phenomena of the initial value problem of system (1.1). A breakthrough has been made by Christodoulou [2], in which he considered the compressible and irrotational relativistic Euler equations with constant entropy. By combining the techniques of geometry and analysis, Christodoulou got the mechanism of the formation of singularities and furthermore constructed the weak solution (shocks) with Lisibach [3] under the assumption of spherical symmetry. Recently, his method has been investigated deeply and applied to various models, see his work with Miao [4] for classical Euler equations, Miao and Yu [13] for a single variational wave equation, Holzegel, Klainerman, Speck and Wong [8] for more general quasilinear wave equations. Besides, there is another way to study the formation of singularities for hyperbolic systems in several space variables. This method is firstly found by Sideris [19], who introduced some averaged quantities arising from the conservation law and studied the evolution equations satisfied by these quantities. Under appropriate assumptions on the fluids and the initial data, he can obtain a class of solutions, which will blowup in finite time. Sideris's method is also widely applied to various models. One can refer to [10,17,23] for classical fluids, [20,25,26] for nonlinear wave equations, [7] and [16] for relativistic fluids.

The results introduced above are on the blowup phenomena of the hyperbolic systems. However, there also exist some other results concerning the global stability of the hyperbolic flows. To the author's best knowledge, there are mainly two structures to ensure the global stability of hyperbolic systems. The first one is the "linearly degenerate" characteristics, and this structure is satisfied by the Eulerian flows such as Chaplygin gas and stiff matter. Based on this structure, Godin [6] proved the global stability of 3D radial solutions to Chaplygin gases with variable entropy. His result is extended to 2D case by Ding, Witt and Yin [5]. By different method, Lei and

Wei [11] extended Godin's result to relativistic case. In [24], the author later proved the global stability of 3D radial solutions to the stiff matter with variable entropy and gave a classification of the fluids with linearly degenerate characteristics based on the strict hyperbolicity. When the entropy is constant, Kong, Liu and Wang [9] proved the global existence of smooth solutions to 2D Chaplygin gases under the assumptions of zero vorticity. The other structure is the damping effect, thanks to the damping terms, Sideris, Thomases and Wang [21] studied the global existence of smooth solutions to 3D damping and isentropic Euler equations. Speck [22] studied the global stability of Einstein–Euler system in the presence of the positive cosmological constant when the equation of state takes $p = c_s^2 \rho$ with $0 \leq c_s \leq \frac{1}{3}$. In the system considered by Speck, the damping term comes from the presence of the positive cosmological constant. He also considered the Euler system in fixed background spacetime in [22] and studied the stabilizing effects of the spacetime to the fluids by pure analysis. By combining the method of conformal geometry and analysis, Oliynyk [15] reconsidered the system in [22] and got the global stability results easily in the case of vacuum background solution. LeFloch and Wei [12] studied the global non-linear stability of Friedmann–Robertson–Walker (FRW) solution to Einstein–Chaplygin fluids by extending Oliynyk's method. Both models in [22,12] get the stability of the relativistic fluids in the background spacetime with exponentially expanding rate.

The main aim of this paper is to study the stabilizing effect of the background spacetime to the relativistic fluids. Clearly speaking, we want to know whether the expanding of the spacetime can prevent the formation of the singularities of the relativistic fluids and how fast of the expanding rate is needed to achieve such global stability.

In this paper, we mainly focus on the stabilizing effect of the power law inflation ($a(t) = t^l$, $l > 1$) to the relativistic Eulerian fluids including isothermal gases ($p = c_s^2 \rho$), generalized Chaplygin gases ($p = -\frac{A}{\rho^\alpha}$, $0 < \alpha \leq 1$) and polytropic gases ($p = A\rho^{\frac{N+1}{N}}$), where N denotes the polytropic index.

In the following two subsections, we give the global existence of the classical solution to isothermal gases and generalized Chaplygin gases by the method of conformal transformation, we also investigate blowup phenomenon for isothermal gases and polytropic gases under suitable assumptions on the initial data and the background solution.

1.1. Global classical solution

At first, we introduce the conformal transformation. For the conformal metric \tilde{g} , we define

$$\tilde{g}_{\mu\nu} = e^{-2\phi} g_{\mu\nu} \quad \text{or} \quad \tilde{g}^{\mu\nu} = e^{2\phi} g^{\mu\nu}, \quad (1.3)$$

where the conformal factor $\phi = -\ln \tau$, in which τ denotes the new time coordinate.

In order to keep the normalization of the conformal velocity field $\tilde{u} = (\tilde{u}^0, \tilde{u}^1, \tilde{u}^2, \tilde{u}^3)$, we define

$$\tilde{u}^\mu = e^\phi u^\mu \quad \text{or} \quad \tilde{u}_\mu = e^{-\phi} u_\mu.$$

Then

$$\tilde{g}_{\mu\nu} \tilde{u}^\mu \tilde{u}^\nu = g_{\mu\nu} u^\mu u^\nu = -1. \quad (1.4)$$

Under above transformation, the initial value problem for the relativistic fluid equation becomes

$$\begin{cases} \tilde{\nabla}_\mu T^{\mu\nu} = -6T^{\mu\nu} \tilde{\nabla}_\mu \phi + \tilde{g}_{\kappa\lambda} T^{\kappa\lambda} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \phi, \\ \tau = 1: \quad \rho(\tau, x) = \rho_1(x), \quad \tilde{u}(\tau, x) = \tilde{u}_1(x), \end{cases} \quad (1.5)$$

where $\tilde{\nabla}$ denotes the covariant derivative with respect to the conformal metric \tilde{g} .

Remark 1.1. The unknown variables for isentropic Euler equations are $(\rho(\tau, x), \tilde{u}(\tau, x))$, and there exists a class of time-dependent solutions $(\bar{\rho}(\tau), \tilde{\tilde{u}}(\tau))$ to (1.5) instead of the constant state solution in flat spacetime. We also remark here that in the following \tilde{u} always means $(\tilde{u}^1, \tilde{u}^2, \tilde{u}^3)$ since \tilde{u}^0 can be obtained by the normalized constraint (1.4).

The main results on the global existence of the relativistic fluids in the background spacetime with power law inflation $(a(t) = t^l, l > 1)$ can be described as follows

Theorem 1.2. Suppose $k \geq 3$, $\rho_1(x) \in H^k(\mathbb{T}^3)$, $\tilde{u}_1(x) \in H^k(\mathbb{T}^3)$, then there exists a small parameter ϵ such that if the initial data satisfies

$$\|\rho_1(x) - \bar{\rho}(1)\|_{H^k} + \|\tilde{u}_1(x) - \tilde{\tilde{u}}(1)\|_{H^k} < \epsilon.$$

Then system (1.5) admits a unique classical solution $(\rho(\tau, x), \tilde{u}(\tau, x)) \in C^1((0, 1] \times \mathbb{T}^3)$ to isothermal fluids ($p = c_s^2 \rho$, $c_s^2 \in (0, \frac{l}{3l+1})$) and generalized Chaplygin gases ($p = -\frac{A}{\rho^\alpha}$, $\alpha \in (0, \frac{l}{3l+1})$) and the solution has the following regularity

$$\rho(\tau, x) \in C^0((0, 1], H^k), \quad \tilde{u}(\tau, x) \in C^0((0, 1], H^k).$$

The solution also satisfies

$$\|\rho(\tau, x) - \bar{\rho}(\tau)\|_{H^k} + \|\tilde{u}(\tau, x)\|_{H^k} \leq C\epsilon$$

for some positive constant C . Moreover, for all $\tau \in [0, 1]$, the solution satisfies the following asymptotic behavior

$$\|\tilde{u}(\bar{\tau}, x)\|_{H^{k-1}} \lesssim C\epsilon \begin{cases} \bar{\tau} & \kappa - C\epsilon > 1 \\ -\bar{\tau} \ln(\bar{\tau}) & \kappa - C\epsilon = 1 \\ \bar{\tau}^{\kappa-C\epsilon} & \kappa - C\epsilon < 1 \end{cases},$$

and

$$\|\psi(\bar{\tau}, x) - \psi(0)\|_{H^{k-1}} \lesssim C\epsilon \begin{cases} \bar{\tau} & \kappa - C\epsilon \geq 1 \\ \bar{\tau} + \bar{\tau}^{2(\kappa-C\epsilon)} & \kappa - C\epsilon < 1 \end{cases},$$

where $\bar{\tau} = \frac{l\tau^{l-1/l}}{l-1}$, $\psi(\bar{\tau}, x) = \zeta(\bar{\tau}, x) - \bar{\zeta}(\bar{\tau})$, $\zeta = \int_1^\rho \frac{d\xi}{\xi+p(\xi)}$, $\kappa = \frac{l-c_s^2(3l+1)}{l-1}$ for isothermal gases and $\kappa = \frac{l-\alpha(3l+1)}{l-1}$ for generalized Chaplygin gases, and above C is free to vary from line to line.

Remark 1.3. Due to the degeneracy of the polytropic gases ($p = A\rho^{\frac{N+1}{N}}$), we could not get the global existence results by the conformal method. In [1], Brauer, Rendall and Reula get the global stability of polytropic gases in the spacetime with exponentially expanding rate. However, the model they investigate is largely simplified, since they consider the Newtonian cosmological models with $T^{\mu\nu} = \rho u^\mu u^\nu + pg^{\mu\nu}$ and $u^0 = 1$. We leave the polytropic fluids to our future's research.

Remark 1.4. In above theorem, we can choose $C\epsilon = \frac{\kappa}{2}$, since the asymptotic behavior comes from the toy model of the energy inequality $x'(t) \leq a(t)x(t) + h(t)$, where $a(t) = \frac{\kappa - C\epsilon}{t}$, so we can choose $C\epsilon = \frac{\kappa}{2}$ when ϵ is suitably small. We can also see the relationship between the speed of the fluids and the expanding rate l to ensure the global stability of the corresponding fluids.

1.2. Formation of singularity

When the initial data satisfies some suitable assumptions, we can obtain the blowup phenomenon for isothermal gases $p = c_s^2 \rho$, $c_s^2 \in [0, \frac{1}{3})$ and polytropic gases ($p = A\rho^{\frac{N+1}{N}}$).

In this case, we consider the original system (1.1), and define the following averaged quantities, which are inspired by [7].

$$E(t) = \int_M (T^{00} - \bar{T}^{00}) a^3(t) dx,$$

and

$$Q(t) = \int_M g_{ij} x^i T^{0j} a^3(t) dx,$$

where $\bar{T}^{0\mu}$ is obtained by inserting background solution $(\bar{\rho}(t), 1, 0, 0, 0)$ and M is the supported domain of the solution.

We also need the following assumptions on the fluids, the initial data and the background solution

H1: $p = p(\rho)$ is non-negative.

H2: $\eta = \sqrt{\frac{dp}{d\rho}} \leq 1$.

H3: η is a non-decreasing function of p .

And

I1: $\bar{\eta}^2 < \frac{1}{3}$.

I2: $E(1) > 0$.

I3:

$$Q(1) > \left(\int_1^\infty \frac{1 - 3\bar{\eta}^2(\tau)}{a^2(\tau)R^2(\tau)[E(1)\exp^{-\int_1^\tau \frac{3\bar{a}(\tau')}{a(\tau')} d\tau'} + \frac{4\pi}{3}\bar{\rho}(\tau)a^3(\tau)R^3(\tau)]} d\tau \right)^{-1}.$$

Where $R(\tau) = 1 + \max_{\tau \in [1, +\infty)} \bar{\eta}(\tau)\tau$. Then we have the following theorem

Theorem 1.5. *The initial value problem to (1.1) cannot have a global classical solution under the assumptions **H1–H3** on the fluids and **I1–I3** on the initial data.*

Remark 1.6. The generalized Chaplygin gases do not satisfy the positivity assumption **H1**. Assumption **H2** is physical, since the speed of the sound is always less than the speed of light. **H2** is also not contradictory to **I1** since we do not consider the small perturbation of the background solution in this case. We also emphasize that the quantity $E(t)$ defined above is not conserved for relativistic fluids in curved spacetime, which is different from [7].

We arrange this paper as follows: In Section 2, we give some preliminaries on the geometry of the metric, the symmetrization of system (1.5) and study the background solution. Section 3 is focused on the properties of the symmetric hyperbolic system in the background spacetime with power law inflation derived in Section 2. In Section 4, we give a global existence result for a class of symmetric hyperbolic system with singular terms and prove the main Theorem 1.2. At last, we investigate the blowup phenomenon in Section 5.

2. Preliminaries

In this section, we mainly give the notations, the principle of the conformal transformation, the symmetrization of (1.5) and study the exact time-dependent solution.

2.1. Notations

As noted before, Greek indices range from 0 to 3, while Latin indices range from 1 to 3. Repeated lower and upper indices means summation with their corresponding metric. For the metrics in this paper, we use \tilde{g} and g to denote the conformal metric and original metric respectively. $\tilde{\Gamma}$ and Γ denote the Christoffel symbols with respect to \tilde{g} and g , respectively. We also use $\tilde{\mathbf{u}}$ to denote the background solution of the unknown \mathbf{u} .

For convenience, we use $A \sim B$ to denote the equivalent relationship between A and B , which means that there exists a positive constant $C > 1$, such that $\frac{A}{C} \leq B \leq CA$. $\tilde{\partial}_\mu = \tilde{\partial}_{x^\mu}$ and $\partial_\mu = \partial_{x^\mu}$ are the partial derivatives of the conformal spacetime and the original spacetime. Similar definitions are used for the covariant derivatives $\tilde{\nabla}$ and ∇ .

The positive constant C used in this paper is free to vary from line to line.

For a function $u(t, x)$, we define the following standard Sobolev norms on torus \mathbb{T}^3

$$\|u(t, x)\|_{L^2(\mathbb{T}^3)} := \left(\int_{\mathbb{T}^3} |u(t, x)|^2 dx \right)^{\frac{1}{2}},$$

$$\|u(t, x)\|_{H^k(\mathbb{T}^3)} := \sum_{l=0}^k \|D^l u(t, x)\|_{L^2(\mathbb{T}^3)},$$

and

$$\|u(t, x)\|_{L^\infty(\mathbb{T}^3)} := \operatorname{ess\,sup}_{x \in \mathbb{T}^3} |u(t, x)|.$$

2.2. Conformal metric and Christoffel symbol

We at first pay our attention to the principle of the choice of conformal factor. Define

$$\tau = \frac{1}{a(t)}.$$

By the monotonic increasing property of $a(t)$, we see that

$$\frac{1}{a(\infty)} \leq \tau \leq \frac{1}{a(1)}.$$

Remark 2.1. Especially, when $a(\infty) = \infty$, then $\tau = 0$, this is the main obstacle for the analysis of the hyperbolic system in the following due to the appearance of $\frac{1}{\tau}$.

Under the new coordinates (τ, x^1, x^2, x^3) , we have

$$g = -d\tau^2 + a^2(t) \sum_{i=1}^3 (dx^i)^2 = \frac{1}{\tau^2} \left(-\frac{1}{\omega^2(t)} d\tau^2 + \sum_{i=1}^3 (dx^i)^2 \right), \quad (2.1)$$

where

$$\omega(t) = \frac{\dot{a}(t)}{a(t)} = \frac{1}{a(t)} \frac{da(t)}{dt}.$$

By conformal transformation (1.3), we can get the conformal metric

$$\tilde{g} = -\frac{1}{\omega^2(t)} d\tau^2 + \sum_{i=1}^3 (dx^i)^2. \quad (2.2)$$

The Christoffel symbol $\tilde{\Gamma}_{\mu\nu}^\gamma$ is defined by

$$\tilde{\Gamma}_{\mu\nu}^\gamma = \frac{1}{2} \tilde{g}^{\gamma\alpha} (\partial_\mu \tilde{g}_{\alpha\nu} + \partial_\nu \tilde{g}_{\mu\alpha} - \partial_\alpha \tilde{g}_{\mu\nu}),$$

so direct calculations give the non-zero coefficient

$$\tilde{\Gamma}_{00}^0 = -\frac{\partial_\tau \omega}{\omega}. \quad (2.3)$$

Remark 2.2. The principle of choosing the conformal factor $\phi = -\ln \tau$ is inspired by (2.1) and (2.2).

2.3. Symmetrization process

In order to do energy estimate on system (1.5), we at first turn (1.5) into a symmetric hyperbolic system. The idea mainly comes from Oliynyk [14]. For self-contained of the paper, we give the main process of the derivation.

By the normalized property of the velocity field (1.4), we have

$$\tilde{u}_\mu \tilde{\nabla}_\nu \tilde{u}^\mu = \tilde{u}_0 \tilde{\nabla}_\nu \tilde{u}^0 + \tilde{u}_i \tilde{\nabla}_\nu \tilde{u}^i = 0,$$

then we have

$$\tilde{\nabla}_\nu \tilde{u}^0 = -\frac{\tilde{u}_i}{\tilde{u}_0} \tilde{\nabla}_\nu \tilde{u}^i. \quad (2.4)$$

Contracting (1.5) with \tilde{u}_ν gives

$$\tilde{u}_\nu \tilde{\nabla}_\mu T^{\mu\nu} = \tilde{u}_\nu (-6T^{\mu\nu} \tilde{\nabla}_\mu \phi + \tilde{g}_{\kappa\lambda} T^{\kappa\lambda} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \phi). \quad (2.5)$$

Expanding (2.5) directly and using $\tilde{u}_\nu \tilde{u}^\nu = -1$, we have

$$\tilde{u}^\nu \partial_\nu \rho + (\rho + p) \tilde{\nabla}_\nu \tilde{u}^\nu = -3(\rho + p) \tilde{u}^\nu \partial_\nu \phi. \quad (2.6)$$

Define the operator L_i^μ orthogonal to \tilde{u}_μ as

$$L_i^\mu = \delta_i^\mu - \frac{\tilde{u}_i}{\tilde{u}_0} \delta_0^\mu,$$

obviously

$$L_i^\mu \tilde{u}_\mu = \tilde{u}_i - \frac{\tilde{u}_i}{\tilde{u}_0} \tilde{u}_0 = 0.$$

We also define

$$L_{i\nu} = \tilde{g}_{\mu\nu} L_i^\mu.$$

Then applying $L_{i\nu}$ to (1.5), we have

$$L_{i\nu} \tilde{\nabla}_\mu T^{\mu\nu} = L_{i\nu} (-6T^{\mu\nu} \tilde{\nabla}_\mu \phi + \tilde{g}_{\kappa\lambda} T^{\kappa\lambda} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \phi). \quad (2.7)$$

(2.7) is equivalent to

$$(\rho + p) L_{i\nu} \tilde{u}^\mu \tilde{\nabla}_\mu \tilde{u}^\nu + L_i^\mu \partial_\mu p = -(\rho + p) L_i^\nu \tilde{\nabla}_\nu \phi. \quad (2.8)$$

Direct calculation shows that

$$L_i^\mu \tilde{\nabla}_\mu \tilde{u}^i = \tilde{\nabla}_i \tilde{u}^i - \frac{\tilde{u}_i \tilde{\nabla}_0 \tilde{u}^i}{\tilde{u}_0} = \tilde{\nabla}_\mu \tilde{u}^\mu,$$

then

$$\begin{aligned}
 L_{i\nu} \tilde{u}^\mu \tilde{\nabla}_\mu \tilde{u}^\nu &= L_{i0} \tilde{u}^\mu \tilde{\nabla}_\mu \tilde{u}^0 + \tilde{u}^\mu L_{ij} \tilde{\nabla}_\mu \tilde{u}^j \\
 &= (L_{ij} - L_{i0} \frac{\tilde{u}_j}{\tilde{u}_0}) \tilde{u}^\mu \tilde{\nabla}_\mu \tilde{u}^j \\
 &= (\tilde{g}_{j\nu} L_i^\nu - \tilde{g}_{0\nu} \frac{\tilde{u}_j}{\tilde{u}_0} L_i^\nu) \tilde{u}^\mu \tilde{\nabla}_\mu \tilde{u}^j \\
 &:= M_{ij} \tilde{u}^\nu \tilde{\nabla}_\nu \tilde{u}^j,
 \end{aligned} \tag{2.9}$$

where

$$M_{ij} = \tilde{g}_{j\nu} L_i^\nu - \tilde{g}_{0\nu} \frac{\tilde{u}_j}{\tilde{u}_0} L_i^\nu = \tilde{g}_{ij} - \frac{\tilde{u}_i}{\tilde{u}_0} \tilde{g}_{j0} - \frac{\tilde{u}_j}{\tilde{u}_0} \tilde{g}_{i0} + \tilde{g}_{00} \frac{\tilde{u}_i \tilde{u}_j}{\tilde{u}_0^2} = \delta_{ij} - \frac{\tilde{u}_i \tilde{u}_j}{\omega^2 \tilde{u}_0^2}.$$

Thus, (2.8) only contains the equations for \tilde{u}^i if we can express \tilde{u}^0 by \tilde{u}^i . This can be obtained directly since

$$\tilde{g}^{00} \tilde{u}_0^2 + 2\tilde{g}^{0i} \tilde{u}_0 \tilde{u}_i + \tilde{g}^{ij} \tilde{u}_i \tilde{u}_j = -1. \tag{2.10}$$

Solving (2.10), we have

$$\tilde{u}_0 = \frac{-\tilde{g}^{0i} \tilde{u}_i + \sqrt{(\tilde{g}^{0i} \tilde{u}_i)^2 - \tilde{g}^{00}(\tilde{g}^{ij} \tilde{u}_i \tilde{u}_j + 1)}}{\tilde{g}^{00}} = -\frac{\sqrt{1 + \sum_{i=1}^3 (\tilde{u}_i)^2}}{\omega}.$$

We also have

$$\tilde{u}^0 = \tilde{g}^{00} \tilde{u}_0 = \omega \sqrt{1 + \sum_{i=1}^3 (\tilde{u}_i)^2} = \omega \sqrt{1 + \sum_{i=1}^3 (\tilde{u}^i)^2}.$$

Introducing the new density

$$\zeta = \zeta(\rho) = \int_1^\rho \frac{d\xi}{\xi + p(\xi)}.$$

Then we get the following symmetric hyperbolic system

Lemma 2.3. *The system (1.5) can be equivalently rewritten as the following symmetric hyperbolic system*

$$A^\mu(U) \partial_\mu U = F(U), \tag{2.11}$$

where $U = (\zeta, \tilde{u}_i)^T$ and

$$A^\mu(U) = \begin{pmatrix} p'(\rho)\tilde{u}^\mu & p'(\rho)L_j^\mu \\ p'(\rho)L_i^\mu & M_{ij}\tilde{u}^\mu \end{pmatrix} \quad (2.12)$$

is symmetric and

$$F(U) = \begin{pmatrix} p'(\rho)(-3\tilde{u}^v\partial_v\phi - \tilde{\Gamma}_{v\gamma}^v\tilde{u}^\gamma) \\ -L_i^v\partial_v\phi - M_{ij}\tilde{\Gamma}_{\mu\gamma}^j\tilde{u}^\mu\tilde{u}^\gamma \end{pmatrix}, \quad (2.13)$$

where $\tilde{\Gamma}_{\mu\nu}^\gamma$ denotes the Christoffel symbol with respect to \tilde{g} .

Proof. The proof is by direct calculation. Dividing (2.6) by $\rho + p$ and using (2.4) and (2.3), we have

$$\tilde{u}^0\partial_\tau\zeta - \frac{\tilde{u}_i}{\tilde{u}_0}\partial_\tau\tilde{u}^i + \tilde{u}^i\partial_i\zeta + \partial_i\tilde{u}^i = -3\tilde{u}^v\partial_v\phi - \tilde{\Gamma}_{v\gamma}^v\tilde{u}^\gamma = \left(\frac{3}{\tau} + \frac{\partial_\tau\omega}{\omega}\right)\tilde{u}^0. \quad (2.14)$$

Dividing (2.8) by $\rho + p$ and inserting (2.9) into (2.8), we have

$$L_i^\mu p'(\rho)\partial_\mu\zeta + M_{ij}\tilde{u}^\mu\partial_\mu\tilde{u}^j = -L_i^\mu\partial_\mu\phi - M_{ij}\tilde{\Gamma}_{\mu\gamma}^j\tilde{u}^\mu\tilde{u}^\gamma = -\frac{\tilde{u}_i}{\tilde{u}_0\tau}.$$

Multiplying (2.14) by $p'(\rho)$, we can get (2.11). \square

Next remark shows the explicit expression of the coefficients of (2.12) and (2.13) in terms of (ζ, \tilde{u}^i) .

Remark 2.4. When the metric \tilde{g} is given by (2.2), then we have the following hyperbolic system

$$A^0 = \begin{pmatrix} p'\omega\sqrt{1+|\tilde{u}|^2} & -\frac{\omega p'\tilde{u}^j}{\sqrt{1+|\tilde{u}|^2}} \\ -\frac{\omega p'\tilde{u}^i}{\sqrt{1+|\tilde{u}|^2}} & \left(\delta_{ij} - \frac{\tilde{u}^i\tilde{u}^j}{1+|\tilde{u}|^2}\right)\omega\sqrt{1+|\tilde{u}|^2} \end{pmatrix}, \quad (2.15)$$

$$A^i = \begin{pmatrix} p'\tilde{u}^i & p'\delta_j^i \\ p'\delta_k^i & \left(\delta_{jk} - \frac{\tilde{u}^j\tilde{u}^k}{1+|\tilde{u}|^2}\right)\tilde{u}^i \end{pmatrix}, \quad (2.16)$$

and

$$F = \begin{pmatrix} p'\left(\frac{3}{\tau} + \frac{\partial_\tau\omega}{\omega}\right)\omega\sqrt{1+|\tilde{u}|^2} \\ \frac{\omega\tilde{u}^i}{\sqrt{1+|\tilde{u}|^2}\tau} \end{pmatrix},$$

where $|\tilde{u}|^2 = \sum_{i=1}^3 (\tilde{u}^i)^2$.

2.4. Background solution

We assume that the background solution of system (2.11) is $(\bar{\zeta}(\tau), 0, 0, 0)^T$, then we can solve (2.11) to obtain

$$\partial_{\tau}\bar{\zeta} = \frac{3}{\tau} + \frac{\partial_{\tau}\omega}{\omega}.$$

Thus, we have

$$\bar{\zeta}(\tau) = \bar{\zeta}(1) + \int_1^{\tau} \left(\frac{3}{\tilde{\tau}} + \frac{\partial_{\tilde{\tau}}\omega}{\omega} \right) d\tilde{\tau}.$$

2.5. Reformulation of the problem

Define

$$\psi(\tau, x) = \zeta(\tau, x) - \bar{\zeta}(\tau),$$

$$v^0(\tau, x) = \tilde{u}^0(\tau, x) - \omega,$$

and

$$v^i(\tau, x) = \tilde{u}^i(\tau, x) - 0.$$

Then in terms of (ψ, \tilde{u}^j) , we have

$$\tilde{u}^0 \partial_{\tau} \psi + \tilde{u}^i \partial_i \psi - \frac{\tilde{u}_i}{\tilde{u}_0} \partial_{\tau} \tilde{u}^i + \partial_i \tilde{u}^i = 0,$$

and

$$L_i^{\mu} p' \partial_{\tau} \psi + L_i^j p' \partial_j \psi + M_{ij} \tilde{u}^{\mu} \partial_{\mu} \tilde{u}^j = -\frac{v_i}{\tilde{u}_0 \tau} + \frac{v_i}{\tilde{u}_0} p' \left(\frac{3}{\tau} + \frac{\partial_{\tau}\omega}{\omega} \right).$$

Then in terms of (ψ, v^i) , we have

$$A^0 \partial_{\tau} \begin{pmatrix} \psi \\ v^i \end{pmatrix} + A^j \partial_j \begin{pmatrix} \psi \\ v^i \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{v_i}{\tilde{u}_0 \tau} + \frac{v_i}{\tilde{u}_0} p' \left(\frac{3}{\tau} + \frac{\partial_{\tau}\omega}{\omega} \right) \end{pmatrix}. \quad (2.17)$$

In which, A^0 and A^i are defined by (2.15) and (2.16) respectively.

3. Power law inflation

In this section, we aim at studying the power law inflation and investigating the properties of the fluids (isothermal gases and generalized gases).

3.1. Power law inflation

We at first study the power law inflation.

Assume

$$a(t) = t^l, \quad (3.1)$$

where $l > 1$ is a constant.

Remark 3.1. The motivation for us to study power law inflation is that we want to know the effect of the expanding rate to the stability of the relativistic Euler fluids. In [18], H. Ringström constructed the power law inflation mechanism to explain the accelerated expansion of the universe when the Einstein equations are coupled to a scalar field with an exponential potential.

Under the assumption (3.1), we can get easily

$$\tau = t^{-l}, \quad \omega = \frac{\dot{a}(t)}{a(t)} = \frac{l}{t} = l\tau^{1/l}$$

and

$$\frac{\partial_\tau \omega}{\omega} = \frac{1}{l\tau}, \quad \zeta(\tau) = \zeta(1) + \left(3 + \frac{1}{l}\right) \ln \tau.$$

Define

$$\bar{\tau} = \frac{\tau^{1-\frac{1}{l}}}{1-1/l}, \quad (3.2)$$

then

$$\partial_{\bar{\tau}} = \partial_\tau \frac{d\tau}{d\bar{\tau}} = \tau^{1/l} \partial_\tau.$$

To avoid the degeneracy of A^0 in (2.17) when $\tau \rightarrow 0$, we can absorb the degenerate term ω by coordinate transformation (3.2). Then (2.17) can be equivalently rewritten as

$$\hat{A}^0 \partial_{\bar{\tau}} \begin{pmatrix} \psi \\ v^i \end{pmatrix} + \hat{A}^j \partial_j \begin{pmatrix} \psi \\ v^i \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{lv^i}{\sqrt{1+|v|^2}(1-1/l)} \left(\frac{1}{\bar{\tau}} - p' \left(\frac{3}{\bar{\tau}} + \frac{1}{l\bar{\tau}} \right) \right) \end{pmatrix}, \quad (3.3)$$

where

$$\hat{A}^0 = \begin{pmatrix} p'l\sqrt{1+|v|^2} & -\frac{lp'v^j}{\sqrt{1+|v|^2}} \\ -\frac{lp'v^i}{\sqrt{1+|v|^2}} & (\delta_{ij} - \frac{v^i v^j}{1+|v|^2})l\sqrt{1+|v|^2} \end{pmatrix},$$

and

$$\hat{A}^i = A^i.$$

In the following we focus on p' of the isothermal gases and generalized Chaplygin gases

3.2. Isothermal gases

The equation of state for the isothermal gases is

$$p = c_s^2 \rho,$$

where c_s denotes the speed of sound. Thus, we have

$$\hat{A}^0 = \begin{pmatrix} c_s^2 l \sqrt{1 + |v|^2} & -\frac{lc_s^2 v^j}{\sqrt{1 + |v|^2}} \\ -\frac{lc_s^2 v^i}{\sqrt{1 + |v|^2}} & (\delta_{ij} - \frac{v^i v^j}{1 + |v|^2}) l \sqrt{1 + |v|^2} \end{pmatrix}, \quad (3.4)$$

and

$$H = \begin{pmatrix} 0 \\ \frac{lv^i}{\sqrt{1 + |v|^2}(1 - 1/l)} \left(\frac{1}{\tau} - c_s^2 \left(\frac{3}{\tau} + \frac{1}{l\tau} \right) \right) \end{pmatrix}.$$

3.3. Generalized Chaplygin gases

The equation of state for generalized Chaplygin gases is given by

$$p = -\frac{A}{\rho^\alpha},$$

where $A \neq 1$ is a positive constant and $0 < \alpha \leq 1$. Then we have

$$\zeta = \int_1^\rho \frac{d\xi}{\xi + p(\xi)} = \int_1^\rho \frac{\xi^\alpha d\xi}{\xi^{1+\alpha} - A} = \frac{1}{1 + \alpha} \ln \left(\frac{\rho^{1+\alpha} - A}{1 - A} \right).$$

By simple calculations, we have

$$\rho^{1+\alpha} = (1 - A)e^{(1+\alpha)\zeta} + A.$$

And

$$p' = \frac{A\alpha}{\rho^{1+\alpha}} = \frac{A\alpha}{(1 - A)e^{(1+\alpha)\zeta} + A}. \quad (3.5)$$

Lemma 3.2. $p'(\bar{\rho}(\tau))$ is uniformly bounded for $\tau \in [0, 1]$ when the initial state satisfies $(1 - A)e^{(1+\alpha)\bar{\zeta}(1)} + A > 0$.

Proof. We see

$$p'(\bar{\rho}(\tau)) = \frac{A\alpha}{(1-A)e^{(1+\alpha)\bar{\zeta}(\tau)} + A},$$

since

$$-\infty \leq \bar{\zeta}(\tau) = \bar{\zeta}(1) + \left(3 + \frac{1}{l}\right) \ln \tau \leq \zeta(1),$$

we have

$$\frac{A\alpha}{(1-A)e^{(1+\alpha)\bar{\zeta}(1)} + A} \leq p'(\bar{\rho}(\tau)) \leq \alpha. \quad \square$$

Remark 3.3. If $A < 1$, $(1-A)e^{(1+\alpha)\bar{\zeta}(1)} + A > 0$ holds obviously, if $A > 1$, then the initial states for $\bar{\zeta}$ should satisfies $\bar{\zeta}(1) > \frac{1}{1+\alpha} \ln(\frac{A}{A-1})$ to ensure the positivity of $p'(\rho)$.

Remark 3.4. $p'(\rho)$ is uniformly bounded provided ψ is sufficiently small.

Proof. By Taylor expansion we have

$$\begin{aligned} p'(\rho) &\sim p'(\bar{\rho}(\tau)) + A\alpha(1-A)e^{(1+\alpha)\bar{\zeta}(\tau)} \sum_{n=1}^{\infty} \left(\frac{((1+\alpha)\psi)^n}{n!} \right) \\ &\sim p'(\bar{\rho}(\tau)) + A\alpha(1-A)e^{(1+\alpha)\bar{\zeta}(1)} \tau^{(1+\alpha)(3+\frac{1}{l})} \sum_{n=1}^{\infty} \left(\frac{((1+\alpha)\psi)^n}{n!} \right) \\ &\sim p'(\bar{\rho}(\tau)) + A\alpha(1-A)e^{(1+\alpha)\bar{\zeta}(1)} ((1-1/l)\bar{\tau})^{(1-1/l)^{-1}(1+\alpha)(3+\frac{1}{l})} \sum_{n=1}^{\infty} \left(\frac{((1+\alpha)\psi)^n}{n!} \right) \\ &\sim p'(\bar{\rho}(\tau)) + A\alpha(1-A)e^{(1+\alpha)\bar{\zeta}(1)} ((1-1/l)\bar{\tau})^{\frac{(3l+1)(1+\alpha)}{l-1}} \sum_{n=1}^{\infty} \left(\frac{((1+\alpha)\psi)^n}{n!} \right). \end{aligned} \quad (3.6)$$

(3.6) shows that p' is uniformly bounded provided ψ is sufficiently small. \square

Remark 3.5. The exponential $\frac{(3l+1)(1+\alpha)}{l-1} \geq 1$ holds for all $\alpha \in (0, 1]$ and $l > 1$ and is very important for analyzing the singular terms in the following. Which means that the matrix \hat{A}^0 is C^1 with respect to $\bar{\tau}$.

Thus, for Chaplygin gases, we have the following symmetric hyperbolic system

$$\hat{A}^0 \partial_{\bar{\tau}} \begin{pmatrix} \psi \\ v^i \end{pmatrix} + \hat{A}^j \partial_j \begin{pmatrix} \psi \\ v^i \end{pmatrix} = \hat{H},$$

where

$$\hat{A}^0 = \begin{pmatrix} p'(\rho)l\sqrt{1+|v|^2} & -\frac{lp'(\rho)v^j}{\sqrt{1+|v|^2}} \\ -\frac{lp'(\rho)v^i}{\sqrt{1+|v|^2}} & (\delta_{ij} - \frac{v^i v^j}{1+|v|^2})l\sqrt{1+|v|^2} \end{pmatrix},$$

$$\hat{A}^j = A^j,$$

$$\hat{H} = \begin{pmatrix} 0 \\ \frac{lv^i}{\sqrt{1+|v|^2}(1-1/l)} (\frac{1}{\tau} - p'(\rho)(\frac{3}{\tau} + \frac{1}{l\tau})) \end{pmatrix}.$$

Here $p'(\rho)$ is defined by (3.5) and is uniformly bounded under assumption that ψ is small enough.

Remark 3.6. For polytropic gases ($p = A\rho^{\frac{N+1}{N}}$), we have

$$p'(\rho) = A(1 + \frac{1}{N}) \frac{e^{\psi/N} e^{\zeta(1)/N} \tau^{(3+1/l)1/N}}{A + 1 - A e^{\psi/N} e^{\zeta(1)/N} \tau^{(3+1/l)1/N}},$$

which is degenerate when τ tends to zero. That means the matrix \hat{A}^0 is degenerate, and this is the main obstacle for the global existence of the classical solution to polytropic gases.

4. Proof of Theorem 1.2

In this section, we at first introduce a class of symmetric hyperbolic system and a main proposition based on this system. Then we prove our global existence results for isothermal gases and generalized Chaplygin gases.

4.1. A class of symmetric hyperbolic system

Consider the following symmetric hyperbolic system.

$$B^\mu \partial_\mu u = \frac{1}{t} \mathbf{B} \mathbf{P} u + F \quad \text{in } [T_0, T_1] \times \mathbb{T}^n,$$

$$u = u_0 \quad \text{in } T_0 \times \mathbb{T}^n,$$

where

- (i) $T_0 < T_1 \leq 0$,
- (ii) \mathbf{P} is a constant, symmetric projection operator, i.e., $\mathbf{P}^2 = \mathbf{P}$, $\mathbf{P}^T = \mathbf{P}$,
- (iii) $u = u(t, x)$ and $F(t, u)$ are \mathbb{R}^N -valued maps, $F \in C^0([T_0, 0], C^\infty(\mathbb{R}^N))$ and satisfies $F(t, 0) = 0$,
- (iv) $B^\mu = B^\mu(t, u)$ and $\mathbf{B} = \mathbf{B}(t, u)$ are $\mathbb{M}_{N \times N}$ -valued maps, $B^\mu, \mathbf{B} \in C^0([T_0, 0], C^\infty(\mathbb{R}^N))$ and they satisfy

$$(B^\mu)^T = B^\mu, \quad [\mathbf{P}, \mathbf{B}] = \mathbf{P}\mathbf{B} - \mathbf{B}\mathbf{P} = 0,$$

(v) there exist constants κ , γ_1 , γ_2 such that

$$\gamma_1 I \leq B^0 \leq \frac{1}{\kappa} \mathbf{B} \leq \gamma_2 I,$$

for all $(t, u) \in [T_0, 0] \times \mathbb{R}^N$,

(vi) for all $(t, u) \in [T_0, 0] \times \mathbb{R}^N$, we have

$$\mathbf{P}^\perp B^0(t, x, \mathbf{P}^\perp u) \mathbf{P} = \mathbf{P} B^0(t, x, \mathbf{P}^\perp u) \mathbf{P}^\perp = 0,$$

where $\mathbf{P}^\perp = \mathbb{I} - \mathbf{P}$ is the orthogonal projection operator,

(vii) there exist constants θ , β_1 , β_2 , β_3 , β_4 and $\omega > 0$ such that

$$\begin{aligned} |\mathbf{P}^\perp [D_u B^0 \cdot (B^0)^{-1} \mathbf{B} \mathbf{P} u] \mathbf{P}^\perp| &\leq |t| \theta + \frac{2\beta_1}{\omega + |\mathbf{P}^\perp u|^2} |\mathbf{P} u|^2, \\ |\mathbf{P}^\perp [D_u B^0 \cdot (B^0)^{-1} \mathbf{B} \mathbf{P} u] \mathbf{P}| &\leq |t| \theta + \frac{2\beta_2}{\sqrt{\omega + |\mathbf{P}^\perp u|^2}} |\mathbf{P} u|, \\ |\mathbf{P}^\perp [D_u B^0 \cdot (B^0)^{-1} \mathbf{B} \mathbf{P} u] \mathbf{P}^\perp| &\leq |t| \theta + \frac{2\beta_3}{\sqrt{\omega + |\mathbf{P}^\perp u|^2}} |\mathbf{P} u|, \\ |\mathbf{P} [D_u B^0 \cdot (B^0)^{-1} \mathbf{B} \mathbf{P} u] \mathbf{P}| &\leq |t| \theta + \beta_4, \end{aligned}$$

for all $(t, x, u) \in [T_0, 0] \times \mathbb{T}^n \times \mathbb{R}^M$,

(viii)

$$u^T \cdot \partial_t B^0 \cdot u \ll u^T \cdot B^0 \cdot u.$$

Remark 4.1. The assumption (viii) does not contained in the paper [15] since $\partial_t B^0 = 0$. While this assumption is needed for the Chaplygin fluids. In assumption (v), we mean that there exists a vector field u such that $\gamma_1 u^T \cdot \mathbb{I} \cdot u \leq u^T \cdot B^0 \cdot u \leq \frac{1}{\kappa} u^T \cdot \mathbf{B} \cdot u \leq \gamma_2 u^T \cdot \mathbb{I} \cdot u$. Assumption (vi) means that the term $\mathbf{P}^\perp B^0 \mathbf{P}$ depends at least linearly on $\mathbf{P} u$.

We will be able to conclude our argument with the help of the following proposition without proof, which comes directly from [14].

Proposition 4.2. Suppose that $k \geq \frac{n}{2} + 1$, $u_0 \in H^k(\mathbb{T}^n)$ and assumptions (i)–(viii) are fulfilled. Then there exists a $T_* \in (T_0, 0)$, and a unique classical solution $u \in C^1([T_0, T_*] \times \mathbb{T}^n)$ that satisfies $u \in C^0([T_0, T_*], H^k) \cap C^1([T_0, T_*], H^{k-1})$ and the energy estimate

$$\|u(t)\|_{H^k}^2 - \int_{T_0}^t \frac{1}{\tau} \|\mathbf{P} u\|_{H^k}^2 \leq C e^{C(t-T_0)} (\|u(T_0)\|_{H^k}^2)$$

for all $T_0 \leq t < T_*$, where $C = C(\|u\|_{L^\infty([T_0, T_*], H^k)}, \gamma_1, \gamma_2, \kappa)$, and can be uniquely continued to a larger time interval $[T_0, T^*)$ for all $T^* \in (T_*, 0]$ provided $\|u\|_{L^\infty([T_0, T_*], W^{1, \infty})} < \infty$.

Moreover, there exists a $\delta > 0$ such that if $\|u_0\|_{H^k} \leq \delta$, then the solution exists on the time interval $[T_0, 0)$ and can be uniquely extended to $[T_0, 0]$ as an element of $C^0([T_0, 0], H^{k-1})$ satisfying

$$\|Pu(\tau)\|_{H^{k-1}} \leq C\delta \begin{cases} -t & \text{if } \kappa > 1, \\ t \ln(\frac{t}{T_0}) & \text{if } \kappa = 1, \\ (-t)^k & \text{if } \kappa < 1 \end{cases}$$

and

$$\|P^\perp u(\tau) - P^\perp u(0)\|_{H^{k-1}} \leq C\delta \begin{cases} -t & \text{if } \kappa \geq 1 \text{ or } [B^0, P] = 0 \\ -t + (-t)^{2\kappa} & \text{if } \kappa < 1 \end{cases}$$

for $T_0 \leq t \leq 0$.

4.2. Global stability for isothermal gases and generalized Chaplygin gases

In the following, we apply above proposition to isothermal gases and generalized Chaplygin gases. It suffices to prove that system (3.3) satisfies the assumptions in above.

Isothermal gases

It is easy to see that $F = 0$, \hat{A}^0 is given by (3.4) and

$$H = \begin{pmatrix} 0 \\ \frac{lv^i}{\sqrt{1+|v|^2}(1-1/l)} \left(\frac{1}{\bar{\tau}} - c_s^2 \left(\frac{3}{\bar{\tau}} + \frac{1}{l\bar{\tau}} \right) \right) \end{pmatrix} = \frac{1}{\bar{\tau}} \mathbf{B} \mathbf{P} \begin{pmatrix} \psi \\ v^j \end{pmatrix}.$$

Where

$$\mathbf{B} = \begin{pmatrix} \frac{c_s^2 l(l - c_s^2(3l+1))}{l-1} & 0 \\ 0 & \frac{l(1 - c_s^2(3+1/l))}{\sqrt{1+|v|^2}(1-1/l)} \delta_{ij} \end{pmatrix},$$

and

$$\mathbf{P} = \begin{pmatrix} 0 & 0 \\ 0 & \delta_j^i \end{pmatrix}.$$

It is easy to see that

$$\hat{A}^0|_{(\psi, v)=0} = \begin{pmatrix} c_s^2 l & 0 \\ 0 & \delta_{ij} l \end{pmatrix}, \quad \mathbf{B}|_{(\psi, v)=0} = \begin{pmatrix} \frac{c_s^2 l(l - c_s^2(3l+1))}{l-1} & 0 \\ 0 & \frac{l(1 - c_s^2(3+1/l))}{1-1/l} \delta_{ij} \end{pmatrix}.$$

The coefficients satisfy all the assumptions and there exists $\kappa = \frac{l - c_s^2(3l+1)}{l-1}$ such that

$$\hat{A}^0 \leq \frac{1}{\kappa} \mathbf{B}.$$

Thus, by Proposition 4.2, under the assumption $1 - c_s^2(3 + 1/l) \geq 0$, we have

$$\|u(\bar{\tau})\|_{H^k}^2 \leq C\epsilon. \quad (4.1)$$

It is also easy to check that β_4 depends on $\mathbf{P}u$ and $\beta_4 \in [0, C\epsilon]$ for some positive constant C , then we have

$$\|v(\bar{\tau})\|_{H^{k-1}} \lesssim C\epsilon \begin{cases} \bar{\tau} & \kappa - C\epsilon > 1 \\ -\bar{\tau} \ln(\bar{\tau}) & \kappa - C\epsilon = 1 \\ \bar{\tau}^{\kappa - C\epsilon} & \kappa - C\epsilon < 1 \end{cases}, \quad (4.2)$$

and

$$\|\psi(\bar{\tau}) - \psi(0)\|_{H^{k-1}} \lesssim C\epsilon \begin{cases} \bar{\tau} & \kappa - C\epsilon \geq 1 \\ \bar{\tau} + \bar{\tau}^{2(\kappa - C\epsilon)} & \kappa - C\epsilon < 1 \end{cases}. \quad (4.3)$$

Generalized Chaplygin gases

Similar to isothermal gases, we have

$$\mathbf{B} = \begin{pmatrix} \frac{p'(\rho)l(l-p'(\rho)(3l+1))}{l-1} & 0 \\ 0 & \frac{l(1-p'(\rho)(3+1/l))}{\sqrt{1+|v|^2}(1-1/l)} \delta_{ij} \end{pmatrix},$$

then setting $\kappa = \min_{\tau \in [0,1]} \frac{l-p'(\bar{\rho}(\tau))(3l+1)}{l-1}$, we can also get the same results (4.1), (4.2) and (4.3).

In order to ensure the non-negativity of κ , we have to give some constraints on the parameter α of the generalized Chaplygin gases. By Lemma 3.2, we have $\frac{l-\alpha(3l+1)}{l-1} \leq \kappa \leq \frac{l[(A-1)e^{(1+\alpha)\bar{\zeta}(1)}+A]-A\alpha(3l+1)}{(l-1)[(A-1)e^{(1+\alpha)\bar{\zeta}(1)}+A]}$. So we need $0 < \alpha < \frac{l}{3l+1}$ to ensure $\frac{l-\alpha(3l+1)}{l-1} > 0$. On the other hand, $\kappa = \frac{l-\alpha(3l+1)}{l-1}$ satisfies all the assumptions of subsection 4.1.

Remark 4.3. Different from the isothermal gases, the speed of the generalized Chaplygin gases $p'(\rho)$ is not a constant, however, it is uniformly bounded provided that $\|u\|_{L^\infty}$ is sufficiently small. Another difference is that \hat{A}^0 also depends on τ , so we add assumption (viii) in last subsection.

5. Proof of Theorem 1.5

In this section, we focus on the formation of singularities for the relativistic fluids under suitable assumptions **H1**, **H2**, **H3** and **I1**, **I2**, **I3** defined in Section 1. The main result can be seen as a direct generalization of Guo's work [7] in Minkowski spacetime to the FLRW spacetime. Here, we do not use the conformal equations (1.5) but the original system (1.1) under the general FLRW metric (1.2)

Direct calculations give us the non-zero components of the Christoffel symbols

$$\Gamma_{ij}^0 = \dot{a}(t)a(t)\delta_{ij}, \quad (5.1)$$

and

$$\Gamma_{0j}^i = \frac{\dot{a}(t)}{a(t)} \delta_{ij}. \quad (5.2)$$

Then expanding the whole system, we get the following equations in local coordinates

$$\begin{cases} \nabla_\mu T^{\mu 0} = \partial_t T^{00} + \partial_i T^{i0} + \Gamma_{\mu\alpha}^\mu T^{0\alpha} + \Gamma_{\mu\alpha}^0 T^{\mu\alpha} = 0 \\ \nabla_\mu T^{\mu i} = \partial_t T^{0i} + \partial_j T^{ji} + \Gamma_{\mu\alpha}^\mu T^{\alpha i} + \Gamma_{\mu\alpha}^i T^{\mu\alpha} = 0. \end{cases} \quad (5.3)$$

Inserting (5.1) and (5.2) into (5.3), we can obtain

$$\begin{cases} \partial_t T^{00} + \partial_i T^{0i} + 3 \frac{\dot{a}(t)}{a(t)} T^{00} + \frac{\dot{a}(t)}{a(t)} ((\rho + p)|u|^2 + 3p) = 0, \\ \partial_t T^{0i} + \partial_j T^{ji} + 5 \frac{\dot{a}(t)}{a(t)} T^{0i} = 0. \end{cases} \quad (5.4)$$

Before studying the blowup phenomena of relativistic fluids, we note the following lemma

Lemma 5.1. *There exists a positive constant $T > 1$ such that the solution of system (5.4) exists on $[1, T)$, and when the initial data sets are supported in finite torus $\mathbb{T}^3 = [-1, 1]^3$, then the solution is supported in $B_{R(t)} := \{x \in M | x^0 = t, |x| \leq R(t) = 1 + \bar{\eta}t\}$, where $\bar{\eta} = \max_{t \in [1, T]} \bar{\eta}(t)$ and $\bar{\eta}(t)$ denotes the speed of the sound of the background solution.*

Proof. The system can be equivalently rewritten as a hyperbolic system as before, see subsection 2.3, so by the standard energy estimates, we can obtain the lemma. For more detailed analysis, one can refer to [16]. \square

Remark 5.2. In fact, the speed of the propagation of the wave is $\frac{\sqrt{\bar{p}'(t)}}{a(t)}$, in above lemma, the speed we have chosen is larger than the real one, which does not affect the whole proof.

We assume that the system (5.4) admits a solution $(\bar{\rho}(t), 1, 0, 0, 0)$ depending only on the time variable. Then we have

$$\begin{cases} \partial_t \bar{T}^{00} + 3 \frac{\dot{a}(t)}{a(t)} \bar{T}^{00} + 3 \frac{\dot{a}(t)}{a(t)} \bar{\rho} = 0, \\ \partial_t \bar{T}^{0i} + 5 \frac{\dot{a}(t)}{a(t)} \bar{T}^{0i} = 0. \end{cases}$$

Define

$$E(t) = \int_M (T^{00} - \bar{T}^{00}) a^3(t) dx = \int_M [(\rho + p)|u|^2 + \rho - \bar{\rho}] a^3(t) dx$$

and

$$Q(t) = \int_M g_{ij} x^i T^{0j} a^3(t) dx = \int_M (\rho + p) \sqrt{1 + |u|^2} \langle x, u \rangle a^5(t) dx.$$

Where \langle, \rangle denotes the usual inner product in Euclidean space, $|u|^2 = \sum_{i=1}^3 a^2(t) (u^i)^2$.

In the following, we deduce the inequality satisfied by the two integral quantities E and F . Taking derivative with t , we have

$$\begin{aligned} \frac{d}{dt}E &= \int_M \left[\partial_t(T^{00} - \bar{T}^{00}) + 3\frac{\dot{a}(t)}{a(t)}(T^{00} - \bar{T}^{00}) \right] a^3(t) dx \\ &= \int_M \left[-\partial_i T^{0i} - \frac{\dot{a}(t)}{a(t)} \left((\rho + p)|u|^2 + 3(p - \bar{p}) \right) \right] a^3(t) dx \\ &= - \int_M \left[\frac{\dot{a}(t)}{a(t)} \left((\rho + p)|u|^2 + 3(p - \bar{p}) \right) \right] a^3(t) dx, \\ \frac{d}{dt}Q &= \int_M [g_{ij}x^i \partial_t T^{0j} + 5\frac{\dot{a}(t)}{a(t)} T^{ij} g_{ij}x^i] a^3(t) dx \\ &= \int_M g_{ij}(T^{ij} - \bar{T}^{ij}) a^3(t) dx \\ &= \int_M \left((\rho + p)|u|^2 + 3(p - \bar{p}) \right) a^3(t) dx. \end{aligned}$$

Then by assumptions **H2** and **H3**, we get

$$\rho - \bar{\rho} = \rho(p) - \rho(\bar{p}) = \int_{\bar{p}}^p \frac{\partial \rho}{\partial p'} dp' = \int_{\bar{p}}^p \frac{1}{\eta^2(p')} dp' \leq \frac{1}{\bar{\eta}^2(t)}(p - \bar{p}), \quad (5.5)$$

and

$$p - \bar{p} = \int_{\bar{p}}^p \eta^2(p') dp' \leq \eta^2(p)(\rho - \bar{\rho}) \leq (\rho - \bar{\rho}).$$

Combining assumption **I1** with **I2**, we can obtain

$$-3\frac{\dot{a}(t)}{a(t)}E \leq \frac{d}{dt}E \leq -\frac{\dot{a}(t)}{a(t)}3\bar{\eta}^2(t) \int_M [(\rho + p)|u|^2 + \rho - \bar{\rho}] a^3(t) dx = -\frac{\dot{a}(t)}{a(t)}3\bar{\eta}^2(t)E.$$

By the comparison theorem of ODE, we have

$$E(1) \exp^{-\int_1^t \frac{3\dot{a}(\tau)}{a(\tau)} d\tau} \leq E(t) \leq E(1) \exp^{-\int_1^t \frac{\dot{a}(\tau)}{a(\tau)} 3\bar{\eta}^2(\tau) d\tau}. \quad (5.6)$$

Thus, $E(t) > 0$, when $1 \leq t < \infty$.

Remark 5.3. The positivity of $E(t)$ is important and $E(t)$ is not conserved in curved background. However, it has the same symbol with the initial data $E(1)$ according to (5.6).

From (5.5), we have

$$\frac{d}{dt}Q(t) \geq 3\bar{\eta}^2(t)E(t) + (1 - 3\bar{\eta}^2(t)) \int_M (\rho + p)|u|^2 a^3(t) dx > 0.$$

Thus, $Q(t) > 0$ if $Q(1) > 0$.

On the other hand, by Hölder inequality, we have

$$Q(t) \leq R(t) \left(\int_M (\rho + p)|u|^2 a^3(t) dx \right)^{\frac{1}{2}} \left(\int_M [(\rho + p)|u|^2 + \rho - \bar{\rho} + \bar{\rho}] a^5(t) dx \right)^{\frac{1}{2}},$$

where we have used $p \geq 0$, then

$$Q^2(t) \leq \frac{R^2(t)a^2(t)}{1 - 3\bar{\eta}^2} Q'(t) \left[E + \frac{4\pi}{3} a^3(t) R^3(t) \bar{\rho}(t) \right].$$

Thus, we have

$$\frac{1}{Q(t)} \leq \frac{1}{Q(1)} - \int_1^t \frac{1 - 3\bar{\eta}^2(\tau)}{a^2(\tau) R^2(\tau) [E(1) \exp^{-\int_1^\tau \frac{3\bar{a}(\tau')}{a(\tau')} d\tau'} + \frac{4\pi}{3} \bar{\rho}(\tau) a^3(\tau) R^3(\tau)]} d\tau.$$

Which contradicts the positivity of $Q(t)$ if

$$Q(1) > \left(\int_1^\infty \frac{1 - 3\bar{\eta}^2(\tau)}{a^2(\tau) R^2(\tau) [E(1) \exp^{-\int_1^\tau \frac{3\bar{a}(\tau')}{a(\tau')} d\tau'} + \frac{4\pi}{3} \bar{\rho}(\tau) a^3(\tau) R^3(\tau)]} d\tau \right)^{-1} := I^{-1}. \quad (5.7)$$

5.1. Application to isothermal fluids

When $p = c_s^2 \rho$, we have

$$\bar{\eta} = c_s, \quad R(t) = 1 + c_s t$$

and

$$\bar{\rho}(t) = \rho(1) \exp^{3(1+c_s^2) \int_1^t \frac{\bar{a}(\tau)}{a(\tau)} d\tau} = \rho(1) t^{3l(1+c_s^2)}.$$

It is easy to see that the integral I of (5.7) is finite and away from 0 provided $c_s^2 < \frac{1}{3}$. Thus, there exists a class of initial data such that the solution must blowup in finite time.

Remark 5.4. When $c_s = 0$, the above blowup result still holds and the blowup phenomenon does not contradict to the global stability results since the assumption on the initial data (5.7) is not small.

5.2. Application to polytropic fluids

When $p = A\rho^\gamma$ with $\gamma = 1 + \frac{1}{N}$. We need to solve the following equation

$$\partial_t \bar{\rho} + 3 \frac{\dot{a}(t)}{a(t)} \bar{\rho} + 3 \frac{A\dot{a}(t)}{a(t)} \bar{\rho}^{1+\frac{1}{N}} = 0.$$

We have

$$\bar{\rho}(t) = \frac{e^{\zeta(1)t^{-1-3I}}}{(A+1 - Ae^{\zeta(1)/N} t^{\frac{-1-3I}{N}})^N}, \quad (5.8)$$

and

$$\bar{\eta}^2(t) = A(1 + \frac{1}{N})\rho^{\frac{1}{N}}(t) = A(1 + \frac{1}{N}) \frac{e^{\frac{\zeta(1)}{N} t^{\frac{-1-3I}{N}}}}{A+1 - Ae^{\zeta(1)/N} t^{\frac{-1-3I}{N}}}. \quad (5.9)$$

In above, $\zeta(1)$ is the initial state of the modified energy density. From (5.8) and (5.9), we see that the integral I of (5.7) is finite. Thus, the solution must blowup in finite time.

Remark 5.5. Brauer, Rendall and Reula [1] also proved the formation of singularity for this fluid in one-dimensional case by the method of characteristics. The philosophy of their work is based on the collision of two characteristics.

Remark 5.6. The initial data sets are not empty, since we have four variables but three constraints.

Acknowledgments

This work was partially supported by NSFC (Grant No. 11701517) and the Scientific Research Foundation of Zhejiang Sci-Tech University (Grant No. 16062021-Y).

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