

Existence of the Entropy Solution for a Viscoelastic Model

Changjiang Zhu

*Wuhan Institute of Physics and Mathematics, The Chinese Academy of Sciences,
Kowloon, Hong Kong, People's Republic of China and
Department of Mathematics, City University of Hong Kong,
Kowloon, Hong Kong, People's Republic of China*

Received May 2, 1996; revised December 19, 1997

In this paper, we consider the Cauchy problem for a viscoelastic model with relaxation

$$u_t + \sigma_x = 0, \quad (\sigma - f(u))_t + \frac{1}{\delta} (\sigma - \mu f(u)) = 0$$

with discontinuous, large initial data, where $0 < \mu < 1$, $\delta > 0$ are constants. When the system is nonstrictly hyperbolic, under the additional assumption $v_{0x} \in L^\infty$, the system is reduced to an inhomogeneous scalar balance law by employing the special form of the system itself. After introducing a definition of entropy solutions to the system, we prove the existence, uniqueness, and continuous dependence of the global entropy solution for the system. When the system is strictly hyperbolic, some special entropy pairs of the Lax type are constructed, in which the progression terms are functions of a single variable, and the necessary estimates for the major terms are obtained by using the theory of singular perturbation of the ordinary differential equations. The special entropy pairs are used to prove the existence of the global entropy solutions for the corresponding Cauchy problem by applying the method of compensated compactness. © 1998 Academic Press

Key Words: viscoelastic model; relaxation; singular perturbation; entropy; compensated compactness; Young measure.

1. INTRODUCTION

In this paper, we consider the existence and uniqueness of the global entropy solutions for a viscoelastic model with relaxation

$$\begin{cases} u_t + \sigma_x = 0, \\ (\sigma - f(u))_t + \frac{\sigma - \mu f(u)}{\delta} = 0, \end{cases} \quad (1.1)$$

with initial data

$$(u(x, 0), \sigma(x, 0)) = (u_0(x), \sigma_0(x)), \quad (1.2)$$

where $0 < \mu < 1$, $\delta > 0$ are constants, $f(u) \in C^2(\mathbf{R})$. Let

$$v(x, t) = \sigma(x, t) - f(u(x, t)); \quad (1.3)$$

then the Cauchy problem (1.1), (1.2) can be rewritten as

$$\begin{cases} u_t + (f(u) - v)_x = 0, \\ v_t + \frac{v - (1 - \mu) f(u)}{\delta} = 0, \end{cases} \quad (1.4)$$

with initial data

$$(u(x, 0), v(x, 0)) = (u_0(x), v_0(x)) = (u_0(x), \sigma_0(x) - f(u_0(x))), \quad (1.5)$$

and in what follows, we only consider the Cauchy problem (1.4), (1.5).

By simple calculation, the two eigenvalues of the system (1.4) are respectively

$$\lambda_1(u, v) = 0, \quad \lambda_2(u, v) = f'(u), \quad (1.6)$$

with the corresponding right eigenvectors

$$r_1(u, v) = (1, f'(u))^T, \quad r_2(u, v) = (1, 0)^T.$$

Riemann invariants are

$$w(u, v) = f(u) - v, \quad z(u, v) = v, \quad (1.7)$$

respectively. Furthermore, we have

$$\nabla \lambda_1(u, v) \cdot r_1(u, v) = 0, \quad \nabla \lambda_2(u, v) \cdot r_2(u, v) = f''(u). \quad (1.8)$$

Therefore, it follows from (1.6), (1.8) that $\lambda_1(u, v) = \lambda_2(u, v)$ on the set $S_1 = \{(u, v) : f'(u) = 0\}$, in which strict hyperbolicity fails to hold, and the first characteristic field is linearly degenerate on the whole $u - v$ plane, the second characteristic field is linearly degenerate on the $S_2 = \{(u, v) : f''(u) = 0\}$, in the sense of Lax (see [18]).

The system (1.1) serves as a model for the equations of motion of a viscoelastic solid with relaxation term, which arises in several areas of physical, such as gas dynamics, elasticity, kinetic theory, multiphase flow, phase transition, etc. (cf. [3, 13, 22, 32]). Hence, the study of the Cauchy problem (1.1), (1.2) is of interest and much investigation has been done on

it. Paper [32] proved that the global smooth solution for the Cauchy problem (1.1), (1.2) exists for smooth initial data provided that the C^0 -norm of the initial data are sufficiently small. However, for a special type of the initial data

$$(u(x, 0), \sigma(x, 0)) = \begin{cases} (u_-, \mu f(u_-)), & x < 0, \\ (0, 0), & x > 0, \end{cases}$$

Greenberg and Hsiao [13] found that even if u_- is sufficiently small to solve the above initial value problem, shock waves and contact discontinuity must be introduced and the corresponding initial value problem does not have globally defined smooth solutions. Therefore, for discontinuous initial data, only discontinuous solutions may exist in the large and one has to seek the global entropy weak solutions for the Cauchy problem (1.4), (1.5). Under the assumption of strict hyperbolicity, Chen and Liu [3] obtained the existence of the global weak solutions for the Cauchy problem (1.4), (1.5).

In this paper, our first goal is to extend the results in [3] to nonstrictly hyperbolic case. After introducing Definition 3.1 of entropy solutions for the Cauchy problem (1.4), (1.5), we prove the existence, uniqueness and continuous dependence of the global entropy solution for the Cauchy problem (1.4), (1.5). For system of hyperbolic conservation laws, there are, however, very few results addressing the question of uniqueness, and continuous dependence. Some uniqueness results for rather general systems are derived by DiPerna [8] and Liu [21]. Stability of the constant state for general systems is studied by Temple [31]. To do this, a key idea is to transform the system (1.4) into an inhomogeneous scalar conservation law by using the special form of the system (1.4). For this purpose, it is sufficient to establish the uniform bounded estimate (2.15) of $v_x^\varepsilon(x, t)$ with respect to the viscous coefficient ε . Thus, the assumption $v_{0x} \in L^\infty$ must be added. Our second goal, under strictly hyperbolic hypothesis, is to construct some special entropy pairs of Lax type, in which the progression terms are functions of a single variable, and establish the necessary estimates for the major progression terms by using the theory of singular perturbation of the ordinary differential equations. These special entropy pairs provide the convergence of the viscosity solutions by applying the method of compensated compactness.

It is worthwhile to point out that, in the strictly hyperbolic case, the system under our consideration is linearly degenerate. To our knowledge, very few results have been obtained for such systems with large initial data (see [2, 3, 9, 13, 20, 28, 33, 34]). We think that the special entropy-entropy flux pairs of Lax type constructed in our paper is a new approach to deal with the strong convergence of the approximate solution sequence to

hyperbolic systems of conservation laws with degenerate characteristic fields (see [33, 34]).

2. THE NONSTRICTLY HYPERBOLIC CASE

In this section, we consider the nonstrictly hyperbolic case and prove the existence, uniqueness, and continuous dependence of the global entropy solution for the Cauchy problem (1.4), (1.5).

First we introduce the following definition on entropy solutions:

DEFINITION 2.1. A pair of functions $(u(x, t), v(x, t))$ is called an entropy solution for the Cauchy problem (1.4), (1.5), if the following conditions are satisfied:

(i) For any $\varphi(x, t) \in C_0^\infty(\mathbf{R} \times \mathbf{R}^+)$,

$$\left\{ \begin{aligned} & \iint_{t>0} \{u\varphi_t + f(u)\varphi_x + v_x\varphi\} dx dt + \int_{\mathbf{R}} u_0(x)\varphi(x, 0) dx = 0, \\ & \iint_{t>0} \left\{v\varphi_t - \frac{v-(1-\mu)f(u)}{\delta}\varphi\right\} dx dt + \int_{\mathbf{R}} v_0(x)\varphi(x, 0) dx = 0, \\ & \iint_{t>0} \left\{s\varphi_t + \frac{\mu(1-\mu)}{\delta^2}\exp\left(\frac{\mu}{\delta}t\right)u\varphi\right\} dx dt + \int_{\mathbf{R}} s_0(x)\varphi(x, 0) dx = 0, \end{aligned} \right. \quad (2.1)$$

where

$$s(x, t) = \left(v_x(x, t) + \frac{1-\mu}{\delta}u(x, t) \right) \exp\left(\frac{\mu}{\delta}t\right);$$

(ii) the entropy inequalities

$$\iint_{t>0} \{ \rho(u)\varphi_t + \theta(u)\varphi_x + \rho'(u)v_x\varphi \} dx dt \geq 0, \quad (2.2)$$

$$\iint_{t>0} \left\{ F(v)\varphi_t - \frac{1}{\delta}F'(v)(v-(1-\mu)f(u))\varphi \right\} dx dt \geq 0, \quad (2.3)$$

and

$$\iint_{t>0} \left\{ G(s)\varphi_t + \frac{\mu(1-\mu)}{\delta^2}\exp\left(\frac{\mu}{\delta}t\right)uG'(s)\varphi \right\} dx dt \geq 0, \quad (2.4)$$

hold for any non-negative function $\varphi(x, t) \in C_0^\infty(\mathbf{R} \times \mathbf{R}^+)$, where (ρ, θ) satisfies

$$\theta'(u) = \rho'(u) f'(u), \quad \rho \in C^2, \quad \rho'' > 0, \quad (2.5)$$

and F, G are arbitrary C^2 -functions with $F'' > 0, G'' > 0$.

Under the above definition, we have the following existence and uniqueness results:

THEOREM 2.2. *Assume $f(u)$ satisfies the following hypotheses:*

(H1) $f(u) \in C^2(\mathbf{R}^+)$ with $f(0) = 0, f'(u) \geq 0$ for $u \geq 0$ and the inverse function $f^{-1}(u)$ of $f(u)$ exists on $u \geq 0$;

(H2) there exists $u_* > 0$ such that

$$f''(u) \leq 0, \quad u \in [0, u_*]; \quad f''(u) \geq 0, \quad u \in [u_*, +\infty).$$

(H3) $\text{meas}\{u : f''(u) = 0\} = 0$.

Then, if $(u_0(x), v_0(x)) \in (L^\infty, W^{1, \infty})$, the Cauchy problem (1.4), (1.5) admits a unique global entropy solution $(u(x, t), v(x, t))$.

For the proof of the existence of entropy solution, we adopt the method of vanishing viscosity and for the proof of the uniqueness, we apply Kruzkov's methods [17] with a slight modification. The proof is divided into three parts.

2.1. Viscosity Solutions

In this subsection we consider the Cauchy problem for the related parabolic system

$$\begin{cases} u_t + (f(u) - v)_x = \varepsilon u_{xx}, \\ v_t + \frac{v - (1 - \mu) f(u)}{\delta} = \varepsilon v_{xx}, \end{cases} \quad (2.6)$$

with initial data

$$(u(x, 0), v(x, 0)) = (u_0^\varepsilon(x), v_0^\varepsilon(x)), \quad (2.7)$$

where $u_0^\varepsilon(x), v_0^\varepsilon(x)$ are the regularizations of $u_0(x), v_0(x)$ respectively

$$\begin{cases} u_0^\varepsilon(x) = \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \rho\left(\frac{x-y}{\varepsilon}\right) u_0(y) dy, \\ v_0^\varepsilon(x) = \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \rho\left(\frac{x-y}{\varepsilon}\right) v_0(y) dy. \end{cases} \quad (2.8)$$

Here $\rho(x)$ is a mollifier; that is $0 \leq \rho(x) \in C_0^\infty(\mathbf{R})$, $\text{supp } \rho(x) \subset [-1, 1]$, and $\int_{\mathbf{R}} \rho(x) dx = 1$.

First, we give *a priori*- L^∞ estimates, independent of ε , to the Cauchy problem (2.6), (2.7) by the framework of positively invariant regions in [4]. The result of [4] can be summarized in the following theorem.

THEOREM 2.3. *Let g_i ($i = 1, 2$) be two smooth functions, $g_i: \mathbf{R}^2 \rightarrow \mathbf{R}$ and $\Sigma = \{(u, v) : g_i(u, v) \leq 0, i = 1, 2\}$. Assume that for all $t > 0$ and $(u_0, v_0) \in \partial\Sigma$, the following conditions hold:*

- (i) ∇g_i is a left eigenvector of $\nabla(f(u) - v, 0)$ corresponding to λ_i ;
- (ii) g_i is quasi-convex at (u_0, v_0) , i.e., for all $\xi \in \mathbf{R}^2$, if whenever $\xi \cdot \nabla g_i = 0$, then $\nabla^2 g_i(\xi, \xi) \geq 0$;
- (iii) $\nabla g_i(F) < 0$ at (u_0, v_0) , for all $t > 0$, where $F = (0, ((1 - \mu)f(u) - v)/\delta)^T$.

Then Σ is an invariant region for (2.6) for all $\varepsilon > 0$; that is, if $(u_0, v_0) \in \Sigma$ for all x , then $(u^\varepsilon(x, t), v^\varepsilon(x, t)) \in \Sigma$, for all $(x, t) \in \mathbf{R} \times \mathbf{R}^+$.

By applying Theorem 2.3, we immediately obtain the following results. The details are omitted.

THEOREM 2.4. *Let (H1), (H2) hold. Then the following regions are positively invariant regions for (2.6) for all $\varepsilon > 0$:*

$$\Sigma_C = \left\{ (u, v) : 0 \leq v \leq \frac{1-\mu}{\mu} C, f^{-1}(v) \leq u \leq f^{-1}(v + C) \right\},$$

where

$$f(u_*) \leq C \leq \frac{\mu}{1-\mu} f(u_*).$$

See Fig. 2.1. The details are omitted.

From Theorem 2.4, the viscosity solutions of the Cauchy problem (2.6), (2.7) have the following uniform *a priori* estimates

$$|u^\varepsilon(x, t)| \leq M, \quad |v^\varepsilon(x, t)| \leq M, \quad (2.9)$$

where M is a positive constant depending only on the initial data, but independent of ε , then we get the following global existence theorem by using the local existence of the solutions for the Cauchy problem (2.6), (2.7), which can be easily obtained by applying the contraction mapping principle to an integral representation of (2.6).

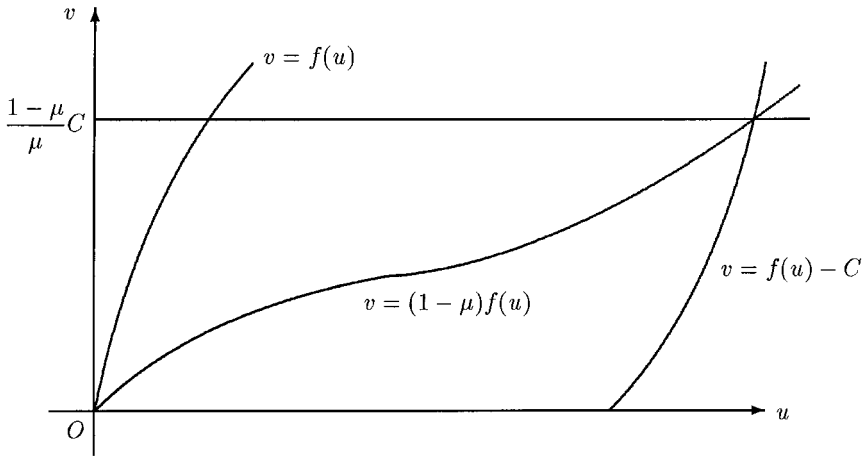


FIGURE 2.1

THEOREM 2.5. *Let (H1)–(H2) hold. Furthermore, assume the initial data $(u_0(x), v_0(x))$ satisfies*

$$(u_0 - \bar{u}, v_0 - \bar{v}) \in L^\infty \cap L^2(-\infty, \infty), \quad (2.10)$$

for some equilibrium state $(\bar{u}, \bar{v}) = (\bar{u}, (1 - \mu)f(\bar{u}))$. Then for any fixed $\varepsilon > 0$, the Cauchy problem (2.6), (2.7) exists a unique global smooth solution $(u^\varepsilon(x, t), v^\varepsilon(x, t))$ satisfies a priori estimates (2.9).

To reduce the system (1.4) to an inhomogeneous scalar balance law, as has been pointed out in the introduction, we have to estimate $v_x^\varepsilon(x, t)$. For simplicity, we shall drop the indices ε of the approximate solutions $(u^\varepsilon(x, t), v^\varepsilon(x, t))$ in the following. Let

$$r(x, t) = \exp\left(\frac{\mu}{\delta}t\right) v(x, t). \quad (2.11)$$

Then (2.6) can be rewritten as

$$\begin{cases} u_t + \left(f(u) - \exp\left(-\frac{\mu}{\delta}t\right) r \right)_x = \varepsilon u_{xx}, \\ r_t + \frac{1-\mu}{\delta} \exp\left(\frac{\mu}{\delta}t\right) \left(\exp\left(-\frac{\mu}{\delta}t\right) r - f(u) \right) = \varepsilon r_{xx}. \end{cases} \quad (2.12)$$

Differentiating the second equation in (2.12) with respect to x , and using the first equation in (2.12), we get

$$s_t - \frac{\mu(1-\mu)}{\delta^2} \exp\left(\frac{\mu}{\delta} t\right) u = \varepsilon s_{xx}, \quad (2.13)$$

where

$$s(x, t) = r_x(x, t) + \frac{1-\mu}{\delta} \exp\left(\frac{\mu}{\delta} t\right) u(x, t). \quad (2.14)$$

By virtue of (2.13) and maximum principle, we obtain the following results:

THEOREM 2.6. *Under the hypotheses of Theorem 2.5, if we assume further that $v_{0x}(x) \in L^\infty$; then any smooth solutions $(u^\varepsilon(x, t), v^\varepsilon(x, t))$ of (2.6), (2.7) on the domain $\pi(T) = \{(x, t) : x \in \mathbf{R}, 0 < t \leq T\}$ satisfy*

$$|v_x^\varepsilon(x, t)| \leq M_1 + \frac{1-\mu}{\delta} M_0, \quad (2.15)$$

where $M_0 = \max(M, \|u_0\|_{L^\infty})$, $M_1 = \|v_{0x}\|_{L^\infty}$.

Proof. From (2.8), we have

$$\left| \frac{d}{dx} v_0^\varepsilon(x) \right| \leq M_1. \quad (2.16)$$

Consider an auxiliary function

$$s(x, t) = \bar{s}(x, t) + M_1 + \frac{1-\mu}{\delta} M_0 \exp\left(\frac{\mu}{\delta} t\right) + \frac{N}{L^2} (x^2 + 3\varepsilon e^t), \quad (2.17)$$

where L is a positive constant and N is the upper bound of $|s(x, t)|$ on $\pi(T)$ (by the local existence, N can be obtained).

Substituting (2.17) into (2.13) and noticing (2.9), (2.11), (2.14), (2.16), one can get the following initial-boundary-value problem

$$\bar{s}_t + \frac{\mu(1-\mu)}{\delta^2} \exp\left(\frac{\mu}{\delta} t\right) (M_0 - u) + \frac{N}{L^2} (3\varepsilon e^t - 2\varepsilon) = \varepsilon \bar{s}_{xx}, \quad (2.18)$$

with initial-boundary-value conditions

$$\begin{cases} \bar{s}(x, 0) = v_x(x, 0) - M_1 + \frac{1-\mu}{\delta} (u(x, 0) - M_0) - \frac{N}{L^2} (x^2 + 3\varepsilon) < 0, \\ \bar{s}(\pm L, t) = s(\pm L, t) - N - M_1 - \frac{1-\mu}{\delta} M_0 \exp\left(\frac{\mu}{\delta} t\right) - \frac{3\varepsilon N}{L^2} e^t < 0. \end{cases} \quad (2.19)$$

It follows that

$$\bar{s}(x, t) < 0 \quad \text{for } (x, t) \in [-L, L] \times [0, T]. \quad (2.20)$$

Otherwise, let

$$\bar{t} = \sup_t \{t : \bar{s}(x, t) < 0, \text{ for any } x \in (-L, L)\};$$

then $0 < \bar{t} \leq T < +\infty$. From the continuity of $\bar{s}(x, t)$, there exists $(\bar{x}, \bar{t}) \in \{(x, t) : -L < x < L, 0 < t \leq T\}$, such that $\bar{s}(\bar{x}, \bar{t}) = 0$, $\bar{s}_t(\bar{x}, \bar{t}) \geq 0$, $\bar{s}_x(\bar{x}, \bar{t}) = 0$, $\bar{s}_{xx}(\bar{x}, \bar{t}) \leq 0$, i.e.,

$$(\bar{s}_t - \varepsilon \bar{s}_{xx})|_{(\bar{x}, \bar{t})} \geq 0.$$

Thus we obtain a contradiction from (2.18) and (2.9), and (2.20) holds.

Equality (2.17) and inequality (2.20) show that

$$s(x, t) < M_1 + \frac{1-\mu}{\delta} M_0 \exp\left(\frac{\mu}{\delta} t\right) + \frac{N}{L^2} (x^2 + 3\varepsilon e^t). \quad (2.21)$$

Letting $L \rightarrow \infty$ in (2.21), we get

$$s(x, t) \leq M_1 + \frac{1-\mu}{\delta} M_0 \exp\left(\frac{\mu}{\delta} t\right). \quad (2.22)$$

From (2.9), (2.11), (2.14), and (2.22), it follows

$$v_x(x, t) \leq \exp\left(-\frac{\mu}{\delta} t\right) s(x, t) \leq M_1 + \frac{1-\mu}{\delta} M_0. \quad (2.23)$$

On the other hand, if we make the transformation

$$s(x, t) = \bar{s}(x, t) - M_1 - \frac{1-\mu}{\delta} M_0 \exp\left(\frac{\mu}{\delta} t\right) - \frac{N}{L^2} (x^2 + 3\varepsilon e^t), \quad (2.24)$$

then, similar to the proof of (2.23), we can get

$$v_x(x, t) \geq -M_1 - \frac{1-\mu}{\delta} M_0. \quad (2.25)$$

Thus the proof of Theorem is completed.

2.2. Existence of Entropy Solutions

In this subsection, we are going to research the existence of entropy solutions for the Cauchy problem (1.4), (1.5). Our technique is to apply the method of compensated compactness.

For our purposes we first give the following lemmas.

LEMMA 2.7. *If the conditions of Theorem 2.5 are satisfied, then for any fixed $\varepsilon > 0$, the solutions $(u^\varepsilon(x, t), v^\varepsilon(x, t))$ of the Cauchy problem (2.6), (2.7) satisfy*

$$\varepsilon^{1/2} \frac{\partial}{\partial x} u^\varepsilon \quad \text{is uniformly bounded in } L^2_{loc}(\mathbf{R} \times (0, \infty)).$$

Proof. Let $K \subset \subset \mathbf{R} \times (0, \infty)$ and choose $\phi \in C_0^\infty(\mathbf{R} \times (0, \infty))$ such that $\phi|_K = 1$, $0 \leq \phi \leq 1$ and $G = \text{supp}\{\phi\}$.

Multiplying the first equation in (2.6) by $u^\varepsilon \phi$, integrating over $\mathbf{R} \times (0, \infty)$ and noticing (2.9), (2.15), we obtain

$$\begin{aligned} & \varepsilon \int_0^\infty \int_{-\infty}^\infty (u_x^\varepsilon)^2 \phi \, dx \, dt \\ &= \int_0^\infty \int_{-\infty}^\infty \left\{ \frac{1}{2} (u^\varepsilon)^2 \phi_t + \left(u^\varepsilon f(u^\varepsilon) - \int_0^{u^\varepsilon} f(s) \, ds \right) \phi_x \right\} dx \, dt \\ & \quad + \int_0^\infty \int_{-\infty}^\infty \left\{ v_x^\varepsilon u^\varepsilon \phi + \frac{1}{2} \varepsilon (u^\varepsilon)^2 \phi_{xx} \right\} dx \, dt \\ &\leq C_1(M_0, \phi) + \frac{1-\mu}{\delta} C_2(M_0, M_1, \phi), \end{aligned} \tag{2.26}$$

where $C_1(M_0, \phi)$, $C_2(M_0, M_1, \phi)$ are constants, independent of ε , which implies that Lemma 2.7 is proved.

LEMMA 2.8. *Under the hypotheses of Theorem 2.5,*

$$\frac{\partial}{\partial t} \rho(u^\varepsilon) + \frac{\partial}{\partial x} \theta(u^\varepsilon) \tag{2.27}$$

lie in a compact set of $H_{loc}^{-1}(\Omega)$ for any C^2 -function pairs $(\rho(u), \theta(u))$ satisfying

$$\theta'(u) = \rho'(u) f'(u), \tag{2.28}$$

where $\Omega \subset \mathbf{R} \times \mathbf{R}^+$ is any bounded open set.

Proof. Multiplying the first equation in (2.6) by $\rho'(u^\varepsilon)$ and using (2.28), it is easy to see that

$$\frac{\partial}{\partial t} \rho(u^\varepsilon) + \frac{\partial}{\partial x} \theta(u^\varepsilon) = \varepsilon \frac{\partial^2}{\partial x^2} \rho(u^\varepsilon) - \varepsilon \rho''(u^\varepsilon) \left(\frac{\partial}{\partial x} u^\varepsilon \right)^2 + v_x^\varepsilon \rho'(u^\varepsilon). \quad (2.29)$$

By applying the result of Murat [24], in order to complete the proof of Lemma 2.8, it is sufficient to prove that

$$\begin{aligned} (2.27) &\in (\text{compact set of } H^{-1}(\Omega) + \text{bounded set of } \mu(\Omega)) \\ &\cap \text{bounded set of } W^{-1, \infty}, \end{aligned} \quad (2.30)$$

where $\mu(\Omega)$ denotes the set of measures. By using (2.26), (2.29) and (2.15), (2.30) can be obtained. The details are omitted. This completes the proof of the Lemma.

Proof of the Existence of Entropy Solutions. Thanks to the estimate (2.15), the system (1.4) has been transformed into an inhomogeneous scalar conservation law. Hence, we first expect to prove the convergence of the viscosity solutions $\{u^\varepsilon(x, t)\}$ defined by (2.6), (2.7), or, equivalently, the corresponding family of Young measures $v_{x, t}$, uniquely determined by $\{u^\varepsilon(x, t)\}$ to be a family of Dirac masses. To do this, we introduce Tartar's commutation relation [30]

$$(P_1) \quad \langle v_{x, t}, \rho_1 \theta_2 - \rho_2 \theta_1 \rangle = \langle v_{x, t}, \rho_1 \rangle \langle v_{x, t}, \theta_2 \rangle - \langle v_{x, t}, \rho_2 \rangle \langle v_{x, t}, \theta_1 \rangle$$

where $(\rho_i, \theta_i) \in C^2$ ($i = 1, 2$) satisfying (2.28). Now we will prove that $v_{x, t}$ can be reduced to a point mass by analyzing the progressing entropy waves of Lax type.

Let Q denote the smallest characteristic rectangle

$$Q = \{u : u^- \leq u \leq u^+\},$$

which contains the support $v_{x, t}$.

By (2.28), for $k = \pm 1, \pm 2, \dots$, one can construct the following C^2 -entropy pairs of Lax type

$$\begin{cases} \rho_k(u) = e^{ku}, \\ \theta_k(u) = e^{ku}(f'(u) + F_k(u)), \end{cases} \quad (2.31)$$

where $F_k(u)$ satisfies the following equation:

$$F'_k + kF_k + f''(u) = 0.$$

Especially if we choose

$$F_k(u) = \begin{cases} e^{-ku} \int_{-M}^u f''(s) e^{ks} ds, & \text{if } k > 0, \\ e^{-ku} \int_M^u f''(s) e^{ks} ds, & \text{if } k < 0, \end{cases}$$

then the following estimate holds

$$|F_k(u)| \leq \frac{C}{|k|}, \quad k = \pm 1, \pm 2, \dots, \quad (2.32)$$

where C is a positive constant depending on f and M , but independent of k .

Equality (2.31) and inequality (2.32) show that

$$\theta_k(u) = \left(f'(u) + O\left(\frac{1}{k}\right) \right) \rho_k(u). \quad (2.33)$$

Now we introduce probability measure μ_k on Q defined by

$$\langle \mu_k, h \rangle = \frac{\langle v_{x,t}, h \rho_k \rangle}{\langle v_{x,t}, \rho_k \rangle}, \quad (2.34)$$

where $h = h(u)$ denotes an arbitrary continuous function. Letting $k \rightarrow +\infty$ and $k \rightarrow -\infty$ in (2.34), it is easy to prove that there exist two point masses $\delta^+ = \delta_{u^+}$ and $\delta^- = \delta_{u^-}$ such that

$$\langle \delta^+, h \rangle = \lim_{k \rightarrow +\infty} \langle \mu_k, h \rangle, \quad \langle \delta^-, h \rangle = \lim_{k \rightarrow -\infty} \langle \mu_k, h \rangle, \quad (2.35)$$

Next, we shall show that δ^+ , δ^- represent the action of $v_{x,t}$ on the fundamental form $\theta(u) - f'(u) \rho(u)$ as follows:

$$\langle \delta^+, \theta - f' \rho \rangle = \langle v_{x,t}, \theta - \lambda^+ \rho \rangle \quad (2.36)$$

and

$$\langle \delta^-, \theta - f' \rho \rangle = \langle v_{x,t}, \theta - \lambda^- \rho \rangle, \quad (2.37)$$

where $\rho, \theta \in C^2$ satisfy (2.28), and

$$\lambda^\pm = \langle \delta^\pm, f' \rangle. \quad (2.38)$$

In fact, from the measure equation (P_1) , we have

$$\langle v_{x,t}, \theta \rangle - \langle v_{x,t}, \rho \rangle \frac{\langle v_{x,t}, \theta_k \rangle}{\langle v_{x,t}, \rho_k \rangle} = \frac{\langle v_{x,t}, \theta \rho_k - \theta_k \rho \rangle}{\langle v_{x,t}, \rho_k \rangle}. \quad (2.39)$$

Noticing (2.33), it follows from (2.39) in the limit

$$\langle v_{x,t}, \theta \rangle - \langle v_{x,t}, \rho \rangle \langle \delta^+, f' \rangle = \langle \delta^+, \theta - f' \rho \rangle$$

and

$$\langle v_{x,t}, \theta \rangle - \langle v_{x,t}, \rho \rangle \langle \delta^-, f' \rangle = \langle \delta^-, \theta - f' \rho \rangle.$$

This proves (2.36) and (2.37). Next, we assert that

$$\lambda^+ = \lambda^- \quad (2.40)$$

and as a consequence

$$\langle \delta^+, \theta - f' \rho \rangle = \langle \delta^-, \theta - f' \rho \rangle,$$

i.e.,

$$\theta(u^+) - f'(u^+) \rho(u^+) = \theta(u^-) - f'(u^-) \rho(u^-), \quad (2.41)$$

for all C^2 -function pairs $(\rho(u), \theta(u))$ satisfying (2.28).

In addition, integrating by parts (2.28) over $[u^-, u^+]$, we have

$$\begin{aligned} \theta(u^+) &= \int_{u^-}^{u^+} \rho'(s) f'(s) ds \\ &= \theta(u^-) + f(u^+) \rho(u^+) - f'(u^-) \rho(u^-) - \int_{u^-}^{u^+} f''(s) \rho(s) ds, \end{aligned}$$

i.e.,

$$\theta(u^+) - f'(u^+) \rho(u^+) = \theta(u^-) - f'(u^-) \rho(u^-) - \int_{u^-}^{u^+} f''(s) \rho(s) ds. \quad (2.42)$$

Combining (2.41) with (2.42), it follows

$$\int_{u^-}^{u^+} f''(s) \rho(s) ds = 0. \quad (2.43)$$

Let

$$B_1 = \{u : f''(u) > 0\}, \quad B_2 = \{u : f''(u) < 0\}, \quad B_3 = \{u : f''(u) = 0\}.$$

Then B_1 and B_2 are open subsets of $\mathbf{R}$. Thus there exist two sequences of nonoverlapping open intervals $\{(b_{i_k}^1, b_{i_k+1}^1)\}$ and $\{(b_{i_k}^2, b_{i_k+1}^2)\}$, at most countable, such that

$$B_1 = \bigcup_k (b_{i_k}^1, b_{i_k+1}^1), \quad B_2 = \bigcup_k (b_{i_k}^2, b_{i_k+1}^2)$$

and

$$(b_{i_j}^1, b_{i_j+1}^1) \cap (b_{i_k}^2, b_{i_k+1}^2) = \emptyset$$

for all j, k . Thus

$$B_3 \subset \bigcup_k \{b_{i_k+1}^1, b_{i_k+1}^2\}$$

and

$$\mathbf{R} \subset \bigcup_{n=1}^2 \left(\bigcup_k (b_{i_k}^n, b_{i_k+1}^n] \right).$$

Now we prove that $u^+ = u^-$. Otherwise, we can let $u^+ > u^-$. For definiteness, assume $u^+ \in (b_{i_k}^1, b_{i_k+1}^1]$ and $f''(u) \geq 0$ for $u \in (b_{i_k}^1, b_{i_k+1}^1]$. Thus, there exist constants $u_1, u_2 \in (b_{i_k}^1, b_{i_k+1}^1)$ and C_1 with $u^- \leq u_1 < u_2 < u^+$ and $C_1 > 0$ such that

$$f''(u) \geq C_1 \quad \text{for } u \in [u_1, u_2].$$

Taking $\rho(u) = e^{ku}$ in (2.43), it follows

$$\begin{aligned} - \int_{u^-}^{u_1} f''(s) e^{ks} ds &= \int_{u_1}^{u_2} f''(s) e^{ks} ds + \int_{u_2}^{u^+} f''(s) e^{ks} ds \\ &\geq \int_{u_1}^{u_2} f''(s) e^{ks} ds \\ &\geq \frac{C_1}{k} (\exp(ku_2) - \exp(ku_1)). \end{aligned} \tag{2.44}$$

On the other hand, we have

$$\left| - \int_{u^-}^{u_1} f''(s) e^{ks} ds \right| \leq \frac{C_2}{k} (\exp(ku_1) - \exp(ku^-)), \tag{2.45}$$

where C_2 is a positive constant independent of k .

Combining (2.44) with (2.45), we get

$$\frac{C_2}{k} (\exp(ku_1) - \exp(ku^-)) \geq \frac{C_1}{k} (\exp(ku_2) - \exp(ku_1)). \quad (2.46)$$

Letting $k \rightarrow +\infty$ in (2.46), we conclude that $u_1 = u_2$, which is a contradiction. Thus $u^+ = u^-$ and $v_{x,t}$ is a point mass.

It remains to prove the assertion (2.40). Indeed, we have from the measure equation (P₁) that

$$\frac{\langle v_{x,t}, \theta_k \rangle}{\langle v_{x,t}, \rho_k \rangle} - \frac{\langle v_{x,t}, \theta_{-k} \rangle}{\langle v_{x,t}, \rho_{-k} \rangle} = \frac{\langle v_{x,t}, \theta_k \rho_{-k} - \theta_{-k} \rho_k \rangle}{\langle v_{x,t}, \rho_k \rangle \langle v_{x,t}, \rho_{-k} \rangle}. \quad (2.47)$$

If $u^+ = u^-$, equality (2.40) is immediate. If not, then for sufficiently small $\alpha > 0$ we have

$$\langle v_{x,t}, \rho_k \rangle \geq \text{const. } e^{k(u^+ - \alpha)}$$

and

$$\langle v_{x,t}, \rho_{-k} \rangle \geq \text{const. } e^{-k(u^- + \alpha)},$$

for large k and the denominator on the right-hand side of (2.47) approaches infinity. The numerator is in general $O(1/k)$. Thus the right-hand side approaches zero while the left approaches $\lambda^+ - \lambda^-$. This proves the assertion (2.40).

According to the argument above, by applying the framework of the theory of compensated compactness in [9, 29], we show that there exists a function $u(x, t) \in L^\infty$ such that

$$u^\varepsilon(x, t) \rightarrow u(x, t) \quad \text{a.e. as } \varepsilon \rightarrow 0. \quad (2.48)$$

The convergence of $v^\varepsilon(x, t)$ and $v_x^\varepsilon(x, t)$ is a consequence of the convergence of $u^\varepsilon(x, t)$.

From the second equation in (2.6), we have

$$\begin{aligned} v^\varepsilon(x, t) &= \exp\left(-\frac{t}{\delta}\right) T_\varepsilon(t) v_0^\varepsilon(x) \\ &\quad + \frac{1-\mu}{\delta} \int_0^t T_\varepsilon(t-\tau) \exp\left(-\frac{t-\tau}{\delta}\right) f(u^\varepsilon(x, \tau)) d\tau, \end{aligned}$$

where

$$T_\varepsilon(t) \varphi(x) = \frac{1}{\sqrt{4\pi\varepsilon t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4\varepsilon t}\right) \varphi(y) dy.$$

It is clear from the above equality that, for the smooth function f , the convergence of $u^\varepsilon(x, t)$ implies that

$$v^\varepsilon(x, t) \rightarrow v(x, t), \quad \text{a.e. as } \varepsilon \rightarrow 0, \quad (2.49)$$

for some L^∞ function $v(x, t)$ satisfying

$$v(x, t) = \exp\left(-\frac{t}{\delta}\right) v_0(x) + \frac{1-\mu}{\delta} \int_0^t \exp\left(-\frac{t-\tau}{\delta}\right) f(u(x, \tau)) d\tau.$$

Now we further prove the convergence of $v_x^\varepsilon(x, t)$.

From (2.13) and noticing (2.11), (2.14), we can get the following integral representation of $v_x^\varepsilon(x, t)$:

$$\begin{aligned} v_x^\varepsilon(x, t) = & \exp\left(-\frac{\mu}{\delta} t\right) T_\varepsilon(t) \left(\frac{d}{dx} v_0^\varepsilon(x) + \frac{1-\mu}{\delta} u_0^\varepsilon(x) \right) - \frac{1-\mu}{\delta} u^\varepsilon(x, t) \\ & + \frac{\mu(1-\mu)}{\delta^2} \int_0^t T_\varepsilon(t-\tau) \exp\left(\frac{\mu}{\delta}(\tau-t)\right) u^\varepsilon(x, \tau) d\tau. \end{aligned}$$

It is clear from the above equality that, the convergence of $u^\varepsilon(x, t)$ implies that

$$v_x^\varepsilon(x, t) \rightarrow D_x v(x, t), \quad \text{a.e. as } \varepsilon \rightarrow 0, \quad (2.50)$$

where

$$\begin{aligned} D_x v(x, t) = & \exp\left(-\frac{\mu}{\delta} t\right) \left(v_{0x}(x) + \frac{1-\mu}{\delta} u_0(x) \right) - \frac{1-\mu}{\delta} u(x, t) \\ & + \frac{\mu(1-\mu)}{\delta^2} \int_0^t \exp\left(\frac{\mu}{\delta}(\tau-t)\right) u(x, \tau) d\tau \in L^\infty(\mathbf{R} \times \mathbf{R}^+). \end{aligned}$$

From (2.49), (2.50), it is easy to prove that $v_x(x, t)$ exists for almost all $(x, t) \in \mathbf{R} \times \mathbf{R}^+$, and

$$v_x(x, t) = D_x v(x, t) \in L^\infty(\mathbf{R} \times \mathbf{R}^+). \quad (2.51)$$

This proves that there exists a subsequence (still labeled) $(u^\varepsilon(x, t), v^\varepsilon(x, t), v_x^\varepsilon(x, t))$ and functions $(u(x, t), v(x, t), v_x(x, t)) \in (L^\infty, L^\infty, L^\infty)$ such that

$$(u^\varepsilon(x, t), v^\varepsilon(x, t), v_x^\varepsilon(x, t)) \rightarrow (u(x, t), v(x, t), v_x(x, t)), \quad \text{a.e. as } \varepsilon \rightarrow 0.$$

From (2.6), (2.11), (2.13), (2.14) and noticing (2.51), it is easy to verify that equality (2.1) holds. Thus $(u(x, t), v(x, t))$ is a weak solution of (1.4), (1.5).

Finally, it is standard to show that the limit function $(u(x, t), v(x, t))$ satisfies the entropy conditions (2.2)–(2.4). This completes the proof of the theorem.

2.3. Uniqueness and Continuous Dependence of Entropy Solution

To complete the proof of Theorem 2.2, in this subsection we prove the uniqueness and continuous dependence of the entropy solution of the Cauchy problem (1.4), (1.5). The methods are essentially due to Kruzkov [17] and Oleinik [25] with a slight modification. As a consequence of our uniqueness argument, we also obtain a continuous dependence result with respect to the initial data $(u_0(x), v_0(x))$.

THEOREM 2.9. *Assume that $(u(x, t), v(x, t))$ (respectively $(\bar{u}(x, t), \bar{v}(x, t))$) is an entropy solution to the system (1.4) with initial data $(u_0(x), v_0(x))$ (respectively $(\bar{u}_0(x), \bar{v}_0(x))$) and $N = \sup |f'(u)|$; then we have, for each $X > 0$, $0 \leq t \leq X/N$,*

$$\begin{aligned} & \int_{-X+Nt}^{X-Nt} |u(x, t) - \bar{u}(x, t)| dx \\ & \leq \int_{-X}^X |u_0(x) - \bar{u}_0(x)| dx + \int_0^{X/N} \int_{-X+Nt}^{X-Nt} |v_x(x, t) - \bar{v}_x(x, t)| dx dt, \end{aligned} \quad (2.52)$$

$$\begin{aligned} & \int_{-X+Nt}^{X-Nt} |v(x, t) - \bar{v}(x, t)| dx \\ & \leq \int_{-X}^X |v_0(x) - \bar{v}_0(x)| dx + \frac{1}{\delta} \int_0^{X/N} \int_{-X+Nt}^{X-Nt} |v(x, t) - \bar{v}(x, t)| dx dt \\ & \quad + \frac{1-\mu}{\delta} N \int_0^{X/N} \int_{-X+Nt}^{X-Nt} |u(x, t) - \bar{u}(x, t)| dx dt, \end{aligned} \quad (2.53)$$

and

$$\begin{aligned}
& \int_{-X+Nt}^{X-Nt} |v_x(x, t) - \bar{v}_x(x, t)| dx \\
& \leq \int_{-X}^X |v_{0x}(x) - \bar{v}_{0x}(x)| dx + \frac{1-\mu}{\delta} \exp\left(\frac{\mu X}{\delta N}\right) \int_{-X+Nt}^{X-Nt} |u(x, t) - \bar{u}(x, t)| dx \\
& \quad + \frac{\mu(1-\mu)}{\delta^2} \int_0^{X/N} \int_{-X+Nt}^{X-Nt} |u(x, t) - \bar{u}(x, t)| dx dt.
\end{aligned} \tag{2.54}$$

Proof. We only prove (2.52); the rest can be treated similarly. For any $X > 0$, let

$$\Omega = \left\{ (x, t) : 0 < t < \frac{X}{N}, |x| < X - Nt \right\}$$

and

$$A = \{ (\rho(u), \theta(u)) : \theta'(u) = \rho'(u) f'(u), \rho \in C^2, \rho'' > 0 \}.$$

Since $(u(x, t), v(x, t))$ is an entropy solution for the Cauchy problem (1.4), (1.5), we have, for any $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$, and any $(\rho(u), \theta(u)) \in A$,

$$\iint_{t>0} \{ \rho(u) \phi_t + \theta(u) \phi_x + \rho'(u) v_x \phi \} dx dt \geq 0. \tag{2.55}$$

Let

$$U(u) = |u - k|, \quad F(u) = \text{sign}(u - k)(f(u) - f(k)),$$

where k is a constant.

Then there exists a sequence $\{(\rho_n(u), \theta_n(u))\} \subset A$ such that

$$\rho_n(u) \rightarrow U(u), \quad \theta_n(u) \rightarrow F(u), \quad \rho'_n(u) \rightarrow \text{sign}(u - k),$$

in $C([-M, M])$.

Hence

$$\begin{aligned}
& \iint_{t>0} |u - k| \phi_t dx dt + \iint_{t>0} \text{sign}(u(x, t) - k)(f(u(x, t)) - f(k)) \phi_x dx dt \\
& \quad + \iint_{t>0} \text{sign}(u(x, t) - k) \phi v_x(x, t) dx dt \geq 0,
\end{aligned} \tag{2.56}$$

for any $\phi \in C_0^\infty(\Omega)$.

Let

$$\psi(x, t, \xi, \tau) = \phi(x, t) j_h(x - \xi, t - \tau) = \phi(x, t) \omega_h(x - \xi) \omega_h(t - \tau), \quad (2.57)$$

where

$$\omega_h(\lambda) = \frac{1}{h} \omega\left(\frac{\lambda}{h}\right),$$

and $\omega(\lambda)$ is a mollifier; that is $0 \leq \omega(\lambda) \in C_0^\infty(\mathbf{R})$, $\text{supp } \omega(\lambda) \subset [-1, 1]$, and $\int_{\mathbf{R}} \omega(\lambda) d\lambda = 1$.

Placing k, ϕ by $\bar{u}(\xi, \tau), \psi$ in (2.56) respectively and integrating (2.56) with respect to ξ, τ over $\mathbf{R} \times \mathbf{R}^+$, we get

$$\begin{aligned} & \iiint_{\mathbf{R}_+^2 \times \mathbf{R}_+^2} |u(x, t) - \bar{u}(\xi, \tau)| \psi_t dx dt d\xi d\tau \\ & + \iiint_{\mathbf{R}_+^2 \times \mathbf{R}_+^2} \text{sign}(u(x, t) - \bar{u}(\xi, \tau))(f(u(x, t)) - f(\bar{u}(\xi, \tau))) \psi_x dx dt d\xi d\tau \\ & + \iiint_{\mathbf{R}_+^2 \times \mathbf{R}_+^2} \text{sign}(u(x, t) - \bar{u}(\xi, \tau)) \psi v_x(x, t) dx dt d\xi d\tau \geq 0. \end{aligned} \quad (2.58)$$

Similarly, we have

$$\begin{aligned} & \iiint_{\mathbf{R}_+^2 \times \mathbf{R}_+^2} |\bar{u}(\xi, \tau) - u(x, t)| \psi_\tau d\xi d\tau dx dt \\ & + \iiint_{\mathbf{R}_+^2 \times \mathbf{R}_+^2} \text{sign}(\bar{u}(\xi, \tau) - u(x, t))(f(\bar{u}(\xi, \tau)) - f(u(x, t))) \psi_\xi d\xi d\tau dx dt \\ & + \iiint_{\mathbf{R}_+^2 \times \mathbf{R}_+^2} \text{sign}(\bar{u}(\xi, \tau) - u(x, t)) \psi \bar{v}_\xi(\xi, \tau) d\xi d\tau dx dt \geq 0. \end{aligned} \quad (2.59)$$

By (2.58) and (2.59), we have

$$\begin{aligned} & \iiint_{\mathbf{R}_+^2 \times \mathbf{R}_+^2} |u(x, t) - \bar{u}(\xi, \tau)| (\psi_t + \psi_\tau) dx dt d\xi d\tau \\ & + \iiint_{\mathbf{R}_+^2 \times \mathbf{R}_+^2} \text{sign}(u(x, t) - \bar{u}(\xi, \tau))(f(u(x, t)) \\ & - f(\bar{u}(\xi, \tau)))(\psi_x + \psi_\xi) dx dt d\xi d\tau \\ & + \iiint_{\mathbf{R}_+^2 \times \mathbf{R}_+^2} \text{sign}(u(x, t) - \bar{u}(\xi, \tau)) \psi (v_x(x, t) - \bar{v}_\xi(\xi, \tau)) dx dt d\xi d\tau \geq 0. \end{aligned} \quad (2.60)$$

By simple calculation, we have

$$\psi_t + \psi_\tau = \phi_t j_h, \quad \psi_x + \psi_\xi = \phi_x j_h. \quad (2.61)$$

Hence letting $h \rightarrow 0^+$ in (2.60) and noticing (2.61), it follows

$$\begin{aligned} & \iint_{t>0} |u(x, t) - \bar{u}(x, t)| \phi_t dx dt \\ & + \iint_{t>0} \text{sign}(u(x, t) - \bar{u}(x, t))(f(u(x, t)) - f(\bar{u}(x, t))) \phi_x dx dt \\ & + \iint_{t>0} \text{sign}(u(x, t) - \bar{u}(x, t)) \phi(v_x(x, t) - \bar{v}_x(x, t)) dx dt \geq 0. \end{aligned} \quad (2.62)$$

Letting

$$\begin{aligned} \alpha_\varepsilon(x) &= \int_{-\infty}^x \omega_\varepsilon(\lambda) d\lambda, \quad \alpha(x) = \int_{-\infty}^x \omega(\lambda) d\lambda, \\ X_\varepsilon(x, t) &= 1 - \alpha_\varepsilon(|x| + Nt - X + \varepsilon), \quad \varepsilon > 0, \end{aligned}$$

and choosing $\phi(x, t) = (\alpha_h(t - t_1) - \alpha_h(t - t_2)) X_\varepsilon(x, t)$, $0 < t_1 < t_2$ in (2.62), we have

$$\begin{aligned} & \iint_{t>0} (\omega_h(t - t_1) - \omega_h(t - t_2)) X_\varepsilon(x, t) |u(x, t) - \bar{u}(x, t)| dx dt \\ & + \iint_{t>0} (\alpha_h(t - t_1) - \alpha_h(t - t_2)) |u(x, t) - \bar{u}(x, t)| \\ & \times \left((X_\varepsilon)_t + \frac{f(u(x, t)) - f(\bar{u}(x, t))}{u(x, t) - \bar{u}(x, t)} (X_\varepsilon)_x \right) dx dt \\ & + \iint_{t>0} \text{sign}(u(x, t) - \bar{u}(x, t))(v_x(x, t) - \bar{v}_x(x, t)) \\ & \times (\alpha_h(t - t_1) - \alpha_h(t - t_2)) X_\varepsilon(x, t) dx dt \geq 0. \end{aligned} \quad (2.63)$$

It is easy to see that

$$(X_\varepsilon)_t + \frac{f(u(x, t)) - f(\bar{u}(x, t))}{u(x, t) - \bar{u}(x, t)} (X_\varepsilon)_x \leq (X_\varepsilon)_t + N |(X_\varepsilon)_x| = 0. \quad (2.64)$$

Letting $\varepsilon \rightarrow 0^+$ in (2.63), we get, from (2.64),

$$\begin{aligned} & \int_0^{X/N} \int_{-X+Nt}^{X-Nt} (\omega_h(t-t_1) - \omega_h(t-t_2)) |u(x, t) - \bar{u}(x, t)| dx dt \\ & \geq - \int_0^{X/N} \int_{-X+Nt}^{X-Nt} |v_x(x, t) - \bar{v}_x(x, t)| dx dt. \end{aligned} \quad (2.65)$$

Letting $h \rightarrow 0^+$ in (2.65), we have

$$\begin{aligned} & \int_{-X+Nt_1}^{X-Nt_1} |u(x, t_1) - \bar{u}(x, t_1)| dx - \int_{-X+Nt_2}^{X-Nt_2} |u(x, t_2) - \bar{u}(x, t_2)| dx \\ & \geq - \int_0^{X/N} \int_{-X+Nt}^{X-Nt} |v_x(x, t) - \bar{v}_x(x, t)| dx dt. \end{aligned} \quad (2.66)$$

Letting $t_1 \rightarrow 0^+$ in (2.66), it follows

$$\begin{aligned} & \int_{-X+Nt_2}^{X-Nt_2} |u(x, t_2) - \bar{u}(x, t_2)| dx \\ & \leq \int_{-X}^X |u_0(x) - \bar{u}_0(x)| dx + \int_0^{X/N} \int_{-X+Nt}^{X-Nt} |v_x(x, t) - \bar{v}_x(x, t)| dx dt. \end{aligned}$$

This proves (2.52) and, hence, the proof of Theorem 2.9 is completed.

From Theorem 2.9, we have the following continuous dependence results:

COROLLARY 2.10. *Assume that $(u(x, t), v(x, t))$ (respectively $(\bar{u}(x, t), \bar{v}(x, t))$) is an entropy solution for the Cauchy problem (1.4) with initial data $(u_0(x), v_0(x))$ (respectively $(\bar{u}_0(x), \bar{v}_0(x))$), then we have for each $X > 0$, $0 \leq t \leq X/N$,*

$$\begin{aligned} & \int_{-X+Nt}^{X-Nt} |u(x, t) - \bar{u}(x, t)| dx \\ & \leq C \int_{-X}^X (|u_0(x) - \bar{u}_0(x)| + |v_{0x}(x) - \bar{v}_{0x}(x)|) dx, \end{aligned}$$

and

$$\begin{aligned} & \int_{-X+Nt}^{X-Nt} |v(x, t) - \bar{v}(x, t)| dx \\ & \leq C \int_{-X}^X (|v_0(x) - \bar{v}_0(x)| + |u_0(x) - \bar{u}_0(x)| + |v_{0x}(x) - \bar{v}_{0x}(x)|) dx, \end{aligned}$$

where C is a positive constant depending only on δ, μ, X, N .

Having obtained the continuous dependence results (Corollary 2.10), we can immediately obtain the uniqueness result. This completes the proof of the uniqueness part of Theorem 2.2.

3. THE STRICTLY HYPERBOLIC CASE

In this section we shall assume that the system (1.1) is strictly hyperbolic and $f(u)$ satisfies the condition (H2) and the following conditions:

(A1) $f(u) \in C^6(\mathbf{R}^+)$ and $f(0) = 0, f'(u) > 0$ for $u \geq 0$;

(A2) there exists $u_* > 0$ such that

$$f''(u) < 0, \quad u \in [0, u_*]; \quad f''(u) \geq 0, \quad u \in [u_*, +\infty);$$

(A3) there exist $u_1, u_2, \dots, u_n$ with $0 \leq u_1 < u_2 < \dots < u_n \leq u_*$ such that $3(f''(u_i))^2 - f'(u_i)f'''(u_i) = 0, i = 1, 2, \dots, n$, and $3(f''(u))^2 - f'(u)f'''(u)$ remains the sign on $u \in [0, M]$, and $3(f''(u))^2 - f'(u)f'''(u) \neq 0$ if $u \neq u_i, i = 1, 2, \dots, n$.

Remark. There exist many functions $f(u)$ satisfying the hypotheses (A1)–(A3). For example:

$$f(u) = \frac{1}{5} \left(u - \frac{1}{5} \right)^5 + u + 5^{-6}, \quad u \in \left[0, \frac{1}{5} + 3^{-1/4} \right],$$

$$f(u) = \arctan(u - 1) + \frac{\pi}{4}$$

and

$$f(u) = e^{u-d} - e^{d-u} + e^d - e^{-d}, \quad u \in [0, d + \tfrac{1}{2} \ln 2], \quad 0 < d \leq \tfrac{1}{2} \ln 2.$$

Under the assumptions (A1)–(A3), directly applying the result of Theorem 2.4, we get the following theorem.

THEOREM 3.1. *Let (A1)–(A3) hold. Furthermore, the initial data $(u_0(x), v_0(x))$ satisfy*

$$\begin{cases} (u_0(x), v_0(x)) \in \Sigma_C, \\ (u_0 - \bar{u}, v_0 - \bar{v}) \in L^\infty \cap L^2(-\infty, \infty), \end{cases} \quad (3.1)$$

for some equilibrium state $(\bar{u}, \bar{v}) = (\bar{u}, (1 - \mu) f(\bar{u}))$. Then, for any fixed $\varepsilon > 0$, the Cauchy problem (2.1), (2.2) admits a unique global solution $(u^\varepsilon(x, t), v^\varepsilon(x, t))$ satisfying the following estimates

$$0 \leq v^\varepsilon(x, t) \leq f(u^\varepsilon(x, t)) \leq M < +\infty, \quad (3.2)$$

where M is a positive constant depending only on the initial data, but independent of ε .

For conciseness, we divide this section into two parts.

3.1. Entropy Waves

This subsection is concerned with entropy waves for the system (1.4). We will construct the Lax entropies and give the required estimates by using the theory of singular perturbation of the ordinary differential equations.

We recall that a pair of real-valued mapping (η, q) is an entropy-entropy flux pair of (1.4) if all smooth solutions satisfy

$$q_u = f'(u) \eta_u, \quad q_v = -\eta_u. \quad (3.3)$$

Eliminating the q from (3.3), we have

$$\eta_{uu} + f'(u) \eta_{uv} = 0. \quad (3.4)$$

From the equation (3.4), we can easily see that the system (1.4) admits a strictly convex entropy

$$\eta^*(u, v) = e^{-v} \left(\int_0^u e^{f(\xi)} d\xi + A \right), \quad (u, v) \in \{(u, v) : 0 \leq v \leq f(u) \leq M\}, \quad (3.5)$$

where $A > (1/B) e^M$, $B = \inf_{[0, f^{-1}(M)]} f'(u) > 0$.

Thus we deduce the following theorem by [9, 30]. The details are omitted.

THEOREM 3.2. *For any C^2 -entropy pair $(\eta(u, v), q(u, v))$ of the system (1.4),*

$$\frac{\partial}{\partial t} \eta(u^\varepsilon, v^\varepsilon) + \frac{\partial}{\partial x} q(u^\varepsilon, v^\varepsilon)$$

is compact in $H_{loc}^{-1}(\mathbf{R} \times \mathbf{R}^+)$.

Theorem 3.2 guarantees that the measure equation (P_2) in Section 3.2 is true. Equation (P_2) is the soul of the theory of compensated compactness and we will use it to deduce the family of Young measures $\nu_{x,t}$ determined by the viscosity solution sequence $(u^\varepsilon, v^\varepsilon)$ for the Cauchy problem (2.6), (2.7) which is a family of Dirac masses.

Now we will construct the entropy-entropy flux pairs of the Lax type. Substituting entropies $\eta_k = e^{kw}(\sum_{n=0}^1 (a_n(u)/k^n) + (a_2(u, k)/k^2))$ into (3.4), we obtain that

$$\begin{aligned} & k(f'(u) a'_0 + f''(u) a_0) + f'(u) a'_1 + f''(u) a_1 + a''_0 \\ & + \frac{1}{k} \left(\frac{a''_2}{k} + f'(u) a'_2 + f''(u) a_2 + a''_1 \right) = 0. \end{aligned}$$

Letting

$$f'(u) a'_0 + f''(u) a_0 = 0 \quad (3.6)$$

and

$$f'(u) a'_1 + f''(u) a_1 + a''_0 = 0, \quad (3.7)$$

then

$$\frac{a''_2}{k} + f'(u) a'_2 + f''(u) a_2 + a''_1 = 0. \quad (3.8)$$

Solving recursively (3.6), (3.7) with respect to a_n ($n=0, 1$), we get

$$\begin{cases} a_0(u) = \frac{1}{f'(u)} > 0, \\ a_1(u) = f''(u)(f'(u))^{-3}. \end{cases} \quad (3.9)$$

and

$$\begin{cases} a'_0(u) = -f''(u)(f'(u))^{-2}, \\ a'_1(u) = (f'(u) f'''(u) - 3f''^2(u))(f'(u))^{-4}. \end{cases} \quad (3.10)$$

In order to get the existence of $a_2(u, k)$ and its uniform boundedness with respect to k , we first establish the following theorem:

THEOREM 3.3. *For sufficiently large k , equation (3.8) exists C^3 -solution $a_2(u, k)$ and $a_2(u, k)$, $a'_2(u, k)$ are uniformly bounded with respect to k .*

Before proving Theorem 3.3, we introduce the following theorem of singular perturbation of the ordinary differential equations (see [14]).

THEOREM 3.4. *Let $Y(x) \in C^2[\alpha, \beta]$ be the solution of the equation*

$$F(x, Y, Y') = 0,$$

and functions $f(x, y, z, \lambda)$, $F(x, y, z)$ be continual on the regions $\alpha \leq x \leq \beta$, $|y - Y(x)| \leq P(x)$, $|z - Y'(x)| \leq Q(x)$ for some positive functions $P(x)$, $Q(x)$ and $\lambda_0 > \lambda > 0$. In addition,

$$\begin{cases} |f(x, y, z, \lambda) - F(x, y, z)| \leq \delta, \\ |F(x, y_2, z) - F(x, y_1, z)| \leq A |y_2 - y_1|, \\ \frac{F(x, y, z_2) - F(x, y, z_1)}{z_2 - z_1} \geq L, \end{cases} \quad (3.11)$$

for some positive constants δ, A, L .

If $y(x) = y(x, \lambda)$ is a solution of the following second-order ordinary differential equation

$$\lambda y'' + f(x, y, y', \lambda) = 0,$$

with $y(x_0) = Y(x_0)$ and $y'(x_0)$ being arbitrary, where $x_0 \in [\alpha, \beta]$, then for sufficiently small $\lambda > 0$, $\delta > 0$, and $B = |y'(x_0) - Y'(x_0)|$, $y(x)$ exists for all $\alpha \leq x \leq \beta$ and satisfies

$$|y(x, \lambda) - Y(x)| < \left\{ \frac{\delta}{A} + \lambda \left(\frac{B}{L} + \frac{D}{A} \right) \right\} \exp \left(\frac{Ax}{L} \right),$$

where $D = \max_{\alpha \leq x \leq \beta} |Y''(x)|$.

Proof of Theorem 3.3. It is easy to find that $Y_1(u) = (f'(u) f'''(u) - 3f''^2(u))/(f'(u))^4$ is a C^2 -solution of equation

$$f'(u) Y'(u) + f''(u) Y(u) + a''_1(u) = 0$$

on $0 \leq u \leq M$.

Choosing

$$F(u, y, y') = f\left(u, y, y', \frac{1}{k}\right) = f'(u) y' + f''(u) y + a_1''(u),$$

$$B_1 = 0, \quad \lambda = \frac{1}{k}, \quad \delta_1 = 0, \quad L_1 = \inf_{0 \leq u \leq M} f''(u) > 0, \quad A_1 = \sup_{0 \leq u \leq M} |f''(u)|,$$

then it is easy to verify (3.11). By using Theorem 3.4, equation (3.8) has solution $a_2(u, k)$ on $0 \leq u \leq M$ and

$$|a_2(u, k) - Y_1(u)| \leq \frac{D_1}{kA_1} \exp\left(\frac{A_1 u}{L_1}\right),$$

where $D_1 = \max_{0 \leq u \leq M} |Y_1''(u)|$. Thus

$$|a_2(u, k)| \leq |Y_1(u)| + \frac{D_1}{A_1} \exp\left(\frac{A_1 u}{L_1}\right).$$

Next we further estimate $a_2'(u, k)$. To this purpose, differentiating equation (3.8) with respect to u , we get

$$\frac{a_2'''(u, k)}{k} + f'(u) a_2''(u, k) + 2f''(u) a_2'(u, k) + f'''(u) a_2(u, k) + a_1'''(u) = 0.$$

Let $Y_2(u)$ be a C^2 -solution of equation

$$f'(u) Y_2'(u) + 2f''(u) Y_2(u) + f'''(u) Y_1(u) + a_1'''(u) = 0.$$

Then $Y_2(u)$ is uniformly bounded with respect to k .

Choosing

$$F(u, y, y') = f'(u) y' + 2f''(u) y + f'''(u) Y_1(u) + a_1'''(u),$$

$$f\left(u, y, y', \frac{1}{k}\right) = f'(u) y' + 2f''(u) y + f'''(u) a_2(u, k) + a_1'''(u),$$

$$\delta_2 = \frac{D_1}{kA_1} \sup_{0 \leq u \leq M} |f'''(u)| \exp\left(\frac{A_1 u}{L_1}\right), \quad \lambda = \frac{1}{k},$$

$$B_2 = 0, \quad A_2 = 2 \sup_{0 \leq u \leq M} |f''(u)|, \quad L_2 = L_1,$$

then it is easy to verify (3.11) and it follows from Theorem 3.4

$$|a'_2(u, k) - Y_2(u)| \leq \frac{D_2}{kA_2} \exp\left(\frac{A_2 u}{L_2}\right),$$

where $D_2 = \sup_{0 \leq u \leq M} |Y_2''(u)|$. Thus

$$|a'_2(u, k)| \leq |Y_2(u)| + \frac{D_2}{A_2} \exp\left(\frac{A_2 u}{L_2}\right).$$

This completes the proof of Theorem 3.3.

Thus using (3.3) and the argument above, a progressing wave of the system (1.4) is provided by

$$\begin{cases} \eta_k = e^{kw} \left(\sum_{n=0}^1 \frac{a_n(u)}{k^n} + \frac{a_2(u, k)}{k^2} \right), \\ q_k = \lambda_2 \eta_k + \frac{1}{k} e^{kw} \left(\sum_{n=0}^1 \frac{a'_n(u)}{k^n} + \frac{a'_2(u, k)}{k^2} \right). \end{cases} \quad (3.12)$$

In a similar way, we can obtain the following entropy-entropy flux pair of Lax type

$$\begin{cases} \eta_{-k} = e^{-kw} \left(\sum_{n=0}^1 \frac{b_n(u)}{k^n} + \frac{b_2(u, k)}{k^2} \right), \\ q_{-k} = \lambda_2 \eta_{-k} - \frac{1}{k} e^{-kw} \left(\sum_{n=0}^1 \frac{b'_n(u)}{k^n} + \frac{b'_2(u, k)}{k^2} \right), \end{cases} \quad (3.13)$$

where $b_n(u)$ ($n=0, 1$) and $b_2(u, k)$ satisfy

$$b_0(u) = a_0(u) > 0, \quad (3.14)$$

$$f'(u) b'_1 + f''(u) b_1 - b''_0 = 0, \quad (3.15)$$

$$\frac{b''_2}{k} - f'(u) b'_2 - f''(u) b_2 + b''_1 = 0. \quad (3.16)$$

From (3.14), (3.15), we have

$$b_1(u) = -f''(u)(f'(u))^{-3}$$

and

$$b'_1(u) = (3(f''(u))^2 - f'(u) f'''(u))(f'(u))^{-4}. \quad (3.17)$$

Using Lemma 3.3 again to the equation (3.16), we can get the existence of $b_2(u, k)$ and the uniformly bounded estimates of $b_2(u, k)$, $b'_2(u, k)$ with respect to k .

3.2. Entropy Solutions

Consider a compactly supported probability measure $\nu_{x,t}$ on $\mathbf{R}^2$ such that

$$(P_2) \quad \langle \nu_{x,t}, \eta_1 q_2 - \eta_2 q_1 \rangle = \langle \nu_{x,t}, \eta_1 \rangle \langle \nu_{x,t}, q_2 \rangle - \langle \nu_{x,t}, \eta_2 \rangle \langle \nu_{x,t}, q_1 \rangle,$$

for all C^2 -entropy pairs (η_i, q_i) ($i = 1, 2$) of the system (1.4).

Let Q denote the smallest characteristic rectangle containing the support of $\nu_{x,t}$

$$Q = \{(u, v) : w^- \leq w(u, v) \leq w^+, v^- \leq v(u, v) \leq v^+\}.$$

As done in [9], we introduce probability measures $\mu_k^\pm$ on Q defined by

$$\langle \mu_k^+, h \rangle = \frac{\langle \nu_{x,t}, h \eta_k \rangle}{\langle \nu_{x,t}, \eta_k \rangle} \quad (3.18)$$

and

$$\langle \mu_k^-, h \rangle = \frac{\langle \nu_{x,t}, h \eta_{-k} \rangle}{\langle \nu_{x,t}, \eta_{-k} \rangle}, \quad (3.19)$$

where $h = h(u, v)$ denotes an arbitrary continuous function. As a consequence of weak-star compactness, there exist probability measures $\mu^\pm$ on Q such that

$$\langle \mu^\pm, h \rangle = \lim_{k \rightarrow \infty} \langle \mu_k^\pm, h \rangle, \quad (3.20)$$

after the selection of an appropriate subsequence. We observe that the measures μ^+ and μ^- are respectively concentrated on the boundary sections of Q associated with w , i.e.,

$$I^\pm = Q \cap \{(u, v) : w = w^\pm\}. \quad (3.21)$$

Noticing (3.11), (3.13), we have as given in [9] that

$$\langle \mu^+, \lambda_2 \eta - q \rangle = \langle \mu^-, \lambda_2 \eta - q \rangle, \quad (3.22)$$

for any C^2 -entropy pairs (η, q) .

From the assumptions (A1)–(A3) and (3.12), (3.22), we can prove that

$$w^+ = w^-. \quad (3.23)$$

Otherwise, let $w^+ > w^-$, taking $\eta = \eta_k$, $q = q_k$ in (3.22), we have from (3.12), (3.22).

$$\begin{aligned} & \left\langle \mu^+, \frac{1}{k} e^{kw^+} \left(\sum_{n=0}^1 \frac{a'_n(u)}{k^n} + \frac{a'_2(u, k)}{k^2} \right) \right\rangle \\ &= \left\langle \mu^-, \frac{1}{k} e^{kw^-} \left(\sum_{n=0}^1 \frac{a'_n(u)}{k^n} + \frac{a'_2(u, k)}{k^2} \right) \right\rangle, \end{aligned}$$

i.e.,

$$e^{kw^+} \left\langle \mu^+, \sum_{n=0}^1 \frac{a'_n(u)}{k^n} + \frac{a'_2(u, k)}{k^2} \right\rangle = e^{kw^-} \left\langle \mu^-, \sum_{n=0}^1 \frac{a'_n(u)}{k^n} + \frac{a'_2(u, k)}{k^2} \right\rangle. \quad (3.24)$$

Now we prove that

$$\langle \mu^+, a'_0(u) \rangle = \langle \mu^-, a'_0(u) \rangle = 0. \quad (3.25)$$

If (3.25) is violated, without loss of generality, we let $\langle \mu^+, a'_0(u) \rangle > 0$, then there exist positive constants c_1, c_2 such that for sufficiently large k

$$e^{kw^+} \left\langle \mu^+, \sum_{n=0}^1 \frac{a'_n(u)}{k^n} + \frac{a'_2(u, k)}{k^2} \right\rangle \geq c_1 e^{kw^+} \quad (3.26)$$

and

$$e^{kw^-} \left\langle \mu^-, \sum_{n=0}^1 \frac{a'_n(u)}{k^n} + \frac{a'_2(u, k)}{k^2} \right\rangle \leq c_2 e^{kw^-}. \quad (3.27)$$

Combining (3.26), (3.27) with (3.24), it is easy to deduce that $w^+ = w^-$. This proves (3.25).

It follows, from (3.24), (3.25),

$$e^{kw^+} \left\langle \mu^+, \frac{a'_1(u)}{k} + O\left(\frac{1}{k^2}\right) \right\rangle = e^{kw^-} \left\langle \mu^-, \frac{a'_1(u)}{k} + O\left(\frac{1}{k^2}\right) \right\rangle, \quad (3.28)$$

the result deduces that

$$\begin{cases} \text{supp } \mu^+ \subset \{(w^+, v_i^+), i = 1, 2, \dots, n\}, \\ \text{supp } \mu^- \subset \{(w^-, v_i^-), i = 1, 2, \dots, n\}, \end{cases} \quad (3.29)$$

where $v_i^+ = f(u_i) - w^+$, $v_i^- = f(u_i) - w^-$, and $v_i^+ < v_i^-$. Otherwise, if (3.29) is violated, without loss of generality, we can assume that there exists $(w^+, v_0^+) \in \{(w^+, v_i^+), i = 1, 2, \dots, n\}$ such that

$$(w^+, v_0^+) \subset \text{supp } \mu^+.$$

Thus from the assumption (A3) and (3.11), there exists $c_3 > 0$, independent of k , such that

$$\left| e^{kw^+} \left\langle \mu^+, \frac{a'_1(u)}{k} + O\left(\frac{1}{k^2}\right) \right\rangle \right| \geq \frac{c_3}{k} e^{kw^+} |3(f''(u_0^+))^2 - f'(u_0^+) f'''(u_0^+)| \quad (3.30)$$

for sufficiently large k , where $u_0^+ = w^+ + v_0^+ \in \{u_1, u_2, \dots, u_n\}$.

On the other hand, there exists $c_4 > 0$, independent of k , such that

$$\left| e^{kw^-} \left\langle \mu^-, \frac{a'_1(u)}{k} + O\left(\frac{1}{k^2}\right) \right\rangle \right| \leq \frac{c_4}{k} e^{kw^-} \quad (3.31)$$

for sufficiently large k .

By using (3.28), (3.30), and (3.31), it follows

$$\frac{c_3}{k} e^{kw^+} |3(f''(u_0^+))^2 - f'(u_0^+) f'''(u_0^+)| \leq \frac{c_4}{k} e^{kw^-}. \quad (3.32)$$

Letting $k \rightarrow +\infty$ in (3.32) and noticing $3(f''(u_0^+))^2 - f'(u_0^+) f'''(u_0^+) \neq 0$, it is easy to get $w^+ = w^-$ and this contradiction proves (3.29).

In addition, (3.29) shows that there exist constants $\alpha_1, \alpha_2, \dots, \alpha_n$ with $\alpha_i \geq 0$, $\sum_{i=1}^n \alpha_i = 1$, such that

$$\left\langle \mu^+, a'_0(u) \right\rangle = \sum_{i=1}^n \alpha_i a'_0(u_i) = - \sum_{i=1}^n \alpha_i \frac{f''(u_i)}{(f'(u_i))^2}. \quad (3.33)$$

(3.25) and (3.33) show that

$$\sum_{i=1}^n \alpha_i \frac{f''(u_i)}{(f'(u_i))^2} = 0. \quad (3.34)$$

From the assumptions (A1), (A2), and (A3), we have

$$\alpha_1 = \alpha_2 = \dots = \alpha_{n-1} = 0, \alpha_n = 1, u_n = u_*$$

and

$$\text{supp } \mu^+ = \{(w^+, v_*^+) : v_*^+ = f(u_*) - w^+\}. \quad (3.35)$$

Similarly

$$\text{supp } \mu^- = \{(w^-, v_*^-) : v_*^- = f(u_*) - w^-\} \quad (3.36)$$

and

$$v_*^+ < v_*^-. \quad (3.37)$$

According to (3.22), (3.35), (3.36), we obtain the condition that

$$f'(u_*) \eta(w^+, v_*^+) - q(w^+, v_*^+) = f'(u_*) \eta(w^-, v_*^-) - q(w^-, v_*^-),$$

for all C^2 -entropy pairs (η, q) .

Especially if we choose $\eta = v$, $q = 0$ in the above equality, then

$$f'(u_*) v_*^+ = f'(u_*) v_*^-. \quad (3.38)$$

Thus we obtain a contradiction from (3.37), (3.38), and (3.23) is proved.

Therefore, it follows from the results in [9, 28, 30]

$$w^\varepsilon(x, t) \rightarrow w(x, t) \quad \text{a.e. as } \varepsilon \rightarrow 0.$$

The convergence of $v^\varepsilon(x, t)$ is a consequence of the convergence of $w^\varepsilon(x, t)$. From the second equation in (2.1) we have

$$\begin{aligned} v^\varepsilon(x, t) = & \exp\left(-\frac{\mu}{\delta} t\right) \left\{ T_\varepsilon(t) v_0^\varepsilon(x) + \frac{1-\mu}{\delta} \int_0^t T_\varepsilon(t-\tau) \right. \\ & \left. \times \exp\left(\frac{\mu}{\delta} \tau\right) w^\varepsilon(x, \tau) d\tau \right\}. \end{aligned}$$

It is clear from the above equality that the convergence of $w^\varepsilon(x, t)$ implies that

$$v^\varepsilon(x, t) \rightarrow v(x, t) \quad \text{a.e. as } \varepsilon \rightarrow 0,$$

for some L^∞ function $v(x, t)$ satisfying

$$v(x, t) = \exp\left(-\frac{\mu}{\delta} t\right) v_0(x) + \frac{1-\mu}{\delta} \int_0^t \exp\left(-\frac{\mu}{\delta} (t-\tau)\right) w(x, \tau) d\tau.$$

Let $u(x, t) = f^{-1}(w(x, t) + v(x, t))$. Then it is clear that $(u(x, t), v(x, t))$ is a global entropy solution of (1.4), (1.5); that is, $(u(x, t), v(x, t)) \in (L^\infty, L^\infty)$ satisfying the following conditions:

(i) for any $\varphi(x, t) \in C_0^\infty(\mathbf{R} \times \mathbf{R}^+)$,

$$\begin{cases} \iint_{t>0} \{u\varphi_t + (f(u) - v)\varphi_x\} dx dt + \int_{\mathbf{R}} u_0(x) \varphi(x, 0) dx = 0, \\ \iint_{t>0} \left\{v\varphi_t - \frac{v - (1-\mu)f(u)}{\delta} \varphi\right\} dx dt + \int_{\mathbf{R}} v_0(x) \varphi(x, 0) dx = 0; \end{cases} \quad (3.39)$$

(ii) the entropy inequality

$$\iint_{t>0} \left\{ \eta(u, v) \varphi_t + q(u, v) \varphi_x - \frac{\partial}{\partial v} \eta(u, v) \frac{v - (1-\mu)f(u)}{\delta} \varphi \right\} dx dt \geq 0, \quad (3.40)$$

holds for any nonnegative function $\varphi(x, t) \in C_0^\infty(\mathbf{R} \times \mathbf{R}^+)$, where (η, q) is any entropy-entropy flux pair of the system (1.4) satisfying (3.3) with convex entropy η .

Thus we conclude with the following theorem:

THEOREM 3.5. *Under the assumptions (A1)–(A3), if the initial data $(u_0(x), v_0(x))$ satisfies (3.1), then the Cauchy problem (1.4), (1.5) admits a global entropy solution $(u(x, t), v(x, t))$ satisfying the integral equality (3.39) and the entropy inequality (3.40).*

Remark. If we replace the conditions (A2) and (A3) in Theorem 3.5 by the following assumptions (A4) and (A5), respectively:

(A4) there exists $u_* > 0$ such that

$$f''(u) \leq 0, \quad u \in [0, u_*]; \quad f''(u) > 0, \quad u \in [u_*, +\infty);$$

(A5) there exist $u_1, u_2, \dots, u_n$ with $u_* \leq u_1 < u_2 < \dots < u_n \leq M$ such that $3(f''(u_i))^2 - f'(u_i)f'''(u_i) = 0$, $i = 1, 2, \dots, n$, and $3(f''(u))^2 - f'(u)f'''(u)$ remains the sign on $u \in [0, M]$, and $3(f''(u))^2 - f'(u)f'''(u) \neq 0$ if $u \neq u_i$, $i = 1, 2, \dots, n$, then the conclusions of Theorem 3.5 still hold.

ACKNOWLEDGMENTS

I am grateful to Professor Ding Xiabi for his encouragement and guidance. This work is supported by National Natural Youth Science Foundation of China #19301038.

REFERENCES

1. J. M. Ball, Convexity conditions and existence theorems in the nonlinear elasticity, *Arch. Rational Mech. Anal.* **63** (1977), 337–403.
2. G. Q. Chen, Propagation and cancellation of oscillations for hyperbolic systems of conservation laws, *Comm. Pure Appl. Math.* **44** (1991), 121–140.
3. G. Q. Chen and T. P. Liu, Zero relaxation and dissipation limits for hyperbolic conservation laws, *Comm. Pure Appl. Math.* **46** (1993), 755–781.
4. K. N. Chuek, C. C. Conley, and J. A. Smoller, Positively invariant regions for systems of nonlinear diffusion equations, *Indiana Univ. Math. J.* **26** (1977), 372–411.
5. C. M. Dafermos, Estimates for conservation laws with little viscosity, *SIAM J. Math. Anal.* **18** (1987), 409–421.
6. C. M. Dafermos, Generalized characteristics and the structure of solutions of hyperbolic conservation laws, *Indiana Univ. Math. J.* **26** (1977), 1097–1119.
7. X. X. Ding, G. Q. Chen, and P. Z. Luo, Convergence of the Lax–Friedrichs scheme for isentropic gas dynamics, I, II, *Acta Math. Sci.* **5** (1985), 415–432, 433–472.
8. R. J. DiPerna, Uniqueness of solutions to hyperbolic conservation laws, *Indiana Univ. Math. J.* **28** (1979), 137–187.
9. R. J. DiPerna, Convergence of approximate solutions to conservation laws, *Arch. Rational Mech. Anal.* **82** (1983), 27–70.
10. R. J. DiPerna, Convergence of the viscosity method for isentropic gas dynamics, *Comm. Math. Phys.* **91** (1983), 1–30.
11. H. Frid and M. M. Santos, The Cauchy problem for the systems $\partial_t z - \partial_x \bar{z}^r = 0$, *J. Differential Equations* **111** (1994), 340–359.
12. J. Glimm, The continuous structure of discontinuities, *Lecture Notes in Phys.* **344** (1989), 177–186.
13. J. M. Greenberg and L. Hsiao, The Riemann problem for system $u_t + \sigma_x = 0$ and $(\sigma - f(u))_t + (\sigma - \mu f(u))_x = 0$, *Arch. Rational Mech. Anal.* **82** (1983), 87–108.
14. E. Kamake, Differentialgleichungen Lösungsmethoden und Lösungen I: Gewöhnliche Differentialgleichungen, 1967.
15. P. T. Kan, “On the Cauchy problem of a 2×2 system of nonstrictly hyperbolic conservation laws,” Ph.D. thesis, Courant Institute, New York Univ., 1989.
16. S. Klainerman and A. Majda, Singular perturbations of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids, *Comm. Pure Appl. Math.* **34** (1981), 481–524.
17. S. N. Kruzkov, First order quasilinear equations with several space variables, *Mat. USSR Sb.* **10** (1970), 217–243.
18. P. D. Lax, Hyperbolic systems of conservation laws, II, *Comm. Pure Appl. Math.* **10** (1957), 537–566.
19. P. D. Lax, Shock waves and entropy, in “Contributions to Nonlinear Functional Analysis” (E. A. Zarantonello, Ed.), pp. 603–634, Academic Press, San Diego, 1971.
20. P. X. Lin, Young measures and an application of compensated compactness to one-dimensional nonlinear elastodynamics, *Trans. Amer. Math. Soc.* **239** (1992), 377–413.
21. T. P. Liu, Uniqueness of weak solutions of the Cauchy problem for general 2×2 conservation laws, *J. Differential Equations* **20** (1976), 369–398.
22. T. P. Liu, Hyperbolic conservation laws with relaxation, *Comm. Math. Phys.* **108** (1987), 153–175.
23. C. S. Morawetz, On a weak solution for a transonic flow problem, *Comm. Pure Appl. Math.* **38** (1985), 797–817.
24. F. Murat, L’injection du cône positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$, *J. Math. Pures Appl.* **60** (1981), 309–322.

25. O. Oleinik, Uniqueness and a stability of the generalized solution of the Cauchy problem for a quasilinear equation, *Amer. Math. Soc. Transl. Ser.* **33** (1964), 285–290.
26. T. Nishida and J. A. Smoller, Solutions in the large for some nonlinear hyperbolic conservation laws, *Comm. Pure Appl. Math.* **26** (1973), 183–200.
27. M. E. Schonbek, Existence of solutions to singular conservation laws, *SIAM J. Math. Anal.* **15** (1984), 1123–1139.
28. D. Serre, La compacité par compensation pour les systèmes hyperboliques non linéaires de deux équations à une dimension d'espace, *J. Math. Pures Appl.* **65** (1986), 423–468.
29. J. A. Smoller, “Shock Waves and Reaction-Diffusion Equations,” Springer-Verlag, Berlin/New York, 1982.
30. L. Tartar, Compensated compactness and applications to partial differential equations, in “Nonlinear Analysis and Mechanics, Heriot-Watt Symposium, IV” (R. J. Knops, Ed.), Pitman Research Notes in Math., pp. 136–192, Longman, Harlow/New York, 1979.
31. B. Temple, Stability and decay in systems of conservation laws, in “Nonlinear Hyperbolic Problems Proceedings” (Carasso, Raviart, and Serre, Eds.), Springer-Verlag, Berlin/New York, 1986.
32. C. J. Zhu, Global resolvability for a viscoelastic model with relaxation, *Proc. Royal Soc. Edinburgh Sect. A* **125** (1995), 1277–1285.
33. C. J. Zhu, Convergence of the viscosity solutions for 2×2 hyperbolic conservation laws with one characteristic field linearly degenerate on some zero measure sets, *Chin. Sci. Bull.* **41** (1996), 11–16.
34. C. J. Zhu, Convergence of the viscosity solutions for the system of nonlinear elasticity, *J. Math. Anal. Appl.* **209** (1997), 585–604.