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On the Hopf Bifurcation in Control Systems with a Bounded Nonlinearity Asymptotically Homogeneous at Infinity

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The paper studies existence, uniqueness, and stability of large-amplitude periodic cycles arising in Hopf bifurcation at infinity of autonomous control systems with bounded nonlinear feedback. We consider systems with functional nonlinearities of Landesman–Lazer type and a class of systems with hysteresis nonlinearities. The method is based on the technique of parameter functionalization and methods of monotone concave and convex operators. © 2001 Academic Press

Key Words: control system; Hopf bifurcation; periodic cycle; orbital stability; monotone operator; concave and convex operators; hysteresis nonlinearity.

1. INTRODUCTION

The study of systems which have a principal linear part at infinity is commonly based on analysis of the asymptotic behavior of the nonlinear terms. For solvability of problems with bounded functional nonlinearities, the asymptotic Landesman–Lazer condition plays a major role ([8, Sects 24, 25]; see also [4, 5, 17, 19]). In the context here, it suggests that the nonlinearity $f(x)$ has finite limits as $x \to \pm\infty$. The superposition operator generated by such a nonlinearity has the following principal property [8],

$$\lim_{\rho \to \infty,\ \mu \to +0} \ \sup_{\|y(t)\|_E \leqslant \mu} \|f(t, \rho u_*(t) + \rho y(t)) - \psi(t)\|_{E_1} = 0, \qquad (1)$$

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with the function $u_*(t)$, determined by the linear part of the problem, and with appropriate choice of the function $\psi(t)$ and the spaces E, E_1. The property (1) makes it possible to calculate important topological characteristics of asymptotically degenerate operator equations. It leads to existence results for boundary-value problems, problems on nonlinear oscillations, bifurcation problems, and the like.

This paper studies Hopf bifurcation at infinity in autonomous control systems, depending on a parameter [3, 6, 11, 12, 18]. Hopf bifurcation at infinity is said to occur if the system has cycles of arbitrarily large amplitudes for parameter values close to a critical point. For control systems with Landesman–Lazer type nonlinearities satisfying some additional regularity assumptions at infinity, we develop a method to study existence and uniqueness of large-amplitude cycles, and to analyze stability of each cycle.

Formulated in operator terms, the Landesman–Lazer condition can be used to study systems with nonlinearities of various types. In particular, it is observed in [1, 9] that analogs of equation (1) hold for many hysteresis nonlinearities: the `stop` operator, some classes of hysterons, the Ishlinskii nonlinearity, the Preisach nonlinearity and so on (for the general theory of hysteresis operators, see [2, 14, 16, 20]). Although hysteresis nonlinearities are not differentiable, those above prove to satisfy regularity assumptions sufficient for our method to be applicable. As an example, Hopf bifurcation in a control system with the `stop` nonlinearity is considered.

The paper is organized as follows. Section 2 contains some preliminary notions and states the problem. Section 3 proves the principal result of the paper on control systems with functional nonlinearities. First the problem is reduced to the analysis of specific operator equations in the phase space $\mathfrak{R}^\ell$ of the system. For their construction, the technique of parameter functionalization is used [7, 15]. In Subsection 2.4, with the aid of the lemmas from Subsections 2.2 and 2.3, monotonicity and concavity (or convexity) of the operators involved are proved. Then, existence, uniqueness, and stability (or instability) of solutions follow from general theorems on monotone concave and convex operators, acting in spaces partially ordered in the sense of Krein [10, 13]. Section 4 proves similar results for systems with a `stop` nonlinearity.

2. PRELIMINARIES

Let the polynomials, of degree ℓ, m, $\ell > m$,

$$L_\lambda(s) = s^\ell + a_1(\lambda)\, s^{\ell-1} + \cdots + a_\ell(\lambda),$$

$$M_\lambda(s) = b_0(\lambda)\, s^m + b_1(\lambda)\, s^{m-1} + \cdots + b_m(\lambda),$$

depend continuously on a real parameter λ. Suppose that the polynomials are coprime and that $b_0(\lambda) \neq 0$ for every λ in some neighborhood of a point λ_0. Consider the scalar equation

$$L_\lambda\left(\frac{d}{dt}\right) x(t) = M_\lambda\left(\frac{d}{dt}\right) f(x(t)) \tag{2}$$

associated with a single-circuit autonomous control system including a linear unit, with rational transfer function $W_\lambda(s) = M_\lambda(s)/L_\lambda(s)$ and non-linear feedback f. It is well-known that Eq. (2) is equivalent to the ℓ-dimensional system

$$\frac{dz}{dt} = A(\lambda)\, z + \gamma(\lambda)\, f(x(t)), \qquad x(t) = c^T z(t), \tag{3}$$

where $c, \gamma(\lambda) \in \mathfrak{R}^\ell$ and the eigenvalues of the square matrix $A(\lambda)$ of order ℓ are zeros of the polynomial $L_\lambda(s)$. The vector $z \in \mathfrak{R}^\ell$ is the state vector of the control system. In particular, Eq. (2) and the system (3) are equivalent if

$$A(\lambda) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_\ell(\lambda) & -a_{\ell-1}(\lambda) & -a_{\ell-2}(\lambda) & \cdots & -a_1(\lambda) \end{pmatrix}, \qquad c = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

and the components of the vector $\gamma = \gamma(\lambda)$ are given by $\gamma_1 = \cdots = \gamma_{\ell-m-1} = 0$,

$$\gamma_{\ell-m} = b_0, \qquad \gamma_{\ell-m+1} + a_1 \gamma_{\ell-m} = b_1, \ldots, \gamma_\ell + a_1 \gamma_{\ell-1} + \cdots + a_m \gamma_{\ell-m} = b_m.$$

In this last, $a_j = a_j(\lambda)$, $b_j = b_j(\lambda)$, $j = 0, \ldots, m$.

In all that follows it is supposed that

(i) *The numbers* $\pm i$ *are simple zeros of the polynomial* $L_{\lambda_0}(s)$, *while every other zero of this polynomial has a negative real part.*

Hence, $\pm i$ are simple eigenvalues of the matrix $A(\lambda_0)$. Denote by E the corresponding two-dimensional invariant subspace of this matrix and let its complementary invariant subspace of $A(\lambda_0)$ in $\mathfrak{R}^\ell$ be E'. Let P be the projection of $\mathfrak{R}^\ell$ onto E. In the sequel, it is also assumed that

(ii) *The inequality* $c^T P \gamma(\lambda_0) \neq 0$ *holds.*

By definition, the plane $E \subset \Re^\ell$ consists of the stable 2π-periodic cycles of the linear system $\dot{z} = A(\lambda_0)\, z$. We study existence and stability of large periodic cycles of the system (3) with the bounded nonlinearity f for λ close to λ_0. Below only cycles of periods T near 2π are considered. More precisely, $T \in (3\pi/2, 5\pi/2)$.

3. SYSTEMS WITH FUNCTIONAL NONLINEARITIES

3.1. *Main Result*

Let the function f satisfy the following conditions:

(j) *There exist finite limits* $f_- = \lim_{x \to -\infty} f(x)$, $f_+ = \lim_{x \to \infty} f(x)$ *and* $f_- \neq f_+$.

(jj) *The function f is globally Lipschitz continuous and moreover,*

$$|f(x_1) - f(x_2)| \leqslant \alpha(r)\, |x_1 - x_2|, \qquad |x_1|, |x_2| \geqslant r, \quad x_1 x_2 > 0,$$

where $\alpha(r) = o(r^{-1})$, $r \to \infty$.

Since the polynomial $L_\lambda(s)$ depends continuously on λ, the condition (i) implies that for every λ close to λ_0 the polynomial L_λ has the simple zeros $\nu(\lambda) \pm i\omega(\lambda)$, where $\nu(\lambda)$, $\omega(\lambda)$ are continuous functions such that $\nu(\lambda_0) = 0$, $\omega(\lambda_0) = 1$. Set

$$\kappa = 2(f_+ - f_-)\, c^T P \gamma(\lambda_0)$$

and define

$$\Lambda^\delta_+ = \{\lambda : \nu(\lambda)\, \kappa \geqslant 0,\ |\lambda - \lambda_0| < \delta\},$$
$$\Lambda^\delta_- = \{\lambda : \nu(\lambda)\, \kappa < 0,\ |\lambda - \lambda_0| < \delta\}, \tag{4}$$

where $\delta > 0$. The same notation $z(t; \lambda)$ is used below both for a periodic solution of the system (3) and for the corresponding cycle in the phase space $\Re^\ell$. We say that the cycle $z(t; \lambda)$ is r_0-large if $\|z(\cdot\,; \lambda)\|_C = \max\{|z(t; \lambda)| : t \in \Re\} \geqslant r_0$.

THEOREM 1. *Let the conditions* (j), (jj) *hold. Then there exist* $r_0 > 0$ *and* $\delta > 0$ *such that the system* (3) *has no* r_0*-large periodic cycles whenever* $\lambda \in \Lambda^\delta_+$. *The system* (3) *has a unique* r_0*-large periodic cycle* $z_*(t; \lambda)$ *for every* $\lambda \in \Lambda^\delta_-$. *The cycle* $z_*(t; \lambda)$ *depends continuously on* λ *and* $\|z_*(\cdot\,; \lambda)\|_C \to \infty$ *as* $\lambda \to \lambda_0$, $\lambda \in \Lambda^\delta_-$.

THEOREM 2. *Under the assumptions of Theorem* 1, *every* r_0-*large cycle* $z_*(t;\lambda)$, $\lambda\in\Lambda_-^\delta$, *is orbitally asymptotically stable if* $\kappa>0$ *and orbitally unstable if* $\kappa<0$.

In particular, Λ_-^δ is either $(\lambda_0-\delta, \lambda_0)$ or $(\lambda_0, \lambda_0+\delta)$ if the function $\nu(\lambda)$ is strictly monotone in the δ-neighborhood of the point λ_0.

The rest of this section is devoted to the proofs of Theorems 1 and 2.

3.2. *Auxiliary Lemma*

Let C, C^1, and L_1 be the spaces of continuous, continuously differentiable, and locally summable scalar-valued functions defined on the semi-axis $t\geqslant 0$. We use the notations $\|x\|_C$, $\|y\|_{C^1}$, $\|w\|_{L_1}$ for the C, C^1, and L_1-norm of the restrictions of the functions $x\in C$, $y\in C^1$, $w\in L_1$ to the segment $[0, 5\pi/2]$; here $\|y\|_{C^1}=|y(0)|+\|y\|_C+\|y'\|_C$. Define

$$\mathscr{K}_{\mu,r}(u_*)=\{x=\rho y : y\in C^1,\ \|y-u_*\|_{C^1}\leqslant\mu,\ \rho\geqslant r\},\qquad \mu, r>0,$$

where $u_*(t)=\sin t$, $t\geqslant 0$. Now introduce the function

$$f^0(\eta)=\tfrac{1}{2}(f_++f_-)+\tfrac{1}{2}(f_+-f_-)\operatorname{sgn}\eta,\qquad \eta\in\mathfrak{R},$$

where sgn denotes the signum function. Consider the superposition operators

$$(\mathscr{F}x)(t)=f(x(t)),\qquad (\mathscr{F}^0x)(t)=f^0(x(t)),\quad x\in C^1.$$

The following lemma is a straightforward consequence of the assumptions (j)–(jj).

LEMMA 1. *For all functions* $x, y\in\mathscr{K}_{\mu,r}(u_*)$ *and every* $\theta>1$,

$$\|\mathscr{F}x-\mathscr{F}^0u_*\|_{L_1}<\varepsilon_1(\mu,r),\qquad \|\mathscr{F}(\theta x)-\mathscr{F}x\|_{L_1}<\varepsilon_2(\mu,r)(\theta-1),\quad (5)$$

$$\|\mathscr{F}x-\mathscr{F}y\|_{L_1}<\varepsilon_3(\mu,r)\ \|x-y\|_C,\quad (6)$$

where $\varepsilon_k(\mu,r)\to 0$ *as* $\mu\to 0$, $r\to\infty$, *for each* $k=1, 2, 3$.

3.3. *Operator of the Periodic Problem*

Fix a nonzero vector $g_0\in E$ such that $c^Tg_0=0$. Note that since the pair g_0, $A(\lambda_0)\,g_0$ is a basis in E, the condition (ii) implies that $p=c^TA(\lambda_0)\,g_0\neq 0$. Define $g=p^{-1}g_0$, $h=p^{-1}A(\lambda_0)\,g_0$. By construction,

$$c^Tg=0,\qquad c^Th=1,\qquad h=A(\lambda_0)\,g,\qquad g=-A(\lambda_0)\,h.$$

Also, the vectors $g,\ h \in E$ are linearly independent, hence for every $z_0 \in \mathfrak{R}^\ell$ there is a unique representation $z_0 = \xi(z_0)\, g + \eta(z_0)\, h + \zeta(z_0)$, where $\xi(z_0)$, $\eta(z_0) \in \mathfrak{R}$, $\zeta(z_0) \in E'$:

$$\eta(z_0) = c^T P z_0, \qquad \eta(z_0)\, h = P z_0 - \xi(z_0)\, g, \qquad \zeta(z_0) = z_0 - P z_0.$$

It follows from the condition (i) that there exists $q < 1$ such that the estimate $|\mu| < q < 1$ holds for every eigenvalue μ of the restriction of the matrix $e^{2\pi A(\lambda_0)}$ to its invariant subspace E'. Therefore there is a norm $|\cdot|_q$ on E' such that

$$|e^{2\pi A(\lambda_0)}\zeta|_q \leqslant q\, |\zeta|_q, \qquad \zeta \in E'. \tag{7}$$

From now on, denote by $|\cdot|$ the extension $|z_0| = |\xi(z_0)| + |\eta(z_0)| + |\zeta(z_0)|_q$ of the norm $|\cdot|_q$ to all of $\mathfrak{R}^\ell$.

Set $\mathfrak{R}^\ell_+ = \{z_0 \in \mathfrak{R}^\ell : \xi(z_0) > 0\}$ and define the functional

$$\tau(z_0) = 2\pi - \arctan \frac{\eta(z_0)}{\xi(z_0)}, \qquad z_0 \in \mathfrak{R}^\ell{}_+. \tag{8}$$

Since f is a bounded Lipschitz function, every initial condition $z(0) = z_0$ determines a unique solution $z(t;\, z_0, \lambda)$ of the system (3) on the semiaxis $t \geqslant 0$. Consider the operator

$$U_1^\lambda(z_0) = z(\tau(z_0);\, z_0, \lambda), \qquad z_0 \in \mathfrak{R}^\ell_+.$$

By definition, U_1^λ is a translation along the trajectories of the system (3) for the time (8) depending on the argument z_0. Therefore, every fixed point z_* of the operator U_1^λ is contained in a T-periodic cycle of the system (3), where $T = \tau(z_*)$. Since cycles do not intersect by uniqueness, z_* is confined to a single cycle.

Write U_1^λ in the form $U_1^\lambda = G_\lambda + Q_\lambda$, where

$$G_\lambda(z_0) = e^{A(\lambda)\, \tau(z_0)} z_0. \tag{9}$$

Equation (8) yields $\tau(\theta z_0) \equiv \tau(z_0)$, $\theta > 0$, and so

$$G_\lambda(\theta z_0) = \theta G_\lambda(z_0), \qquad z_0 \in \mathfrak{R}^\ell_+, \quad \theta > 0, \tag{10}$$

that is, the operator G_λ is positively homogeneous of order 1. From

$$z(t;\, z_0, \lambda) = e^{A(\lambda)\, t} z_0 + \int_0^t e^{A(\lambda)(t-s)} \gamma(\lambda)\, f(c^T z(s;\, z_0, \lambda))\, ds, \qquad t \geqslant 0, \tag{11}$$

it follows that

$$Q_\lambda(z_0) = \int_0^{\tau(z_0)} e^{A(\lambda)(\tau(z_0)-s)} \gamma(\lambda)\, f(c^T z(s;\, z_0, \lambda))\, ds.$$

LEMMA 2. *For every λ sufficiently close to λ_0, the nonlinear operator G_λ has eigenvector g_λ, $G_\lambda(g_\lambda)=\beta_\lambda g_\lambda$, where g_λ, β_λ depend continuously on λ and $g_{\lambda_0}=g$, $\beta_{\lambda_0}=1$. Moreover*, $\operatorname{sgn}(\beta_\lambda-1)=\operatorname{sgn} v(\lambda)$.

Proof. Denote by E_λ the two-dimensional invariant subspace of the matrix $A(\lambda)$ corresponding to the eigenvalues $v(\lambda)\pm i\omega(\lambda)$. Consider the intersection $\mathscr{L}_\lambda$ of the subspace E_λ with the hyperplane $R_\lambda^{\ell-1}=\{z_0\in\Re^\ell:\eta(z_0)=-\xi(z_0)\tan(2\pi/\omega(\lambda))\}$. Put $\mathscr{L}_\lambda^+=\mathscr{L}_\lambda\cap\Re_+^\ell$. First note that $E_{\lambda_0}=E$, $R_{\lambda_0}^{\ell-1}=\{z_0\in\Re^\ell:\eta(z_0)=0\}$, and so $\mathscr{L}_{\lambda_0}$ is the line $\{\xi g:\xi\in\Re\}$ and $\mathscr{L}_{\lambda_0}^+$ is the ray $\{\xi g:\xi>0\}$. Since the subspaces E_λ, $R_\lambda^{\ell-1}$ depend continuously on λ, it follows that $\mathscr{L}_\lambda$ is a line and $\mathscr{L}_\lambda^+$ is a ray for every λ close to λ_0 and both $\mathscr{L}_\lambda$ and $\mathscr{L}_\lambda^+$ depend continuously on λ.

Denote by g_λ the unit vector generating the ray $\mathscr{L}_\lambda^+=E_\lambda\cap\Re_\lambda^{\ell-1}\cap\Re_+^\ell$. Note that $\tau(z_0)=2\pi/\omega(\lambda)$ for every $z_0\in\Re_\lambda^{\ell-1}\cap\Re_+^\ell$. Furthermore,

$$e^{2\pi A(\lambda)/\omega(\lambda)}z_0=e^{2\pi v(\lambda)/\omega(\lambda)}z_0,\qquad z_0\in E_\lambda,$$

hence

$$G_\lambda(g_\lambda)=e^{A(\lambda)\,\tau(g_\lambda)}g_\lambda=e^{2\pi A(\lambda)/\omega(\lambda)}g_\lambda=e^{2\pi v(\lambda)/\omega(\lambda)}g_\lambda,$$

that is, g_λ is the eigenvector of the eigenvalue $\beta_\lambda=e^{2\pi v(\lambda)/\omega(\lambda)}$ for the operator G_λ. By definition, g_λ, β_λ depend continuously on λ, $g_{\lambda_0}=g$, and $\operatorname{sgn}(\beta_\lambda-1)=\operatorname{sgn} v(\lambda)$. ∎

Clearly, the operator $G_\lambda(z_0)$ is differentiable at each point $z_0\in\Re_+^\ell$ and its derivative, the Jacobian matrix $G'_\lambda(z_0)$, depends continuously on z_0, λ. By Lemma 2, $G_{\lambda_0}(g)=g$. From (10), this yields $G'_{\lambda_0}(g)\,g=g$, whence g is the eigenvector of the eigenvalue 1 for the matrix $G'_{\lambda_0}(g)$. Furthermore, by direct calculation

$$G'_{\lambda_0}(g)\,h=0,\qquad G'_{\lambda_0}(g)\,\zeta=e^{2\pi A(\lambda_0)}\zeta,\quad \zeta\in E'. \tag{12}$$

Therefore $\Pi=\{z_0\in\Re^\ell:\xi(z_0)=0\}$ is an invariant subspace for the matrix $G'_{\lambda_0}(g)$ and all the spectrum of $G'_{\lambda_0}(g)$, except for the simple eigenvalue 1, lies in the circle $\{|\mu|<q\}$ of the complex plane. Equations (7), (12) imply the important estimate

$$|G'_{\lambda_0}(g)\,z_0|\leqslant q\,|z_0|,\qquad z_0\in\Pi. \tag{13}$$

Now, consider the operator Q_λ. Since f is bounded, it follows that

$$M_\delta=\sup\{|Q_\lambda(z_0)|:z_0\in\Re_+^\ell,\ |\lambda-\lambda_0|\leqslant\delta\}<\infty,\qquad \delta>0. \tag{14}$$

Introduce the cone

$$K_\sigma = \{z_0 \in \Re^\ell : \xi(z_0) \geqslant 0,\ |z_0 - \xi(z_0)\, g| \leqslant \sigma \xi(z_0)\}, \qquad \sigma \geqslant 0.$$

Note that the domain $\Re^\ell_+$ of U_1^λ contains the set $K_\sigma(\rho) = \{z_0 \in K_\sigma : \xi(z_0) \geqslant \rho\}$ for each $\rho > 0$. Define

$$d = \int_0^{2\pi} e^{A(\lambda_0)(2\pi - s)} \gamma(\lambda_0)\, f^0(u_*(s))\, ds.$$

LEMMA 3. *For every $z_0, z_{01}, z_{02} \in K_\sigma(\rho)$ and every $\theta > 1$,*

$$|Q_\lambda(z_0) - d| < \varepsilon_1(\sigma, \rho, \lambda), \qquad |Q_\lambda(\theta z_0) - Q_\lambda(z_0)| < \varepsilon_2(\sigma, \rho, \lambda)(\theta - 1), \tag{15}$$

$$|Q_\lambda(z_{01}) - Q_\lambda(z_{02})| < \varepsilon_3(\sigma, \rho, \lambda)\, |z_{01} - z_{02}|, \tag{16}$$

where $\varepsilon_k(\sigma, \rho, \lambda) \to 0$ as $\sigma \to 0$, $\rho \to \infty$, $\lambda \to \lambda_0$ for each $k = 1, 2, 3$.

Proof. Let $z_0 = \xi g + \eta h + \zeta \in K_\sigma(\rho)$, $z(t) = z(t; z_0, \lambda)$. Equation (11) yields

$$\dot z(t) = A(\lambda)\, e^{A(\lambda)\, t} z_0 + \int_0^t A(\lambda)\, e^{A(\lambda)(t-s)} \gamma(\lambda)\, f(c^T z(s))\, ds + \gamma(\lambda)\, f(c^T z(t)),$$
$$t \geqslant 0,$$

hence

$$\begin{aligned}
&\xi^{-1} c^T \dot z(t) - c^T A(\lambda)\, e^{A(\lambda)\, t} g \\
&\quad = c^T A(\lambda)\, e^{A(\lambda)\, t} (\xi^{-1} z_0 - g) \\
&\qquad + \xi^{-1} \left[\int_0^t c^T A(\lambda)\, e^{A(\lambda)(t-s)} \gamma(\lambda)\, f(c^T z(s))\, ds + c^T \gamma(\lambda)\, f(c^T z(t)) \right].
\end{aligned}$$

The inclusion $z_0 \in K_\sigma(\rho)$ implies that $|\xi^{-1} z_0 - g| \leqslant \sigma$, $\xi^{-1} \leqslant \rho^{-1}$, so taking into account the continuity of $A(\lambda)$ and the boundedness of f, obtain

$$\|\xi^{-1} c^T \dot z(t) - c^T A(\lambda_0)\, e^{A(\lambda_0)\, t} g\|_C < \varepsilon(\sigma, \rho, \lambda), \qquad z_0 \in K_\sigma(\rho),$$

where $\varepsilon(\cdot, \cdot, \cdot) \to 0$ as $\sigma \to 0$, $\rho \to \infty$, $\lambda \to \lambda_0$. But

$$\begin{aligned}
c^T A(\lambda_0)\, e^{A(\lambda_0)\, t} g &= c^T A(\lambda_0)(g \cos t + h \sin t) \\
&= c^T (h \cos t - g \sin t) = \cos t = \dot u_*(t),
\end{aligned}$$

that is, $\|\xi^{-1} c^T \dot z(t) - \dot u_*(t)\|_C < \varepsilon(\sigma, \rho, \lambda)$. At the same time,

$$|\xi^{-1} c^T z(0) - u_*(0)| = |\xi^{-1} c^T z_0| = |c^T (\xi^{-1} z_0 - g)| \leqslant c_0 \sigma,$$

where c_0 is the norm of the linear functional $c^T z$. Therefore, for every $z_0 \in K_\rho(\sigma)$ we have $\|\xi^{-1} c^T z(t) - u_*(t)\|_{C^1} < \varepsilon(\sigma, \rho, \lambda) + c_0 \sigma$. Since $\xi \geqslant \rho$, this implies that

$$c^T z(t; z_0, \lambda) \in \mathscr{K}_{\mu, \rho}(u_*) \quad \text{for every} \quad z_0 \in K_\sigma(\rho), \tag{17}$$

where $\mu = \mu(\sigma, \rho, \lambda) \leqslant \varepsilon(\sigma, \rho, \lambda) + c_0 \sigma$ and hence $\mu(\cdot, \cdot, \cdot) \to 0$ as $\sigma \to 0$, $\rho \to \infty$, $\lambda \to \lambda_0$. Now the first of the estimates (5) yields

$$\sup\{\|f(c^T z(t; z_0, \lambda)) - f^0(u_*(t))\|_{L_1} : z_0 \in K_\sigma(\rho)\} \to 0$$
$$\text{as} \quad \sigma \to 0,\ \rho \to \infty,\ \lambda \to \lambda_0.$$

In addition, $|\eta(z_0)| \leqslant \sigma \xi(z_0)$ and so $|\tau(z_0) - 2\pi| \leqslant \arctan \sigma$ for every $z_0 \in K_\sigma(\rho)$, thus

$$\sup_{z_0 \in K_\sigma(\rho)} \left| \int_0^{\tau(z_0)} e^{A(\lambda_0)(\tau(z_0) - s)} \gamma(\lambda)\, f(c^T z(t; z_0, \lambda))\, ds - d \right|$$
$$= \sup_{z_0 \in K_\sigma(\rho)} |Q_\lambda(z_0) - d| \to 0$$

as $\sigma \to 0$, $\rho \to \infty$, $\lambda \to \lambda_0$, that is, the first of the estimates (15) holds.

By definition of Q_λ, there are numbers $M_1, M_2 > 0$ such that

$$|Q_\lambda(z_{01}) - Q_\lambda(z_{02})|$$
$$\leqslant M_1 \|f(c^T z(t; z_{01}, \lambda)) - f(c^T z(t; z_{02}, \lambda))\|_{L_1} + M_2 |\tau(z_{01}) - \tau(z_{02})|$$

for $z_{01}, z_{02} \in \Re^\ell_+$. Let $z_{01}, z_{02} \in K_\sigma(\rho)$. Then $c^T z(t; z_{0i}, \lambda) \in \mathscr{K}_{\mu, \rho}(u_*)$, $i = 1, 2$ and by (6),

$$\|f(c^T z(t; z_{01}, \lambda)) - f(c^T z(t; z_{02}, \lambda))\|_{L_1}$$
$$< \varepsilon_3(\mu, \rho)\, \|c^T z(t; z_{01}, \lambda) - c^T z(t; z_{01}, \lambda)\|_C.$$

Let $a > 0$ be fixed but otherwise arbitrary. Since the function $f(x)$ is globally Lipschitz, there is a $L > 0$ such that $|z(t; z_{01}, \lambda) - z(t; z_{02}, \lambda)| \leqslant L\, |z_{01} - z_{02}|$ for all $z_{01}, z_{02} \in \Re^\ell$, $t \in [0, 5\pi/2]$, $|\lambda - \lambda_0| \leqslant a$. So

$$\|f(c^T z(t; z_{01}, \lambda)) - f(c^T z(t; z_{02}, \lambda))\|_{L_1} < \varepsilon_3(\mu(\sigma, \rho, \lambda), \rho)\, L c_0 |z_{01} - z_{02}|$$

and further,

$$|Q_\lambda(z_{01}) - Q_\lambda(z_{02})| < \varepsilon(\sigma, \rho, \lambda)\, |z_{01} - z_{02}| + M_2\, |\tau(z_{01}) - \tau(z_{02})|,$$
$$z_{01}, z_{02} \in K_\sigma(\rho), \tag{18}$$

where $\varepsilon(\cdot, \cdot, \cdot) \to 0$ as $\sigma \to 0$, $\rho \to \infty$, $\lambda \to \lambda_0$. By the mean value theorem,

$$\tau(z_{01}) - \tau(z_{02}) = (\xi^2(s) + \eta^2(s))^{-1} (\eta(s)[\xi_1 - \xi_2] - \xi(s)[\eta_1 - \eta_2]),$$

where $\xi_i = \xi(z_{0i})$, $\eta_i = \eta(z_{0i})$ and $\xi(s) = s\xi_1 + (1-s)\,\xi_2$, $\eta(s) = s\eta_1 + (1-s)\,\eta_2$ for some $s \in [0, 1]$. But $\xi_i \geqslant \rho$, $|\eta_i| \leqslant \sigma\xi_i$, hence $\xi(s) \geqslant \rho$, $|\eta(s)| \leqslant \sigma\xi(s)$ and therefore

$$|\tau(z_{01}) - \tau(z_{02})| \leqslant \xi^{-1}(s)(\sigma\,|\xi_1 - \xi_2| + |\eta_1 - \eta_2|) \leqslant \rho^{-1}(\sigma + 1)\,|z_{01} - z_{02}|.$$

Combining this with (18), we obtain (16).

Finally set $z(t) = z(t; z_0, \lambda)$, $z_\theta(t) = z(t; \theta z_0, \lambda)$, where $\theta > 1$, $z_0 \in K_\sigma(\rho)$. Recall that $\tau(\theta z_0) = \tau(z_0)$, and so

$$|Q_\lambda(\theta z_0) - Q_\lambda(z_0)| \leqslant M_1 \,\| f(c^T z_\theta(t)) - f(c^T z(t)) \|_{L_1}. \tag{19}$$

But $c^T z(t) \in \mathscr{K}_{\mu, \rho}(u_*)$ and therefore $\theta c^T z(t) \in \mathscr{K}_{\mu, \rho}(u_*)$. Also, $\theta z_0 \in K_\sigma(\rho)$ yields $c^T z_\theta(t) \in \mathscr{K}_{\mu, \rho}(u_*)$. It follows from the second of the estimates (5) and from (6) that

$$\| f(\theta c^T z(t)) - f(c^T z(t)) \|_{L_1} < (\theta - 1)\,\varepsilon_2,$$
$$\| f(c^T z_\theta(t)) - f(\theta c^T z(t)) \|_{L_1} < \varepsilon_3 \,\| c^T z_\theta(t) - \theta c^T z(t) \|_C,$$

hence

$$\| f(c^T z_\theta(t)) - f(c^T z(t)) \|_{L_1} \leqslant (\theta - 1)\,\varepsilon_2 + \varepsilon_3 \,\| c^T z_\theta(t) - \theta c^T z(t) \|_C, \tag{20}$$

where $\varepsilon_i = \varepsilon_i(\mu(\sigma, \rho, \lambda), \rho)$, $i = 2, 3$. Multiplying (11) by θ and subtracting from

$$z_\theta(t) = \theta e^{A(\lambda)\, t} z_0 + \int_0^t e^{A(\lambda)(t-s)} \gamma(\lambda)\; f(c^T z_\theta(s))\, ds,$$

we obtain

$$z_\theta(t) - \theta z(t) = \int_0^t e^{A(\lambda)(t-s)} \gamma(\lambda) [f(c^T z_\theta(s)) - \theta f(c^T z(s))]\, ds.$$

Set $M = \sup\{|c^T e^{A(\lambda)\, t} \gamma(\lambda)| : t \in [0, 5\pi/2],\ |\lambda - \lambda_0| \leqslant a\}$, $\bar{f} = 5\pi \sup |f(x)|/2$. Then

$$\| c^T z_\theta(t) - \theta c^T z(t) \|_C \leqslant M \,\| f(c^T z_\theta(t)) - f(c^T z(t)) \|_{L_1} + (\theta - 1)\, M\bar{f}.$$

Now (20) gives $(1-\varepsilon_3 M)\,\|f(c^T z_\theta(t))-f(c^T z(t))\|_{L_1}\leqslant(\theta-1)(\varepsilon_2+\varepsilon_3 M\bar{f})$. But $\varepsilon_2,\varepsilon_3\to 0$, so

$$\|f(c^T z_\theta(t))-f(c^T z(t))\|_{L_1}<\varepsilon(\sigma,\rho,\lambda)(\theta-1),\qquad z_0\in K_\sigma(\rho),\quad \theta>1,$$

where $\varepsilon(\cdot,\ \cdot,\cdot)\to 0$ as $\sigma\to 0$, $\rho\to\infty$, $\lambda\to\lambda_0$. Hence from (19) the second of the estimates (15) follows. This completes the proof of Lemma 3. ▌

3.4. *Existence, Uniqueness, and Stability of a Fixed Point*

Let K be a cone in $\mathfrak{R}^\ell$, that is, $K+K=K$, $-K\cap K=\{0\}$, and $\theta K=K$ for every $\theta>0$. The cone K generates a partial order in $\mathfrak{R}^\ell$: write $z_1\geqslant z_2$ if $z_1-z_2\in K$. If K has nonempty interior, also write $z_1>z_2$ whenever $z_1-z_2\in \operatorname{int} K$.

An operator B is said to be monotone on a set $\Omega\subset K$ if

$$B(z_1)\geqslant B(z_2)\qquad\text{for every}\qquad z_1\geqslant z_2;\quad z_1,z_2\in\Omega.$$

Let K have nonempty interior. Suppose that $\Omega\subseteq \operatorname{int} K$ and $\theta\Omega=\Omega$ for every $\theta>1$. Then an operator B is called strongly concave on the set Ω if

$$B(\theta z_0)<\theta B(z_0)\qquad\text{for every}\qquad z_0\in\Omega,\quad\theta>1.$$

An operator B is called strongly convex on Ω if

$$B(\theta z_0)>\theta B(z_0)\qquad\text{for every}\qquad z_0\in\Omega,\quad\theta>1.$$

Further constructions are based on the well-known fact that any monotone operator, strongly concave on Ω, has at most one fixed point z_* in Ω. Moreover, if Ω has nonempty interior and $z_*\in\operatorname{int}\Omega$, then the fixed point z_* is asymptotically stable. On the other hand, every fixed point $z_*\in\operatorname{int}\Omega$ of a monotone operator, strongly convex on Ω, is unstable. Note that a convex operator may have more than one fixed point. These are simple cases of general theorems [10] on monotone concave and convex operators, acting in partially ordered Banach spaces with cones of various types.

From the formula

$$e^{A(\lambda_0)\,t}(\xi g+\eta h)=(\xi\cos t-\eta\sin t)\,g+(\xi\sin t+\eta\cos t)\,h,$$

it follows that

$$\begin{aligned}\xi(d)&=\int_0^{2\pi}(\xi(\gamma_0)\cos s+\eta(\gamma_0)\sin s)\ f^0(\sin s)\,ds\\&=(f_+-f_-)\int_0^{\pi}(\xi(\gamma_0)\cos s+\eta(\gamma_0)\sin s)\,ds,\end{aligned}$$

where $\gamma_0 = \gamma(\lambda_0)$, that is $\xi(d) = 2(f_+ - f_-)\,\eta(\gamma_0) = 2(f_+ - f_-)\,c^T P \gamma_0 = \kappa$. By assumptions (ii) and (j), $\kappa \neq 0$ and so $\kappa d \in \mathfrak{R}^\ell_+$. Fix $\sigma_0 > 0$ large enough so that $\kappa^{-1} d \in \operatorname{int} K_{\sigma_0}$. In what follows, $\mathfrak{R}^\ell$ is regarded as partially ordered by the cone K_{σ_0}. The symbols $\geqslant$, $>$ and the terms monotone, concave, convex are given the meaning corresponding to this order.

Define D by $Dz_0 = \xi(z_0)\,g$ and write

$$U^\lambda_{-1}(z_0) = (2D - I)(2Dz_0 - U^\lambda_1(z_0)), \qquad z_0 \in \mathfrak{R}^\ell_+,$$

where I is the identity. Since $D^2 = D$, $(2D - I)^{-1} = 2D - I$, the equations $z_0 = U^\lambda_{-1}(z_0)$ and $z_0 = U^\lambda_1(z_0)$ are equivalent, that is, the fixed points of U^λ_1 and U^λ_{-1} are the same.

THEOREM 3. *There are positive numbers δ, ρ_1, σ_1, with $\sigma_1 < \sigma_0$, such that for every $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$ the operators U^λ_1 and U^λ_{-1} are monotone on the set $K_{\sigma_1}(\rho_1)$. Moreover, the operator U^λ_k is strongly concave on $K_{\sigma_1}(\rho_1)$, while the operator U^λ_{-k} is strongly convex on $K_{\sigma_1}(\rho_1)$, where $k = \operatorname{sgn}\kappa$.*

THEOREM 4. *Let $\delta > 0$ be sufficiently small. Then there are no fixed points of the operator U^λ_1 in the set $K_{\sigma_1}(\rho_1)$ whenever $\lambda \in \Lambda^\delta_+$, while for every $\lambda \in \Lambda^\delta_-$ the operator U^λ_1 has a fixed point $z_*(\lambda)$, depending continuously on λ, such that*

$$(1 - \beta_\lambda)\,\xi_*(\lambda) \to \kappa, \quad \xi_*^{-1}(\lambda)\,z_*(\lambda) \to g \qquad \textit{as} \qquad \lambda \to \lambda_0, \quad \lambda \in \Lambda^\delta_-, \tag{21}$$

where $\xi_*(\lambda) = \xi(z_*(\lambda))$. Here, by Lemma 2, $(1 - \beta_\lambda)\,\kappa > 0$, $\lambda \in \Lambda^\delta_-$, and $\beta_\lambda \to 1$ as $\lambda \to \lambda_0$.

Equations (21) imply that for each $\lambda \in \Lambda^\delta_+$ sufficiently close to λ_0 the estimates $\xi_*(\lambda) > \rho_1$, $|z_*(\lambda) - \xi_*(\lambda)\,g| < \sigma_1 \xi_*(\lambda)$ hold, that is $z_*(\lambda) \in \operatorname{int} K_{\sigma_1}(\rho_1)$. By Theorem 3, the operator U^λ_k, where $k = \operatorname{sgn}\kappa$, is monotone and strictly concave on $K_{\sigma_1}(\rho_1)$. Therefore the solution $z_*(\lambda)$ of the equivalent equations $z_0 = U^\lambda_k(z_0)$ and $z_0 = U^\lambda_{-k}(z_0)$ is unique in $K_{\sigma_1}(\rho_1)$. Moreover, $z_*(\lambda)$ is an asymptotically stable fixed point of the concave operator U^λ_k, while it is an unstable fixed point of the convex operator U^λ_{-k}.

Proof of Theorem 3. Let $z_0 \neq 0$, $z_0 \in K_\sigma$, that is, $\xi(z_0) > 0$, $|\xi^{-1}(z_0)\,z_0 - g| \leqslant \sigma$. Since the Jacobian matrix $G'_\lambda(z_0)$ depends continuously on z_0, λ, we have

$$|G'_\lambda(\xi^{-1}(z_0)\,z_0) - G'_{\lambda_0}(g)| < \varepsilon_0(\sigma, \lambda), \qquad z_0 \neq 0, \quad z_0 \in K_\sigma,$$

where $\varepsilon_0(\cdot,\ \cdot) \to 0$ as $\sigma \to 0$, $\lambda \to \lambda_0$ and $|\cdot|$ denotes the matrix norm. Equation (10) implies the identity $G'_\lambda(\theta z_0) \equiv G'_\lambda(z_0)$ for all $z_0 \in \Re^\ell_+$, $\theta > 0$. Hence

$$|G'_\lambda(z_0) - G'_{\lambda_0}(g)| < \varepsilon_0(\sigma, \lambda), \qquad z_0 \neq 0, \quad z_0 \in K_\sigma. \tag{22}$$

Let $z_{0i} \in K_\sigma(\rho)$ and so $z_{0i} \in K_\sigma \backslash \{0\}$, $i = 1, 2$. By definition $K_\sigma \backslash \{0\}$ is a convex set, hence it contains the segment $s z_{01} + (1 - s)\, z_{02}$, $0 \leqslant s \leqslant 1$. It follows from the formula

$$G_\lambda(z_{01}) - G_\lambda(z_{02}) - G'_{\lambda_0}(g)(z_{01} - z_{02})$$
$$= \int_0^1 (G'_\lambda(s z_{01} + (1 - s)\, z_{02}) - G'_{\lambda_0}(g))(z_{01} - z_{02})\, ds$$

and from the estimate (22) that

$$|G_\lambda(z_{01}) - G_\lambda(z_{02}) - G'_{\lambda_0}(g)(z_{01} - z_{02})| < \varepsilon_0(\sigma, \lambda)\, |z_{01} - z_{02}|. \tag{23}$$

Since $G_\lambda = U^\lambda_1 - Q_\lambda$, then (16), (23) give

$$|U^\lambda_1(z_{01}) - U^\lambda_1(z_{02}) - G'_{\lambda_0}(g)(z_{01} - z_{02})| \leqslant \varepsilon(\sigma, \rho, \lambda)\, |z_{01} - z_{02}|, \tag{24}$$

where $\varepsilon = \varepsilon_0(\cdot,\ \cdot) + \varepsilon_3(\cdot,\ \cdot, \cdot) \to 0$ as $\sigma \to 0$, $\rho \to \infty$, $\lambda \to \lambda_0$. Using the representations

$$z_{01} - z_{02} = \xi_0 g + y_0, \qquad U^\lambda_1(z_{01}) - U^\lambda_1(z_{02}) = \xi_1 g + y_1,$$
$$G'_{\lambda_0}(g)(z_{01} - z_{02}) = \xi_2 g + y_2,$$

where $\xi_i \in \Re$, $y_i \in \Pi = \{z_0 \in \Re^\ell \colon \xi(z_0) = 0\}$, $i = 0, 1, 2$, we can rewrite (24) as $|(\xi_1 - \xi_2)\, g + y_1 - y_2| \leqslant \varepsilon\, |\xi_0 g + y_0|$. But $|\xi g + y| = |\xi| + |y|$, $\xi \in \Re$, $y \in \Pi$, consequently

$$|\xi_1 - \xi_2| \leqslant \varepsilon\, |\xi_0| + \varepsilon\, |y_0|, \qquad |y_1 - y_2| \leqslant \varepsilon\, |\xi_0| + \varepsilon\, |y_0|.$$

Recall that Π is an invariant subspace of the matrix $G'_{\lambda_0}(g)$ and that $G'_{\lambda_0}(g)\, g = g$. Therefore $\xi_2 = \xi_0$, $y_2 = G'_{\lambda_0}(g)\, y_0$ and by (13), $|y_2| \leqslant q\, |y_0|$. Hence

$$|\xi_1 - \xi_0| \leqslant \varepsilon\, |\xi_0| + \varepsilon\, |y_0|, \quad |y_1| \leqslant |y_1 - y_2| + |y_2| \leqslant \varepsilon\, |\xi_0| + \varepsilon\, |y_0| + q\, |y_0|. \tag{25}$$

Now consider the vector $U_{-1}^{\lambda}(z_{01}) - U_{-1}^{\lambda}(z_{02}) = \xi_{-1} g + y_{-1}$, where $\xi_{-1} \in \mathfrak{R}$, $y_{-1} \in \Pi$, that is $\xi_{-1} g = D(U_{-1}^{\lambda}(z_{01}) - U_{-1}^{\lambda}(z_{02}))$, $y_{-1} = (I-D)(U_{-1}^{\lambda}(z_{01}) - U_{-1}^{\lambda}(z_{02}))$. Note that

$$DU_{-1}^{\lambda}(z_0) = 2Dz_0 - DU_1^{\lambda}(z_0),$$

$$(I-D)\, U_{-1}^{\lambda}(z_0) = (I-D)\, U_1^{\lambda}(z_0) \qquad \text{for all} \quad z_0 \in \mathfrak{R}_+^{\ell},$$

hence $\xi_{-1} = 2\xi_0 - \xi_1$, $y_{-1} = y_1$. Therefore the estimates (25) hold also for ξ_{-1}, y_{-1} in place of ξ_1, y_1. If $z_{01} \geqslant z_{02}$, then $\xi_0 \geqslant 0$, $|y_0| \leqslant \sigma_0 \xi_0$ and so

$$|\xi_i - \xi_0| \leqslant \varepsilon \xi_0 (1+\sigma_0), \qquad |y_i| \leqslant \varepsilon \xi_0 (1+\sigma_0) + q\sigma_0 \xi_0, \quad i = \pm 1.$$

Since $q<1$, this gives $|y_i| \leqslant \sigma_0 \xi_i$, $i = \pm 1$ whenever $\varepsilon = \varepsilon(\sigma, \rho, \lambda)$ is sufficiently small. Equivalently, there is a large $\rho > 0$ and small $\sigma, \delta > 0$ such that $z_{01} \geqslant z_{02}$ implies $U_i^{\lambda}(z_{01}) \geqslant U_i^{\lambda}(z_{02})$ for every $z_{01}, z_{02} \in K_\sigma(\rho)$, $|\lambda - \lambda_0| < \delta$, $i = \pm 1$. Thus the operators U_1^{λ}, U_{-1}^{λ} are monotone on the set $K_\sigma(\rho)$.

Now let $z_0 \in K_\sigma(\rho)$, $\theta > 1$. By (10), $\theta U_1^{\lambda}(z_0) - U_1^{\lambda}(\theta z_0) = \theta Q_\lambda(z_0) - Q_\lambda(\theta z_0)$. Hence

$$\begin{aligned}(\theta-1)^{-1}(\theta U_1^{\lambda}(z_0) - U_1^{\lambda}(\theta z_0)) - d \\ = Q_\lambda(z_0) - d + (\theta-1)^{-1}(Q_\lambda(z_0) - Q_\lambda(\theta z_0)).\end{aligned}$$

The estimates (15) give

$$|\kappa(\theta-1)^{-1}(\theta U_1^{\lambda}(z_0) - U_1^{\lambda}(\theta z_0)) - \kappa d| < \kappa \varepsilon_1(\sigma, \rho, \lambda) + \kappa \varepsilon_2(\sigma, \rho, \lambda).$$

But $\kappa d \in \operatorname{int} K_{\sigma_0}$. Therefore $\kappa(\theta U_1^{\lambda}(z_0) - U_1^{\lambda}(\theta z_0)) > 0$ whenever $\varepsilon_1(\cdot, \cdot, \cdot)$ and $\varepsilon_2(\cdot, \cdot, \cdot)$ are sufficiently small. Finally note that

$$\theta U_{-1}^{\lambda}(z_0) - U_{-1}^{\lambda}(\theta z_0) = -(2D-I)(\theta U_1^{\lambda}(\theta z_0) - U_1^{\lambda}(\theta z_0))$$
$$\text{for every} \quad z_0 \in \mathfrak{R}_+^{\ell}.$$

So taking into account that the operator $2D - I$ maps the set $\operatorname{int} K_{\sigma_0}$ into itself, from $\kappa(\theta U_1^{\lambda}(z_0) - U_1^{\lambda}(\theta z_0)) > 0$ we obtain $\kappa(\theta U_{-1}^{\lambda}(z_0) - U_{-1}^{\lambda}(\theta z_0)) < 0$. That is, for $k = \operatorname{sgn} \kappa$,

$$\theta U_k^{\lambda}(z_0) > U_k^{\lambda}(\theta z_0), \qquad \theta U_{-k}^{\lambda}(z_0) < U_{-k}^{\lambda}(\theta z_0), \qquad z_0 \in K_\sigma(\rho), \quad \theta > 1.$$

That is, the operator U_k^{λ} is strongly convex on $K_\sigma(\rho)$, while the operator U_{-k}^{λ} is strongly concave on $K_\sigma(\rho)$ for every sufficiently small σ, ρ^{-1} and $|\lambda - \lambda_0|$, which completes the proof. ∎

Proof of Theorem 4. Let $z_0 \in K_\sigma(\rho)$, $\xi_0 = \xi(z_0)$. The estimate (23), where we take $z_{01} = z_0$, $z_{02} = \xi_0 g$, gives $|G_\lambda(z_0) - G_\lambda(\xi_0 g)| \leqslant |G'_{\lambda_0}(g)(z_0 - \xi_0 g)| + \varepsilon_0(\sigma, \lambda)\, |z_0 - \xi_0 g|$. Also, (13) implies that $|G'_{\lambda_0}(g)(z_0 - \xi_0 g)| \leqslant q\, |z_0 - \xi_0 g|$ and hence

$$|G_\lambda(z_0) - G_{\lambda_0}(\xi_0 g)| \leqslant |G_\lambda(\xi_0 g) - G_{\lambda_0}(\xi_0 g)| + (q + \varepsilon_0(\sigma, \lambda))\, |z_0 - \xi_0 g|.$$

From $G_{\lambda_0}(g) = g$ and (10), this can be rewritten as

$$|G_\lambda(z_0) - \xi_0 g| \leqslant \xi_0 \varepsilon(\lambda) + (q + \varepsilon_0(\sigma, \lambda))\, |z_0 - \xi_0 g|,$$

where $\varepsilon(\lambda) = |G_\lambda(g) - G_{\lambda_0}(g)| \to 0$ as $\lambda \to \lambda_0$. By (14), $|Q_\lambda(z_0)| \leqslant M_\delta$, hence

$$\begin{aligned}&|U_1^\lambda(z_0) - \xi_0 g|\\ &\qquad \leqslant |G_\lambda(z_0) - \xi_0 g| + M_\delta \leqslant \xi_0 \varepsilon(\lambda) + (q + \varepsilon_0(\sigma, \lambda))\, |z_0 - \xi_0 g| + M_\delta\end{aligned}$$

for every $|\lambda - \lambda_0| \leqslant \delta$. But $\xi_0 \geqslant \rho$, $|z_0 - \xi_0 g| \leqslant \sigma \xi_0$, consequently

$$\begin{aligned}&|\xi_0^{-1}\, U_1^\lambda(z_0) - g|\\ &\qquad \leqslant \varepsilon(\lambda) + (q + \varepsilon_0(\sigma, \lambda))\, \sigma + \rho^{-1} M_\delta, \qquad z_0 \in K_\sigma(\rho), \quad |\lambda - \lambda_0| < \delta. \end{aligned} \tag{26}$$

Now note that the ball $B_\sigma = \{z_0 \in \Re^\ell : |z_0 - g| \leqslant \sigma\}$ is contained in K_σ for each $\sigma \leqslant 1$. By (26), given any small σ, there is a sufficiently large $\rho = \rho(\sigma)$ and a $\delta = \delta(\sigma)$ such that for all $z_0 \in K_\sigma(\rho)$, $|\lambda - \lambda_0| < \delta$ the inclusion $\xi_0^{-1} U_1^\lambda(z_0) \in B_\sigma$ holds and hence $U_1^\lambda(z_0) \in K_\sigma$. Also note that $\xi_0^{-1}(2Dz_0 - U_1^\lambda(z_0)) - g = g - \xi_0^{-1}\, U_1^\lambda(z_0)$. Therefore $\xi_0^{-1} U_1^\lambda(z_0) \in B_\sigma$ gives $\xi_0^{-1}(2Dz_0 - U_1^\lambda(z_0)) \in B_\sigma$ and so $2Dz_0 - U_1^\lambda(z_0) \in K_\sigma$. Further, since $(2D - I)\, K_\sigma = K_\sigma$, it follows that $U_{-1}^\lambda(z_0) = (2D - I)(2Dz_0 - U_1^\lambda(z_0)) \in K_\sigma$. Thus, each of the operators U_1^λ, U_{-1}^λ maps the set $K_\sigma(\rho)$ into the cone K_σ whenever $|\lambda - \lambda_0| < \delta$.

By Lemma 2, $g_\lambda \to g$ as $\lambda \to \lambda_0$. Therefore, $g_\lambda > 0$ for λ close to λ_0. Moreover, for any σ, $\rho > 0$ we have $xg_\lambda \in K_\sigma(\rho)$ whenever x is large enough and $|\lambda - \lambda_0|$ is sufficiently small. So the first of the estimates (15) gives $Q_\lambda(xg_\lambda) \to d$ as $x \to \infty$, $\lambda \to \lambda_0$. Equivalently,

$$\kappa^{-1} Q_\lambda(xg_\lambda) = \kappa^{-1}(U_1^\lambda(xg_\lambda) - G_\lambda(xg_\lambda)) = \kappa^{-1}(U_1^\lambda(xg_\lambda) - \beta_\lambda x g_\lambda) \to \kappa^{-1} d.$$

But $\kappa^{-1} d > 0$. Therefore there are x_0, $\delta > 0$ such that $\kappa^{-1}(U_1^\lambda(xg_\lambda) - \beta_\lambda x g_\lambda) \geqslant 0$, that is $\kappa^{-1}(U_1^\lambda(xg_\lambda) - xg_\lambda) \geqslant \kappa^{-1}(\beta_\lambda - 1)\, xg_\lambda$ for $x \geqslant x_0$, $|\lambda - \lambda_0| < \delta$. By Lemma 2, if $\lambda \in \Lambda_+^\delta$, then $(\beta_\lambda - 1)\, \kappa \geqslant 0$. Hence $\kappa^{-1}(\beta_\lambda - 1)\, xg_\lambda \geqslant 0$ and so

$$\kappa^{-1}(U_1^\lambda(xg_\lambda) - xg_\lambda) \geqslant 0, \qquad x \geqslant x_0, \quad |\lambda - \lambda_0| < \delta. \tag{27}$$

Note that $U^{\lambda}_{-1}(z_0)-z_0=(2D-I)(z_0-U^{\lambda}_1(z_0))$, $z_0\in\mathfrak{R}^{\ell}_+$. But $(2D-I)K_{\sigma_0}=K_{\sigma_0}$, whence

$$\begin{aligned} U^{\lambda}_1(z_0)\leqslant z_0 \quad &\text{gives} \quad U^{\lambda}_{-1}(z_0)\geqslant z_0,\\ U^{\lambda}_1(z_0)\geqslant z_0 \quad &\text{gives} \quad U^{\lambda}_{-1}(z_0)\leqslant z_0, \end{aligned} \tag{28}$$

for every $z_0\in\mathfrak{R}^{\ell}_+$. Therefore, (27) implies that

$$U^{\lambda}_k(xg_\lambda)\geqslant xg_\lambda, \qquad x\geqslant x_0, \quad \lambda\in\Lambda^{\delta}_+. \tag{29}$$

Take σ_1,ρ_1, with $\sigma_1<\sigma_0$, such that the conclusions of Theorem 3 hold. Without loss of generality, assume that $g_\lambda\in K_{\sigma_1}$, $|\lambda-\lambda_0|<\delta$. Let $z_0\in K_{\sigma_1}(\rho_1)$, $\lambda\in\Lambda^{\delta}_+$. Since $\sigma_1<\sigma_0$, then $K_{\sigma_1}(\rho_1)\subset\operatorname{int}K_{\sigma_0}$ and hence $z_0>0$. Also, $g_\lambda>0$, and so there are positive numbers θ_1,θ_2 such that $0<\theta_1 g_\lambda\leqslant z_0\leqslant\theta_2 g_\lambda$. But K_{σ_0} is a closed set, so the number $\theta_0=\max\{\theta>0: z_0\geqslant\theta g_\lambda\}$ is well defined. Take x large enough for the inclusion $xg_\lambda\in K_{\sigma_1}(\rho_1)$ and the estimates $x>x_0$, $x>\theta_0$ to hold. By Theorem 3, the operator U^{λ}_k is strictly concave and monotone on $K_{\sigma_1}(\rho_1)$. Hence the estimates $x\theta_0^{-1}>1$ and $x\theta_0^{-1}z_0\geqslant xg_\lambda$ imply that $x\theta_0^{-1}U^{\lambda}_k(z_0)>U^{\lambda}_k(x\theta_0^{-1}z_0)$ and $U^{\lambda}_k(x\theta_0^{-1}z_0)\geqslant U^{\lambda}_k(xg_\lambda)$. Combining this with (29), we obtain $x\theta_0^{-1}U^{\lambda}_k(z_0)>xg_\lambda$. Therefore, $\theta_0 g_\lambda<U^{\lambda}_k(z_0)$, that is $U^{\lambda}_k(z_0)-\theta_0 g_\lambda\in\operatorname{int}K_{\sigma_0}$. At the same time, by definition of θ_0, the point $z_0-\theta_0 g_\lambda$ lies on the boundary of K_{σ_0}, hence $z_0\neq U^{\lambda}_k(z_0)$. This means that the operator U^{λ}_k does not have fixed points in $K_{\sigma_1}(\rho_1)$ whenever $\lambda\in\Lambda^{\delta}_+$.

Now let $\lambda\in\Lambda^{\delta}_-$. Recall that $\xi(\kappa^{-1}d)=1$. Hence there are numbers c_1,c_2 such that

$$0<c_1<1<c_2, \qquad c_1 g<\kappa^{-1}d<c_2 g. \tag{30}$$

By Lemma 2, $(1-\beta_\lambda)\kappa>0$, $\lambda\in\Lambda^{\delta}_-$. Set $\alpha_i(\lambda)=(1-\beta_\lambda)^{-1}\kappa c_i$, $i=1,2$. Since $\beta_\lambda\to1$, then $\alpha_i(\lambda)\to\infty$ as $\lambda\to\lambda_0$, $\lambda\in\Lambda^{\delta}_-$, $i=1,2$. So the first of the estimates (15) gives $Q_\lambda(\alpha_i(\lambda)\,g_\lambda)\to d$. It follows from the equalities

$$Q_\lambda(\alpha_i(\lambda)\,g_\lambda)=U^{\lambda}_1(\alpha_i(\lambda)\,g_\lambda)-G_\lambda(\alpha_i(\lambda)\,g_\lambda)=U^{\lambda}_1(\alpha_i(\lambda)\,g_\lambda)-\beta_\lambda\alpha_i(\lambda)\,g_\lambda$$

that $U^{\lambda}_1(\alpha_i(\lambda)\,g_\lambda)-\alpha_i(\lambda)\,g_\lambda=Q_\lambda(\alpha_i(\lambda)\,g_\lambda)-\kappa c_i g_\lambda$ and therefore that

$$\kappa^{-1}[U^{\lambda}_1(\alpha_i(\lambda)\,g_\lambda)-\alpha_i(\lambda)\,g_\lambda]\to\kappa^{-1}d-c_i g \qquad \text{as} \quad \lambda\to\lambda_0,\ \lambda\in\Lambda^{\delta}_-.$$

Hence (30) gives $\kappa^{-1}[U^{\lambda}_1(\alpha_1(\lambda)\,g_\lambda)-\alpha_1(\lambda)\,g_\lambda]\geqslant0$, while $\kappa^{-1}[U^{\lambda}_1(\alpha_2(\lambda)\,g_\lambda)-\alpha_2(\lambda)\,g_\lambda]\leqslant0$ for every $\lambda\in\Lambda^{\delta}_-$ close to λ_0. So using (28), we obtain

$$\alpha_1(\lambda)\,g_\lambda\leqslant U^{\lambda}_k(\alpha_1(\lambda)\,g_\lambda), \qquad U^{\lambda}_k(\alpha_2(\lambda)\,g_\lambda)\leqslant\alpha_2(\lambda)\,g_\lambda.$$

However, given any small $\sigma > 0$, there are $\rho, \delta > 0$ such that $U_k^\lambda K_\sigma(\rho) \subseteq K_\sigma$, $|\lambda - \lambda_0| < \delta$. Take $\sigma \in (0, \sigma_1)$ and $\rho \in (\rho_1, \infty)$ for this inclusion to hold. Denote by Ω_λ the intersection of the cone K_σ with the conical interval

$$\langle \alpha_1(\lambda)\, g_\lambda; \alpha_2(\lambda)\, g_\lambda \rangle = \{ z_0 \in \Re^\ell : \alpha_1(\lambda)\, g_\lambda \leqslant z_0 \leqslant \alpha_2(\lambda)\, g_\lambda \}.$$

By definition, Ω_λ is a convex compact set. Note that

$$\alpha_1(\lambda)\, \xi(g_\lambda) \leqslant \xi(z_0) \leqslant \alpha_2(\lambda)\, \xi(g_\lambda) \qquad \text{for every} \quad z_0 \in \langle \alpha_1(\lambda)\, g_\lambda; \alpha_2(\lambda)\, g_\lambda \rangle. \tag{31}$$

Therefore $\xi(z_0) \geqslant \rho$, $z_0 \in \Omega_\lambda$, whenever $\lambda \in \Lambda_-^\delta$ is sufficiently close to λ_0 and hence

$$\begin{aligned} \Omega_\lambda &= \Omega_\lambda \cap \{ z \in \Re^\ell : \xi(z_0) \geqslant \rho \} \\ &= \langle \alpha_1(\lambda)\, g_\lambda; \alpha_2(\lambda)\, g_\lambda \rangle \cap K_\sigma \cap \{ z_0 \in \Re^\ell : \xi(z_0) \geqslant \rho \}, \end{aligned}$$

that is, $\Omega_\lambda = \langle \alpha_1(\lambda)\, g_\lambda; \alpha_2(\lambda)\, g_\lambda \rangle \cap K_\sigma(\rho)$. Since the operator U_k^λ is monotone on the subset $K_\sigma(\rho)$ of $K_{\sigma_1}(\rho_1)$ and $\alpha_i(\lambda)\, g_\lambda \in K_\sigma(\rho)$, $i = 1, 2$ for $\lambda \in \Lambda_-^\delta$ close to λ_0, then

$$\begin{aligned} \alpha_1(\lambda)\, g_\lambda &\leqslant U_k^\lambda(\alpha_1(\lambda)\, g_\lambda) \leqslant U_k^\lambda(z_0) \\ &\leqslant U_k^\lambda(\alpha_2(\lambda)\, g_\lambda) \leqslant \alpha_2(\lambda)\, g_\lambda, \qquad z_0 \in \Omega_\lambda, \end{aligned}$$

which means that $U_k^\lambda \Omega_\lambda \subset \langle \alpha_1(\lambda)\, g_\lambda; \alpha_2(\lambda)\, g_\lambda \rangle$. But $U_k^\lambda \Omega_\lambda \subseteq U_k^\lambda K_\sigma(\rho) \subseteq K_\sigma$, so the operator U_k^λ maps the set Ω_λ into itself. By the Brouwer fixed point principle, the equation $z_0 = U_k^\lambda(z_0)$ has a solution $z_*(\lambda)$ in Ω_λ.

By (31), $c_1 \leqslant (1 - \beta_\lambda)\, \kappa^{-1} \xi^{-1}(g_\lambda)\, \xi_*(\lambda) \leqslant c_2$, where $\xi_*(\lambda) = \xi(z_*(\lambda))$. Recall that the partial order in $\Re^\ell$ is generated by the cone K_{σ_0} with σ_0 fixed, but arbitrarily large. Clearly, given $\varepsilon > 0$, there is a σ_0 such that the estimates (30) hold for $c_1 = 1 - \varepsilon$, $c_2 = 1 + \varepsilon$. Therefore, $|(1 - \beta_\lambda)\, \kappa^{-1} \xi^{-1}(g_\lambda)\, \xi_*(\lambda) - 1| \leqslant \varepsilon$ for all $\lambda \in \Lambda_-^\delta$ if δ is sufficiently small. Since $\xi(g_\lambda) \to 1$ as $\lambda \to \lambda_0$, this gives the first of equations (21). Also, for every given $\sigma > 0$, we have $z_*(\lambda) \in K_\sigma$ and so $|\xi_*^{-1}(\lambda)\, z_*(\lambda) - g| \leqslant \sigma$ whenever $\lambda \in \Lambda_-^\delta$ is close to λ_0, which implies the second of Eqs. (21).

Finally, take $\delta > 0$ small enough for $z_*(\lambda) \in \operatorname{int} K_{\sigma_1}(\rho_1)$ to hold for every $\lambda \in \Lambda_-^\delta$. Clearly, given a $\lambda \in \Lambda_-^\delta$ and a $\varepsilon > 0$, there is a θ close to 1, $\theta > 1$, such that the conical interval $\langle \theta^{-1} z_*(\lambda), \theta z_*(\lambda) \rangle = \{ z_0 \in \Re^\ell : \theta^{-1} z_*(\lambda) \leqslant z_0 \leqslant \theta z_*(\lambda) \}$ is included in the neighborhood $\{ z_0 \in \Re^\ell : |z_0 - z_*(\lambda)| < \varepsilon \} \cap \operatorname{int} K_{\sigma_1}(\rho_1)$ of the point $z_*(\lambda)$. Since the operator U_k^λ is strictly concave on $K_{\sigma_1}(\rho_1)$, then

$$\begin{gathered} U_k^\lambda(\theta z_*(\lambda)) < \theta U_k^\lambda(z_*(\lambda)) = \theta z_*(\lambda), \\ U_k^\lambda(\theta^{-1} z_*(\lambda)) > \theta^{-1} U_k^\lambda(z_*(\lambda)) = \theta^{-1} z_*(\lambda). \end{gathered}$$

Besides, U_k^λ is monotone on $K_{\sigma_1}(\rho_1)$, so for each $z_0 \in \langle \theta^{-1} z_*(\lambda), \theta z_*(\lambda) \rangle$ we have

$$\theta^{-1} z_*(\lambda) < U^\lambda(\theta^{-1} z_*(\lambda)) \leqslant U_k^\lambda(z_0) \leqslant U_k^\lambda(\theta z_*(\lambda)) < \theta z_*(\lambda),$$

that is the operator U_k^λ maps the conical interval $\langle \theta^{-1} z_*(\lambda), \theta z_*(\lambda) \rangle$ into its interior. By continuity, every operator $U_1^{\lambda_1}$, where $\lambda_1 \in \Lambda_-^\delta$ is sufficiently close to λ, maps this conical interval into itself. Therefore a unique fixed point $z_*(\lambda_1)$ of $U_k^{\lambda_1}$ in the set $K_{\sigma_1}(\rho_1)$ satisfies $z_*(\lambda_1) \in \langle \theta^{-1} z_*(\lambda), \theta z_*(\lambda) \rangle \subset \{ z_0 \in \Re^\ell : |z_0 - z_*(\lambda)| < \varepsilon \}$, which proves the continuity of the curve $z_*(\lambda)$, $\lambda \in \Lambda_-^\delta$, and completes the proof of Theorem 4. ∎

3.5. *Proof of Theorems* 1, 2

Set $w_*(t) = g \cos t + h \sin t$, $t \in \Re$. That is, $w_*(t)$ is a 2π-periodic solution of the system $\dot z = A(\lambda_0) z$; $\|w_*\|_C = \sqrt{2}$. It can be easily shown that for every large-amplitude periodic solution $z(t; \lambda)$ of the system (3) there is a number $\varphi \in [0, 2\pi)$ such that

$$\|\sqrt{2} r^{-1} z(t; \lambda) - w_*(2\pi t/T + \varphi)\|_C < \varepsilon_1(r, \lambda), \qquad |T - 2\pi| < \varepsilon_2(r, \lambda), \tag{32}$$

where T is the period of the solution, $r = \|z(t; \lambda)\|_C$, and $\varepsilon_i(r, \lambda) \to 0$ as $r \to \infty$, $\lambda \to \lambda_0$, $i = 1, 2$. Denote by Γ_σ the arc $w_*(s)$, $-\arctan \sigma \leqslant s \leqslant \arctan \sigma$, of the ellipse w_*. If $\sigma \in (0, \sigma_1)$ is fixed, then $\Gamma_\sigma \subset \operatorname{int} K_{\sigma_1}$. By (8), the functional $\tau(\cdot)$ maps the arc Γ_σ onto the segment $[2\pi - \sigma, 2\pi + \sigma]$. So the *a priori* estimates (32) imply that the equation $\tau(z_0) = T$ has a solution $z_T \in \sqrt{2} r^{-1} z(t; \lambda) \cap \operatorname{int} K_{\sigma_1}$ whenever the numbers r^{-1} and $|\lambda - \lambda_0|$ are sufficiently small. Therefore the point $z_* = r z_T / \sqrt{2}$ lies on the T-periodic cycle $z(t; \lambda)$. Since $\tau(z_*) = \tau(z_T) = T$, it follows that z_* is a fixed point of the translation operator U_1^λ. Now note that $z_T \in K_{\sigma_1}$ gives $z_* \in K_{\sigma_1}$, which implies $|z_* - \xi(z_*) g| \leqslant \sigma_1 \xi(z_*)$ and hence $(1 + \sigma_1)^{-1} |z_*| \leqslant \xi(z_*) \leqslant |z_*|$. Also, by (32), $\sqrt{2}\, r^{-1} \min_t |z(t; \lambda)| \geqslant \underline{w} - \varepsilon_1(r, \lambda)$, where $\underline{w} = \min_t |w_*(t)| > 0$; therefore

$$\begin{aligned}\xi(z_*) &\geqslant (1 + \sigma_1)^{-1} |z_*| \geqslant (1 + \sigma_1)^{-1} \min_t |z(t; \lambda)| \\ &\geqslant r(\sqrt{2}\,(1 + \sigma_1))^{-1} (\underline{w} - \varepsilon_1(r, \lambda)).\end{aligned}$$

It follows that $\xi(z_*) \geqslant \rho_1$ and so $z_* \in K_{\sigma_1}(\rho_1)$ if r is sufficiently large and λ is close to λ_0. Thus, there are numbers $r_0, \delta > 0$ such that every r_0-large periodic cycle $z(t; \lambda)$, $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, of the system (3) intersects the set $K_{\sigma_1}(\rho_1)$ and moreover, the intersection contains a fixed point of the operator U_1^λ.

If $\lambda \in \Lambda_+^\delta$, then by Theorem 4, there are no fixed points of the operator U_1^λ in $K_{\sigma_1}(\rho_1)$ and hence the system (3) has no r_0-large periodic cycles.

If $\lambda \in \Lambda_{-}^{\delta}$, then by Theorems 3 and 4, the translation operator U_1^{λ} has a fixed point $z_*(\lambda)$ which is unique in $K_{\sigma_1}(\rho_1)$, depends continuously on λ, and satisfies (21). Therefore the solution $z_*(t; \lambda)$ originating from the point $z_*(\lambda)$ is a $\tau(z_*(\lambda))$-periodic cycle of the system (3). Moreover, this cycle depends continuously on λ. Also, the first of equations (21) gives $\xi_*(\lambda) \to \infty$ as $\lambda \to \lambda_0$. It follows from $\|z_*(t; \lambda)\|_C \geqslant |z_*(\lambda)| \geqslant \xi_*(\lambda)$ that $\|z_*(t; \lambda)\|_C \to \infty$ as $\lambda \to \lambda_0$. So $\|z_*(t; \lambda)\|_C \geqslant r_0$, $\lambda \in \Lambda_{-}^{\delta}$, if δ is sufficiently small. The uniqueness of the fixed point $z_*(\lambda)$ in $K_{\sigma_1}(\rho_1)$ implies that $z_*(t; \lambda)$ is a unique r_0-large cycle for each $\lambda \in \Lambda_{-}^{\delta}$, which completes the proof of Theorem 1. Finally, if $\kappa > 0$, then the fixed point $z_*(\lambda)$ of the translation operator U_1^{λ} is asymptotically stable and hence the cycle $z_*(t; \lambda)$ is orbitally asymptotically stable. If $\kappa < 0$, the fixed point $z_*(\lambda)$ is unstable, therefore the cycle $z_*(t; \lambda)$ is orbitally unstable. This proves Theorem 2.

Note that Eqs. (21) imply the asymptotic formula

$$\|(1-\beta_\lambda)\, z_*(t; \lambda) - \kappa w_*(t)\|_C \to 0 \qquad \text{as} \qquad \lambda \to \lambda_0,\ \lambda \in \Lambda_{-}^{\delta}$$

for the periodic cycle $z_*(t; \lambda)$ of the system (3).

4. SYSTEMS WITH HYSTERESIS NONLINEARITIES

4.1. *System with* `stop`

In this section we study system of the form (2) with a simple hysteresis nonlinearity called `stop` [14] in the feedback. The inputs of `stop` are arbitrary continuous scalar-valued functions $x(t)$, $t \geqslant t_0$. The `stop` state space is the interval $[-1, 1]$. For every given initial state v_0 and input $x(t)$, $t \geqslant t_0$, the input–state operator

$$v(t) = S[t_0, v_0]\, x(t) \tag{33}$$

determines the state of `stop` at each instant $t \geqslant t_0$. The function $v(t)$, $t \geqslant t_0$ is also the output of `stop`. For monotone inputs

$$S[t_0, v_0]\, x(t) = \begin{cases} \min\{1,\ v_0 + x(t) - x(t_0)\} & \text{for } x(t) \text{ nondecreasing,} \\ \max\{-1,\ v_0 + x(t) - x(t_0)\} & \text{for } x(t) \text{ nonincreasing.} \end{cases} \tag{34}$$

For each locally piecewise monotone input the output is defined using the semigroup property

$$S[t_0, v_0]\, x(t) = S[\tau, S[t_0, v_0]\, x(\tau)]\, x(t), \qquad t_0 \leqslant \tau \leqslant t.$$

Furthermore, it can be shown that for every pair of such inputs the outputs satisfy

$$\max_{[t_0,t]} |S[t_0,v_0]\,x_1(\tau)-S[t_0,v_0]\,x_2(\tau)| \leqslant 2 \max_{[t_0,t]} |x_1(\tau)-x_2(\tau)|, \qquad t\geqslant t_0. \tag{35}$$

Then, for any continuous input $x(t)$, $t\geqslant t_0$, the output (33) is defined by

$$S[t_0,v_0]\,x(t)=\lim_{n\to\infty} S[t_0,v_0]\,x_n(t), \qquad t\geqslant t_0,$$

where $x_n(\cdot)$ is an arbitrary sequence of locally piecewise monotone inputs such that $\max\{|x_n(\tau)-x(\tau)| : t_0\leqslant\tau\leqslant t\}\to 0$ for every $t\geqslant t_0$. By construction, the function (33) is continuous. The estimate (35) holds for every pair of inputs. Further details can be found in [14].

Consider the equation

$$L_\lambda\left(\frac{d}{dt}\right)x(t)=M_\lambda\left(\frac{d}{dt}\right)S[0,v(0)]\,x(t)$$

and the equivalent autonomous system

$$\frac{dz}{dt}=A(\lambda)\,z+\gamma(\lambda)\,v(t), \qquad v(t)=S[0,v(0)](c^Tz(t)), \tag{36}$$

where $t\geqslant 0$. It is supposed that the conditions (i)–(ii) hold for the system. The phase space of the system (36) is $\Re^\ell\times[-1,1]$. For every initial condition

$$z(0)=z_0, \qquad v(0)=v_0 \tag{37}$$

the Cauchy problem (36)–(37) has a unique solution

$$\{z(t),v(t)\}=\{z(t;z_0,v_0,\lambda),v(t;z_0,v_0,\lambda)\}$$

on the whole semiaxis $t\geqslant 0$. This solution depends continuously on the initial data $z_0\in\Re^\ell$, $v_0\in[-1,1]$, and the parameter λ. Moreover, it is globally Lipschitz in z_0, v_0.

4.2. *Existence and Stability of Large Cycles*

The properties of the superposition operator $\mathscr{F}$ summarized in Lemma 1 turn out to be common for some classes of functional and hysteresis nonlinearities. In particular, this is the case for the input–output operator (33) of `stop`.

LEMMA 4. *For every* $x, y \in \mathcal{K}_{\mu,r}(u_*)$ *and every* $v_0 \in [-1,1]$, $\theta > 1$

$$\|S[0, v_0]\, x - \operatorname{sgn} \dot{u}_* \|_{L_1} < \varepsilon_1(\mu, r),$$

$$\|S[0, v_0](\theta x) - S[0, v_0]\, x\|_{L_1} < \varepsilon_2(\mu, r)(\theta - 1), \tag{38}$$

$$\|S[0, v_0]\, x - S[0, v_0]\, y\|_{L_1} < \varepsilon_3(\mu, r)\, \|x - y\|_C. \tag{39}$$

In addition, for $t \in [0, 5\pi/2]$,

$$S[0, v_0]\, x(t) = \operatorname{sgn} \dot{u}_*(t) \qquad \textit{whenever} \quad |t - \pi n/2| > \varepsilon_4(\mu, r),$$
$$n = 0, 1, 3, 5. \tag{40}$$

Here, each $\varepsilon_k(\mu, r) \to 0$ *as* $\mu \to 0$, $r \to \infty$.

Proof. Take an arbitrarily small $\varepsilon > 0$. Since

$$\dot{u}_*(t) > 0 \qquad \text{if} \quad |t - 2\pi n| < \pi/2,$$
$$\dot{u}_*(t) < 0 \qquad \text{if} \quad |t - \pi - 2\pi n| < \pi/2,$$

where $n = 0, 1$, then there are numbers $\alpha = \alpha(\varepsilon) > 0$ and $\mu = \mu(\varepsilon) > 0$ such that for every function y from the ball $B_\mu(u_*) = \{ y \in C^1 : \|y - u_*\|_{C^1} \leqslant \mu\}$ the estimates

$$\dot{y}(t) \geqslant \alpha \qquad \text{if} \qquad |t - 2\pi n| \leqslant \pi/2 - \varepsilon,$$
$$\dot{y}(t) \leqslant -\alpha \qquad \text{if} \quad |t - \pi - 2\pi n| \leqslant \pi/2 - \varepsilon$$

hold. Let $x \in \mathcal{K}_{\mu,r}(u_*)$, $r \geqslant 2/(\alpha\varepsilon)$. Then $x = \rho y$, where $y \in B_\mu(u_*)$, $\rho \geqslant 2/(\alpha\varepsilon)$. Hence,

$$\begin{aligned} \dot{x}(t) &\geqslant 2/\varepsilon \qquad \text{for} \quad t \in [0, t_1 - \varepsilon] \cup [t_3 + \varepsilon, t_5 - \varepsilon], \\ \dot{x}(t) &\leqslant -2/\varepsilon \qquad \text{for} \quad t \in [t_1 + \varepsilon, t_3 - \varepsilon], \end{aligned} \tag{41}$$

where $t_n = \pi n/2$. Now take an arbitrary $v_0 \in [-1, 1]$ and consider the `stop` output $v(t) = S[0, v_0]\, x(t)$, $t \geqslant 0$. By the first of the estimates (41), the function $x(t)$ increases on the segment $[0, t_1 - \varepsilon]$. Moreover, $x(t) - x(0) + v_0 \geqslant 2\varepsilon^{-1} t + v_0 \geqslant 2\varepsilon^{-1} t - 1 \geqslant 1$ for $t \in [\varepsilon, t_1 - \varepsilon]$, so (34) gives $v(t) \equiv 1$ on the segment $t \in [\varepsilon, t_1 - \varepsilon]$. Similarly, $v(t) = S[t_3 + \varepsilon, v(t_3 + \varepsilon)]\, x(t) \equiv 1$ for all $t \in [t_3 + 2\varepsilon, t_5 - \varepsilon]$. On the other hand, by the second of the estimates (41), the input $x(t)$ decreases on the segment $[t_1 + \varepsilon, t_3 - \varepsilon]$. Also, $x(t) - x(0) + v_0 \leqslant -1$ and hence $v(t) \equiv -1$ for every $t \in [t_1 + 2\varepsilon, t_3 - \varepsilon]$. Thus,

$$v(t) \equiv 1 \qquad \text{for} \quad t \in [\varepsilon, t_1 - \varepsilon] \cup [t_3 + 2\varepsilon, t_5 - \varepsilon],$$
$$v(t) \equiv -1 \qquad \text{for} \quad t \in [t_1 + 2\varepsilon, t_3 - \varepsilon].$$

Equivalently,

$$S[0, v_0]\, x(t) = \operatorname{sgn} \dot{u}_*(t),$$

$$t \in [\varepsilon, t_1 - \varepsilon] \cup [t_1 + 2\varepsilon, t_3 - \varepsilon] \cup [t_3 + 2\varepsilon, t_5 - \varepsilon]$$

for every $x \in K_{\mu, r}(u_*)$, $v_0 \in [-1, 1]$, which proves (40).

Equation (40) immediately implies the first of the estimates (38). Further, combining (35) with (40) gives the estimate (39).

Finally, take $b > 0$, $v^0 \in [-b, b]$ and consider the operator defined by

$$S_b[t_0, v^0]\, x(t) = bS[t_0, b^{-1}v^0](b^{-1}x(t)), \qquad t \geqslant t_0,$$

acting in the space of continuous functions $x(t)$, $t \geqslant t_0$. If $b = 1$, it is the `stop` input–output operator (33). An important fact (see, for example, [14]) is that the estimate

$$|S_{b_1}[t_0, v_1^0]\, x(t) - S_{b_2}[t_0, v_2^0]\, x(t)| \leqslant |b_1 - b_2|, \qquad t \geqslant t_0$$

holds for any input x whenever $|v_1^0 - v_2^0| \leqslant |b_1 - b_2|$. In particular,

$$|S[0, v_0](\theta x(t)) - S_\theta[0, \theta v_0](\theta x(t))| \leqslant \theta - 1, \qquad t \geqslant 0,\ \theta > 1$$

for every $x \in C$, $v_0 \in [-1, 1]$. Equivalently, $|S[0, v_0](\theta x(t)) - \theta S[0, v_0]\, x(t)| \leqslant \theta - 1$, $t \geqslant 0$. But $\|S[0, v_0]\, x(t)\|_C \leqslant 1$, hence

$$\|S[0, v_0](\theta x) - S[0, v_0]\, x\|_C \leqslant 2(\theta - 1), \qquad x \in C,\ v_0 \in [-1, 1]. \tag{42}$$

In addition, if $x \in K_{\mu, r}(u_*)$, then $\theta x \in K_{\mu, r}(u_*)$. By (40), $S[0, v_0](\theta x(t)) = S[0, v_0]\, x(t)$ for every $t \in [0, 5\pi/2]$ such that $|t - \pi n/2| > \varepsilon_4(\mu, r)$, $n = 0, 1, 3, 5$. So (42) gives the second of the estimates (38) and the proof is complete. ∎

Only slight modifications to the methods of Section 2 are required to prove the analogs of Theorems 1 and 2 for the system (36). Let $\{z_*(t; \lambda), v_*(t; \lambda)\}$ be a periodic solution of this system. The same notation is used for the cycle in the phase space $\mathfrak{R}^\ell \times [-1, 1]$. Put

$$\begin{aligned} &\chi(\{z_0, v_0\}, \{z_*(\cdot\,; \lambda), v_*(\cdot\,; \lambda)\}) \\ &\qquad = \min\{|z_0 - z_*(\tau; \lambda)| + |v_0 - v_*(\tau; \lambda)| : \tau \in \mathfrak{R}\}. \end{aligned}$$

The solution $\{z_*(t;\lambda), v_*(t;\lambda)\}$ is called orbitally stable if for every $\varepsilon > 0$ there is a $\delta > 0$ such that the estimate $\chi(\{z_0, v_0\}, \{z_*(\cdot;\lambda), v_*(\cdot;\lambda)\}) < \delta$ implies

$$\chi(\{z(t; z_0, v_0, \lambda), v(t; z_0, v_0, \lambda)\}, \{z_*(\cdot;\lambda), v_*(\cdot;\lambda)\}) < \varepsilon \qquad \text{for every} \quad t \geqslant 0.$$

The definition of orbital instability of a solution follows in the usual way.

An orbitally stable periodic solution (a cycle) $\{z_*(t;\lambda), v_*(t;\lambda)\}$ is called orbitally asymptotically stable if there exists a $a > 0$ such that

$$\chi(\{z(t; z_0, v_0, \lambda), v(t; z_0, v_0, \lambda)\}, \{z_*(\cdot;\lambda), v_*(\cdot;\lambda)\}) \to 0 \qquad \text{as} \quad t \to \infty$$

whenever $\chi(\{z_0, v_0\}, \{z_*(\cdot;\lambda), v_*(\cdot;\lambda)\}) < a$. We say that a periodic cycle is r_0-large if its first component is r_0-large, that is $\|z_*(\cdot;\lambda)\|_C \geqslant r_0$.

Set

$$\tilde{\kappa} = 4c^T PA(\lambda_0)\,\gamma(\lambda_0).$$

A direct calculation gives $\tilde{\kappa} = \xi(\tilde{d})$, where

$$\tilde{d} = \int_0^{2\pi} e^{A(\lambda_0)(2\pi - s)}\gamma(\lambda_0)\,\mathrm{sgn}\,\dot{u}_*(s)\,ds.$$

In this section, the number $\tilde{\kappa}$ plays the same role as does κ in Section 2. Suppose $\tilde{\kappa} \neq 0$. Here the same notations Λ^δ_+ and Λ^δ_- are used respectively for the sets $\{\lambda \in \mathfrak{R} : v(\lambda)\,\tilde{\kappa} \geqslant 0, |\lambda - \lambda_0| < \delta\}$ and $\{\lambda \in \mathfrak{R} : v(\lambda)\,\tilde{\kappa} < 0, |\lambda - \lambda_0| < \delta\}$ as in (4).

THEOREM 5. *There exist $r_0 > 0$ and $\delta > 0$ such that for every $\lambda \in \Lambda^\delta_+$ the system* (36) *has no r_0-large periodic cycles, while for every $\lambda \in \Lambda^\delta_-$ the system has a unique r_0-large periodic cycle $\{z_*(t;\lambda), v_*(t;\lambda)\}$. This cycle depends continuously on λ and $\|z_*(\cdot;\lambda)\|_C \to \infty$ as $\lambda \to \lambda_0$, $\lambda \in \Lambda^\delta_-$. Moreover, if $\tilde{\kappa} > 0$ the cycle is orbitally asymptotically stable. If $\tilde{\kappa} < 0$ the cycle is orbitally unstable.*

4.3. *Proof of Theorem* 5

First, consider the problem (36)–(37) with fixed initial value $v_0 = 1$ of the `stop` state. Define the operator $W_1^\lambda : \mathfrak{R}^\ell_+ \to \mathfrak{R}^\ell$ by

$$W_1^\lambda(z_0) = z(\tau(z_0); z_0, 1, \lambda), \qquad z_0 \in \mathfrak{R}^\ell_+.$$

Then $W_1^\lambda = G_\lambda + \tilde{Q}_\lambda$, where G_λ is the operator (9) and

$$\tilde{Q}_\lambda(z_0) = \int_0^{\tau(z_0)} e^{A(\lambda)(\tau(z_0)-s)}\gamma(\lambda)\, S[0,1](c^T z(s; z_0, 1, \lambda))\, ds.$$

It was proved in Section 2 that the estimates (5), (6) of Lemma 1 imply the estimates (15), (16) of Lemma 3. In much the same way, Lemma 4 implies the analog of Lemma 3 for the operator $\tilde{Q}_\lambda$. More precisely, the estimates (15), (16), where Q_λ is replaced with $\tilde{Q}_\lambda$ and d is replaced with $\tilde{d}$, hold for every $z_0, z_{01}, z_{02} \in K_\sigma(\rho)$, $\theta > 1$. This leads to the analogs of Theorems 3 and 4 for the operators W_1^λ and $W_{-1}^\lambda = (2D - I)(2D - W_1^\lambda)$. To obtain the exact statements, it suffices to replace U_i^λ with W_i^λ for $i = \pm 1$ and to take $k = \operatorname{sgn} \tilde{\kappa}$ in place of $k = \operatorname{sgn} \kappa$ in the formulations of Theorems 3, 4. Therefore, there are $\delta, \sigma_1, \rho_1 > 0$ such that the operator W_1^λ does not have fixed points in $K_{\sigma_1}(\rho_1)$ for $\lambda \in \Lambda_+^\delta$, while W_1^λ has a unique fixed point $z_*(\lambda)$ in $K_{\sigma_1}(\rho_1)$ for $\lambda \in \Lambda_-^\delta$. This fixed point depends continuously on λ and satisfies Eq. (21). Moreover, it is asymptotically stable if $\tilde{\kappa} > 0$, and unstable if $\tilde{\kappa} < 0$.

Take $\sigma \in (0, \sigma_1)$, $\rho \in (\rho_1, \infty)$ and consider the solution of the Cauchy problem (36), (37) for arbitrary $v_0 \in [-1, 1]$, $z_0 \in K_\sigma(\rho)$. Arguing as in the proof of Lemma 3, obtain the inclusion

$$c^T z(t; z_0, v_0, \lambda) \in \mathscr{K}_{\mu,\rho}(u_*) \quad \text{for every} \quad z_0 \in K_\sigma(\rho),$$
$$v_0 \in [-1, 1],\ |\lambda - \lambda_0| < \delta,$$

in similar fashion to (17). Here, $\mu = \mu(\sigma, \rho, \lambda) \to 0$ as $\sigma \to 0$, $\rho \to \infty$, $\lambda \to \lambda_0$. But $z_0 \in K_\sigma(\rho)$ implies that $|\tau(z_0) - 2\pi| \leqslant \arctan \sigma$. Therefore

$$|\tau(z_0) - \pi n/2| \geqslant |2\pi - \pi n/2| - |\tau(z_0) - 2\pi| \geqslant \pi/2 - \arctan \sigma, \qquad n = 0, 1, 3, 5.$$

If σ, ρ^{-1}, $|\lambda - \lambda_0|$ are sufficiently small, then $|\tau(z_0) - \pi n/2| > \varepsilon_4(r, \mu)$, $n = 0, 1, 3, 5$, and by (40), $v(\tau(z_0); z_0, v_0, \lambda) = S[0, v_0](c^T z(\tau(z_0); z_0, v_0, \lambda))$ $= \operatorname{sgn} \dot{u}_*(\tau(z_0)) = \operatorname{sgn} \cos \tau(z_0)$, so $\tau(z_0) \in (3\pi/2, 5\pi/2)$ gives

$$v(\tau(z_0); z_0, v_0, \lambda) = 1, \qquad z_0 \in K_\sigma(\rho),\ v_0 \in [-1, 1],\ |\lambda - \lambda_0| < \delta. \quad (43)$$

Let $\{z(t; \lambda), v(t; \lambda)\}$ be a r_0-large periodic cycle of the system (36) for $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$. Since the nonlinearity (33) is bounded, the *a priori* estimates (32) hold for the first component of the periodic cycle and for its period T. This implies that the curve $z(t; \lambda)$ contains a point $z^* \in K_\sigma(\rho)$ such that $\tau(z^*) = T$ if r_0 is large enough and δ is sufficiently small. Choose t' so that $z^* = z(t'; \lambda)$ and take $v^* = v(t'; \lambda)$, then by periodicity, $z(\tau(z^*); z^*, v^*, \lambda) = z^*$, $v(\tau(z^*); z^*, v^*, \lambda) = v^*$. Now (43) gives $v^* = 1$ and hence $z^* = z(\tau(z^*); z^*, 1, \lambda)$. This means that every r_0-large cycle of the system (36) contains the point $\{z^*, 1\}$ such that $z^* = W_1^\lambda(z^*) \in K_\sigma(\rho)$.

If $\lambda \in \Lambda_+^\delta$, then there are no fixed points of the operator W_1^λ in $K_\sigma(\rho) \subset K_{\sigma_1}(\rho_1)$ and therefore the system (36) has no r_0-large periodic cycles. If $\lambda \in \Lambda_-^\delta$ and λ is sufficiently close to λ_0, then Eqs. (21) imply that the fixed point $z_*(\lambda) \in K_{\sigma_1}(\rho_1)$ of the operator W_1^λ lies in $K_\sigma(\rho)$, that is $z_*(\lambda) = z(T_\lambda; z_*(\lambda), 1, \lambda) \in K_\sigma(\rho)$, where $T_\lambda = \tau(z_*(\lambda))$. Furthermore, (43) gives $v(T_\lambda; z_*(\lambda), 1, \lambda) = 1$ and hence

$$\{z_*(t; \lambda), v_*(t; \lambda)\} = \{z(t; z_*(\lambda), 1, \lambda), v(t; z_*(\lambda), 1, \lambda)\}, \qquad \lambda \in \Lambda_-^\delta,$$

is a T_λ-periodic cycle of the system (36). From continuity of $z_*(\lambda)$ it follows that this cycle depends continuously on λ. Equations (21) imply that $\|z_*(\cdot; \lambda)\|_C \to \infty$ as $\lambda \to \lambda_0$, $\lambda \in \lambda_-^\delta$. Also note that the cycles of the system (36) do not intersect each other. Since every r_0-large cycle contains a point $\{z_0^*, 1\}$, where $z_0^* = W_1^\lambda(z_0^*) \in K_\sigma(\rho)$, then uniqueness of the fixed point $z_*(\lambda)$ of the operator W_1^λ in $K_\sigma(\rho)$ implies that $\{z_*(t; \lambda), v_*(t; \lambda)\}$ is a unique r_0-large cycle for every $\lambda \in \Lambda_-^\delta$ close to λ_0.

Finally, consider the operator $H_\lambda : \mathfrak{R}_+^\ell \times [-1, 1] \to \mathfrak{R}^\ell \times [-1, 1]$ of translation along the trajectories of the system (36) by the time $\tau(z_0)$ of (8),

$$H_\lambda\{z_0, v_0\} = \{z(\tau(z_0); z_0, v_0, \lambda), v(\tau(z_0); z_0, v_0, \lambda)\},$$
$$z_0 \in \mathfrak{R}_+^\ell, \ v_0 \in [-1, 1].$$

Take arbitrary $z_0 \in K_\sigma(\rho)$, $v_0 \in [-1, 1]$ and consider the sequence of iterations $\{z_{n+1}, v_{n+1}\} = H_\lambda\{z_n, v_n\}$, $n = 0, 1, 2, \ldots$. If $z_n \in K_\sigma(\rho)$ for every n, then by (43),

$$v_1 = v_2 = \cdots = v_n = \cdots = 1,$$
$$z_{n+1} = z(\tau(z_n); z_n, 1, \lambda) = W_1^\lambda(z_n), \qquad n = 1, 2, \ldots$$

So $\{z_{n+1}, v_{n+1}\} = \{W_1^\lambda(z_n), 1\}$, $n = 1, 2, \ldots$. In addition, if $v_0 = 1$, then also $\{z_1, v_1\} = \{W_1^\lambda(z_0), 1\}$. Therefore, asymptotic stability of the fixed point $z_*(\lambda) \in \operatorname{int} K_\sigma(\rho)$ of the operator W_1^λ implies asymptotic stability of the fixed point $\{z_*(\lambda), 1\}$ of the translation operator H_λ. Consequently, the cycle $\{z_*(t; \lambda), v_*(t; \lambda)\}$, $\lambda \in \Lambda_-^\delta$ is orbitally asymptotically stable if $\tilde{\kappa} > 0$. On the other hand, if $\tilde{\kappa} < 0$, then the fixed point $z_*(\lambda) \in \operatorname{int} K_\sigma(\rho)$ of W_1^λ is unstable and so is the fixed point $\{z_*(\lambda), 1\}$ of the translation operator H_λ. Thus, the cycle $\{z_*(t; \lambda), v_*(t; \lambda)\}$, $\lambda \in \Lambda_-^\delta$ is orbitally unstable for $\tilde{\kappa} < 0$. This completes the proof of Theorem 5.

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